DEDEKIND DOMAINS AND GENERALIZATIONS

by

Mark A. Goddard

Submitted in Partial Fulfillment of the Requirements

for the Degree of

Master of Science

in the

Mathematics

Program

Samuel Floyd Barger 8/14/84

Advisor Date

Sally M. Hotchkiss August 30, 198

Dean of the Graduate School Date

Youngstown State University

August, 1984

ABSTRACT

DEDEKIND DOMAINS AND GENERALIZATIONS

Mark A. Goddard
Waster of Science
Youngstown State University, 1984

The unique factorization theorem for integers may in some ways be generalized to a commutative ring R and an R-module M. This paper considers, with varying degrees of success, attempts to factor submodules of M, ideals of R (considered as submodules of R) and principal ldeals of K.

If every submodule of M is finitely generated, then any submodule of M may be factored as an intersection of submodules of M. Under a number of restrictions some limited uniqueness results can be obtained but only under extreme conditions may the factorization be considered truly unique.

If R is a type of integral domain known as a Dedekind domain and only ideals of R are considered then unique factorization can be achieved. A Dedekind domain is an integral domain in which every ideal is finitely generated and in a sense invertible within the quotient field of R. Dedekind domains can be related to other domains such as principal ideal domains and GCD-domains but the important property of

Dedekind domains in this thesis is the fact that every ideal in a Dedekind domain can be expressed as a unique product of prime ideals.

Another less restrictive class of domains known as π-domains preserves the factorization property of Dedekind domains but only for principal ideals. Actually the generalization from Dedekind domains to π-domains is not a considerable jump since under certain conditions π-domains and Dedekind domains are equivalent. As a matter of fact if the factorization property is extended to ideals generated by two elements, then the domain is Dedekind.

ACKNOWLEDGEMENTS

I would like to thank Dr. S. Floyd Barger and Mr. James Poggione for their assistance in the development of this paper. Their opinions and suggestions were extremely helpful.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

The unique factorization theorem for integers guarantees that any positive integer greater than one may be factored as a unique product of prime numbers. As a generalization of this concept one might consider, for a commutative ring with identity R and an R-module M, under what conditions may a submodule Q of M be uniquely "factored". First the concept of factoring an R module must be defined. We might consider three possibilities: the expression of an R-module as a union, intersection or product of R-modules. In chapter II we consider the factorization of an R-module as an intersection of R-nodules. While the uniqueness results obtained are intriguing, they are not entirely satisfactory. Chapter III is devoted to the derivation of a class of integral domains known as Dedekind domains. If R is a Dedekind domain and we consider R as an R-module and ideals of R as submodules then we can indeed generalize the unique factorization theorem, for an ideal in a Dedekind domain can be uniquely factored as a product of prime ideals. Chapter IV considers an alternative approach to the definition of Dedekind domains and discusses domains related to Dedekind domains.

At this point several general definitions and concepts from the theory of commutative rings, which shall be used in the following chapters, will be introduced. In the remainder of this chapter R shall be a commutative ring with identity and M an R-module.

If N is a submodule of M and I is an ideal of R then N:I is defined to be the set of elements x in M such that Ix is contained in N. N:I is referred to as the residual of N by I.[1]

M is said to be Noetherian if every ascending chain of submodules of M is finite. An equivalent condition is that every submodule of M must be finitely generated.[2] R is Noetherian if every ascending chain of ideals in R is finite or equivalently if every ideal of R is finitely generated. It can easily be shown that if every prime ideal of R is finitely generated then R is a Noetherian ring. If R has only one maximal ideal then R is said to be quasi local. A local ring is a ring which is both quasi local and Noetherian.[3]

The final concept to be introduced in this chapter is that of localization. In the general sense if S is a multiplicatively closed set in R and N is an R-module then N_S is defined as the set of elements n/s such that n is in N and s is in S. n_1/s_1 and n_2/s_2 are said to be equivalent if there is an s in S such that $s(n_1 s_2 - n_2 s_1) = 0$. If we restrict ourselves to the R_S-module M containing N then N_S is the set of elements m in M such that there exists an s in S with sm in N provided $S \cap Z(M)$ is empty. If P is a prime ideal of R then R_P is defined to be R_S where S = R - P. In the case of R as an R-module, R_S is the ring generated by making all of the elements of S invertible

---

[1] Max D. Larsen and Paul J. McCarthy, Multiplicative Theory of Ideals (New York: Academic Press, 1971), p. 38.

[2] Larsen and McCarthy, p. 9.

[3] Irving Kaplansky, Commutative Rings (Chicago: The University of Chicago Press, 1974), p. 5.

with respect to multiplication. It can be shown that there is a one to one correspondence between the prime ideals of R_S and the prime ideals of R disjoint from S.[4]

[4] Kaplansky, pp. 22–23.

CHAPTER II

PRIMARY DECOMPOSITION

Given a commutative ring R and an R–nodule M, a submodule Q of M is said to be primary if for all a in R and x in M, ax in Q and x not in Q imply that a M is contained in Q for some positive integer n. A related concept is that of an irreducible submodule. A submodule N is said to be irreducible if the fact that N is equal to the intersection of two submodules L and K implies that L = N or K = N.[5]

Lemma 1. If M is Noetherian then any irreducible submodule Q of M is primary.

Proof: Let a be an element of R and x an element of M such that ax is in Q and x is not an element of Q. We have an ascending chain of submodules $Q:(a^1)$ contained in $Q:(a^2)$ contained in $Q:(a^3)$, ... Since M is Noetherian, it satisfies the ascending chain condition. Thus $Q:(a^n) = Q:(a^{n+1}) = \ldots$ for some n. We claim that $Q = (Q+a^n M) \cap (Q+Rx)$ for if y is an element of $(Q+a^n M) \cap (Q+Rx)$ then $y = z+a^n u = q+rx$ with q, z in Q, u in M, and r in R. Since aq+arx is in Q, ay is in Q. Thus $a^{n+1}u = ay - az$ is in Q. Therefore u is in $Q:(a^{n+1})$ and as a result u is in $Q:(a^n)$; hence $a^n u$ is in Q and y is in Q. Containment in the opposite direction is immediate so we have $Q=(Q+a^n M) \cap (Q+Rx)$. Since Q is irreducible, Q is equal to Q+Rx or $Q+a^n M$.

---

[5]Larsen and McCarthy, p. 39.

Since x is not in Q, $Q \neq Q+Rx$. Therefore $Q = Q+a^nM$ and we have a^nM contained in Q. Thus Q is primary.[6]

It is worth noting that the converse of this lemma is not true. If $R = Z[x]$ for example, then $(4,2x,x^2)$ is a primary ideal. The fact that $(4,2x,x^2)$ is equal to the intersection of the ideals $(4,x)$ and $(2,x^2)$ shows that $(4,2x,x^2)$ is not irreducible however.[7]

Lemma 2. If M is Noetherian then every submodule of M can be written as a finite intersection of irreducible submodules of M.

Proof: Let X be the set of submodules which violate the above conclusion. This set may be partially ordered by inclusion and every totally ordered set has a maximal element since M is Noetherian. If X is nonempty, we may apply Zorn's lemma to find a maximal element N of X. Since N cannot be irreducible we can find submodules L and K such that N is the intersection of L and K but N is properly contained in L and K. Since N is maximal, L and K are not in X. Therefore L and K may be expressed as finite intersections of irreducible submodules of M. This implies that N is a finite intersection of irreducible submodules of M which contradicts the fact that N is in X. Therefore X must be empty.[8]

It is easy to see that every prime ideal is primary. As tempting as it appears, a primary ideal cannot always be expressed as a power of a prime ideal; nor must a power of a prime ideal be primary.[9]

---

[6] Larsen and McCarthy, p. 39.

[7] Larsen and McCarthy, p. 57.

[8] Larsen and McCarthy, p. 40.

[9] Larsen and McCarthy, pp. 56–57.

As stated in the following theorem, which is a direct result of the preceding lemmas, a submodule of a Noetherian module can indeed be "factored" into primary submodules.

Theorem 3. If M is Noetherian, then every submodule N of M can be written as a finite intersection of primary submodules of M.[10]

This finite intersection of primary submodules in known as a primary decomposition of N. Clearly this decomposition is not unique. (M may appear In the intersection as many times as desired, for example.) With two additional restrictions, we may force the decomposition to be at least In a sense unique but first we need an additional definition.

Given an ideal I in R, define Rad(I) to be the set of all elements a in R such that a^n is in I for some positive integer n. This definition can be extended to a submodule N of an R-module M be setting Rad(N) = Rad(N:M). Rad(N:M) Is called the radical of N. For any ideal I of R, Rad(I) is contained in every prime ideal containing I. For any primary submodule Q of M, Rad(Q) itself is a prime ideal. An ideal with a prime radical does not need to be primary however. If R = K[x,y] for some field K, and I = (x^2,xy) for example, then Rad(I) = (x), a prime ideal. (x^2,xy) is not a primary ideal however. If Q is primary and Rad(Q) = P then Q is said to be P-primary.[11]

If a submodule N of M has primary decomposition $N=Q_1 \cap \cdots \cap Q_n$, then this decomposition is called reduced if:

1) No Q_j contains the intersection of all the other Q_j

[10] Larsen and McCarthy, p. 40.

[11] Larsen and McCarthy, pp. 41-56.

2) For all $i \neq j$, $Rad(Q_i) \neq Rad(Q_j)$

Before considering the question of uniqueness, it is necessary to verify that each submodule with a primary decompostion has a reduced primary decomposition. The first condition may easily be satisfied for if Q_i contains the intersection of the remaining Q_j for some i then that Q_i may simply be removed from the set of intersecting submodules. If $Rad(Q_{i1}) = \ldots = Rad(Q_{im})$ then by use of the following proposition the intersection of Q_{i1} throught Q_{im} may be replaced by a single primary submodule Q. By repeated use of this proposition the second condition for a reduced primary decomposition may also be satisfied.[12]

<u>Proposition 4.</u> If $Q_1, \ldots, Q_k$ are P-primary submodules of an K-module M then the intersection Q of Q_1 through Q_k is also P-primary.

Even a reduced primary decomposition need not he unique. For example let R=K[x,y] where K is a field. $(x^2,xy) = (x) \cap (y-cx, x^2)$ is a reduced primary decomposition of (x^2,xy) for every c in K. Certain features of reduced primary decompositions can be shown to be unique however.[13]

One of the unique features of a reduced primary decomposition of a submodule N is the set of prime divisors of N. If $N = Q_1 \cap \ldots \cap Q_k$ is a reduced primary decomposition of N and $P_i = Rad(Q_i)$ then $P_1, \ldots, P_k$ are said to be the prime divisors of N. The fact that the set of prime divisors of N is unique is a direct result of theorem 6 which is stated after the following lemma.[14]

---

[12] Larsen and McCarthy, pp. 48-49.

[13] Larsen and McCarthy, pp. 49-56.

[14] Larsen and McCarthy, pp. 49-50.

Lemma 5. If M is an R-module, P a prime ideal of R, L a P-primary submodule of M and N a submodule of M and if N is not contained in L then $L:N$ is P-primary.

Theorem 6. Let N be a submodule of M with reduced primary decomposition $N = Q_1 \cap \ldots \cap Q_k$. Let $P_i = Rad(Q_i)$ for each i. For any prime ideal P of R, $P = P_i$ for some i if and only if there exists an x not in N such that $N:Rx$ is a P-primary ideal of R.

Proof: Let $P = P_1$. Since the primary decomposition is reduced we can find an element x in every Q_i except Q_1 (if $N = Q_1$ then x need only be outside of N). By lemma 5, $Q_1:Rx$ is P-primary. Since $Q_i:Rx=R$ for $i \neq 1$, $N:Rx = Q_1:Rx$. Therefore $N : Rx$ is P-primary and x is not in N. If we assume that $N : Rx$ is P-primary for some x not in N then $P = Rad(N:Rx)$, which is the intersection of the radicals of $Q_i:Rx$ for each i. If x is in Q_i then $Rad(Q_i:Rx)=R$ and if x is not in Q_i then $Rad(Q_i:Rx)=P_i$. Since x is not in N, $Rad(Q_i:Rx)=P_i$ for at least one i. Therefore P can be written as an intersection of sets from $P_1, \ldots, P_k$. Since P and each P_i is prime $P = P_i$ for some i.[15]

If $N = Q_1 \cap \ldots \cap Q_k$ is a reduced primary decomposition of N and $P_i=Rad(Q_i)$ for each i, then a set of these ideals $\{P_{i1}, \ldots, P_{ij}\}$, which satisfies the property that every P_i, contained in one of the ideals in this set, is itself in the set, is called an isolated set of prime divisors of N. The intersection of Q_{i1} through Q_{ij} is called an isolated component of N. The following two propositions establish that the isolated components of N are unique and independent of the reduced primary decomposition chosen for N.[16]

---

[15]Larsen and McCarthy, pp. 49–50.

Proposition 7. Let N be a submodule of M with primary decomposition $N = Q_1 \cap \cdots \cap Q_k$. Let S be a multiplicatively closed set in R and let $P_i = Rad(Q_i)$ for each i. If the intersection of P_i and S is empty for i = 1 to h and the intersection of P_i and S is nonempty for i=h+1 to k then $N_S = Q_1 \cap \cdots \cap Q_h$.

Proof: Let x be element of N_S. Therefore sx is in N for some s In S and this implies that sx is in Q_i for i = 1 to h. x not in Q_i for some i between 1 and h would imply that s^n is in Q_i for some n since Q_3 is primary. This would put s in both S and P_i contradicting the hypothesis. Thus x must be in each Q_i for i = 1 to h. On the other hand if x is in each Q_i for i = 1 to h, then if h = k we are done. If not, choose s_j from the intersection of S and P_j for j = h+1 to k. There is a positive integer n such that $(s_{h+1} \cdots s_k)^n$ is in $Q_j:M$ for j = h+1 to k. Therefore $(s_{h+1} \cdots s_k)\, x$ is in $Q_{h+1} \cap \cdots \cap Q_k$. Since x is in Q_i for i = 1 to h, $(s_{h+1} \cdots s_k)^n x$ is in each Q_i for i=1 to k and thus in N. By definition x is in N_S.[17]

Proposition 8. An isolated component of N depends only on N and not the reduced primary decomposition of N.

Proof: Let $\{P_{i1}, \ldots, P_{ir}\}$ be a set of prime divisors of N associated with an isolated component of N. Let $S=R-(P_{i1} \cup \cdots \cup P_{ir})$. Since S is closed, it suffices to show that $N_S = Q_{i1} \cap \cdots \cap Q_{ir}$. Clearly the intersection of S and Q_{ij} is empty for j=1 to r. If $i \neq i_j$ for any j then P_i is not contained in P_{ij} for any j. Therefore P_i Is not contained in $P_{i1} \cup \cdots \cup P_{ir}$ and as a result the intersection of

---

[16] Larsen and McCarthy, pp. 51–52.

[17] Larsen and McCarthy, p. 51.

P_i and S is nonempty. We can apply proposition 7 to obtain the desired result.[18]

The following theorem is a direct result of propositions 7 and 8. It offers another uniqueness feature of reduced primary decompositions.

<u>Theorem 9.</u> If Q appears in a reduced primary decomposition of a submodule N of M and Rad(Q) is a minimal prime divisor of N then Q appears in every reduced primary decomposition of N.[19]

<u>Proof</u>: Let P = Rad(Q). Then $\{P\}$ is an isolated set of prime divisors of N and $Q = N_S$ where $S = R - P$. If $N = Q_1 \cap \ldots \cap Q_k$ is a reduced primary decomposition of N with the intersection of Q_i and S empty for i = 1 to h and the intersection nonempty for i = h+1 to k. By proposition 7, $Q = N_S = Q_1 \cap \ldots \cap Q_h$ so $P = P_1 \cap \ldots \cap P_h$. Thererfore $P = P_i$ for some i and by proposition 8, $Q_i = Q$.

The search for a generalization of the unique factorization theorem for integers appears to have been only partially successful. In order to guarantee any reduced primary decomposition at all, it is necessary to require that the R-module M be Noetherian. As shown this decomposition need not be unique. The best that can be said is that the radicals of the sets in the decomposition are unique and that those sets in the reduced primary decomposition of N whose radicals are minimal prime divisors of N occur in each decompositjon.

---

[18] Larsen and McCarthy, p. 51.

[19] Larsen and McCarthy, p. 52.

CHAPTEK III

DERIVATION AND PROPERTIES OF DEDEKIND DOMAINS

Integral Elements

Before defining the integral closure of a commutative ring R, it is necessary to introduce the concept of an R-algebra. An R-algebra T is a ring which is an P,-module and satifsies the requirement that $r(st) = (rs)t = s(rt)$ for all r in R and s and t in T. An element u of this ring T is said to be integral over R if u satisfies a polynomial equationwith coefficients in R and highest coefficient a unit. T itself is integral over R if all of its elements are integral over R.[20]

Proposition 10. An element u in T is integral over R if and only if there is a finitely generated R-submodule A of T such that uA is contained in A, 1 is an element of A, and the only element of T which annihilates A on the left is 0.

Proof: If u is integral over R then it satisfies an equation of the form

$$u^n + r_1 u^{n-1} + \dots + r_n = 0 \qquad (1)$$

with r_i in R. If A is defined to be the R-module spanned by $1, u, \dots, u^{n-1}$ then the requirements of the proposition are satisfied. On the other hand if we have a finitely generated R submodule A so

---

[20] Kaplansky, p.9.

that uA is contained in A and the left annihilator of A in T is 0 then A is spanned by a finite number of elements, say $a_1, \ldots, a_n$. Therefore for i=1 to n

$$ua_i = r_{i1}a_1 + \ldots + r_{in}a_n \tag{2}$$

with r_{ij} in R. Bringing all the terms of (2) to the left hand side yields

$$\begin{bmatrix} u-r_{11} & -r_{12} & \cdots & -r_{1n} \\ -r_{21} & u-r_{22} & \cdots & -r_{2n} \\ & & & \\ -r_{n1} & -r_{n2} & \cdots & u-r_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \\ a_n \end{bmatrix} = 0$$

If d is the determinant of the above matrix then d is a left annihilator of $a_1, \ldots, a_n$ and thus d left annihilates all of A. Therefore d = 0 and expanding the above determinant yields a monic polynomial equation satisfied by u.[21]

By using this proposition we can state an equivalent defintion for an integral element. If R and T are rings and R is contained in T then a in T is integral over R if there is a subring L of T such that a is in L and L is a finitely generated R-module. If the requirement that L be a ring is removed, we are left with a slightly weaker condition. An element a in T is said to be almost integral if there exists a finitely generated R-submodule L of T such that a^n is in L for every positive integer n.[22]

Theorem 11. If T is a commutative algebra then the elements of T which are integral over R form a subring of T.

---

[21] Kaplansky, p. 9.

[22] Larsen and McCarthy, pp. 82-92.

Proof: Let s and t be integral over R. There are R modules A and B such that sA is contained in A and tB is contained in B which satisfy the hypotheses of proposition 10. We can say that A is generated by $a_0, a_1, \ldots, a_k$ and B is generated by $b_0, b_1, \ldots, b_L$ where $a_0 = b_0 = 1$ which implies that AB is generated by $\{a_i b_j : 0 \le i \le k, 0 \le j \le L\}$. Clearly (u+v)AB and (uv)AB are contained in AR so by proposition 10, u+v and uv are integral over R.[23]

The subring of T guaranteed by theorem 11 is known as the integral closure of R in T. If R itself is the integral closure of R in T then R is said to be integrally closed.[24] The following proposition shows that the integral properties discussed above are transitive.

Proposition 12. Let R be a commutative ring and T and K–algebra integral over R. If u is an element of a ring containing T and u is integral over T, then u is integral over R.

Proof: Since u is integral over T we can say

$$u^n + t_1 u^{n-1} + \ldots + t_n = 0$$

with t_i in T. Since each t_i is integral over R, proposition 10 implies that $R[t_1, \ldots, t_n]$ is finitely generated over R. Another application of proposition 10 yields $R[t_1, \ldots, t_n, u]$ as a finitely generated K–module since u is integral over $R[t_1, \ldots, t_n]$. By applying proposition 10 in the opposite direction, u is integral over R.[25]

---

[23]Kaplansky, p. 10.

[24]Kaplansky, p. 27.

[25]Kaplansky, p. 28.

If R and T are rings with T integral over R, a number of relationships can be established between the prime ideals of R and those of T. In particular maximal chains of prime ideals in T may be matched to corresponding maximal chains in R.

Lemma 13. Let R and T be rings such that T is integral over R. Let P and P_0 be primes in R such that P_0 contains P and let Q be prime in T such that the intersection of Q and R is P. Then there exists a prime ideal Q_0 in T such that the intersection of Q_0 and R is P_0.

Proof: Let $S = R-P_0$. S is disjoint from Q so Q can be expanded to a maximal prime ideal Q_0 disjoint from S. If the intersection of Q_0 and R is P then we are done; if not then the intersection of Q_0 and R is strictly contained in P. In the latter case, let u be in P but not in Q_0 or R. Therefore (Q_0,u) properly contains Q_0 and must therefore intersect S at an element s. This element may be expressed $s = q+tu$ with q in Q_0 and t in T. Since t is integral over R, t satisfies

$$t^n + r_1t^{n-1} + \dots + r_n = 0 \tag{3}$$

with r_i in R. Multiplying (3) by u^n yields

$$(tu)^n + r_1u(tu)^{n-1} + \dots + r_nu^n = 0$$

Because $tu = s-q$, we can say that $(tu)^n = s \pmod{Q_0}$. Thus

$$s^n + r_1us^{n-1} + \dots + r_nu^n = 0 \pmod{Q_0} \tag{4}$$

This implies that the left hand side of (4) is in Q_0 and hence also in P. u in P implies that each term except the first lies in P so s^n lies in P. Thus s is in P and we have a contradiction.[26]

---

[26] Kaplansky, pp. 28–30.

Lemma 14. Let R and T be rings such that T is integral over R. Let Q and Q_0 be distinct primes in T with the same contraction P in R. Then Q and Q_0 are incomparable.

Proof: If we assume that Q is contained in Q_0 then let q be in Q_0 but not in Q. q in T implies that q is integral over R so we may pick a monic polynomial, with coefficients in R, of least degree, which equals zero (mod Q) when evaluated at q, i.e.

$$q^n + r_1 q^{n-1} + \dots + r_n = 0 \pmod{Q} \qquad (5)$$

Since 1 is not in Q and the polynomial is monic, n is positive. The left hand side of (5) lies in Q and thus also in Q_0. Every term on the left hand side of (5) lies in Q_0 because q is in Q_0. Therefore r_n lies in Q_0 and R and as a result lies in their intersection P and thus also in Q. As a result,

$$q(q^{n-1} + r_1 q^{n-2} + \dots + r_{n-1})$$

lies In Q. The second factor above cannot lie in Q since its degree is less than n and the first factor was chosen not in Q. Hence we have contradicted the fact that Q is prime.[27]

The following lemma and theorem require a pair of definitions. The corank of a prime ideal P is defined to be the supremum of the lengths of all chains of prime ideals ascending from P. The dimension of a commutative ring R is the supremum of the lengths of all chains of prime ideals in R. Equivalently the dimension of R may be defined as the supremum of the coranks of all prime ideals, or just all minimal prime ideals, of R. The theorem is a direct result of lemma 15.[28]

---

[27] Kaplansky, pp. 26–30.

[28] Kaplansky, pp. 31–32.

Lemma 15. Let R and T be rings such that T is integral over R. Let Q be a prime ideal in T and let $P=Q\cap R$. Then corank (P) = corank(Q).

Proof: If we are given a chain of prime ideals ascending from P, a chain of prime ideals of equal length ascending from Q can be generated by repeatedly using lemma 13. Hence corank(P) $\leq$ corank(Q). On the other hand any chain of prime ideals ascending from Q must contract to a chain of prime ideals of equal length in P by lemma 14. Hence corank(p)=corank(Q).[29]

Theorem 16. If R and T are rings such that T is integral over R then the dimension of T equals the dimension of R.[30]

Up to this point, we have been concerned with a commutative ring R and an R-algebra T. For the rest of this chapter, the emphasis will be upon integral domain rather than simply commutative rings. An integral domain is defined to be integrally closed if it is integrally closed within its quotient field.[31]

Lemma 17. If R is an integrally closed integral domain and S is a multiplicatively closed set in R then R_S is integrally closed.

Proof: Let u be integral over R_S. Therefore u satisfies

$$u^n + (r_1/s_1)u^{n-1} + \dots + (r_n/s_n) = 0$$

with r_i in R and s_i in S. Let $s=s_1 s_2 \dots s_n$ and let $t_i=s/s_i$. We have

$$s^n u^n + t_1 r_1 s^{n-1} + \dots + t_n r_n s^{n-1} = 0 \qquad (6)$$

[29] Kaplansky, p. 31.

[30] Kaplansky, p. 32.

[31] Kaplansky, p. 32.

Regrouping (6) yields

$$(su)^n + t_1r_1(su)^{n-1} + \dots + t_nr_ns^{n-1} = 0$$

so su is integral over R. Since R is integrally closed, su is in R and thus u is in R_S.[32]

Lemma 18. The intersection of a family of integrally closed domains all contained in one large domain is integrally closed,

The proof of lemma 18 is immediate from lemma 17. The following corollary is an immediate consequence of lemmas 17 and 18 and the fact that $R=\cap R_M$ where the intersection ranges over all maximal ideals M of R. Theorem 21 gives us a much stronger result however.[33]

Corollary 19. R is integrally closed if and only if R_M is integrally closed for each maximal ideal M of R.

Lemma 20. If R is an integral domain then $R=\cap R_P$ with the intersection ranging over all maximal prime ideals of principal ideals.

Proof: Clearly R is contained in $\cap R_P$. Let u be in $\cap R_P$ so u=r/t with r and t in R. Let I be the set of elements y in R such that yr is in (t). If I≠R then 1 Is in I and as a result r is in (t) and u=r/t is in R. If I = R then, since I is contained in the zero divisors of R/(b), I can be expanded to a maximal prime ideal P of (t). u in R_P implies that u=a/s with a in R and s in S=R-P. Thus a/s = r/t and at=sr so we have sr as an element of (t) and s in I. Therefore s is in P, a contradiction.[34]

---

[32] Kaplansky, p. 33.

[33] Kaplansky, p. 34.

[34] Kaplansky, pp. 34-35.

Theorem 21. If R is an integral domain then R is integrally closed if and only if Rp is integrally closed for each maximal prime ideal P of a principal ideal.

Proof: The result is immediate from lemmas 17,113, and 20.[35]

A valuation domain is defined as an integral domain R such that for all a and b in R, a divides b or b divides a. Theorem 26 states a result similar to lemma 20 using valuation domains if R is integrally closed but first a number of preliminary results are needed.[36]

Proposition 22. Let R be an integral domain with quotient field K. R is a valuation domain if and only if for all u in K, u or 1/u is in R.

Lemma 23. Let R and T be rings such that T contains R and let u be a unit in T. If I is an ideal in R then $IR[u] \neq R[u]$ or $IR[1/u] \neq R[1/u]$

Proof: Suppose $IR[u] = R[u]$ and $IR[1/u] = R[1/u]$. Therefore $ir = 1$ and $jt = 1$ for some i and j in I, r in R[u] and t in R[1/u]. Thus we have

$$i_0 + i_1 u + \dots + i_n u^n = 1 \quad (7)$$
$$j_0 + j_1(1/u) + \dots + j_m(1/u)^m = 1 \quad (8)$$

with i_k and j_k in I. Without loss of generality say $n \geq m$ and n has been chosen to be minimal. If equation (8) is multiplied by u^n and reordered we have

$$(1-j_0)u^n = j_1 u^{n-1} + \dots + j_m u^{n-m} \quad (9)$$

Multiplying equation (7) by $1-j_0$ and substituting $(1-j_0)u^n$ from equation (9) yields a new equation of the same form as (7) but with a

---

[35] Kaplansky, pp. 35-36.

[36] Kaplansky, pp. 35-36.

smaller maximum exponent of u than n contradicting the minimality of n.[37]

Lemma 24. If K is a field, R a subring of K and I an ideal of R then there exists a valuation domain V such that R is contained in V, V is contained in K, K is the quotient field of V and $IV \neq V$.

Proof: Let X be the set of all pairs (R_Z, I_Z) with R contained in R_Z, R_Z contained in K, I contained in I_Z and I_Z an ideal of R_Z. X can be partially ordered by (R_Y, I_Y) contained in (R_Z, I_Z) if R_Y is contained in R_Z and I_Y is contained in I_Z. Clearly every totally ordered subset of X has a maximal element and X is nonempty so we may apply Zorn's lemma to find a maximal pair (V, J). Let u be an element of K. By lemma 23 either $JV[u] \neq V[u]$ or $JV[1/u] \neq V[1/u]$. Thus $JV[u]$, $V[u]$ is in X or $JV[1/u]$, $V[1/u]$ is in X but since (J,V) is maximal this implies that $V=V[u]$ or $V=V[1/u]$. Therefore u or u^{-1} is in V which implies that V is a valuation domain with quotient field K.[38]

Lemma 25. Let R and T be commutative rings with R contained in T. If u is in T then $1/u$ is integral over R if and only if $1/u$ is in $R[u]$.

Proof: Assume $1/u$ is integral over R. Then $1/u$ satisfies a polynomial equation:

$$(1/u)^n + r_1(1/u)^{n-1} + \ldots + r_n = 0 \qquad (10)$$

with r_i in R. Multiplying equation 10 by u^n and rearranging yields:

$$u(r_1 + r_2 u + \ldots + r_n u^{n-1}) = -1$$

[37] Kaplansky, p. 35.

[38] Kaplansky, p. 36.

Therefore $1/u$ is an element of $R[u]$. The proof may be reversed to establish the other direction.[39]

Theorem 26. If R is an integrally closed domain with quotient field K then R is the intersection of the valuation domains between R and K.

Proof: Let y be in every valuation domain between R and K. If y is integral over R then y is in R and we are done. If not then let $u=1/y$ so $1/u$ is not integral over R. By lemma 25 $1/u$ is not an element of $R[u]$ so $(u) = R[u]$. We can find a valuation domain V between $R[u]$ and K such that $(u)V = V$ by lemma 24. $y=1/u$ is in V which implies that $(u)V=V$. Thus we have the required contradiction.[40]

We will now examine a class of integral domains known as GCD-domains. A GCD-domain is an integral domain in which any two elements have a greatest common divisor. The notation $[a,b]$ will be used for the greatest common divisor of a and b. It will be shown in theorem 28 that any GCD-domain is integrally closed.[41]

Proposition 27. In a GCD-domain

1) $[ab,ac] = a[b,c]$

2) If $g = [a,b]$ then $[a/g,b/g] = 1$

3) If $[a,b] = [a,c] = 1$ then $[a,bc] = 1$ [42]

Theorem 28. If R is a GCD-domain then R is integrally closed.

---

[39] Kaplansky, p. 10.

[40] Kaplansky, p. 36.

[41] Kaplansky, p. 32.

[42] Kaplansky, p. 32.

Proof: Let K be the quotient field of R and let an element u of K be integral over R. We have

$$u^n + r_1u^{n-1} + \dots + r_n = 0 \qquad (11)$$

with r_i in R. Since u is in K we may write u as s/t with s and t in R. Since R is a GCD-domain we may say without loss of generality that $[s,t] = 1$. Multiplying (11) by t^n yields:

$$s^n + r_1s^{n-1}t + \dots + r_nt^n = 0$$

which implies that t divides s^n. Since $[s^n,t] = 1$ by proposition 27 (part 3), t must be a unit and hence u must be in R.[43]

Clearly all valuation domains are GCD-domains. Before considering other classes of domains related to GCD-domains it is necessary to examine the concept of an invertible ideal. If R is an integral domain with quotient field K then I^{-1} is the set of elements x in K with xI contained in R. I is called invertible if $II^{-1} = R$.[44]

Proposition 29. If I is an invertible ideal then I is finitely generated.

Proof: $II^{-1} = R$ implies that $a_1b_1 + \dots + a_nb_n = 1$ with a_i in I and b_i in I^{-1}. For i in I

$$i = i(a_1b_1 + \dots + a_nb_n) = b_1ia_1 + \dots + b_nia_n$$

with $b_j i$ in R so $\{a_1, \dots, a_n\}$ generates I.[45]

Theorem 30. If R is an integral domain with a finite number of maximal ideals and I is an invertible ideal of R then I is principal.

---

[43] Kaplansky, p. 33.

[44] Kaplansky, p. 37.

[45] Kaplansky, p. 37.

Proof: Let $M_1, \ldots, M_n$ be the maximal ideals of R. Since $II^{-1} = R$ we may choose a_i in I, b_i in I^{-1} such that $a_i b_i$ is not in M_i for each i. Since no M_i can contain the intersection of the remaining maximal ideals for each i we may choose u_i not in M_i, such that u_i is in M_j for every $j \neq i$. If we let $v = u_1 b_1 + \ldots + u_n b_n$ then v is in I^{-1} so vI is contained in R. Suppose vI is contained in M_i. Therefore va_i is in M_i but

$$va_i = u_1 b_1 a_i + \ldots + u_n b_n a_i$$

and every term on the right hand side lies in M_i except $u_i b_i a_i$ which is a contradiction. Thus vI lies in no maximal ideal and hence must equal R. Therefore I = (1/v) is principal.[46]

Lemma 31. If I is an invertible ideal and S a multiplicatively closed set in an integral domain R then I_S is invertible in R_S.

Proof: Let r/s be in R_S. Since $II^{-1} = R$ we can write $r = a_1 b_1 + \ldots + a_n b_n$ with a_i in I and b_i in I^{-1}. Hence

$$r/s = (a_1/s)b_1 + \ldots + (a_n/s)b_n$$

with a_i/s in I_S and b_i in I_S^{-1} so r/s is in $I_S I_S^{-1}$.

Theorem 32. If R is an integral domain and I is a finitely generated ideal of R then I is invertible if and only if I_M is principal for every maximal ideal M of R.

Proof: Assume that I_M is principal for each maximal ideal M. If II^{-1} is properly contained in R, it is contained in some maximal ideal M. Let $(i/s^*) = I_S$ with i in I and s^* in S and let $a_1, \ldots, a_n$ be the generators of I. For each i we can find s_i in R-M such that $s_i a_i$

[46]Kaplansky, p. 38.

is in (i). Let $s=s_1 \dots s_n$ and we have s/i in I^{-1}. However since $s=(s/i)i$, s is in $I^{-1}I$ and hence s is in M which is a contradiction. The converse is an immediate consequence of theorem 30 and lemma 31.[47]

Prufer Domains

We may now define two classes of integral domains closely related to GCD–domains. A Prufer domain is defined as an integral domain in which a nonzero ideal is invertible if and only if it is finitely generated. A slightly more restrictive class is the class of Bezout domains which consists of integral domains in which every finitely generated ideal is principal. It can easily be shown that any Bezout domain is a GCD–domain and that the greatest common divisor of two elements in a Bezout domain is a linear combination of the elements.[48] The following propositions relate Prufer domains and Bezout domains to valuation domains. The proof of proposition 33 is easy and may be found in Kaplansky (p.39).

Proposition 33. A quasi local domain is a valuation domain if and only if it is a Bezout domain.

Proposition 34. An integral domain R is Prufer if and only if R satisfies one of the following:

1) For every prime ideal P of R, R_P is a valuation domain.

2) For every maximal ideal M of R, R_M is a valuation domain

[47] Kaplansky, p. 38.

[48] Kaplansky, pp. 32,38.

Proof: If R is Prufer and J is a nonzero finitely generated ideal of R_P then $J = (r_1/s_1, \ldots, r_n/s_n)$ with r_i in R and s_i in R–P. If $I = (r_1, \ldots, r_n)$ then $I_P = J$. Since R is Prufer, I is invertible and by lemma 31, J is invertible. By theorem 30, J is principal so R_P is Bezout. Thus we can apply proposition 33 to establish that R_P is a valuation domain. Clearly the first condition implies the second, and by theorem 32 the second condition implies that R is Prufer.[49]

The following proposition gives a sufficient condition for an integral domain to be a Prufer domain. This proposition is useful in the construction of Prufer domains.

Proposition 35. If R is an integral domain satisfying the condition:

(*) If a and b are in R and $a^2 + b^2 \neq 0$ then $a^2/(a^2+b^2)$ and $ab/(a^2+b^2)$ are in R

then R is a Prufer domain.

Proof: Let K be the quotient field of R. Let i be the square root of –1. If i is in K then any element in K except 0 and 1/2 may be expressed in the form $1/(1-ix)$ for some x in K. Since $(1+ix)/(1+x^2) =$ / (– x) we can express any element of K except 0 and 1/2 as the product shown for some x in K. Applying this fact to one and using (*) yields the fact that i is in R. As a result it can easily be shown that R=K and thus R is a Prufer domian.

On the other hand if i is not in K then let P be a prime ideal in R and let u be in K. Since $u \neq i$, $1+u^2 \neq 0$ and by (*), $x = 1/(1+u^2)$ and $y = u/(1+u^2)$ are in R. Thus $1-x = u^2/(1+u^2)$ is in R. Since 1 is

---

[49] Kaplansky, p. 39.

not in P, P cannot contain both x and 1−x. Therefore $y/x=u$ or $y/(1-x) = 1/u$ is in R_P and as a result R_P is a valuation domain. By proposition 34, R is a Prufer domain.[50]

As a result of this proposition it is possible to enlarge any integral domain R to a Prufer domain. All that is necessary is that R be expanded to an integral domain containing $x/(1+x^2)$ and $1/(1+x^2)$ for all x in K such that $x^2 = -1$. It can be shown using this construction that for every positive integer n there is an n dimensional Prufer domain R_n with an ideal I_n which can be generated by n+1 elements but no less. If we let R be the real numbers and

$$B_n = R[x_0, \ldots , x_n]/(x_0^2 + \ldots + x_n^2-1)$$

then R_n is the Prufer domain generated by the above procedure from the R−subalgebra of B_n generated by all combinations $x_i x_j$. The ideal I_n is generated by $x_0^2, x_0x_1, \ldots , x_0x_n$.[51]

Dedekind Domains

One of the siinplest of the many equivalent definitions is that a Dedekind domain is a Noetherian Prufer damain. By proposition 29 this definition is clealy equivalent to the requirement that every nonzero ideal in R be invertible.[52]

---

[50] Richard G. Swan, "n−Generator Ideals In Prufer Domains" Pacific Journal of Mathematics, Vol III, No. 2 1984, 434−435.

[51] Swan, pp. 433−35.

[52] Kaplansky, p. 67.

Proposition 36. For an integral domain R the following are equivalent:

(1) R is a Dedekind domain

(2) R is Noetherian and for every maximal ideal M, every ideal of R_M is principal.

Proof: For (1) implies (2), proposition 34 says that R_M is a valuation domain. Since R is Noetherian, R_M is Noetherian and as a result every ideal of R_M is finitely generated. Since R_M is a valuation domain every ideal is hence principal. The converse is immediate from theorem 32.[53]

Note that every ideal of R itself need not be principal. If we let K equal the square root of −5, R=Z(k) and I=(3,1+2k) for example, then R is a Dedekind domain but I Is not principal. It is possible however to show that every ideal in a Dedekind domain can be generated by at most two elements. In the process we shall establish the result sought in chapter I, that every nonzero ideal in a Dedekind domain may be uniquely factored as a product of prime ideals.

Lemma 37. If K is a Dedekind domain then the dimension of R is less than or equal to one.

Proof: If P and M are prime ideals of R with P contained in M then P_M is a prime ideal contained In M_M in the local valuation domain R_M. We may write $P_M=(r)$ and $M_M = (x)$. r may be expressed as ux^n with u, a unit. Since P_M is prime this implies that x Is in P_M so $P_M = M_M$ and hence P=M.

[53]Kaplansky, pp. 67–68.

Lemma 38. In a Dedekind domain R, any nonzero ideal I may be expressed as a unique product of prime ideals.

Proof: Let X be the set of nonzero ideals which may not be expressed as a product of prime ideals. If we order X by inclusion we can find maximal element A of X by Zorn's lemma, if X is nonempty, since the fact that R is Noetherian guarantees that every totally ordered set in X has a maximal element. If M is a maximal ideal containing A then $A \neq M$ since A is in X. A is contained in AM^{-1} which is contained in R. If $A = AM^{-1}$ then $AM = A$ and thus $M = A^{-1}AM = R$ which is a contradiction. On the other hand if $A \neq AM^{-1}$ then AM^{-1} is not in X so AM^{-1} may be expressed as a product of primes. Therefore $A = AM^{-1}M$ may also be expressed as a product of primes which is also a contradiction.[54]

If $I = P_1 \cdots P_m = Q_1 \cdots Q_n$ are two prime factorizations of I then $Q_1 \cdots Q_n$ is contained in P_1. Therefore some Q_i is contained in P_1. By lemma 37 we have $P_1 = Q_i$. Since P_1 is invertible, it may cancelled on both sides and the process may he continued to show uniqueness.[55]

Theorem 39. If R is a Dedekind domain then every ideal A of R can be generated by at most two elements.

Proof: If A=(0) we are done. If not then let $b \neq 0$ be in A. Let $B = bA^{-1}$. Since B is an ideal of R and R is Dedekind we may write $B = P_1 \cdots P_n$ with P_i prime by lemma 38. Thus $b = p_1 \cdots p_n$ with p_i in P_i. Therefore b is in P_1 and $K-P_1$ consists of elements of the form

[54]Larsen and McCarthy, pp. 136–37.

[55]Kaplansky, p. 68.

$u(1-b)^n$ where u is a unit. By lemma 37 A localized at P_1 is principal and hence equal to (a). We can find a positive integer n such that $(1-b)^n/a$ is in A^{-1} and thus $(1-b)^n$ is in aA^{-1}. As a result 1 is in $B+aA^{-1}$ so $B+aA^{-1} = R$. Hence $A = AB+AaA^{-1} = AbA^{-1}+(a) = (b)+(a) = (a,b)$.

The example following proposition 35 shows that theorem 39 cannot be generalized to Prufer domains. The last result in this chapter reutrns to the question of factorizations of ideals raised in chapter I. Not only is every ideal in a Dedekind domain uniquely factorable as a product of primes but every domain in which all ideals are uniquely factorable is a Dedekind domain.

Theorem 40. An integral domain R is a Dedekind domain if and only if every nonzero ideal of R can be expressed as unique product of prime ideals.

Proof: The first half of the proof is presented in lemma 38. If every ideal of R is a unique product of primes, then first we must show that every invertible prime ideal in R in maximal. If P is an invertible prime which is not maximal then $P+(a) \neq R$ for some a in R–P. Thus $P+(a) = P_1 \cdots P_k$ and $P+(a^2) = Q_1 \cdots Q_n$ with P_i and Q_i prime. If we let $R^* = R/P$, $P_i^* = P_i/P$, $Q_i^* = Q_i/P$ and $a^* = a+P$ then $a^*R^* = P_1^* \cdots P_k^*$ and $(a^*)^2R^* = Q_1^* \cdots Q_n^*$. Since a^*R^* and $(a^*)^2R^*$ are invertible each P_i^* and Q_i^* is invertible so we have $(P_1^*)^2 \cdots (P_k^*)^2 = Q_1^* \cdots Q_n^*$ with $P_i^* = Q_{2i-1}^* = Q_{2i}^*$. Therefore $(P+(a))^2 = P+(a^2)$ and as a result P is contained in $(P+(a))^2$. Thus for all p in P, $p = q+ra$ with q in P^2 and r in R. Since p and q are in P but a is not, r must be in P so P is contained in P^2+Pa. Therefore $R = PP^{-1}$ is contained in $P^2P^{-1}+PaP^{-1} = P+(a)$ which contradicts our choice of a so P must be maximal.[56]

If Q is a nonzero prime ideal in R and $a \neq 0$ is in Q then $(a)=P_1 \cdots P_k$ with each P_i a prime ideal. Since (a) is invertible, each P_i is invertible and hence maximal. Since $P_1 \cdots P_k$ is contained in Q, P_i is contained in Q for some i and thus $P_i=Q$ and Q is invertible. If I is a nonzero ideal of R then $I = P_1 \cdots P_n$ with P_i prime. Since each P_i is invertible, I is also invertible. Therefore R is a Dedekind domain.[57]

---

[56] Larsen and McCarthy, pp. 135–36.

[57] Larsen and McCarthy, p. 135.

CHAPTER IV

GENERALIZATIONS OF DEDEKIND DOMAINS

In this chapter, we shall begin by considering another equivalent definition of a Dedekind domain. First we must consider some definitions.

If R is a commutative ring and A an R-module then an ordered sequence of elements $x_1, \ldots, x_n$ in R is called an R-sequence on A if $(x_1, \ldots, x_n)A \neq A$ and x_i is not in the zero divisors of $A/(x_1, \ldots, x_{i-1})A$ for any i. In the case of $A = R$, the sequence $x_1, \ldots, x_n$ is an R-sequence if each x_i is neither a unit nor a zero divisor in $R/(x_1, \ldots, x_{i-1})$. It can be shown that if R is Noetherian and I is an ideal of R then all of the maximal R-sequences on R contained in I have the same length. In this case, the grade of the ideal I is defined as the common length of the maximal R-sequences.[58]

This definition can be generalized for the case of grade 1 to non-Noetherian domains by defining an ideal I in a domain R to have grade 1 if there is a nonzero element x in I such that I is contained in the zero divisors of $R/(x)$. For our purposes it will be necessary to use a slightly stronger definition of grade 1. We shall define an

---

[58] Kaplansky, pp. 84-89.

ideal I in a domain R to have grade 1 if for any nonzero i in I there exists an r not in (i) such that rI is contained in (i). The following two propositions show that these two definitions of grade 1 are equivalent if R is Noetherian. The proof of the first proposition is presented in Kaplansky(pp.55–56).

Proposition 41. If R is Noetherian, A a finitely generated R–module and S a subring contained in the zero divisors of A then there is a nonzero element a in A such that Sa = 0.

Proposition 42. If I is a nonzero finitely generated ideal of R $(I=(t_1, \ldots, t_n)\ t_1 \neq 0)$ and if there exists r not in (t_1) such that rI is contained in (t_1) then for all $a \neq 0$ in I there is a b not in (a) such that bI is contained in (a).[59]

Proof: If a is a nonzero element of I then $ra=bt_1$ for some b in R. We claim that b is not in (a) for if $b=r_1a$ then $ra = r_1 at_1$. Cancelling a on each side would yield r in $(t_1$, a contradiction. Since $bt_1I = arI$ which is contained in $a(t_1)$ we have bI contained in (t_1).

We can now relate grade to Dedekind domains. A local domain R with maximal ideal M is defined to be a discrete valuation ring (DVK) if R is integrally closed and M has grade 1. Theorem 45 which is a direct result of the following lemmas and theorem 32 relates Dedekind domains and discrete valuation rings.

Lemma 43. If R is a Dedekind domain then for every maximal ideal M of R, R_M is a DVR.

---

[59]S. Floyd Barger, "Invertible Ideals and Theory of Grade" (unpublished work, Youngstown State University, 1974), p. 1.

Proof: By proposition 34, R_M is a valuation domain and hence integrally closed by theorem 28. (Note that by theorem 21, R itself is integrally closed.) By theorem 30, M is principal and thus has grade 1.[60]

Lemma 44. If R is a DVR then every ideal in R is principal.

Proof: Since the maximal ideal M has grade 1 there are elements x in M and y in R such that y is not in (x) but My is contained in (x). Therefore y/x is in M^{-1} but not in R. Since M is maximal and MM^{-1} contains M and R contains MM^{-1}, MM^{-1} = M or R. If MM^{-1} = M then by proposition 10, M^{-1} is integral over R and hence equal to R since every DVR is integrally closed. Since this is a contradiction, MM^{-1} = R so M invertible and thus principal by theorem 30.[61]

If we let M = (x) then every element of R may be written ux^n where u is a unit and n is a positive integer. Clearly for all r and s in R, r divides s or s divides r. As a result R is a valuation domain and since R is Noetherian all of its ideals must therefore be principal.

Theorem 45. A Noetherian domain is Dedekind if and only if for every maximal ideal M of R, R_M is a DVR.

The result of theorem 45 can be generalized to domains which are not quasi local as shown in theorem 46. The "if" portion of the proof of theorem 46 is very similar to the proof of lemma 44 and has been omitted.

---

[60] Kaplansky, pp. 67–68.

[61] Kaplansky, p. 67.

Theorem 46. A Noetherian domain R is Dedekind if and only if R is integrally closed and every ideal of R has grade 1.

Proof: If R is a Dedekind domain then R is integrally closed (see proof of lemma 43). Let I be an ideal of R. Since $II^{-1} = R$, I^{-1} strictly contains R. Let r/s be an element of I^{-1} not in R. Therefore rI is contained in (s) but r is not in (s). If I+(s) = R then

$$1 = i + ts \tag{12}$$

with i in I and t in K. Multiplying both sides by r yields two terms on the right hand side in (s). This would force r into (s), a contradiction. Therefor I+(s) is an ideal of R. Clearly r(I+(s)) is contained in (s) so by proposition 42, I+(s) has grade 1. For all $i \neq 0$ in I, i is in I+(s) so there exists a b not in (i) such that b(I+(s)) is contained in (a). Thus bI is contained in (a).[62]

There is no immediate generalization of theorem 45 to non-Noetherjan domains. If K is an algebraically closed field and K(x,y) the ring of rational functions in two variables over K and if R consists of all rational functions of the form p(x,y)/q(x,y) such that p(0,y)/q(0,y) is in K and x divides q(x,y) then R has grade 1, and is integrally closed and quasi-local. R is not Noetherian and not a Prufer domain however.

We will now consider a class of domains less restrictive than Dedekind domains. A π-domain is an integral domain in which every principal ideal is a product of prime ideals. Clearly every Dedekind domain is a π-domain. If we add the restriction that every nonzero ideal has grade 1 then π-domains and Dedekind domains are equivalent.

---

[62] Barger, pp. 1–2.

Lemma 47. If P is an invertible prime ideal properly contained in an ideal I which is properly contained in R then $I^{-1} = R$.

Proof: Let M be a maximal ideal containing I. Suppose there is a z in I^{-1} such that z is not in R. If $z = r/s$ with r in R, s in R-M, then $(r/s)I$ is contained in R so I is contained in (s). Therefore P is strictly contained in (s). In R_P, P and (s) are both principal and P is prime so s must be a unit which contradicts the fact that z is not in R. Thus z is in I_M^{-1} and hence P_M^{-1} but not in R_M. Since P_M is principal by theorem 32 we can say $P_M = (x)$ for some x in R_M. There is some r in R_M such that $zx = r$. Since z is not in R_M, x does not divide r. If we let y be in I_M but not in P_M however then $yz=r_1$ in R_M which implies that yr is in $(x) = P_M$. This leads to the contradiction that x does divide r. Therefore $I^{-1} = R$.

Theorem 48. If R is a π-domain in which every nonzero ideal has grade 1 then R is Dedekind.

Proof: Let P be a prime ideal in R and a an element of P, $a \neq 0$. We can write $(a) = P_1 \cdots P_n$ and since (a) is invertible each P_i is invertible. We claim that each P_i is maximal for if not then there is a b not in P_i such that $P_i+(b)$ is properly contained in R. By lemma 47 $(P+(b))^{-1}=R$ but as shown in the proof of lemma 44, $(P+(b))^{-1}$ properly contains R since $P+(b)$ is finitely generated and has grade 1 so we have a contradiction. Since $P_1 \cdots P_n$ is contained in P some P_i is contained in and thus equal to P. Therefore P is maximal and invertible. As a result R_P is a valuation domain and since every prime ideal in R is finitely generated, R is Noetherian. Therefore by proposition 34, R is a Prufer domain and hence a Dedekind domain.[63]

[63] Barger, pp. 2–3.

The class of π-domains may be generalized to a larger class known as Krull domains. An integral domain R is a Rrull domain if for every minimal prime P of R, R_P is a DVR, $R = \bigcap R_P$ where the intersection ranges over all minimal primes of R, and any nonzero element of R lies in only finitely many minimal prime ideals. It can be shown that any integrally closed Noetherian domain is a Krull domain.[64]

As a conclusion to this chapter, we shall consider another means of strengthening a π-domain to make it equivalent to a Dedekind domain, It is sufficient to require that every ideal generated by two elements be a product of prime ideals.

<u>Lemma 49.</u> If R is a domain in which every ideal generated by two elements is a product of prime ideals, then every R-sequence has length 1.

<u>Proof</u>: If a, b is an R-sequence then write $(a,h) = P_1 \cdots P_n$. If $R^* = R/(a)$ and $(b^*) = (b+(a)) = P_1^* \cdots P_n^*$ then b^* is not a zero divisor so (b^*) is invertible. Thus each P_i^* is invertible. $(b^*)^2 = (P_1^*)^2 \cdots (P_n^*)^2$ but if $(a,b^2) = Q_1 \cdots Q_m$ then we also have $(b^*)^2 = Q_1^* \cdots Q_m^*$ with each Q_i^* invertible. Since P_i^* and Q_i^* are invertible, $m = 2n$ and $P_i = Q_{2i-1} = Q_{2i}$. Therefore $(a,b)^2 = (a,b^2)$ and as a result a is in $(a,b)^2$. Thus we may write $a = r_1 a^2 + r_2 ab + r_3 b^2$ which forces $r_3 b^2$ into (a). Since b is not a zero divisor of $R/(a)$, r_3 must be in (a). Therefore $a = r_1 a^2 + (r_2 a + rb)ab$ so (a) is contained in $(a)^2 + (a)(b)$.

---

[64] Kaplansky, p. 82.

Multiplying by $(a)^{-1}$ yields R contained in (a)+(b) = (a,b), a contradiction.[65]

Theorem 50. If R is a domain in which every ideal generated by two elements is a product of prime ideals then R is Dedekind.

Proof: Let P be a minimal prime of R and let x be in P, $x \neq 0$. We can write $(x) = P_1 \ldots P_n$ so $P_1 \ldots P_n$ is contained in P. Therefore some P_i is contained in and hence equal to P. Thus P is invertible and so P_M is principal where M is a maximal ideal containing P.[66] Let (p) = P. For all a not in (p),p,a is not an R–sequence by lemma 49 so (p,a) = R_M. Therefore (p) is maximal in R_M and hence equal to M_M.[67] Therefore R_M is Noetherian and M_M has grade 1. As shown in the proof of lemma 44, R_M is a valuation domain since M_M is principal. Therefore is integrally closed and hence a DVR so by theorem 45, R is a Dedekind domain.

---

[65] Barger, p. 3.

[66] Robert Gilmer, Multiplicative Ideal Theory (New York: Marcel Dekker, Inc., 1972), p. 573.

[67] Barger, p. 4.

BIBLIOGRAPHY

Books

Gilmer, Robert. Multiplicative Ideal Theory. New York: Marcel Dekker, Inc., 1972.

Larsen, Max D. and McCarthy, Paul J. Multiplicative Theory of Ideals. New York: Academic Press, 1971.

Kaplansky, Irving. Commutative Rings. Chicago: The University of Chicago Press, 1974.

Articles

Barger, S. Floyd. "Invertible Ideals and Theory of Grade." Unpublished work, Youngstown State University, 1374

Swan, Richard G. "n-Generator Ideals in Prufer Domains." Pacific Journal of Mathematics, Vol III, No. 2. 1984, *433-435*