THE STUDY OF NESTED TOPOLOGIES THROUGH FUZZY TOPOLOGY

by

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TABLE OF CONTENTS

CHAPTER I

Preliminaries

In this paper L will denote a complete, completely distributive lattice with an order reversing involution. We will distinguish the two following subsets of L:
$L^c=\{\alpha\in L: \beta\in L \Rightarrow \alpha\leq\beta \text{ or } \beta\leq\alpha\}$, $L^a=\{\alpha\in L^c: \alpha<\beta \text{ and } \alpha<\gamma \Rightarrow \alpha<\beta\wedge\gamma\}$.
$0\in L^a$ will be assumed throughout this paper. Let X be a set. An L-fuzzy topology $\mathbb{T}$ is a collection of L-fuzzy sets (mappings from X into L) which is closed under arbitrary suprema and finite infima. Let P(X) be the set of crisp subsets of X, $c: P(X)\to P(X)$ is a semi-closure operator provided that $c(\emptyset)=\emptyset$, $c(X)=X$, $A\subset c(A)$ for $A\in P(X)$, and $c(A\cup B)=c(A)\cup c(B)$ for A,B in P(X). For $A\in P(X)$, $\mu(A)$ denotes the characteristic function of A. By an L-fuzzy topology in the Lowen sense, we mean a fuzzy topology including the constant maps as open sets. For further references on basic definitions and properties of L-fuzzy topologies see [9].

CHAPTER II

Introduction

Using Rodabaugh's definition of α-closure [6], Klein has defined L-fuzzy topology producing collection of operators (L-FTP). Using this concept and the related results, we will study how a finite family of nested topologies indexed by a lattice generates a fuzzy topology, Moreover, we will examine how topological properties are transmitted from the nested topologies to the induced fuzzy topologies and vice-versa. In particular, we will refine Klein's result in [4] about the equivalence of fuzzy continuity and level continuity and prove it to hold for the Lowen topology. We will show that plenty of suitable closed (open) sets are at our disposition, a result that may be significant with regard to the Tietze' extension problem. Finally, we will categorically embed topologies generated by a finite lattice in topologies generated by the unit interval, generalising the results obtained so far. As a general remark, it will appear that properties involving only open sets (closed) behave relatively well (e.g. compactness, hausdorff) but that properties requiring in their definition both open and closed sets are somewhat more elusive to track down.

CHAPTER III

L-FTP families of topologies

We now give a summary of the results obtained by Klein in [3,4].

Definition 3.1. Let $\alpha \in L-\{1\}$ and let A be a crisp subset of X. The 4-closure of A, denoted by $c_\alpha(A)$, is given by:

$$c_\alpha(A)=\{x: \text{if } G\in T \text{ and } G(x)>\alpha \text{, then } G\wedge 4(A)\neq 0\}.$$

It was shown in [3] that c_α is a semi-closure operator if $\alpha\in L^a-\{1\}$.

Definition 3.2. Let $\alpha\in L^a-\{1\}$ and let $G\in T$. $\alpha(G)=\{x: G(x)>\alpha\}$. By lemma 2.2 in [3], $\{\alpha(G): G\in T\}$ is a topology for X which we denote by T_α.

Definition 3.3. Let X be a set and let $C=\{k_\alpha : \alpha\in L^a-\{1\}\}$ be a collection of operators on P(X). C is L-fuzzy topology producing (L-FTP) provided :

(a) for every $\alpha\in L^a-\{1\}$, k_α is a semi-closure operator,

(b) if $\emptyset\neq\Gamma\subset \{1\}$, $\alpha=\wedge\{\beta:\beta\in\Gamma\}$, and $A\in P(X)$, then $k_\alpha(A)=\cap\{k_\beta(A) : \beta\in\Gamma\}$,

(c) if $A,B\in P(X)$ and $A\subset k_0(B)$, then $k_\alpha(A)\subset k_\alpha(B)$ for every $\alpha\in L^a-\{1\}$.

Definition 3.4. Let C be an L-FTP collection.

(a) For $A\in P(X)$, G_A is the L-fuzzy set defined by $G_A(x)=\wedge\{\beta: x\ k_\beta(A)\}$. (By convention $\wedge\emptyset=1$).

(b) $\Gamma(C)$ is the L-fuzzy topology with basis $\{G_A: A\in P(X)\}$.

The α-closure operators generated by $T(C)$ are the operators in C.

The class of α-closure operators induced by an L-fuzzy topological space was shown to be an L-FTP collection in[4].

We are ready now to prove our first lemma.

Lemma 3.5. Let $C=\{k_\alpha : \alpha \in L^a - \{1\}\}$ with L^a finite be a family of closure operators such that if $\alpha \leq \beta$ then $T_\beta \subseteq T_\alpha$ (where T_α, T_β are the topologies with k_α, k_β as closure operators respectively.) We have :

(a) if $\alpha \leq \beta$ $\alpha,\beta \in L$ and $A \in P(X)$ then $k_\alpha(A) \subseteq k_\beta(A)$,

(b) if $\emptyset \neq \Gamma \subset L^a - \{1\}$,$\alpha = \wedge\{\beta : \beta \in \Gamma\}$ and $A \in P(X)$, then $k_\alpha(A) = \cap\{k_\beta(A) : \beta \in \Gamma\}$

(c) if A,B and $A \subset k_o(B)$ then $k_\alpha(A) \subset k_\alpha(B)$.

Proof. (a) Suppose $x \notin k_\beta(a)$. Then $x \in X - k_\beta(A) = U_\beta$, an open set in T_β. Since $T_\beta \subseteq T_\alpha$, U_β is open in T_α and $X-U_\beta$ is a closed set in T_α . Thus we have $x \notin X - U_\beta \supset A$ and so $x \notin k\ (A)$.

(b) Since Γ is finite, $\alpha \in \{\beta : \beta \in \Gamma\}$ and the conclusion follows from part (a).

(c) Let A,B in P(X) and $A \subset k_o(B)$. We have $k_\alpha(A) \subset k_\alpha(k_o(B))$. Also $k_o(B) \subset k_\alpha(B) \Rightarrow k_\alpha(k_o(B)) \subset k_\alpha(k_\alpha(B)) = k_\alpha(B)$. Thus $k_\alpha(A) \subset k_\alpha(B)$.

As an immediate consequence of lemma 3.5 we have the following theorem.

Theorem 3.6. If $\mathcal{A} = \{k_\alpha : \alpha \in L^a - \{1\}\}$ is a finite family of closure operators generating topologies T_γ such that if $\beta \leq \alpha$ then $T_\alpha \subseteq T_\beta$, we have that $\mathcal{A}$ is an L-FPP family.

Definition 3.7. Let L be a lattice with $L^a-\{1\}=\{\alpha_1,\alpha_2,\dots,\alpha_n\}$ and $0=\alpha_1<\alpha_2<\dots<\alpha_n$. A family of topologies $\mathcal{A}$ is L-FTP iff :

(1) $\mathcal{A}=\{W_{\alpha_i}\}$ with $\alpha_i\in L^a-\{1\}$

(2) $W_{\alpha_1}\supset W_{\alpha_2}\supset\dots\supset W_{\alpha_n}$

As a notational convention, $C=\{k_{\alpha_i}\}$ will denote the associated family of closure operators.

Remark. Given an L-FTP family of topologies, the fuzzy topology $\Gamma(C)$ will be used on X unless another topology is explicitly mentioned.

We are now going to prove a computational lemma, which will introduce a construction used throughout this paper.

Lemma 3.8. Let $\mathcal{A}$ be an L-FTP family of topologies. Then for any A in P(X) we have :

$G_A(x)=1$ iff $x\in X-k_{\alpha_n}(A)$

$G_A(x)=\alpha_n$ iff $x\in k_{\alpha_n}(A)-k_{\alpha_{n-1}}(A)$

$G_A(x)=\alpha_{n-k}$ iff $x\in k_{\alpha_{n-k}}(A)-k_{\alpha_{n-k-1}}(A)$

$G_A(x)=0$ iff $x\in k_{\alpha_1}(A)$

Proof. Denote $B=\{\alpha_i\in L^a-\{1\} : x\in k_{\alpha_i}(A)\}$. We have :

$B=\emptyset$ or $\{\alpha_n\}$ or $\{\alpha_n,\alpha_{n-1}\}$ or ... or $\{\alpha_{n-k},\alpha_{n-k+1},\dots,\alpha_n\}$ or $\{\alpha_1,\alpha_2,\dots,\alpha_m\}$

$B=\emptyset \Leftrightarrow G_A(x)=1 \Leftrightarrow x\notin k_{\alpha_n}(A) \Leftrightarrow x\in X-k_{\alpha_n}(A)$

$B=\{\alpha_n\}\Leftrightarrow G_A(x)=\alpha_n \Leftrightarrow x\in k_{\alpha_n}(A)-k_{\alpha_{n-1}}(A)$

$B=\{\alpha_{n-k},\alpha_{n-k+1},\dots,\alpha_m\} \Leftrightarrow G_A(x)=\alpha_{n-k} \Leftrightarrow x\in k_{\alpha_{n-k}}(A)-k_{\alpha_{n-k+1}}(A)$

$B=\{\alpha_1,\dots,\alpha_m\} \Leftrightarrow G_A(x)=\alpha_1=0 \Leftrightarrow x\in k_{\alpha_1}(A)$

Theorem 3.9 Let $\mathcal{J}$ be an L-FTP family of topologies. Then $\forall \alpha_i \in L^a - \{1\}$ $W_{\alpha_i} = T_{\alpha_i}$ (where W_{α_i}, T_{α_i} are the topologies of Def.3.7 and Def.3.2 respectively).

Proof. By Theo. 2.4 in [3] we have for $\alpha_i \in L^a - \{1\}$: $W_{\alpha_i} \subseteq T_{\alpha_i}$. So it is sufficient to show $T_{\alpha_i} \subseteq W_{\alpha_i}$. Let $A \in P(X)$. By lemma 3.8 $\alpha_i(G) = \{x : G_A(x) > \alpha_i\} = X - k_{\alpha_i}(A)$, therefore $T_{\alpha_i} \subseteq W_{\alpha_i}$ and $T_{\alpha_i} = W_{\alpha_i}$ for each $\alpha_i \in L^a - \{1\}$.

Using the definition of the α-property in [6]:

Definition 3.10. Let $\alpha \in L - \{1\}$. (X, T) has the α-property provided, for $A \in X$, $c_\alpha(A) = A$ if and only if there is U with $A = \{x : U(x) \leq \alpha\}$.

Applying Theorem 2.4 in [3] we have the following corollary:

Corollary 3.11. Let $\mathcal{J}$ be an L-FTP family of topologies. Then (C) has the α-property for all $\alpha \in L^a - \{1\}$.

We will need several notions first introduced in [4].

Definition 3.12. Let $C = \{k_\alpha : \alpha \in L^a - \{1\}\}$ be an L-FTP collection on P(X). $\mathcal{F}(C)$ denotes the set of fuzzy topologies for X which induce k_α as α-closure operator when $\alpha \in L^a - \{1\}$. $\mathcal{F}(C)$ need not be a singleton or closed under finite intersections but is closed under suprema. For further details refer to [4].

Definition 3.13. (a) T_c denotes the collection of fuzzy subsets of X which are constant maps from X into L.

(b) T_p denotes the collection of fuzzy subsets of X which are either $\mu(\emptyset)$ or a map from X into $L - \{0\}$.

We have for $T\in\mathcal{F}(C)$: $\sup\{T, T_c\}$ is in $\mathcal{F}(C)$ (corollary 2.8 in [4]) and $\sup\mathcal{F}(C) = \sup\{T(C), T_c\}$ (theorem 2.9 in [4]) .

Proposition 3.14. Let $\mathcal{A} = \{W_{\alpha_i} : \alpha_i \in L^a - \{1\}\}$ be an L–FTP family of topologies. Then $W_{\alpha_i} = \{\alpha_i(G) : G \in T(C) \vee T_c\}$.

Proof. Any basis element of $T(C)\vee T_c$ can be written as $G_A \wedge b$, for $G_A \in T(C)$ and $b \in L$ (b also represents the constant map with value b) . Let $F = \{\alpha_i \in L^a - \{1\}$ such that $\alpha_i \geqslant b\}$ and let $\wedge F = \alpha_s$.

Applying lemma 3.8, we have :

$G_A \wedge b(x) = b$ iff $x \in X - k_{s-1}(A)$

$G_A \wedge b(x) = \alpha_{s-1}$ iff $x \in k_{s-1}(A) - k_{s-2}(A)$

$G_A \wedge b(x) = 0$ iff $x \in k_0(A)$

Thus, for $1 \leqslant i \leqslant s-1$ $\{x : G_A \wedge b(x) > \alpha_i\} = X - k_{\alpha_i}(A) \in W_{\alpha_i}$ and for $s \leqslant i$ $\{x : G_A \wedge b(x) > \alpha_i\} = \emptyset$ since $\alpha_i \geqslant b$.

Remark. For L linearly ordered, $\sup\{T(C), T_c\}$ is the smallest. topology in the Lowen sense in $\mathcal{F}(C)$ [4] . In this paper, this topology will be called the Lowen minimum.

Corollary 3.15. Let $\mathcal{A}$ be an L–FTP collection of topologies. Then the Lowen minimum has the α -property for all $\alpha \in L^a - \{1\}$.

In general, $\sup\mathcal{F}(C)$ does not have the α-property for d in $L^a - \{1\}$. It may in some cases. For example, if T_{α_i} is discrete for all i $\sup\mathcal{F}(C)$ does have the α-property for all $\alpha \in L^a - \{1\}$.

Example. Let $X = \mathbb{R}$, $T_{\frac{1}{2}}$ = usual topology, T_0 = discrete topology and $L^a = \{0, \frac{1}{2}, 1\}$. We have :

$G_{(0,1)} = \frac{1}{2} \iff x \in [0,1]$

$G_{(0,1)} = 0 \iff x \in (0,1)$

Take $H \in T_\rho$ such that $H(x)=\frac{1}{2}$ on $X-[2,3]$ and $H(x)=1$ on $[2,3]$. Then $\{x : G_{(0,1)} \wedge H(x) > \frac{1}{2}\} = [2,3] \cap X-[0,1] = 3$ which is not open in $T_{\frac{1}{2}}$.

Remark, $G_A = G_B$ does not imply $A=B$.

Example. $X=\mathbb{R}$, $T_{\frac{1}{2}}$=indiscrete topology, T_0 =usual topology then $G_{(0,1)}=G_{[0,1)}$. This fact, of course, stems from : $\bar{A}=\bar{B} \not\Rightarrow A=B$.

We will need the following fact in a later section:

Lemma 3.16. $\{x : G(x) > \alpha_{i-1}\} = \{x : G(x) \geq \alpha_i\}$.

Lemma 3.17. Let $\mathcal{A}$ be an L-FTP collection of topologies. Then for T in $\mathcal{F}(C)$ we have $T(C) \subset T$ if L is linearly ordered.

Proof. This follows directly from Theorem 2.3 in [4].

CHAPTER IV

Hausdorff and α-Hausdorff properties

We will need the following notion first defined in [6].

Definition 4.1. $(X,\mathbb{T})$ is α-Hausdorff (α^*-Hausdorff) for $\alpha \in L$ if for each $x,y \in X$ such that $x \neq y$, there are $u,v \in \mathbb{T}$ such that $u(x)>\alpha$ $(u(x)\geq\alpha)$, $v(y)>\alpha$ $(v(y)\geq\alpha)$ and $u \wedge v=0$.

Proposition 4.2. Let $\mathcal{A}$ be an L–FTP family of topologies. If $\mathbb{T}_{\alpha_n}$ is Hausdorff then for any α_i in $L^a-\{1\}$, $\mathbb{T}(C)$ is α_i-Hausdorff (α_i^*-Hausdorff and also 1^*-Hausdorff).

Proof. Let $x \neq y$. Since $\mathbb{T}_{\alpha_n}$ is Hausdorff, there are $U(x),U(y)$ in $\mathbb{T}$ such that $x \in U(x), y \in U(y)$ and $U(x) \cap U(y)=\emptyset$. Since $X-U(x)$ and $X-U(y)$ are closed in each $\mathbb{T}_{\alpha_i}$,

$$G_{X-U(x)}(a)=\begin{cases}1 & \text{if } a \in U(x)\\ 0 & \text{if } a \in X-U(x)\end{cases}$$

$$G_{X-U(y)}(a)=\begin{cases}1 & \text{if } a \in U(y)\\ 0 & \text{if } a \in X-U(y)\end{cases}$$

We have $G_{X-U(x)}(x)=1$, $G_{X-U(y)}(y)=1$ and $G_{X-U(x)} \wedge G_{X-U(y)}=0$, therefore $\mathbb{T}(C)$ is α_i-Hausdorff (α_i^*-Hausdorff) for any $\alpha_i \in L^a-\{1\}$.

Proposition 4.3. If $\mathbb{T}(C)$ is α_i-Hausdorff for α_i in $L^a-\{1\}$ then $\mathbb{T}_{\alpha_i}$ is Hausdorff. If $\mathbb{T}(C)$ is α_i^*-Hausdorff for α_i in $L^a-\{0,1\}$ then $\mathbb{T}_{\alpha_{i-1}}$ is Hausdorff.

Proof. Let $x_0 \neq y_0$ be in X. Since $\mathbb{T}(C)$ is α_i-Hausdorff (α_i^*-Hausdorff) there are G,H in $\mathbb{T}(C)$ such that $G(x_0)>\alpha_i$, $H(x_0)>\alpha_i$ $(G(x_0)\geq\alpha_i$, $H(x_0)\geq\alpha_i)$ and $G \wedge H=0$. Consider $\{x : G(x)>\alpha_i\}$ and $\{x : H(x)>\alpha_i\}$.

$(\{x: G(x) \geq \alpha_i\}$ and $\{x: H(x) \geq \alpha_i\})$. Suppose $z \in \{x: G(x) > \alpha_i\} \cap \{x: H(x) > \alpha_i\}$ $(z \in \{x: G(x) \geq \alpha_i\} \cap \{x: H(x) \geq \alpha_i\})$. Then $G(z) > \alpha_i$ and $H(z) > \alpha_i$. Since $\alpha \in L^a - \{1\}$, $G \wedge H(z) > \alpha$; we have a contradiction with the fact, that $G \wedge H = 0$. $(G(z) \geq \alpha_i$, $H(z) \geq \alpha_i$, and therefore we have $G(z) \wedge H(z) \geq \alpha_i$, a contradiction with $G \wedge H = 0)$. Since, clearly $x_0 \in \{x: G(x) > \alpha_i\}$, $y_0 \in \{x: H(x) > \alpha_i\}$ and both of these sets are open in T_{α_i}, $T(C)$ having the α-property for all α, we have T_{α_i} is Hausdorff. $(x_0 \in \{x: G(x) \geq \alpha_i\}$, $y_0 \in \{x: H(x) \geq \alpha_i\}$ and these sets are open in $T_{\alpha_{i-1}}$ by Lemma 3.16, hence $T_{\alpha_{i-1}}$ is Hausdorff).

Remark. This result depends in an essential way on the fact that $T(C)$ has the α-property for all α.

One can weaken the hypothesis of proposition 4.2 and prove a slightly more general result.

Proposition 4.4. Let $\mathcal{A}$ be an L-FTY family of topologies. Let T_{α_j} be Hausdorff. Then for $\alpha_i \in L^a - \{1\}$ with $\alpha_i \leq \alpha_j$, $T(C)$ is α_i-Hausdorff (α_i^*-Hausdorff).

Proof. Similar to 4.2.

Corollary 4.5. Let $\alpha_i \in L^a - \{1\}$. Then $T(C)$ is α_i-Hausdorff iff T_{α_i} is Hausdorff. Let $\alpha_i \in L^a - \{0\}$. Then $T(C)$ is α_i^*-Hausdorff iff $T_{\alpha_{i-1}}$ is Hausdorff.

Corollary 4.6. Proposition 4.2 and 4.4 are still true if $T(C)$ is replaced by any T in $\mathcal{F}(C)$. Proposition 4.3 is still true if L is linearly ordered and $T(C)$ is replaced by any T in $\mathcal{F}(C)$ having the α-property.

Proof. A direct application of Lemma 3.17.

CHAPTER V

Compactness and the different notions of fuzzy compactness

lhe definitions of d-compactness (α^*-compactness) are due to Gantner and Steinlage [11].

Definition 5.1. Let (X,T) be an L-fuzzy space, and let $\alpha \in L$. A collection $\mathcal{U} \subset T$ will be called an α-shading (resp. α^*-shading) of X if, for each x in X, there exists a U in T with $U(x) > \alpha$ (resp. $U(x) \geq \alpha$). A subcollection $\mathcal{V}$ of an α-shading (resp. 4*-shading) of X that is also and-shading (resp. α^*-shading) is called an α-subshading (resp. α^*-subshading) of $\mathcal{U}$. (X,T) will be calledd-compact (resp. α^*-compact) if each#-shading (resp. α^*-shading) of X has a finite α-subshading (resp. α^*-subshading).

Proposition 5.2. Let $\mathcal{A}$ be an L-FTP family of topologies. Let $\alpha_i \in L^a - \{1\}$. Then T_{α_i} is compact iff T(C) is 4;-compact.
Proof. To prove sufficiency, let $\{U_\gamma\}_\gamma$ be an open covering in T_{α_i}. By Theorem 3.9, $U_\gamma = \{x : G_\beta(x) > \alpha_i\}$ for some G_β in T(C). Obviously, the G_β constituteand;-shading of X, which is reducible to a finite α_i-subshading since T(C) is α_i-compact, and therefore $\{U_\gamma\}_\gamma$ is reducible to a finite subcovering. For necessity, let $\{G_\beta\}_\beta$ be an α_i-shading of X. Consider $\{\{x : G_\beta(x) > \alpha_i\}\}$. Clearly, it is an open covering of X.

Since T_{α_i} is compact, it is reducible to a finite subcovering and therefore $\{G_j\}_j$ is reducible to a finite α_i-subshading.

Corollary 5.3 T(C) is α_i-compact for all α_i in $L^a-\{1\}$ iff , is compact.

Remark 1. This proposition and its ensuing corollary depend on T(C) having the α-property.

Remark 2. A closely related functorial proof of Froposition 5.2 can be found in Theorem 3.1 of [6].

From now on we suppose $\mathcal{A}$, an L-FTP family of topologies, given.

Proposition 5.4. (X,T(C)) is α_k^*-compact iff (X,T(C)) is α_{k-1}-compact for α in $L^a-\{0\}$.

Proof. To prove sufficiency, let G be an α_{k-1}-shading of T(C). - Then we have $\bigvee_j G_j \geq \alpha_k$ and since T(C) is α_k^*-compact, there exists a finite subfamily $\mathcal{L}$ of $\mathcal{J}$ such that $\bigvee_{\mathcal{L}} G_j \geq \alpha_k$ that is $\bigvee_{\mathcal{L}} G_j > \alpha_{k-1}$. - Hence, $\mathcal{J}$ is reducible to a finite $\alpha_k^{7\mathcal{L}}$-subshading.

For necessity, let $\{H_j\}_j$ be an α_k^*-shading of T(C). Then $\bigvee_j H_j \geq \alpha_k$ implies $\bigvee_j H_j > \alpha_{k-1}$, hence $\{H_j\}$ is an α_{k-1}-shading of T(C). Since T(C) is α_{k-1}-compact, there exists a finite subfamily $\mathcal{L}$ such that $\bigvee_{\mathcal{L}} H_j \geq \alpha_k$. Therefore, $\{H_j\}_{\mathcal{L}}$ is a finite α_k^*-subshading of T(C).

Remark 3. By Corollary 3.15 , Corollary 5.3 and proposition 5.4 are still true for the Lowen minimum.

CHAPTER VI

Connectivity and 4-connectivity

In this chapter I use Rodabaugh's definition of α-connectivity from [7].

Definition 6.1. Let $(X,\mathcal{T})$ be a fuzzy topological space. $(X,\mathcal{T})$ is α-connected if there do not exist U,V in $\mathcal{T}-\{0,1\}$ such that $U \vee V > \alpha'$ and $U \wedge V = 0$. $(X,\mathcal{T})$ is α-disconnected if there are U,V in $\mathcal{T}-\{0,1\}$ such that $U \vee V > \alpha$ and $U \wedge V = 0$.

Proposition 6.2. Let $\mathcal{J}$ be an L-FTP family of topologies. Then for α_i in $L^a - \{1\}$, if $\mathcal{T}(C)$ is α_i-disconnected, then $\mathcal{T}_{\alpha_i}$ is disconnected.

Proof. Suppose $\mathcal{T}(C)$ is α_i-disconnected. Therefore there exist G,H in $\mathcal{T}(C)$ such that $G \vee H > \alpha_i$ and $G \wedge H = 0$. By theorem 3.9, the sets $U = \{x : G(x) > \alpha_i\}$, $V = \{x : H(x) > \alpha_i\}$ are open. Obviously, $U \cup V = X$. Suppose z is in $U \cap V$. $G(z) > \alpha_i$ and $H(z) > \alpha_i$ and since $\alpha_i \in L^a - \{1\}$, $H(z) \wedge G(z) > \alpha_i$ which contradicts the fact that $G \wedge H = 0$. therefore $U \cap V = \emptyset$, which proves that $(X, \mathcal{T}_{\alpha_i})$ is disconnected.

Proposition 6.3. Let $\mathcal{J}$ be an L-FTP family of topologies. Then $\mathcal{T}_{\alpha_1}$ disconnected implies $\mathcal{T}(C)$ not 1-connected.

Proof. Suppose $\mathcal{T}_{\alpha_1}$ is disconnected. 'There exist U,V in $\mathcal{T}$ such that $U \cup V = X$, $U \cap V = 0$. By theorem 3.9, $U = \{x : G(x) > 0\}$ and $V = \{x : H(x) > 0\}$ for some G,H in $\mathcal{T}(C)$. Clearly $G \vee H > 0$ and

$G \wedge H=0$. Thus $T(C)$ is not 1-connected.

Remark 1. This proposition depends in an essential way upon $T(C)$ having the α-property.

Proposition 6.4. Let $\mathcal{A} = \{T_i\}_{\mathcal{J}}$ be an L-FTP family of topologies with T_{α_1} connected. If $G \in T(C)$ and $G=\bar{G}$, then $G=0$ or $G=1$.

Proof. Let $G=\bar{G}$ and $G \neq 0,1$. $\{x : G(x) > 0\}$ is open in T_{α_1}, which implies $\{x : G(x)=0\}$ is closed in T_{α_1}. Since $G=\bar{G}$, G' is open and $\{x : G'(x)=1\} = \{x : G'(x) > \alpha_n\}$ is open in $T_{\alpha_n} \subseteq T_{\alpha_1}$. Hence, $\{x : G(x)=0\}$ is open in T_{α_1}. Therefore, T_{α_1} is disconnected.

Remark 2. These propositions are still true for any topologies having the α-property in $\mathcal{F}(C)$, including Lowen's minimum.

CHAPTER VII

Continuity and L-fuzzy continuity

Definition 7.1. Let $(X,\mathcal{T}),(Y,\Gamma)$ be two topological spaces. A function $F:(X,\mathcal{T})\to(Y,\Gamma)$ is said to be L-fuzzy continuous if for any H in Γ $F^{-1}(H)$ is in $\mathcal{T}$. ($F^{-1}(H)=H\circ F$).

Theorem 7.2. Let $\mathcal{A}=\{(X,\mathcal{T}_{\alpha_i})\}$, $\mathcal{B}=\{(Y,\Gamma_{\alpha_i})\}$ be two L-FTP families of topological spaces. Let c_α, k_α be their α-closure operators in X,Y, respectively, and T(C),T(D) the generated fuzzy topologies.

Let $f:(X,\mathcal{T}_{\alpha_i})\to(Y,\Gamma_{\alpha_i})$. We have:

(1) If $f:(X,T(C))\to(Y,T(D))$ is L-fuzzy continuous then $f:(X,\mathcal{T}_{\alpha_i})\to(Y,\Gamma_{\alpha_i})$ is continuous for all α_i in $L^a-\{1\}$.

(2) The converse is true if $f:(X,\mathcal{T}_{\alpha_i})\to(Y,\Gamma_{\alpha_i})$ is a homeomorphism for all α_i in $L^a-\{1\}$.

Proof. (1) Since f is L-fuzzy continuous, we can use Lemma 2.11 in [4]: Let (X,T) and (Y,T) be L-fuzzy topologies and let $f:X\to Y$ be L-fuzzy continuous. For α_i in $L^a-\{1\}$, let c_{α_i} and k_{α_i} be the α-closure operators in X,Y respectively. Then for every A in P(X), $f(c_{\alpha_i}(A))\subset k_{\alpha_i}(f(A))$. Hence, f is continuous at each level.

(2) It is sufficient to show that for H_A, a basis element of T(D), $f^{-1}(H_A)$ is an open set in T(C). For y in Y and A in P(Y) the general form of a basis element in T(D) is :

$H_A(y)=1$ iff $y \in Y-k_{\alpha_n}(A)$

$H_A(y)=\alpha_i$ iff $y \in k_{\alpha_i}(A)-k_{\alpha_{i-1}}(A)$

$H_A(y)=0$ iff $y \in k_o(A)$.

Now let $x \in X$. We have $H_A(f(x))=1$ iff $x \in X-f^{-1}(k_{\alpha_n}(A))$

$H_A(f(x))=\alpha_i$ iff $x \in f^{-1}(k_{\alpha_i}(A))-f^{-1}(k_{\alpha_{i-1}}(A))$

$H_a(f(x))=0$ iff $x \in f^{-1}(k_{\alpha_0}(A))$.

Since f is a homeomorphism for each α_i in $L^a-\{1\}$, we have $f^{-1}(k_{\alpha_i}(A))=c_{\alpha_i}(f^{-1}(A))$, therefore $H_A(f(x))=H_{f^{-1}(A)}(x)$, which shows that f is L-fuzzy continuous.

Corollary 7.3. Let $\mathcal{A}=\{X,\mathcal{T}_{\alpha_i}\}$, $\mathcal{B}=\{Y,\Gamma_{\alpha_i}\}$ be two L-FTP families of topological spaces, we have :

f is an L-fuzzy homeomorphism iff for each α_i in $L^a-\{1\}$ $f:(X,\mathcal{T}_{\alpha_i})\rightarrow(Y,\Gamma_{\alpha_i})$ is a homeomorphism.

In Pheorem 2.12 in[4], it was shown that level continuity was equivalent to fuzzy continuity if instead of T(C) and T(D), we take Sup($\mathcal{F}$(C)) and Sup($\mathcal{F}$(D)). We shall see in the following example that it is possible to find a smaller topology in $\mathcal{F}$(C) such that this conclusion still holds.

Example. Let $X=Y=\mathbb{R}$ and let $L=\{0,\frac{1}{2},1\}$.

$\frac{1}{2}$-level	$(X,T_{\frac{1}{2}})$=usual topology on $\mathbb{R}$	$(Y,T_{\frac{1}{2}})$=indiscrete
0-level	(X,Γ_o)=usual topology on $\mathbb{R}$	(Y,T_o)=usual topology on $\mathbb{R}$

For $A \in P(X)$ and $B \in P(Y)$ and $B \neq \emptyset$, we have;

$G_A(x)=1$ iff $x \in X-k_{\frac{1}{2}}(A)$ $\qquad$ $G_B(y)=1$ never

$G_A(x)=\frac{1}{2}$ never $\qquad$ $G_B(y)=\frac{1}{2}$ iff $y \in Y-c_o(B)$

$G_A(x)=0$ iff $x \in k_o(A)=k_{\frac{1}{2}}(A)$ $\qquad$ $G_B(y)=0$ iff $y \in c_o(B)$

Let f be any function from X into Y. Then $f^{-1}(G_B)=G_B of$ will take only two values: $0, \frac{1}{2}$. Hence, the inverse image of G_B cannot be written as a supremum of characteristic functions. In other words, no map is fuzzy continuous from T(C) into T(D). Suppose now, that f is continuous at each level. We claim that for any $B \neq \emptyset$ in P(Y), $G_B of = G_{f^{-1}(c_0(B))} \wedge \frac{1}{2}$.

$G_B of(x)=1$ never

$G_B of(x)=\frac{1}{2}$ iff $x \in X - f^{-1}(c_0(B))$

$G_B of(x)=0$ iff $x \in f^{-1}(c_0(B))$

Therefore, it is easy to see that $G_B of = G_{f^{-1}(c_0(A))} \wedge \frac{1}{2}$. Hence, f is fuzzy continuous from $(X, T(C) \vee T_c)$ into $(Y, T(D))$.

To conclude, let us show that $T_c \vee T(C) \neq \mathrm{Sup}\,\mathcal{F}(C)$. Let $A=[0,1)$. Define $G(x)=1$ iff $x \in X-A$ and $G(x)=\frac{1}{2}$ iff $x \in A$. Then $G \in T_\rho$. Suppose G is in $T_c \vee T(C)$, then $\{x : G(x)=1\}$ is open in $T_0 = T_{\frac{1}{2}}$ because $T_c \vee T(C)$ has the α-property. To summarize, we have exhibited a fuzzy topology different from $\mathrm{Sup}(\mathcal{F}(C))$ for which level continuity is equivalent to fuzzy continuity. Our next theorem will generalize this example. From now on, we will denote $T_c \vee T(C)$ by $T(K)$.

Theorem 7.4. Let $\mathcal{A}, \mathcal{B}$ be two L-P1P families of topological spaces as given in Theorem 7.2. Then continuity at each level is equivalent to fuzzy continuity from $(X, T(K))$ into $(Y, T(D))$.

Proof. We only need to show sufficiency. Let f be continuous at each level α_i, and let G_A be in $T(D)$. We claim that $G_A of = H$, where $H = G_{f^{-1}(k_{\alpha_n}(A))} \vee [G_{f^{-1}(k_{\alpha_{n-1}}(A))} \wedge \alpha_n] \ldots \vee [G_{f^{-1}(k_{\alpha_1}(A))} \wedge \alpha_2] \vee G_{f^{-1}(k_0(A))}$

Note that $G_{f^{-1}(k_{\alpha_n}(A))}$ is the characteristic function of $X-f^{-1}(k_{\alpha_n}(A))$ because $f^{-1}(k_{\alpha_n}(A))$ is closed in each T_{α_i}.
Moreover, for any r such that $1 \leq r \leq n-1$, an easy computation shows that $G_{f^{-1}(k_{\alpha_n}(A))}{}^{\wedge}\alpha_{r+1}$ takes only two values α_{r+1} and 0.
More precisely,

$$G_{f^{-1}(k_{\alpha_r}(A))}{}^{\wedge}\alpha_{r+1}(x) = \alpha_{r+1} \text{ iff } x \in X-f^{-1}(k_{\alpha_r}(A))$$

$$G_{f^{-1}(k_{\alpha_r}(A))}{}^{\wedge}\alpha_{r+1}(x) = 0 \text{ iff } x \in f^{-1}(k_{\alpha_r}(A))$$

On $X-f^{-1}(k_{\alpha_n}(A))$, $G_A of(x)=H(x)=1$. On $f^{-1}(k_{\alpha_{r+1}}(A))-f^{-1}(k_{\alpha_r}(A)) = f^{-1}(k_{\alpha_{r+1}}(A)-k_{\alpha_r}(A))$, $G_A of(x)=\alpha_{r+1}$ and for any $j>r$, $f^{-1}(k_{\alpha_j}(A)) \supseteq f^{-1}(k_{\alpha_{r+1}}(A))$, that is:

$$\left.\begin{array}{l} G_{f^{-1}(k_{\alpha_j}(A))}{}^{\wedge}\alpha_{j+1}(x)=0 \\ G_{f^{-1}(k_{\alpha_r}(A))}{}^{\wedge}\alpha_{r+1}(x)=\alpha_{r+1} \end{array}\right. \text{ for } x \in f^{-1}(k_{\alpha_{r+1}}(A)-f^{-1}(k_{\alpha_r}(A)))$$

On $f^{-1}(k_0(A))$, everything is 0. In conclusion, $H=G_A of$.
Level continuity is equivalent to fuzzy continuity using the Lowen minimum for both domain and range,

Remark. We can slightly generalize this result by using a countable chain for L^a (rather than a finite chain) in the definition of an L-FTP family of topological spaces, It is easy to see that Lemma 3.8, 'Theorem 3.9 are still true and that Theorem 7.2, 7.4 still hold.

Corollary 7.5. Let $\mathcal{A}, \mathcal{B}$ be two L-FPP families of topological spaces as given in Theorem 7.2. Let $f: X \to Y$. Let T_1, T_2 be L-fuzzy topologies such that $T(C) \leq T_1 \subseteq T(K)$ and $T(D) \leq T_2$.
If there is an α in $L^a-\{1\}$ such that $f:(X,T_\alpha) \to (Y,T_\alpha)$ is discontinuous, then $f:(X,T_1) \to (Y,T_2)$ is fuzzy discontinuous.
Proof. Suppose there is an α in $L^a-\{1\}$ such that f is dis-

continuous. By Theorem 7.4, $f:(X,\mathbb{T}(K))\to(Y,\mathbb{T}(D))$ is fuzzy discontinuous. Therefore, $f:(X,\mathbb{T}_1)\to(Y,\mathbb{T}(D))$ is fuzzy discontinuous and so $f:(X,\mathbb{T}_1)\to(Y,\mathbb{T}_2)$ is fuzzy discontinuous.

CHAPTER VIII

Suitability

First, recall from [7] the definitions of a suitable space and of a fuzzy retract. For both (X,T) is an L-fuzzy topological space.

Definition 8.1. If $A \subset X$, then A is non-trivial iff $\emptyset \subsetneq A \subsetneq X$. A is a suitable open set in (X,T) iff A is non-trivial and $\mu(A)$ is an L-fuzzy open set in (X,). (X,T) is suitable iff (X,T) has a suitable open set.

Definition 8.2. Let $A \subset X$. A is an L-fuzzy retract of X in (X,T) if there is a function $r:(X,T) \to (A,T_A)$ such that $r(x)=x$ for each x in A and r is fuzzy continuous.

Theorem 8.3. Let $\mathcal{A}$ be an L-FTP family of topological spaces. Then we have: A is suitable open iff for each α_i in $L^a-\{1\}$, A is open in T_{α_i}.

Proof. Suppose A is suitable open in $\Gamma(C)$. Let $\mu(A)=\bigvee_j G_{A_i}$. We have $A=\bigcup_j \{x:G_{A_i}(x)=1\}=\bigcup\{X-c_{\alpha_n}(A_i)\}$, which is open -- in T_{α_n}, therefore open in T_{α_i}, for each α_i in $L^a-\{1\}$.

Now, for sufficiency, let A in T_{α_i}, for each α_i in $L^a-\{1\}$. Denote $C=X-A$, we have $c_{\alpha_j}(C)=c_{\alpha_i}(C)=C$, for each α_i in $L^a-\{1\}$. Therefore, we have:

$$G_C(x)=1 \text{ iff } x \in X-c_{\alpha_n}(C) \text{ iff } x \in X-C.$$

$$G_C(x)=\alpha_i \text{ iff } x \in c_{\alpha_i}(C)-c_{\alpha_{i-1}}(C)=C-C=\emptyset$$

$$G_C(x)=0 \text{ iff } x \in c_{\alpha_1}(C)=C$$

Therefore, $G_C=\mu(X-C)=\mu(A)$ and A is a suitable open set.

Corollary 8.4. Given $\mathcal{A}$, an L-FTP family of topological spaces, we have that the set of suitable open sets of $T(C)$ is equal to $T_{\alpha_n}-\{\emptyset,X\}$.

Corollary 8.5. Let (X,T) be a topological space. We can associate to (X,T) a fuzzy topological space $(X,\mathcal{T})$ in a natural way: f is in $\mathcal{T}$ iff $f=\mu(A)$ for A in T. Let $(X,\mathcal{T}_{\alpha_n})$ be the fuzzy topological space associated with T_{α_n}, the coarsest topology in $\mathcal{A}$. Then $T(C)\supset\mathcal{T}_{\alpha_n}$.

Remark 8.6. Corollary 8.5 gives us another proof of Proposition 4.2.

Corollary 8.7. Let $|L|\geqslant 3$. Then $\sup(\mathcal{F}(C))=T_C\vee T(C)=T(K)$ iff T_{α_n} is discrete.

Proof. For necessity, let $O_{\alpha_i}=\{x:G(x)=\alpha_i\}$, and $O_1=\{x:G(x)=1\}$ for some chosen G in T_p. The O_{α_i} are pairwise disjoint. Denote $H=\vee(\mu(O_{\alpha_i})\wedge\alpha_i)$. We have $H(x)=\alpha_i$ iff $x\in O_{\alpha_i}$ ($\mu(O_{\alpha_i})$ is in $T(C)$ by Theorem 8.3), that is $H=G$ and G is in $T(K)$.

To prove sufficiency, let A be in $P(X)$. Define G by $G(x)=1$ iff $x\in A$, $\alpha_i\neq 0$ otherwise. G is in $T_{p'}$ and by Lemma 3.8 $\{x:G(x)=1\}$ is open in T_{α_n} and hence A is open in T_{α_n}.

Remark 8.8. The condition on the cardinality of L is indispensable in the above corollary. If $|L|=2$, $T_p=\{\mu(X)\}$ and $T(C)=\sup(\mathcal{F}(C))$ for any L-FTP family of topological spaces.

CHAPTER VIII

Suitability

First, recall from [7] the definitions of a suitable space and of a fuzzy retract. For both $(X,\mathbb{T})$ is an L-fuzzy topological space.

Definition 8.1. If $A\subset X$, then A is non-trivial iff $\emptyset\subsetneq A\subsetneq X$. A is a suitable open set in $(X,\mathbb{T})$ iff A is non-trivial and $\mathcal{M}(A)$ is an L-fuzzy open set in $(X,\mathbb{T})$. $(X,\mathbb{T})$ is suitable iff $(X,\mathbb{T})$ has a suitable open set.

Definition 8.2. Let $A\subset X$. A is an L-fuzzy retract of X in $(X,\mathbb{T})$ if there is a function $r:(X,\mathbb{T})\rightarrow(A,\mathbb{T}_A)$ such that $r(x)=x$ for each x in A and r is fuzzy continuous.

Pheorem 8.3. Let $\mathcal{J}$ be an L-FTP family of topological spaces. Then we have: A is suitable open iff for each α_i in $L^a-\{1\}$, A is open in $\mathbb{T}_{\alpha_i}$.

Proof. Suppose A is suitable open in $\mathbb{T}(C)$. Let $\mathcal{M}(A)=\bigvee_{\eta} G_{A_i}$. We have $A=\bigcup_{\eta}\{x:G_{A'}(x)=1\}=\bigcup\{X-c_{\alpha_n}(A_i)\}$, which is open -- in $\mathbb{T}_{\alpha_n}$, therefore open in $\mathbb{T}_{\alpha_i}$, for each α_i in $L^a-\{1\}$.

Now, for sufficiency, let A in $\mathbb{T}_{\alpha_i}$, for each α_i in $L^a-\{1\}$. Denote $C=X-A$, we have $c_{\alpha_j}(C)=c_{\alpha_i}(C)=C$, for each α_i in $L^a-\{1\}$.

'Therefore, we have:

$$G_C(x)=1 \text{ iff } x\in X-c_{\alpha_n}(C) \text{ iff } x\in X-C.$$

$$G_C(x)=\alpha_i \text{ iff } x\in c_{\alpha_i}(C)-c_{\alpha_{i-1}}(C)=C-C=\emptyset$$

$$G_C(x)=0 \text{ iff } x\in c_{\alpha_1}(C)=C$$

Therefore, $G_C=\mu(X-C)=\mu(A)$ and A is a suitable open set.

Corollary 8.4. Given $\mathcal{A}$, an L-FTP family of topological spaces, we have that the set of suitable open sets of $T(C)$ is equal to $T_{\alpha_n}-\{\emptyset, X\}$.

Corollary 8.5. Let (X,T) be a topological space. We can associate to (X,T) a fuzzy topological space (X,τ) in a natural way: f is in τ iff $f=\mu(A)$ for A in T. Let $(X,\tilde{\tau}_{\alpha_n})$ be the fuzzy topological space associated with T_{α_n}, the coarsest topology in $\mathcal{A}$. Then $T(C)\supset\tilde{\tau}_{\alpha_n}$.

Remark 8.6. Corollary 8.5 gives us another proof of Proposition 4.2.

Corollary 8.7. Let $|L|\geq 3$. Then $\sup(\tilde{\mathcal{F}}(C))=T_c \vee T(C)=T(K)$ iff T_{α_n} is discrete.

Proof. For necessity, let $O_{\alpha_i}=\{x:G(x)=\alpha_i\}$, and $O_1=\{x:G(x)=1\}$ for some chosen G in T_p. The O_{α_i} are pairwise disjoint. Denote $H=\vee(\mu(O_{\alpha_i})\wedge\alpha_i)$. We have $H(x)=\alpha_i$ iff $x\in O_{\alpha_i}$ ($\mu(O_{\alpha_i})$ is in $T(C)$ by Theorem 8.3), that is H=G and G is in $T(K)$.

To prove sufficiency, let A be in P(X). Define G by $G(x)=1$ iff $x\in A$, $\alpha_i\neq 0$ otherwise. G is in T_p, and by Lemma 3.8 $\{x:G(x)=1\}$ is open in T_{α_n} and hence A is open in T_{α_n}.

Remark 8.8. The condition on the cardinality of L is indispensable in the above corollary. If $|L|=2$, $T_p=\{\mu(X)\}$ and $T(C)=\sup(\tilde{\mathcal{F}}(C))$ for any L-FTP family of topological spaces.

Let $B\subset X$ and $\mathcal{A}=\{(X,T_{\alpha_i})\}_\eta$ a family of topological spaces be given. $\mathcal{B}=\{(B,T_{\alpha_i}\cap B)\}_\eta$ is also an L-FTP family of topological spaces, the generated fuzzy topology will be denoted by $T(B,C)$. By $T_B(C)$, we understand the fuzzy subspace topology induced on $T(C)$ by B, that is $T_B(C)=\{G|_B : G\in T(C)\}$.

Lemma 8.9. Let $B\subset X$ be suitable closed and let $\mathcal{A}$ be an L-FTP family of topological spaces. Then $T(B,C)\subseteq T_B(C)\subseteq T(B,K)$.

Proof. (1) $T(B,C)\subseteq T_B(C)$ $(T(B,K)\subseteq T_B(K))$

Denote by c_α the closure operator of $\mathcal{A}$ and k_α the closure operator of $\mathcal{B}=\{(B,T_{\alpha_i}\cap B)\}$. By definition of a subspace topology, we have: for $A\in P(B)$, $k_{\alpha_i}(A)=c_{\alpha_i}(A)\cap B$. Let $A\subseteq B$, $G_A\in T(B,C)$, $H_A\in T(C)$. For sets U,V,S, $S\cap(U-V)=S\cap U-S\cap V$. Then

$$H_A|_B(x)=1 \text{ iff } x\in X-c_{\alpha_n}(A)\cap B=B-k_{\alpha_n}(A)$$
$$\vdots$$
$$=0 \text{ iff } x\in c_{\alpha_1}(A)\cap B=k_{\alpha_1}(A).$$

Hence $H_A|_B=G_A$ and $T(B,C)\subseteq T_B(C)$ $(T(B,K)\subseteq T_B(K))$.

(2) $T_B(C)\subseteq T(B,K)$ $(T_B(K)\subseteq T(B,K))$.

Let $A\subseteq X$, $G_A\in T(C)$ and such that $\exists\, i\in L^a-\{1\}$, $c_{\alpha_i}(A)\cap B\neq\emptyset$ and $c_{\alpha_{i-1}}(A)\cap B=\emptyset$. We have

$$G_A|_B(x)=1 \text{ iff } x\in B-k_{\alpha_n}(A)$$
$$\vdots$$
$$=\alpha_i \text{ iff } x\in k_{\alpha_i}(A)$$
$$\vdots \quad \text{never}$$
$$=0$$

Claim: $G_A|_B=H=\left[\bigvee_{r=i}^{n}(G_{B\cap c_{\alpha_r}(A)}\wedge\alpha_{r+1})\right]\Big|_B\vee\alpha_i$

As in the proof of Pheorem 7.4, $G_{B\cap c_{\alpha_r}(A)}\wedge\alpha_{r+1}$ takes only two values a_{r+1} and 0. More precisely,

$$G_{B\cap c_{\alpha_r}(A)}\wedge\alpha_{r+1}|_B(x)=\alpha_{r+1} \text{ iff } x\in(X-c_{\alpha_r}(B\cap c_{\alpha_r}(A)))\cap B$$
$$\text{iff } x\in B-(c_{\alpha_r}(B)\cap c_{\alpha_r}(A))\cap B$$
$$\text{iff } x\in B-k_{\alpha_r}(A) \text{ (since } B \text{ is suitable closed)}$$

A similar computation shows that $G_{B\cap c_{\alpha_r}(A)}\wedge\alpha_{r+1}|_B(x)=0$ iff $x\in k_{\alpha_r}(a)$. Hence, $G_A|_B(x)=H(x)$ for $x\in B-k_{\alpha_i}(A)$. Let $x\in k_{\alpha_i}(A)$, then for any $r\geq i$, $G_{B\cap c_{\alpha_r}(A)}\wedge\alpha_{r+1}(x)=0$ thus, $H(x)=\alpha_i$.
Conclusion: $H(x)=G_A|_B(x)$ on B and since $B\cap c_{\alpha_r}(A)\subseteq B$ for any r, $H\in T(B,K)$.

Remark. If for $A\in P(X)$ and $c_{\alpha_i}(A)\cap B=\emptyset$ for each i, then $H_A|_B$ is identically one. Let $i:(B,T_{\alpha_i}\cap B)\to(X,T_{\alpha_i})$ be the injection. Then $G_A|_B=G_A\circ i$ on $B-k_{\alpha_i}(A)$.

Corollary 8.10. $T(B,K)=T_B(K)$ for B suitable closed.

Theorem 8.11. Let $B\subset X$ be suitable closed and $\mathcal{A}=\{(X,T_{\alpha_i})\}_3$, $\mathcal{B}=\{(B,T_{\alpha_i}\cap B)\}_3$ be two L-FTP families of topological hausdorff spaces. Then we have:

$\alpha_i\in L$ $r:(X,T_{\alpha_i})\to(B,T_{\alpha_i}\cap B)$ is a retraction iff
$r:(X,T(K))\to(B,T_B(K))$ is a fuzzy retraction.

Proof. This is a simple application of Theorem 7.4 and Corollary 8.10.

Remark 8.12. Since every problem of extension can be reduced to a problem of retraction (see Hu [1] for the ordinary case for example, and Rodabaugh [7] for the fuzzy case) we have, in fact, an extension property related to the Tietze extension property.

Theorem 8.13. Let $B\subseteq X$, $\mathcal{A}=\{(X,T_{\alpha_i})\}$, be an L-FTP family of topological spaces. If there exists a continuous map $r:(X,T_{\alpha_n})\to(B,T_{\alpha_1}\cap B)$ such that $r(x)=x$ for $x\in B$ then $(B,T_B(K))$ is a fuzzy retract of $(X,T(K))$.

Proof. Let $r:(X,T_{\alpha_n})\to(B,T_{\alpha_i}\cap B)$ be a continuous map such that $r(x)=x$ for $x\in B$. Then for each $\alpha_i\in L^a-\{1\}$, $r:(X,T_{\alpha_i})\to(B,T_{\alpha_i}\cap B)$ is a retraction, hence by Theorem 8.11, $(B,T_B(K))$ is a fuzzy retract of $(X,T(K))$.

CHAPTER IX

K_α as a Semi-closure Operator or a Closure Operator for $\alpha \in L-L^a$

$T(C)$ generates α-closure operators for α in $L-L^a$. In Proposition 2.10 [4], Klein shows that for α in L-1 with $\alpha \notin L^a$ and T in $\mathcal{F}(C)$ with c_α, k_α the α-closure operators generated by T, T(C) respectively, we have for every A in P(X) $k_\alpha(A) \subset c_\alpha(A)$. In this chapter, we will find conditions where this inclusion becomes an equality.

Definition 9.1. In a partially ordered set $(P, \leq)$, an element y in P is said to cover an element x of P if $x \leq y$ and if there does not exist any element $z \neq y$ in P such that $x \leq z$ and $z \leq y$.

Lemma 9.2. For G in $T(C)$, G only takes values in L^a.
Proof. This is clear for any basis element G_A. For an arbitrary sup of basis elements, the conclusion of the lemma holds because of the finiteness of L^a.

Proposition 9.3. Given $\mathcal{A}$ an L-FTP family of topological spaces, we have:

(i) $\forall \beta$ in $L-L^a$, k_β is a semi-closure operator

(ii) if β covers 0, k_β is a closure operator

(iii) $W_\beta = W_{\alpha_i}$, where $\alpha_i = \vee\{\alpha_j \in L^a - \{1\}$ such that $\beta > \alpha_j\}$

(α_i is in L^a since L^a is finite)

Proof. (i) It is sufficient to show: $\forall$ A,B in P(X), $k_\beta(A\cup B)\subseteq k_\beta(A)\cup k_\beta(B)$. Let $x\notin k_\beta(A)\cup k_\beta(B)$. Then there are G,H in T(C) such that:

$$G(x)>\beta \text{ and } G\wedge\mu(A)=0$$

$$H(x)>\beta \text{ and } H\wedge\mu(B)=0$$

We have $G(x)=\alpha_i$, $H(x)=\alpha_j$ with $\alpha_i,\alpha_j\in L^a-\{1\}$. Without loss of generality, we have $G(x)\wedge H(x)=\alpha_i>\beta$ and $G\wedge H\wedge\mu(A)=0$, $G\wedge H\wedge\mu(B)=0$, which implies $(G\wedge H\wedge\mu(A))\vee(G\wedge H\wedge\mu(B))=0$. Hence, $(G\wedge H)\wedge\mu(A\cup B)=0$, so $x\notin k_\beta(A\cup B)$ **and** k_β is a semi-closure operator.

(ii) Let A be $\in P(X)$. It is sufficient to show that:

$$k_\beta(k_\beta(A))\subseteq k_\beta(A).$$

Let $x_0\in X-k_\beta(A)$. Then there exists G in T(C) such that $G(x_0)>\beta$ and $G\wedge\mu(A)=0$. Let us consider $G\wedge\mu(k_\beta(A))$. For x in $X-k_\beta(A)$, we have $G\wedge\mu(k_\beta(A))(x)=0$. For x in A, $G(x)=0$, hence $G\wedge\mu(k_\beta(A))(x)=0$. For x in $k_\beta(A)-A$, we have $G(x)\not>\beta$, that is $G(x)<\beta$ by Lemma 9.2 and $\beta\in L-L^a$. Since β covers 0, $G(x)=0$. Hence, $\forall\, x\in X$, $G\wedge\mu(k_\beta(A))(x)=0$. So $x_0\notin k_\beta(k_\beta(A))$ and k_β is a closure operator.

(iii) Let $\alpha_i=\vee\{\alpha_j\in L^a-\{1\}$ such that $\beta>\alpha_j\}$. By Lemma 9.2, we have for any G in T(C), $\{x: G(x)>\beta\}=\{x: G(x)>\alpha_i\}$, so $W_{\alpha_i}=W_\beta$.

CHAPTER X

Normality

All the topological properties we have considered so far transfer rather nicely to the fuzzy topology $\mathbb{T}(C)$ generated by an L-FTP family of topological spaces. This was due to the fact that $\mathbb{T}(C)$ had the α-property. For fuzzy normality, we do not have, so far, such a direct correspondence.

Definition 10.1, $(X,\mathbb{T})$ is pseudo-fuzzy normal iff for any A,B closed in $\mathbb{T}$ such that $A\wedge B=0$, there exist U,V in $\mathbb{T}$ such that $A\leq U$, $B\leq V$ and $U\wedge V=0$.

Theorem 10.2. Given an L-FTP family of topological spaces, we have: $\mathbb{T}(C)$ is pseudo-normal iff T_{α_n} is normal.

Proof. Let A,B be closed in T_{α_n} and such that $A\cap B=\emptyset$. We have $\mu'(A),\mu'(B)$ in $\mathbb{T}(C)$ (cf. suitability) and $\mu(A)\wedge\mu(B)=0$. Hence, there exist H,G in $\mathbb{T}(C)$ such that $\mu(A)\leq H$, $\mu(B)\leq G$ and $H\wedge G=0$. We have $\{x:H(x)=1\}$, $\{x:G(x)=1\}$ are in T_{α_n}, and $\{x:\mu(A)=1\}\subset\{x:H(x)=1\}=U$ and $U\cap V=\emptyset$.

$\{x:\mu(B)=1\}\subset\{x:G(x)=1\}=V$

Therefore, T_{α_n} is normal.

For the converse, let F',K' be in $\mathbb{T}(C)$ and let $F\wedge K=0$. We have $A=\{x:F(x)>0\}$ is closed in T_{α_n} ($\{x:F(x)=0\}=\{x:F'(x)=1\}$ is open). $B=\{x:K(x)>0\}$ is closed in T_{α_n}. $A\cap B=\emptyset$. Therefore, there are U,V in T_{α_n} such that $A\subset U$, Proof. (i) It is sufficient to show: $\forall$ A,B in P(X), $k_\beta(A\cup B)\subseteq k_\beta(A)\cup k_\beta(B)$. Let $x\notin k_\beta(A)\cup k_\beta(B)$. Then there are G,H in T(C) such that:

$$G(x)>\beta \text{ and } G\wedge\mu(A)=0$$

$$H(x)>\beta \text{ and } H\wedge\mu(B)=0$$

We have $G(x)=\alpha_i$, $H(x)=\alpha_j$ with $\alpha_i,\alpha_j\in L^a-\{1\}$. Without loss of generality, we have $G(x)\wedge H(x)=\alpha_i>\beta$ and $G\wedge H\wedge\mu(A)=0$, $G\wedge H\wedge\mu(B)=0$, which implies $(G\wedge H\wedge\mu(A))\vee(G\wedge H\wedge\mu(B))=0$. Hence, $(G\wedge H)\wedge\mu(A\cup B)=0$, so $x\notin k_\beta(A\cup B)$ and k_β is a semi-closure operator.

(ii) Let A be $\in P(X)$. It is sufficient to show that:

$$k_\beta(k_\beta(A))\subseteq k_\beta(A).$$

Let $x_0\in X-k_\beta(A)$. Then there exists G in T(C) such that $G(x_0)>\beta$ and $G\wedge\mu(A)=0$. Let us consider $G\wedge\mu(k_\beta(A))$. For x in $X-k_\beta(A)$, we have $G\wedge\mu(k_\beta(A))(x)=0$. For x in A, $G(x)=0$, hence $G\wedge\mu(k_\beta(A))(x)=0$. For x in $k_\beta(A)-A$, we have $G(x)\not>\beta$, that is $G(x)<\beta$ by Lemma 9.2 and $\beta\in L-L^a$. Since β covers 0, $G(x)=0$. Hence, $\forall$ $x\in X$, $G\wedge\mu(k_\beta(A))(x)=0$. So $x_0\notin k_\beta(k_\beta(A))$ and k_β is a closure operator.

(iii) Let $\alpha_i=\bigvee\{\alpha_j\in L^a-\{1\}$ such that $\beta>\alpha_j\}$. By Lemma 9.2, we have for any G in T(C), $\{x:G(x)>\beta\}=\{x:G(x)>\alpha_i\}$, so $W_{\alpha_i}=W_\beta$.

CHAPTER X

Normality

All the topological properties we have considered so far transfer rather nicely to the fuzzy topology $T(C)$ generated by an L-FTP family of topological spaces. This was due to the fact that $T(C)$ had the α-property. For fuzzy normality, we do not have, so far, such a direct correspondence.

Definition 10.1. (X,T) is pseudo-fuzzy normal iff for any A,B closed in T such that $A\wedge B=0$, there exist U,V in T such that $A\leq U$, $B\leq V$ and $U\wedge V=0$.

Theorem 10.2. Given an L-FTP family of topological spaces, we have: $T(C)$ is pseudo-normal iff T_{α_n} is normal.

Proof. Let A,B be closed in T_{α_n} and such that $A\cap B=\emptyset$. We have $\mu'(A),\mu'(B)$ in $T(C)$ (cf. suitability) and $\mu(A)\wedge\mu(B)=0$. Hence, there exist H,G in $T(C)$ such that $\mu(A)\leq H$, $\mu(B)\leq G$ and $H\wedge G=0$. We have $\{x:H(x)=1\}$, $\{x:G(x)=1\}$ are in T_{α_n}, and $\{x:\mu(A)=1\}\subset\{x:H(x)=1\}=U$ and $U\cap V=\emptyset$.

$\{x:\mu(B)=1\}\subset\{x:G(x)=1\}=V$

Therefore, T_{α_n} is normal.

for the converse, let F',K' be in $T(C)$ and let $F\wedge K=0$. We have $A=\{x:F(x)>0\}$ is closed in T_{α_n} ($\{x:F(x)=0\}=\{x:F'(x)=1\}$ is open). $B=\{x:K(x)>0\}$ is closed in T_{α_n}. $A\cap B=\emptyset$. Therefore, there are U,V in T_{α_n} such that $A\subset U$,

$B \subset V$ and $U \cap V = \emptyset$. Consider $\mu(U)$, $\mu(V)$. They both are in $\mathcal{T}(C)$. We have $F \leq \mu(U)$, since if x is in A, $F(x) > 0$ and $\mu(U) = 1$, while if x is in X-A, $F(x) = 0$. Similarly $K \leq \mu(V)$. Clearly $\mu(U) \wedge \mu(V) = 0$. Therefore $\mathcal{T}(C)$ is pseudo-normal.

Definition 10.3. (Hutton) A fuzzy topological space is normal if for every closed set K and open set U such that $K \leq U$, there exists a set V such that : $K \leq \overset{\circ}{V} \leq \bar{V} \leq U$.

Remark 10.4. This definition is more interesting than the preceding one, since we can prove a fuzzy Urysohn's lemma using it. See [2].

Theorem 10.5. Let $\mathcal{C}$ be an L-FTP family of topological spaces, let $L = \{0 = \alpha_1 < \alpha_2 < \alpha_3 = 1\}$ and let $N(\alpha_i) = \{x : N(x) \geq \alpha_i\}$. Then we have the following:

$*\left[\forall F, G \text{ such that } F', G \in \mathcal{T}(C) \text{ and } F \leq G, \exists H \in \mathcal{T}(C) \text{ such that } F(\alpha_2) \subseteq H(\alpha_3) \subseteq \dot{\bar{H}}(\alpha_2) \subseteq G(\alpha_3)\right]$ implies $(X, \mathcal{T}(C))$ is fuzzy normal.

Proof. Suppose $*$ true. Let F', G be in $\mathcal{T}(C)$ such that $F \leq G$. By hypothesis, there is an H in $\mathcal{T}(C)$ such that:

$$\{x : F(x) \geq \alpha_2\} \subseteq \{x : H(x) = 1\} \subseteq \{x : \bar{H}(x) \geq \alpha_2\} \subseteq \{x : G(x) = 1\}.$$

We have $F \leq H \leq \bar{H} \leq G$, that is $(X, \mathcal{T}(C))$ is fuzzy normal.

APPENDIX

The following definitions can be found in [9].

Definition A.1. The objects of FUZZ are of the form $(X,L,\mathcal{T})$, where $(X,\mathcal{T})$ is L-fts.

Definition 8.2. A morphism from $(X,L_1,\mathcal{T}_1)$ to $(X,L_2,\mathcal{T}_2)$ is a pair (f,ϕ) satisfying the following conditions:

(i) $f:X\to X$ is a function

(ii) $\phi^{-1}:L_2\to L_1$ is a function preserving $\vee$, $\wedge$, $'$ (we call ϕ^{-1} a lattice morphism;) and

(iii) v implies $\phi^{-1}ovof\in\mathcal{T}_1$.

We only assume ϕ is a relation. We may speak of (f,ϕ) as being F-continuous. We say (f,ϕ) is an (F-) homeomorphism if f and ϕ^{-1} are bijections and each of (f,ϕ) and (f^{-1},ϕ^{-1}) are morphisms.

Proposition A.3. Let L_1, L_2 be linearly ordered and such that $|L_1|=|L_2|$. Let $\{T_{\alpha_i}\}_{\alpha_i\in L_1}$ be an L-FTP family of topologies. Then the L-FTP family of topologies $\{T_{\beta_i}\}_{\beta_i\in L_2}$ where $T_{\alpha_i}=T_{\beta_i}$ generates a fuzzy topology T_2 homeomorphic to the fuzzy topology T_1 generated by $\{T_{\alpha_i}\}$: $(X,T_1,L_1)\simeq(X,T_2,L_2)$.

Proof. (i),(ii) of Definition A.2 are trivially satisfied, with $f=Id_X$ and $\phi : L_1\to L_2$ by $\phi(\alpha_i)=\beta_i$. Let H_A be in T_2, then $H_A(x)=\wedge\{\beta\in L_2$ such that $x\in k_\beta(A)\}$. We have $\phi^{-1}(H_A(x))=\wedge\{\phi^{-1}(\beta)\in L_1$ such that $x\in k_\beta(A)\}$ and since by hypothesis $k_\beta(A)$ (A), $\phi^{-1}(H_A(x))=\wedge\{\alpha\in L_1$ such that $x\in k_\alpha(A)\}=G_A(x)\in$

Hence (iii) is satisfied. The proof that $(i_X^{-1}$, is a morphism is similar.

As an application of this proposition, we are going to construct a fuzzy topology with the unit interval as the lattice. (For $\alpha \in L$, $\alpha' = 1-\alpha$)

Let (X, T_1, L_1) be a fuzzy topological space generated by the L-FTP family of topological spaces $\{(X, T_{\alpha_i})\}_{\alpha_i \in L_1}$ and $\varphi : L_1 \to I$ be a lattice morphism defined by $\varphi(\alpha_i) = \beta_i$. For $\varphi(\alpha_i) \leq \beta < \varphi(\alpha_{i+1})$, put $T_\beta = T_{\varphi(\alpha_i)} = T_{\alpha_i}$. Since $\alpha_i' = (\alpha_{n-i+2})'$, we have that:
$\varphi(\alpha_i') = \varphi(1-\alpha_i) = \varphi(\alpha_{n-i+2}) = \beta_{n-1+2} = \beta_i' = 1-\varphi(\alpha_i)$. (Recall that $L_1 = \{\alpha_1 = 0 \leq \alpha_2 \ldots \leq \alpha_n \leq 1\}$.)

Lemma A.4. $\{(X, T_\beta)\}_{\beta \in I}$ is an L-FTP family of topological spaces.
Proof. The family $\mathcal{C} = \{c_\beta\}_\beta$ of closure operators satisfies clearly the conditions of Definition 3.3.

Remark A.5. Denote by (X, T', I) the fuzzy topological space generated by $\{(X, T_\beta)\}_\beta$ $(T' = T(C))$. In (X, T', I) all fuzzy sets are finite valued.

Theorem A.6. Given a lattice isomorphism $\phi : L_1 \to L_2$ defined by $\phi(\alpha_i) = \beta_i$, there exists an isomorphism (i_X, Γ) such that the following diagram commutes:

$$\begin{array}{ccc} (X, T_1, L_1) & \xrightarrow[\simeq]{(i_X, \phi)} & (X, T_2, L_2) \\ (i_X, \varphi) \downarrow & & \downarrow (i_X, \psi) \\ (X, T_1', I) & \xrightarrow[\simeq]{(i_X, \Gamma)} & (X, T_2', I) \end{array}$$

Proof. Let $a = \dfrac{\psi(\phi(\alpha_{i+1})) - \psi(\phi(\alpha_i))}{\varphi(\alpha_{i+1}) - \varphi(\alpha_i)}$. For $\varphi(\alpha_i) \leq \gamma < \varphi(\alpha_{i+1})$, define Γ by $\Gamma(\gamma) = a(\gamma - \varphi(\alpha_i)) + \psi(\phi(\alpha_i))$. Clearly Γ is a monomorphism. Now let $\xi \in I$. If $\xi = \psi(\beta_i)$ for some $\beta_i \in L_2$, then $\xi = \Gamma(\alpha_i)$. If $\psi(\beta_i) < \xi < \psi(\beta_{i+1})$ then $\xi = \Gamma(\eta)$ where $\eta = -\psi(\phi(\alpha_i)) + a\varphi(\alpha_i)$. So Γ is a bijection. Since Γ is cbntinuous, it preserves $\vee, \wedge$.

Let $\varphi(\alpha_i) < \gamma < \varphi(\alpha_{i+1})$, then $\varphi(\alpha_{n-i+2}) < \gamma' < \varphi(\alpha_{n-i+1})$.

$$a = \frac{\psi(\phi(\alpha_{n-i+1})) - \psi(\phi(\alpha_{n-i+2}))}{\varphi(\alpha_{n-i+1}) - \varphi(\alpha_{n-i+2})}, \quad b = \psi(\phi(\alpha_{n-i+2})),$$

$$a = \frac{(1 - \psi(\phi(\alpha_{i+1}))) - (1 - \psi(\phi(\alpha_i)))}{1 - \varphi(\alpha_{i+1}) - 1 + \varphi(\alpha_i)}, \quad b = 1 - \psi(\phi(\alpha_i)),$$

$$a = \frac{\psi(\phi(\alpha_{i+1})) - \psi(\phi(\alpha_i))}{\varphi(\alpha_{i+1}) - \varphi(\alpha_i)}, \quad \gamma - \varphi(\alpha_{n-1+2}) = (\varphi(\alpha_i) - \gamma).$$

Hence,
$$\Gamma(\gamma') = \Gamma(1-\gamma) = a(\varphi(\alpha_i) - \gamma) + 1 - \psi(\phi(\alpha_i))$$
$$= 1 - a(\gamma - \varphi(\alpha_i)) - \psi(\phi(\alpha_i)).$$

Therefore, Γ preserves the involution.

Now, $\Gamma^{-1}(\gamma) = \frac{1}{a}(\gamma + a(\varphi(\alpha_i)) - \psi(\phi(\alpha_i)))$ and

$$\Gamma^{-1}(1-\gamma) = \frac{1}{a}(1 - \gamma + a - a\varphi(\alpha_i) - 1 + \psi(\phi(\alpha_i)))$$
$$= 1 - \frac{1}{a}(\gamma + a(\varphi(\alpha_i)) - \psi(\phi(\alpha_i))).$$

The case where $\gamma = \varphi(\alpha_i)$ is similar but simpler. In conclusion. Γ is an isomorphism.

Remark A.7. Γ by construction is not unique, that is, -- our diagram is not universal in the categorical sense.

The interest of this construction is that it allows us to use Lowen's definition of fuzzy compactness without any modifications.[5]

Definition A.8. (X,δ) is fuzzy compact in the Lowen sense **iff** for each family $\beta\subset\delta$ such that $\bigvee_{\beta} u\geq\alpha$ and for each $\varepsilon\in(0,\alpha]$ there exists a finite subfamily β_0 of β such that $\bigvee_{\beta_0} u\geq\alpha-\varepsilon$.

Definition A.9. (X,T,L_1) is fuzzy compact in the Lowen sense **iff** (X,T',I) is fuzzy compact in the Lowen sense.

Proposition A.10. (X,T,L_1) is fuzzy compact in the Lowen sense iff $\forall\alpha_i\in L_1$, (X,T) is α_i^*-compact.

Proof. To show sufficiency, let $\alpha_i\leq\alpha<\alpha_{i+1}$.

<u>case 1</u>. $\alpha=\alpha_i$. Let $\{u\}_\beta$ be an α^*-shading of X. Then $\bigvee_\beta u\geq\alpha$ implies $\bigvee_{\beta_0} u\geq\alpha_i$ for some finite subfamily β_0 of β.
Let $0<\varepsilon\leq\alpha_i$, then $\bigvee_{\beta_0} u\geq\alpha_i-\varepsilon$.

<u>case 2</u>. $\alpha>\alpha_i$. $\bigvee_\beta u\geq\alpha$ implies $\bigvee_{\beta_0} u\geq\alpha_{i+1}$ by the Remark A.5.
Let $0<\varepsilon\leq\alpha$ then $\bigvee_{\beta_0} u\geq\alpha_{i+1}>\alpha-\varepsilon$.

For necessity, let $\varepsilon=\alpha_i-\alpha_{i-1}/2$ and $\{u\}_\beta$ be an α_i^*-shading of X. Then $\bigvee_\beta u\geq\alpha_i$ implies there exists a finite subfamily β_0 of β such that $\bigvee_{\beta_0} u\geq\alpha_i-\varepsilon>\alpha_{i-1}$. By the Remark A.5, $\bigvee_{\beta_0} u\geq\alpha_i$.
Now let $\alpha_i<\alpha<\alpha_{i+1}$ and $\bigvee_\beta u\geq\alpha$ then in particular for $\varepsilon=\alpha-\alpha_i/2$, there exists a finite subfamily β_0 of β such that $\bigvee_{\beta_0} u\geq\alpha-\varepsilon$, that is $\bigvee_{\beta_0} u\geq\alpha_{i+1}$ or $\bigvee_{\beta_0} u\geq\alpha$.

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