A Character Theory Free Proof of Burnside's $p^a q^b$ Theorem

by

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ABSTRACT

In 1904, George Burnside [2] proved that any group G with $|G| = p^a q^b$ where p and q are primes and a and b are positive integers is solvable. Burnside accomplished this through the use of character theory, i.e., the interaction between a group and a vector space.

Since then, group theorists began to try to prove this theorem without the use of character theory. They wanted a proof that relied only on group theoretical principles. This was finally achieved in 1972 by Helmut Bender [1].

However, in 1970, David M. Goldschmidt [3] supplied a group theoretic proof of Burnside's Theorem but only when the order of the group, G, was odd. Then in 1972, Hiroshi Matsuyama [4] supplied a group theoretic proof of Burnside's Theorem when the order of the group, G, was even. Ironically, Bender's and Matsuyama's results occurred independently and simultaneously. Therefore, both papers were published even though Bender's proof was more general.

The goal of this paper is to present the background knowledge and the more general proof of Burnside's Theorem.

I would like to thank my friends and family for their support. I would especially like to thank my advisor, Dr. Flowers, for all his insight and for inspiring me to learn about group theory.

Contents

1 Preliminaries

In this section, we will introduce some background concepts and ideas. These ideas will build up the tools needed for the proof of Burnside's Theorem. We begin by introducing the idea of a group.

Definition 1.1. *A* **group** *is a nonempty set G along with a binary operation $*$ such that*

1. *(closure): $a * b \in G$ for all $a, b \in G$;*
2. *(associativity): $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$;*
3. *(identity): there exists $e \in G$ such that $e * a = a * e = a$ for all $a \in G$;*
4. *(inverse): for all $a \in G$ there exists $b \in G$ such that $a * b = b * a = e$.*

Definition 1.2. *Let $(G, *)$ be a group. A subset $H \subseteq G$ is called a* **subgroup of G** *if $(H, *)$ is a group. We write $H \leq G$.*

Theorem 1.1 (Subgroup Test)**.** *Let G be a group and $\emptyset \neq H \subseteq G$. Then $H \leq G$ if and only if $ab^{-1} \in H$ for all $a, b \in H$.*

Definition 1.3. *Let G be group, $a \in G$, and $H \leq G$. Then the following are subgroups of G:*

1. *$Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$. We call this the* **center of G**.
2. *$\langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}$. We call this the* **cyclic subgroup generated by a**.
3. *$C_G(a) = \{g \in G \mid ag = ga\}$. We call this the* **centralizer of a**.
4. *$N_G(H) = \{g \in G \mid gHg^{-1} \in H\}$. We call this the* **normalizer of H**.

Definition 1.4. *Let G be a group, $H \leq G$ and $g \in G$. The* **left coset of $\boldsymbol{H}$ in $\boldsymbol{G}$ containing $\boldsymbol{g}$** *is*

$$gH = \{gh \mid h \in H\}.$$

Theorem 1.2. *Let G be a group, $H \leq G$, and $a, b \in G$. Then $aH = bH$ if and only if $b^{-1}a \in H$.*

Definition 1.5. *Let G_1 and G_2 be groups and $\phi : G_1 \to G_2$. Then ϕ is a* **homomorphism** *if*

$$\phi(ab) = \phi(a)\phi(b) \quad \text{for all } a, b \in G.$$

If, in addition, ϕ is 1-1 and onto, we call ϕ an isomorphism and we write $G_1 \cong G_2$.

Theorem 1.3. *Let G_1, G_2 be groups and $\phi : G_1 \to G_2$ be a homomorphism. Define the* **kernel of $\boldsymbol{\phi}$** *by*

$$kern\, \phi = \{g \in G_1 \mid \phi(g) = 1\}.$$

Then $kern\, \phi \leq G_1$.

Definition 1.6. *Let G be a group and $H \leq G$. Then H is a* **normal subgroup of $\boldsymbol{G}$** *if $ghg^{-1} \in H$ for all $g \in G$ and for all $h \in H$. We write $H \trianglelefteq G$.*

Theorem 1.4. *Let G be a group and $H \trianglelefteq G$. Define*

$$\frac{G}{H} = \{gH \mid g \in G\}.$$

Then $\frac{G}{H}$ is a group under the operation

$$aHbH = abH \ \text{ for all } aH, bH \in \frac{G}{H}.$$

We call $\frac{G}{H}$ the quotient group.

Lemma 1.1. *Let G be a group. Then $\frac{G}{\{1\}} \cong G$.*

Proof. Define $\phi : G \to \frac{G}{\{1\}}$ by $\phi(g) = g\{1\}$ for all $g \in G$. We want to show that ϕ is a homomorphism. Let $a, b \in G$. Then

$$\begin{aligned} \phi(ab) &= ab\{1\} \\ &= a\{1\}b\{1\} \\ &= \phi(a)\phi(b). \end{aligned}$$

Thus, ϕ is a homomorphism. We want to show that ϕ is onto. Let $g\{1\} \in \frac{G}{\{1\}}$. Then $g \in G$ and $\phi(g) = g\{1\}$. Thus, ϕ is onto. We want to show that ϕ is 1-1. Suppose $a, b \in G$ such that $\phi(a) = \phi(b)$. Then $a\{1\} = b\{1\}$ or $b^{-1}a \in \{1\}$. So $b^{-1}a = 1$ or $a = b$. Thus, ϕ is 1-1. Therefore, ϕ is an isomorphism and so $\frac{G}{\{1\}} \cong G$.

□

Lemma 1.2. *Let G be a group and $H \leq G$ such that $\frac{|G|}{|H|} = 2$ then $H \trianglelefteq G$.*

Proof. Let $g \in G$ and $h \in H$. We want to show that $ghg^{-1} \in H$. If $g \in H$ then $ghg^{-1} \in H$ since $H \leq G$. If $g \notin H$ then $gH \neq 1H$. Then since $\frac{|G|}{|H|} = 2$ we get $G = 1H \cup gH$. Now $ghg^{-1} \in G$ and so $ghg^{-1} \in 1H$ or $ghg^{-1} \in gH$. If $ghg^{-1} \in gH$ then there exists $h_1 \in H$ such that $ghg^{-1} = gh_1$. Then $hg^{-1} = h_1$ or $g = h_1^{-1}h \in H$, which contradicts $g \notin H$. Therefore $ghg^{-1} \in 1H = H$ and so $H \trianglelefteq G$.

□

Theorem 1.5 (1^{st} Isomorphism Theorem)**.** *Let G_1, G_2 be groups and $\phi : G_1 \to G_2$ be a homomorphism. Then*

$$\frac{G_1}{kern\ \phi} \cong \phi(G_1).$$

Proof. Let $K = kern\,\phi$. Define $\theta : \frac{G_1}{K} \to \phi(G_1)$ by $\theta(aK) = \phi(a)$ for all $aK \in \frac{G_1}{K}$. We want to show that θ is a homomorphism. Let $aK, bK \in \frac{G_1}{K}$. Then

$$\begin{aligned}\theta(aKbK) &= \theta(abK)\\ &= \phi(ab)\\ &= \phi(a)\phi(b)\\ &= \theta(aK)\theta(bK)\end{aligned}$$

and so θ is a homomorphism. We want to show that θ is 1-1. Let $aK, bK \in \frac{G_1}{K}$ such that $\theta(aK) = \theta(bK)$. Then $\phi(a) = \phi(b)$ or $\phi(b)^{-1}\phi(a) = 1$. Thus, $\phi(b^{-1})\phi(a) = 1$ or $\phi(b^{-1}a) = 1$. Hence $b^{-1}a \in kern\,\phi = K$. Thus, $aK = bK$ and so θ is 1-1. We want to show that θ is onto. Let $\phi(x) \in \phi(G_1)$ where $x \in G_1$. Then $xK \in \frac{G_1}{K}$ and $\theta(xK) = \phi(x)$. Hence, θ is onto and so θ is an isomorphism. Therefore, $\frac{G_1}{kern\,\phi} \cong \phi(G_1)$.

□

Theorem 1.6 (2^{nd} Isomorphism Theorem)**.** *Let G be a group, $N \trianglelefteq G$, and $H \leq G$. Then*

$$\frac{HN}{N} \cong \frac{H}{H \cap N}.$$

Proof. Define $\phi : H \to \frac{HN}{N}$ by $\phi(h) = hN$ for all $h \in H$. We want to show that ϕ is a homomorphism. Let $a, b \in H$. Then

$$\begin{aligned}\phi(ab) &= abN\\ &= aNbN\\ &= \phi(a)\phi(b)\end{aligned}$$

and so ϕ is a homomorphism. We want to show that ϕ is onto. Let $hnN \in \dfrac{HN}{N}$. Then

$$\begin{aligned}\phi(h) &= hN \\ &= hnN \quad \text{as } (hn)^{-1}h = n^{-1} \in N\end{aligned}$$

and so ϕ is onto. We claim that the $kern\, \phi = H \cap N$. Now,

$$h \in kern\, \phi \Leftrightarrow \phi(h) = 1N \Leftrightarrow hN = 1N \Leftrightarrow 1^{-1}h \in N \Leftrightarrow h \in N \Leftrightarrow h \in H \cap N.$$

Thus, $kern\, \phi = H \cap N$. By Theorem 1.5, $\dfrac{H}{kern\, \phi} \cong \phi(H)$. Thus, $\dfrac{H}{H \cap N} \cong \phi(H)$ and, since ϕ is onto, we get $\dfrac{H}{H \cap N} \cong \dfrac{HN}{N}$.

$\square$

Theorem 1.7 (3^{rd} Isomorphism Theorem)**.** *Let G be a group, $N \trianglelefteq G$, $H \trianglelefteq G$ such that $N \leq H$. Then*

$$\frac{G/N}{H/N} \cong \frac{G}{H}.$$

Proof. Define $\phi : \dfrac{G}{N} \to \dfrac{G}{H}$ by $\phi(gN) = gH$ for all $gN \in \dfrac{G}{N}$. We want to show that ϕ is well-defined. Let $aN, bN \in \dfrac{G}{N}$ such that $aN = bN$. Then $a = a1 \in aN = bN$ and so there exists $n \in N$ such that $a = bn$. Then

$$\begin{aligned}\phi(aN) &= aH \\ &= bnH \\ &= bH \\ &= \phi(bN)\end{aligned}$$

and so ϕ is well-defined. We want to show that ϕ is a homomorphism. Let $aN, bN \in \frac{G}{N}$. Then

$$\begin{aligned}\phi(aNbN) &= \phi(abN)\\ &= abH\\ &= aHbH\\ &= \phi(aN)\phi(bN)\end{aligned}$$

and so ϕ is a homomorphism. We want to show that ϕ is onto. Let $gH \in \frac{G}{H}$. Then $gN \in \frac{G}{N}$ and $\phi(gN) = gH$. Thus, ϕ is onto. We claim that the $kern\, \phi = \frac{H}{N}$. Then,

$$gN \in kern\, \phi \Leftrightarrow \phi(gN) = 1H \Leftrightarrow gH = 1H \Leftrightarrow g \in H \Leftrightarrow gN \in \frac{H}{N}.$$

Thus, $kern\, \phi = \frac{H}{N}$. By Theorem 1.5, $\frac{G/N}{kern\, \phi} \cong \phi(G/N)$ or $\frac{G/N}{H/N} \cong \phi(G/N)$ and, since ϕ is onto, we get $\frac{G/N}{H/N} \cong \frac{G}{H}$.

$\square$

Theorem 1.8. *Let G be a group and $N \trianglelefteq G$. Define $\phi : G \to \frac{G}{N}$ by $\phi(g) = gN$ for all $g \in G$. We call ϕ the natural map. The following are true:*

1. *ϕ is a homomorphism*
2. *$kern\, \phi = N$*
3. *If $H \leq G$, then $\phi(H) = \frac{HN}{N}$*
4. *If $H \leq G$, then $\phi^{-1}\left(\frac{HN}{N}\right) = HN$*
5. *If $L \leq \frac{G}{N}$, then $L = \frac{K}{N}$ where $N \leq K \leq G$*

Proof. For (1), let $a, b \in G$. Then

$$\begin{aligned}\phi(ab) &= abN \\ &= aNbN \\ &= \phi(a)\phi(b).\end{aligned}$$

Thus, ϕ is a homomorphism. For (2), let $n \in kern\, \phi$. Then,

$$n \in kern\, \phi \Leftrightarrow \phi(n) = 1N \Leftrightarrow nN = 1N \Leftrightarrow 1^{-1}n \in N \Leftrightarrow n \in N.$$

Thus, $kern\, \phi = N$. For (3), $\phi(H) \subseteq \frac{HN}{N}$. Let $h \in H, n \in N$, and $\phi(h) \in \phi(H)$. Then

$$\begin{aligned}\phi(h) &= hN \\ &= h1N \\ &\in \frac{HN}{N}.\end{aligned}$$

Hence, $\phi(H) \subseteq \frac{HN}{N}$. Next, $\frac{HN}{N} \subseteq \phi(H)$. Let $hnN \in \frac{HN}{N}$. Then

$$\begin{aligned}\phi(h) &= hN \\ &= hnN \quad \text{as } h^{-1}hn \in N.\end{aligned}$$

Hence, $hnN \in \phi(H)$ and so $\frac{HN}{N} \subseteq \phi(H)$. Therefore, $\phi(H) = \frac{HN}{N}$. For (4),

$HN \subseteq \phi^{-1}\left(\frac{HN}{N}\right)$. Let $hn \in HN$. Then

$$\begin{aligned}\phi(hn) &= hnN \\ &\in \frac{HN}{N}.\end{aligned}$$

Thus, $hn \in \phi^{-1}\left(\frac{HN}{N}\right)$ and $HN \subseteq \phi^{-1}\left(\frac{HN}{N}\right)$. Next, we want to show that $\phi^{-1}\left(\frac{HN}{N}\right) \subseteq HN$. Let $g \in \phi^{-1}\left(\frac{HN}{N}\right)$. Then $\phi(g) \in \frac{HN}{N}$ or $gN \in \frac{HN}{N}$. Thus, there exists $h \in H$ and $n \in N$ such that $gN = hnN$. Then $g = g1 \in gN = hnN$ and so there exists $n_1 \in N$ such that $g = hnn_1 \in HN$. Thus, $\phi^{-1}\left(\frac{HN}{N}\right) \subseteq HN$ and so $\phi^{-1}\left(\frac{HN}{N}\right) = HN$. For (5), we know $\phi^{-1}(L) \leq G$. If $n \in N$ then $\phi(n) = 1N \in L$ and so $n \in \phi^{-1}(L)$. Thus $N \leq \phi^{-1}(L)$. We claim that $\frac{\phi^{-1}(L)}{N} = L$. Let $gN \in L$. Then $\phi(g) \in L$ and so $g \in \phi^{-1}(L)$. Hence, $gN \in \frac{\phi^{-1}(L)}{N}$ and so $L \leq \frac{\phi^{-1}(L)}{N}$. Let $xN \in \frac{\phi^{-1}(L)}{N}$. Then $x \in \phi^{-1}(L)$ and so $\phi(x) \in L$. But $\phi(x) = xN$ and so $xN \in L$. Thus, $\frac{\phi^{-1}(L)}{N} \leq L$ and so $L = \frac{\phi^{-1}(L)}{N}$.

$\square$

Theorem 1.9. *Let G be any group and $S \subseteq G$. Define*

$$\langle S \rangle = \{s_1^{n_1} s_2^{n_2} \cdots s_k^{n_k} \mid s_i \in S, n_i \in \mathbb{Z}, \textit{ for all } 1 \leq i \leq k, k \in \mathbb{Z}^+\}.$$

Then $\langle S \rangle \leq G$ and is called the **subgroup generated by S**.

Proof. Let $s \in S$. Then $s = s^1 \in \langle S \rangle$ and so $\langle S \rangle \neq \emptyset$. Let

$$s_1^{n_1} s_2^{n_2} \cdots s_k^{n_k}, r_1^{m_1} r_2^{m_2} \cdots r_l^{n_l} \in \langle S \rangle$$

where $s_i \in S$ and $r_i \in S$ for all i and $n_i \in \mathbb{Z}$ and $m_i \in \mathbb{Z}$ for all i and $k, l \in \mathbb{Z}^+$. Then

$$(s_1^{n_1} s_2^{n_2} \cdots s_k^{n_k})(r_1^{m_1} r_2^{m_2} \cdots r_l^{n_l})^{-1} = s_1^{n_1} s_2^{n_2} \cdots s_k^{n_k} r_l^{-m_l} r_{l-1}^{-m_{l-1}} \cdots r_1^{-m} \\ \in \langle S \rangle.$$

Hence $\langle S \rangle \leq G$ by the Subgroup Test.

$\square$

Definition 1.7. *Let G be a group, $a, b \in G$, $H \leq G$, and $K \leq G$. Then*

1. $[a, b] = aba^{-1}b^{-1}$ *is called the* **commutator of a and b**
2. $[H, K] = \langle \{[h, k] \mid h \in H \text{ and } k \in K\} \rangle$ *is called the* **commutator subgroup generated by H and K**
3. $G' = \langle \{[a, b] \mid a, b \in G\} \rangle$ *is called the* **commutator subgroup of G**

Lemma 1.3. *Let G be a group, $N \trianglelefteq G$, $H \leq G$, and $a, b \in G$. Then*

1. $[a, b] = 1$ *if and only if* $ab = ba$
2. $G' \trianglelefteq G$
3. $\frac{G}{G'}$ *is abelian*
4. $\frac{G}{N}$ *is abelian if and only if* $G' \leq N$
5. *If* $G' \leq H$ *then* $H \trianglelefteq G$

Proof. For (1),

$$[a, b] = 1 \Leftrightarrow aba^{-1}b^{-1} = 1 \Leftrightarrow ab = ba.$$

For (2), let $x \in G'$ and $g \in G$. Then, $x = \prod_{i=1}^{k} [a_i, b_i]$ and so

$$\begin{aligned} gxg^{-1} &= g(\prod_{i=1}^{k}[a_i, b_i])g^{-1} \\ &= \prod_{i=1}^{k} g[a_i, b_i]g^{-1} \\ &= \prod_{i=1}^{k}[ga_ig^{-1}, gb_ig^{-1}] \\ &\in G' \end{aligned}$$

and so $G' \trianglelefteq G$. For (3), let $aG', bG' \in \frac{G}{G'}$. Then,

$$\begin{aligned} [aG', bG'] &= [a, b]G' \\ &= 1G' \quad \text{as } 1^{-1}[a, b] = [a, b] \in G'. \end{aligned}$$

Therefore, $\frac{G}{G'}$ is abelian. For (4),

$$\frac{G}{N} \text{ is abelian} \Leftrightarrow [aN, bN] = 1N \text{ for all } a, b \in G$$

$$\Leftrightarrow [a, b]N = 1N \Leftrightarrow [a, b] \in N \Leftrightarrow G' \leq N \text{ since} N \leq G.$$

For (5), let $g \in G$ and $h \in H$. Then $[h^{-1}, g] \in G' \leq H$ and so $[h^{-1}, g] \in H$. Let $[h^{-1}, g] = h_1$ where $h_1 \in H$. Then $h^{-1}g(h^{-1})^{-1}g^{-1} = h_1$. Thus, $h^{-1}ghg^{-1} = h_1$ implying $ghg^{-1} = hh_1 \in H$. Therefore, $H \trianglelefteq G$.

$\square$

Definition 1.8. *Let G be a group and p a prime. Then G is called a* ***p*-group** *if $|G| = p^r$ for some $r \in \mathbb{Z}^+ \cup \{0\}$.*

Lemma 1.4. *Let G be a group and $H \trianglelefteq G$. Then $Z(H) \trianglelefteq G$.*

Theorem 1.10 (Cauchy's Theorem for Abelian Groups)**.** *Let G be abelian and p be a prime such that $p \mid |G|$. Then G has an element of order p.*

Definition 1.9. *The group consisting of the set S_n of all permutations on $A = \{1, 2, \ldots, n\}$, under the operation of permutation multiplication is called the* **symmetric group** *of degree n.*

Definition 1.10. *Let G be a group and $S \neq \{\}$ be a set. Then G* **acts** *on S if there exists a homomorphism $\phi : G \to Sym(S)$.*

Definition 1.11. *Let G be a group, S be a set, and $a \in S$. The* **orbit of *S* containing *a*** *is*

$$Ga = \{ga \mid g \in G\}.$$

Definition 1.12. *A group G acts* **transitively** *on a set S, if there is only one orbit; i.e., $S = Ga$ for all $a \in S$; i.e., for all $c, d \in S$ there exists $g \in G$ such that $cg = d$.*

Definition 1.13. *Let G be a group, p be a prime, and $n \in \mathbb{Z}^+ \cup \{0\}$ such that $p^n \mid |G|$ but $p^{n+1} \nmid |G|$. Then*

1. *$|G|_p = p^n$ is called the p^{th} part of G.*

2. *A subgroup $H \leq G$ is called a* **sylow *p*-subgroup** *if $|H| = |G|_p$.*

3. *$Syl_p(G)$ is the set of all sylow p-subgroups of G.*

Theorem 1.11 (Sylow's Theorem)**.** *Let G be a group, p be any prime, $H \leq G$ be a p-group, and $n_p = |Syl_p(G)|$. Then*

1. *$Syl_p(G) \neq \{\}$*

2. *There exists $P \in Syl_p(G)$ such that $H \leq P$. Moreover, G acts transitively on $Syl_p(G)$ by conjugation*

3. *$n_p \mid |G|$ and $n_p \equiv 1 (mod\, p)$.*

2 Solvable Groups

We next need to introduce what it means for a group to be solvable and will discover some important properties about solvability.

Definition 2.1. *A group G is* **solvable** *if there exists a subnormal series*

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = 1$$

such that $\dfrac{G_i}{G_{i+1}}$ is abelian for all $0 \leq i \leq n-1$.

Example 2.1. *S_3 is a solvable group.*

Proof. Consider the subnormal series

$$S_3 \trianglerighteq A_3 \trianglerighteq 1.$$

Now, $\left|\dfrac{S_3}{A_3}\right| = \dfrac{|S_3|}{|A_3|} = \dfrac{6}{3} = 2$ and so $\dfrac{S_3}{A_3} \cong \mathbb{Z}_2$ is abelian. Next, $\left|\dfrac{A_3}{\{1\}}\right| = \dfrac{|A_3|}{|\{1\}|} = 3$ and so $\dfrac{A_3}{\{1\}} \cong \mathbb{Z}_3$ is abelian. Therefore S_3 is solvable.

$\square$

Lemma 2.1. *Let G be an abelian group. Then G is solvable.*

Proof. Consider the subnormal series

$$G = G_0 \trianglerighteq 1.$$

Then by Lemma 1.1 we know $\frac{G}{\{1\}} \cong G$. Since G is abelian, $\frac{G}{\{1\}}$ is abelian and so G is solvable.

$\square$

Example 2.2. *The abelian groups $\mathbb{Z}_n$ and $\mathbb{Z}_a \times \mathbb{Z}_b \times \cdots \times \mathbb{Z}_c$ are solvable groups by Lemma 2.1.*

Lemma 2.2. *Let G be solvable and $H \leq G$. Then H is solvable.*

Proof. Since G is solvable we know there exists a subnormal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = 1$$

such that $\frac{G_i}{G_{i+1}}$ is abelian. Consider the series

$$H = H_0 \geq H \cap G_1 \geq H \cap G_2 \geq \cdots \geq H \cap G_n = 1.$$

We want to show that $H \cap G_{i+1} \trianglelefteq H \cap G_i$. Let $x \in H \cap G_{i+1}$ and $g \in H \cap G_i$. Then $gxg^{-1} \in G_{i+1}$ since $x \in G_{i+1}$ and $G_{i+1} \trianglelefteq G_i$. Also, $gxg^{-1} \in H$ since $g \in H$ and $x \in H$. Thus, $gxg^{-1} \in H \cap G_{i+1}$. Hence $H \cap G_{i+1} \trianglelefteq H \cap G_i$ for all $0 \leq i \leq n-1$. Therefore,

$H = H_0 \trianglerighteq H \cap G_1 \trianglerighteq H \cap G_2 \trianglerighteq \cdots \trianglerighteq H \cap G_n = 1$ is a subnormal series. Now

$$\begin{aligned}
\frac{H \cap G_i}{H \cap G_{i+1}} &= \frac{H \cap G_i}{H \cap G_i \cap G_{i+1}} \\
&\cong \frac{(H \cap G_i)G_{i+1}}{G_{i+1}} \quad \text{by } 2^{nd} \text{ Isomorphism Theorem} \\
&\leq \frac{G_i}{G_{i+1}}.
\end{aligned}$$

Since $\frac{G_i}{G_{i+1}}$ is abelian we get $\frac{H \cap G_i}{H \cap G_{i+1}}$ is abelian for all $0 \leq i \leq n-1$. Thus H is solvable.

□

Lemma 2.3. *Let G be solvable and $N \trianglelefteq G$. Then $\frac{G}{N}$ is solvable.*

Proof. Since G is solvable we know there exists a subnormal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = 1$$

such that $\frac{G_i}{G_{i+1}}$ is abelian for all $0 \leq i \leq n-1$. Taking the image of this series under the natural map we get

$$\frac{G}{N} = \frac{G_0}{N} \geq \frac{G_1 N}{N} \geq \frac{G_2 N}{N} \geq \cdots \geq \frac{G_n N}{N} = 1N.$$

We claim that $\frac{G_{i+1}N}{N} \trianglelefteq \frac{G_iN}{N}$. Let $g_{i+1}n_1N \in \frac{G_{i+1}N}{N}$ and $g_in_2N \in \frac{G_iN}{N}$. Then

$$\begin{aligned}
(g_in_2N)(g_{i+1}n_1N)(g_in_2N)^{-1} &= (g_in_2N)(g_{i+1}n_1N)(n_2^{-1}g_i^{-1}N) \\
&= g_in_2g_{i+1}n_1n_2^{-1}g_i^{-1}N \\
&= g_in_2g_i^{-1}g_ig_{i+1}g_i^{-1}g_in_1n_2^{-1}g_i^{-1}N \\
&= g_in_2g_i^{-1}g_ig_{i+1}g_i^{-1}N \quad \text{since } g_in_1n_2^{-1}g_i^{-1} \in N \\
&\in \frac{G_{i+1}N}{N} \quad \text{since } g_in_2g_i^{-1} \in G_{i+1} \text{ and } g_ig_{i+1}g_i^{-1} \in N
\end{aligned}$$

Thus,

$$\frac{G}{N} = \frac{G_0}{N} \trianglerighteq \frac{G_1N}{N} \trianglerighteq \frac{G_2N}{N} \trianglerighteq \cdots \trianglerighteq \frac{G_nN}{N} = 1N$$

is a subnormal series. Then

$$\begin{aligned}
\frac{G_iN/N}{G_{i+1}N/N} &\cong \frac{G_iN}{G_{i+1}N} \quad \text{by } 3^{rd} \text{ Isomorphism Theorem} \\
&= \frac{G_iG_{i+1}N}{G_{i+1}N} \\
&\cong \frac{G_i}{G_I \cap G_{i+1}N} \quad \text{by } 2^{nd} \text{ Isomorphism Theorem} \\
&\cong \frac{G_i/G_{i+1}}{G_I \cap G_{i+!}N/G_{i+1}} \quad \text{by } 3^{rd} \text{ Isomorphism Theorem}
\end{aligned}$$

Since quotients of abelian groups are abelian, we get $\frac{G_iN/N}{G_{i+1}N/N}$ is abelian for all $0 \leq i \leq n-1$. Therefore, $\frac{G}{N}$ is solvable.

$\square$

Theorem 2.1. *Let G be a p-group. Then G is solvable.*

Proof. Use induction on $|G|$. If $|G| = 1$ then $G = \{1\}$ is abelian and therefore solvable. Assume the theorem holds for all p-groups of order less than $|G|$. Without loss of generality, $G \neq 1$. Since G is a p-group we know $Z(G) \neq 1$. Then $\left|\frac{G}{Z(G)}\right| = \frac{|G|}{|Z(G)|} < |G|$ and $\frac{G}{Z(G)}$ is a p-group. Thus $\frac{G}{Z(G)}$ is solvable by induction and so there exists a subnormal series

$$\frac{G}{Z(G)} = \frac{G_0}{Z(G)} \trianglerighteq \frac{G_1}{Z(G)} \trianglerighteq \cdots \trianglerighteq \frac{G_n}{Z(G)} = Z(G)$$

such that $\frac{G_i/Z(G)}{G_{i+1}/Z(G)}$ is abelian for all $0 \leq i \leq n-1$. Taking the pre-image of this series under the natural map we get

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq Z(G) \trianglerighteq 1.$$

Then $\frac{G_i}{G_{i+1}} \cong \frac{G_i/Z(G)}{G_{i+1}/Z(G)}$ is abelian by the 3^{rd} Isomorphism Theorem and $\frac{Z(G)}{\{1\}} \cong Z(G)$ which is abelian. Therefore, G is solvable and every p-group is solvable by induction.

$\square$

Definition 2.2. *Let G be a group. Define the* **derived series of G** *by*

$$G^{(0)} = G, G^{(1)} = (G^{(0)})' = G', G^{(2)} = (G^{(1)})' = G''$$

and inductively define

$$G^{(n)} = (G^{(n-1)})'.$$

By Lemma 1.3 we have a subnormal series

$$G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq G^{(2)} \trianglerighteq G^{(3)} \trianglerighteq \cdots$$

Theorem 2.2. *Let G be a group. Then G is solvable if and only if there exists $n \in \mathbb{Z}^+$ such that $G^{(n)} = 1$.*

Proof. ($\Leftarrow$) Suppose there exists $n \in \mathbb{Z}^+$ such that $G^{(n)} = 1$. Consider the derived series

$$G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq G^{(2)} \trianglerighteq \cdots \trianglerighteq G^{(n)} = 1$$

Then $\frac{G^{(i)}}{G^{(i+1)}} = \frac{G^{(i)}}{(G^{(i)})'}$ is abelian by Lemma 1.3 for all $0 \leq i \leq n-1$. Therefore, G is solvable. ($\Rightarrow$) Suppose G is solvable. Then there exists a subnormal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = 1$$

such that $\frac{G_i}{G_{i+1}}$ is abelian. We claim that $G^{(i)} \leq G_i$. Use induction on i. If $i = 0$ then $G^{(0)} = G \leq G = G_0$. Suppose $G^{(i)} \leq G_i$. We want to show $G^{(i+1)} \leq G_{i+1}$. Now

$$\begin{aligned} G^{(i+1)} &= (G^{(i)})' \\ &\leq (G_i)' \quad \text{by induction hypothesis} \\ &\leq G_{i+1} \quad \text{since } \frac{G_i}{G_{i+1}} \text{ is abelian and by Lemma 1.3} \end{aligned}$$

Thus, the claim holds. Hence $G^{(n)} \leq G_n = 1$ and so $G^{(n)} = 1$.

□

Theorem 2.3. *Let G be a group and $H \trianglelefteq G$ such that H and $\frac{G}{H}$ are solvable. Then G is solvable.*

Proof. Since H and $\frac{G}{H}$ are solvable then there exist $m, n \in \mathbb{Z}^+$ such that $H^{(m)} = 1$ and $\left(\frac{G}{H}\right)^{(n)} = 1H$. Then $\frac{G^{(n)}H}{H} = 1H$ by the claim in Lemma 2.3. Let $ahH \in \frac{G^{(n)}H}{H}$ where $a \in G^{(n)}$ and $h \in H$. Then $ahH = 1H$ and so, by Theorem 1.2, $1^{-1}ah = ah \in H$

and so there exists $h_1 \in H$ such that $ah = h_1$ or $a = h_1 h^{-1} \in H$. Thus, $G^{(n)} \leq H$. Now by the claim in Lemma 2.3 we get $G^{(n+m)} = (G^{(n)})^{(m)} \leq H^{(m)} = 1$. Thus, $G^{(n+m)} = 1$ and so G is solvable.

$\square$

Definition 2.3. *Let G be a group and $\phi : G \to G$ be a map. Then ϕ is an* **automorphism** *if ϕ is 1-1, onto, and a homomorphism. Let*

$$Aut(G) = \{\phi \mid \phi \textit{ is an automorphism}\}.$$

Definition 2.4. *Let G be a group and $H \leq G$. Then H is a* **characteristic subgroup of *G*** *if $\phi(H) \leq H$ for all $\phi \in Aut(G)$. We write H char G.*

Lemma 2.4. *Let G be a group, $H \leq G$, and $K \leq G$ such that H char K and K char G. Then H char G.*

Proof. Let $\phi \in Aut(G)$. Since K char G we know $\phi(K) \leq K$. If $x, y \in K$ such that $\phi(x) = \phi(y)$ then since ϕ is 1-1 we get $x = y$. Thus, $|\phi(K)| = |K|$ and so $\phi(K) = K$. But then $\phi|_K \in Aut(K)$ since H char K we get $\phi|_K(H) \leq H$ and so $\phi(H) \leq H$. Thus, H char G.

$\square$

Lemma 2.5. *Let G be a group, $H \leq G$, and $K \leq G$ such that H char K and $K \trianglelefteq G$. Then $H \trianglelefteq G$.*

Proof. Let $g \in G$ and $h \in H$. We want to show that $ghg^{-1} \in H$. Define $\phi_g : K \to K$ by $\phi_g(k) = gkg^{-1}$ for all $k \in K$. First, we need to show that ϕ_g is a homomorphism.

Let $x, y \in K$. Then,

$$\begin{aligned}\phi_g(xy) &= gxyg^{-1} \\ &= gxg^{-1}gyg^{-1} \\ &= \phi_g(x)\phi_g(y).\end{aligned}$$

Next, we need to show that ϕ_g is 1-1. If $\phi_g(x) = \phi_g(y)$ then $gxg^{-1} = gyg^{-1}$ implying $x = y$. Finally, we need to show that ϕ_g is onto. Let $x \in K$. Since $K \trianglelefteq G$ we know $g^{-1}xg = (g^{-1})x(g^{-1})^{-1} \in K$ and $\phi_g(g^{-1}xg) = g(g^{-1}xg)g^{-1} = x$. Thus, $\phi_g \in Aut(K)$. Since H char K we get $\phi_g(h) \in H$ or $gxg^{-1} \in H$. Therefore, $H \trianglelefteq G$.

□

Lemma 2.6. *Z(G) char G.*

Proof. Let $\phi \in Aut(G)$, $z \in Z(G)$, and $g \in G$. We want to show that $\phi(z) \in Z(G)$. Since $\phi \in Aut(G)$ there exists $g_1 \in G$ such that $\phi(g_1) = g$. Now,

$$\begin{aligned}\phi(z)g &= \phi(z)\phi(g_1) \\ &= \phi(zg_1) \quad \text{since } \phi \in Aut(G) \\ &= \phi(g_1 z) \quad \text{since } z \in Z(G) \\ &= \phi(g_1)\phi(z) \\ &= g\phi(z).\end{aligned}$$

Thus, $\phi(z) \in Z(G)$ and so $Z(G)$ char G.

□

Definition 2.5. *A group G is* **characteristically simple** *if $\{1\}$ and G are its only characteristic subgroups.*

Definition 2.6. *Let G be a group and $\{H_i\}_{i=1}^n$ be a collection of subgroups of G. We say $G = H_1 \times H_2 \times \cdots \times H_n$ if*

1. $G = \prod_{i=1}^{n} H_i$
2. $H_i \cap \prod_{j \neq i} H_i = 1$ *for all* $1 \leq i \leq n$
3. $H_i \trianglelefteq G$ *for all* $1 \leq i \leq n$

Theorem 2.4. *Let G be a characteristically simple group. Then $G = G_1 \times G_2 \times \cdots \times G_n$ where G_is are simple isomorphic groups.*

Proof. Let $1 \neq G_1 \trianglelefteq G$ such that $|G|$ is minimal and $H = \prod_{i=1}^{n} G_i$ such that

1. $G_i \cong G_1$ for all $1 \leq i \leq n$
2. $G_i \trianglelefteq G$ for all $1 \leq i \leq n$
3. $G_i \cap \prod_{j \neq i} G_j = 1$ for all $1 \leq i \leq n$
4. n is maximal

Clearly, $H \trianglelefteq G$ since $G_i \trianglelefteq G$ for all $1 \leq i \leq n$. If H is not a characteristic subgroup of G then there exists $\phi \in Aut(G)$ and $1 \leq i \leq n$ such that $\phi(G_i) \not\leq H$. Since $G_i \trianglelefteq G$ and $\phi \in Aut(G)$ we know $\phi(G_i) \trianglelefteq G$. Also $\phi(G_i) \cong G_i \cong G_1$ and so $\phi(G_i) \cong G_1$. Now $H \cap \phi(G_i) \trianglelefteq G$ and $H \cap \phi(G_i) < \phi(G_i)$. Thus, $|H \cap \phi(G_i)| < |\phi(G_i)| = |G_i| = |G_1|$. Hence, $H \cap \phi(G_i) = 1$ by the minimality of $|G_1|$. But then $\phi(G_i) \cap \prod_{i=1}^{n} G_i = \phi(G_i) \cap H = 1$. Therefore, the subgroups $\{G_1, G_2, \cdots, G_n, \phi(G_i)\}$

satisfy (1), (2), and (3), a contradiction, since n is maximal. Therefore, H char G. Since G is characteristically simple we get $G = H = \prod_{i=1}^{n} G_i = G_1 \times G_2 \times \cdots \times G_n$ where G_i's are isomorphic groups. Suppose $N \trianglelefteq G_i$ for some $1 \leq i \leq n$. If $j \neq i$ and $x \in G_i$ and $y \in G_j$ then, $xyx^{-1}y^{-1} \in G_j \cap G_i \leq G_j \cap \prod_{i \neq j}^{n} G_i = 1$. Hence $xy = yx$. Now let $g_1 g_2 \cdots g_n \in G$ where $g_i \in G_i$ for all $1 \leq i \leq n$ and $n \in N$. Then,

$$\begin{aligned} g_1 g_2 \cdots g_n n (g_1 g_2 \cdots g_n)^{-1} &= g_1 g_2 \cdots g_n n g_n^{-1} g_{n-1}^{-1} \cdots g_1^{-1} \\ &= g_i n g_i^{-1} \\ &\in N \quad \text{since } N \trianglelefteq G_i. \end{aligned}$$

Thus, $N \trianglelefteq G$. But, $|N| < |G_i| = |G_1|$. Hence, $N = 1$ or $N = G_i$ by the minimality of $|G_1|$. Therefore, each G_i is simple for all $1 \leq i \leq n$.

□

Definition 2.7. *Let G be a group and $N \leq G$. Then N is a* **minimal normal subgroup of *G*** *if*

1. *$N \trianglelefteq G$*

2. *If there exists a $L \leq N$ such that $L \trianglelefteq G$ then $L = 1$ or $L = N$.*

Definition 2.8. *A group G is called an* **elementary abelian *p*-group** *if $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ where p is a prime.*

Theorem 2.5. *Let G be a group and N be a minimal normal subgroup of G. Then N is an elementary abelian p-group for some prime p or $N = N_0 \times N_1 \times \cdots \times N_n$ where $N_i s$ are nonabelian simple isomorphic groups.*

Proof. If K char N, then by Lemma 2.5, since $N \trianglelefteq G$ we get $K \trianglelefteq G$. But then, $K = 1$ or $K = N$ since N is a minimal normal subgroup. Hence, N is characteristically simple. Then, by Theorem 2.4, $N = N_1 \times N_2 \times \cdots \times N_n$ where N_i's are simple isomorphic groups.

Case 1 N_i is nonabelian for all $0 \leq i \leq n$. Then $N = N_1 \times N_2 \times \cdots \times N_n$ and N_is are nonabelian simple isomorphic groups.

Case 2 N_is are abelian for all $0 \leq i \leq n$. Then N_i is simple and abelian for all $0 \leq i \leq n$. Then the only subgroups of N_i are $\{1\}$ and N_i for all $0 \leq i \leq n$. If N_i is not a p-group then there exists a prime q such that $q \mid |N_i|$ and $q \neq p$. By Sylow's Theorem there exists $Q \in Syl_q(N_i)$. Then $Q \leq N_i$ and $Q \neq 1$ and $Q \neq N_i$. Thus, N_i is a p-group for some prime p. Let $|N_i| = p^n$. If $n > 1$ then by Cauchy's Theorem for Abelian Groups, there exists $1 \neq x \in N_i$ such that $x^p = 1$. Then, $\langle x \rangle \leq N_i$ and $|\langle x \rangle| = p < |N_i|$. Therefore, $\langle x \rangle \neq 1$ and $\langle x \rangle \neq N_i$. Hence, $n = 1$ and $|N_i| = p$. Now we know that N_i is cyclic and so $N_i \cong \mathbb{Z}_p$. Thus, $N \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ is an elementary abelian p-group.

Therefore, N is an elementary abelian p-group for some prime p or $N = N_0 \times N_1 \times \cdots \times N_n$ where N_i's are nonabelian simple isomorphic groups.

□

Theorem 2.6. *Let G be solvable and N be a minimal normal subgroup of G. Then N is an elementary abelian p-group for some prime p.*

Proof. By Theorem 2.5, N is an elementary abelian p-group for some prime p or $N = N_1 \times N_2 \times \cdots \times N_n$ such that N_is are simple nonabelian isomorphic groups. Hence N_1 is simple. Then the only subnormal series N_1 has is $N_1 \trianglerighteq 1$ by simplicity.

But, $\dfrac{N_1}{\{1\}} \cong N$ is nonabelian. Therefore, N_1 is not solvable. But, $N_1 \leq G$ and G is solvable, a contradiction. Thus, N is an elementary abelian p-group for some p.

□

3 Nilpotent Groups

We now introduce the idea of nilpotent groups. This allows us to explore important properties of nilpotent groups and will let us build the structures of these groups.

Definition 3.1. *Let G is a group. Define the* **upper central series of G** *by*

$$Z_0(G) = 1, Z_1(G) = Z(G), \frac{Z_2(G)}{Z_1(G)} = Z\left(\frac{G}{Z_1(G)}\right), \frac{Z_3(G)}{Z_2(G)} = Z\left(\frac{G}{Z_2(G)}\right), \cdots$$

and inductively define

$$\frac{Z_n(G)}{Z_{n-1}(G)} = Z\left(\frac{G}{Z_{n-1}(G)}\right) \textit{ for all } n \in \mathbb{Z}^+.$$

Lemma 3.1. *Let G be a group. Then $Z_i(G) \trianglelefteq G$ for all i and $Z_i(G) \leq Z_{i+1}(G)$ for all i.*

Proof. Use induction on i. If $i = 0$, then $Z_0(G) = \{1\} \trianglelefteq G$. Assume $Z_n(G) \trianglelefteq G$. Then $\dfrac{Z_{n+1}(G)}{Z_n(G)} = Z\left(\dfrac{G}{Z_n(G)}\right) \trianglelefteq \dfrac{G}{Z_n(G)}$ and so taking pre-images we get $Z_{n+1}(G) \trianglelefteq G$. Hence, $Z_i(G) \leq Z_{i+1}(G)$ for all i.

□

Definition 3.2. *A group G is* **nilpotent** *if there exists $n \in \mathbb{Z}^+ \cup \{0\}$ such that $G = Z_n(G)$.*

Definition 3.3. *Let G be a group. Define the* **lower central series of G** *by*

$$K_0(G) = G, K_1(G) = [K_0(G), G] = [G, G] = G', K_2(G) = [K_1(G), G], \cdots$$

and inductively define

$$K_n(G) = [K_{n-1}, G].$$

Lemma 3.2. *Let G be group. Then $K_i(G) \trianglelefteq G$ for all i and $K_{i+1}(G) \leq K_i(G)$ for all i.*

Proof. Use induction on i. If $i = 0$ then $K_0(G) = G \trianglelefteq G$. Suppose $K_i(G) \trianglelefteq G$. Then, since $G \trianglelefteq G$ we get $K_{i+1}(G) = [K_i(G), G] \trianglelefteq G$ as conjugation is a homomorphism. Next, we know that $K_i(G) \trianglelefteq G$. Thus, $K_{i+1}(G) = [K_i(G), G] \leq K_i(G)$ for all i.

$\square$

Theorem 3.1. *Let G be a group. Then G is nilpotent if and only if there exists $n \in \mathbb{Z}^+ \cup \{0\}$ such that $K_n(G) = 1$.*

Proof. ($\Rightarrow$) Let G be nilpotent. Then there exists $n \in \mathbb{Z}^+ \cup \{0\}$ such that $Z_n(G) = G$. We claim that $K_i(G) \leq Z_{n-i}(G)$ for all i. Use induction on i. If $i = 0$ then $K_0(G) = G \leq G = Z_n(G) = Z_{n-0}(G)$. Suppose $K_i(G) \leq Z_{n-i}(G)$. Then,

$$\begin{aligned} K_{i+1}(G) &= [K_i(G), G] \\ &\leq [Z_{n-i}(G), G] \quad \text{since } K_i(G) \leq Z_{n-i}(G) \\ &\leq Z_{n-i-1}(G) \quad \text{since } \frac{Z_{n-i}(G)}{Z_{n-i-1}(G)} = Z\left(\frac{G}{Z_{n-i-1}(G)}\right) \\ &= Z_{n-(i+1)}(G). \end{aligned}$$

Thus, the claim hold by induction. But then, $K_n(G) = Z_{n-n}(G) = Z_0(G) = 1$ and so $K_n(G) = 1$. ($\Leftarrow$) Suppose there exists a $n \in \mathbb{Z}^+ \cup \{0\}$ such that $K_n(G) = 1$. We claim that $K_{n-i}(G) \leq Z_i(G)$ for all i. Use induction on i. If $i = 0$ then $Z_0(G) = 1 \geq 1 = K_n(G) = K_{n-0}(G)$. Suppose $K_{n-i}(G) \leq Z_i(G)$. Now, $[K_{n-i-1}, G] = K_{n-i}(G) \leq Z_i(G)$. Hence, $\frac{K_{n-i-1}(G)Z_i(G)}{Z_i(G)} \leq Z(\frac{G}{Z_i(G)}) = \frac{Z_{i+1}}{Z_i(G)}$. Taking pre-images we get $K_{n-i-1}(G) \leq K_{n-i-1}(G)Z_i(G) \leq Z_{i+1}(G)$ or $K_{n-(i+1)}(G) \leq Z_{i+1}(G)$. Thus, the claim holds. But then, $Z_n(G) \geq K_{n-n}(G) = K_0(G) = G$ and so $Z_n(G) = G$ and so G is nilpotent.

□

Lemma 3.3. *Let G be a group, $N \trianglelefteq G$, and $H \leq G$ such that $N \leq H$. If $\frac{H}{N} \leq Z\left(\frac{G}{N}\right)$ if and only if $[G, H] \leq N$.*

Proof. $\frac{H}{N} \leq Z\left(\frac{G}{N}\right) \Leftrightarrow [hN, gN] = N$ for all $h \in H$ and for all $g \in G$. Then $hNgN(hN)^{-1}(gN)^{-1} = N \Leftrightarrow hNgNh^{-1}g^{-1} = N \Leftrightarrow hgh^{-1}g^{-1}N = N \Leftrightarrow [h, g] = N$ $\Leftrightarrow [h, g] \in N \Leftrightarrow [G, H] \leq N$.

□

Theorem 3.2. *Let G be nilpotent. Then $Z(G) \neq 1$.*

Proof. Suppose $Z(G) = 1$. Since G is nilpotent there exists $n \in \mathbb{Z}^+ \cup \{0\}$ such that $Z_n(G) = G$. Notice $Z_1(G) = Z(G) = 1$. Suppose $Z_i(G) = 1$. Then

$$\frac{Z_{i+1}(G)}{Z_i(G)} = Z\left(\frac{G}{Z_i}\right) = Z\left(\frac{G}{\{1\}}\right) \cong Z(G) = 1.$$

Then, $\left|\frac{Z_{i+1}(G)}{Z_i(G)}\right| = 1$ or $\frac{|Z_{i+1}(G)|}{|Z_i(G)|} = 1$ and so $|Z_{i+1}(G)| = |Z_i(G)|$. But, $Z_i(G) \leq Z_{i+1}(G)$ and so $Z_{i+1}(G) = Z_i(G) = 1$. Thus, by induction $Z_i(G) = 1$ for all i. But then we get $G = Z_n(G) = 1$, a contradiction. Therefore, $Z(G) \neq 1$.

□

Theorem 3.3. *Let G be nilpotent and $1 \neq H \trianglelefteq G$. Then $H \cap Z(G) \neq 1$.*

Proof. Since G is nilpotent there exists $n \in \mathbb{Z}^+$ such that $Z_n(G) = G$. Define

$$H_0 = H, H_1 = [H_0, G] = [H, G]$$

and inductively define

$$H_n = [H_{n-1}, G].$$

Since $H \trianglelefteq G$ we get $H = H_0 \geq H_1 \geq H_2 \geq \cdots$. We claim that $H_i \leq Z_{n-i}(G)$ for all i. If $i = 0$ then $H_0 = H \leq G = Z_n(G) = Z_{n-0}(G)$. Assume $H_i \leq Z_{n-i}(G)$. Then,

$$\frac{H_i Z_{n-i-1}(G)}{Z_{n-i-1}(G)} \leq \frac{Z_{n-i}(G)}{Z_{n-i-1}(G)} = Z\left(\frac{G}{Z_{n-i-1}(G)}\right).$$

By Lemma 3.3, $[H_i Z_{n-i-1}(G), G] \leq [Z_{n-i-1}(G)]$. Hence, $[H_i, G] \leq [H_i Z_{n-i-1}(G), G] \leq Z_{n-i-1}(G)$. Thus, $H_{i+1} = [H_i, G] \leq Z_{n-i-1}(G) = Z_{n-(i+1)}(G)$. Therefore, the claim holds by induction. But then $H_n = [H_{n-1}, G] \leq Z_{n-n}(G) = Z_0(G) = 1$ and so $H_n = 1$. Let $0 \leq k \leq n$ be minimal such that $H_k = 1$. Then $H_{k-1} \neq 1$ and $1 = H_k = [H_{k-1}, G]$ and so $1 \neq H_{k-1} \leq H \cap Z(G)$.

□

Theorem 3.4. *Let G be nilpotent and $H \leq G$. Then $H < N_G(H)$.*

Proof. Since G is nilpotent there exists $n \in \mathbb{Z}^+$ such that $Z_n(G) = G$. Let i be minimal such that $Z_i(G) \not\leq H$. Then $Z_{i-1}(G) \leq H$. Also, $[H, Z_i(G)] \leq [G, Z_i(G)] \leq Z_{i-1}(G)$ since $\dfrac{Z_i(G)}{Z_{i-1}(G)} = Z\left(\dfrac{G}{Z_{i-1}(G)}\right)$. Thus, $[H, Z_i(G)] \leq H$. Hence, $Z_i(G) \leq N_G(H) \setminus H$ and so $H < N_G(H)$.

□

Definition 3.4. *Let G be a group and $M \leq G$. Then M is a* **maximal subgroup** *if*

1. *$M \neq G$*

2. *whenever there exists $H \leq G$ such that $M \leq H \leq G$ then $H = M$ or $H = G$*

Theorem 3.5. *Let G be nilpotent and M be a maximal subgroup of G. Then $M \trianglelefteq G$.*

Proof. Since M is a maximal subgroup of G we know $M < G$. By Theorem 3.4, $M < N_G(M) \leq G$. Hence $G = N_G(M)$ by the maximality of M. Therefore, $M \trianglelefteq G$.

□

Lemma 3.4. *Let G be a group, $P \in Syl_p(G)$, and $N \trianglelefteq G$. Then $P \cap N \in Syl_p(N)$.*

Lemma 3.5 (Frattini Argument)**.** *Let G be a group, $N \trianglelefteq G$, $P \in Syl_p(G)$. Then, $G = N_G(P \cap N)N$.*

Proof. Clearly, $N_G(P \cap N)N \subseteq G$ since G is a group. Let $g \in G$. Since $N \trianglelefteq G$ and $P \in Syl_p(G)$ by Lemma 3.4 $P \cap N \in Syl_p(N)$. Since $N \trianglelefteq G$ and $P \cap N \leq N$ we get $g^{-1}P \cap Ng \leq g^{-1}Ng = N$. Now, $|g^{-1}(P \cap N)g| = |P \cap N|$ and so $g^{-1}(P \cap N)g \in Syl_p(N)$. By Sylow's Theorem there exists $n \in N$ such that $ng^{-1}(P \cap N)gn^{-1} = P \cap N$. Hence $ng^{-1} \in N_G(P \cap N)$ and so there exists $x \in N_G(P \cap N)$ such that $ng^{-1} = x$. But then $g = x^{-1}n \in N_G(P \cap N)N$. Thus, $G \subseteq N_G(P \cap N)N$. Hence $G = N_G(P \cap N)N$.

□

Lemma 3.6. *Let G be nilpotent and $P \in Syl_p(G)$ then $P \trianglelefteq G$.*

Proof. Suppose P is not normal in G. Then $N_G(P) < G$. Hence, there exists a maximal subgroup M of G such that $N_G(P) \leq M$. Since G is nilpotent, by Theorem 3.5, $M \trianglelefteq G$. Now, $P \leq N_G(P) \leq M$ and so $P \leq M$. By Lemma 3.5,

$$G = N_G(P \cap M)M = N_G(P)M = M.$$

Thus, $G = M$, a contradiction, since M is maximal. Hence, $P \trianglelefteq G$.

$\square$

Lemma 3.7. *Let G be a group, $H \trianglelefteq G$, $K \trianglelefteq G$ such that H and K are nilpotent. Then $HK \trianglelefteq G$ and HK is nilpotent.*

Proof. Use induction on $|G|$. Since $H \trianglelefteq G$ and $K \trianglelefteq G$ clearly $HK \trianglelefteq G$. If $HK < G$ then $H \trianglelefteq HK$ and $K \trianglelefteq HK$. Also, H and K are nilpotent. Thus, by induction HK is nilpotent. We may assume $G = HK$. Since H is nilpotent by Theorem 3.2, $Z(H) \neq 1$. Let $N = [Z(H), K]$.

Case 1 If $N = 1$. Then $[Z(H), K] = 1$. But also $[H, Z(H)] = 1$. Hence, $[G, Z(H)] = 1$ since $G = HK$. Thus $1 \neq Z(H) \leq Z(G)$. Thus, $\frac{G}{Z(G)}$ is a group and $\left|\frac{G}{Z(G)}\right| = \frac{|G|}{|Z(G)|} < |G|$ since $Z(G) \neq 1$. Since, $H \trianglelefteq G$ and $K \trianglelefteq G$ we get $\frac{HZ(G)}{Z(G)} \trianglelefteq \frac{G}{Z(G)}$ and $\frac{KZ(G)}{Z(G)} \trianglelefteq \frac{G}{Z(G)}$. Then,

$$\frac{HZ(G)}{Z(G)} \cong \frac{H}{H \cap Z(G)} \quad \text{by } 2^{nd} \text{ Isomorphism Theorem}$$
$$\frac{KZ(G)}{Z(G)} \cong \frac{K}{K \cap Z(G)} \quad \text{by } 2^{nd} \text{ Isomorphism Theorem}$$

and $\frac{HZ(G)}{Z(G)}$ is nilpotent by induction hypothesis.

Case 2 If $N \neq 1$. Since $K \trianglelefteq G$ we know $N \leq K$. Also, since $H \trianglelefteq G$ by Lemma 1.4 $Z(H) \trianglelefteq G$. Therefore, since $K \trianglelefteq G$ we get $N = [Z(H), G] \trianglelefteq G$. Hence $N \trianglelefteq K$. Since $N \neq 1$ and K is nilpotent we get $1 \neq N \cap Z(K)$. Now $Z(H) \trianglelefteq G$ implies $N \leq Z(H)$. Thus, we get $1 \neq N \cap Z(K) \leq Z(H) \cap Z(K)$. Since $G = HK$ we have $Z(H) \cap Z(K) \leq Z(G)$. Thus, $Z(G) \neq 1$ and we get HK is nilpotent by Case 1.

□

Lemma 3.8. *Let G be a group and $N \trianglelefteq G$ such that $N \leq Z_i(G)$ for all $i \in \mathbb{Z}^+$. Then $Z_i\left(\frac{G}{N}\right) = \frac{Z_i(G)}{N}$ for all i.*

Theorem 3.6. *Let G be nilpotent and $N \trianglelefteq G$. Then $\frac{G}{N}$ is nilpotent.*

Proof. Since G is nilpotent there exists $n \in \mathbb{Z}^+ \cup \{0\}$ such that $Z_n(G) = G$. We claim that $\frac{Z_i(G)N}{N} \leq Z_i\left(\frac{G}{N}\right)$ for all i. Use induction on i. If $i = 0$ then

$$\begin{aligned}\frac{Z_0(G)N}{N} &= \frac{1N}{N} \\ &= \frac{N}{N} \\ &= 1N \\ &= Z_0\left(\frac{G}{N}\right).\end{aligned}$$

Assume the claim holds. Since $Z\left(\frac{G}{Z_i(G)}\right) = \frac{Z_{i+1}(G)}{Z_i(G)}$, by Lemma 3.3, $[G, Z_{i+1}(G)] \leq Z_i(G)$. Then $\frac{[G, Z_{i+1}(G)]N}{N} \leq \frac{Z_i(G)N}{N}$. Since $N \trianglelefteq G$ we get $[G, Z_{i+1}(G)]N = [G, Z_{i+1}(G)N]$. But, $\left[\frac{G, Z_{i+1}(G)N}{N}\right] = \left[\frac{G}{N}, \frac{Z_{i+1}(G)N}{N}\right]$.

Therefore, $\left[\frac{G}{N}, \frac{Z_{i+1}(G)N}{N}\right] \leq \frac{Z_i(G)N}{N} \leq Z_i\left(\frac{G}{N}\right)$. Then by Lemma 3.8,

$$\frac{\frac{Z_{i+1}(G/N)N}{N}}{Z_i(G/N)} \leq Z\left(\frac{G/N}{Z_i(G/N)}\right)$$

or

$$\frac{\frac{Z_{i+1}(G/N)N}{N}}{Z_i(G/N)} \leq \frac{Z_{i+1}(G/N)}{Z_i(G/N)}$$

and taking pre-images we get $\frac{Z_{i+1}(G/N)N}{N} \leq Z_{i+1}(G/N)$. Thus the claims holds by induction. Then $Z_n\left(\frac{G}{N}\right) \leq \frac{Z_n(G)N}{N} = \frac{GN}{N} = \frac{G}{N}$. Therefore, $Z_n\left(\frac{G}{N}\right) = \frac{G}{N}$. Hence, $\frac{G}{N}$ is nilpotent.

$\square$

Theorem 3.7. *Let G be nilpotent. Then $G = \prod P$ where the product runs over all $P \in Syl_p(G)$ and $p \mid |G|$.*

Proof. By Lemma 3.6 $P \trianglelefteq G$ for all $P \in Syl_p(G)$. Therefore, $\prod P \leq G$ where $P \in Syl_p(G)$ and $p \mid |G|$. Since $P \cap \prod\limits_{P \neq Q} Q = 1$ for all P when $q \neq p$ where $Q \in Syl_q(G)$ we get $|\prod P| = \prod |P| = |G|$. Thus, $G = \prod P$.

$\square$

Definition 3.5. *Let G be a group. Define the* **fitting group** *$F(G) = \prod\limits_{N \trianglelefteq G} N$ and N nilpotent. Then $F(G)$ is the unique maximal normal nilpotent subgroup of G.*

Definition 3.6. *Let G be a group and p be a prime. Define $O_p(G)$ by $O_p(G) = \prod\limits_{P \trianglelefteq G} P$ and P is a p-group. Then $O_p(G)$ is the unique maximal normal p-subgroup of G.*

Definition 3.7. *Let G be a group and p be a prime. Define $O_{p'}(G) = \prod\limits_{Q \trianglelefteq G} Q$ and Q is a p'-subgroup. Then $O_{p'}(G)$ is the unique maximal normal p'-subgroup of G.*

Theorem 3.8. *Let G be a group. Then $F(G) = \prod O_p(G)$ where $p \mid |G|$.*

Proof. As $p \mid |G|$ and $O_p(G)$ is a p-group, we know $O_p(G)$ is nilpotent. Since $O_p(G) \trianglelefteq G$ we get $O_p(G) \leq F(G)$. By Sylow's Theorem there exists $P \in Syl_p(F(G))$ such that $O_p(G) \leq P$. By Lemma 3.6 we know $P \trianglelefteq G$. Hence since P is a p-group we get $P \leq O_p(G)$. Thus, $O_p(G) = P \in Syl_p(F(G))$. Since $F(G)$ is nilpotent, by Theorem 3.7, $F(G) = \prod P = \prod O_p(G)$.

□

Lemma 3.9. *Let G be a group and $H \trianglelefteq G$. Then $C_G(H) \trianglelefteq G$.*

Lemma 3.10. *Let G be a group, $H \leq G$, $K \leq G$, and $L \leq G$ such that $[H, K] = 1$. Then $[H, KL] = [H, L]$.*

Lemma 3.11. *Let G be a group, $H \leq G$, $K \leq G$, and $L \leq G$ such that $K \leq H$. Then $H \cap KL = K(H \cap L)$.*

Theorem 3.9. *Let G be solvable. Then $C_G(F(G)) \leq F(G)$.*

Proof. Let $F = F(G)$ and $C = C_G(F)$. Suppose $C \not\leq F$. By Lemma 3.9, since $F \trianglelefteq G$ we know $C \trianglelefteq G$. Then, $\frac{CF}{F} \trianglelefteq \frac{G}{F}$. Also, since $C \not\leq F$ we know $\frac{CF}{F} \neq 1F$. Then there exists $1 \neq \frac{N}{F} \leq \frac{CF}{F}$ such that $\frac{N}{F}$ is a minimal normal subgroup of $\frac{G}{F}$. Since G is solvable we know $\frac{G}{F}$ is solvable. Hence by Theorem 2.6, $\frac{N}{F}$ is an elementary abelian p-group. Hence, $\left(\frac{N}{F}\right)' = \frac{N'}{F} = 1$ and so $N' \leq F$. Since $\frac{N}{F} \leq \frac{CF}{F}$ we get $N \leq CF$. But then, $N = N \cap CF = F(N \cap C)$. We claim that $K_i(N) \leq K_{i-1}(F)$ for all $i \geq 1$. Use induction on i. If $i = 1$ then, $K_1(N) = [N, N] = N' \leq F = K_0(F)$. Assume

$K_i(N) \leq K_{i-1}(F)$ for all $i \geq 1$. Then

$$\begin{aligned} K_{i+1}(N) &= [K_i(N), N] \\ &\leq [K_{i-1}(F), N] \\ &= [K_{i-1}(F), F(N \cap C)] \\ &= [K_{i-1}(F), F] \quad \text{by Lemma 3.10} \\ &= K_i(F) \end{aligned}$$

Thus, the claim holds. Since $F = F(G)$ in nilpotent there exists $n \in \mathbb{Z}^+ \cup \{0\}$ such that $K_n(F) = 1$. Then $K_{n+1}(N) \leq K_n(F) = 1$ and so $K_{n+1}(N) = 1$. Thus N is nilpotent. Since $\frac{N}{F} \trianglelefteq \frac{G}{F}$ we have $N \trianglelefteq G$. Thus, $N \leq F$. But then, $\frac{N}{F} = 1$, a contradiction. Therefore, $C_G(F(G)) \leq F(G)$.

□

Lemma 3.12. *Let G be a group and $P \in Syl_p(F(G))$. Then $P \trianglelefteq G$.*

Proof. Let $g \in G$. Since $P \leq F(G)$ we get $gPg^{-1} \leq gF(G)g^{-1}$. Since $F(G) \trianglelefteq G$, $gF(G)g^{-1} \leq F(G)$ and so $gPg^{-1} \leq F(G)$. Now $F(G)$ is nilpotent implies $P \trianglelefteq F(G)$. Thus by Sylow's Theorem,

$$n_p = \frac{|F(G)|}{|N_{F(G)}(P)|} = \frac{|F(G)|}{|F(G)|} = 1.$$

Also $|gPg^{-1}| = |P|$ and so $gPg^{-1} \in Syl_p(F(G))$. Since $n_p = 1$ we get $P = gPg^{-1}$ and so $P \trianglelefteq G$.

□

Theorem 3.10. *Let G be a group, $P \leq G$ be a p-group, and $N \trianglelefteq G$ be a p'-group. Then*

$$\frac{N_G(P)}{P} = N_{G/N}\left(\frac{PN}{N}\right)$$

Proof. Let $xN \in \frac{N_G(P)}{N}$ where $x \in N_G(P)$. Then

$$\begin{aligned}
xN\left(\frac{PN}{N}\right)x^{-1}N &= \frac{x(PN)x^{-1}}{N} \\
&= \frac{xPx^{-1}xNx^{-1}}{N} \\
&= \frac{PN}{N} \quad \text{since } x \in N_G(P) \text{ and } N \trianglelefteq G
\end{aligned}$$

Hence, $xN \in N_{G/N}\left(\frac{PN}{N}\right)$ and so $\frac{N_G(P)N}{N} \leq N_{G/N}\left(\frac{PN}{N}\right)$. Let $xN \in N_{G/N}\left(\frac{PN}{N}\right)$. Then $xN\left(\frac{PN}{N}\right)x^{-1}N = \frac{PN}{N}$ and so as before we get $\frac{xPx^{-1}}{N} = \frac{PN}{N}$ Taking pre-images we get $xPx^{-1} = PN$. Since N is a p'-group we get $P, xPx^{-1} \in Syl_p(PN)$. By Sylow's Theorem there exists $n \in N$ such that $nxPx^{-1}n^{-1} = P$ or $nxP(nx)^{-1} = P$. Thus, $nx \in N_G(P)$. But then $xN = nxN \in \frac{N_G(P)N}{N}$. Thus, $N_{G/N}\left(\frac{PN}{N}\right) \leq \frac{N_G(P)N}{N}$. Therefore, $\frac{N_G(P)N}{N} = N_{G/N}\left(\frac{PN}{N}\right)$.

□

4 Groups Acting on Groups

We know look at how groups act on groups and will prove important Theorems about co-prime actions.

Definition 4.1. *Let G and H be groups. Then G acts on H if there exists a homomorphism ϕ such that $\phi : G \to Aut(H)$.*

Theorem 4.1. *Let G and H be p-groups such that G acts on H. Then there exists $1 \neq h \in H$ such that $G = G_h$.*

Proof. Since G acts on H we know G acts on $S = H \setminus \{1\}$. Since H is a p-group and $p \not| 1$ we know $p \not| |S|$. Since G is a p-group by the Fixed Point Theorem there exists $s \in S$ such that $G = G_s$. But then, $1 \neq s \in H$.

$\square$

Theorem 4.2. *Let G be a group, $A \leq G$, $B \leq G$, and $C \leq G$ such that $[A, B, C] = 1$ and $[B, C, A] = 1$. Then $[C, A, B] = 1$.*

Proof. Let $a \in A$, $b \in B$, and $c \in C$. Notice

$$b[a^{-1}, b, c^{-1}]b^{-1}c[b^{-1}, c, a^{-1}]c^{-1}a[c^{-1}, a, b^{-1}]a^{-1} = 1$$

Now, $[a^{-1}, b, c^{-1}] = 1$ and so $b[a^{-1}, b, c^{-1}]b^{-1} = 1$ since $[A, B, C] = 1$. Similarly, $c[b^{-1}, c, a^{-1}]c^{-1} = 1$ since $[B, C, A] = 1$. Thus, $a[c^{-1}, a, b^{-1}]a^{-1} = 1$ and so $[c^{-1}, a, b^{-1}] = 1$. Therefore, $[C, A, B] = 1$.

$\square$

Theorem 4.3. *Let $A \leq Aut(P)$ be a p'-group and P be a p-group such that there exists a subnormal series*

$$P \trianglerighteq P_1 \trianglerighteq P_2 \trianglerighteq \cdots \trianglerighteq P_n = 1$$

such that P_i is A-invariant and A acts trivially on $\frac{P_i}{P_{i+1}}$ for all $1 \leq i \leq n-1$. Then A acts trivially on P.

Proof. Use induction on $|P|$. Since $|P_1| \leq |P|$ we get A acts trivially on P_1. If A does not act trivially on P there exists $\phi \in A$ and $x \in P$ such that $\phi(x) \neq x$. Since A acts trivially on $\frac{P}{P_1}$ we get $\phi(xP_1) = xP_1$ or $\phi(x)P_1 = xP_1$. Hence, there exists a $y \in P_1$ such that $\phi(x) = xy$. Then,

$$\begin{aligned}\phi(\phi(x)) &= \phi(xy) \\ &= \phi(x)\phi(y) \\ &= xyy \\ &= xy^2.\end{aligned}$$

Since A acts trivially on P_1 we get $x = \phi^{|\phi|}(x) = xy^{|\phi|}$. But then, $y^{|\phi|} = 1$ and so $|y| \mid |\phi|$. Since P is a p-group and A is a p'-group we get $\gcd(|y|, |\phi|) = 1$. Thus, $|y| = 1$ and so $y = 1$. But then $\phi(x) = x1 = x$, a contradiction. Therefore, A acts trivially on P.

$\square$

Theorem 4.4. *Suppose $A \times B$ acts on P such that A is a p'-group and B and P are p-groups. If A acts trivially on $C_P(B)$ then A acts trivially on P.*

Proof. Let $C_P(B) \leq Q \leq P$ where Q is a maximal $A \times B$-invariant subgroup of P such that A acts trivially on Q. If $Q < P$. Now by Theorem 3.4, $Q < N_P(Q) = R$ and $Q \trianglelefteq R$. Since P and Q are $A \times B$-invariant we know R is $A \times B$-invariant. Thus, $A \times B$ acts on $\frac{R}{Q}$ and so B acts on $\frac{R}{Q}$. Let $1Q \neq \frac{S}{Q} \leq \frac{R}{Q}$ be a minimal $A \times B$-invariant subgroup of $\frac{R}{Q}$. Now, B acts on $\frac{S}{Q}$ and they are both p-groups. Therefore, by Theorem 4.1, $1 \neq C_{S/Q}(B) \leq \frac{S}{Q}$. Since $\frac{S}{Q}$ is $A \times B$-invariant we get $C_{S/Q}(B)$ is $A \times B$-invariant. By the minimality of $\frac{S}{Q} = C_{S/Q}(B)$. But then, $[S, B] \leq Q$.

Now, $[S, B, A] \leq [Q, A] = 1$ since A acts trivially on Q. Hence, $[S, B, A] = 1$. Also, $[B, A] = 1$ implies $[B, A, S] = 1$. By Theorem 4.2, $[A, S, B] = 1$. Thus, $[A, S] \leq C_P(B) \leq Q$. Now we have a subnormal series $S \trianglelefteq Q \trianglelefteq 1$ and A acts trivially on $\frac{S}{Q}$ and on $\frac{Q}{\{1\}}$. Since A is a p'-group and S is a p-group by Theorem 4.3, A acts trivially on S. Now, $Q < S$ since $\frac{S}{Q} \neq 1Q$ and S is $A \times B$-invariant, this contradicts the maximality of Q. Hence, $P = Q$ and A acts trivially on P.

$\square$

Definition 4.2. *Let G be a group, $A \leq Aut(G)$, $g \in G$, and $\phi \in A$. Then*

1. *$[g, \phi] = g^{-1}\phi(g)$ is the commutator of g and ϕ*

2. *$[G, A] = \langle\{[g, \phi] \mid g \in G \, for \, all \, \phi \in A\}\rangle$*

3. *$C_G(A) = \{g \in G \mid \phi(g) = g \, for \, all \, \phi \in A\}$.*

Theorem 4.5. *Let $A \leq Aut(P)$ be a p'-group and P be an abelian p-group. Then $P = C_P(A) \times [P, A]$.*

Proof. Let $|A| = n$ and writing P additively define $\theta = \frac{1}{n} \sum_{\phi \in A} \phi$. Then $\theta : P \to P$ is a homomorphism since P is abelian. We want to show the following,

1. $\theta\phi_1 = \theta$ for all $\phi_1 \in A$

2. $\theta^2 = \theta$

3. $\theta(P) = C_P(A)$

4. $[P, A] = \{-x + \theta(x) \mid x \in P\}$

5. $P = \theta(P) \times B$ where $B = [P, A]$

For (1), let $x \in P$. Then

$$\begin{aligned}\theta\phi_1(X) &= \frac{1}{n}\sum_{\phi\in A}\phi\phi_1(x)\\ &= \frac{1}{n}\sum_{\phi\in A}\phi(x) \quad \text{since } P \text{ is abelian}\\ &= \theta(x).\end{aligned}$$

For (2), let $x \in P$. Then,

$$\begin{aligned}\theta^2(x) &= \theta(\frac{1}{n}\sum_{\phi\in A}\phi(x))\\ &= \frac{1}{n}\sum_{\phi\in A}\theta\phi(x)\\ &= \frac{1}{n}\sum_{\phi\in A}\theta(x)\\ &= \frac{1}{n}n\theta(x)\\ &= \theta(x).\end{aligned}$$

For (3), let $\theta(x) \in \theta(P)$ and $\phi \in A$. Then $\phi\theta(x) = \theta(x)$. Hence, $\theta(x) \in C_P(A)$ and so $\theta(P) \leq C_P(A)$. Let $x \in C_P(A)$. Then,

$$\begin{aligned}\theta(x) &= \frac{1}{n}\sum_{\phi\in A}\phi(x)\\ &= \frac{1}{n}\sum_{\phi\in A}x\\ &= \frac{1}{n}nx\\ &= x.\end{aligned}$$

Hence, $x \in \theta(P)$ and so $C_P(A) \leq \theta(P)$. Thus, $\phi(P) = C_P(A)$. For (4), let $x \in P$. Then,

$$\begin{aligned} -x + \theta(x) &= -x + \frac{1}{n} \sum_{\phi \in A} \phi(x) \\ &= \frac{1}{n} \sum_{\phi \in A} -x + \theta(x) \quad \text{since } P \text{ is abelian} \\ &\in [P, A] \quad \text{since } -x + \theta(x) \in [P, A] \text{ for all } x \in P \text{ and } \phi \in A. \end{aligned}$$

Hence, $\{-x + \theta(x) \mid x \in P\} \subseteq [P, A]$. Let $x \in P$ and $\phi \in A$. Then,

$$\begin{aligned} [x, \phi] &= -x + \phi(x) \\ &= -x + \phi(x) + 0 \\ &= -x + \phi(x) + \theta(x + -\phi(x)) \quad \text{by (1)} \\ &\in \{x + \theta(x) \mid x \in P\}. \end{aligned}$$

Hence, $[P, A] \subseteq \{-x + \theta(x) \mid x \in P\}$ since $\{-x + \theta(x) \mid x \in P\}$ is closed. Therefore, $[P, A] = \{-x+\theta(x) \mid x \in P\}$. For (5), let $x \in P$. Then, $x = \theta(x)+x+-\theta(x) \in \theta(P)B$. Thus, $P = \theta(P)B$. Suppose, there exists $u \in \phi(P) \cap B$. Then, there exists $x, y \in P$

such that $u = \theta(x)$ and $u = -y + \theta(y)$. Then,

$$\begin{aligned} u &= \theta(x) \\ &= \theta^2(x) \quad \text{by (2)} \\ &= \theta(\theta(x)) \\ &= \theta(u) \\ &= \theta(-y + \theta(y)) \\ &= -\theta(y) + \theta^2(y) \\ &= -\theta(y) + \theta(y) \\ &= 0. \end{aligned}$$

Hence, $\theta(B) \cap B = 0$ and so $P = \theta(P)B = C_P(A)[P, A] = C_P(A) \times [P, A]$.

□

Lemma 4.1. *Let G be a group and $A \leq Aut(G)$. Then $[G, A] \trianglelefteq G$ and $[G, A]$ is A-invariant.*

Lemma 4.2. *Let G be a group, $A \leq Aut(G)$, and $N \trianglelefteq G$ be A-invariant. Then A acts on $\frac{G}{N}$ by $\phi(gN) = \phi(g)N$ for all $gN \in \frac{G}{N}$ and for all $\phi \in A$.*

Theorem 4.6. *Let $A \leq Aut(P)$ be a p'-group and P be a p-group. Then*

$$P = C_P(A)[P, A].$$

Proof. Let $H = [P, A]$.

Case 1 $H \leq Z(P)$. Let $\phi \in A$. Define $\alpha_\phi : P \to [P, \phi]$ by $\alpha_\phi = [x, \phi]$ for all $x \in P$. If $x, y \in P$ then

$$\begin{aligned}
\alpha_\phi(xy) &= [xy, \phi] \\
&= (xy)^{-1}\phi(xy) \\
&= y^{-1}x^{-1}\phi(x)\phi(y) \\
&= x^{-1}\phi(x)y^{-1}\phi(y) \quad \text{since } x^{-1} \in \phi(x) \leq Z(P) \\
&= [x, \phi][y, \phi] \\
&= \alpha_\phi(x)\alpha_\phi(y).
\end{aligned}$$

Hence, α_ϕ is a homomorphism. Now, $Kern\, \alpha_\phi = C_P(\phi)$ and α_ϕ is onto. By Theorem 1.5, $\dfrac{P}{Kern\, \alpha_\phi} \cong \alpha_\phi(P)$. Thus, $\dfrac{P}{C_P(\phi)} \cong [P, \phi]$ since α_ϕ is onto. Since $[P, \phi] \leq H \leq Z(P)$ we get $\dfrac{P}{C_P(\phi)}$ is abelian. Hence, by Lemma 1.3, $P' \leq C_P(\phi)$. Therefore, $P' \leq C_P(A)$. Now A acts on $\dfrac{P}{P'}$ which is an abelian p-group. Then, $\dfrac{P}{P'} = C_{P/P'}(A)\left[\dfrac{P}{P'}, A\right] = C_{P/P'}\dfrac{[P, A]P'}{P'}$. Let $\dfrac{C}{P'} = C_{P/P'}(A)$. Then, $\dfrac{P}{P'} = \dfrac{C}{P'}\dfrac{[P, A]P'}{P'}$ and taking pre-images we get $P = C[P, A]P'$ or $P = C[P, A]$. Now, $[P', C] = P'$ since $\dfrac{C}{P'} = C_{P/P'}(A)$. Hence, we have a subnormal series

$$C \trianglerighteq P' \trianglerighteq 1$$

and A acts trivially on $\dfrac{C}{P'}$ and $\dfrac{P'}{\{1\}}$. Since A is a p'-group and C is a p-group by Theorem 4.3, A acts trivially on C. Thus, $P = C[P, A] \leq C_P(A)[P, A] \leq P$ and so $P = C_P(A)[P, A]$.

Case 2 $H \not\leq Z(P)$. Use induction on $|P|$. Since P is nilpotent and $H = [P, A] \trianglelefteq P$

by Theorem 3.3, $1 \neq H \cap Z(P)$. Now, $K = H \cap Z(P) \trianglelefteq P$ and so $\frac{P}{K}$ is a p-group. Also, K is A-invariant and so A acts on $\frac{P}{K}$. Also, $\left|\frac{P}{K}\right| < |P|$ and so by induction $\frac{P}{K} = C_{P/K}(A)\left[\frac{P}{K}, A\right] = C_{P/K}(A)\frac{[P,A]K}{K}$. Let $\frac{C}{K} = C_{P/K}(A)$. Then, $\frac{P}{K} = \frac{C}{K}\frac{[P,A]K}{K}$ and taking pre-images we get $P = C[P,A]K = C[P,A]$. If $P = C$ then $\frac{P}{K} = \frac{C}{K} = C_{P/K}(A)$. Hence, $\left[\frac{P}{K}, A\right] = K$ and so $[P,A] \leq K$. But then $H = [P,A] \leq K \leq Z(P)$, a contradiction. Therefore, $P \neq C$. Thus, $C < P$ and so $|C| < |P|$. Hence, by induction $C = C_C(A)[C,A]$. But then, $P = C_C(A)[C,A][P,A] \leq C_P(A)[P,A] \leq P$ and so $P = C_P(A)$.

□

Theorem 4.7. *Let $A \leq Aut(P)$ be a p'-group, P be a p-group, and $N \trianglelefteq P$ be A-invariant. Then $C_{P/N}(A) = \frac{C_P(A)N}{N}$.*

Proof. Let $cN \in \frac{C_P(A)N}{N}$ and $\phi \in A$. Then, $\phi(cN) = \phi(c)N = cN$ since $c \in C_P(A)$. Thus, $\frac{C_P(A)N}{N} \leq C_{P/N}(A)$. Let $\frac{C}{N} = C_{P/N}(A)$. Then, $N \leq C \leq P$ and C is A-invariant. Also $[C,A] \subseteq N$. Hence, by Theorem 4.6, $C = C_P(A)[C,A] \leq C_P(A)N$. Therefore, by taking pre-images we get $C_{P/N}(A) = \frac{C}{N} \leq \frac{C_P(A)N}{N}$. Hence, $C_{P/N}(A) = \frac{C_P(A)N}{N}$.

□

Definition 4.3. *Let G be a group then $O_{p'}(G) = \prod_{N \trianglelefteq G} N$ where N is a p'-group and is the largest normal p'-subgroup of G.*

Theorem 4.8. *Let G be solvable and $P \leq G$ be a p-subgroup. Then, $O_{p'}(N_G(P)) \leq O_{p'}(G)$.*

Proof. Let $A = O_{p'}(N_G(P))$ and $B = O_{p'}(G)$.

Case 1 $O_{p'}(G) = 1$. We want to show that $A = 1$. Suppose, $A \neq 1$. Then, $A \trianglelefteq N_G(P)$ and $P \trianglelefteq N_G(P)$. Hence, $AP \trianglelefteq N_G(P)$. Since A is a p'-group and P is a p-group we get $|A \cap P| = 1$. Since $A \trianglelefteq N_G(P)$ and $P \trianglelefteq N_G(P)$ we get $[A, P] = 1$. Since $B \trianglelefteq G$ we get $A \times P$ acts on B by conjugation. Now, $C_B(P) \leq N_G(P)$ and $A \trianglelefteq N_G(P)$. Since $B \trianglelefteq G$ we get $[A, C_B(P)] \leq A \cap B = 1$. Thus, A acts trivially on $C_B(P)$. By Theorem 4.4, A acts trivially on B. But then, $A \leq C_G(B)$ and so $C_G(B)$ is not a p-group. Since $B \trianglelefteq G$ we know $C_G(B) \trianglelefteq G$. Then, $\frac{C_G(B)B}{B} \trianglelefteq \frac{G}{B}$. If $\frac{C_G(B)B}{B} = 1$ we get $C_G(B) \leq B$. But then, since B is a p-group we get $C_G(B)$ is a p-group, a contradiction. Thus, $1 \neq \frac{C_G(B)}{B} \trianglelefteq \frac{G}{B}$. Hence, there exists $1 \neq \frac{N}{B} \leq \frac{C_G(B)}{B}$ such that $\frac{N}{B}$ is a minimal subgroup of $\frac{G}{B}$. Since G is solvable by Theorem 2.3, $\frac{G}{B}$ is solvable. By Theorem 2.6, $\frac{N}{B}$ is an elementary q-group. Suppose $p = q$. Since $\frac{N}{B} \trianglelefteq \frac{G}{B}$ we get $N \trianglelefteq G$. Also, $|N| = \frac{|N|}{|B|}$ and $|B| = \left|\frac{N}{B}\right| |B|$ is a power of P and so N is a p-group. Hence, $N \leq B = O_p(G)$ and we get $\frac{N}{B} = 1$, a contradiction. Therefore, $p \neq q$. Let $Q \in Syl_q(N)$. Then, $\frac{QB}{B} \in Syl_q\left(\frac{N}{B}\right)$. Since $\frac{N}{B}$ is a p-group we get $\frac{N}{B} = \frac{QB}{B}$. Taking pre-images we get $N = QB$. Since $\frac{N}{B} \leq \frac{C_G(B)}{B}$ we get $N \leq C_G(B)$. Hence, $Q \leq N \leq C_G(B)B$ and so $\frac{QC_G(B)}{C_G(B)} \leq \frac{C_G(B)B}{C_G(B)}$. But, $\frac{C_G(B)B}{C_G(B)} \cong \frac{B}{B \cap C_G(B)}$ is a p-group. Thus, $\frac{QC_G(B)}{C_G(B)}$ is a p-group. But, $\frac{QC_G(B)}{C_G(B)} \cong \frac{Q}{Q \cap C_G(B)}$ is a q-group. Thus, $\frac{QC_G(B)}{C_G(B)} = 1$ and so $Q \leq C_G(B)$. Since $N = QB$ we get $Q \trianglelefteq N$. Therefore, Q is the only Sylow q-subgroup of N by Sylow's Theorem. Now, since $N \trianglelefteq G$ we get $Q \trianglelefteq G$. Since $p \neq q$ we know Q is a p'-group and so $Q \leq O_{p'}(G) = 1$ and so $N = QB = B$ and we get $\frac{N}{B} = \frac{B}{B} = 1$, a contradiction. Thus, $A = 1$

Case 2 $O_{p'}(G) \neq 1$. Then, $\frac{G}{O_{p'}(G)}$ is solvable and $\frac{PO_{p'}(G)}{O_{p'}} \leq \frac{G}{O_{p'}}(G)$ is a p-group.

Finally, $O_{p'}\left(\dfrac{G}{O_{p'}(G)}\right) = 1$ by Case 1 we get $O_{p'}\left(N_{G/O_{p'}(G)}\left(\dfrac{PO_{p'}(G)}{O_{p'}(G)}\right)\right) = 1$. Then, $O_{p'}\left(\dfrac{N_G(P)O_{p'}(G)}{O_{p'}}\right) = 1$ by Lemma 3.10. But, $\dfrac{O_{p'}(N_G(P))O_{p'}(G)}{O_{p'}(G)} \leq O_{p'}\left(\dfrac{N_G(P)O_{p'}(G)}{O_{p'}}\right)$ and so $\dfrac{O_{p'}(N_G(P))O_{p'}(G)}{O_{p'}(G)} = 1$ which implies $O_{p'}(N_G(P)) \leq O_{p'}(G)$.

□

Definition 4.4. *Let G be a group. Define the* **Franttini Subgroup** *by $\Phi(G) = \bigcap M$ where M is a maximal subgroup of G.*

Theorem 4.9. *Let P be a p-group. Then, $\dfrac{P}{\Phi(P)}$ is an elementary abelian p-group. In particular, if $\Phi(P) = 1$ then P is an elementary p-group.*

Proof. Let M be a maximal subgroup of P and let $x \in P$. Since P is nilpotent we get $M \trianglelefteq P$ by Theorem 3.5. Since M is maximal we know $\{1\}$ and $\dfrac{P}{M}$ are the only subgroups of $\dfrac{P}{M}$. Thus, $\dfrac{P}{M} \cong \mathbb{Z}_p$ is abelian. Thus, $P' \leq M$. Also, $(xM)^p = x^pM = 1M$ since $\dfrac{P}{M} \cong \mathbb{Z}_p$. Thus, $x^p \in M$ but then, $P' \leq \Phi(P)$ and $x^p \in \Phi(P)$ for all $x \in P$ which implies all the elements have order p or 1. By the Fundamental Theorem of Finite Abelian Groups we get $\dfrac{P}{\Phi(P)} \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ is an elementary abelian p-group. In particular, if $\Phi(P) = 1$ then $P \cong \dfrac{P}{\{1\}} \cong \dfrac{P}{\Phi(P)}$ is an elementary p-group.

□

Definition 4.5. *A group A acts* **regularly** *on a group G if $C_G(\alpha) = 1$ for all $1 \neq \alpha \in A$.*

Theorem 4.10. *Suppose an elementary p-group A acts regularly on a q-group V. Then $A \cong \mathbb{Z}_p$.*

Proof. Use contradiction. Suppose $A \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Then all elements of A have order P. Hence, $H = \bigcup_{i=1}^{p+1} A_i$ such that $|A_i| = p$ for all i and $A_i \cap A_j = 1$ for all $i \neq j$. Let $1 \neq v \in V$ and $1 \neq a_0 \in A$. Then $a_0(\prod_{a \in A} av) = \prod_{a \in A} a_0av = \prod_{a \in A} av$ since as a runs over A so does a_0a and V is abelian. Since A acts regular on V we get $\prod av = 1$. Similarly, $\prod_{a_i \in A_i} a_iv = 1$. Hence,

$$\begin{aligned}
1 &= \prod_{i=1}^{p+1} \prod_{a_i \in A_i} a_iv \\
&= v^p \prod_{a \in A} av \quad \text{since } V \text{ is abelian} \\
&= v^p 1 \\
&= v^p.
\end{aligned}$$

Hence, $v^p = 1$ and so $|v| = p$ since $v \neq 1$. But then, $p = |v| \mid |V|$ which implies $p \mid q^a$, a contradiction. Thus, $A \cong \mathbb{Z}_p$.

□

Theorem 4.11. *Let $G = BV$ be a group such that $B \trianglelefteq G$ is a p-group and V is an elementary abelian q-group. Then*

$$B = \langle C_B(U) \mid U \leq V, \frac{|V|}{|U|} = q \rangle$$

Proof. Use induction on $|G|$. Let $A = \langle C_B(U) \mid U \leq V, \frac{|V|}{|U|} = q \rangle$. If $A < B$ then since B is nilpotent we get $A < N_B(A)$. Since $V \leq N_G(B)$ and V is abelian we know $V \leq N_G(A)$. Hence, $V \leq N_B(N_B(A))$ and so $VN_B(A) \leq G$. If $VN_B(A) < G$ then by induction we get $N_B(A) = \langle C_{N_B(A)}(U) \mid U \leq V, \frac{|V|}{|U|} = q \rangle \leq A$, a contradiction.

Hence, $G = VN_B(A)$ and so $A \trianglelefteq G$. Thus, $\dfrac{G}{A} = \dfrac{B}{A}\dfrac{VA}{A}$ is a group. If $A \neq 1$, then $\left|\dfrac{G}{A}\right| < |G|$ and $\dfrac{B}{A} \trianglelefteq \dfrac{G}{A}$ is a p-group and $\dfrac{VA}{A}$ is an elementary q-group. Hence, by induction $\dfrac{B}{A} = \langle C_{B/A}(\dfrac{U}{A}) \mid \dfrac{U}{A} \leq \dfrac{VA}{A}, \dfrac{|VA|}{|U|} = q\rangle$. Since $A < B$ we know $\dfrac{B}{A} \neq 1A$. Hence, there exists $\dfrac{U}{A} \leq \dfrac{VA}{A}$ such that $\dfrac{|VA|}{|U|} = q$ and $C_{B/A}(\dfrac{U}{A}) \neq 1A$. Let $U_0 \in Syl_q(U)$. Then $\dfrac{U_0A}{A} \in Syl_q(\dfrac{U}{A})$. Since $\dfrac{U}{A}$ is a q-group we get $\dfrac{U}{A} = \dfrac{U_0A}{A}$ Hence, $C_{B/A}(\dfrac{U_0A}{A}) \neq 1A$. Since U_0 and $\dfrac{U_0A}{A}$ act on $\dfrac{B}{A}$ in the same way we get $C_B(U_0) \neq 1$ by Theorem 4.7 the q-group U_0 acts on the p-group $\dfrac{B}{A}$ and we get $1 \neq C_{B/A}(U_0) = \dfrac{C_B(U_0)A}{A}$. Hence, $C_B(U_0) \not\leq A$. Now,

$$\begin{aligned} q &= \frac{|VA|}{|U|} \\ &= \frac{|VA|}{|U_0A|} \\ &= \frac{|VU_0A|}{|U_0A|} \\ &= \frac{\frac{|V||U_0A|}{|V \cap U_0A|}}{|U_0A|} \\ &= \frac{|V|}{|V \cap U_0A|}. \end{aligned}$$

Hence, $\dfrac{|V|}{|V \cap U_0A|} = q$. Now, $V \cap U_0A \leq U_0A$ and $V \cap U_0A$ is a q-group. Since $U_0 \in Syl_q(U_0A)$ by Sylow's Theorem there exists $a \in A$ such that $V \cap U_0A \leq aU_0a^{-1}$. Then $V \cap U_0A \leq V \cap aU_0a^{-1}$. But $V \cap aU_0a^{-1} \leq V \cap U_0A$ and so $V \cap U_0A = V \cap aU_0a^{-1}$. Hence, $\dfrac{|V|}{|V \cap aU_0a^{-1}|} = q$ and so $C_B(V \cap aU_0a^{-1}) \leq A$. Now, $C_B(U_0) \not\leq A$, a contradiction. Hence, $A = 1$. As, $\Phi(B) < B$ and $B \trianglelefteq G$, we get $\Phi(B) \trianglelefteq G$. Then $\Phi(B)V < BV = G$. Hence, $|\Phi(B)| < |G|$ and so, by induction, $\Phi(B) = \langle C_{\Phi(B)}(U) \mid U \leq V, \dfrac{|V|}{|U|} = q\rangle \leq A = 1$. Thus, $\Phi(B) = 1$ which implies by Theorem 4.9 B is an elementary abelian p-group. Let $1 \neq b \in B$ and $\langle b^G \rangle = \langle gbg^{-1} \mid g \in G\rangle$. Then

$\langle b^G\rangle \trianglelefteq G$ and so V acts on $\langle b^G\rangle$ by conjugation. Now, since $G = BV$ and B is abelian we get $\langle b^G\rangle = \langle b^V\rangle$. Moreover, since V is abelian, $\dfrac{V}{C_V(b)}$ acts on $\langle b^V\rangle$ regularly by conjugation. Then, by Theorem 4.10, $\dfrac{V}{C_V(b)} \cong \mathbb{Z}_q$ and so $|\dfrac{V}{C_V(b)}| = q$. Now, $1 \neq b \in C_B(C_V(b)) \leq A = 1$, a contradiction. Thus, $B = A = \langle C_B(U) \mid U \leq V, \dfrac{|V|}{|U|}\rangle$.

□

Theorem 4.12. *Let $G = AB$ where A is a p-group and B is a q-group. Further suppose there exists $1 \neq A_0 \trianglelefteq A$ and $1 \neq B_0 \trianglelefteq B$ such that $\langle A_0^{B_0}\rangle$ is a p-group. Then G is not simple.*

Proof. Let $\langle A_0^{B_0}\rangle \leq P_0 \leq G$ such that P_0 is maximal with respect to P_0 being a p-group, P_0 generated by conjugates of A_0, and $B_0 \leq N_G(P_0)$. By Sylow's Theorem there exists $P \in Syl_p(G)$ such that $P_0 \leq P$. We want to show that $P_0 \trianglelefteq P$. Suppose not. Then, $N_P(P_0) < P$. Since P is nilpotent we get $N_P(P_0) < N_P(N_P(P_0))$. Let $x \in N_P(N_P(P_0)) \setminus N_P(P_0)$. Then $xP_0x^{-1} \neq P_0$ and so $xP_0x^{-1} \nleq P_0$. Hence, there exists $g \in G$ such that $xgA_0(xg)^{-1} \nleq gA_0g^{-1} \leq P_0$. Let $H = \langle P_0(xgA_0(xg)^{-1})^{B_0}\rangle$. Then $P_0 \leq H$. Also, H is generated by conjugates of A_0 and since $B_0 \leq N_G(P_0)$ we know $B_0 \leq N_B(H)$. Now, $gA_0g^{-1} \leq P_0 \leq N_P(P_0)$ and so $xgA_0(xg)^{-1} \leq N_P(P_0)$ since $x \in N_P(N_P(P_0))$. Thus, $(xgA_0(xg)^{-1})^{B_0} \leq N_G(P_0)$. Therefore, $H = \langle P_0, (xgA_0(xg)^{-1})^{B_0}\rangle = P_0\langle (xgA_0(xg)^{-1})^{B_0}\rangle$. Now, since $A_0 \trianglelefteq A$ and $B_0 \trianglelefteq B$ and $G = AB$ we get $\langle (xgA_0(xg)^{-1})^{B_0}\rangle \leq \langle A_0^{B_0}\rangle^b$ since $g = ba$ and $b \in B$ and $a \in A$. But since $\langle A_0^{B_0}\rangle$ is a p-group we get $\langle A_0^{B_0}\rangle^b$ is a p-group. Therefore, $\langle (xgA_0(xg)^{-1})^{B_0}\rangle$ is a p-group. Since P_0 is a p-group we get $H = P_0\langle (xgA_0(xg)^{-1})^{B_0}\rangle$ is a p-group, a contradiction to the maximality of P_0. So, $P_0 \trianglelefteq P$. Now since $G = AB$ and $P \in Syl_p(G)$ we get $G = PB$ so then since $B_0 \trianglelefteq B$, $B_0 \leq N_P(P_0)$, and $P \leq N_G(P_0)$ we get $1 \neq B_0 \leq \bigcap_{b\in B} bN_G(P_0)b^{-1} = \bigcap_{g\in G} gN_G(P_0)g^{-1} \trianglelefteq G$. If $\bigcap_{g\in G} gN_G(P_0)g^{-1} \neq G$ we get G

is not simple. If $G = \bigcap_{g \in G} gN_G(P_0)g^{-1}$ then $G = N_G(P_0)$. But then $1 \neq P_0 \trianglelefteq G$ and $P_0 = G$ since P_0 is a p-group. Hence, G is not simple.

$\square$

Definition 4.6. *Let G be a group and p be a prime. Define*

$$\Omega_1(G) = \langle x \in G \mid x^p = 1 \rangle.$$

Definition 4.7. *Let G be group and $P \leq G$ be a p-group. Define*

$$J(P) = \langle A \mid A \leq P \textit{ is abelian and } |A| \textit{ is maximal} \rangle.$$

Then $J(P)$ is called the **Thompson Subgroup**.

Theorem 4.13 (Baer)**.** *Let G be a group and $H \leq G$ such that $\langle H, gHg^{-1} \rangle$ is a p-group for all $g \in G$. Then $H \leq O_p(G)$.*

Theorem 4.14. *Let G be a group and $x \notin O_2(G)$ such that $x^2 = 1$. Then there exists $y \in G$ such that $|y|$ is odd and $xyx^{-1} = y^{-1}$.*

Theorem 4.15. *Let G be a group such that $|G| = p^a q^b$ for odd primes p and q and $P \in Syl_p(G)$ such that $C_G(\Omega_1(Z(P))) = P$. Then $J(P) \trianglelefteq G$.*

Definition 4.8. *Let G be a group and $H \leq G$. Then H* **is a p-central subgroup of G** *is there exists $P \in Syl_p(G)$ such that $H \leq Z(P)$. We write H p-central $\leq G$.*

5 Burnsides $p^a q^b$ Theorem

We now have all the group theoretical tools needed to begin our proof of Burnside's Theorem.

Theorem 5.1 (Burnside's Theorem)**.** *Let G be a group such that $|G| = p^a q^b$. Then G is solvable.*

Proof. Assume the theorem is false and let G be a minimal counterexample. We to prove the following about G,

1. G is simple.

   Assume not. There exists $1 \neq N \trianglelefteq G$ and $N \neq G$. Then $\frac{G}{N}$ is a group such that $\left|\frac{G}{N}\right| < |G|$ and N is a group such that $|N| < |G|$ and they are both pq groups. Hence, by the minimality of G we get $\frac{G}{N}$ and N are solvable and so by Theorem 2.3, G is solvable, a contradiction. Thus, G is simple.

2. If M is a maximal subgroup of G then $F(M)$ is a p or q group.

   Suppose $p \mid |F(M)|$ and $q \mid |F(M)|$. Let $Z = Z_p Z_q$ where $Z_p = \Omega_1(Z(O_p(M)))$ and $Z_q = \Omega_1(Z(O_q(M)))$. Then $Z_p \trianglelefteq M$ and so $M \leq N_G(Z_p) \leq G$. By the maximality of M we get $M = N_G(Z_p)$ or $G = N_G(Z_p)$. If $G = N_G(Z_p)$ then we get $Z_p \trianglelefteq G$, a contradiction since G is simple. Therefore, $M = N_G(Z_p)$ and similarly $M = N_G(Z_q)$. We claim that M is the unique maximal subgroup of G such that $Z \leq M$. Suppose $Z \leq H$ and H is a maximal subgroup of G. Then $O_p(M) \cap H \trianglelefteq M \cap H$ is a q-group. But then,

$$\begin{aligned}
O_p(M) \cap H &\leq O_p(M \cap H) \\
&= O_p(N_G(Z_q) \cap H) \\
&= O_p(N_H(Z_q)) \\
&\leq O_p(H).
\end{aligned}$$

Similarly, using $M = M_G(Z_p)$ we get $O_q(M) \cap H \leq O_q(H)$. Hence,

$$\begin{aligned} F(M) \cap H &= O_p(M)O_q(M) \cap H \\ &= (O_p(M) \cap H)(O_q(M) \cap H) \\ &\leq O_p(H)O_q(H) \\ &= F(H). \end{aligned}$$

Thus, $F(M) \cap H \leq F(H)$. Now, $Z = Z_q Z_q \leq F(M) \cap H \leq F(H)$. Hence, by Sylow's Theorem $Z_p \leq O_p(H)$ and $Z_q \leq O_q(H)$. Now, $[Z_p, O_q(H)] \leq [O_p(H), O_q(H)] \leq O_p(H) \cap O_q(H) = 1$ and so $[Z_p, O_q(H)] = 1$. Thus, $O_q(H) \leq C_G(Z_p) \leq N_G(Z_p) = M$. Similarly, $O_p(H) \leq M$ and so $F(M) = O_p(H)O_q(H) \leq M$. Since $Z_p \leq O_p(H)$ and $Z_q \leq O_q(H)$ we get $p \mid |F(H)|$ and $q \mid |F(H)|$. Similarly, since H is maximal subgroup, using $Z_p^* = \Omega_1(Z(O_p(H)))$ and $Z_q^* = \Omega_1(Z(O_q(H)))$ we get $F(M) \cap M \leq F(M)$ and $F(M) \leq H$. But then $F(M) = F(M) \cap H \leq F(H) = F(H) \cap M \leq F(M)$. Thus, $F(M) = F(H)$. Now since M and H are maximal and G is simple we get $M = N_G(F(M)) = N_G(F(H)) = H$. We claim M does not contain a Sylow p-subgroup of G. Let $M_p \in Syl_p(M)$. If $M_p \in Syl_p(G)$ then by Sylow's Theorem there exists $G_q \in Syl_q(G)$ such that $O_q(M) \leq G_q$. Then $G = M_p G_q$. Now since $O_q(M) \trianglelefteq M$ we get $1 \neq O_q(M) \leq \bigcap_{x \in M_p} x G_q x^{-1} = \bigcap_{x \in G} x G_q x^{-1} \trianglelefteq G$. Thus, $1 \neq \bigcap_{x \in G} x G_q x^{-1} \trianglelefteq G$ but $\bigcap_{x \in G} x G_q x^{-1} \leq G_q < G$, a contradiction since G is simple. Hence, M does not contain a Sylow p-subgroup of G and similarly M does not contain a Sylow q-subgroup of G. Let $M_p \in Syl_p(M)$. Then there exists $G_p \in Syl_p(G)$ such that $M_p < G_p$. Since G is nilpotent, by Theorem 3.4, $M_p < N_{G_p}(M_p)$. Let $x \in N_{G_p}(M_p) \setminus M_p$. Since $Z_p \trianglelefteq M$ is a p-group by Sylow's Theorem

$Z_p \leq M_p$. Hence, $Z_p \leq M_p = xM_px^{-1} \leq xMx^{-1}$. Now since $Z_q \trianglelefteq M$ we get $xZ_qx^{-1} \trianglelefteq xMx^{-1}$. Since $Z(O_p(M))$ is abelian we know $Z_p = \Omega_1(Z(O_p(M)))$ is an elementary abelian p-group. From the action of Z_p and xZ_qx^{-1}, by Theorem 4.11, we get $xZ_qx^{-1} = \langle C_{xZ_qx^{-1}}(U) \mid U \leq Z_p, \dfrac{|Z_p|}{|U|} = p\rangle$. Let $U \leq Z_p$ such that $\dfrac{|Z_p|}{|U|} = p$. Since Z is abelian we get $Z_p \leq C_G(U) < G$. By the uniqueness of M we have $C_G(U) \leq M$. Thus, since $M\ C_{xZ_qx^{-1}}(U) \leq C_G(U)$ we get $xZ_qx^{-1} \leq M$. But then $Z_q \leq x^{-1}Mx$. Hence, $Z = Z_pZ_q \leq x^{-1}Mx$. Again by the uniqueness of M we get $M = x^{-1}Mx$. Hence, $x \in N_G(M)$. But since M is maximal and G is simple we have $M = N_G(M)$. Thus, $x \in M$ and so $x \in M \cap G_p$. Now by Sylow's Theorem there exists $m \in M$ such that $G_p \cap M \leq mM_pm^{-1}$. Hence we get $M_p \leq G_p \cap M \leq mM_pm^{-1}$. and so $M_p = G_p \cap M$. Thus we get $x \in G_p \cap M = M_p$, a contradiction. Hence, $Z_p \cong \mathbb{Z}_p$ and $\{1\}$ is the only subgroup of Z_p with index p thus Z_p is cyclic. Similarly, $Z_q \cong \mathbb{Z}_q$ is cyclic. Since $Z_p \leq xMx^{-1}$ and $xZ_qx^{-1} \trianglelefteq xMx^{-1}$ we know $H = Z_pxZ_qx^{-1}$ is a subgroup. Without loss of generality, $p > q$. Then $n_p = 1$ and $n_q = 1$. Hence, $Z_p \trianglelefteq H$ and $xZ_qx^{-1} \trianglelefteq H$. But then $[Z_p, xZ_qx^{-1}] \leq Z_p \cap xZ_qx^{-1} = 1$. Thus, $xZ_qx^{-1} \leq C_G(Z_p) \leq N_G(Z_p) = M$ and so $x \in M_p$. Also, $Z_p \leq M_p \leq M$ so $xZx^{-1} = xZ_px^{-1}xZ_qx^{-1} \leq M$ or $Z = x^{-1}Mx$ and $M = x^{-1}Mx$. Thus, $x \in N_G(M) = M$, a contradiction.

3. Let M be a maximal subgroup of G then M cannot contain a p-central subgroup of G and a q-central subgroup of G.

   By (2) we may assume $F(M)$ is a p-group. By Sylow's Theorem there exists $M_p \in Syl_p(M)$ such that $F(M) \leq M_p$ and there exists $G_p \in Syl_p(G)$ such that $M_p \leq G_p$. Thus, $F(M) \leq G_p$. If $C_G(F(M)) \not\leq M$ then $M < \langle C_G(F(M)), M\rangle \leq$

G. Hence, by the maximality of M we get $G = \langle C_G(F(M)), M\rangle$. But then $F(M) \trianglelefteq \langle C_G(F(M)), M\rangle = G$, a contradiction since G is simple. Thus, $C_G(F(M)) \leq M$ and so $C_G(F(M)) = C_M(F(M))$. Now, $Z(G_p) \leq C_G(F(M)) = C_M(F(M)) \leq F(M)$ by Theorem 3.9 since M is solvable. Hence, $Z(G_p) \leq M$ and $Z(G_p)$ is a p-central subgroup of G. Suppose $H \leq M$ such that H is a q-central subgroup of G. Then there exists $G_q \in Syl_q(G)$ such that $H \leq Z(G_q)$. Then $G = G_pG_q$, $Z(G_p) \trianglelefteq G_p$, $H \trianglelefteq G_q$, and $\langle Z(G_p)^H\rangle \leq \langle F(M)^H\rangle \leq F(M)$ since $F(M) \trianglelefteq M$ and $F(M)$ is a p-group. Hence $\langle Z(G_p)^H\rangle$ is a p-group. But then by Theorem 4.12 we get G is not simple, a contradiction.

4. A p-central subgroup of G cannot normalize a q-subgroup of G.

   Suppose H is a p-central subgroup of G and $Q \leq G$ is a q-group such that $H \leq N_G(Q)$. By Sylow's Theorem, there exists $G_q \in Syl_q(G)$ such that $Q \leq G_q$. Since $N_G(Q) < G$ there exists a maximal subgroup M of G such that $N_G(Q) \leq M$. Now $Z(G_q) \leq C_G(Q) \leq N_G(Q) \leq M$. Also, $H \leq N_G(Q) \leq M$. But $Z(G_q)$ is a q-central subgroup of G and H is a p-central subgroup of G, which contradicts (3). Thus, a p-central subgroup of G cannot normalize a q-subgroup of G.

5. $|G|$ is odd.

   Suppose not. Then, $2 \mid |G|$ and so by Sylow's Theorem there exists $G_2 \in Syl_2(G)$. Then by Theorem 3.2, $Z(G_2) \neq 1$. Hence, by Theorem 1.10, there exists $1 \neq x \in Z(G_2)$ such that $x^2 = 1$. Since G is simple we get $O_2(G) = 1$. Hence $x \notin O_2(G)$. By Theorem 4.14, there exists $y \in G$ such that $xyx^{-1} = y^{-1}$ and $|y|$ is odd. Hence, we get $\langle x\rangle \leq N_G(\langle y\rangle)$. But $\langle x\rangle$ is a 2-central subgroup of G and $\langle y\rangle$ is a q-group, which contradicts (4). Therefore, $|G|$ is odd.

6. Let M be a maximal subgroup of G such that $F(M)$ is a p-group and $M_p \in Syl_p(M)$ such that $F(M) \leq M_p$. Then $J(M_p) \trianglelefteq M$ and $M_p \in Syl_p(G)$.

   We want to show that $J(M_p) \trianglelefteq M$. By Theorem 4.15 it is enough to show $C_M(\Omega_1(Z(M_p))) = M_p$. Since $\Omega_1(Z(M_p)) \leq Z(M_p)$ we get $M_p \leq C_M(\Omega_1(Z(M_p)))$. Let $G_p \in Syl_p(G)$ such that $M_p \leq G_p$. Then $F(M) \leq M_p \leq G_p$. Thus,

$$\begin{aligned} Z(G_p) &\leq C_G(F(M)) \\ &= C_M(F(M)) \\ &\leq F(M) \quad \text{since } M \text{ is solvable} \\ &\leq M_p. \end{aligned}$$

   Thus, $Z(G_p) \leq M_p$ and so $Z(G_p) \leq Z(M_p)$. Hence, $\Omega_1(Z(G_p)) \leq \Omega_1(Z(M_p))$ and so $C_M(\Omega_1(Z(M_p))) \leq C_G(\Omega_1(Z(G_p)))$. But, by (4) $C_G(\Omega_1(Z(G_p)))$ has no q-subgroups and so $C_G(\Omega_1(Z(G_p)))$ is a p-group. Hence, $C_M(\Omega_1(Z(M_p)))$ is a p-group. But $M_p \leq C_M(\Omega_1(Z(M_p)))$ and $M_p \in Syl_p(M)$. Hence, $M_p = C_M(\Omega_1(Z(M_p)))$ and so Theorem 4.15 $J(M_p) \trianglelefteq M$. If $M_p < G_p$ then since G_p is nilpotent, by Theorem 3.4, $M_p < N_{G_p}(M_p) = H$. Now, $M_p \trianglelefteq H$. If $H \leq M$ then $H \leq G_p \cap M$. But $G_p \cap M = M_p$ and so we get $H \leq M_p$, a contradiction. Thus, $H \not\leq M$. Bu then $M < \langle M, H \rangle \leq G$ and so $G = \langle M, H \rangle$ by the maximality of M. Now, $J(M_p) \trianglelefteq M$. also $M_p \trianglelefteq H$ we get $J(M_p) \trianglelefteq H$. Hence, $J(M_p) \trianglelefteq \langle M, H \rangle$, a contradiction, since G is simple. Thus, $M_p = G_p \in Syl_p(G)$.

Let

$$C_p = \{M \mid M \text{ is a maximal subgroup of G and } F(M) \text{ is a } p\text{-group}\}$$

and

$$C_q = \{M \mid M \text{ is a maximal subgroup of G and } F(M) \text{ is a } q\text{-group}\}$$

Let $M_1, M_2 \in C_p$ and $P_1 \in Syl_p(M_1)$ and $P_2 \in Syl_p(M_2)$. By (6) $P_1, P_2 \in Syl_p(G)$. By Sylow's Theorem there exists $g \in G$ such that $gP_1g^{-1} = P_2$. If $gM_1g^{-1} \neq M_2$ then $M_2 < \langle gM_1g^{-1}, M_2\rangle \leq G$. Hence we get $G = \langle gM_1g^{-1}, M_2\rangle$ since M_2 is maximal. By (6) we know $J(P_2) = J(gP_1g^{-1}) \trianglelefteq \langle gM_1g^{-1}, M_2\rangle = G$, a contradiction since G is simple. Thus, $gM_1g^{-1} = M_2$ and so G acts transitively on C_p by conjugation. Similarly, G acts transitively on C_q by conjugation. Let $M_1, M_2 \in C_p$ such that $|M_1 \cap M_2|_p$ is maximal. If $|M_1 \cap M_2|_p \neq 1$ then let $P \in Syl_p(M_1 \cap M_2)$. If $P \in Syl_p(M_1)$ then, by (6), we get $P \in Syl_p(G)$. Hence, since $P \leq M_2$ we get $P \in Syl_p(M_2)$. Now, by (6), we get $J(P) \trianglelefteq \langle M_1, M_2\rangle = G$, a contradiction since G is simple. Hence, $P \notin Syl_p(M_1)$ and similarly $P \notin Syl_p(M_2)$. Therefore, $P < N_{M_1}(P) \leq N_G(P)$ and $P < N_{M_2}(P) \leq N_G(P)$. Since $N_G(P) < G$, there exists a maximal subgroup R of G such that $N_G(P) \leq R$. If $F(R)$ is a q-group let $G_p \in Syl_p(G)$ such that $P \leq G_p$. Then, $Z(G_p) \leq C_G(P) \leq N_G(P) \leq R$ and $F(R) \trianglelefteq R$. Hence $Z(G_p) \leq N_G(F(R))$, but $Z(G_p)$ is a p-central subgroup of G and $F(R)$ is a q-group, a contradiction of (4).

Thus, $F(R)$ is a p-group and $R \in C_p$. Now

$$\begin{aligned} |M_1 \cap R|_p &\geq |M_1 \cap N_G(P)|_p \\ &= |N_{M_1}(P)|_p \\ &> |P| \\ &= |M_1 \cap M_2|_p. \end{aligned}$$

Hence, by the maximality of $|M_1 \cap M_2|_p$ we get $R = M_1$. Also, similarly $R = M_2$. Thus, $M_1 = R = M_2$, a contradiction since M_1 and M_2 are distinct. Therefore, $|M_1 \cap M_2|_p = 1$ and similarly $|H_1 \cap H_2|_q = 1$ for all $H_1, H_2 \in C_q$. Suppose $p^a > q^b$. Let $M_1, M_2 \in C_p$ be distinct and $P_1 \in Syl_p(M_1)$ and $P_2 \in Syl_p(M_2)$. Then $P_1 \cap P_2 \leq M_1 \cap M_2$ is a p-group and $|M_1 \cap M_2|_p = 1$. Hence $|P_1 \cap P_2| = 1$. But then we get

$$\begin{aligned} p^a q^b &= |G| \\ &\geq |P_1 P_2| \\ &= \frac{|P_1||P_2|}{|P_1 \cap P_2|} \\ &= \frac{p^a p^a}{1} \\ &= p^{2a} \\ &> p^a q^b \end{aligned}$$

a contradiction. Similarly, we get a contradiction if $q^b > p^a$. Therefore, G is solvable.

□

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