

# ECCENTRICITY SEQUENCE OF 2

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# ECCENTRICITY SEQUENCE OF 2

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## **Abstract**

We attempt to construct graphs with eccentricity sequence of 2. By looking at the degree sequence of a graph, it is shown that some properties can be found that describe graphs with eccentricity sequence of 2. The main result of this research is that the minimum graph with eccentricity sequence of 2 has degree sum of  $2(2n - 5)$ . This enables us to count the number of degree sums of graphs with eccentricity sequence of 2.

## ACKNOWLEDGMENTS

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# Chapter 1

## Introduction

This research work contributes to the question asked by Frank Harary<sup>1</sup> and Fred Buckley<sup>2</sup> in [3] whether a given nonnegative sequence of integers, say  $S$ , is eccentric, and if so, whether a graph, say  $G$ , can be constructed using the given sequence  $S$ . Hence this paper shall serve as a reference to the progress that has been made in this area with some original work presented.

As simple as the question may seem, numerous contributions have been made in order to answer this unsolved problem. Most notably, Linda Lesniak<sup>3</sup> in [6] proved that a sequence  $S$  is **eccentric** if and only if a subsequence  $S_n$  is eccentric. That is to say, if a given graph  $G$  with an eccentric sequence is  $\{2, 2, 3, 3, 3\}$  then a subsequence  $\{2, 2, 3, 3\}$  is also eccentric. The inspiration from her proof is used in this research work. Another notable contribution was made by R. Nandakumar and described in [3]. Nandakumar defined that an eccentric sequence is **minimal** if it has no proper eccentric subsequence with the same number of distinct eccentricities. In other words, he found and computed examples of minimal eccentric sequences with least eccentricity 2. In addition, we will introduce some theorems from other authors who have made some form of contribution to this unsolved problem.

In this paper, we focus our work in understanding and constructing graphs which have an eccentricity sequence of 2. Although we could focus on graphs with eccentricity sequence of 1, we find that there aren't many properties that can be derived, as illustrated in Chapter 2. Therefore, we do hope that this work would "inspire"

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<sup>1</sup>Department of Electrical and Computer Engineering, University of Missouri

<sup>2</sup>Department of Mathematics, Baruch College, City University of New York

<sup>3</sup>Proof found in appendix



other contribution to this unsolved problem.

## 1.1 Notation

To ease the reader's understanding of the material presented, we denote our use of standard and nonstandard notations below with many illustrations of graphs for clarification. In addition, this work is written in such a way that any mathematical student whose discipline is not in Graph Theory would be able to understand the material presented. A graph, henceforth denoted  $G$ , is assumed to be an **undirected connected** graph [See Definition 1.2.3]. By this we mean that if we start at a node, there is always a path to any node in the graph  $G$ . In addition, the reader should keep in mind that our idea of a graph is different from that of a traditional graph with axes usually introduced in an elementary algebra course. Continuing, each node shall not have any loops (i.e. an edge cannot start and end at the same node). Also, the position of all nodes with respect to the universe is arbitrary.

In the coming sections regarding to the eccentricity sequence, we will denote our set in compressed form. By this, we mean that if a number appears more than once, then it is written only once in the sequence. Hence all numbers are "grouped" together in the sequence. Also, each number in the sequence is a natural number (i.e.  $x \in \mathbb{N}$ ).

## 1.2 A Connected Graph: Definition

Let us define our graph  $G$  in a more formal way.

**Definition 1.2.1.** Let  $G$  be a graph. If for each pair of nodes, say  $v_x, v_y \in V(G)$ , there is a link that connects them, then it is an **edge**.

**Definition 1.2.2.** Let  $V(G) = \{v_1, \dots, v_m\}$  be the set of nodes and  $E(G) = \{e_1, \dots, e_n\}$  be the set of undirected edges. Then we say  $G$  is a **graph** if  $G = (V, E)$ .

Furthermore, in this paper, we restrict our attention to connected graphs.

**Definition 1.2.3.** (See [1]) Let  $G$  be a graph. We say that  $G$  is a **connected graph** if for each pair of nodes  $v_x$  and  $v_y$ , there is a path that joins  $v_x$  and  $v_y$ .

**Definition 1.2.4.** (See [1]) Let  $G$  be a connected graph. Then the number of nodes in the set  $V(G)$ , denoted as  $n$ , is the **order** of the graph.

To illustrate these definitions, consider the following example.

**Example 1.2.1.** Let  $G$  be the undirected connected graph shown in Figure 1.1.

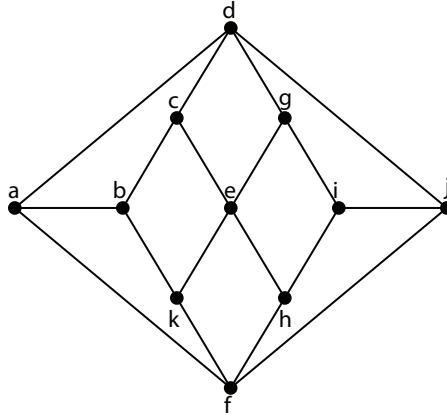


Figure 1.1: An example of a connected graph (See [1], Pg 450.)

As we can see from Figure 1.1, there are no nodes in  $G$  that are alone and we can travel from any node to any node and so  $G$  is a connected graph. Also, as stated above, the position of the nodes in the graph is irrelevant as long as the appropriate edges between nodes are maintained.

Another important property that will be used in this work is the distance between any two nodes in a graph. Consider the following definition.

**Definition 1.2.5.** Let  $G$  be a connected graph. Let  $v_x, v_y \in V(G)$  be nodes of  $G$ . Then the **distance** between  $v_x$  and  $v_y$ , denoted  $d(v_x, v_y)$ , is the length of the shortest path between  $v_x$  and  $v_y$ .

Consider the following example.

**Example 1.2.2.** Let  $G$  be the following graph shown in Figure 1.2. Then we show the distance from each node to other nodes in the graph  $G$ . For example, a node, say  $v_x$ , that has a distance sequence  $\{1, 2, 2, 3\}$  means that there are 4 nodes it connects to, and for each of these nodes, the distance from  $v_x$  to these other 4 nodes is represented in the sequence.

**Definition 1.2.6.** Let  $G$  be a connected graph and let  $v_x, v_y \in V(G)$  be nodes of  $G$ . If there exists an edge  $e \in E(G)$  such that it connects to  $v_x$  and  $v_y$  at its endpoints, then we say  $v_x$  and  $v_y$  are **adjacent** to each other and the length of an edge has an

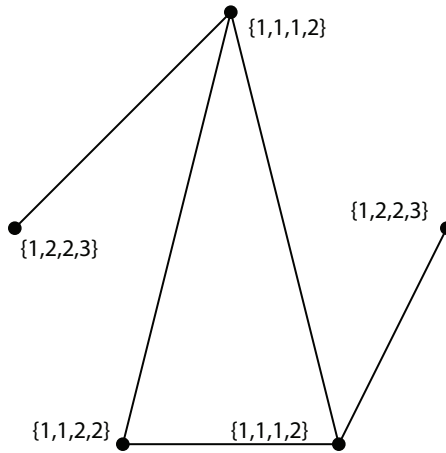


Figure 1.2: A connected graph with each node having its distance to other nodes represented in a set

arbitrary unit of 1. That is, the length of the edge is irrelevant for the purposes of our work.

### 1.3 Radius and Diameter of distances

From the last section, we noted that a graph  $G$  is composed of nodes and edges [See Definition 1.2.2]. Two nodes in a graph may be connected by many different paths. We will be, however, concerned with the minimum and maximum distances.

**Definition 1.3.1.** Let  $G$  be a connected graph. The **radius** of  $G$  is the minimum over all nodes  $v$  of the maximum distance from  $v$  to another node.

**Definition 1.3.2.** Let  $G$  be a connected graph. The **diameter** of  $G$  is the maximum distance between any two nodes of  $G$ .

**Example 1.3.1.** For the graph in Figure 1.2, we can see that the radius is 2 and the diameter is 3.

We make use of the minimum and maximum distances in order to explain some lemmas and theorems in Chapter 3.

# Chapter 2

## Eccentricity

### 2.1 Eccentricity

In this section, we turn our focus on the eccentricity of nodes in a graph  $G$ , which is defined as follows.

**Definition 2.1.1.** Let  $G$  be a connected graph and  $v \in V(G)$  be a node of  $G$ . We say  $e(v)$  is the **eccentricity** of  $v$  if it is the distance to a node, say  $u \in V(G)$ , that is farthest from  $v$ . Thus

$$e(v) = \max \{d(u, v) : u \in V(G)\}$$

**Example 2.1.1.** Let  $G$  be the following graph with its eccentricities shown. Recall

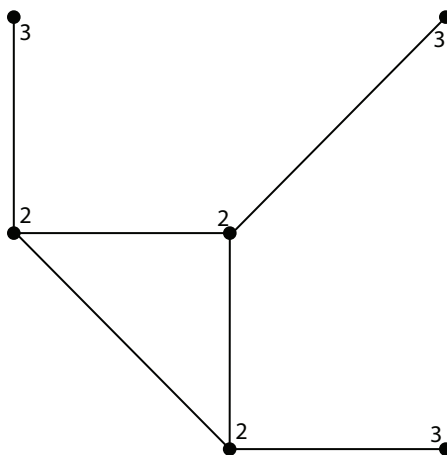


Figure 2.1: A connected graph with its eccentricities

that in Figure 1.2, where the distances are shown, the eccentricity of each node in

that graph is the biggest number in its set of distances.

In order to understand the eccentricity of a node in a graph, we will introduce some examples about some known graphs and their eccentricities later. Also the reader should notice that there is some relation/similarity regarding the eccentricity of a node and the distance as explained in the previous chapter. This brings us to the concept of minimum and maximum eccentricity of nodes since a graph has many nodes.

**Definition 2.1.2.** Let  $G$  be a connected graph and  $v_i \in V(G)$  for  $i = 1, \dots, n$  be the nodes of  $G$ . We say  $r(G)$  is the **radius** of  $G$  if it is the minimum eccentricity of the set of eccentricities in  $G$ . So,

$$r(G) = \min\{e(v_i)\}_{i=1}^n$$

**Definition 2.1.3.** Let  $G$  be a connected graph and  $v_i \in G$  for  $i = 1, \dots, n$  be the nodes of  $G$ . We say  $d(G)$  is the **diameter** of  $G$  if it is the maximum eccentricity of the set of eccentricities in  $G$ . So

$$d(G) = \max\{e(v_i)\}_{i=1}^n$$

### 2.1.1 Eccentricity Sequence

With the definition of the eccentricity of a node in hand, consider a graph  $G$  that has many nodes. In this case we would have a collection of nodes, and in turn the collection of their eccentricities would form a sequence of nondecreasing positive integers. The following definition explains this concept formally.

**Definition 2.1.4.** Let  $G$  be a connected graph and  $v_i \in V(G)$ , for  $i = 1, \dots, n$ , be the nodes of  $G$ . Then the **eccentricity sequence** of  $G$ , denoted  $e(G)$ , is the set of  $e(v_i)$ . Hence

$$e(G) = \{e(v_1), \dots, e(v_n)\}$$

Consider the following examples of some known graphs where their radius, diameter, and eccentricity sequence are found.

**Example 2.1.2.** Let  $G = C_n$  where  $C_n$  is a cycle with  $n$  nodes for  $n \geq 3$ . Then  $r(G) = \lfloor \frac{n}{2} \rfloor$  and  $d(G) = \lfloor \frac{n}{2} \rfloor$ . Below we give some examples of specific cycles with

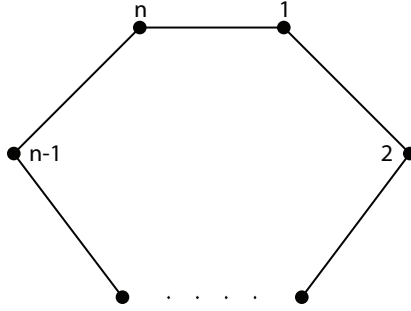


Figure 2.2: A cycle with  $n$  nodes

their eccentricity sequences shown.

$$C_3 \Rightarrow e(G) = \underbrace{\{1, \dots, 1\}}_3$$

$$C_4 \Rightarrow e(G) = \underbrace{\{2, \dots, 2\}}_4$$

$$C_5 \Rightarrow e(G) = \underbrace{\{2, \dots, 2\}}_5$$

$$C_6 \Rightarrow e(G) = \underbrace{\{3, \dots, 3\}}_6$$

$$C_7 \Rightarrow e(G) = \underbrace{\{3, \dots, 3\}}_7$$

⋮

$$C_n \Rightarrow e(G) = \underbrace{\left\{ \left\lfloor \frac{n}{2} \right\rfloor, \dots, \left\lfloor \frac{n}{2} \right\rfloor \right\}}_n$$

**Example 2.1.3.** Let  $G = P_n$  where  $P_n$  is a path with  $n$  nodes where  $n \geq 2$ . If  $n$  is even then  $r(G) = \frac{n}{2}$  and  $d(G) = n - 1$ . If  $n$  is odd then  $r(G) = \frac{n-1}{2}$  and  $d(G) = n - 1$ . So we have the following:

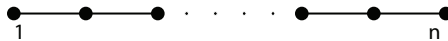


Figure 2.3: A path with  $n$  nodes

$$\begin{aligned}
P_2 &\Rightarrow e(G) = \{1\} \\
P_3 &\Rightarrow e(G) = \{1, 2\} \\
P_4 &\Rightarrow e(G) = \{2, 3\} \\
P_5 &\Rightarrow e(G) = \{2, 3, 4\} \\
P_6 &\Rightarrow e(G) = \{3, 4, 5\} \\
&\vdots \\
P_n &\Rightarrow \begin{cases} e(G) = \left\{ \frac{n}{2}, \dots, n-1 \right\}, & \text{if } n \text{ is even} \\ e(G) = \left\{ \frac{n-1}{2}, \dots, n-1 \right\}, & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

In Example 2.1.2, the eccentricity sequence was written for each node in  $G$ . However, from this point on, we will be write our sequence in a compressed form. That is to say that if more than two nodes have the same eccentricity, it will be written only once. For example if  $e(G) = \{1, 1, 2, 2, 2\}$  for some graph  $G$  then it will be rewritten as  $e(G) = \{1, 2, \}$ .

## 2.2 Eccentricity Sequence of 1

In this section, we look at graphs that have an eccentricity sequence of 1. This means that the graphs we'll be looking at have a radius of 1 and a diameter of 1.

**Example 2.2.1.** Let  $G = K_{m,n}$  where  $K_{m,n}$  is a bipartite graph shown in Figure 2.4. If  $m = 1 = n$ , then we find that the radius  $r(G) = 1$  and the diameter  $d(G) = 1$ . Hence the sequence is  $e(G) = \{1\}$ .



Figure 2.4: A bipartite graph with  $m = n = 1$

**Example 2.2.2.** Let  $G = K_n$  where  $K_n$  is a complete graph with  $n$  nodes, where  $n \geq 2$ . Then  $r(G) = 1$  and  $d(G) = 1$  and

$$\begin{aligned} K_2 &\Rightarrow e(G) = \underbrace{\{1, \dots, 1\}}_2 \\ K_3 &\Rightarrow e(G) = \underbrace{\{1, \dots, 1\}}_3 \\ K_4 &\Rightarrow e(G) = \underbrace{\{1, \dots, 1\}}_4 \\ &\vdots \\ K_n &\Rightarrow e(G) = \underbrace{\{1, \dots, 1\}}_n \end{aligned}$$

In Example 2.2.1 and Example 2.2.2, notice that  $e(G) = \{1\}$ . Now let's look at the graph from a different point of view. We focus our attention on each node and investigate the number of edges that terminate at each node. To do this, we need the following definition.

### 2.2.1 Degree of a node

**Definition 2.2.1.** Let  $G$  be a connected graph and let  $v \in V(G)$  be a node of graph of  $G$ . The **degree** of  $v$ , denoted  $\deg(v)$ , is the number of edges that terminate at  $v$ .

**Definition 2.2.2.** Let  $G$  be a connected graph. The **degree sequence** of  $G$ , denoted  $\deg(G)$ , is the set

$$\deg(G) = \{\deg(v) : v \in V(G)\}$$

Armed with this information, we state a lemma regarding the degree of each node in  $G$  in relation to the eccentricity. Although the lemma may seem obvious, we still state it for formality.

**Lemma 2.2.1.** *Let  $G$  be a connected graph of order  $n \geq 2$  and let  $v \in V(G)$  be a node of  $G$ . Then  $\deg(v) = n - 1$  if and only if  $e(v) = 1$ .*

*Proof.* ( $\rightarrow$ ). Suppose  $\deg(v) = n - 1$ . Then  $v$  is connected to all nodes except itself. Then for all  $v_x \in V(G) - \{v\}$  we have that  $d(v, v_x) = 1$  which implies  $e(v) = 1$ .

( $\leftarrow$ ). Suppose  $e(v) = 1$ . Then  $v$  is adjacent to all nodes except itself. Then clearly,  $\deg(v) = n - 1$ .

■



### 2.2.2 $K_n$ : Complete Graphs

So far we have studied 3 types of graphs that have an eccentricity sequence of 1 [See Example 2.2.1, Example 2.2.2, and  $C_3$ ]. However, the only graph that has an eccentricity sequence of 1 is the complete graph  $K_n$ . We mean that in a complete graph, every node is adjacent to every other node. That is,  $K_n$  has  $n$  nodes, each adjacent to the  $n - 1$  other nodes. Thus we introduce the following theorem.

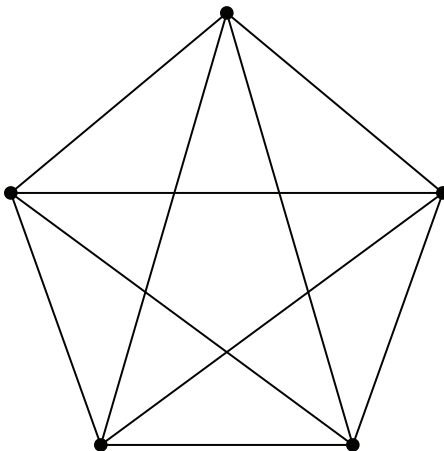


Figure 2.5:  $K_5$ ; A complete graph with 5 nodes

**Theorem 2.2.1.** *Let  $G$  be a connected graph. Then  $e(G) = \{1\}$  if and only if  $G = K_n$  for  $n \geq 2$ .*

*Proof.* ( $\rightarrow$ ). Suppose  $e(G) = \{1\}$ . Then for all  $v \in V(G)$ ,  $e(v) = 1$ . Let  $v_x \in V(G) - \{v\}$  be an arbitrary node. Then  $d(v, v_x) = 1$ . Since  $v$  is not connected to itself, it follows that  $\deg(v) = n - 1$  where  $n \geq 2$ . Now the only graph with  $\deg(v) = n - 1$  for all  $v \in V(G)$  is  $K_n$ . Hence  $G = K_n$ .

( $\leftarrow$ ). Suppose  $G = K_n$  for  $n \geq 2$ . Then for all  $v \in V(G)$ ,  $v$  is adjacent to  $n - 1$  nodes. So,  $\deg(v) = n - 1$ . But, by Lemma 2.2.1,  $e(v) = 1$  for all  $v \in V(G)$ , which implies that  $e(G) = \{1\}$ . ■

## 2.3 Eccentricity Sequence of 1, 2

In the last section we studied complete graphs and found that they have an eccentricity sequence of 1 [See Theorem 2.2.1]. In this section we focus on graphs that have an eccentricity sequence of 1, 2 and produce some new results. An eccentricity

sequence of 1, 2 implies that there exists a pair of nodes such that their distance is 2 and all other nodes are either adjacent to each other or have a distance of 2.

**Lemma 2.3.1.** *Let  $G$  be a connected graph. If  $r(G) = 1$  then  $d(G) \leq 2$ .*

*Proof.* Suppose  $r(G) = 1$ . Then there exists  $v \in V(G)$  such that  $v$  is adjacent to all other nodes. Then by Lemma 2.2.1,  $e(v) = 1$ . Since  $r(G) = 1$  then for all  $v_x, v_y \in V(G) - \{v\}$ , the distance  $d(v, v_x) = r(G) = 1 = r(G) = d(v, v_y)$ . Since  $G$  is a connected graph then  $d(v_x, v_y) = d(G)$ . So by the Triangle Inequality,

$$\begin{aligned} d(v_x, v_y) &\leq d(v, v_x) + d(v, v_y) \\ d(G) &\leq r(G) + r(G) \\ d(G) &\leq 1 + 1 \\ d(G) &\leq 2 \end{aligned}$$

Hence  $d(G) \leq 2$ . ■

Consider some examples.

**Example 2.3.1.** Let  $G = K_{m,n}$  be a complete bipartite graph as shown in Figure 2.6. If  $m = 1$  and  $n > 1$ . Then  $e(G) = \{1, 2\}$ .

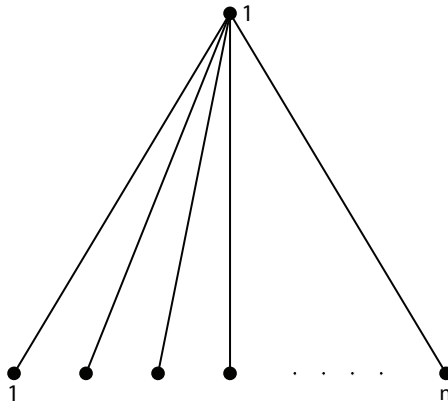


Figure 2.6:  $K_{1,n}$ : A complete bipartite graph

**Example 2.3.2.** Let  $G = K_4 - e$ . Then  $r(G) = 1$  and  $d(G) = 2$ . And so  $e(G) = \{1, 2\}$  as shown in Figure 2.7.

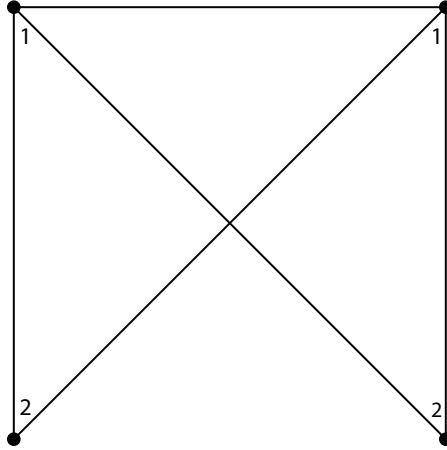


Figure 2.7:  $K_4 - e$  graph

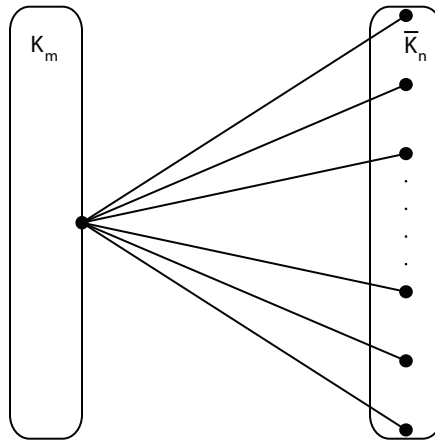


Figure 2.8: A connected graph  $G$  in which  $G = K_m + \overline{K}_n$ .

**Example 2.3.3.** Let  $G$  be a connected graph such that  $G = K_m + \overline{K}_n$ . Then  $G$  has  $m$  nodes with degree  $m + n - 1$ , and so these nodes have eccentricity of 1, and has  $n$  nodes where their eccentricity is 2. This implies that  $e(G) = \{1, 2\}$ .

Now we provide a relationship between the radius and the diameter of a given graph  $G$ . Before we do so, we need to introduce some additional definitions.

**Definition 2.3.1.** Let  $G$  be a connected graph and  $v \in V(G)$  be a node of  $G$ . Then we say  $v$  is a **central node** if

$$r(G) = e(v)$$

**Definition 2.3.2.** Let  $G$  be a connected graph and  $v \in V(G)$  be a node of  $G$ . Then

we say  $v$  is a **peripheral node** if

$$d(G) = e(v)$$

The following lemma was given as an exercise in [2]. We give a proof here.

**Lemma 2.3.2.** *Let  $G$  be a connected graph. Then*

$$r(G) \leq d(G) \leq 2r(G)$$

*Proof.* Let  $G$  be a connected graph. Then by definition of  $r(G)$  and  $d(G)$ , it follows that  $r(G) \leq d(G)$ . Now let  $v_i \in V(G)$  be a central node and  $v_j, v_k \in V(G)$  be peripheral nodes such that  $d(v_j, v_k) = d(G)$ . Since  $G$  is connected,  $v_i$  is connected to  $v_j$  and  $v_k$  for  $i \neq j \neq k$  thus  $d(v_i, v_j) \leq r(G)$  and  $d(v_i, v_k) \leq r(G)$  by definition of  $r(G)$ . Then by the Triangle Inequality,

$$\begin{aligned} d(v_j, v_k) &\leq d(v_i, v_j) + d(v_i, v_k) \\ d(G) &\leq r(G) + r(G) \\ d(G) &\leq 2r(G) \end{aligned}$$

Hence  $r(G) \leq d(G) \leq 2r(G)$ . ■

In Lemma 2.3.2, we see that the diameter of a graph  $G$  would never be bigger than twice its radius. Chartrand in [4] states a similar idea but from a different point of view. But before we state his theorem, we need some definitions.

**Definition 2.3.3.** Let  $G$  be a connected graph. Let  $v \in V(G)$  be a node of  $G$ . Then we say  $v$  is an **eccentric node** of a node  $u \in V(G)$  if  $d(u, v) = e(u)$ .

**Definition 2.3.4.** Let  $G$  be a connected graph. Let  $v \in V(G)$  be a node of  $G$ . Then  $v$  is an **eccentric node** of  $G$  if  $v$  is an eccentric node of some node in  $G$ .

**Definition 2.3.5.** Let  $G$  be a connected graph. Then we say  $G$  is an **eccentric graph** if for all  $v \in V(G)$ ,  $v$  is an eccentric node.

**Theorem 2.3.1** (See [4]). *For each integer  $k \geq 2$ , if  $G$  is a graph with  $r(G) = k$  and  $d(G) = k + 1$ , then  $G$  is an eccentric graph if and only if for each central node  $u$  there exists a central node  $v$  such that  $d(u, v) = k$ .*

*Proof.* Let  $k \geq 2$  be an integer and let  $G$  be an eccentric graph with  $r(G) = k$  and  $d(G) = k + 1$ , and let  $u$  be a central node of  $G$ . That  $u$  must be an eccentric node of some central node  $v$  with  $d(u, v) = k$  is immediate. Conversely, assume that for each central node  $u$  there exists a central node  $v$  with  $d(u, v) = k$ . Thus, every central node is eccentric. Since remaining nodes are peripheral and peripheral nodes are eccentric,  $G$  is an eccentric graph. ■

Hrnčiar and Monoszová in [5] made a conjecture in which they described, in general, minimal eccentric sequence which have the form  $\{r^\alpha, (r + 1)^\beta\}$ . Before their result is shown, we need some definitions.

**Definition 2.3.6.** (See [7]) Let  $G$  be a connected graph. We say  $B$  is a **block** of  $G$  if there exists a maximal biconnected subgraph.

**Definition 2.3.7.** Let  $G$  be a connected graph. Let  $v \in V(G)$  be a node of  $G$ . We say  $v$  is a **cut-node** if removing  $v$  and the edges that terminate at  $v$  would result in more components than  $G$ .

**Definition 2.3.8.** Let  $G$  be a connected graph. Then the **circumference** of  $G$ , denoted  $c(G)$ , is the length of any longest cycle of  $G$ .

**Theorem 2.3.2** (See [5]). *Let  $G$  be a connected graph. Let  $r(G) \geq 3$  and  $e(G) = \{r^\alpha, (r + 1)^\beta\}$ . Then*

- i. there exists a block  $B$  of  $G$  which contains all cut-nodes of  $G$  and moreover with the property that for every  $u \in V(G) - V(B)$  it holds  $d(u, B) = 1$ ,*
- ii. for circumference of  $G$  and for the block  $B$  from the previous it holds  $c(G) \geq c(B) \geq 2r(G) - 2$ ,*
- iii. if  $c(G) < 2r(G)$  then  $\alpha \geq 2r(G) - 2$ ,*

Since the proof is quite long, we invite the reader to read the proof as given in the reference.

# Chapter 3

## Eccentricity Sequence of 2

In the last chapter, we looked at graphs that have an eccentricity sequence of 1 and 1, 2. Now we turn our focus on graphs having eccentricity sequence of 2. We will introduce a theorem about minimal graphs with eccentricity sequence 2. This theorem will permit us to count how many families of graphs there are that have eccentricity sequence of 2. We will use the following lemmas, illustrated by examples to develop the ideas that lead up to the main theorem.

### 3.1 General Graphs of Eccentricity Sequence of 2

#### 3.1.1 $K_{m,n}$ : Complete Bipartite Graphs

**Example 3.1.1.** Let  $G = K_{m,n}$  for  $m, n \geq 2$  as shown in Figure 3.1. Then  $r(G) = 2$  and  $d(G) = 2$ . Thus  $e(G) = \{2\}$ .

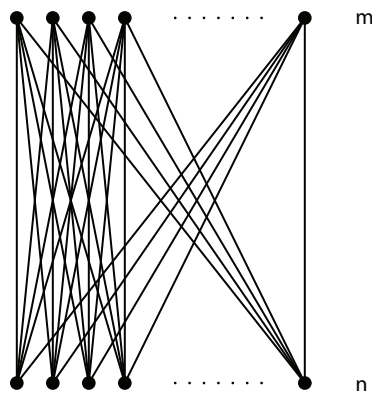


Figure 3.1:  $K_{m,n}$ : A complete bipartite graph

Notice that from Example 3.1.1, complete bipartite graphs are all graphs with eccentricity sequence of 2. Now to formally state that, we introduce the next lemma.

**Lemma 3.1.1.** *If  $G = K_{m,n}$  for  $m, n \geq 2$ , then  $e(G) = \{2\}$ .*

*Proof.* Suppose  $G = K_{m,n}$  with bipartition  $V(X) \subset G$  and  $V(Y) \subset G$  for  $m, n \geq 2$ . Then for all  $v_{x_1}, v_{x_2} \in X$ , it is true that  $d(v_{x_1}, v_{x_2}) \neq 1$ . But  $m, n \geq 2$ , so there exists  $v_{y_1}, v_{y_2} \in V(Y)$  with  $d(v_{y_1}, v_{y_2}) \neq 1$ . Since  $G = K_{m,n}$  then  $v_{x_1}, v_{x_2} \in V(X)$  are both connected to  $v_{y_1}, v_{y_2} \in Y$ , which implies that  $d(v_{x_1}, v_{x_2}) = 2 = d(v_{y_1}, v_{y_2})$ . Therefore,  $e(v_{x_1}) = e(v_{x_2}) = 2 = e(v_{y_1}) = e(v_{y_2})$ . So for all  $v_x \in V(X)$  and for all  $v_y \in V(Y)$ , we have  $e(v_x) = 2 = e(v_y)$ . Hence  $e(G) = \{2\}$ . ■

This lemma produces the following corollary.

**Corollary 3.1.1** (to Lemma 3.1.1). *Let  $G$  be a connected graph of order  $n$ . If  $n \geq 6$  and there exist two nodes, say  $v_x$  and  $v_y$ , such that*

- i.  $\deg(v_x) = n - 2 = \deg(v_y)$*
- ii.  $d(v_x, v_y) \neq 1$*
- iii. for all  $v \in V(G) - \{v_x, v_y\}$ ,  $\deg(v) = 2$*

*then  $e(G) = \{2\}$ .*

### 3.1.2 Another example

**Example 3.1.2.** Let  $G = C_4$  be the graph as shown in Figure 3.2. Then the radius  $r(G) = 2$  and the diameter  $d(G) = 2$ . Thus eccentricity sequence is  $e(G) = \{2\}$ .

## 3.2 AO-Graphs: $\deg(v) \in [2, n - 2]$

In this section, we look at the number of edges that terminate at each node. We also know that each node cannot have a degree of  $n - 1$  in a graph with eccentricity sequence of 2. So each node must have degree at most  $n - 2$ . Furthermore, notice that a node cannot have a degree of 1, for otherwise it would introduce eccentricities that are not equal to 2 [See the path  $P_4$  as an example].

**Lemma 3.2.1.** *Let  $G$  be a connected graph and let  $v \in V(G)$  be a node of  $G$ . If  $e(G) = \{2\}$ , then  $\deg(v) \neq 1$  for all  $v \in V(G)$ .*

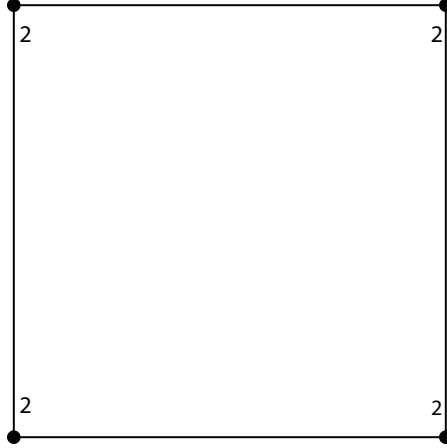


Figure 3.2: A cycle graph with 4 nodes

*Proof.* Let  $G$  be a connected graph of order  $n$ . Suppose  $e(G) = \{2\}$ . Then  $e(v) = 2$  for all  $v \in V(G)$ . Now suppose  $\deg(v) = 1$ . Then  $v$  is connected to only one node, say  $y \in V(G)$ . But  $e(v) = 2$  implies that  $\deg(y) = n - 1$ . This implies, by Lemma 2.2.1, that  $e(y) = 1$ . This is a contradiction since for all  $v \in V(G)$ ,  $e(v) = 2$ . Thus,  $\deg(v) \neq 1$ . ■

Since  $\deg(v) \neq 1$  then  $\deg(v) \geq 2$ . But earlier we mentioned that  $\deg(v) \neq n - 1$ . Thus  $\deg(v) \in [2, n - 2]$ . This motivates the following definition.

**Definition 3.2.1.** We define  $G$  as an **AO-Graph**<sup>1</sup> if for all  $v \in V(G)$

$$2 \leq \deg(v) \leq n - 2,$$

for  $n \geq 4$ , where  $n = |V(G)|$ .

Lemma 3.2.1 implies that all graphs that have their eccentricity sequence of 2 are AO-Graphs. So the degree of each node of a graph with eccentricity sequence of 2 must be in the interval  $[2, n - 2]$ .

**Lemma 3.2.2.** Let  $G$  be a connected graph with order  $n \geq 4$ . If for all  $v \in V(G)$ ,

$$n - 1 > \deg(v) \geq \left\lfloor \frac{n + 1}{2} \right\rfloor$$

then  $e(G) = \{2\}$ .

---

<sup>1</sup>Graph named after the author of this paper



*Proof.* Let  $G$  be a connected graph. Let  $v \in V(G)$  be a node of  $G$ . Suppose for all  $v \in V(G)$ ,

$$n - 1 > \deg(v) \geq \left\lfloor \frac{n + 1}{2} \right\rfloor$$

for  $n \geq 4$ . Then  $v$  is connected to at least  $\left\lfloor \frac{n + 1}{2} \right\rfloor$  nodes. Since  $\deg(v) \in [2, n - 2]$  then  $G$  is an AO-Graph. Let  $X$  be the set of nodes such that  $d(v, v_x) = 1$ . For all  $v_x \in V(X)$ ,

$$\deg(v_x) \geq \left\lfloor \frac{n + 1}{2} \right\rfloor.$$

Let  $v_y \in V(G) - \{v, V(X)\}$ . Since  $\deg(v_y) \geq \left\lfloor \frac{n + 1}{2} \right\rfloor$ , there exists  $v_x \in V(X)$  such that  $d(v_x, v_y) = 1$ , which implies that  $d(v, v_y) = 2$  thus  $e(v) = 2$ . Hence for all  $v \in V(G)$ ,  $e(G) = \{2\}$ . ■

**Corollary 3.2.1.** *Let  $G$  be a connected graph of order  $n$ , where  $n \geq 4$  is an even integer. For all  $v \in V(G)$ ,*

$$\deg(v) = \frac{n}{2},$$

*then  $e(G) = \{2\}$ .*

*Proof.* Let  $G$  be a connected graph of order  $n \geq 4$  and  $v \in V(G)$  be a node. Now suppose  $\deg(v) = \frac{n}{2}$ . Then  $v$  is connected to  $\frac{n}{2}$  nodes and  $v$  is not adjacent to  $\frac{n}{2} - 1$  nodes. Let  $X \subset G - \{v\}$  be a subgraph such that  $d(v, v_x) = 1$ , for all  $v_x \in V(X)$ . Let  $Y \subset G - X$  such that  $d(v, v_y) \neq 1$  for all  $v_y \in V(Y)$ . Now, we know  $\deg(v_y) = \frac{n}{2}$ . But the number of nodes in  $Y$  is  $\frac{n}{2} - 1$ . This implies that there exists  $v_x \in V(X)$  such that  $d(v_x, v_y) = 1$ . But  $v$  is adjacent to all nodes in  $X$ , so  $d(v, v_y) = 2$ . This implies that  $e(v) = 2$ . Hence,  $e(v) = 2$  for all  $v \in V(G)$ , which implies that  $e(G) = \{2\}$ . ■

A good example of the use of Corollary 3.2.1 would be the cycle graph with 4 nodes shown in Figure 3.2.

**Remark 3.2.1.** Suppose we have a graph with eccentricity sequence of 2. Then all nodes would have at degree at least 2. We can now add edges  $e \in E(G)$ . Continuing this process allows us to cycle through the interval given in Definition 3.2.1. If the

number of nodes  $n$  in  $G$  is even, then as we add edges, the maximum number of edges each node can have is  $n - 2$  and so the total is  $n(n - 2)$  edges. But if  $n$  is odd, then we have  $n - 1$  of  $n - 2$  nodes and one remaining node such that the degree is  $n - 3$ . This is due to the fact that as an edge is added, it picks up an even number of nodes (See [1]). Hence, the total is  $(n - 1)(n - 2) + (n - 3)$  edges.

**Example 3.2.1.** Let  $G$  be the following connected graph shown in Figure 3.3. Then  $G$  has 6 nodes and  $\deg(v) = n - 2 = 6 - 2 = 4$  for all  $v \in V(G)$ .

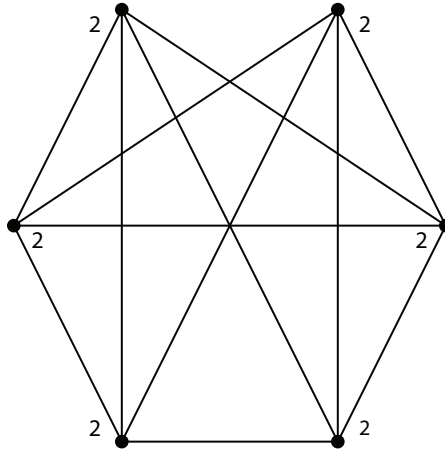


Figure 3.3: A sample graph with degree of  $n - 2$

**Lemma 3.2.3.** Let  $G$  be a connected graph of even order  $n \geq 4$ . Suppose

$$\deg(v) = n - 2,$$

for all  $v \in V(G)$ , then,  $e(G) = \{2\}$ .

*Proof.* Let  $G$  be a connected graph of order  $n$ . Suppose  $\deg(v) = n - 2$  for all  $v \in V(G)$ . Then by Definition 3.2.1,  $G$  is an AO-Graph. Then there exists  $v_x \in V(G)$  such that  $d(v, v_x) \neq 1$ . Since  $\deg(v_x) = n - 2$ , then  $v_x$  is adjacent to all nodes except  $v$ . Also  $n \geq 4$ , implies that there exists  $v_y \in V(G)$ , such that  $d(v, v_y) = 1 = d(v_y, v_x)$ . Therefore,  $d(v, v_x) = 2$  implies that  $e(v) = 2 = e(v_x)$ . So, for all  $v \in V(G)$ , we conclude that  $e(G) = \{2\}$ . ■

**Lemma 3.2.4.** *Let  $G$  be a connected graph of order  $n \geq 5$ . Suppose*

$$\deg(v) = n - 3,$$

*for all  $v \in V(G)$ , then,  $e(G) = \{2\}$ .*

*Proof.* Let  $G$  be a connected graph of order  $n$  and let  $v \in V(G)$  be a node of  $G$ . Suppose  $\deg(v) = n - 3$ , for all  $v \in V(G)$ . Then by Definition 3.2.1,  $G$  is an AO-Graph. So  $v$  is connected to  $n - 3$  nodes. Since  $n \geq 5$ , there exists  $v_x, v_y \in V(G) - \{v\}$ , such that  $d(v, v_x) \neq 1$  and  $d(v, v_y) \neq 1$ . However  $\deg(v_x) = n - 3 = \deg(v_y)$ , so  $v_x, v_y$  are also connected to  $n - 3$  nodes. Let  $X \subseteq G - \{v, v_x, v_y\}$  be a connected subgraph. Let  $\{v_1, \dots, v_m\}$  be the set of nodes in  $X$ . Since  $\deg(v) = n - 3$  for all  $v \in V(G)$  and  $X \subseteq G - \{v, v_x, v_y\}$ , then for all  $v_i \in V(X)$ ,  $\deg(v_i) = n - 3$  for  $i = 1, \dots, m$ . Then for all  $v_i \in V(X)$ ,  $d(v, v_i) = 1$ . But  $v_x$  is adjacent to at least  $n - 4$  nodes in  $X$ . Since  $n \geq 5$  there exists  $v_i \in V(X)$  such that  $d(v_x, v_i) = 1 = d(v_i, v)$ . Thus,  $d(v, v_x) = 2$ . Similarly,  $d(v, v_y) = 2$  and therefore  $e(v) = 2$ . Hence for all  $v \in V(G)$ ,  $e(v) = 2$  and we conclude that  $e(G) = \{2\}$ . ■

Consider the following corollaries.

**Corollary 3.2.2.** *If  $G = K_n - C_n$  where  $K_n$  is a complete graph and  $C_n$  is a cycle graph, then  $e(G) = \{2\}$ .*

*Proof.* Suppose  $G = K_n - C_n$  for  $n \geq 5$ . Then for all  $v \in V(G)$ ,  $\deg(v) = n - 3$ . But Lemma 3.2.4 implies that  $e(v) = 2$  for all  $v \in V(G)$ . We conclude that  $e(G) = \{2\}$ . ■

**Corollary 3.2.3.** *If  $G = K_n - P_n$  where  $K_n$  is a complete graph and  $P_n$  is a path, then  $e(G) = \{2\}$ .*

*Proof.* Suppose  $G = K_n - P_n$  for  $n \geq 5$ . Then there exists  $v_x, v_y \in V(G)$  such that  $\deg(v_x) = n - 2 = \deg(v_y)$  and for all  $v \in V(G) - \{v_x, v_y\}$ ,  $\deg(v) = n - 3$ . Let  $X \subset G - \{v_x, v_y\}$  such that for all  $x \in V(X)$ ,  $\deg(x) = n - 3$ . Then by Lemma 3.2.4,  $e(X) = \{2\}$ . But by Lesniak [6], we can add nodes and edges and still have an eccentricity sequence of 2. Therefore,  $e(G) = \{2\}$ . ■

Before stating the next corollary, we need the following definition.

**Definition 3.2.2.** Let  $G$  be a connected graph. A **matching** of a graph  $G$ , denoted  $M$ , is a collection of edges  $e \in E(G)$  such that no two edges share the same node; That is the edges are non-adjacent.

**Corollary 3.2.4** (to Lemma 3.2.3). *If  $G = K_n - M$  for every even integer  $n \geq 4$  then  $e(G) = \{2\}$  for all  $v \in V(G)$ .*

*Proof.* Suppose  $G = K_n - M$ . Then for all  $v \in V(G)$ ,  $\deg(v) = n - 2$ . By Lemma 3.2.3,  $e(v) = 2$ , for all  $v \in V(G)$ . We conclude that  $e(G) = \{2\}$ . ■

### 3.3 Minimum Graphs with $e(G) = \{2\}$

In this section we introduce the idea of minimal graphs. Nandakumar in [3] defined that a graph is eccentric and its sequence is **minimal** if it has no proper eccentric subsequence with the same number of distinct eccentricities. Nandakumar computed minimal eccentric sequences with least eccentricity 2, but in our case, we studied graphs with minimum degree sums. With this idea in mind we proceed to the following examples. Note that our goal is to find what minimum graphs of eccentricity 2 look like. We make use of the degree sequence and the summation of these degrees. This produces an interesting pattern.

**Example 3.3.1.** Let  $G$  be a connected graph such that  $G = C_5$ . Then we can see that for each  $v \in V(G)$ ,  $\deg(v) = 2$ . This implies that  $\deg(v) = n - 3$ . By Lemma 3.2.4  $e(G) = \{2\}$ . So  $\sum_{i=1}^5 \deg(v_i) = 10$ . Then if we remove an edge, then there would be 2 nodes in  $G$  such that their degrees is 1. But, by Lemma 2.2.1, their degrees cannot be 1 since  $e(G) = \{2\}$ . Then clearly we cannot construct a graph with eccentricity sequence of 2 such that  $\sum_{i=1}^5 \deg(v_i) < 10$ . Thus  $\sum_{i=1}^5 \deg(v_i) = 10$  is the minimum graph.

**Example 3.3.2.** Let  $G$  be a connected graph such that  $|V(G)| = 6$ . We want to show that  $\sum_{i=1}^6 \deg(v_i) = 14$  is the minimum graph with  $e(G) = \{2\}$ . Suppose  $\sum_{i=1}^6 \deg(v_i) = 12$  is the minimum graph. Then our degree sequence would be

$$\deg(G) = \{2, 2, 2, 2, 2, 2\}$$

such that for all  $v \in V(G)$ ,  $\deg(v) \in [2, n - 2]$ . But  $v$  can connect to a maximum of 4 nodes where the distance  $\leq 2$ . This implies that there exists  $y \in V(G) - \{v\}$  such

that  $d(v, y) > 2$ . This is a contradiction. Thus,

$$\sum_{i=1}^6 \deg(v_i) = 14$$

is the minimum graph.

**Example 3.3.3.** We want to show that there exists a graph  $G$  with  $\sum_{i=1}^n \deg(v_i) = 4n - 10$  for all  $v_i \in V(G)$  and  $e(G) = \{2\}$ . Let  $G$  be the connected graph such that  $|V(G)| \geq 6$  and nodes are adjacent as shown in Figure 3.4. Then,

$$\deg(G) = \underbrace{\{2, \dots, 2\}}_{n-2}, n-3, n-3$$

So we have

$$\begin{aligned} 2(n-2) + 2(n-3) &= 2n - 4 + 2n - 6 \\ &= 4n - 10 \end{aligned}$$

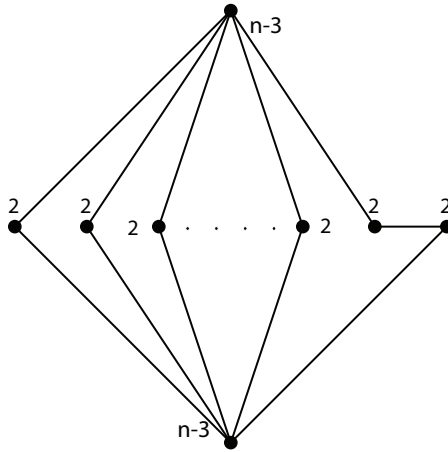


Figure 3.4: A minimal graph with eccentricity sequence of 2

**Example 3.3.4.** Let  $G$  be a connected graph such that  $|V(G)| = 7$ . We want to show that  $\sum_{i=1}^7 \deg(v_i) = 18$  is the minimum graph with  $e(G) = \{2\}$ . Now, suppose  $\sum_{i=1}^7 \deg(v_i) = 16$  is the minimum graph. Then our degree sequences for  $n = 7$  would

be

$$\deg(G) = \{2, 2, 2, 2, 2, 3, 3\} \text{ or } \deg(G) = \{2, 2, 2, 2, 2, 2, 4\}.$$

where  $\deg(G)$  are the only arrangement since  $\deg(v_i) \in [2, n - 2]$ . Let's look at some cases.

i. Suppose

$$\deg(G) = \{2, 2, 2, 2, 2, 2, 4\}$$

Let  $A = \{a, b, c, d, e, f\}$  be nodes such that each element has 2 edges and  $B = \{x\}$  such that  $\deg(x) = 4$ . Since  $x$  can only connect to 4 other nodes in  $A$ , there are 2 nodes left. Thus, there exists a node, say  $t \in A$  such that it is adjacent to 2 nodes in  $A$ . Now since  $\deg(t) = 2$ , we know that  $t$  can connect to a maximum of 4 nodes, where the distance  $\leq 2$ . But, there are a total of 7 nodes, which implies that there exists a node, say  $s$ , such that  $d(t, s) > 2$ . This is a contradiction.

ii. Suppose

$$\deg(G) = \{2, 2, 2, 2, 2, 3, 3\}$$

Let  $A = \{a, b, c, d, e\}$  be nodes such that each element has 2 edges and  $B = \{x, y\}$  such that  $\deg(x) = 3 = \deg(y)$ . Consider these two cases:

(a)  **$x$  and  $y$  are adjacent**

Since  $x$  and  $y$  are adjacent, then they both can connect to a maximum of 4 nodes. But, there are 5 nodes in  $B$ , which implies that there exists a node, say  $z \in A$ , such that it is connected to 2 nodes in  $A$ . But,  $\deg(z) = 2$ , which means that  $z$  can connect to a maximum of 4 nodes, where the distance is  $\leq 2$ . However, there are 7 nodes total, which implies that there exists a node, say  $v$  such that  $d(z, v) > 2$ . This is a contradiction.

(b)  **$x$  and  $y$  are not adjacent**

Since  $x$  and  $y$  are not adjacent and  $\deg(x) = 3 = \deg(y)$ , there are a total of 6 edges. But there are, however, only 5 nodes in  $A$  to connect to, which implies that there exists a node, say  $v \in A$ , such that it is not connected to both  $x$  and  $y$ . So  $v$  can connect to a maximum of 5 nodes, where the distance is  $\leq 2$ . But, there are 7 nodes total which implies that there exists a node, say  $r$  such that  $d(r, v) > 2$ . This is a contradiction.

Thus,

$$\sum_{i=1}^7 \deg(v_i) = 18$$

is the minimum graph.

**Example 3.3.5.** Let  $G$  be a connected graph  $|V(G)| = 8$ . We want to show that

$$\sum_{i=1}^8 \deg(v_i) = 22$$

is the minimum graph  $e(G) = \{2\}$ . Now suppose

$$\sum_{i=1}^8 \deg(v_i) = 20$$

is the minimum graph. Then our degree sequences for  $n = 8$  would be:

$$\deg(G) = \{2, 2, 2, 2, 2, 2, 2, 6\} \text{ or}$$

$$\deg(G) = \{2, 2, 2, 2, 2, 2, 3, 5\} \text{ or}$$

$$\deg(G) = \{2, 2, 2, 2, 2, 2, 4, 4\} \text{ or}$$

$$\deg(G) = \{2, 2, 2, 2, 2, 3, 3, 4\} \text{ or}$$

$$\deg(G) = \{2, 2, 2, 2, 3, 3, 3, 3\}.$$

i. Suppose

$$\deg(G) = \{2, 2, 2, 2, 2, 2, 2, 6\}.$$

The proof is similar to the first case above for  $n = 7$ .

ii. Suppose

$$\deg(G) = \{2, 2, 2, 2, 2, 2, 3, 5\}.$$

Let  $A = \{a, b, c, d, e, f\}$  and  $B = \{x, y\}$ . Let  $x \in B$  such that  $\deg(x) = 5$ . Then there exists  $a \in A$  such that  $d(x, a) \neq 1$ , but  $d(a, y) = 1 = d(a, b)$ . This implies that  $a$  can connect to a maximum of 5 nodes, where the distance is  $\leq 2$ . There are, however, 8 nodes so there exists a node, say  $z$ , such that  $d(a, z) > 2$ . This is a contradiction.

iii. Suppose

$$\deg(G) = \{2, 2, 2, 2, 2, 2, 4, 4\}.$$

The proof is similar to the second case above for  $n = 7$ .

iv. Suppose

$$\deg(G) = \{2, 2, 2, 2, 2, 3, 3, 4\}.$$

Let  $A = \{a, b, c, d, e\}$  and  $B = \{x, y, z\}$ . Let  $\deg(x) = 4$ ,  $\deg(y) = 3 = \deg(z)$ . Now, since  $x$  can connect to a maximum of 4 nodes, there exists a node, say  $t \in A$  such that  $d(t, y) = 1 = d(t, z)$ . Then  $t$  can connect to a maximum of 6 nodes such that the distance is  $\leq 2$ . But there are a total of 8 nodes which implies that there exists a node, say  $d$ , such that  $d(t, d) > 2$ . This is a contradiction.

v. Suppose

$$\deg(G) = \{2, 2, 2, 2, 3, 3, 3, 3\}.$$

Let  $A = \{a, b, c, d\}$  where  $a = b = c = d = 2$  and  $B = \{w, x, y, z\}$  where  $w = x = y = z = 3$ . If all elements in  $B$  are connected to all elements in  $A$  then there will be too many edges. Thus, some elements in  $B$  must be connected with each other. Now let  $a \in A$  such that  $d(a, w) = 1 = d(a, x)$ . Then  $a$  can connect to a maximum of 6 nodes where the distance is  $\leq 2$ . There are, however, 8 nodes so there exists a node, say  $f$ , such that  $d(a, f) > 2$ . This is a contradiction. Hence,

$$\sum_{i=1}^8 \deg(v_i) = 22$$

is the minimum graph.

With these 4 examples, we state our theorem that investigates the minimality of graphs that have an eccentricity sequence of 2.

**Theorem 3.3.1.** *Let  $G$  be a connected graph such that  $e(G) = \{2\}$ . If*

$$\sum_{i=1}^n \deg(v_i) = 2(2n - 5)$$

*for  $n \geq 5$ , then  $G$  is the minimum graph.*



*Proof.* Suppose not. Then  $G$  is a connected graph such that  $e(G) = \{2\}$ , for  $n \geq 5$  and

$$\sum_{i=1}^n \deg(v_i) < 2(2n - 5).$$

Now, suppose

$$\sum_{k=1}^n \deg(v_k) = 2(2n - 5) - 2,$$

where  $v_k \in V(G)$ . Let  $X = \deg(G)$  be the degree sequence of  $G$ , where

$$X = \{x_1, x_2, \dots, x_n\},$$

such that for all  $x \in X$ ,  $\deg(x) \in [2, n - 2]$ . These assumptions produce 2 cases.

**Case 1**  $n \in \{5, 6, 7, 8\}$ .

See Examples 3.3.1, 3.3.2, 3.3.4, and 3.3.5.

**Case 2**  $n > 8$ .

Let

$$X = \{x_1, \dots, x_{n-2}, x_{n-1}, x_n\}$$

be a degree sequence for  $G$  such that for all  $x \in X$ ,  $\deg(x) \in [2, n - 2]$ .

Suppose  $\deg(x_n) = n - 2$ . Then, there exists a node, say  $x_{n-1} \in X$ , such that  $d(x_n, x_{n-1}) \neq 1$ . Let  $W = \{x_1, \dots, x_{n-2}\} \subset X$  be the set of nodes that are adjacent to  $x_n$ . Then for all  $w_i \in W$ ,

$$\begin{aligned} \deg(x_{n-1}) + \sum_{i=1}^{n-2} \deg(w_i) &= 4n - 12 - (n - 2) \\ &= 3n - 10 \end{aligned}$$

But,  $\deg(w_i) \geq 2$  for all  $i$ , which implies that  $\deg(x_{n-1}) \leq 3n - 10 - 2(n - 2) = n - 6$ . Now, suppose

$$\deg(x_{n-1}) = n - 6.$$

Then  $x_{n-1}$  is adjacent to some  $w \in W$ . So there exists at least 4 nodes in  $W$  such that they are not adjacent to  $x_{n-1}$ . But these other nodes are adjacent to  $x_n$ , and

their degrees are 2. Thus,  $x_{n-1}$  connects to  $n - 5$  nodes, where the distance is  $\leq 2$ . But the distance from  $x_{n-1}$  to other 4 nodes is  $> 2$ . This is a contradiction.

Now, suppose

$$\deg(x_{n-1}) < n - 6.$$

We can remove a maximum of  $n - 8$  edges from  $x_{n-1}$ . But, there are  $n - 2$  nodes in  $W$  which implies that  $x_{n-1}$  does not have enough edges to cover all nodes in  $W$  such that the distance is  $\leq 2$ . There exists, however, at least 4 nodes that are still adjacent to  $x_n$  but not connected to  $x_{n-1}$  where the distance from  $x_{n-1}$  to these nodes is  $> 2$ . This is a contradiction.

Now suppose

$$\deg(x_n) < n - 2.$$

Let  $Y$  be the set of nodes that are not adjacent to  $x_n$ . Then if you remove an edge from  $x_n$ , that edge must be used to connect the node it disconnects from  $x_n$  to a node in  $W$ , so that the node has distance at most 2 from  $x_n$ . As above, there must be  $y \in Y$ , not adjacent to all  $w \in W$ , such that there exists  $t \in W$  with  $d(y, t) > 2$ . Since there are not enough edges to connect these within distance 2, a contradiction occurs. This completes our proof. ■

### 3.3.1 Counting of graphs with $e(G) = \{2\}$

We now know the degree sum formula for a minimal eccentric graph with its sequence being all 2's and we also know the maximum each node can be depending on whether  $n$  is even or odd. We can also find the number of different degree sums of graphs with eccentricity sequence of 2.

**Example 3.3.6.** Let  $G$  be a connected graph such that  $e(G) = \{2\}$ . Suppose  $n = 5$ . Then by Theorem 3.3.1,  $G$  is minimal when all degrees add up to 10. But  $n$  is odd so the total number of edges it can have is  $(n - 1)(n - 2) + (n - 3)$  or  $4 * 3 + 2 = 14$ . Thus, we have 3 types of graphs (See Figure 3.5) that we can construct such that  $\deg(v) \in [2, n - 2]$ , for all  $v \in V(G)$ .

**Example 3.3.7.** Let  $G$  be a connected graph such that  $e(G) = \{2\}$ . Suppose  $n = 6$ . Then, by Theorem 3.3.1,  $G$  is minimal when all degrees add up to 14. But  $n$  is even so the maximum degree each node can be is  $\deg(v) = n - 2$  for all  $v \in V(G)$ . So the

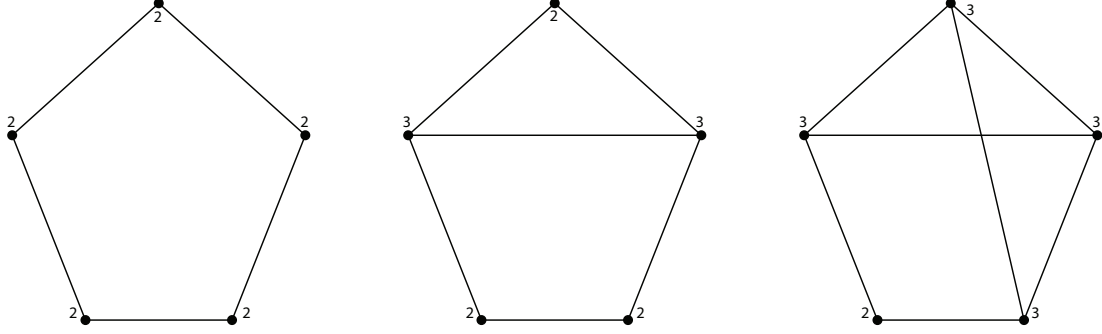


Figure 3.5: Graphs with eccentricity sequence of 2 with its degrees shown

total number of edges it can be is  $n(n - 2) = 6(6 - 2) = 24$ . So we have 6 degree sums of graphs that we can construct a graph with  $\deg(v) \in [2, n - 2]$ , for all  $v \in V(G)$ .

Consider the following lemma.

**Lemma 3.3.1.** *Let  $G$  be a connected graph such that  $e(G) = \{2\}$ . Then*

$$\left| \sum G \right| = \begin{cases} \frac{n^2 - 6n + 12}{2}, & \text{if } n \text{ is even} \\ \frac{n^2 - 6n + 11}{2}, & \text{if } n \text{ is odd} \end{cases},$$

where  $\left| \sum G \right| =$  number of different types of degree sequence sums of graphs with  $e(G) = \{2\}$  for  $n \geq 5$ .

*Proof.* Let  $G$  be an AO-Graph<sup>2</sup> such that  $e(G) = \{2\}$  for  $n \geq 5$ . We'll divide our proof into 2 cases.

1.  **$n$  is even**

Suppose by Theorem 3.3.1  $G$  is minimal graph with  $e(G) = \{2\}$ . By Remark 3.2.1, we can add edges in  $G$  and still keep an eccentricity sequence of 2. Thus the maximum number of edges each node can have is  $n - 2$ . Thus, the total for each  $v \in V(G)$  is  $n(n - 2)$ . Note that each edge  $e \in E(G)$  picks up 2 nodes, to count each graph, we take the maximum and subtract the minimum and divide

---

<sup>2</sup>See Definition 3.2.1

by 2. Since we lose the initial value so we have to add 1. Thus, we have

$$\begin{aligned}
|\sum G| &= \frac{n(n-2) - (4n-10)}{2} + 1 \\
&= \frac{n^2 - 2n - 4n + 10}{2} + 1 \\
&= \frac{n^2 - 6n + 10}{2} + 1 \\
&= \frac{n^2 - 6n + 10 + 2}{2} \\
&= \frac{n^2 - 6n + 12}{2}.
\end{aligned}$$

## 2. $n$ is odd

Suppose by Theorem 3.3.1  $G$  is minimal graph with  $e(G) = \{2\}$ . By Remark 3.2.1, we can add edges in  $G$  and still keep an eccentricity sequence of 2. If we continue the same process as when  $n$  is even, we find that  $n-1$  of each  $v \in V(G)$  can have  $n-2$  number of edges. Therefore, we have a total of  $(n-1)(n-2)$ . Since  $n$  is odd, there exists a node such that it has  $n-3$  edges. So in total, we have  $(n-1)(n-2) + n-3$ . Thus, we have

$$\begin{aligned}
|\sum G| &= \frac{(n-1)(n-2) + n-3 - (4n-10)}{2} + 1 \\
&= \frac{n^2 - 3n + 2 + n - 3 - 4n + 10}{2} + 1 \\
&= \frac{n^2 - 6n + 9}{2} + 1 \\
&= \frac{n^2 - 6n + 9 + 2}{2} \\
&= \frac{n^2 - 6n + 11}{2}.
\end{aligned}$$

Hence, combining both cases we have

$$|\sum G| = \begin{cases} \frac{n^2 - 6n + 12}{2}, & \text{if } n \text{ is even} \\ \frac{n^2 - 6n + 11}{2}, & \text{if } n \text{ is odd} \end{cases}.$$

This completes our proof.

■

To combine both cases when  $n$  is even or odd, we produce the following corollary.

**Corollary 3.3.1.** *Let  $G$  be a connected graph such that  $e(G) = \{2\}$ . Then*

$$\left| \sum G \right| = \left\lfloor \frac{2n^2 - 12n + 23}{4} \right\rfloor,$$

for  $n \geq 5$ .

# Chapter 4

## Conclusion

### 4.1 Research Summary

We embarked in finding graphs that have an eccentricity sequence of 2. Looking at their degrees for each node enables us to understand the rearrangement of these edges that produces an eccentricity sequence of 2. In addition, we found that we can count the number of degree sums graphs by understanding the pattern the sum of their degrees makes. This counting of these graphs with eccentricity sequence of 2 is not limited to just well known graphs such as complete graphs, cycles, etc. but also non-standard graphs.

### 4.2 Unsolved Problems

As mentioned at the beginning of this paper, this work was a contribution to an unsolved problem asked by Frank Harary and Fred Buckley in [3]. In addition, we propose the following questions in relation to the unsolved problem.

**Question 4.2.1.** Find the smallest degree sum where every AO-Graph has  $e(G) = \{2\}$ .

**Question 4.2.2.** Find and characterize graphs with an eccentricity sequence of 3.

# Appendix A

## Theorems

### A.1 Linda Lesniak's Theorem

**Theorem A.1.1.** *A nondecreasing sequence  $S = \{a_1, \dots, a_n\}$  with  $m$  distinct values is eccentric if and only if some subsequence, say  $S_m$ , with  $m$  distinct values is eccentric.*

*Proof.* If  $S$  is eccentric, then it is an eccentric subsequence with  $m$  distinct values. For the converse, suppose  $S_m$  is an eccentric subsequence with  $m$  distinct values. Let  $G$  be a graph with eccentricity sequence  $S_m$  and let  $\{t_1, \dots, t_m\}$  be the distinct values that occur in  $S_m$ . For each  $t_i$ ,  $1 \leq i \leq m$ , select a vertex  $w_i \in G$  whose eccentricity in  $G$  is  $t_i$ . For each  $i$ ,  $1 \leq i \leq m$ , let  $n_i$  equal one more than the number of occurrences of  $t_i$  in  $S$  less the number of occurrences of  $t_i$  in  $S_m$ . In  $G$ , replace  $w_1$  with a copy of  $K_{n_1}$  and join each vertex of  $K_{n_1}$  to all vertices adjacent to  $w_1$  in  $G$ . Call this graph  $G_1$ . In  $G_1$ , replace  $w_2$  with a copy of  $K_{n_2}$  and join each vertex of  $K_{n_2}$  to all vertices adjacent to  $w_2$  in  $G_1$ . Call this graph  $G_2$ . Continue in this fashion to obtain the graph  $G_m$ . Then  $S$  is the eccentricity sequence of  $G_m$ . ■

# References

- [1] Richard Brualdi, *Introductory combinatorics*, 4th ed., Prentice Hall, 2004.
- [2] Fred Buckley and Frank Harary, *Distance in graphs*, Perseus Press, 1990.
- [3] Fred Buckley and Frank Harary, *Unsolved problems on distance in graphs*, Electronic Notes in Discrete Mathematics **11** (2002), 89 – 97, The Ninth Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms and Applications.
- [4] Gary Chartrand, Weizhen Gu, Michelle Schultz, and Steven J. Winters, *Eccentric graphs*, Networks **34** (1999), no. 2, 115–121.
- [5] Pavel Hrnčiar and Gabriela Monoszová, *Minimal eccentric sequences with two values*, (2005), no. 12, 43–65, Acta Universitatis Matthiae Belli.
- [6] Linda Lesniak, *Eccentric sequences in graphs*, Periodica Mathematica Hungaria **6** (1975), 287–293.
- [7] Eric W Weinstein, *Block*, (2010), <http://mathworld.wolfram.com/Block.html>.



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