On the Existence of Solutions to Discrete, Two Point, Non-linear Boundary Value Problems

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Damon F. Haught

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Damon F. Haught

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Signature:

Damon F. Haught, Student　　Date

Approvals:

Dr. Padraic Taylor, Thesis Advisor　　Date

Dr. David Pollack, Committee Member　　Date

Dr. Richard Goldthwait, Committee Member　　Date

Peter J. Kasvinsky, Dean of School of Graduate Studies & Research　　Date

ABSTRACT

Within this treatise we establish conditions for the existence of solutions to two-point, discrete, non-linear boundary value problems. We will be examining two different variations of the problem. First, we will be examining generalized discrete non-linear systems of the form

$$x(t+1) = Ax(t) + f(x(t)),\ t \in \{0, 1, ..., N-1\}$$

subject to

$$Bx(0) + Dx(N) = 0.$$

We demonstrate the existence of solutions to this type of problem when the associated linear, homogeneous boundary value problem has only the trivial solution, and the nonlinear element exhibits sublinear growth.

Next, we will consider scalar, two-point, nonlinear boundary value problems of the form

$$y(t+n) + a_{n-1}y(t+n-1) + \cdots + a_0 y(t) = g(y(t)),$$

for $t \in \{0, 1, ..., N-1\}$, subject to

$$\sum_{j=1}^{n} b_{ij} y(j-1) + \sum_{j=1}^{n} d_{ij} y(j+N-1) = 0,$$

for $i = 1, 2, ..., n$.

In this case, we assume the associated linear homogeneous boundary value problem has a one-dimensional solution space and establish criteria that guarantee the existence of solutions by analyzing the relationship between the nonlinear element and the solution space of the associated linear boundary value problem through a projection scheme.

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1 Introduction

In this thesis, we attempt to establish conditions for the existence of solutions to non-linear, discrete, two point boundary value problems of the form:

$$x(t+1) = Ax(t) + f(x(t)), t \in \{0, 1, ..., N-1\} \tag{1}$$

subject to

$$Bx(0) + Dx(N) = 0. \tag{2}$$

For the purpose of this paper, we will make the following assumptions:

1. $x(t)$ is a vector in $\mathbb{R}^n$ for each $t \in \{0, 1, ..., N\}$.
2. A is an invertible $n \times n$ matrix.
3. $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map.
4. B and D are constant $n \times n$ matrices.
5. N is some fixed integer larger than two.

While this problem may appear benign at first glance, the non-linearity of the equation poses quite a unique challenge. Standard solution techniques commonly utilized for linear difference equations and boundary value problems do not yield conclusive results. Thus, a different plan of attack is necessary for determining if solutions exist to our boundary value problem, and for establishing sufficient conditions for such a solution to exist.

Our approach to this problem relies heavily on both linear and non-linear operators on Banach spaces. Thus, chapter 1 will focus on a variety of preliminary notions and theorems that will be vital to the remainder of the work done in this manuscript. We will state and prove a few theorems concerning the dimension of vector spaces, as well as the continuity and boundedness of linear operators. We will then spend some time introducing the concept of degree theory, and utilize it to prove the Brouwer fixed point theorem, a powerful theorem that will prove to be a highly useful tool in solving our problem. We will next reformulate the problem using operators. Using these tools vastly simplifies the problem, and allows us to make use of their properties to further analyze it. To be more specific, we will introduce the following spaces:

$$X = \{\phi : \{0, 1, \cdots, N\} \to \mathbb{R}^n, \ B\phi(0) + D\phi(N) = 0\}$$

and

$$Y = \{\gamma : \{0, 1, \cdots, (N-1)\} \to \mathbb{R}^n\}.$$

Then, we will rewrite the problem as follows:

$$Lx = Fx$$

where $L : X \to Y$ is defined by

$$(Lx)(t) = x(t+1) - Ax(t),$$

and $F : X \to Y$ is defined by

$$(Fx)(t) = f(x(t)).$$

During the course of the chapter, we will demonstrate the equivalence of this new form of the problem to the original. Following this convenient reformulation, we can examine the problem from two distinct perspectives:

1. If L is invertible.
2. If L is not invertible.

In Chapter 2, we will examine the case where L is invertible. We will impose specific restrictions on the nonlinear element that will allow us to demonstrate the existence of solutions to (1)-(2). This restriction will be an extension of previous work found in my senior research project **[3]**, and is similar to results found in **[2]**. We will examine the structure of the linear operator L and its relationship to the main problem. This will lead to an important conclusion about the nature of the dimension of the kernel of L, namely that $\dim\{\ker(L)\} = \dim\{\ker[B + DA^N]\}$. From here, we will see that the invertibility of L is directly dependent on the invertibility of the matrix $[B + DA^N]$. Once these criteria have been established, we will demonstrate the existence of solutions to the problem given these conditions using both degree theory and the Brouwer fixed point theorem.

In Chapter 3, we will consider a more specific family of problems. Namely, scalar, discrete, nonlinear boundary value problems of the form

$$y(t+n) + a_{n-1}y(t+n-1) + \cdots + a_0 y(t) = g(y(t))$$

subject to

$$\sum_{j=1}^{n} b_{ij} y(j-1) + \sum_{j=1}^{n} d_{ij} y(j+N-1) = 0,$$

for $i = 1, 2, ..., n$.

We will see that this version of the problem can also be written in the form (1)-(2). However, for this problem we will no longer assume that L is invertible. More specifically, we will impose the restriction that the dimension of the kernel of L be one. Using a projection scheme, we will work towards a set of viable conditions that will allow us to demonstrate the existence of solutions, again through the use of degree theory and the Brouwer fixed point theorem. This treatment is similar to work found in **[2]**,**[8]**.

Finally, we will present a concrete example. We will utilize all of the concepts and tools presented in this manuscript to demonstrate that this example does indeed have a solution.

2 Preliminaries

2.1 Important Theorems

The plan of attack for our problem relies heavily on a variety of notions and theorems from Linear Algebra and Analysis. This manuscript assumes that the reader has some background in these areas. Concepts such as linear operators, vector spaces, Banach spaces, completeness, kernel, image, and theorems related to these concepts are assumed to be a part of the reader's knowledge base. In addition, basic knowledge of solution processes for linear, homogeneous, discrete equations are considered prior knowledge for this paper. For a concise exposition of any of these specific notions as they related to this treatise, please see (**[1], [4], [5], [6].**).

There are, however, a few important theorems from functional analysis and analysis of difference equations, crucial to our work, whose statement and proof are useful exercises. We begin with a look at the solution process to linear, non-homogeneous systems of difference equations.

Proposition 2.1. The solution to the non-homogeneous system $x(t+1) = Ax(t) + y(t)$, $t \in 0, 1, ...$, where $x(t)$, $y(t)$ are vectors in $\mathbb{R}^n$, and A is an invertible $n \times n$ constant matrix, is

$$x(t) = A^t x(0) + A^t \sum_{i=0}^{t-1} A^{-(i+1)} y(i). \tag{2.1}$$

Proof. Clearly, the solution satisfies the initial condition. Now, using 2.1,

$$x(t+1) = A^{t+1} x(0) + A^{t+1} \sum_{i=0}^{t} A^{-(i+1)} y(i).$$

We can then remove the last element of the summation, when $i = t$, to obtain

$$x(t+1) = A^{t+1} x(0) + A^{t+1} \sum_{i=0}^{t-1} A^{-(i+1)} y(i) + y(t).$$

Factoring out A, we have

$$x(t+1) = A \left(A^t x(0) + A^t \sum_{i=0}^{t-1} A^{-(i+1)} y(i) \right) + y(t) = Ax(t) + y(t).$$

$\square$

Next, we examine a few important properties of linear operators. We begin with a definition and lemma concerning norms on finite dimensional vector spaces.

Definition 2.1. Let X be a vector space, and $\|\cdot\|_\alpha$, $\|\cdot\|_\beta$ be two norms on X. $\|\cdot\|_\alpha$, and

$\|\cdot\|_\beta$ are said to be *equivalent* if $\exists m, M > 0$ such that

$$m\|x\|_\alpha \leq \|x\|_\beta \leq M\|x\|_\alpha$$

$\forall x \in X$.

Lemma 2.1. In a finite dimensional vector space, any two norms are equivalent.

Proof. Let X be a finite dimensional vector space. Let $\|\cdot\|_\alpha$, $\|\cdot\|_\beta$ be two norms on X. Let $B = \{b_1, ..., b_n\}$ be a basis for X. So, for any $x \in X$, we can write

$$x = c_1 b_1 + \cdots + c_n b_n$$

for scalars $c_1, .., c_n$. For $x \in X$, define

$$\|x\|_\infty = \max\{|c_j|; 1 \leq j \leq n\}.$$

So, if we can find $m, M > 0$ such that

$$m\|x\|_\infty \leq \|x\|_\alpha \leq M\|x\|_\infty,$$

and similarly for $\|x\|_\beta$, then we will have our result.
Now,

$$\begin{aligned} \|x\|_\alpha &\leq \sum_{i=1}^{n} \|c_i b_i\|_\alpha \\ &= \sum_{i=1}^{n} |c_i| \|b_i\|_\alpha \\ &\leq \left(\sum_{i=1}^{n} \|b_i\|_\alpha\right) \|x\|_\infty \end{aligned}$$

Thus, $\|x\|_\alpha \leq M\|x\|_\infty$, for $M = \sum\limits_{i=1}^{n} \|b_i\|_\alpha$.

Now, consider the unit sphere $S = \{x \in X : \|x\|_\infty = 1\}$ in $(X, \|\cdot\|_\infty)$. Let $d = \inf\{\|x\|_\alpha : x \in S\}$. Now, since S is compact, we know there is a sequence $\{y_k\}_{k=1}^\infty$ of unit vectors in S such that $\|y_k\|_\alpha \to d$. We know each y_k can be written as

$$y_k = c_{1_k} b_1 + \cdots c_{n_k} b_n.$$

Now, since y_k is in S, we know $|c_{j_k}| \leq 1$ for all $c_{j_k}, 1 \leq j \leq n$. Since j is finite, and $\{c_{j_k}\} \in [-1, 1]$, we may find a subsequence $k_1, k_2, \cdots$ such that $\{c_{j_{k_m}}\}$ converges as m approaches infinity.

So, let $c_j = \lim\limits_{m \to \infty} \{c_{j_{k_m}}\}$.

We know $\|y_{k_m}\|_\alpha \to d$. Let $y_0 = \sum_{i=1}^{n} c_j b_j$.

We next seek to show that $y_{k_m} \to y_0$ in $\|\cdot\|_\infty$.
Consider

$$\begin{aligned}
\|y_{k_m} - y_0\|_\infty &= \left\| \sum_{j=1}^{\infty} c_{j_{k_m}} b_j - \sum_{j=1}^{\infty} c_j b_j \right\| \\
&= \left\| \sum_{j=1}^{\infty} \left(c_{j_{k_m}} - c_j \right) b_j \right\| \\
&= \max\{|c_{j_{k_m}} - c_j| : 1 \le j \le n\}.
\end{aligned}$$

But $c_j = \lim_{m\to\infty} \{c_{j_{k_m}}\}$. So $\|y_{k_m} - y_0\|_\infty \to 0$, which shows that $y_{k_m} \to y_0$ in $\|\cdot\|_\infty$.
Thus, since each y_k is in S, we have $\|y_0\|_\infty = 1$. Therefore, $y_0 \neq 0$.
We know $\|y_{k_m} - y_0\|_\alpha \le M\|y_{k_m} - y_0\|_\infty \to 0$. Thus, $\|y_0\|_\alpha = \lim_{m\to\infty} \|y_{k_m}\|_\alpha = d$. Thus, $d \neq 0$.

Now, for $x \in X$, $x \neq 0$, $\frac{x}{\|x\|_\infty} \in S$. So,

$$\|y_0\|_\alpha \le \left\| \frac{x}{\|x\|_\infty} \right\|_\alpha .$$

So we have

$$\|x\|_\infty \|y_0\|_\alpha \le \|x\|_\alpha$$

$$\Rightarrow d\|x\|_\infty \le \|x\|_\alpha$$

Thus, we have for $m = d$,

$$m\|x\|_\infty \le \|x\|_\alpha \le M\|x\|_\infty.$$

□

Next, we introduce an important property concerning linear operators on finite-dimensional vector spaces.

Proposition 2.2. Every linear map from a finite-dimensional normed vector space into a normed vector space is bounded.

Proof. Let $T : X \to Y$ be a linear operator, with X being a finite-dimensional normed vector space, and Y being a normed vector space. Let $B = \{b_1, ..., b_n\}$ be a basis for X. As in the proof for the previous proposition, let $\|\cdot\|_\infty$ be a second norm on X. We know the map T is bounded with respect to the original norm if and only if it is bounded with respect to the equivalent norm $\|\cdot\|_\infty$. Consider for x in X

$$\begin{aligned}
\|Tx\| &= \left\| T\left(\sum_{i=1}^{n} c_j b_j\right)\right\|_Y \\
&\leq \sum_{i=1}^{n} |c_j| \|Tb_j\|_Y \\
&\leq \left(\max_{1\leq j\leq n} |c_j|\right)\left(\sum_{i=1}^{n} \|Tb_j\|_Y\right) \\
&= \left(\sum_{i=1}^{n} \|Tb_j\|_Y\right)\|x\|_\infty
\end{aligned}$$

Thus, we have established the boundedness of T. □

Finally, we state a crucial theorem regarding the relationship between continuity and boundedness of linear operators.

Proposition 2.3. If $T : X \to Y$ is a linear map from a normed vector space X into a normed vector space Y, then the following are equivalent.
a) T is continuous at a point x_0 in X.
b) T is continuous.
c) T is bounded.

Proof. **a $\Rightarrow$ b.** Suppose T is continuous at some point x_0 in X. So, given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|Tx - Tx_0\| < \epsilon, \text{ whenever } \|x - x_0\| < \delta.$$

Now, let y, z be in X, with $\|y - z\| < \delta$. This means that

$$\|x_0 - (x_0 + z - y)\| < \delta.$$

This implies that

$$\|Ty - Tz\| = \|T(y - z)\| = \|T[x_0 - (x_0 + z - y)]\| = \|Tx_0 - T(x_0 + z - y)\| < \epsilon,$$

which tells us that T is continuous on all of X.

b $\Rightarrow$ c. Let $\epsilon = 1$. Now, since T is continuous, we know there exists a $\delta > 0$ such that

$$\|Tx - T0\| = \|Tx\| \leq \epsilon = 1, \text{ whenever } \|x - 0\| < \delta, \forall x \in X$$

Now, suppose x_0 is in X, with $\|x_0\| = 1$. This means that $\|\delta x_0\| = \delta\|x_0\| = \delta$. This gives us that

$$\|T(\delta x_0)\| = \delta\|Tx_0\| \leq 1.$$

This, in turn, gives us

$$\|Tx_0\| = \frac{\|Tx_0\|}{\|x_0\|} \leq \frac{1}{\delta}.$$

Thus, $\|T\| \leq M$, for $M = \frac{1}{\delta}$.

c $\Rightarrow$ **a.** For x_0 in X, and $\epsilon > 0$, if

$$\|x - x_0\| < \delta, \text{ where } \delta = \frac{\epsilon}{1 + \|T\|},$$

then

$$\|Tx - Tx_0\| \leq \|T\| \|x - x_0\| < \epsilon.$$

□

Now that we have established these crucial properties, we can move into another realm of analysis that will prove vital to our problem's solution process.

2.2 Introduction to Degree Theory

We now introduce perhaps the most important piece of the puzzle necessary to solve the problem at hand. We will devote the next section of this treatise to the development of degree theory. Degree theory is a powerful tool within analysis that allows us to ascertain various properties of functions. Furthermore, degree theory can be utilized to prove a variety of important and useful results within mathematics, such as the Brouwer fixed point theorem. More importantly, it will serve as the key that will guarantee the existence of a solution to our problem.

Prior to officially defining the degree of a map, we need to establish some preliminary assumptions and concepts. Following **[9]**, let $\mathbf{D}$ be an open, bounded subset of $\mathbb{R}^n$. Next, define a map $f : \overline{\mathbf{D}} \to \mathbb{R}^n$. We assume this map is $\mathcal{C}^1$, which implies that it is continuously differentiable. In addition, we assume $f(\vec{x}) \neq 0$ for all $\vec{x} \in \partial\mathbf{D}$, where $\partial\mathbf{D}$ represents the boundary of the set $\mathbf{D}$. We now define two important sets $\mathbf{A}_f = \{\vec{x} \in \overline{\mathbf{D}} : \mathbf{f}(\vec{x}) = \vec{0}\}$, and $\mathbf{B}_f = \{\vec{x} \in \mathbf{D} : J_{\mathbf{f}}(\vec{x}) = 0\}$. For our purposes, $J_{\mathbf{f}}(\vec{x})$, which represents the Jacobian of the function f evaluated at some vector $\vec{x}$, will be defined as the determinant of the matrix containing all of the first-order partial derivatives of our function. We note that if $\mathbf{A}_f$ is non-empty, then it is compact. Finally, we assume that $\mathbf{A}_f \cap \mathbf{B}_f = \emptyset$. This simply means that when $f(\vec{x}) = 0$, $J_{\mathbf{f}}(\vec{x}) \neq 0$. When this condition holds, we say that the function f is non-degenerate. With these hypotheses in place, it can be shown that $\mathbf{A}_f$ is a finite set. For more information on these preliminaries, and a more in-depth examination into degree theory, see **[9]**.

Now that these preliminary notions have been established, we can officially define the degree of a map.

Definition 2.2. The topological degree of a map f, with respect to $\mathbf{D}$ and $\vec{0}$, denoted by $d[\mathbf{f}, \mathbf{D}, \vec{0}]$ is given by:

$$d[\mathbf{f}, \mathbf{D}, \vec{0}] = \sum_{\vec{x} \in \mathbf{A}_f} \operatorname{sign} J_{\mathbf{f}}(\vec{x}).$$

At this point, clarification of what we mean by $\operatorname{sign} J_{\mathbf{f}}(\vec{x})$ may be helpful. Recall that we have defined the Jacobian to be the determinant of the matrix of first-order partial derivatives of a function evaluated at a point. Certainly, this determinant will have a sign, be it positive or negative. We assign to each sign a particular value. If the sign of the determinant is positive, we assign it a value of 1. If the sign of the determinant is negative, we assign it a value of -1. From this definition we can extrapolate an extremely important result that will be useful throughout the remainder of this treatise.

Proposition 2.4. If $d[\mathbf{f}, \mathbf{D}, \vec{0}] \neq 0$, $\mathbf{A}_f \neq \emptyset$.

Proof. This follows directly from the definition. We simply consider the contrapositive. That is, if $\mathbf{A}_f = \emptyset$, then clearly we have no elements to sum, and thus the degree of our map is 0. Therefore, if $d[\mathbf{f}, \mathbf{D}, \vec{0}] \neq 0$, $\mathbf{A}_f \neq \emptyset$. □

Before presenting the main result of our exploration into degree theory, we need to introduce one final notion, which will prove crucial to our main problem.

Definition 2.3 (**Homotopy**). If X and Y are subsets of $\mathbb{R}^n$, and g and h are continuous maps from X to Y, then g is **homotopic** to h if there is a continuous map $\Phi : X \times [0, 1] \rightarrow Y$ such that $\Phi(x, 0) = g(x)$ and $\Phi(x, 1) = h(x), \forall x \in X$. Such a map is called a **homotopy**.

Armed with this definition, we can state the following theorem:

Theorem 2.1 (**Theorem of Invariance with Respect to Homotopy**). Let $F : \overline{\mathbf{D}} \times [0, 1] \rightarrow \mathbb{R}^n$ be a continuous function, with $F(\vec{x}, \lambda) \neq 0, \forall (\vec{x}, \lambda) \in \partial\mathbf{D} \times [0, 1]$. Define $\mathbf{f}_\lambda : \overline{\mathbf{D}} \rightarrow \mathbb{R}^n$ by $\mathbf{f}_\lambda(\vec{x}) = F(\vec{x}, \lambda)$. Then $d[\mathbf{f}_\lambda, \mathbf{D}, \vec{0}]$ is independent of λ.

For a proof of this theorem, see **[10]**. In essence, this theorem allows us to establish to the existence of a zero for a function we know little about by relating it to another function that we know has a zero.

2.3 The Brouwer Fixed Point Theorem

With these important preliminaries taken care of, we can set our sights on stating and proving the Brouwer fixed point theorem.

Theorem 2.2 (The Brouwer Fixed Point Theorem). Let $B_1 = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| < 1\}$. If $f : \overline{B}_1 \to \overline{B}_1$ is a continuous map, then f has at least one fixed point in $\overline{B}_1$.

Proof. Let $B_1 = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| < 1\}$. Let $f : \overline{B}_1 \to \overline{B}_1$ be a continuous function. We will show that $\exists \hat{x} \in \overline{B}_1$ such that $f(\hat{x}) = \hat{x}$.

Our plan of attack will center around the Theorem of Invariance with Respect to Homotopy. So, we begin by constructing a homotopy.

First, let $T : \overline{B}_1 \to \mathbb{R}^n$ be defined by $T(\vec{x}) = \vec{x} - f(\vec{x})$. Note that T is continuous, and that if $T(\vec{x}) = \vec{0}$, then $f(\vec{x}) = \vec{x}$. Therefore, the remainder of this proof will focus on finding a zero for this new map T. Next, we let $g : \overline{B}_1 \to \mathbb{R}^n$ be the identity map. That is, $g(\vec{x}) = \vec{x}$. The identity map presents a few useful properties. For instance, note that:

i) $\vec{0} \in \overline{B}_1$;
ii) $g(\vec{0}) = \vec{0}$;
iii) $\vec{0}$ is the unique zero for g;
iv) $J_g(\vec{0}) = 1$.

From these properties, we know $d[g, B_1, \vec{0}] = 1$.

Equipped with these maps, we set about to construct a homotopy to relate g and T. We let $F : \overline{B}_1 \times [0,1] \to \mathbb{R}^n$ be defined by $F(\vec{x}, \lambda) = \vec{x} - \lambda f(\vec{x})$. Clearly, F is a continuous map. In addition, we note that $F(\vec{x}, 0) = \vec{x} = g(\vec{x})$ and $F(\vec{x}, 1) = \vec{x} - f(\vec{x}) = T(\vec{x})$.

Our goal here is to show

$$d[T, B_1, \vec{0}] = d[g, B_1, \vec{0}].$$

In order to demonstrate this, we must show $F(\vec{x}, \lambda) \neq \vec{0}, \forall (\vec{x}, \lambda) \in \partial B_1 \times [0,1]$. To do this, we consider two cases:

Case I: $\lambda = 1$.
Let $\vec{x} \in \partial B_1$. So we have

$$F(\vec{x}, 1) = \vec{x} - f(\vec{x}).$$

Thus, if $F(\vec{x}, 1) = \vec{0}$, then $f(\vec{x}) = \vec{x}$, and we have our fixed point. In this case, no homotopy argument is required.

Case II: $\lambda \in [0,1), F(\vec{x}, 1) \neq \vec{0}$.
Once again, we let $\vec{x} \in \partial B_1$. So we have

$$F(\vec{x}, \lambda) = \vec{x} - \lambda f(\vec{x}).$$

Since $f(\vec{x}) \in \overline{B}_1, \|f(\vec{x})\| \leq 1$. And certainly, since $\lambda \in [0,1), \lambda\|f(\vec{x})\| < 1$.
Next, we let $f_\lambda(\vec{x}) = F(\vec{x}, \lambda)$. So we have:

$$\begin{aligned} \|f_\lambda(\vec{x})\| &= \|\vec{x} - \lambda f(\vec{x})\| \\ &\geq \|\vec{x}\| - \lambda\|f(\vec{x})\| \\ &= 1 - \lambda\|f(\vec{x})\| \\ &> 0. \end{aligned}$$

Since $\|f_\lambda(\vec{x})\| > 0$, we know $F(\vec{x}, \lambda) \neq \vec{0}$, when $\vec{x} \in \partial B_1, \forall \lambda \in [0,1]$. Therefore, we can use our theorem on the invariance of degree with respect to homotopy. This means that $d[f_\lambda, B_1, \vec{0}]$ is independent of λ. So we have

$$d[T, B_1, \vec{0}] = d[f_1, B_1, \vec{0}] = d[f_0, B_1, \vec{0}] = d[g, B_1, \vec{0}] = 1.$$

Therefore, there exists an $\hat{x} \in B_1$ such that $f(\hat{x}) = \hat{x}$. □

2.4 Extension of Brouwer

For the purposes of our problem, this basic incarnation of the Brouwer fixed point theorem will be insufficient. While the theorem is quite useful within its original context, it does not provide ample conditions for use in our situation. Thus, we need to establish an extension of the theorem to other sets. We begin with a definition:

Definition 2.4. Let A and B be subsets of $\mathbb{R}^n$. If $\phi : A \to B$ is continuous, bijective, and ϕ^{-1} is also continuous, then we say ϕ is a **homeomorphism**, and A is **homeomorphic** to B.

With this definition, we present an important proposition:

Proposition 2.5. A compact, convex subset of $\mathbb{R}^n$ with nonempty interior is homeomorphic to the unit ball.

For a proof of this proposition, see **[3]**.

Armed with these tools, we can now present an official extension to the Brouwer fixed point theorem:

Theorem 2.3. Let K be a compact, convex subset of $\mathbb{R}^n$ with nonempty interior. If $f : K \to K$ is continuous, then f has a fixed point in K.

Proof. Since K is a compact, convex subset of $\mathbb{R}^n$ with nonempty interior, there exists a homeomorphism $\phi : \overline{B}_1 \to K$. Let $g = \phi^{-1} f \phi$, then g is continuous and $g : \overline{B}_1 \to \overline{B}_1$. By the Brouwer fixed point theorem, g has a fixed point, denoted by $\hat{x}$. From here we see that

$$\begin{aligned} g(\hat{x}) &= \hat{x} \\ \Rightarrow \phi^{-1} f \phi(\hat{x}) &= \hat{x} \\ \Rightarrow f \phi(\hat{x}) &= \phi(\hat{x}). \end{aligned}$$

This means that $\phi(\hat{x})$ is our fixed point for f. $\square$

3 Discrete, Nonlinear, Two-Point Boundary Value Problems

3.1 Introduction

We devote this chapter to demonstrating the existence of solutions to two-point, discrete, nonlinear boundary value systems of the form

$$x(t+1) = Ax(t) + f(x(t)),\ t \in \{0, 1, ..., N-1\} \tag{3.1}$$

subject to

$$Bx(0) + Dx(N) = 0. \tag{3.2}$$

We remind the reader of the basic assumptions of this problem, namely:

1. $x(t)$ is a vector in $\mathbb{R}^n$ for each $t \in \{0, 1, ..., N\}$.
2. A is an invertible $n \times n$ matrix.
3. $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map.
4. B and D are constant $n \times n$ matrices.
5. N is some fixed integer larger than two.

Utilizing the results from the previous sections, we will first rewrite the problem in terms of operators L and F on finite-dimensional function spaces X and Y. From here, we will discuss the relationship between the linear homogeneous boundary value problem and the operator L.

Using these criteria, we will prove the existence of solutions to (3.1) and (3.2) with the following conditions:

i) L is invertible;

ii) F is a sublinear map.

More specifically, we will present two distinct proofs. One will utilize a degree theory argument. The other will center on the Brouwer fixed point theorem.

3.2 Rewriting the Problem Using Operators

We can rewrite (3.1) and (3.2) using operators on finite dimensional function spaces. To this end, we introduce the following spaces. Let

$$X = \{\phi : \{0, 1, ..., N\} \to \mathbb{R}^n,\ B\phi(0) + D\phi(N) = 0\}$$

and

$$Y = \{\gamma : \{0, 1, ..., (N-1)\} \to \mathbb{R}^n\}.$$

We define norms on these spaces as follows:
For $x \in X$:

$$\|x\| = \sup_{t \in 0,1,...,N} |x(t)|.$$

For $y \in Y$:

$$\|y\| = \sup_{t \in 0,1,\ldots,(N-1)} |y(t)|,$$

where $|\cdot|$ denotes the any norm on $\mathbb{R}^n$. These spaces are clearly finite-dimensional Banach spaces.

With these definitions in hand, we can now define our maps. First, we define $L : X \to Y$ by

$$(Lx)(t) = x(t+1) - Ax(t).$$

Comparing L to the linear homogeneous boundary value problem

$$x(t+1) = Ax(t),$$

subject to

$$Bx(0) + Dx(N) = 0,$$

we can see that $\ker(L)$ and the solution space to the linear homogeneous problem have the same dimension.

Claim. The solution space to the linear homogeneous boundary value problem and $\ker([B + DA^N])$ have the same dimension.

Proof. We know the solution space to the linear homogeneous problem has the same dimension as $\ker(L)$. Observe that $x \in \ker(L)$ if and only if

$$x(t+1) = Ax(t),\ t \in 0, 1, \ldots, (N-1)$$

and

$$Bx(0) + Dx(N) = 0.$$

This occurs if and only if there is some vector $c \in \mathbb{R}^n$ such that

$$x(t) = A^t c,$$

and

$$Bc + DA^N c = 0.$$

And this occurs if and only if

$$x(t) = A^t c,$$

and

$$c \in \ker([B + DA^N])$$

$\square$

Therefore, the solution space of the linear homogeneous boundary value problem, $\ker(L)$, and $\ker([B + DA^N])$ all have the same dimension.

Next, we can define $F : X \to Y$ by

$$(Fx)(t) = f(x(t)).$$

Observe that x is a solution to (3.1)- (3.2) if and only if x is a solution to

$$Lx = Fx.$$

3.3 The Invertibility of L

This convenient reformulation presents a variety of useful results that allow us to present conditions for the existence of a solution to (3.1)-(3.2). Perhaps the most significant result that follows from this reformulation deals with the invertibility of our linear operator L.

Recall that we are examining this problem under the constraint that L is invertible. Thus, we must ascertain the conditions under which this will occur. To illustrate this, we present the following proposition:

Proposition 3.1. L is invertible if and only if the matrix $\left(B + DA^N\right)$ is invertible.

Proof. Let $y \in Y$. Based on how we have created our linear operator L, we know that $Lx = y$ if and only if

$$x(t+1) = Ax(t) + y(t),\ t \in 0, 1, ..., (N-1),$$

and

$$Bx(0) + Dx(N) = 0.$$

From here, we can utilize the Variation of Constants formula we examined earlier to determine that $Lx = y$ if and only if

$$x(t) = A^t x(0) + A^t \sum_{i=0}^{t-1} A^{-(i+1)} y(i), \tag{3.3}$$

for some $x \in X$, and

$$Bx(0) + Dx(N) = 0.$$

Combining these two equations yields $Lx = y$ if and only if

$$Bx(0) + D\left(A^N x(0) + A^N \sum_{i=0}^{N-1} A^{-(i+1)} y(i)\right) = 0.$$

With a little rearranging we find that

$$Bx(0) + DA^N x(0) = -\left(DA^N \sum_{i=0}^{N-1} A^{-(i+1)} y(i) \right).$$

Finally, we have

$$\left(B + DA^N\right) x(0) = -\left(DA^N \sum_{i=0}^{N-1} A^{-(i+1)} y(i) \right). \tag{3.4}$$

Herein lies the proof of our claim. We note that $Lx = y$ if and only if there exists an $x(0) \in \mathbb{R}^n$ that satisfies (3.4). If such an $x(0)$ exists, we can thus explicitly express an $x(t)$ for equation (3.3), and we will have $Lx = y$. Using the fundamental theorem of invertible matrices, we know that if $\left(B + DA^N\right)$ is invertible, then there is a unique $x(0)$ that will solve (3.4). This implies that there is one and only one x that will solve $Lx = y$. Therefore, L will be invertible.

Conversely, if L is invertible, then there is one and only one x that will solve $Lx = y$. This will lead to a unique $x(0)$ that will solve (4). Hence, $\left(B + DA^N\right)$ must be invertible. □

3.4 Main Theorem

Now that we have rewritten our original problem in terms of operators defined on finite-dimensional operators, and have established conditions for which our linear operator L will be invertible, we are now ready for the coup de grâce. We will prove that (3.1)-(3.2) has a solution using two methods. First, we will show existence using a degree theory argument. Then, we will demonstrate our solution via the Brouwer fixed point theorem.

Theorem 3.1. Let $[B + DA^N]$ be invertible. If there exist $M_1, M_2 \geq 0$, and $\alpha \in \mathbb{R}$, with $0 \leq \alpha < 1$, such that$|f(x)| \leq M_1 + M_2|x|^\alpha$ for all $x \in \mathbb{R}$, then there is at least one solution to (1)-(2).

Proof (Method 1)**.** Assume $[B + DA^N]$ is invertible. This implies that L is invertible. This tells us that

$$Lx = F(x)$$

if and only if

$$x = L^{-1}F(x).$$

We seek to establish existence by utilizing a homotopy argument. More specifically, we will attempt to use the theorem of invariance of degree with respect to homotopy. To this end, we begin by defining $T : X \to X$ by $T(x) = x - L^{-1}F(x)$. Notice that since L is a linear map on a finite-dimensional space, it is continuous. Hence, L^{-1} is also continuous. Thus, by the properties of continuous functions, we know that T is also continuous. Also note that if $T(x) = 0$, then $L^{-1}F(x) = x$, and we will have shown that (1)-(2) has a solution. Thus, we will devote the remainder of this proof to establishing the existence of an x such that $T(x) = 0$.

Next, we note that if f meets the stated criteria, $\|F(x)\| = \sup_{t\in 0,1,...,(N-1)} |f(x(t))|$, and for each $t \in \{0, 1, ..., N-1\}$:

$$|f(x(t))| \leq M_1 + M_2|x(t)|^\alpha \leq M_1 + M_2\|x\|^\alpha,$$

for some $M_1, M_2 \in \mathbb{R}$.
With this in mind, and prior to defining our homotopy map, we define the set

$$B = \{x \in X : \|x\| < M, M \in \mathbb{R}\}.$$

This set will play a crucial role in the development of this proof.

We now define $g : \overline{B} \to \overline{B}$ by $g(x) = x$. As was the case in our proof of the Brouwer fixed point theorem, we utilize this map, because we know that $d[g, B, \vec{0}] = 1$.

Next, we construct our homotopy. We let $H : \overline{B} \times [0, 1] \to X$ be defined by $H(x, \lambda) = x - \lambda L^{-1}F(x)$. Clearly, H is a continuous map. As before, we note that $H(x, 0) = x = g(x)$, and $H(x, 1) = x - L^{-1}F(x) = T(x)$. We seek to demonstrate that

$$d[T, B, \vec{0}] = d[g, B, \vec{0}].$$

To accomplish this, we need to show that $H(x, \lambda) \neq \vec{0}, \forall (x, \lambda) \in \partial B \times [0, 1]$. Once again, we examine two cases:

Case I: $\lambda = 1$.
We let $x \in \partial B$. So we have

$$H(x, 1) = x - L^{-1}F(x).$$

We see that if $H(x, 1) = \vec{0}$, then $L^{-1}F(x) = x$, and we will have shown the existence of our solution.

Case II: $\lambda \in [0, 1),\ H(x, 1) \neq 0$.
As before, we let $x \in \partial B$. So once again we have

$$H(x, \lambda) = x - \lambda L^{-1}F(x).$$

Now, we know that $x \in \partial B$. Thus we know $\|x\| = M$, for some $M \in \mathbb{R}$. Also, $\lambda \in [0, 1)$.

So we have

$$
\begin{aligned}
\|H_\lambda(x)\| &= \|x - \lambda L^{-1}F(x)\| \\
&\geq \|x\| - \|\lambda L^{-1}F(x)\| \\
&\geq \|x\| - \lambda\|L^{-1}F(x)\| \\
&> \|x\| - \|L^{-1}\|(M_1 + M_2\|x\|^\alpha) \\
&= M - \|L^{-1}\|(M_1 + M_2\|M\|^\alpha).
\end{aligned}
$$

This tells us that

$$\frac{\|H_\lambda(x)\|}{M} > 1 - \|L^{-1}\| \left(\frac{M_1}{M} + \frac{M_2}{M^{1-\alpha}}\right).$$

Thus, for M sufficiently large, $\frac{\|H_\lambda\|}{M} > 0$. Hence, for M sufficiently large, $\|H_\lambda(x)\| > 0$. With this, we have shown that $H(x,\lambda) \neq \vec{0}, \forall (x,\lambda) \in \partial B \times [0,1]$, and for all $x \in \partial B$. As a result, we can see that

$$d[T, B, \vec{0}] = d[H_1, B, \vec{0}] = d[H_0, B, \vec{0}] = d[g, B, \vec{0}] = 1.$$

Therefore, there exists an $x \in B$ such that $Tx = 0$, and thus there exists an $x \in X$ such that $L^{-1}F(x) = x$. □

We next prove our main theorem using a more concise method. This method relies on the Brouwer fixed point theorem we proved earlier:

Proof (Method 2)**.** As before, we assume $[B + DA^N]$ be invertible. This again implies that L will be invertible, and thus we have

$$x = L^{-1}F(x).$$

For this proof we introduce the map $\Phi : X \to X$, defined by $\Phi(x) = L^{-1}F(x)$. By how we define composition of functions, Φ is a continuous map. We seek to demonstrate that there exists an $x \in X$ such that $\Phi(x) = x$.

Recall that $F(x)$ is a bounded function. As such, we can see that

$$\|\Phi(x)\| = \|L^{-1}F(x)\| \leq \|L^{-1}\|(M_1 + M_2\|x\|^\alpha)$$

for some non-negative real numbers M_1, M_2. Thus, note that if we let x reside in our set B, which was defined earlier in Version I of the proof, then certainly

$$\|\Phi(x)\| \leq \|L^{-1}\|(M_1 + M_2M^\alpha).$$

Thus,

$$\frac{\|\Phi(x)\|}{M} \leq \|L^{-1}\| \left(\frac{M_1}{M} + \frac{M_2}{M^{1-\alpha}}\right).$$

So, for M sufficiently large, $\frac{\|\Phi(x)\|}{M} \leq 1$. So, there exists an $M > 0$ such that for all $x \in X$, if $\|x\| \leq M$, $\|\Phi(x)\| \leq M$. With this, we see that Φ maps $B = \{x \in X : \|x\| < M\}$ onto itself. In addition, we know that B is a compact, convex subset of $\mathbb{R}^n$ with non-empty interior. Thus, we know that Φ has a fixed point in B.

4 Scalar, Nonlinear, Two-Point Boundary Value Problems

4.1 Introduction

We now consider a more specific type of nonlinear, discrete boundary value problem. These scalar problems are of the form

$$y(t+n) + a_{n-1}y(t+n-1) + \cdots + a_0 y(t) = g(y(t)) \tag{4.1}$$

for $t \in \{0, 1, ...N-1\}$, subject to

$$\sum_{j=1}^{n} b_{ij} y(j-1) + \sum_{j=1}^{n} d_{ij} y(j+N-1) = 0, \tag{4.2}$$

for $i = 1, 2, ..., n$. In similar fashion to the previous section, we will assume g is a continuous, real-valued function, and N is some integer larger than two. In addition, the constants b_{ij}, d_{ij}, and $a_0, ..., a_{n-1}$ are all real-valued, with $a_0 \neq 0$.

Our exploration into the existence of solutions to (4.1)-(4.2) will involve a process of transforming the scalar problem into a discrete, nonlinear system of the form

$$x(t+1) = Ax(t) + f(x(t)),\ t \in \{0, 1, ...N-1\}$$

subject to

$$Bx(0) + Dx(N) = 0.$$

We will then rewrite the problem using linear and nonlinear operators, again transforming the problem into the form

$$Lx = F(x).$$

In this section, we will consider the case where L is not invertible. More specifically, we will examine the problem under the assumption that the dimension of the kernel of L is 1. Then, using a projection scheme, we will establish conditions for the existence of solutions to (4.1)-(4.2) that will rely on the end behavior of the function g, and on the behavior of the linear component. The reader should note that much of the following exposition follows closely with the work presented by Dr. Padraic Taylor, as well as earlier work presented by Dr. Debra Etheridge and Dr. Jesus Rodriguez. For more information on these works, see **[8]**, and **[2]**

4.2 Rewriting the Problem

Recall that we are considering the existence of solutions to

$$y(t+n) + a_{n-1}y(t+n-1) + \cdots + a_0 y(t) = g(y(t))$$

for $t \in \{0, 1, ...N-1\}$, subject to

$$\sum_{j=1}^{n} b_{ij} y(j-1) + \sum_{j=1}^{n} d_{ij} y(j+N-1) = 0,$$

for $i = 1, 2, ..., n$.

We now work to rewrite the problem as an $n \times n$ system of the form

$$x(t+1) = Ax(t) + f(x(t)),\ t \in \{0, 1, ...N-1\} \tag{4.3}$$

subject to

$$Bx(0) + Dx(N) = 0. \tag{4.4}$$

First, we state the form of our $n \times n$ matrix A:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}.$$

Next, define the matrix $B = (b_{ij})$, and the matrix $D = (d_{ij})$, for $i \in 1, 2, ..., n$, and $j \in 1, 2, ..., n$. Then we set the vector

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y(t+1) \\ \vdots \\ y(t+n-1) \end{bmatrix},$$

and define the function $f : \mathbb{R}^n \to \mathbb{R}^n$ by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(x_1) \end{bmatrix}.$$

With these definitions, (4.1)-(4.2) can be analyzed as (4.3)-(4.4).

As before, we wish to make use of linear and nonlinear operators to further study

this problem. To this end, we remind the reader of the following spaces and maps:
Let

$$X = \{\phi : \{0, 1, ..., N\} \to \mathbb{R}^n,\ B\phi(0) + D\phi(N) = 0\}$$

and

$$Y = \{\gamma : \{0, 1, ..., (N-1)\} \to \mathbb{R}^n\}.$$

We define norms on these spaces as follows:
For $x \in X$:

$$\|x\| = \sup_{t \in 0,1,\ldots,N} |x(t)|.$$

For $y \in Y$:

$$\|y\| = \sup_{t \in 0,1,\ldots,(N-1)} |y(t)|,$$

where $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$.
Define $L : X \to Y$ by

$$(Lx)(t) = x(t+1) - Ax(t).$$

Define $F : X \to Y$ by

$$(Fx)(t) = f(x(t)).$$

Recall that x is a solution to (4.1)- (4.2) if and only if x is a solution to

$$Lx = Fx.$$

4.3 Analysis of the Linear Operator

We now examine the structure of the linear component of our problem. We first examine the linear homogeneous problem of the form

$$y(t+n) + a_{n-1}y(t+n-1) + \cdots + a_0 y(t) = 0 \tag{4.5}$$

subject to

$$\sum_{j=1}^{n} b_{ij} y(j-1) + \sum_{j=1}^{n} d_{ij} y(j+N-1) = 0, \tag{4.6}$$

for $i = 1, 2, ..., n$. Recall that for x to be a solution to the linear homogeneous boundary value problem, $x(t)$ must be of the form $A^t x(0)$, and $x(0)$ must be in $\ker[B + DA^N]$. Additionally, we saw that that $x \in \ker(L)$ if and only if

$$x(t+1) = Ax(t),\ t \in 0, 1, ..., (N-1)$$

and

$$Bx(0) + Dx(N) = 0.$$

Therefore, $\ker(L)$ is equivalent to the solution space of the linear homogeneous problem. In our problem, we are considering the case where the dimension of the kernel of L is one. From here we define the following map $S : \{0, 1, ..., N\} \to \mathbb{R}^n$ by

$$S(t) = A^t d,$$

where d is a unit vector in $\ker(B + DA^N)$.

Clearly, we can see that x is in the kernel of L if and only if $x(t) = S(t)c$ for some $c \in \mathbb{R}$.

We now begin constructing our projection scheme by introducing projections onto the kernel and image of L.

Let d be a unit vector in $\ker(B + DA^N)$. Define $P : X \to X$ by

$$(Px)(t) = S(t)d^T x(0).$$

Claim. P is a projection onto the kernel of L.

Proof. The linearity and boundedness of P are clear. Thus, we need to demonstrate that $P^2 = P$, and that $Im(P) = \ker(L)$. we begin with showing that $P^2 = P$. Consider

$$\begin{aligned}(P(Px))(t) &= P(S(\cdot)d^T x(0))(t) \\ &= S(t)d^T S(0)d^T x(0) \\ &= S(t)(d^T d)d^T x(0) \\ &= S(t)d^T x(0) \\ &= (Px)(t).\end{aligned}$$

Next, we need to show that $Im(P) = \ker(L)$. First, let $x \in X$. So we have

$$\begin{aligned}(Px)(t) &= S(t)d^T x(0) \\ &= S(t)\alpha,\end{aligned}$$

where $\alpha = d^T x(0)$. Thus, $Px \in \ker(L)$. So, $Im(P) \subseteq \ker(L.)$
Now, let $x \in \ker(L)$. so $x = S(t)\beta$, for some $\beta \in \mathbb{R}$. So we have

$$\begin{aligned}(Px)(t) &= P(S(t)\beta)(t) \\ &= \beta(P(S(t)))(t) \\ &= \beta S(t)d^T S(0) \\ &= \beta S(t)d^T d \\ &= \beta S(t) \\ &= x(t).\end{aligned}$$

Thus, if $x \in \ker(L), x \in Im(P)$.
So, $\ker(L) \subseteq Im(P)$.
Thus, $Im(P) = \ker(L)$. □

Next, we begin constructing a projection onto the image of L. This turns out to be quite an arduous process, and involves utilizing a few other important maps to lay a framework. To this end, we introduce the following:
Let c be a vector in $\mathbb{R}^n$, with $\ker([B + DA^N]^T) = \text{span}\{c\}$. Define $\psi : \{0, 1, ..., N-1\} \to \mathbb{R}^n$ by

$$\psi(t) = [DA^N A^{-(t+1)}]^T c.$$

With this definition, we state a crucial lemma:

Lemma 4.1. ψ is the zero map if and only if $\ker(B^T) \bigcap \ker(D^T) \neq 0$.

Proof. Assume there is some nonzero vector v in $\ker(B^T) \bigcap \ker(D^T)$. This tells us that

$$B^T v = 0,$$

and

$$D^T v = 0.$$

This tells us that

$$(A^N)^T (D^T v) = 0$$
$$\Rightarrow (DA^N)^T v = 0.$$

From here, we obtain

$$0 = 0 + 0$$
$$\Rightarrow 0 = B^T v + (DA^N)^T v$$
$$\Rightarrow 0 = (B + DA^N)^T v$$

Thus, v is in $\ker([B+DA^N]^T)$. So, $v = \alpha c$, where $c \in \ker([B+DA^N]^T)$, and $\alpha \in \mathbb{R}, \alpha \neq 0$. Therefore,

$$c = \frac{1}{\alpha} v.$$

And so

$$\psi(t) = [DA^N A^{-(t+1)}]^T \frac{1}{\alpha} v = 0, \; \forall t \in 0, 1, ..., N-1.$$

Next, assume that ψ is the zero map for all $t \in 0, 1, ..., N-1$. So we have

$$\psi(N-1) = 0$$
$$\Rightarrow [DA^N A^{-(N-1+1)}]c = 0$$
$$\Rightarrow D^T c = 0.$$

Thus, $c \in \ker(D^T)$. But, we also know that $c \in \ker([B + DA^N]^T)$. So, $c \in \ker(B^T)$. Therefore, $c \in \ker(B^T) \bigcap \ker(D^T)$. □

Now, in order to properly create a projection onto the image of L, we need to get a sense of what that space looks like. To that end, we introduce the following proposition:

Proposition 4.1. Let $h \in Y$. Then $h \in Im(L)$ if and only if

$$\sum_{i=0}^{N-1} \psi^T(i) h(i) = 0.$$

Proof. Let $h \in Y$. We know from Proposition (3.1) that h is in the image of L if and only if

$$Bx(0) + DA^N x(0) = -\left(DA^N \sum_{i=0}^{N-1} A^{-(i+1)} y(i) \right).$$

This occurs if and only if

$$DA^N \sum_{i=0}^{N-1} A^{-(i+1)} h(i) \in Im([B + DA^N]).$$

We note at this point that $Im([B + DA^N])$ is orthogonal to $\ker([B + DA^N]^T)$. So, we have

$$[DA^N \sum_{i=0}^{N-1} A^{-(i+1)} h(i)]^T \beta = 0$$

$\forall \beta \in \ker([B + DA^N]^T)$.
With some rearranging, we see this is equivalent to

$$\sum_{i=0}^{N-1} h^T(i)[DA^N A^{-(i+1)}]^T c = 0,$$

where c spans $\ker([B + DA^N])$.
Thus, h is in the image of L if and only if $\sum_{i=0}^{N-1} h^T(i)\psi(i) = 0$, or equivalently

$$\sum_{i=0}^{N-1} \psi^T(i) h(i) = 0.$$

$\square$

Armed with this definition, we are ready to create our projection onto the image of L. We define $W : Y \to Y$ by

$$(Wh)(t) = \psi(t) \left(\sum_{i=0}^{N-1} |\psi(i)|^2 \right)^{-1} \sum_{i=0}^{N-1} \psi^T(i) h(i).$$

We can use this definition, in conjunction with the knowledge that if $\ker(B^T) \bigcap \ker(D^T) = 0$, then ψ is not identically the zero map, to officially construct our projection onto the image of L.

Proposition 4.2. Assume $\ker(B^T) \bigcap \ker(D^T) = 0$. Then $E = I - W$ is a projection onto the image of L.

Proof. We must first demonstrate that W is a projection. We can clearly see that W is linear. Thus, since W is a linear map defined on a finite-dimensional vector space, we know that W is bounded. It remains to be shown that $W^2 = W$. So, consider

$$\begin{aligned}
(W(Wh))(t) &= W\left(\psi(\cdot)\left(\sum_{i=0}^{N-1}|\psi(i)|^2\right)^{-1}\sum_{i=0}^{N-1}\psi^T(i)h(i)\right)\\
&= \psi(t)\left(\sum_{k=0}^{N-1}|\psi(k)|^2\right)^{-1}\sum_{k=0}^{N-1}\psi^T(k)\psi(k)\left(\sum_{i=0}^{N-1}|\psi(i)|^2\right)^{-1}\sum_{i=0}^{N-1}\psi^T(i)h(i)\\
&= \psi(t)\left(\sum_{k=0}^{N-1}|\psi(k)|^2\right)^{-1}\sum_{k=0}^{N-1}|\psi(k)|^2\left(\sum_{i=0}^{N-1}|\psi(i)|^2\right)^{-1}\sum_{i=0}^{N-1}\psi^T(i)h(i)\\
&= \psi(t)\left(\sum_{i=0}^{N-1}|\psi(i)|^2\right)^{-1}\sum_{i=0}^{N-1}\psi^T(i)h(i)\\
&= (Wh)(t).
\end{aligned}$$

This shows that W is indeed a projection. Hence, $E = I - W$ must also be a projection. Next, we seek to show that $Im(E) = Im(L)$. Recall that we previously demonstrated that $y \in Im(L)$ when $\sum_{i=0}^{N-1}\psi^T(i)y(i) = 0$. So, let $h \in Y$, and consider

$$\begin{aligned}
\sum_{i=0}^{N-1}\psi^T(i)(Eh)(i) &= \sum_{i=0}^{N-1}\psi^T(i)(h(i)-(Wh)(i))\\
&= \sum_{i=0}^{N-1}\psi^T(i)h(i) - \sum_{i=0}^{N-1}\psi^T(i)(Wh)(i)\\
&= \sum_{i=0}^{N-1}\psi^T(i)h(i) - \sum_{i=0}^{N-1}\psi^T(i)\left(\psi(i)\left(\sum_{k=1}^{N-1}|\psi(k)|^2\right)^{-1}\sum_{k=1}^{N-1}\psi^T(k)h(k)\right)\\
&= \sum_{i=0}^{N-1}\psi^T(i)h(i) - \sum_{i=0}^{N-1}|\psi(i)|^2\left(\sum_{i=0}^{N-1}|\psi(i)|^2\right)^{-1}\sum_{k=1}^{N-1}\psi^T(k)h(k)\\
&= \sum_{i=0}^{N-1}\psi^T(i)h(i) - \sum_{i=0}^{N-1}\psi^T(k)h(k)\\
&= 0
\end{aligned}$$

Thus, $Im(E) \subseteq Im(L)$. Now, let $h \in Im(L)$. Then

$$(Eh)(t) = h(t) - \psi(t)\left(\sum_{i=0}^{N-1}|\psi(i)|^2\right)^{-1}\sum_{i=0}^{N-1}\psi^T(k)h(k) = h(t).$$

Therefore, $Eh = h$, which tells us that $Im(E) = Im(L)$. □

We conclude our analysis of the linear component by noting some important properties of elements of our space X. Since P is a continuous projection, we know for

$x \in X, x = P(x) + (I - P)(x)$. Furthermore, if we denote $X_P = Im(P)$ and $X_{I-P} = Im(I - P)$, we see that $X_P \bigcap X_{I-P} = \{0\}$. Thus, $X = X_P \oplus X_{I-P}$. Since $Im(P) = \ker(L)$, we can see that $L : X_{I-P} \to Im(L)$ is a bijection. This tells us that there exists a bounded linear map $M : Im(L) \to X_{I-P}$ such that

a. $LM(h) = h, \forall h \in Im(L),$

and

b. $ML(x) = (I - P)(x), \forall x \in X.$

4.4 Main Theorem

We are now ready to tackle the main problem. With the tools we have constructed, we can work towards establishing the necessary criteria that guarantee solutions to (4.1)-(4.2). Recall that we have reformulated the problem using operators in the form

$$Lx = F(x).$$

From here, we take our first step towards establishing our conditions with a proposition:

Proposition 4.3. Let $[x]_k$ denote the k^{th} component of the vector x. $Lx = F(x)$ if and only if there exists $\alpha \in \mathbb{R}$ such that $x = \alpha S + MEFx$, and $\sum_{i=0}^{N-1} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1) = 0$.

Proof. Denote x_P, x_{I-P} by $x_P = P(x)$, and $x_{I-P} = (I - P)(x)$. Then, consider the following:

$$\begin{aligned}
Lx = F(x) &\Leftrightarrow E(Lx - F(x)) = 0 \text{ and } (I - E)(Lx - F(x)) = 0 \\
&\Leftrightarrow ELx - F(x) = 0 \text{ and } Lx - ELx - (I - E)F(x) = 0 \\
&\Leftrightarrow Lx - EF(x) = 0 \text{ and } (I - E)F(x) = 0 \\
&\Leftrightarrow L(x_P + x_{I-P}) - EF(x) = 0 \text{ and } F(x) \in Im(L) \\
&\Leftrightarrow L(x_{I-P}) - EF(x) = 0 \text{ and } \sum_{i=0}^{N-1} \psi^T(i)(F(x))(i) = 0 \\
&\Leftrightarrow MLx_{I-P} = MEF(x) \text{ and } \sum_{i=0}^{N-1} \psi^T(i)(F(x))(i) = 0 \\
&\Leftrightarrow x_{I-P} = MEF(x) \text{ and } \sum_{i=0}^{N-1} \psi^T(i)(F(x))(i) = 0 \\
&\Leftrightarrow x = x_P + MEF(x) \text{ and } \sum_{i=0}^{N-1} \psi^T(i) f(x(i)) = 0 \\
&\Leftrightarrow x = \alpha S + MEF(x) \text{ and } \sum_{i=0}^{N-1} \psi^T(i) \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ g(x_1(i)) \end{bmatrix}^T = 0
\end{aligned}$$

for some $\alpha \in \mathbb{R}$. This occurs if and only if

$$x = \alpha S + MEF(x) \text{ and } \sum_{i=0}^{N-1} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1) = 0$$

$\square$

We are nearly ready to establish the final conditions that guarantee the existence of solutions to our problem. Prior to this, we need to introduce a few important concepts. We begin by letting $\lim_{x\to\infty} g(x) = g(\infty)$, and $\lim_{x\to-\infty} g(x) = g(-\infty)$. So, we introduce the following sets: Let

$$\mathcal{U}_0 = \{i \in \{0, 1, ..., N-1\} : [S(i)]_1 = 0\},$$

$$\mathcal{U}_1 = \{i \in \{0, 1, ..., N-1\} : [S(i)]_1 > 0\},$$

$$\mathcal{U}_2 = \{i \in \{0, 1, ..., N-1\} : [S(i)]_1 < 0\}.$$

Next, define H_1, and H_2 by

$$H_1 = g(\infty) \sum_{i\in\mathcal{U}_1} [\psi(i)]_2 + g(-\infty) \sum_{i\in\mathcal{U}_2} [\psi(i)]_2,$$

and

$$H_2 = g(-\infty) \sum_{i\in\mathcal{U}_1} [\psi(i)]_2 + g(\infty) \sum_{i\in\mathcal{U}_2} [\psi(i)]_2.$$

Finally, define R_+, and R_- by

$$R_+ = \sum_{i=0}^{N-1} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1),$$

and

$$R_- = \sum_{i=0}^{N-1} [\psi(i)]_n g([-\alpha S(i) + MEFx(i)]_1).$$

Armed with these definitions, we present the following corollary.

Proposition 4.4. Assume $\mathcal{U}_0 = \emptyset, H_1 \neq 0, H_2 \neq 0$. Also, assume $g(\infty)$, and $g(-\infty)$ exist. Then, there exists $\alpha_0 > 0$ such that for $\alpha \geq \alpha_0$, H_1 has the same sign as R_+, and H_2 has the same sign as R_-.

Proof. Let $\epsilon > 0$. We show that for sufficiently large α, H_1 has the same sign as R_+. The proof for H_2, R_- is similar.

Recall that each of the maps P, S, M, and E are all linear operators on finite-dimensional spaces, and are therefore bounded. Also, F is bounded by assumption. Thus, $[S(i) + MEFx(i)]$ is bounded for each i. Now, since $\mathcal{U}_0$ is empty, we can find $\alpha_0 > 0$ large enough that

$$g(\infty) - \epsilon < g([\alpha S(i) + MEFx(i)]_1) < g(\infty) + \epsilon,$$

for $i \in \mathcal{U}_1$, and

$$g(-\infty) - \epsilon < g([\alpha S(i) + MEFx(i)]_1) < g(-\infty) + \epsilon$$

for $i \in \mathcal{U}_2, \forall \alpha \geq \alpha_0$.

Now, to simplify the problem, we introduce the following spaces:

$$\tau_0 = \{i \in \mathcal{U}_1 : [\psi(i)]_n = 0\},$$

$$\tau_1 = \{i \in \mathcal{U}_1 : [\psi(i)]_n > 0\},$$

$$\tau_2 = \{i \in \mathcal{U}_1 : [\psi(i)]_n < 0\},$$

$$\mathcal{V}_0 = \{i \in \mathcal{U}_2 : [\psi(i)]_n = 0\},$$

$$\mathcal{V}_1 = \{i \in \mathcal{U}_2 : [\psi(i)]_n > 0\},$$

$$\mathcal{V}_2 = \{i \in \mathcal{U}_2 : [\psi(i)]_n < 0\}.$$

So, for $\alpha \geq \alpha_0$, we have

$$\sum_{i \in \mathcal{U}_1} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1)$$

$$= \sum_{i \in \tau_1} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1) + \sum_{i \in \tau_2} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1).$$

So we have

$$(g(\infty) - \epsilon) \sum_{i \in \tau_1} [\psi(i)]_n + (g(\infty) + \epsilon) \sum_{i \in \tau_2} [\psi(i)]_n$$

$$< \sum_{i \in \mathcal{U}_1} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1)$$

$$< (g(\infty) + \epsilon) \sum_{i \in \tau_1} [\psi(i)]_n + (g(\infty) - \epsilon) \sum_{i \in \tau_2} [\psi(i)]_n.$$

Similarly, we have for $\alpha \geq \alpha_0$,

$$\sum_{i \in \mathcal{U}_2} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1)$$

$$= \sum_{i \in \mathcal{V}_1} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1) + \sum_{i \in \mathcal{V}_2} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1).$$

This yields

$$(g(-\infty) - \epsilon) \sum_{i \in \mathcal{V}_1} [\psi(i)]_n + (g(-\infty) + \epsilon) \sum_{i \in \mathcal{V}_2} [\psi(i)]_n$$

$$< \sum_{i \in \mathcal{U}_2} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1)$$

$$< (g(-\infty) + \epsilon) \sum_{i \in \mathcal{V}_1} [\psi(i)]_n + (g(-\infty) - \epsilon) \sum_{i \in \mathcal{V}_2} [\psi(i)]_n.$$

We can combine all of these inequalities to obtain

$$(g(\infty)-\epsilon)\sum_{i\in\tau_1}[\psi(i)]_n+(g(\infty)+\epsilon)\sum_{i\in\tau_2}[\psi(i)]_n+(g(-\infty)-\epsilon)\sum_{i\in\mathcal{V}_1}[\psi(i)]_n+(g(-\infty)+\epsilon)\sum_{i\in\mathcal{V}_2}[\psi(i)]_n$$

$$<\sum_{i\in\mathcal{U}_1}[\psi(i)]_n g([\alpha S(i)+MEFx(i)]_1)+\sum_{i\in\mathcal{U}_2}[\psi(i)]_n g([\alpha S(i)+MEFx(i)]_1)$$

$$<(g(\infty)+\epsilon)\sum_{i\in\tau_1}[\psi(i)]_n+(g(\infty)-\epsilon)\sum_{i\in\tau_2}[\psi(i)]_n+(g(-\infty)+\epsilon)\sum_{i\in\mathcal{V}_1}[\psi(i)]_n+(g(-\infty)-\epsilon)\sum_{i\in\mathcal{V}_2}[\psi(i)]_n$$

After some clever factoring and rearranging, we obtain

$$H_1-\epsilon\left(\sum_{i\in\tau_1}[\psi(i)]_n-\sum_{i\in\tau_2}[\psi(i)]_n+\sum_{i\in\mathcal{V}_1}[\psi(i)]_n-\sum_{i\in\mathcal{V}_2}[\psi(i)]_n\right)<R_+,$$

and

$$R_+<H_1+\epsilon\left(\sum_{i\in\tau_1}[\psi(i)]_n-\sum_{i\in\tau_2}[\psi(i)]_n+\sum_{i\in\mathcal{V}_1}[\psi(i)]_n-\sum_{i\in\mathcal{V}_2}[\psi(i)]_n\right).$$

This tells us that

$$R_+\in\left(H_1-\epsilon\left(\sum_{i=0}^{N-1}|[\psi(i)]_n|\right),H_1+\epsilon\left(\sum_{i=0}^{N-1}|[\psi(i)]_n|\right)\right).$$

Now, since $H_1\neq 0$, we know there exists some $i\in\{0,1,...,N-1\}$ such that $[\psi(i)]_n\neq 0$. So, our interval described above is non-empty. Since ϵ was chosen arbitrarily, then there is an $\alpha_0>0$ such that H_1 and R_+ have the same sign for all $\alpha\geq\alpha_0$. A similar argument demonstrates the result for H_2 and R_- ☐

We now state our main theorem, and as before, use two different arguments to prove the result.

Theorem 4.1. Assume the kernel of $(B+DA^N)$ is one-dimensional, and that $\ker(B^T)\bigcap\ker(D^T)=\{0\}$. If
a.) g is a continuous, bounded function;
b.) $g(\infty)$ and $g(-\infty)$ exist;
c.) $U_0=\emptyset$;
d.) and $H_1H_2<0$,
then there is at least one solution to (4.1)-(4.2).

Proof (Method 1)**.** Without loss of generality, suppose $H_1>H_2$. As from the last proposition, let α_0 be sufficiently large that for $\alpha>\alpha_0$, H_1 and R_+ have the same sign, and H_2, R_- have the same sign. Additionally, let $\alpha_0>mN\|\psi\|$, where $m=\sup\limits_{t\in\mathbb{R}}|g(t)|$. We seek to utilize the Brouwer fixed point theorem to demonstrate our result. To that end, we introduce the following maps:
Let $G_1:X\times\mathbb{R}\to X$ be defined by

$$G_1(x,\alpha)=\alpha S+MEFx.$$

Let $G_2 : X \times \mathbb{R} \to \mathbb{R}$ be defined by

$$G_2(x, \alpha) = \alpha - \sum_{i=0}^{N-1} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1),$$

and let $G : X \times \mathbb{R} \to X \times \mathbb{R}$ be defined by

$$G(x, \alpha) = (G_1(x, \alpha), G_2(x, \alpha)).$$

Our goal is to construct a nonempty, closed, convex set $B \subseteq X \times \mathbb{R}$ such that $G(B) \subseteq B$. To that end, we construct the following set:

$$B = \{(x, \alpha) : \|x\| \leq \delta\|S\| + m\|ME\|, \text{ and } |\alpha| \leq \delta, \text{where } \delta = \alpha_0 + mN\|\psi\|\}.$$

Now, we know $H_1 H_2 < 0$, and $H_1 > H_2$. So, for $\alpha \geq \alpha_0$ we know

$$R_+ = \sum_{i=0}^{N-1} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1) > 0,$$

and

$$R_- = \sum_{i=0}^{N-1} [\psi(i)]_n g([-\alpha S(i) + MEFx(i)]_1) < 0.$$

So, if $\alpha_0 \leq \alpha \leq \delta$,

$$G_2(x, \alpha) = \alpha - \sum_{i=0}^{N-1} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1) \leq \alpha,$$

and

$$G_2(x, -\alpha) = -\alpha - \sum_{i=0}^{N-1} [\psi(i)]_n g([-\alpha S(i) + MEFx(i)]_1) \geq -\alpha.$$

Now, we know

$$|\sum_{i=0}^{N-1} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1)| \leq mN\|\psi\|.$$

So, since $\alpha \geq \alpha_0 > mN\|\psi\|$,

$$G_2(x, \alpha) = \alpha - \sum_{i=0}^{N-1} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1) \geq 0.$$

Similarly, $G_2(x, -\alpha) \leq 0$. So, for any x, and any $\alpha \in [\alpha_0, \delta]$, (or $-\alpha \in [-\delta, -\alpha_0]$) $G_2(x, \alpha) \in [-\delta, \delta]$.

Now, consider $\alpha \in [0, \alpha_0)$. So we have

$$\begin{aligned}
|G_2(x,\alpha)| &= |\alpha - \sum_{i=0}^{N-1} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1)| \\
&\le |\alpha| + mN\|\psi\| \\
&< \alpha_0 + mN\|\psi\| \\
&= \delta.
\end{aligned}$$

Similarly, $|G_2(x, -\alpha)| < \delta$. Thus, for any x, and any $\alpha \in [0, \alpha_0)$, (or $-\alpha \in (-\alpha_0, 0]$), $G_2(x,\alpha) \in [-\delta, \delta]$. So, for any (x, α) in B, $G_2(x, \alpha)$ is in $[-\delta, \delta]$.
Now, let $(x, \alpha) \in B$. We have

$$\|G_1(x,\alpha)\| = \|\alpha S + MEFx\| \le \delta\|S\| + m\|ME\|.$$

So, for $(x, \alpha) \in B$, $G_1(x, \alpha) \in B$.
Therefore, for any $(x, \alpha) \in B$, $G(x, \alpha) \in B$. Now, since G is continuous, by the Brouwer fixed point theorem, G has at least one fixed point in B. That is, if $(\hat{x}, \hat{\alpha})$ is a fixed point in B, $G(\hat{x}, \hat{\alpha}) = (\hat{x}, \hat{\alpha})$. This tells us that

$$\hat{x} = \hat{\alpha} S + MEF\hat{x},$$

and

$$\sum_{i=0}^{N-1} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1) = 0.$$

So, by proposition 4.3, we know $L\hat{x} = F(\hat{x})$.
Thus, (4.1)-(4.2) has at least one solution. □

Proof (Method 2)**.** In this proof, we will utilize a homotopy argument to demonstrate our main result. We begin with the same assumptions as those in method 1. And, as before, we assume $H_1 > H_2$. We utilize a slightly modified set

$$B = \{(x,\alpha) : \|x\| \le \delta\|S\| + m\|ME\|, \text{and} |\alpha| \le \delta, \text{where } \delta = \alpha_0 + mN\|\psi\|\},$$

and modify our function G in the following way:
Let $G_1 : X \times \mathbb{R} \to X$ be defined by

$$G_1(x, \alpha) = x - \alpha S + MEFx.$$

Let $G_2 : X \times \mathbb{R} \to \mathbb{R}$ be defined by

$$G_2(x,\alpha) = \sum_{i=0}^{N-1} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1),$$

and let $G : X \times \mathbb{R} \to X \times \mathbb{R}$ be defined by

$$G(x, \alpha) = (G_1(x, \alpha), G_2(x, \alpha)).$$

For a homotopy argument we need to find an (x, α) in B such that $G(x, \alpha) = (0, 0)$. This would then show that

$$\alpha S + MEFx = x,$$

and

$$\sum_{i=0}^{N-1} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1 = 0).$$

From here, we define $\phi : B \to B$ by

$$\phi(x, \alpha) = (x, \alpha).$$

We note that the degree of this map is 1.

We now work towards creating our homotopy map. We first define $T_X : B \times \mathbb{R} \times [0, 1] \to X$ by

$$T_X(x, \alpha, \lambda) = x - \lambda[\alpha S + MEFx],$$

then define $T_{\mathbb{R}} : B \times \mathbb{R} \times [0, 1] \to \mathbb{R}$ by

$$T_{\mathbb{R}}(x, \alpha, \lambda) = (1 - \lambda)\alpha + \lambda \sum_{i=0}^{N-1} [\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1),$$

and let $T : B \times \mathbb{R} \times [0, 1] \to B \times \mathbb{R}$ by

$$T(x, \alpha, \lambda) = (T_1(x, \alpha, \lambda), T_2(x, \alpha, \lambda)).$$

We note that $T(x, \alpha, 0) = (x, \alpha) = \phi(x, \alpha)$, and that $T(x, \alpha, 1) = G(x, \alpha)$. As with previous homotopy arguments throughout this article, we introduce the following notation: Let $T_\lambda(x, \alpha) = T(x, \alpha, \lambda)$.

We seek to show that

$$d[\phi, \overline{B}, \vec{0}] = d[G, \overline{B}, \vec{0}].$$

To accomplish this, we need to demonstrate that $T(x, \alpha, \lambda) \neq (0, 0)$ for all (x, α) in ∂B, and for all λ in $[0, 1]$. As before, we examine two cases:

Case 1: $\lambda = 1,\ (x, \alpha) \in \partial B$

If $\lambda = 1$, we know $T(x, \alpha, 1) = G(x, \alpha)$. If $T(x, \alpha, 1) = (0, 0)$, then $G(x, \alpha) = (0, 0)$, and we will have established the existence of our solution.

Case 2: $\lambda \in [0, 1),\ (x, \alpha) \in \partial B.$

Now, we know that since $(x, \alpha) \in \partial B$, $\|x\| = \delta\|S\| + m\|ME\|$, and $|\alpha| = \delta = \alpha_0 +$

$mN\|\psi\|$. Additionally, we recall that $\|Fx\| = \sup_{t\in\{0,1,\ldots,N-1\}} |f(x(t))| \leq m$. So, consider

$$\begin{aligned}
\lambda\|\alpha S + MEFx\| &< \|\alpha S + MEFx\| \\
&\leq \|\alpha S\| + \|MEFx\| \\
&\leq \delta\|S\| + m\|ME\| \\
&= \|x\|.
\end{aligned}$$

From here we see that

$$\begin{aligned}
\|T_X(x,\alpha,\lambda)\| &= \|x - \lambda[\alpha S + MEFx]\| \\
&\geq \|x\| - \lambda\|\alpha S + MEFx\| \\
&> 0.
\end{aligned}$$

Also, just as in method 1 of our proof, we know there exists an $\alpha_0 > 0$ such that for $\alpha \geq \alpha_0$, $R_+ > 0$, and $R_- < 0$. We can then see that for $\lambda \in [0,1)$,

$$T_{\mathbb{R}}(x,\delta,\lambda) = (1-\lambda)\delta + \lambda\sum_{i=0}^{N-1}[\psi(i)]_n g([\delta S(i) + MEFx(i)]_1) > 0,$$

and

$$T_{\mathbb{R}}(x,-\delta,\lambda) = (1-\lambda)(-\delta) + \lambda\sum_{i=0}^{N-1}[\psi(i)]_n g([-\delta S(i) + MEFx(i)]_1) < 0.$$

We note that in the case where $H_1 < H_2$, we would need a slightly modified homotopy. In this instance, we would modify $T_{\mathbb{R}}$ by letting

$$T_{\mathbb{R}}(x,\alpha,\lambda) = (1-\lambda)\alpha - \lambda\sum_{i=0}^{N-1}[\psi(i)]_n g([\alpha S(i) + MEFx(i)]_1).$$

With this modification, the proof will be analogous to that found above.
Thus, $T_\lambda(x,\alpha) = T(x,\alpha,\lambda) \neq (0,0)$ for all (x,α) in ∂B, and for all λ in $[0,1]$. This tells us that

$$d[G,\overline{B},\vec{0}] = d[T_1,\overline{B},\vec{0}] = d[T_0,\overline{B},\vec{0}] = d[\phi,\overline{B},\vec{0}].$$

Therefore, there exists an $(x,\alpha) \in B$ such that $G(x,\alpha) = (0,0)$. Hence, we have $Lx = F(x)$, and, as in method 1 of the proof, we achieve the desired result. $\square$

5 Example

We now introduce a concrete example to illustrate the utility of the tools we have constructed. Consider

$$y(t+2) + 2y(t+1) + y(t) = g(y(t)), \tag{5.1}$$

for $t \in \{0, 1, ...N-1\}$, subject to

$$y(0) + y(1) = 0, \tag{5.2}$$

and

$$y(N) + y(N+1) = 0, \tag{5.3}$$

and where

$$g(x) = \begin{cases} \frac{2}{\pi}\arctan(x) + 1 & x \geq 0 \\ -\frac{4}{\pi}\arctan(x) + 1 & x < 0. \end{cases}$$

For this problem, we assume that N is some odd integer greater than six.

First, we convert this scalar problem into a two-point, discrete system of the form

$$x(t+1) = Ax(t) + f(x(t)),\ t \in \{0, 1, ...N-1\} \tag{5.4}$$

subject to

$$Bx(0) + Dx(N) = 0, \tag{5.5}$$

by letting

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix},$$

and letting

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \text{and } D = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}.$$

Additionally, we have

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y(t+1) \end{bmatrix},$$

and

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ g(x_1) \end{bmatrix}.$$

With this reformulation, we begin checking all of the conditions that were discussed in section 4. First we consider

$$\ker(B^T) = \ker\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\},$$

and

$$\ker(D^T) = \ker\left(\begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}.$$

So, clearly $\ker B^T \bigcap \ker D^T = \{0\}$.
Next, we use Maple to see that

$$A^N = \begin{bmatrix} (-1)^N - (-1)^N N & (-1)^{N+1} N \\ (-1)^N N & (-1)^N + (-1)^N N \end{bmatrix}.$$

From here, we find that

$$(B + DA^N) = \begin{bmatrix} 1 & 1 \\ (-1)^{N+1} & (-1)^{N+1} \end{bmatrix},$$

so

$$\ker\left\{(B + DA^N)\right\} = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}.$$

This also tells us that the kernel of $(B + DA^N)$ is one-dimensional.
Next, we have

$$S(t) = A^t \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} (-1)^{t+1} \\ (-1)^t \end{bmatrix},$$

for all $t \in \{0, 1, ..., N\}$. This allows us to build the following important sets:

$$\mathcal{U}_0 = \{i : [S(i)]_1 = 0\} = \emptyset;$$

$$\mathcal{U}_1 = \{i : [S(i)]_1 > 0\} = \{1, 3, ..., N-2\};$$

$$\mathcal{U}_2 = \{i : [S(i)]_1 < 0\} = \{0, 2, ..., N-1\}.$$

Next, we consider

$$(B + DA^N)^T = \begin{bmatrix} 1 & (-1)^{N+1} \\ 1 & (-1)^{N+1} \end{bmatrix},$$

which means

$$\ker\{(B + DA^N)^T\} = \text{span}\left\{\begin{bmatrix} (-1)^N \\ 1 \end{bmatrix}\right\}.$$

From here, we define

$$\psi(t) = (DA^{N-t-1})^T \begin{bmatrix} (-1)^N \\ 1 \end{bmatrix} = \begin{bmatrix} (-1)^{N-t} \\ (-1)^{N-t} \end{bmatrix},$$

for all $t \in \{0, 1, ..., N-1\}$. Now, we know

$$\lim_{x \to \infty} g(x) = g(\infty) = 2,$$

and

$$\lim_{x \to -\infty} g(x) = g(-\infty) = 3.$$

From here we use ψ and these limits to check that $H_1 H_2 < 0$. So,

$$\begin{aligned} \sum_{i \in \mathcal{U}_1} [\psi(i)]_2 &= \sum_{\substack{i \in \mathcal{U}_1 \\ i=1}}^{N-2} (-1)^{N-i} \\ &= (-1)^{N-1} + (-1)^{N-3} + \cdots + (-1)^2 \\ &= \frac{N-1}{2}. \end{aligned}$$

Likewise,

$$\begin{aligned} \sum_{i \in \mathcal{U}_2} [\psi(i)]_2 &= \sum_{\substack{i \in \mathcal{U}_2 \\ i=0}}^{N-1} (-1)^{N-i} \\ &= (-1)^{N} + (-1)^{N-2} + \cdots + (-1)^1 \\ &= -\frac{N+1}{2}. \end{aligned}$$

So we have

$$\begin{aligned} H_1 &= g(\infty) \sum_{i \in \mathcal{U}_1} [\psi(i)]_2 + g(-\infty) \sum_{i \in \mathcal{U}_2} [\psi(i)]_2 \\ &= 2\left(\frac{N-1}{2}\right) - 3\left(\frac{N+1}{2}\right) \\ &= \frac{2N - 2 - 3N - 3}{2} \\ &= \frac{-N-5}{2} \\ &< 0, \text{when } \; N \geq 3. \end{aligned}$$

Similarly,

$$\begin{aligned}
H_2 &= g(-\infty)\sum_{i\in\mathcal{U}_1}[\psi(i)]_2 + g(\infty)\sum_{i\in\mathcal{U}_2}[\psi(i)]_2 \\
&= 3\left(\frac{N-1}{2}\right) - 2\left(\frac{N+1}{2}\right) \\
&= \frac{3N-3-2N-2}{2} \\
&= \frac{N-5}{2} \\
&> 0, \text{when } \ N \geq 7.
\end{aligned}$$

Thus, $H_1 H_2 < 0$. Therefore, we have met all of the necessary criteria to guarantee the existence of a solution to (5.1)-(5.3).

6 References

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