METRIC PRESERVING FUNCTIONS

by

Klotilda Lazaj

Submitted in Partial Fulfillment of the Requirements

for the Degree of

Master of Science

in the

Mathematics

Program

YOUNGSTOWN STATE UNIVERSITY

[August 2009]

Metric Preserving Functions

Klotilda Lazaj

I hereby release this thesis to the public. I understand that this thesis will be made available from the OhioLINK ETD Center and the Maag Library Circulation Desk for public access. I also authorize the University or other individuals to make copies of this thesis as needed for scholarly research.

Signature:

Klotilda Lazaj, Student   Date

Approvals:

Zbigniew Piotrowski, Thesis Advisor   Date

Eric Wingler, Committee Member   Date

Jamal Tartir, Committee Member   Date

Peter J. Kasvinsky, Dean of School of Graduate Studies & Research   Date

Abstract. *The primary topic of this paper is distance (or "metric") preserving functions. In particular, the paper will focus on the least integer function - a step function, also referred to as the ceiling function. Herein, the author will provide information about the ceiling function, as well as a proof that it is indeed metric preserving, supported by Wilson's Theorem and the Borsik-Doboš Theorem. In addition, the paper will show that the amenable condition and triangle triplet condition guarantee that a function is distance preserving.*

Symbols

$\forall$	= for all
$\exists$	= there exists
$\Rightarrow$	= implies
$\Leftrightarrow$	= one side implies the other
$\lceil \cdot \rceil$	= the least integer function
$\mathrm{d}(x, y)$	= the distance from x to y, or the $\lvert x - y \rvert$
(a, b, c)	= the point with coordinates a, b, and c
$B_O(x, r)$	= the open ball centered at x with radius r
$B_C(x, r)$	= the closed ball centered at x with radius 0
(X, d)	= metric space

In the textbook "Metric Spaces" (*Sierpinski*, 9), Sierpinski considers the following problem:

> *"Does there exist an infinite subset of the plane such that the distance between any two different points is a natural number, and this set is* ***not*** *colinear (i.e., contained in a common straight line)?"*

It was shown that such a set does not exist. (*Sierpinski*, 59-60) However, for every natural number n, there is a set of n noncollinear points in the plane such that all of its distances are natural numbers. All these considerations were done assuming the usual metric on the plane.

One can "reverse" the question and ask:

> *"Does there exist a metric on $\Re^k$ such that the distance between any two different points is a natural number?"*

The obvious examples of such metrics are discrete metrics and their combinations. Sierpinski considered *non-trivial* cases - namely, functions which are composed of the values of the original metrics, quantifying the "*old metric*".

Example 1: You are in New York City, and just got a cab - you get in. The meter shows \$1.50. This is a flat rate as the driver says. Now every $\frac{1}{8}$ of a mile will cost you \$0.25. The graph is shown below: ($Dobo\check{s}$ and $Piotrowski$, 513)

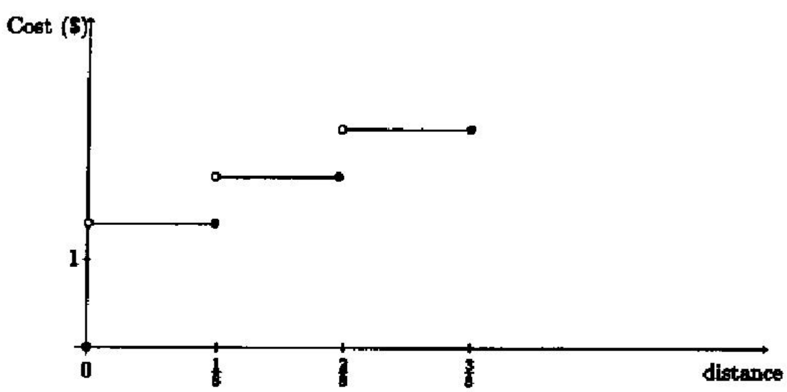

Figure 1

Example 2: Have you called your friend in Paris from New York? The current AT&T rate is \$1.71 for the first minute, and \$1.08 for every additional minute. The graph is illustrated below: ($Dobo\check{s}$ and $Piotrowski, 513$)

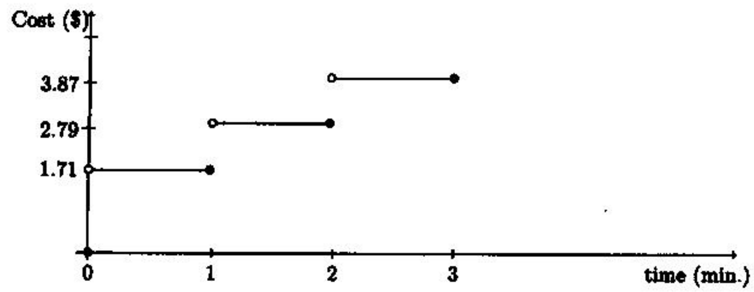

Figure 2

By replacing the distance (in miles) on the x-axis, and the cost (in \$) on the y-axis, then the function $f{:}X \rightarrow Y$ is a **step function**.

One may notice a common theme with these two examples: that the variables distance and time, respectively, may be stated in terms of cost. Moreover, each function presented is a **step function** (i.e., f only assumes discrete values). Without loss of generality, one may assume that the values of f are **natural numbers**.

Definition 1: Let X be a nonempty set. We say that the function $d : X \times X \rightarrow \mathbf{R}^{+} \cup \{0\}$ is a *distance function*, or a *metric*, if the following axioms are met for any x, y, and z in X:

Axiom 1: $d(x, y) = 0$ *if and only if* $x = y$,
Axiom 2: $d(x, y) = d(y, x)$,
Axiom 3: $d(x, y) \leq d(x, z) + d(z, y)$ *(the triangle inequality).*

The pair (X, d) is called a *metric space*.

Given a distance d, we shall refer to d as "the old metric", or "the old distance". We will consider these functions $f : [0, +\infty) \to [0, +\infty)$ such that the composition e is defined by:

$$e(x, y) = f(d(x, y)),$$

which is the "new distance".

So, we will quantify the values of the old metrics by "shuffling" the in-between numbers to nearby integer numbers. This, in turn, makes f a step function.

In terms of our previous two examples, the *distance* (*Example* 1) and *time* (*Example* 2) would be our "old metric", d, such that when the appropriate function, f, composed with d, the "new metric", e - in terms of cost - results.

Lemma 1. Let $f : [0, +\infty) \to [0, +\infty)$ be defined by $f(u) = \lceil u \rceil$, where $\lceil \cdot \rceil$ denotes the least integer function. Then f preserves distances.

Proof:

Axiom 1: $e(x, y) = 0 \Leftrightarrow x = y$.
$e(x, y) = \lceil d(x, y) \rceil = 0 \Leftrightarrow d(x, y) = 0$.
Since d is a metric, then $d(x, y) = 0 \Leftrightarrow x = y$.
Thus, $e(x, y) = 0 \Leftrightarrow x = y$.

Axiom 2: $e(x, y) = e(y, x)$.
Since d is a metric, then
$d(x, y) = d(y, x) \Leftrightarrow \lceil d(x, y) \rceil = \lceil d(y, x) \rceil$.
Clearly,
$e(x, y) = \lceil d(x, y) \rceil = \lceil d(y, x) \rceil = e(y, x)$.
Thus, $e(x, y) = e(y, x)$.

Axiom 3: $e(x, y) \leq e(x, z) + e(z, y)$.
Clearly, *
$d(x, z) \leq \lceil d(x, z) \rceil$ and $d(z, y) \leq \lceil d(z, y) \rceil$.
Adding both inequalities yields:
$d(x, z) + d(z, y) \leq \lceil d(x, z) \rceil + \lceil d(z, y) \rceil$.
Thus,
$\lceil d(x, z) + d(z, y) \rceil \leq \lceil \lceil d(x, z) \rceil + \lceil d(z, y) \rceil \rceil$,
which implies that
$\lceil d(x, z) + d(z, y) \leq \lceil d(x, z) + \lceil d(z, y) \rceil$.
Now, since d is a metric, then
$d(x, y) \leq d(x, z) + d(z, y)$.

Applying the least integer function gives us:
$\lceil d(x,y) \rceil \leq \lceil d(x,z) + d(z,y) \rceil \overset{*}{\leq} \lceil d(x,z) \rceil + \lceil d(z,y) \rceil$,
which implies:
$\lceil d(x,y) \rceil \leq \lceil d(x,z) \rceil + \lceil d(z,y) \rceil$.
Therefore, $e(x,y) \leq e(x,z) + e(z,y)$.
Hence, it follows that e is a metric.

It is easy to see that the metric e which quantifies the "old" distance is a *majorizing* metric (i.e., if d the original metric for e, then for any two points $x, y \in \Re$:

$$e(x,y) \geq d(x,y). \ \square$$

It is also clear that $(\Re, e)$ is not *isometric* to $(\Re, \epsilon)$, since e takes only integer values, and the Euclidean metric takes non-negative real values.

Open and Closed Balls

In $(\Re, e)$, consider open balls $B_O(x,r)$ and closed balls $B_C(x,r)$.
In general,

$$\text{card } B_O(x,r) \neq \text{ card } B_C(x,r)$$

because it is possible for card $B_C(x,r) = \mathbf{C}$, whereas card $B_O(x,r) = 1$.

Completeness

$(\Re, e)$ is a *complete metric space*, that is if a sequence $\{x_n\}$ satisfies Cauchy's condition, then $\{x_n\}$ is convergent.

Proof:

Claim: the only sequences $\{x_n\}$ satisfies Cauchy's condition are sequences that are eventually constant. In fact, assume that $\{x_n\}$ satisfies the Cauchy condition. Choose $\epsilon > 0$, such that $0 < \epsilon < n$. Then there exist K_0 such that for any x_l, x_m with $l, m \geq K_0$ we have $e(x_1, x_m) < \epsilon$. But, if $x_1 \neq x_m$ then $\inf_{x_m, x_1 \in \Re}(e(x_1, x_m)) = n$; hence, it follows from the inequality $n > \epsilon$ that $x_1 = x_n$. Therefore, the sequence $\{x_k\}$ is constant, for $k \geq k_0$. Thus, if a sequence satisfies Cauchy's conditions, then this sequence is eventually constant, hence convergent. $\square$
$(Piotrowski, 6)$

Equivalent Metrics

Definition: Two metrics d_1 and d_2 on a set X are equivalent if the following conditions hold:

(1) $\forall\ x \in X$ and $\forall\ r_1 > 0$, $\exists\ r_2 > 0$ such that $B_2(x, r_2) \subset B_1(x, r_1)$, and
(2) $\forall\ x \in X$ and $\forall\ r_2 > 0$, $\exists\ r_1 > 0$ such that $B_1(x, r_1) \subset B_2(x, r_2)$ where B_i is a ball in the space (X, d_i) $i = 1, 2, \ldots$.

Problem: Let $X = R^2$ and let d_1, d_2, and d_3 respectively denote:

(1) $d_1(x, y) = |y_1 - x_1| + |y_2 - x_2|$
(2) $d_2(x, y) = \max\{|y_1 - x_1|, |y_2 - x_2|\}$
(3) The discrete metric.

Are d_1, d_2, d_3 equivalent metrics?

Solution: $d_2(x, y) = \max\{|y_1 - x_1|, |y_2 - x_2|\} \le |y_1 - x_1| + |y_2 - x_2| = d_1(x, y)$.

Clearly, $B_1(x, r) \subset B_2(x, r)$
$B_1(0, r) \subset B_2(0, r)$
$x \in B_1(0, r) \Rightarrow d_1(x, 0) < r$
$\Rightarrow d_2(x, 0) \le d_1(x, 0) < r$
$\Rightarrow x \in B_2(0, r)$.

Next, $d_1(x, y) = |y_1 - x_1| + |y_2 - x_2| \le 2\max\{|y_1 - x_1|, |y_2 - x_2|\} = 2d_2(x, y)$
$x \in B_2(0, r) \Rightarrow d_2(x, 0) < r$
$\Rightarrow d_1(x, 0) \le 2d_2(x, 0) < 2r$
$\Rightarrow x \in B_1(0, 2r)$

Thus, d_1 and d_2 are equivalent metrics.

For any $r > 0$, $\frac{r}{2} \in B_1(0, r)$ and $d_3(\frac{r}{2}, 0) = 1 > \frac{1}{2}$. So $\frac{r}{2} \notin B_3\left(0, \frac{1}{2}\right)$.

Therefore, d_1 and d_3 are not equivalent metrics.

Theorem 1. Let f be a real-valued function defined for non-negative numbers, and such that f is continuous, (the continuity of f is needed only for the equivalence of d with e), non-decreasing, and satisfying the following two conditions (*Kelley*, 131):

(1.) $f(a) = 0 \Leftrightarrow a = 0$
(2.) $f(a + b) \le f(a) + f(b)$ (subadditivity).

Let (Z, d) be a metric space and let $e(x, y) = f(d(x, y))$ for all $x, y \in X$.
Then (X, e) is a metric space and the metrics d and e are equivalent. (*Kelley*, 131)

Theorem 2. Suppose $f : [0, +\infty) \to [0, +\infty)$ is non-decreasing and subadditive. Then f is metric preserving. $(Dobo\check{s}, 10)$

Clearly, the function $f : [0, +\infty) \to [0, +\infty)$ given by $f(u) = \lceil u \rceil$ is non-decreasing, and (1) and (2) also hold for f and any non-negative numbers a and b.

Proof: Let (X, d) be a metric space; we show that $f \circ d$ is a metric. *Axioms* 1 and 2 (of *Definition* 1) are easy to check. For *Axiom* 3, let $x, y, z \in X$, and let

$$a = d(x, z),\ b = d(z, y),\ \text{and } c = d(x, y).$$

it suffices to show that $f(a) + f(b) \geq f(c)$. But

$$\begin{aligned} f(a) + f(b) &\geq f(a+b) && (subadditive) \\ &\geq f(c) && (nondecreasing)\ \square \end{aligned}$$

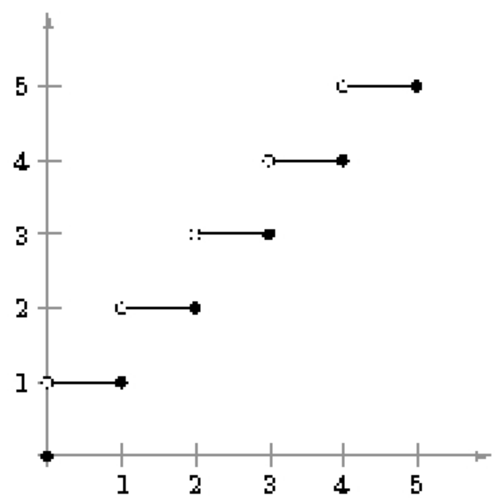

Figure 1: $f(x) = \lceil x \rceil$, $x \geq 0$

Examples of Theorem 2:

(1.) Let a be a number between any two integers x and $x + 1$ on the positive x-axis such that $x < a < x + 1$. By the ceiling function $f(x) = \lceil x \rceil < f(a) = \lceil a \rceil = \lceil x + 1 \rceil = f(x + 1)$. Thus, f is non-decreasing. Also, from the graph above, we see that $f(1) = 1 < f(1.5) = 2 = f(2)$.

(2.) Now, $f(a + b) = \lceil a + b \rceil \overset{*}{\leq} \lceil a \rceil + \lceil b \rceil = f(a) + f(b)$.
Thus, f is subadditive.
Since f is both non-decreasing and subadditive, then it is metric preserving.

A metric preserving function satisfies the following axioms:

(F1) $f : [0, +\infty) \to [0, +\infty)$

(F2) f is non-decreasing

(F3) $f(a) = 0 \Leftrightarrow a = 0$

(F4) $f(a + b) \le f(a) + f(b)$ *(subadditivity)* $(Doboš, 10)$

Theorem 3: If (X, d) is a metric space, and the function f satisfies *(F1)-(F4)*, and the function e satisfies the condition $e(x, y) = f(d(x, y))$ for all $x, y \in X$, then (X, e) is a metric space.

Proof: We shall now check the conditions *Axiom 1 - Axiom 3* for the function e.

Axiom 1. $e(x, y) = 0 \overset{def.e}{\Longleftrightarrow} f(d(x, y)) = 0 \overset{F3}{\Longleftrightarrow} d(x, y) = 0 \overset{Axiom1}{\Longleftrightarrow} x = y$.

Axiom 2. $\forall\, x, y \in X,\ e(x, y) = f(d(x, y)) = f(d(y, x)) = e(y, x)$.

Axiom 3.
$$
\begin{aligned}
\forall\, x, y \in X,\ e(x, y) + e(y, z) &= f(d(x, y)) + f(d(y, z)) && \textit{(by def. e)}\\
&\ge f(d(x, y) + d(y, z)) && \textit{(by (F4)}\\
&\ge f(d(x, z)) && \textit{(by (F2) \& Axiom 3)}\\
&= e(x, z) && \textit{(by def. e)}
\end{aligned}
$$

Based on the theorem above we shall prove that e is a metric. In order to do so, we are to check that the function f such that $f(a) = \lceil a \rceil$ where $a \in [0, +\infty)$ satisfies conditions *(F1)-(F4)*.

Condition $(F1)$ follows from the definition.

Condition $(F2)$. Let $a < b$. Clearly, $a < b \le \lceil b \rceil \Rightarrow \lceil a \rceil \le \lceil b \rceil$.

Condition $(F3)$. Since $f(a) = \lceil a \rceil$ then $f(a) = 0 \Leftrightarrow \lceil a \rceil = 0$.
Then $\lceil a \rceil = 0 = \lceil 0 \rceil \Leftrightarrow \lceil a \rceil = \lceil 0 \rceil \Leftrightarrow a = 0$.

Condition $(F4)$. $f(a + b) = \lceil a + b \rceil \le \lceil a \rceil + \lceil b \rceil = f(a) + f(b)$.
Thus $f(a + b) \le f(a) + f(b)$. $\square$

Definition 2: Let a, b, and c be positive real numbers.
We call the triplet a *triangle triplet* iff

$a \le b + c$, $b \le a + c$, and $c \le a + b$;

Equivalently,

$|a + b| \le c \le a + b$;

i.e.,

$a + b + c \ge 2\max\{a, b, c\}$.

Wilson's Theorem

Let $f : [0, +\infty) \to [0, +\infty)$ be such that $a, b, c \ge 0$, and $a \le b + c$ imply $f(a) \le f(b) + f(c)$. Then f is metric preserving. $(Doboš, 10)$

Example of Wilson's Theorem:

Given $f(x) = 2^x - 1$ does this function satisfiy the hypothesis of Wilson's Theorem?

Solution: Let $(a, b, c) = (2, 3, 4)$, and clearly $4 \leq 2 + 3 = 5$.
Knowing that $f(2) = 3$, $f(3) = 7$, and $f(4) = 15$,
we can check and see that:
$f(4) = 15 > f(2) + f(3) = 3 + 7 = 10$.
Also, let $(a, b, c) = (3, 2, 1)$, then $(f(a), f(b), f(c)) = (7, 3, 1)$.
Thus, $f(a) = 7 > f(b) + f(c) = 3 + 1 = 4$.
Evidently, $7 \nleq 4$. Therefore, $f(x) = 2^x - 1$ does *not* satisfy the hypothesis of Wilson's Theorem.

Borsik-Doboš Theorem Let f: $[0, +\infty) \rightarrow [0, +\infty)$ Then the following are equivalent:

(1.) f is metric preserving
(2.) If (a, b, c) is a triangle triplet then so is $(f(a), f(b), f(c))$
(3.) If (a, b, c) is a triangle triplet then $f(a) \leq f(b) + f(c)$, and
(4.) $\forall x, y \in [0, +\infty) : max\{f(z) : |x - y| \leq z \leq x + y\} \leq f(x) + f(y)$. $(Dobo\check{s}, 11)$

Example 3: Let $f(x) = 2^x - 1$.

(1.) f is not metric preserving
(2.) $(1, 2, 3)$ is a triangle triplet since $1 \leq 2 + 3$, $2 \leq 3 + 1$, $3 \leq 2 + 1$, but $(f(1), f(2), f(3))$ is not a triangle triplet since $7 > 3 + 1 = 4$.
(3.) $(1, 2, 3)$ is a triangle triplet, but $f(3) = 7 > f(2) + f(1) = 3 + 1 = 4$
(4.) Consider $x = 4$, $y = 6$; then $max\{f(z) : |4 - 6| \leq z \leq 4 + 6\} \leq f(4) + f(6)$
$\max\{f(10) : 2 \leq z \leq 10\} > f(4) + f(6)$.
So, $f(10) = 1024 > f(4) + f(6) = 16 + 64 = 80$

Theorem 4. Let f:$[0, +\infty) \rightarrow [0, +\infty)$.
Then f is distance preserving if and only if:
(a) f vanishes exactly at the origin, and
(b) if (a, b, c) is a triangle triplet, then so is $(f(a), f(b), f(c))$.
$(Dobo\check{s}$ and $Piotrowski, 517)$

Example 4: Let the function h be given by:

$$h(x) = \begin{cases} 0 & \text{if } x = 0, \\ 3 & \text{if } 0 < x \leq 1, \\ 1 & \text{if } x > 1. \end{cases}$$

h is not a distance-preserving. In fact, $(1, 2, 3)$ is a triangle triplet, while $((f(1), f(2), f(3))$ is not:

$$3 = f(1) > f(2) + f(3) = 1 + 1 = 2.$$

References:

(1.) Sierpinski, W., "Metric Spaces", Warsaw (1954).

(2.) Kelley, J., *General Topology*, Springer-Verlag, New York (1955).

(3.) Doboš, J., "Metric Preserving Functions".

(4.) Doboš, J. and Piotrowski, Z., "When Distance Means Money", *Internat. J. Math. Ed. Sci. Tech.* 28 (1997) 513518.

(5.) Piotrowski, Z., O Metrykach Calkowito-Liczbowych" (1974).