Analytic Number Theory and the Prime Number Theorem

by

Dan Buchanan

Submitted in Partial Fulfillment of the Requirements

for the Degree of

Master of Science

in the

Mathematics Program

YOUNGSTOWN STATE UNIVERSITY
May, 2018

Analytic Number Theory and the Prime Number Theorem

Dan Buchanan

I hereby release this thesis to the public. I understand that this thesis will be made available from the OhioLINK ETD Center and the Maag Library Circulation Desk for public access. I also authorize the University or other individuals to make copies of this thesis as needed for scholarly research.

Signature:

Dan Buchanan, Student Date

Approvals:

Dr. Eric Wingler, Thesis Advisor Date

Dr. Thomas Smotzer, Committee Member Date

Dr. Padraic Taylor, Committee Member Date

Dr. Salvatore A. Sanders, Dean of Graduate Studies Date

ABSTRACT

Analytic Number Theory is the cross between Real and Complex Analysis as well as Number Theory. We will examine results involving a function whose power series expansion has Fibonacci coefficients and another with Catalan number coefficients. The divisor function will help in finding an approximation of the number of divisors of a number. Our main focus will be on the Prime Number Theorem and the techniques needed to prove it. We will finish by examining how this proof helped to shape modern complex analysis, as well as discussing the Riemann Hypothesis.

ACKNOWLEDGEMENTS

I would like to thank Dr. Eric Wingler for his constant help, support, guidance and leadership. I would also like to thank my thesis committee members, Dr. Thomas Smotzer and Dr. Padraic Taylor, as well as the entire YSU Department of Mathematics & Statistics for pushing me to new levels, and always keeping my mind open to new ideas. Also, thank you to my mother Virginia Buchanan and girlfriend Cassie Dodson for their help and support along the way.

Contents

1 Introduction

Analytic Number Theory is the combination of two branches of Mathematics: Analysis and Number Theory. "Elementary" Number Theory is a course that is often taught at an undergraduate level. Topics include the following: Introducing the student to the notion of modulus, working with prime numbers, several functions and algorithms that reveal properties of the positive integers, and looking at some historic and important results. The study of positive integers (natural numbers $\mathbb{N}$) is (arguably) the oldest branch of Mathematics. When one ponders why Mathematics came about, it becomes clear that its origins were not centered around finding properties of a triangle or finding an equation for the oscillation of a vibrating string. Mathematics came to be because people needed a convenient way to describe and categorize what they needed in their day-to-day lives. Throughout time, there have been several alterations to how we count and understand numbers, but there is now an almost universal understanding of numbers. In trying to figure out how to represent quantities, people developed what was the beginning of Number Theory.

In this text, we will define the set of natural numbers (or positive integers) to be

$$\mathbb{N} = \{1, 2, 3, ...\}.$$

These are sometimes called the "counting numbers" as they are the traditional ones we use to count how many elements a given set (or collection of objects) has. Most people have been working with natural numbers since they were kids. One of the first things we are taught is how to count and how to use and interpret these numbers. We will not focus on constructing the natural numbers, but the reader is encouraged to study how they came about.

Number Theory deals with studying properties of natural numbers and functions whose domain is the natural numbers (arithmetical functions). Analytic Number Theory also focuses on these topics, but uses analytic techniques to figure out solutions or approximate certain results. Many mathematicians enjoy Number Theory because of the seemingly simple statement of certain problems. However, most questions in Analytic Number Theory have a similarly simple statement, but the proof (or attempt of a proof) is anything but "Elementary." Some of the most popular unsolved problems in mathematics come from Number Theory. Listed below are some of these unsolved problems (note that some of these can be explained to most students in a pre-calculus class, however, have not yet been solved by even the best mathematicians).

Is there an even number greater than 2 that is not the sum of two primes?

Are there infinitely many primes of the form $x^2 + 1$?

Are there infinitely many primes of the form $2^p - 1$ (Mersenne Primes)?

Are there infinitely many twin primes (prime numbers only two numbers apart)?

Note that many of the unsolved problems in Number Theory deal with prime numbers. This is because there is no known formula that generates all of the prime numbers (and only the prime numbers) in a sequential order. Obviously a function such as $f(x) = x$ has a range that contains all of the prime numbers, but there are many other values in the range other than the primes. So this is not a good way to represent the primes. Thus, the difficulty with most of these problems is how to get around the need for a formula that produces all of the prime numbers. When we examine the Prime Number Theorem, we will have to do a lot of work to come up with an expression that is equivalent but doesn't involve the need for a formula that produces all of the prime numbers in order. This is something that has puzzled Mathematicians for centuries, and will continue to do so until a formula is found.

1.1 Background Information

Before reading this paper, the reader should be familiar with concepts from Calculus and Algebra and have been exposed to problems in Real Analysis, Complex Analysis, Number Theory, Measure Theory, and Abstract Algebra courses (however, many of the results needed from these disciplines will be explained when needed). First, one should be familiar with what a function is. Note that we say a function $f : X \to Y$ has domain X and codomain Y. We will be working with functions involving complex numbers in a later section.

Next, one should be comfortable with complex numbers. The set of all complex (or imaginary) numbers, $\mathbb{C}$, is defined as

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}, \text{ and } i = \sqrt{-1}\}.$$

One can also think of the complex numbers as points in the plane, but the x-axis is replaced with the real-axis, and the y-axis is replaced with the imaginary-axis. So the complex number $a + ib$ can be thought of as the point (a, b). Note that from this, we can see a one-to-one correspondence from $\mathbb{C}$ onto $\mathbb{R}^2$, where $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$.

There will be several theorems used from Complex Analysis, Real Analysis and Measure Theory in this text. The term "prime number" has been used several times, but what is a prime number? A prime number, p, is an integer greater than 1 such that the only divisors of p are p and 1 (i.e., for any $n \in \mathbb{N}$ other than p and 1, p/n is not an integer). Some examples of prime numbers are 2, 3, 5, 7, 11, 13, 17, and 19. It has been proven (in several different ways) that there are infinitely many prime numbers. The most universal, and commonly taught, proof was done by Euclid [300 b.c.] in Book IX of *Euclid's Elements*.

Throughout this text, we will be calling on results and assistance from other books, articles, and websites. The main book that will be used is *Introduction to Analytic Number Theory* by Tom M. Apostol [1]. We will also call on the help of certain texts on Analysis for reference on important theorems or definitions, namely [4] and [5]. Some websites and clips of articles will be used to expand on some ideas and how they have historically come about. While it is assumed that the reader can follow along with some results from Calculus, we will make note of results as needed.

2 Introductory Problems

In the following subsections, we will do some problems that are number theoretic in nature but require analysis techniques to ease into the subject. The first problem involves an interesting result connecting the Fibonacci Numbers and the Golden Ratio, $\varphi = \frac{1+\sqrt{5}}{2}$. The next problem introduces the Catalan Numbers and a series associated with these coefficients. Lastly, we will introduce a function known as the *divisor function* which counts the number of divisors of any number.

2.1 Fibonacci Numbers and The Golden Ratio

We start off by defining the Fibonacci Numbers.

Definition 1. *Let $F_1 = 1$, and $F_2 = 1$. Define $F_{n+1} = F_{n-1} + F_n$. These are called the* Fibonacci Numbers*.*

So the first few numbers are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...$$

These numbers, interestingly, arise in nature from time to time. They are sometimes referred to as "Natures numbering system [2]."

We will show a relation involving the Fibonacci Numbers and the Golden Ratio, defined as

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618033....$$

Like the Fibonacci numbers, the Golden Ratio sometimes pops up in nature, and life.

Given that these two ideas arise in nature and are used to sometimes describe similar objects / events, one might conjecture they have a connection. We will show that they do in fact have a correlation, using an infinite series with Fibonacci coefficients.

Now define $f(z) = \sum_{n=1}^{\infty} F_n z^n$ for $z \in \mathbb{C}$. We wish to find an exact expression for $f(z)$. First we must show that $f(z)$ is convergent in some neighborhood of 0 with radius $R > 0$. First, we see that

$$\begin{aligned}
F_{n+1} &= F_n + F_{n-1} \\
&\leq F_n + F_n, \text{ since the Fibonacci numbers are an increasing sequence} \\
&= 2F_n \\
&= 2(F_{n-1} + F_{n-2}) \\
&\leq 2^2(F_{n-1}), \text{ by the same argument as above} \\
&\leq 2^3(F_{n-2}) \\
&\leq ... \\
&\leq 2^n F_1 \\
&= 2^n.
\end{aligned} \tag{1}$$

We can use this to show that this series must have a positive radius of convergence. Now the radius of convergence R is,

$$\begin{aligned}
R &= \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|F_n|}} \\
&= \frac{1}{\limsup_{n\to\infty} \sqrt[n]{F_n}}, \text{ since the Fibonacci numbers are non-negative} \\
&\geq \frac{1}{\limsup_{n\to\infty} \sqrt[n]{2^n}}, \text{ by the argument from (1)} \\
&= \frac{1}{\limsup_{n\to\infty} 2^{n/n}} \\
&= \frac{1}{2}.
\end{aligned} \tag{2}$$

So we know that there exists some $R \geq 1/2$ such that this series must converge for all z with $|z| < R$. Knowing this we can "legally" rearrange terms if needed, by properties of absolute convergence from [4] and [5].

Now we can find an exact expression for $f(z)$ by noticing that

$$
\begin{aligned}
f(z) &= \sum_{n=1}^{\infty} F_n z^n \\
&= z + z^2 + \sum_{n=3}^{\infty} F_n z^n \\
&= z + z^2 + \sum_{n=3}^{\infty} (F_{n-1} + F_{n-2}) z^n \\
&= z + z^2 + \sum_{n=3}^{\infty} F_{n-1} z^n + \sum_{n=3}^{\infty} F_{n-2} z^n \\
&= z + z^2 + \sum_{n=1}^{\infty} F_{n+1} z^{n+2} + \sum_{n=1}^{\infty} F_n z^{n+2} \\
&= z + z^2 + z\sum_{n=1}^{\infty} F_{n+1} z^{n+1} + z^2 \sum_{n=1}^{\infty} F_n z^n \\
&= z + z^2 + z(f(z) - z) + z^2 f(z) \\
&= z + zf(z) + z^2 f(z).
\end{aligned} \tag{3}
$$

Solving for $f(z)$ we see that

$$f(z) = \frac{z}{1 - z - z^2}.$$

Since f is analytic at 0, its power series at 0 will converge in a disc whose radius is the distance from 0 to the closest singularity of f. We can see that a problem arises when $z = \frac{1 \pm \sqrt{5}}{-2}$ by solving $1 - z - z^2 = 0$ using the quadratic formula. Since this value of z will define our radius of convergence, we are only concerned with the smaller of the two numbers $\left|\frac{1-\sqrt{5}}{-2}\right|$ and $\left|\frac{1+\sqrt{5}}{-2}\right|$. Noting that $\left|\frac{1-\sqrt{5}}{-2}\right| \leq \left|\frac{1+\sqrt{5}}{-2}\right|$ tells us that our radius of convergence is $R = \frac{1-\sqrt{5}}{-2} = \frac{\sqrt{5}-1}{2} = \frac{1}{\varphi}$. Note that this agrees with our estimate for the radius of convergence, since $\frac{1}{\varphi} \approx 0.618 > \frac{1}{2}$.

So we have shown that $f(z) = \sum_{n=1}^{\infty} F_n z^n = \frac{z}{1-z-z^2}$ and has radius of convergence $R = \frac{1}{\varphi}$.. Note that with this finding, we can find an approximation for the n^{th} Fibonacci number. We see that

$$
\begin{aligned}
R &= \frac{1}{\varphi} \\
&= \frac{1}{\limsup_{n\to\infty} \sqrt[n]{F_n}}.
\end{aligned}
\tag{4}
$$

So by taking reciprocals and raising to the n, we get that $F_n = \mathcal{O}(\varphi^n)$. Using other techniques, we will show that $F_n = \frac{1}{\sqrt{5}}\left(\varphi^n + \frac{(-1)^n}{\varphi^n}\right)$. Here we wish to demonstrate how complex analysis can be used to address this problem.

Consider $f(z) = \sum_{n=1}^{\infty} F_n z^n$. We have shown above that $f(z) = \frac{z}{1-z-z^2}$. We also found that the roots of $1 - z - z^2$ are $z = -\varphi$ and $z = \varphi^{-1}$. Note that $\varphi + \varphi^{-1} = \sqrt{5}$. Also note that for $|z| < 1$,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + z^4 + ...$$

Therefore

$$
\begin{aligned}
\frac{z}{1-z-z^2} &= \frac{-z}{z^2+z-1} \\
&= z\left(\frac{-1}{(z+\varphi)(z-\varphi^{-1})}\right) \\
&= -z\left(\frac{\frac{1}{-\varphi-\varphi^{-1}}}{z+\varphi} + \frac{\frac{1}{\varphi^{-1}+\varphi}}{z-\varphi^{-1}}\right) \\
&= -z\left(\frac{\frac{1}{-\varphi(\varphi+\varphi^{-1})}}{1+z/\varphi} - \frac{\frac{1}{\varphi^{-1}(\varphi^{-1}+\varphi)}}{1-z/\varphi^{-1}}\right) \\
&= z\left(\left(\frac{1}{\varphi\sqrt{5}}\right)\frac{1}{1+z/\varphi} + \left(\frac{1}{\varphi^{-1}\sqrt{5}}\right)\frac{1}{1-z/\varphi^{-1}}\right) \\
&= z\left(\frac{\varphi^{-1}}{\sqrt{5}}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{\varphi}\right)^n + \frac{\varphi}{\sqrt{5}}\sum_{n=1}^{\infty} z^n\varphi^n\right) \\
&= z\left(\frac{\varphi^{-1}}{\sqrt{5}}\left(1 - \frac{z}{\varphi} + \frac{z^2}{\varphi^2} - \frac{z^3}{\varphi^3} + ...\right) + \frac{\varphi}{\sqrt{5}}\left(1 + z\varphi + z^2\varphi^2 + z^3\varphi^3 + ...\right)\right) \\
&= z\left(\left(\frac{\varphi^{-1}+\varphi}{\sqrt{5}}\right) + \frac{z}{\sqrt{5}}\left(\varphi^2 - \frac{1}{\varphi^2}\right) + \frac{z^2}{\sqrt{5}}\left(\varphi^3 + \frac{1}{\varphi^3}\right) + \frac{z^3}{\sqrt{5}}\left(\varphi^4 - \frac{1}{\varphi^4}\right) + ...\right) \\
&= \frac{1}{\sqrt{5}}\sum_{n=1}^{\infty} z^n\left(\varphi^n + (-1)^{n-1}(\varphi^{-1})^n\right).
\end{aligned}
\tag{5}
$$

We now make use of the following theorem:

Theorem 2. *Let $f(z)$ be a complex-valued function. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f(z) = \sum_{n=0}^{\infty} b_n z^n$. Then $a_n = b_n$ for all $n = 0, 1, 2, 3, ...$*

Since we have two series representations of $f(z) = \frac{z}{1-z-z^2}$ their coefficients must be equal.

Note when $n = 1$, then $F_1 = 1$ and $\frac{\varphi+\varphi^{-1}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$.

When $n = 2$, then $F_2 = 1$ and $\frac{\varphi^2-\varphi^{-2}}{\sqrt{5}} = \frac{\left(\varphi-\frac{1}{\varphi}\right)\left(\varphi+\frac{1}{\varphi}\right)}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$.

When $n = 3$, then $F_3 = 2$ and

$$\begin{aligned} \frac{\varphi^3+\varphi^{-3}}{\sqrt{5}} &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^3 + \left(\frac{\sqrt{5}-1}{2}\right)^3}{\sqrt{5}} \\ &= \frac{\frac{1+3\sqrt{5}+3\sqrt{5}^2+\sqrt{5}^3+\sqrt{5}^3-3\sqrt{5}^2+3\sqrt{5}-1}{8}}{\sqrt{5}} \\ &= \frac{16\sqrt{5}}{8\sqrt{5}} \\ &= 2. \end{aligned} \tag{6}$$

This will work for all $n = 1, 2, 3, 4, ...,$ and each case can be solved by expanding $\left(\frac{1+\sqrt{5}}{2}\right)^n$ and $\left(\frac{\sqrt{5}-1}{2}\right)^n$ using the Binomial Theorem.

Therefore

$$F_n = \frac{\varphi^n + \left(-1\right)^{n-1}\left(\varphi^{-1}\right)^n}{\sqrt{5}}.$$

2.2 A Problem Involving the Catalan Numbers

Consider a set with a binary operation $*$ defined on it. Assume that $*$ is not associative, meaning that the values of $(a*b)*c$ and $a*(b*c)$ may be different. Let c_n be the number of possible values of $a_1*a_2*...*a_n$. Then c_n is the number of ways of placing parentheses around these elements in order to specify the order in which the operations are to be performed. Clearly, c_1 and c_2 are both 1, since no parentheses are required.

Now consider c_3. So we see that there are two ways to uniquely put parentheses around a list of 3 numbers. Specifically, they are **(1.)** $(a * b) * c$, and **(2.)** $a * (b * c)$. Note that $(a * b * c) = (a * b) * c$. So $c_3 = 2$.

Now consider c_4. Here, we see that there are 5 ways of putting parentheses around 4 numbers. Notably, **(1.)** $((a * b) * c) * d$, **(2.)** $(a * (b * c)) * d$, **(3.)** $a * (b * c) * d$, **(4.)** $a * ((b * c) * d)$, and **(5.)** $a * (b * (c * d))$. So $c_4 = 5$.

Computing c_5, c_6, ... by hand can become time consuming. So we will simply list them as follows:

$c_1 = 1$

$c_2 = 1$

$c_3 = 2$

$c_4 = 5$

$c_5 = 14$

$c_6 = 42$

$c_7 = 132$

We can see that these numbers grow very rapidly, and are seemingly arbitrary. However, if we look more carefully at each number and the process to obtain the next, we will see a pattern develop. Notice that

$c_1 = 1$

$c_2 = 1 = c_1 c_1$

$c_3 = 2 = c_1c_2 + c_2c_1$

$c_4 = 5 = c_1c_3 + c_2c_2 + c_3c_1$

$c_5 = 14 = c_1c_4 + c_2c_3 + c_3c_2 + c_4c_1$

$c_6 = 42 = c_1c_5 + c_2c_4 + c_3c_3 + c_4c_2 + c_5c_1$

$c_7 = 132 = c_1c_6 + c_2c_5 + c_3c_4 + c_4c_3 + c_5c_2 + c_6c_1$

.

.

.

$c_n = \sum_{i=1}^{n-1} c_ic_{n-i}$.

This formula arises in the way we compute c_n for large n. Consider $\{a_1, a_2, ..., a_n\}$. Then we can break this set up as $\{a_1\}$ and $\{a_2, a_3, ..., a_n\}$, or $\{a_1, a_2\}$ and $\{a_3, a_4, ..., a_n\}$, etc. Thus in general, we can break up $\{a_1, a_2, ..., a_n\}$ as $\{a_1, ..., a_k\}$ and $\{a_{k+1}, ..., a_n\}$. Note that $\{a_1, a_2, ..., a_k\}$ corresponds to c_k which we have already determined. So we can use the previous results to help us determine c_n.

So we have found a recursive formula for the Catalan numbers, c_n. Now we want to find an exact expression for a function of the form

$$f(x) = \sum_{n=1}^{\infty} c_nx^n,$$

where the coefficients are the Catalan numbers. To do this, we will first do some manipulations with the series $\sum_{n=1}^{\infty} c_nx^n$ that may or may not be “legal”. We will assume that $f(x)$ is analytic at $x = 0$, and use this assumption to find an expression for $f(x)$.

$$
\begin{aligned}
(f(x))^2 &= \left(\sum_{n=1}^{\infty} c_n x^n\right)\left(\sum_{n=1}^{\infty} c_n x^n\right) \\
&= (c_1x + c_2x^2 + c_3x^3 + c_4x^4 + ...)(c_1x + c_2x^2 + c_3x^3 + c_4x^4 + ...) \\
&= c_1c_1x^2 + (c_1c_2 + c_2c_1)x^3 + (c_1c_3 + c_2c_2 + c_3c_1)x^4 + ... \\
&= c_2x^2 + c_3x^3 + c_4x^4 + ... \\
&= (c_1x + c_2c^2 + c_3x^3 + c_4x^4 + ...) - c_1x \\
&= f(x) - x.
\end{aligned} \tag{7}
$$

Hence, $(f(x))^2 - f(x) + x = 0$. Solving this quadratic equation for $f(x)$ we get $f(x) = \frac{1 \pm \sqrt{1-4x}}{2}$. Now to figure out whether the numerator should be $1 + \sqrt{1-4x}$ or $1 - \sqrt{1-4x}$, we will use the fact that $f(0) = 0$ (since $f(0) = c_1(0) + c_2(0) + ... = 0$). So it must be that

$$f(x) = \frac{1 - \sqrt{1-4x}}{2}.$$

The above function is analytic in the disc $\{x : |x| < 1/4\}$ and this disc can be no larger because of the singularity at 1/4. Hence, it will have a power series $\sum_{n=1}^{\infty} a_n x^n$ with radius of convergence 1/4. Because of our initial assumption, we will need to verify that $a_n = c_n$ for all n.

In order to find a Taylor Series for the function $f(x) = \frac{1-\sqrt{1-4x}}{2}$, we will first find a power series expression for $\sqrt{1-x}$. Using the formula $g(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n$, for some analytic function $g(x)$, we get the following:

$$
\begin{aligned}
\sqrt{1-x} &= 1 - \frac{1}{2}x - \frac{\frac{1}{2}\cdot\frac{1}{2}}{2!}x^2 - \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}}{3!}x^3 - \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{4!}x^4 - ... \\
&= 1 - \sum_{n=1}^{\infty} \frac{1\cdot 1\cdot 3\cdot 5\cdot ...\cdot(2n-3)}{2^n n!} x^n.
\end{aligned} \tag{8}
$$

Now,

$$\sqrt{1-4x} = 1 - \sum_{n=1}^{\infty} \frac{1\cdot 1\cdot 3\cdot 5\cdot ...\cdot(2n-3)}{n!} 2^n x^n. \tag{9}$$

Thus,

$$\begin{aligned} 1-\sqrt{1-4x} &= 1-(1-\sum_{n=1}^{\infty}\frac{1\cdot 1\cdot 3\cdot 5\cdot ...\cdot(2n-3)}{n!}2^n x^n \\ &= \sum_{n=1}^{\infty}\frac{1\cdot 1\cdot 3\cdot 5\cdot ...\cdot(2n-3)}{n!}2^n x^n. \end{aligned} \tag{10}$$

Dividing by 2, we get that

$$\begin{aligned} \frac{1-\sqrt{1-4x}}{2} &= \sum_{n=1}^{\infty}\frac{1\cdot 1\cdot 3\cdot 5\cdot ...\cdot(2n-3)}{n!}2^{n-1} x^n \\ &= \sum_{n=1}^{\infty}\frac{(2n-2)!}{2^{n-1}(n-1)!\ n!}2^{n-1}x^n \\ &= \sum_{n=1}^{\infty}\frac{(2n-2)!}{(n-1)!\ n!}x^n. \end{aligned} \tag{11}$$

Now, we need to show that the coefficients $\frac{(2n-2)!}{(n-1)!\ n!}$ are the Catalan numbers.

Proof.

Let $a_n = \frac{(2n-2)!}{(n-1)!n!}$. We have the following:

$$\frac{1-\sqrt{1-4x}}{2} = \sum_{n=1}^{\infty} a_n x^n;$$

$$\frac{1-2\sqrt{1-4x}+1-4x}{4} = \left(\sum_{n=1}^{\infty} a_n x^n\right)^2 = \sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1} a_k a_{n-k}\right)x^n;$$

$$\frac{1-\sqrt{1-4x}}{2} - x = \sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1} a_k a_{n-k}\right)x^n;$$

Hence,

$$\sum_{n=1}^{\infty} a_n x^n = \frac{1-\sqrt{1-4x}}{2} = x + \sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1} a_k a_{n-k}\right)x^n,$$

Equating the coefficients, we get $a_1 = 1$ and $a_n = \sum_{k=1}^{n-1} a_k a_{n-k}$ for each $n > 1$.

Note that $a_1 = 1 = c_1$. Since $a_n = \sum_{i=1}^{n-1} a_i a_{n-i}$ and $c_n = \sum_{k=1}^{n-1} c_k c_{n-k}$ (they are defined the same way) and $a_1 = c_1$, it follows that $a_n = c_n$. □

This shows that $f(x) = \sum_{n=1}^{\infty} c_n x^n$ is analytic in a neighborhood of 0 since $f(x) = \frac{1-\sqrt{1-4x}}{2}$.

Since $R = \frac{1}{4} = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{c_n}}$, then $c = \mathcal{O}(4^n)$. However, this is a rather crude estimate, and we can find a better one using the fact that $c_n = \frac{(2n-2)!}{(n-1)!\ n!}$. For this we will use Stirling's approximation to estimate $n!$. This approximation says $n! \sim n^n e^{-n}\sqrt{2\pi n}$. So,

$$\begin{aligned}
c_n &= \frac{(2n-2)!}{(n-1)!\ n!} \\
&\sim \frac{(2n-2)^{2n-2}e^{2-2n}\sqrt{2\pi(2n-2)}}{(n-1)^{n-1}e^{1-n}\sqrt{2\pi(n-1)}\ n^n e^{-n}\sqrt{2\pi n}} \\
&= \frac{(2n-2)^{2n-2}e\sqrt{2\pi(2n-2)}}{(n-1)^{n-1}\sqrt{2\pi(n-1)}\ n^n\sqrt{2\pi n}} \\
&= \frac{2^{2n-2}(n-1)^{n-1}e\sqrt{2\pi(2n-2)}}{\sqrt{2\pi(n-1)}\ n^n\sqrt{2\pi n}} \\
&= \frac{\sqrt{2}\ 2^{2n-2}e(n-1)^{n-1}}{n^n\sqrt{2n\pi}} \\
&= \frac{2^{2n-2}e(n-1)^{n-1}}{n^n\sqrt{n\pi}} \\
&\sim \frac{4^{n-1}}{n\sqrt{n\pi}}.
\end{aligned} \tag{12}$$

This gives us a better estimate for the growth of the Catalan numbers.

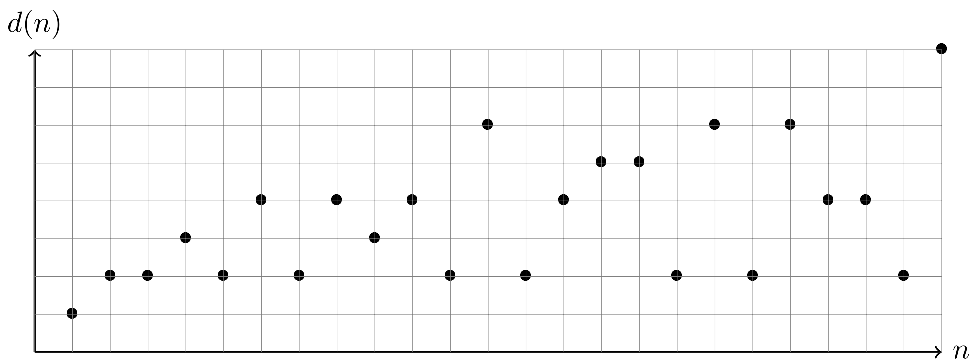
2.3 The Divisor Function

We say that a number d *divides* a number n, written $d|n$, if for some integer k, $n = dk$. Also, when we refer to a "number" we will mainly be referring to positive integers (or natural numbers).

For this problem, we will make use of the divisor function, which is defined as $d(n) = \sum_{d|n} 1$. This function counts the number of divisors of n. For example, the divisors of 6 are 1, 2, 3, 6. So $d(6) = 4$.

Note that if p is a prime number, then $d(p) = 2$.

Clearly the values of this function seem to fluctuate. Our goal is to find an estimate for the number of divisors of n. To do this, we will utilize some graphs involving $d(n)$. First we will plot the graph of $d(n)$.



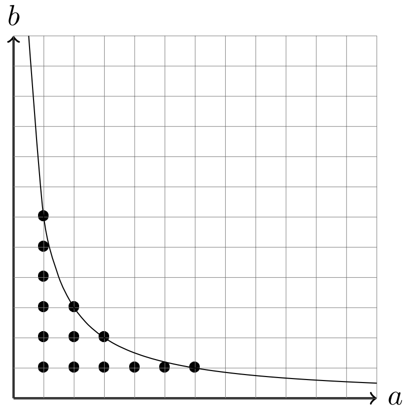
We see that the graph of $d(n)$ is oscillatory with $max_{1\leq k\leq n} d(k)$ increasing. Trying to find a line of best-fit will be difficult since the graph of $d(n)$ doesn't appear to be linear.

In order to better understand the average behavior of $d(n)$, we will examine $\frac{1}{n}\sum_{x=1}^{n} d(x)$. Note that this sum resembles a Riemann Sum. Also note that

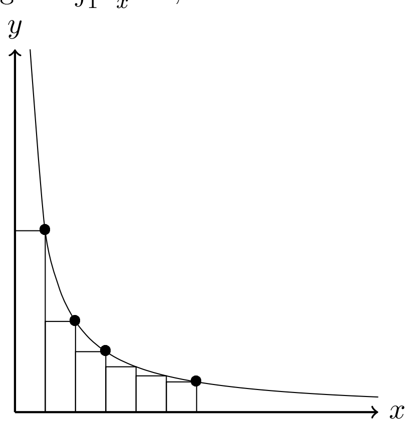
$$\sum_{x=1}^{n} d(x) = \sum_{x=1}^{n} \sum_{d|x} 1,$$

and counting the divisors of x is equivalent to counting the number of pairs a, b such that $x = ab$. So we are adding one for every pair of numbers (a, b) such that $ab = x$. Thus when we examine $\sum_{x=1}^{n} d(x)$, we are adding one for every lattice point (coordinate pair of integers) on the graph of $ab = x$ or $b = \frac{x}{a}$. Therefore $\sum_{x=1}^{n} d(x)$ is summing all lattice points (a, b) such that $ab \leq n$. Consider the following example for $n = 6$.

For $n = 6$ we will be finding all lattice points such that $ab \leq 6$. Note that the curve is $b = \frac{6}{a}$.

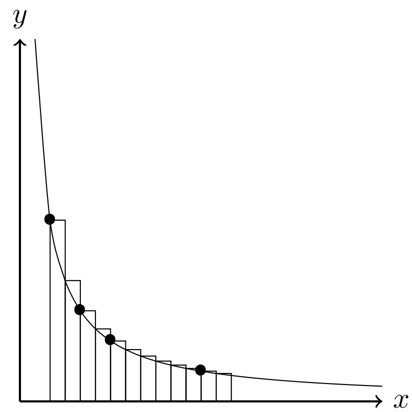


Without loss of generality, we are looking at a graph of $y = \frac{n}{x}$, and counting the number of lattice points on or below this graph. The region each graph represents all numbers x and y that multiply to result in n. Since we noted that $\frac{1}{n}\sum_{x=1}^{n} d(x)$ resembles a Riemann sum, to find the average number of divisors of n we will integrate $f(x) = \frac{n}{x}$, but we need to consider what endpoints to integrate from. Note that this integral will approximate $\sum_{i=1}^{n} d(i)$. If we integrate $\int_1^n \frac{n}{x}\, dx$, we will be considering the following graphical representation (for $n = 6$):



From this picture it is clear that $\sum_{i=1}^{n} d(i) < n + \int_1^n \frac{n}{x} dx$.

Now, if we consider $\int_1^{n+1} \frac{n}{x}\, dx$, we will have the following graphical representation (for $n = 6$):



We see that this integral gives an underestimate on the value of $\sum_{i=1}^{n} d(i)$. Hence, $\sum_{i=1}^{n} d(i) \geq \int_{1}^{n+1} \frac{n}{x}\ dx$.

So we can combine these results to find an estimate for $\sum_{i=1}^{n} d(i)$. So,

$$\int_{1}^{n+1} \frac{n}{x}\ dx \leq \sum_{i=1}^{n} d(i) \leq \int_{1}^{n} \frac{n}{x}\ dx + n.$$

Evaluating these integrals, we get that,

$$n \ln(n+1) \leq \sum_{i=1}^{n} d(i) \leq n \ln(n) + n.$$

Dividing by n yields,

$$\ln(n+1) \leq \frac{1}{n} \sum_{i=1}^{n} d(i) \leq \ln(n) + 1.$$

Note that $\ln(n+1) = \ln(n) + \ln(\frac{n+1}{n}) = \ln(n) + \ln(1 + \frac{1}{n})$. As n grows larger, $\ln(1+\frac{1}{n})$ will tend to 0. So to represent this, we say that $\ln(n+1) = \ln(n) + \mathcal{O}(\frac{1}{n})$, where we define this big Oh notation as follows:

Definition 3. *For two functions $f(z)$ and $g(z)$ we say that $f(z) = \mathcal{O}(g(z))$ as $z \to z_0$ if $|f(z)/g(z)|$ is bounded in a neighborhood of a point z_0.*

Dividing our previous result by $\ln(n)$, we have,

$$\frac{\ln(n+1)}{\ln(n)} = \frac{\frac{1}{n}\sum_{i=1}^{n} d(i)}{\ln(n)} = \frac{1+\ln(n)}{\ln(n)}.$$

Therefore,

$$\lim_{n\to\infty} \frac{\frac{1}{n}\sum_{i=1}^{n} d(i)}{\ln(n)} = 1,$$

or $\frac{1}{n}\sum_{i=1}^{n} d(i) \sim \ln(n)$.

To more thoroughly show this, we turn to *Introduction to Analytic Number Theory* by Tom Apostol. He shows that

$$\sum_{n\le x} d(n) = x\ln(x) + (2\gamma - 1)x + \mathcal{O}(\sqrt{x})\ [1].$$

Here, note that γ is Euler's constant, $\gamma = \lim_{n\to\infty}\big((\sum_{k=1}^{n}\frac{1}{k}) - \ln(n)\big) \approx 0.577$.

In order to prove this, we will need to use the following results that have been previously proven in [1]:

1.) $\sum_{n\le x/d} 1 = \frac{x}{d} + \mathcal{O}(1)$.

2.) $\sum_{n\le x}\frac{1}{n} = \ln(x) + \gamma + \mathcal{O}(\frac{1}{x})$.

Proof. (Note that this proof comes from [1]).

Since $d(n) = \sum_{d|n} 1$, we can rewrite $\sum_{n\le x} d(n)$ as,

$$\sum_{n\le x} d(n) = \sum_{n\le x}\sum_{d|n} 1.$$

For any divisor d of n, we can say that $n = dr$, for some positive integer r. So we can rewrite the sum so that it sums over the pairs of positive integers r, and d; that is,

$$\sum_{n\le x} d(n) = \sum_{n\le x}\sum_{d|n} 1 = \sum_{n\le x}\sum_{dr=n} 1 = \sum_{dr\le x} 1. \qquad (*)$$

By the argument given above, we are considering all lattice points $(x, y) \in \mathbb{N}^2$ such that $xy \le n$.

We will first consider for each $d \le x$, the lattice points on the horizontal line segment

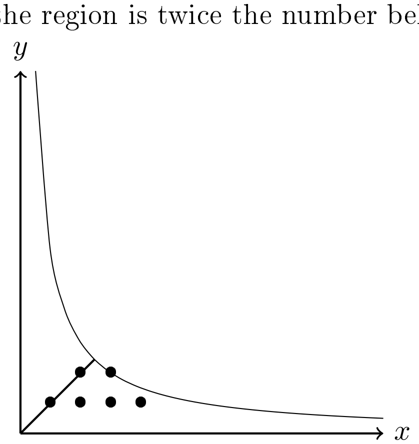
$1 \leq r \leq x/d$, and then sum over all $d \leq x$. So we can rewrite the above sum as,

$$\sum_{n=1}^{x} d(n) = \sum_{d=1}^{x} \sum_{r=1}^{x/d} 1.$$

So we can use the two previously proven results from [1] to see that,

$$\begin{aligned} \sum_{n=1}^{x} d(n) &= \sum_{d=1}^{x} \left(\frac{x}{d} + \mathcal{O}(1) \right) \\ &= x \sum_{d=1}^{x} \frac{1}{d} + \mathcal{O}(x) \\ &= x\big(\ln(x) + C + \mathcal{O}(1/x)\big) + \mathcal{O}(x) \\ &= x \ln(x) + \mathcal{O}(x). \end{aligned} \tag{13}$$

Recall that (*) counts the number of lattice points in the region under the hyperbola. We can use the line $x = y$ to slice this region in half. So the total number of lattice points in the region is twice the number below this line plus the number on the line segment.



Here the dots represent lattice points. So referring to this picture, we see that

$$\sum_{i=1}^{n} d(i) = 2 \sum_{d \leq \sqrt{n}} \left[\frac{n}{d} - d \right] + [\sqrt{n}].$$

Using the relation $[\sqrt{n}] = \sqrt{n} + \mathcal{O}(1)$ and the equations given prior to the proof, we see that

$$\begin{aligned}
\sum_{i \le n} d(i) &= 2 \sum_{d \le \sqrt{n}} \left(\frac{n}{d} - d + \mathcal{O}(1) \right) + \mathcal{O}(\sqrt{n}) \\
&= 2n \sum_{d \le \sqrt{n}} \frac{1}{d} - 2 \sum_{d \le \sqrt{n}} d + \mathcal{O}(\sqrt{n}) \\
&= 2n\big(\log(\sqrt{n}) + \gamma + \mathcal{O}(\frac{1}{\sqrt{n}})\big) - 2\big(\frac{n}{2} + \mathcal{O}(\sqrt{n})\big) + \mathcal{O}(\sqrt{n}) \\
&= n \log(n) + 2n\gamma + 2n\mathcal{O}(\frac{1}{\sqrt{n}}) - n - 2\mathcal{O}(\sqrt{n}) + \mathcal{O}(\sqrt{n}) \\
&= n \log(n) + (2\gamma - 1)n + \mathcal{O}(\sqrt{n}).
\end{aligned} \tag{14}$$

□

Note that over the years, the term $\mathcal{O}(\sqrt{n})$ has been improved. "In 1903 Voronoi proved that the error is $\mathcal{O}(n^{1/3}\ln(n))$; in 1922 van der Corput improved this to $\mathcal{O}(n^{33/100})$. The best estimate to date is $\mathcal{O}(n^{(12/37)+\epsilon})$ for every $\epsilon > 0$, obtained by Kolesnik in 1969." [1]

Figuring out the greatest lower bound for $\mathcal{O}(n^{\theta})$ for all θ is an unsolved problem, called Dirichlet's divisor problem. In 1915, Hardy and Landau showed that infemum of $\theta \ge 1/4$ [1].

We will now examine a table of n, $d(n)$, $\frac{\sum_{i=1}^{n} d(i)}{n}$ and $\ln(n)$. We will round $\frac{\sum_{i=1}^{n} d(i)}{n}$ and $\ln(n)$ to 3 decimal places, for convenience.

n	$d(n)$	$\frac{\sum_{i=1}^{n} d(i)}{n}$	$\ln(n)$
1	1	1	0
2	2	1.5	0.693
4	3	2	1.386
8	4	2.5	2.079
16	5	3.125	2.773
32	6	3.719	3.466
64	7	4.375	4.159
128	8	5.039	4.852
256	9	5.727	5.545
512	10	6.406	6.238
1024	11	7.092	6.931

Looking at this table reveals that the two columns furthest to the right seem to be asymptotic, which is what we verified above.

3 Introduction to The Prime Number Theorem

Before diving into the proof and analysis of The Prime Number Theorem (PNT), we will begin by giving a brief overview of the problem, its history, strategies we will use, and some facts that will be needed during the proof.

3.1 History and Motivation

Prime Numbers have been an area of study and wonder for hundreds of years. How frequently do they occur? Are there infinitely many of a certain form? Do they have an average distribution? All of these seemingly simple questions are very difficult to answer, and many remain unsolved to this day. One of the biggest problems is the lack of a formula to represent them or an algorithm to check if a given number is prime. So many of the results that help us understand prime numbers are estimates, or approximations.

The Prime Number Theorem is arguably the most important result as far as development and application of Analytic Number Theory. The theorem states that the number of primes less than a positive integer x (denoted $\pi(x)$) is approximately $\frac{x}{\ln(x)}$. There have been several proofs of this theorem. We will be examining two of them: An "Elementary" Proof and an analytic proof involving Complex Analysis techniques.

The Prime Number Theorem was first hypothesized by Gauss (1792) and Legendre (1798) when they wrote down a table of primes less than or equal to a number x. They made an estimate indicating that $\pi(x)$ seems to be trending similar to $x/\ln(x)$. The table below shows some of these values:

x	$\pi(x)$	$\frac{x}{\ln(x)}$
2	1	2.885
4	2	2.885
8	4	3.847
16	6	5.771
32	11	9.233
64	18	15.389
128	31	26.381
256	54	46.166
512	97	82.073
1024	172	147.732
2048	309	268.604
4096	564	492.440

Their result was first verified by Hadamard and de la Vallee Poussin in 1896, using techniques from complex analysis and properties of the Riemann Zeta Function. Many thought that an elementary proof didn't exist and that the only way to prove the theorem was by analysis. However, in 1949, an elementary proof was presented by Erdös and Selberg. Their proof didn't involve any techniques from analysis and can be read and understood by a calculus student; however, their proof is very detailed and requires the definition of many other functions and the verification of many other results.

We will give a detailed proof of the analytic method and will skim through the elementary method, as some results from this are needed for the analytic proof, while referring the reader to [1] and [3] for detailed proofs of the necessary results.

3.2 Elementary Method

For the Elementary proof, our goal is to show that

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\ln(x)}} = \lim_{x\to\infty} \frac{\pi(x)\ln(x)}{x} = 1.$$

However, in order to do this, we will need to define several functions and state several results. We will focus on the main ideas and results needed for the analytic method, while giving a sketch of the elementary method.

Define the Mangoldt function as

$$\Lambda(n) = \begin{cases} \ln(p), & \text{if } n = p^m \text{ where } p \text{ is prime and } m \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

We will later show that if

$$\sum_{n=1}^{x} \Lambda(n) \sim x \text{ as } x \to \infty,$$

then the prime number theorem will hold.

Let

$$\psi(x) = \sum_{n=1}^{x} \Lambda(n).$$

Our goal is to show that $\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$ and $\lim_{x\to\infty} \frac{\pi(x)\ln(x)}{x} = 1$ are equivalent. Note that since $\Lambda(n) = 0$ if $n \neq p^m$, then we can rewrite $\psi(x)$ as

$$\psi(x) = \sum_{n\leq x} \Lambda(n) = \sum_{m=1}^{\infty} \sum_{p^m \leq x} \Lambda(p^m) = \sum_{m=1}^{\infty} \sum_{p\leq x^{1/m}} \ln(p).$$

Note that since we are only considering $n \leq x$, we are taking the sum over finitely many terms, so the sum on m is finite. So we can say that

$$\sum_{m=1}^{\infty} \sum_{p\leq x^{1/m}} \ln(p) = \sum_{m=1}^{k} \sum_{p\leq x^{1/m}} \ln(p),$$

where k is the highest power of a prime p that is needed. Note that $\psi(x) = 0$ if $x < 2$ since we have defined $\Lambda(1) = 0$. Note that $x^{1/m} \leq x$ for $m = 2, 3, 4, ..., k$, if $x \geq 2$. Thus this is true for $x^{1/m} < 2$. In this case $\ln(x^{1/m}) < \ln(2)$, so $\frac{1}{m}\ln(x) < \ln(2)$, thus $m > \frac{\ln(x)}{\ln(2)} = \log_2(x)$. So we can say that,

$$\psi(x) = \sum_{m=1}^{\log_2(x)} \sum_{p \leq x^{1/m}} \ln(p).$$

Definition 4. *For $x > 0$, define Chebyshev's ϑ (theta) function by*

$$\vartheta(x) = \sum_{p \leq x} \ln(p),$$

where p runs through all primes less than or equal to x.

With this new notation, we can say that

$$\psi(x) = \sum_{m=1}^{\log_2(x)} \vartheta(x^{1/m}).$$

From [1], we have the following theorem:

Theorem 5. *For $x > 0$,*

$$\lim_{x\to\infty} \left(\frac{\psi(x)}{x} - \frac{\vartheta(x)}{x} \right) = 0$$

This result will help with a future theorem.

We now wish to find a relation between $\vartheta(x)$ and $\pi(x)$. It is worth noting that both of these functions have jumps at the primes (hence they behave like step functions) but $\pi(x)$ jumps by 1 at each prime, and $\vartheta(x)$ jumps by $\ln(p)$ at each prime.

In order to find a relation between $\vartheta(x)$ and $\pi(x)$ we use Abel's Identity.

Definition 6. ***Abel's Identity:*** *For any arithmetical function $a(n)$ let $A(x) = \sum_{n=1}^{x} a(n)$, where $A(x) = 0$ if $x < 1$. Assume f has a continuous derivative on the interval $[x, y]$, $0 < x < y$. Then,*

$$\sum_{x<n\leq y} a(n)f(n) = A(y)f(y) - A(x)f(x) - \int_x^y A(t)f'(t)dt.$$

In the following theorem, we will relate $\pi(x)$ and $\vartheta(x)$ using Abel's Identity.

Theorem 7. *Let $x \geq 2$. Then*

$$\vartheta(x) = \pi(x)\ln(x) - \int_2^x \frac{\pi(t)}{t}dt$$

and

$$\pi(x) = \frac{\vartheta(x)}{\ln(x)} + \int_2^x \frac{\vartheta(t)}{t\ln^2 t}dt.$$

The proof of this comes from Abel's Identity, letting $a(n)$ (the arithmetic function) be $X_p(n)$, the characteristic function of the primes, defined by

$$X_p(n) = \begin{cases} 1, & \text{if } n \text{ is prime;} \\ 0, & \text{otherwise.} \end{cases}$$

With this we can rewrite $\pi(x)$ as

$$\pi(x) = \sum_{p=2}^{x} 1 = \sum_{1<n\leq x} X_p(n),$$

and $\vartheta(x)$ as

$$\vartheta(x) = \sum_{p=2}^{x} \ln(p) = \sum_{1<n\leq x} X_p(n)\ln(n).$$

So by Abel's Identity, letting $f(x) = \ln(x)$ and $y = 1$, we get the result for $\vartheta(x)$.

Then, letting $f(x) = \frac{1}{\ln(x)}$ and $y = 3/2$, we get the result for $\pi(x)$.

Now that we have a relation between $\psi(x)$ and $\vartheta(x)$, and between $\vartheta(x)$ and $\pi(x)$, we can state the following theorem.

Theorem 8. *The following are equivalent:*

$$1. \lim_{x\to\infty} \frac{\pi(x)\ln(x)}{x} = 1 \tag{15}$$

$$2. \lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1 \tag{16}$$

$$3. \lim_{x\to\infty} \frac{\psi(x)}{x} = 1. \tag{17}$$

Note that this theorem does not prove the Prime Number Theorem, but this shows that in order to prove the Prime Number Theorem, it suffices to show that either (2) or (3) holds.

Proof.

Note that by Theorem 5, we have already seen that (2) holds if and only if (3) holds.

(1) ⇒ (2)

Assume that

$$\lim_{x\to\infty} \frac{\pi(x)\ln(x)}{x} = 1.$$

By the previous theorem, we have that

$$\frac{\vartheta(x)}{x} = \frac{\pi(x)\ln(x)}{x} - \frac{1}{x}\int_2^x \frac{\pi(t)}{t}dt,$$

and

$$\frac{\pi(x)\ln(x)}{x} = \frac{\vartheta(x)}{x} + \frac{\ln(x)}{x}\int_2^x \frac{\vartheta(t)dt}{t\ln^2 t}.$$

In order to show that

$$\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1,$$

we need to show that

$$\lim_{x\to\infty} \frac{\vartheta(x)}{x} = \lim_{x\to\infty} \left(\frac{\pi(x)\ln(x)}{x} - \frac{1}{x}\int_2^x \frac{\pi(t)}{t}dt\right) = 1.$$

Since we are assuming (1) holds, it suffices to show only that

$$\lim_{x\to\infty} \frac{1}{x}\int_2^x \frac{\pi(t)}{t}dt = 0.$$

Since we are assuming that $\lim_{x\to\infty} \frac{\pi(x)\ln(x)}{x} = 1$, then $\lim_{x\to\infty} \frac{\pi(x)}{x} = \lim_{x\to\infty} \frac{1}{\ln(x)}$. Since

$$\frac{\pi(x)}{x} = \mathcal{O}\left(\frac{1}{\ln(x)}\right),$$

we have

$$\frac{\int_2^x \frac{\pi(t)}{t} dt}{x} = \mathcal{O}\left(\frac{\int_2^x \frac{1}{\ln(t)} dt}{x}.\right)$$

By L'Hospital's Rule, $\lim_{x\to\infty} \frac{\int_2^x \frac{1}{\ln(t)} dt}{x} = 0$. Thus,

$$\lim_{x\to\infty} \frac{\int_2^x \frac{\pi(t)}{t} dt}{x} = 0.$$

(2) ⇒ (1).

Now assume that $\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1$. By the previous theorem, we must show that

$$\lim_{x\to\infty} \frac{\pi(x)\ln(x)}{x} = \lim_{x\to\infty} \left(\frac{\vartheta(x)}{x} + \frac{\ln(x)}{x} \int_2^x \frac{\vartheta(t)dt}{t\ln^2 t}\right) = 1.$$

By assumption, it suffices to show that $\lim_{x\to\infty} \frac{\ln(x)}{x} \int_2^x \frac{\vartheta(t)dt}{t\ln^2 t} = 0$. Now

$$\frac{\ln(x)}{x} \int_2^x \frac{\vartheta(t)dt}{t\ln^2 t} = \mathcal{O}\left(\frac{\ln(x)}{x} \int_2^x \frac{dt}{\ln^2 t}\right),$$

since $\vartheta(t) = O(t)$. Using L'Hospitals rule, we get

$$\lim_{x\to\infty} \frac{\ln(x)}{x} \int_2^x \frac{dt}{\ln^2 t} = \lim_{x\to\infty} \frac{\frac{1}{\ln^2 x}}{\frac{\ln(x)-1}{\ln^2 x}} = \lim_{x\to\infty} \frac{1}{\ln(x)-1} = 0.$$

Therefore

$$\lim_{x\to\infty} \frac{\pi(x)\ln(x)}{x} = 1.$$

□

We can now outline the elementary proof. Note that there is much work still needed, and many parts will be omitted, however we want to focus on the analytic proof. The reader is encouraged to see [3] and [1] for more specifics on the elementary proof.

Outline of Elementary Proof.

This proof will use Selberg's asymptotic formula, and part of Shapiro's Tauberian Theorem.

Theorem 9. *(Selberg) Let F be a real or complex-valued function defined on $(0,\infty)$, and let $G(x) = \ln(x)\sum_{n=1}^{x} F(x/n)$. Then,*

$$F(x)\ln(x) + \sum_{n=1}^{x} F\left(\frac{x}{n}\right)\Lambda(n) = \sum_{d\leq x} \mu(d) G\left(\frac{x}{d}\right),$$

where $\mu(n)$ is the Möbius function defined by $\mu(1) = 1$, and for p_i prime

$$\mu(n) = \begin{cases} (-1)^k, & \text{if } n = p_1^{a_1}...p_k^{a_k}, \text{ and } a_1 = ... = a_k = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 10. *(Sharpiro's) Let $\{a_n\}$ be a non-negative sequence such that*

$$\sum_{n=1}^{x} a_n \left\lfloor \frac{x}{n} \right\rfloor = x\ln(x) + O(x).$$

Then, for all $x \geq 1$,

$$\sum_{n=1}^{x} \frac{a(n)}{n} = \ln(x) + O(1).$$

Letting $F_1(x) = \psi(x)$, then by Selberg's formula,

$$\psi(x)\ln(x) + \sum_{n=1}^{x} \Lambda(n)\psi\left(\frac{x}{n}\right) = 2x\ln(x) + O(x).$$

This relation is a result of a very involved proof using Selberg's formula, Euler's constant γ, and many other theorems from [1] and [3].

Then we define a new function $\sigma(x) = e^{-x}\psi(e^x) - 1$. In order to show that the prime number theorem holds, one can relate $\sigma(x)$ and Selberg's formula, and then show that

$\limsup_{x\to\infty} |\sigma(x)| = 0$.

This, again, is a result of a very involved proof, and the reader is again referred to [1] and [3] for further clarification.

End of Sketch.

As we can see, the proof of this seemingly simple result is anything but simple. The elementary proof uses no results from complex analysis, and some very basic ideas from real analysis and measure theory (moving limits inside of integrals, limsup, etc.), but there are many functions that need to be defined, many rules of calculus that are needed, and many conclusions to be drawn during the proof. The analytic proof, while it involves more difficult and advanced techniques, will prove to be shorter and “easier" to follow. We will give a full, and detailed proof of the Prime Number Theorem using the analytic technique in the next section.

3.3 Riemann Zeta Function

Definition 11. *We define the* Riemann Zeta Function $\zeta(s)$ *by,*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

if $Re(s) > 1$*, and*

$$\zeta(s) = \lim_{x\to\infty} \left(\sum_{n\le x} \frac{1}{n^s} - \frac{x^{1-s}}{1-x} \right)$$

for $0 < Re(s) < 1$.

The number $s \in \mathbb{C}$ is often written as $s = \sigma + it$.

The zeta function is connected with the prime numbers in that the following holds:

$$\begin{aligned}
\prod_{\text{p prime}} \left(1-\frac{1}{p^s}\right)^{-1} &= \prod_{\text{p prime}} \frac{1}{1-\frac{1}{p^s}}, \text{ note that this is a geometric series}\\
&= \prod_{\text{p prime}} \left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+...\right)\\
&= \left(1+\frac{1}{2^s}+\frac{1}{2^{2s}}+\frac{1}{2^{3s}}+...\right)\left(1+\frac{1}{3^s}+\frac{1}{3^{2s}}+\frac{1}{3^{3s}}+...\right)\\
&\quad \left(1+\frac{1}{5^s}+\frac{1}{5^{2s}}+\frac{1}{5^{3s}}+...\right)... \qquad (18)\\
&= 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{2^{2s}}+\frac{1}{5^s}+\frac{1}{2^s3^s}+\frac{1}{7^s}+...\\
&= 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\frac{1}{7^s}+...\\
&= \sum_{n=1}^{\infty}\frac{1}{n^s}\\
&= \zeta(s).
\end{aligned}$$

In the above argument, we use the fact that every number has a unique prime factorization (Fundamental Theorem of Arithmetic).

From these equations for $\zeta(s)$ we can get an expression for $\zeta'(s)$. Since $\zeta(s)$ is defined for $\text{Re}(s) > 1$, we can do a term-by-term differentiation of $\zeta(s)$ within this domain of convergence. So we get

$$\begin{aligned}
\zeta'(s) &= \frac{d}{ds}\left(\sum_{n=1}^{\infty}\frac{1}{n^s}\right)\\
&= \sum_{n=1}^{\infty}\frac{d}{ds}\left(\frac{1}{n^s}\right) \qquad (19)\\
&= -\sum_{n=1}^{\infty}\frac{\ln(n)}{n^s}.
\end{aligned}$$

You can do this because the series $\sum_{n=1}^{\infty}\frac{1}{n^s}$ is uniformly convergent on $\{s \mid \text{Re}(s) \geq d\}$ for all $d > 1$. We later show that this converges for all $\sigma > 1$.

For the Prime Number Theorem, we will need to work with $\frac{\zeta'(s)}{\zeta(s)}$. This means we will want to examine,

$$\begin{aligned}\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\sum_{n=1}^{\infty}\ln(n)n^{-s}}{\sum_{n=1}^{\infty}n^{-s}} \\ &= -\frac{0+\ln(2)2^{-s}+\ln(3)3^{-s}+\ln(4)4^{-s}+...}{1+2^{-s}+3^{-s}+4^{-s}+...}.\end{aligned} \tag{20}$$

For the purposes of the Prime Number Theorem, we will want to find the singularities of this. However, this becomes difficult in the form we have it in right now, as that would mean finding the zeros of the Riemann Zeta Function, which is the premise of the Riemann Hypothesis (which has not been proven). This means we need a form that is easier to work with. Consider

$$\begin{aligned}\ln(\zeta(s)) &= \ln\left(\prod_{p \text{ prime}}\left(1-\frac{1}{p^s}\right)^{-1}\right) \\ &= \sum_{p \text{ prime}}\ln\left((1-\frac{1}{p^s})^{-1}\right) \\ &= -\sum_{p \text{ prime}}\ln\left(1-\frac{1}{p^s}\right).\end{aligned} \tag{21}$$

From this, we see that

$$\begin{aligned}\frac{d}{ds}\left(\ln\zeta(s)\right) &= \frac{\zeta'(s)}{\zeta(s)} \\ &= \frac{d}{ds}\left(-\sum_{p \text{ prime}}\ln\left(1-\frac{1}{p^s}\right)\right) \\ &= -\sum_{p \text{ prime}}\frac{d}{ds}\ln\left(1-\frac{1}{p^s}\right) \\ &= -\sum_{p \text{ prime}}\frac{-\ln(p)\ p^{-s}}{1-\frac{1}{p^s}} \\ &= \sum_{p \text{ prime}}\frac{\ln(p)\ p^{-s}}{1-\frac{1}{p^s}}.\end{aligned} \tag{22}$$

Note that this holds for $Re(s) > 1$.

When working with the proof of the Prime Number Theorem, we will use many estimates for $|\zeta(s)|$ and $|\zeta'(s)|$, as well as properties of $\zeta(s)$ near the line $\sigma = 1$. Recall from our definition of the Riemann zeta function, it is not defined for $\sigma = 1$. We will show that

$-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$ is analytic on the line $\sigma = 1$. Essentially, the pole at $s = 1$ is being subtracted off, thus making this expression analytic.

The following argument makes use of the Dirichlet Eta function, given by

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \dots \, ,$$

for a complex number s with $\text{Re}(s) > 0$. Note that $\eta(1)$ is defined, as $\eta(1)$ gives the alternating harmonic series, which converges to $\ln(2)$.

Our goal is to show that $\zeta(s)$ has a singularity at $s = 1$.

We start by noting that

$$\begin{aligned}
\zeta(s) - \eta(s) &= \left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots\right) - \left(1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \dots\right) \\
&= 2\left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \dots\right) \\
&= \frac{1}{2^{s-1}}\left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots\right) \\
&= \frac{1}{2^{s-1}}\zeta(s).
\end{aligned} \tag{23}$$

Thus,

$$\eta(s) = \zeta(s)\left(1 - \frac{1}{2^{s-1}}\right).$$

Therefore,

$$\zeta(s) = \frac{\eta(s)}{1 - \frac{1}{2^{s-1}}}.$$

Now we can let s approach 1 and see that $\zeta(s)$ has a pole at $s = 1$. Note that by the coincidence principle, we can legally do the above manipulations.

We now want to show that $\zeta(s)$ has a residue of 1 at the pole $s = 1$.

$$\begin{aligned}\lim_{s\to 1}(s-1)\zeta(s) &= \eta(1)\lim_{s\to 1}\frac{s-1}{1-\frac{1}{2^{s-1}}}\\ &= \eta(1)\lim_{s\to 1}\frac{2^{s-1}(s-1)}{2^{s-1}-1}\\ &= \ln(2)\lim_{s\to 0}\frac{2^s s}{2^s-1}\\ &= \ln(2)\lim_{s\to 0}\frac{s}{2^s-1}\\ &= \ln(2)\cdot\frac{1}{\ln(2)} \text{ by definition of a derivative}\\ &= 1.\end{aligned} \tag{24}$$

Therefore $\zeta(s)$ has a residue of 1 at $s = 1$.

To show that $-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$ is analytic at $s = 1$, we use the following Lemma from [1]:

Lemma 12. *If $f(s)$ has a pole of order k at $s = a$ then the quotient $f'(s)/f(s)$ has a first order pole at $s = a$ with residue $-k$.*

Proof. Note that we can rewrite $f(s)$ as

$$f(s) = \frac{g(s)}{(s-a)^k},$$

where g is analytic at a, and $g(a) \neq 0$, since f has a pole of order k at $s = a$. Using the quotient rule, we get

$$\begin{aligned}f'(s) &= \frac{g'(s)(s-a)^k - k(s-a)^{k-1}g(s)}{(s-a)^{2k}}\\ &= \frac{g'(s)(s-a)^k}{(s-a)^{2k}} - \frac{k(s-a)^{k-1}g(s)}{(s-a)^{2k}}\\ &= \frac{g'(s)}{(s-a)^k} - \frac{kg(s)}{(s-a)^{k+1}}\\ &= \frac{g(s)}{(s-a)^k}\left[\frac{g'(s)}{g(s)} - \frac{k}{s-a}\right].\end{aligned} \tag{25}$$

Therefore,

$$\frac{f'(s)}{f(s)} = \frac{g(s)}{(s-a)^k}\left[\frac{g'(s)}{g(s)} - \frac{k}{s-a}\right]\frac{(s-a)^k}{g(s)} = \frac{g'(s)}{g(s)} - \frac{k}{s-a}.$$

Since $g(s) \neq 0$ and is analytic, it follows that $\frac{g'(s)}{g(s)}$ is analytic at $s = a$ and is equal to $\frac{f'(s)}{f(s)} + \frac{k}{s-a}$ too.

□

Thus, since $-\frac{\zeta'(s)}{\zeta(s)}$ and $\frac{1}{s-1}$ both have a first order pole at $s = 1$ with residue 1, then their difference is analytic at $s = 1$.

3.4 Idea of the Proof

For our proof of the Prime Number Theorem, we will first outline the proof without much elaboration on the ideas used or the results found. The subsequent sections will then give more details. The purpose of this is so that the proof is as straightforward as it can be, while still explaining all of the results.

When we come across a line that needs further explanation, the reader is referred to a later subsection, where the result will be explained.

Much of this proof comes from [1], but some details that are not featured in that text, will be featured in this. The goal is to have the reader follow as much of the proof as possible.

When one does a search on "proof of the Prime Number Theorem," there are many results, but almost none of these results give a straightforward proof. They prove many results and ideas, and then the PNT follows from these. Our goal is to first prove it, and then elaborate when necessary.

4 Analytic Proof of The Prime Number Theorem

Theorem 13. (The Prime Number Theorem) *Let $\pi(x)$ denote the number of primes less than or equal to x. Then $\pi(x) \sim \frac{x}{\ln(x)}$; that is, there are approximately $\frac{x}{\ln(x)}$ primes less than or equal to x.*

Proof.

The goal is to show that

$$\lim_{x\to\infty} \frac{\pi(x)\ln(x)}{x} = 1.$$

In our discussion on the elementary method, we showed that if the limit does tend to 1, then the following limits also equal 1:

$$(1) \lim_{n\to\infty} \frac{\psi(n)}{n}$$

$$(2) \lim_{n\to\infty} \frac{\vartheta(n)}{n}.$$

While much of the elementary method focuses on (2), we will work with (1) for the analytic method. Our goal is to show that $\psi(x) \sim x$, as $x \to \infty$.

Recall that we defined $\psi(x)$ as

$$\psi(x) = \sum_{n=1}^{x} \Lambda(n).$$

Note that $\psi(x)$ is a step function, growing by $\ln(p)$ whenever one reaches a power of a prime. We define a new function $\psi_1(x)$ by

$$\psi_1(x) = \int_1^x \psi(t)dt.$$

We will show that $\psi_1(x) \sim \frac{1}{2}x^2$ as $x \to \infty$.

The first result we get is that,

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds,$$

where $c > 1$. Note that this is a contour integral along the line $Re(s) = c$. This result is verified and elaborated on in section 4.1.

Working with this contour integral, we can also show that

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right)ds,$$

for $c > 1$, and $x \geq 1$ (this result is also verified in section 4.1).

Recall that our goal is to show that as $x \to \infty$, then $\psi_1(x) \sim \frac{1}{2}x^2$. So we want to show the above integral is 0. Our main problem with the above integral is when $Re(s) = 1$, since our current definition of $\zeta(s)$ is not defined for $s = 1$. Thus we need some results about $\zeta(s)$, $\zeta'(s)$, $|\zeta(s)|$, and $|\zeta'(s)|$ near $\sigma = 1$. Section 4.2 will be devoted to finding these relations, but our main result is that $\zeta(1 + it) \neq 0$, for any $t \in \mathbb{R}$.

This is also proven in section 4.2.

Recall from section 3.3 that $\zeta(s)$ has a pole at $s = 1$, but the difference $-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$ is analytic at $s = 1$.

For convenience, let $h(s) = \frac{1}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right)$.

We will devote section 4.3 to showing that

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(1 + it)e^{it\ln(x)}dt.$$

Note that we get the change in integrals by letting $\sigma = 1$ in our original integral. It will then be shown that we only need to be concerned with the integral along the line $\sigma = 1$.

Specifics are covered in section 4.3, but our general method is to do a contour integral around a rectangle $R = \{(1-iT), (c-iT), (c+iT), (1+iT)\}$ and let $T \to \infty$. We show that the integral around R must be 0, by Cauchy's Theorem. We then show that on the upper and lower lines of the rectangle, the integral is 0 and we come to this conclusion by taking bounds on $|h(s)|$. We then show that the two integrals along the vertical line segments are the same and they cancel each other. In doing so, we show that these integrals (which end up being our $h(s)$ from above) are bounded.

We can use the Riemann-Lebesgue Lemma to show that the entire integral must be 0, since we have shown that the integrals are bounded.

With our findings on $\zeta(s)$ from sections 3.3 and 4.2, we can apply the Riemann-Lebesgue Lemma to the integral, even if $c = 1$.

Therefore,

$$\frac{\psi_1(x)}{x^2} \sim \frac{1}{2}\left(1 - \frac{1}{x}\right)^2.$$

Thus as $x \to \infty$, $\psi_1(x) \sim \frac{x^2}{2}$.

Recall that $\psi_1(x) = \int_1^x \psi(t)dt$. Also recall $\psi(1) = 0$. By a lemma from [1] we can differentiate both sides of the above to get

$$\frac{d}{dx}\big(\psi_1(x)\big) = \psi(x) \sim \frac{d}{dx}\left(\frac{x^2}{2}\right) = x.$$

Therefore, as $x \to \infty$, $\psi(x) \sim x$. From our result in section 3.2, we get that since $\psi(x) \sim x$ as $x \to \infty$, then $\pi(x)\ln(x) \sim x$ as $x \to \infty$.

Hence $\pi(x) \sim \frac{x}{\ln(x)}$.

$\square$

4.1 The Contour Integral

The following Lemma comes by use of Abel's Identity.

Lemma 14. *For any arithmetical function $a(n)$, let $A(x) = \sum_{n=1}^{x} a(n)$, where $A(x) = 0$ if $x < 1$. Then*

$$\sum_{n=1}^{x}(x-n)a(n) = \int_{1}^{x} A(t)dt.$$

Proof. By Abel's identity,

$$\sum_{n=1}^{x} a(n)f(n) = A(x)f(x) - \int_{1}^{x} A(t)f'(t)dt,$$

if f has a continuous derivative on $[1, x]$ (smooth). Let $f(t) = t$.

Note that on $[1, x]$ $f(t) = t$ is smooth. Using Abel's Identity,

$$\sum_{n=1}^{x} a(n)f(n) = \sum_{n=1}^{x} a(n)n.$$

Also,

$$A(x)f(x) = x\sum_{n=1}^{x} a(n).$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{x} a(n)f(n) &= \sum_{n=1}^{x} na(n) \\ &= A(x)f(x) - \int_{1}^{x} A(t)f'(t)dt \qquad (26) \\ &= x\sum_{n=1}^{x} a(n) - \int_{1}^{x} A(t)dt. \end{aligned}$$

Thus,

$$\int_{1}^{x} A(t)dt = \sum_{n=1}^{x}(x-n)a(n). \qquad (27)$$

□

We can use this result to find a relation for $\psi_1(x)$.

Theorem 15.

$$\psi_1(x) = \sum_{n=1}^{x} (x-n)\Lambda(n).$$

This is because $\psi(x) = \sum_{n=1}^{x} \Lambda(n)$, and $\psi_1(x) = \int_1^x \psi(t)dt$. So we can use the previous lemma with $A(t) = \psi(t)$, and $a(n) = \Lambda(n)$, since the Mangoldt function is an arithmetical function.

We would like to express $\psi_1(x)$ in terms of a contour integral, as contour integrals can often be simplified and give a result that can be worked with. The following lemma will be useful in finding the contour integral.

Lemma 16. *If $c > 0$, and $u > 0$, then for every integer $k \geq 1$ we have*

$$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{u^{-z}}{z(z+1)\cdots(z+k)}dz = \begin{cases} \frac{1}{k!}(1-u)^k & \text{if } 0 < u \leq 1 \\ 0 & \text{if } u > 1, \end{cases}$$

where the integral is absolutely convergent.

For this proof, we will make use of the Gamma function, defined as

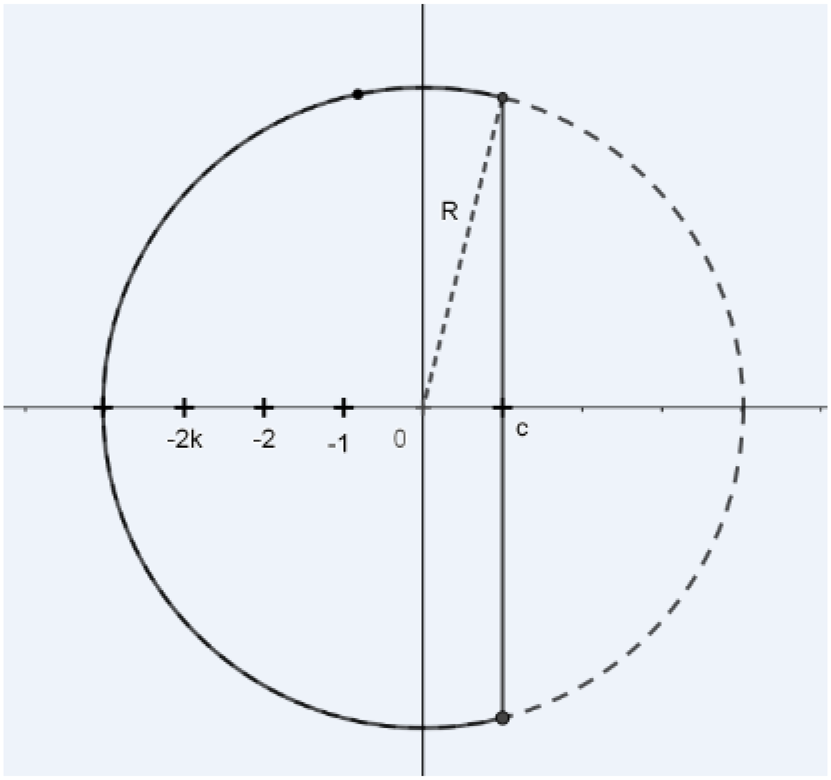
$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx.$$

Note that for any $n \in \mathbb{N}$, $\Gamma(n) = n!$.

Proof. Note that $\Gamma(z+1) = z\Gamma(z)$. So,

$$\begin{aligned} \frac{u^{-z}\Gamma(z)}{\Gamma(z+k+1)} &= \frac{u^{-z}\Gamma(z)}{(z+k)\Gamma(z+k)} \\ &= \frac{u^{-z}\Gamma(z)}{(z+k)(z+k-1)\Gamma(z+k-1)} \\ &= \ldots \\ &= \frac{u^{-z}\Gamma(z)}{(z+k)(z+k-1)\cdots z\Gamma(z)} \\ &= \frac{u^{-z}}{(z+k)(z+k-1)\cdots z}. \end{aligned} \tag{28}$$

We can now apply Cauchy's Residue Theorem to $\frac{1}{2\pi i}\int_{C(R)} \frac{u^{-z}\Gamma(z)}{\Gamma(z+k+1)}dz$, where $C(R)$ is a



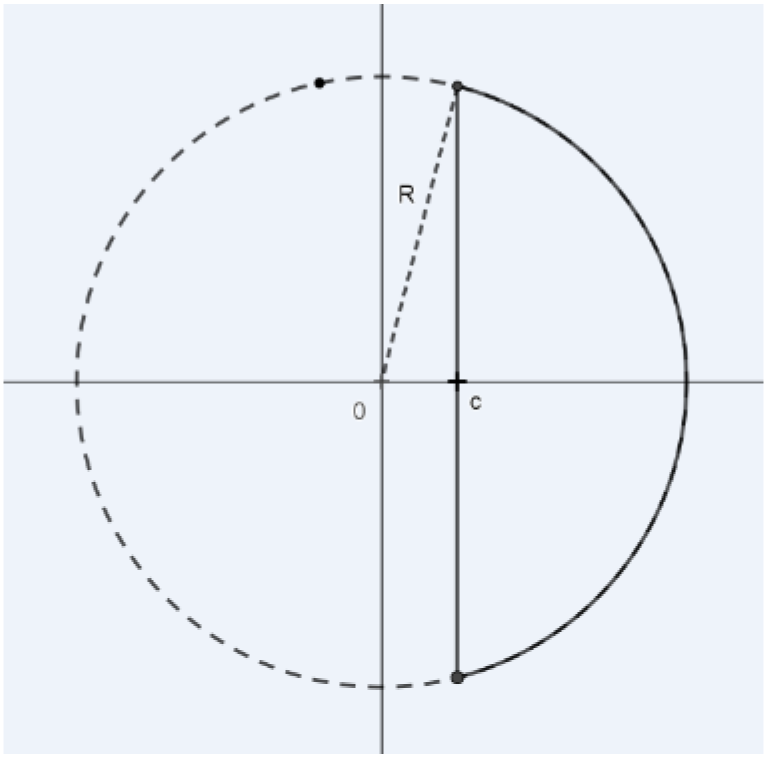
contour we will define momentarily.

Recall that Cauchy's Residue Theorem states that if C is a simple, closed, positively oriented contour in the complex plane, and a function f is analytic except for some sequence of points $\{z_1, z_2, ..., z_n\}$ inside C, then

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} Res_f(z_k).$$

For this problem, we will have two different contours $C(R)$ to consider, depending on the value of u.

This is a circle, where we are cutting off at the line of $\text{Re}(z) = c$, and integrating the left side counterclockwise (positively). This will be used for $0 < u \leq 1$.



This is a circle also cut off at real part $\mathrm{Re}(z) = c$, but we are integrating the right side in a counterclockwise direction. This will be used for $u > 1$.

The reason we choose these contours is so that the circular parts will be zero. We need the circle to miss the poles of the Gamma function, which occur at negative integers. So we need to make sure the circle's radius is strictly greater than $2k + c$, as to miss these points. We will show that as the radius $R \to \infty$, the integrals along the circular arcs go to zero.

Let $z = x + iy$ and $|z| = R$. Since $u > 0$ we have,

$$\begin{aligned} \left| \frac{u^{-z}}{z(z+1)\cdots(z+k)} \right| &= \frac{|u^{-x}||u^{-iy}|}{|z||z+1|\cdots|z+k|} \\ &= \frac{|u^{-x}|}{|z||z+1|\cdots|z+k|} \\ &\leq \frac{u^{-c}}{R|z+1|\cdots|z+k|}, \end{aligned} \tag{29}$$

where the inequality holds because u^{-x} is increasing on $0 < u \leq 1$ and decreasing for $u > 1$.

Let $1 \leq n \leq k$. Then we have

$$|z+n| \geq |z| - n = R - n \geq R - k \geq R/2,$$

since $R > 2k + c$.

Now we have,

$$\begin{aligned} \left| \frac{u^{-z}}{z(z+1)\cdots(z+k)} \right| &\leq \frac{u^{-c}}{R|z+1|\cdots|z+k|} \\ &\leq \frac{u^{-c}}{R(R/2)\cdots(R/2)} \\ &= \frac{u^{-c}}{R(R/2)^k}. \end{aligned} \tag{30}$$

From this we get the following bound for the integral around the circular arc, C:

$$\begin{aligned} \left| \int_C \frac{u^{-z}}{z(z+1)\cdots(z+k)} dz \right| &\leq \left| \int_C \frac{u^{-z}}{z(z+1)\cdots(z+k)} R dz \right| \\ &\leq \int_0^{2\pi} \frac{u^{-c}}{R(R/2)^k} R dz \\ &\leq \frac{2\pi u^{-c}}{(R/2)^k}, \end{aligned} \tag{31}$$

by directly evaluating the last integral.

Letting $R \to \infty$, this tends to 0. So we will examine how these contour integrals behave for different values of u.

Case 1: Consider $u > 1$.

Then inside $C(R)$, the integrand is analytic; therefore $\int_{C(R)} f(z) = 0$. So letting $R \to \infty$, we have that the integral is 0.

Case 2: Consider $0 < u \leq 1$. We will use Cauchy's Residue Theorem (stated above).

Note that this integral has poles at $n = 0, -1, -2, ..., -k$. Thus,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{C(R)} \frac{u^{-z}}{z(z+1)\cdots(z+k)}dz &= \sum_{n=0}^{k} \text{Res}_{z=-n}\frac{u^{-z}}{z(z+1)\cdots(z+k)} \\
&= \sum_{n=0}^{k} \lim_{z\to -n}(z+n)\frac{u^{-z}}{z(z+1)\cdots(z+k)} \\
&= \sum_{n=0}^{k} \frac{u^{n}}{(-n)(-n+1)\cdots(-1)\cdot 1\cdot 2\cdots(k-n)} \\
&= \sum_{n=0}^{k} \frac{u^{n}}{(-1)^{n}n!(k-n)!} \\
&= \sum_{n=0}^{k} \frac{u^{n}(-1)^{n}}{n!(k-n)!} \\
&= \sum_{n=0}^{k} \binom{k}{n}\frac{u^{n}(-1)^{n}}{k!} \\
&= \frac{1}{k!}\sum_{n=0}^{k} \binom{k}{n}(-u)^{n} \\
&= \frac{1}{k!}(1-u)^{k}.
\end{aligned} \tag{32}$$

□

We can now get a contour integral for $\frac{\psi_1(x)}{x^2}$.

Theorem 17. *If $c > 1$ and $x \geq 1$, then*

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds.$$

Proof.

By a previously proven theorem,

$$\frac{\psi_1(x)}{x} = \sum_{n=1}^{x}\left(\frac{1-n}{x}\right)\Lambda(n).$$

Now by the previous lemma, letting $k = 1$ and $u = n/x$ with $n \leq x$ we get

$$\begin{aligned} 1 - \frac{n}{x} &= \frac{1}{1!}(1 - n/x)^1 \\ &= \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{(x/n)^s}{s(s+1)} ds. \end{aligned} \tag{33}$$

Thus,

$$\Lambda(n) - \Lambda(n)\frac{n}{x} = \Lambda(n)\left(1 - \frac{n}{x}\right) = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{\Lambda(n)(x/n)^s}{s(s+1)} ds.$$

Recall from a previous result that $\psi_1(x)/x = \sum_{n=1}^{x} \Lambda(n)(1 - (n/x))$. So,

$$\frac{\psi_1(x)}{x} = \sum_{n=1}^{x} \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{\Lambda(n)(x/n)^s}{s(s+1)} ds = \sum_{n=1}^{x} \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{\Lambda(n)(x/n)^s}{s(s+1)} ds.$$

We want to be able to interchange the summation and integration symbols. Note that the partial sums satisfy

$$\begin{aligned} \sum_{k=1}^{N} \left| \int_{c-\infty i}^{c+\infty i} \frac{\Lambda(k)(x/k)^c}{s(s+1)} ds \right| &= \sum_{k=1}^{N} \int_{c-\infty i}^{c+\infty i} \frac{\Lambda(k)(x/k)^c}{|s||s+1|} ds \\ &= \sum_{k=1}^{N} \frac{\Lambda(k)}{k^c} \int_{c-\infty i}^{c+\infty i} \frac{x^c}{|s||s+1|} ds \\ &\leq \Omega \sum_{k=1}^{N} \frac{\Lambda(k)}{k^c}, \end{aligned} \tag{34}$$

where Ω is a constant.

Since the partial sums are bounded absolutely, the series must converge absolutely, and uniformly converge. So we can interchange the summation and integration symbols. Therefore,

$$\begin{aligned}
\frac{\psi_1(x)}{x} &= \frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{c-\infty i}^{c+\infty i}\frac{\Lambda(n)(x/n)^s}{s(s+1)}ds \\
&= \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\sum_{n=1}^{\infty}\frac{\Lambda(n)(x/n)^s}{s(s+1)}ds \\
&= \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{x^s}{s(s+1)}\left(\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\right)ds \\
&= \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{x^s}{s(s+1)}\sum_{p \text{ prime}}\left(\sum_{\alpha=1}^{\infty}\frac{\ln(p)}{p^{\alpha s}}\right)ds \\
&= \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{x^s}{s(s+1)}\sum_{p \text{ prime}}\ln(p)\frac{1/p^s}{1-1/p^s}ds \\
&= \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{x^s}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds.
\end{aligned} \tag{35}$$

Dividing by x yields the desired result.

□

Theorem 18. *If $c > 1$ and $x \geq 1$ we have*

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1-\frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} x^{s-1}h(s)ds,$$

where

$$h(s) = \frac{1}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right).$$

Proof.

$$\begin{aligned}
\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 &= \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds - \frac{1}{2}\left(1-\frac{1}{x}\right)^2 ds \\
&= \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds - \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{(1/x)^{-s}}{s(s+1)(s+2)}ds \\
&= \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) - \frac{x^{s-1}}{(s-1)(s)(s+1)}ds \\
&= \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right)ds \\
&= \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} x^{s-1}h(s)ds.
\end{aligned} \tag{36}$$

Note that we get from line 1 to line 2 by using a previous lemma for $k = 2$ and $u = 1/x$, and from lines 2 to 3 by replacing s with $s - 1$.

□

So we need to show that as $x \to \infty$ the integrand $\to 0$. Having the integrand tend to 0 as $x \to \infty$ will then imply that $\psi_1(x) \sim x^2/2$ as $x \to \infty$. Letting $c = 1$, we have

$$\int_{1-\infty i}^{1+\infty i} h(1+it)e^{it\ln(x)}dt.$$

Thus we need to examine $\zeta(s)$ and $\zeta'(s)$ near $\sigma = 1$. However, our current definition of the Riemann Zeta Function isn't valid for $\sigma = 1$. We will expand on this in the next section.

4.2 The Riemann Zeta Function Near s=1

This subsection will be devoted to finding bounds, estimates, and properties of $\zeta(s)$ and $\zeta'(s)$ near the line $\sigma = 1$.

We first prove the following lemma:

Lemma 19.

$$\zeta(s) = \sum_{n=1}^{N} -s \int_N^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}} dx + \frac{N^{1-s}}{s-1}.$$

Proof.

Euler's summation formula states that if f is smooth (continuous derivative) on $[a, b]$, $0 < a < b$, then

$$\sum_{a \le n \le b} f(n) = \int_a^b f(t)dt + \int_a^b \big(t - \lfloor t \rfloor\big) f'(t)dt + f(b)\big(\lfloor b \rfloor - b\big) - f(a)\big(\lfloor a \rfloor - a\big).$$

Taking $f(n) = n^{-s}$, $a = N$, and $b \to \infty$, we get

$$\begin{aligned}
\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \\
&= \sum_{n=1}^{N} \frac{1}{n^s} + \sum_{n=N+1}^{\infty} \frac{1}{n^s} \\
&= \sum_{n=1}^{N} \frac{1}{n^s} + \int_N^\infty \frac{1}{t^s} dt + \int_N^\infty \big(t - \lfloor t \rfloor\big)\big(-st^{-s-1}\big)dt + \lim_{x\to\infty} \frac{\lfloor x \rfloor - x}{x^s} - \frac{\lfloor N \rfloor - N}{N^s} \\
&= \sum_{n=1}^{N} \frac{1}{n^s} + \int_N^\infty \frac{1}{t^s} dt - s\int_N^\infty \frac{t - \lfloor t \rfloor}{t^{s+1}} dt + \lim_{x\to\infty} \frac{\lfloor x \rfloor - x}{x^s} \\
&= \sum_{n=1}^{N} \frac{1}{n^s} + \left[\frac{1}{t^{s-1}(s-1)}\right]_N^\infty - s\int_N^\infty \frac{t - \lfloor t \rfloor}{t^{s+1}} dt + \lim_{x\to\infty} \frac{\lfloor x \rfloor - x}{x^s} \\
&= \sum_{n=1}^{N} \frac{1}{n^s} - \frac{1}{N^{s-1}(s-1)} - s\int_N^\infty \frac{t - \lfloor t \rfloor}{t^{s+1}} dt + \lim_{x\to\infty} \frac{\lfloor x \rfloor - x}{x^s} \\
&= \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{-(s-1)}}{s-1} - s\int_N^\infty \frac{t - \lfloor t \rfloor}{t^{s+1}} dt + \lim_{x\to\infty} \frac{\lfloor x \rfloor - x}{x^s} \\
&= \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s\int_N^\infty \frac{t - \lfloor t \rfloor}{t^{s+1}} dt + \lim_{x\to\infty} \frac{\lfloor x \rfloor - x}{x^s}.
\end{aligned} \tag{37}$$

Since

$$\left|\frac{\lfloor x\rfloor - x}{x^s}\right| \leq \left|\frac{1}{x^s}\right|$$

and $\lim_{x\to\infty} \frac{1}{x^s} = 0$, the result holds.

□

Using this representation of $\zeta(s)$, we can see that

$$\zeta'(s) = -\sum_{n=1}^{N} \frac{\ln(n)}{n^s} + s\int_N^\infty \frac{\big(x - \lfloor x\rfloor\big)\ln(x)}{x^{s+1}}dx - \int_N^\infty \frac{x - \lfloor x\rfloor}{x^{s+1}}dx - \frac{N^{1-s}\ln(N)}{s-1} - \frac{N^{1-s}}{(s-1)^2}.$$

We can now use these results, along with discussion from section 3.3 to find approximations and results for the Riemann zeta function near the line $\mathrm{Re}(s) = 1$.

This representation can be used to show that for all $\alpha > 0$, $|\zeta(s)| = \mathcal{O}(\ln(t))$ and $|\zeta'(s)| = \mathcal{O}(\ln^2(t))$ for $\sigma \geq 1/2$ and $t \geq e$.

The following results come from [1].

To show that $\zeta(1+it) \neq 0$, we use the following:

Theorem 20. *For $\sigma > 1$,*

$$\zeta^3(\sigma)\big|\zeta(\sigma+it)\big|^4\big|\zeta(\sigma+2it)\big| \geq 1.$$

To prove this, we make use of a result from [1] which states that for $\mathrm{Re}(s) > 1$, $\zeta(s) = e^{F(s)}$, where $F(s) = \sum_{n=2}^\infty \frac{\Lambda(n)}{\ln(n)}\frac{1}{n^s}$.

Some manipulation can be done with this representation, and also the fact that $\sum_{n=2}^\infty \frac{\Lambda(n)}{\ln(n)}\frac{1}{n^s} = \sum_{p \text{ prime}}\sum_{m=1}^\infty \frac{1}{mp^{ms}}$, for $\sigma > 1$.

Then,

$$\zeta^3(\sigma) = \exp\left(\sum_p\sum_{m=1}^\infty \frac{1}{mp^{m\sigma}}\right)^3 = \exp\left(\sum_p\sum_{m=1}^\infty \frac{3}{mp^{m\sigma}}\right).$$

Also, $|\zeta(s)| = \exp\left(\sum_p\sum_{m=1}^\infty \frac{\cos(mt\ln(p))}{mp^{m\sigma}}\right)$.

So,

$$\big|\zeta(\sigma+it)\big|^4 = \exp\left(\sum_p\sum_{m=1}^\infty \frac{4\cos\big(mt\ln(p)\big)}{mp^{m\sigma}}\right).$$

Using this we also get

$$\left|\zeta(\sigma+2it)\right| = \left(\sum_p \sum_{m=1}^{\infty} \frac{\cos\big(2mt\ln(p)\big)}{mp^{m\sigma}}\right).$$

Multiplying these three results means we are multiplying exponentials, hence are adding the fractions in the exponents. Our fraction looks like

$$\frac{3+4\cos\big(mt\ln(p)\big)+\cos\big(2mt\ln(p)\big)}{mp^{m\sigma}}.$$

Letting $\theta = mt\ln(p)$, we use the fact that,

$$\begin{aligned} 3+4\cos\theta+\cos 2\theta &= 3+4\cos\theta+2\cos^2\theta-1 \\ &= 2\cos^2\theta+4\cos\theta+2 \\ &= 2\big(\cos^2\theta+2\cos\theta+1\big) \\ &= 2\big(\cos\theta+1\big)^2 \\ &\geq 0. \end{aligned} \tag{38}$$

So,

$$\zeta^3(\sigma)\left|\zeta(\sigma+it)\right|^4\left|\zeta(\sigma+2it)\right| \geq \exp\left(\sum_p \sum_{m=1}^{\infty} \frac{0}{mp^{m\sigma}}\right) = e^0 = 1.$$

Therefore the result will hold.

We now have everything we need to prove our goal result.

Theorem 21. $\zeta(1+it) \neq 0$ *for any* $t \in \mathbb{R}$.

Proof.

We can divide the above result by $\sigma-1$ for $\sigma>1$, to get

$$\frac{\zeta^3(\sigma)\left|\zeta(\sigma+it)\right|^4\left|\zeta(\sigma+2it)\right|}{\sigma-1} \geq \frac{1}{\sigma-1}$$

$$\frac{(\sigma-1)^3\zeta^3(\sigma)\left|\zeta(\sigma+it)\right|^4\left|\zeta(\sigma+2it)\right|}{(\sigma-1)^4} \geq \frac{1}{\sigma-1}$$

$$\big((\sigma-1)\zeta(\sigma)\big)^3\left|\frac{\zeta(1+it)}{\sigma-1}\right|^4\left|\zeta(1+2it)\right| \geq \frac{1}{\sigma-1}.$$

We examine first $\lim_{\sigma\to 1^+}((\sigma-1)(\zeta(\sigma))^3$.

Note that since $\zeta(s)$ has a residue 1 at a pole $s=1$, then,

$$\lim_{\sigma\to 1^+}(\sigma-1)\zeta(s)=1.$$

Also,

$$\lim_{\sigma\to 1^+}\zeta(\sigma+2it)=\zeta(1+2it).$$

In order to show that $\zeta(1+it)\neq 0$, assume for a contradiction that $\zeta(1+it)=0$ for some $t\in\mathbb{R}$.

Then,

$$\left|\frac{\zeta(\sigma+it)}{\sigma-1}\right|^4=\left|\frac{\zeta(\sigma+it)-\zeta(1+it)}{\sigma-1}\right|^4,$$

by assumption. Taking the limit of the above as $\sigma\to 1^+$ gives us the derivative of $\zeta(1+it)$. So,

$$\lim_{\sigma\to 1^+}\left|\frac{\zeta(\sigma+it)-\zeta(1+it)}{\sigma-1}\right|^4=\left|\zeta'(1+it)\right|^4.$$

Using the above results, we see that

$$\lim_{\sigma\to 1^+}\frac{\zeta^3(\sigma)\left|\zeta(\sigma+it)\right|^4\left|\zeta(\sigma+2it)\right|}{\sigma-1}=\left|\zeta'(1+it)\right|^4\left|\zeta(1+2it)\right|;$$

however $\lim_{\sigma\to 1^+}\frac{1}{\sigma-1}\to\infty$.

These two results contradict one another. Therefore $\zeta(1+it)\neq 0$, for any $t\in\mathbb{R}$.

□

This is the main result we need in order to prove the Prime Number Theorem, but one more result is necessary.

Theorem 22. *For $\sigma\geq 1$ and $t\geq e$, there exists $M>0$, such that*

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right|<M\ln^9(t).$$

We will not present the proof of this here, but it is the result of previous results, and the reader is referred to [1] for further clarification.

4.3 The Final Step

We now want to show that as $x \to \infty$

$$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}-\frac{1}{s-1}\right)\to 0.$$

We can use information from sections 3.3, 4.1 and 4.2 to show this.

Theorem 23. *[1] For $x \geq 1$,*

$$\frac{\psi_1(x)}{x^2}-\frac{1}{2}\left(1-\frac{1}{x}\right)^2=\frac{1}{2\pi}\int_{-\infty}^{\infty}h(1+it)e^{it\ln(x)}dt,$$

where $\int_{-\infty}^{\infty}|h(1+it)|dt$ converges. Thus, by the Riemann-Lebesgue Lemma, $\psi_1(x) \sim x^2/2$, and thus $\psi(x) \sim x$ as $x \to \infty$.

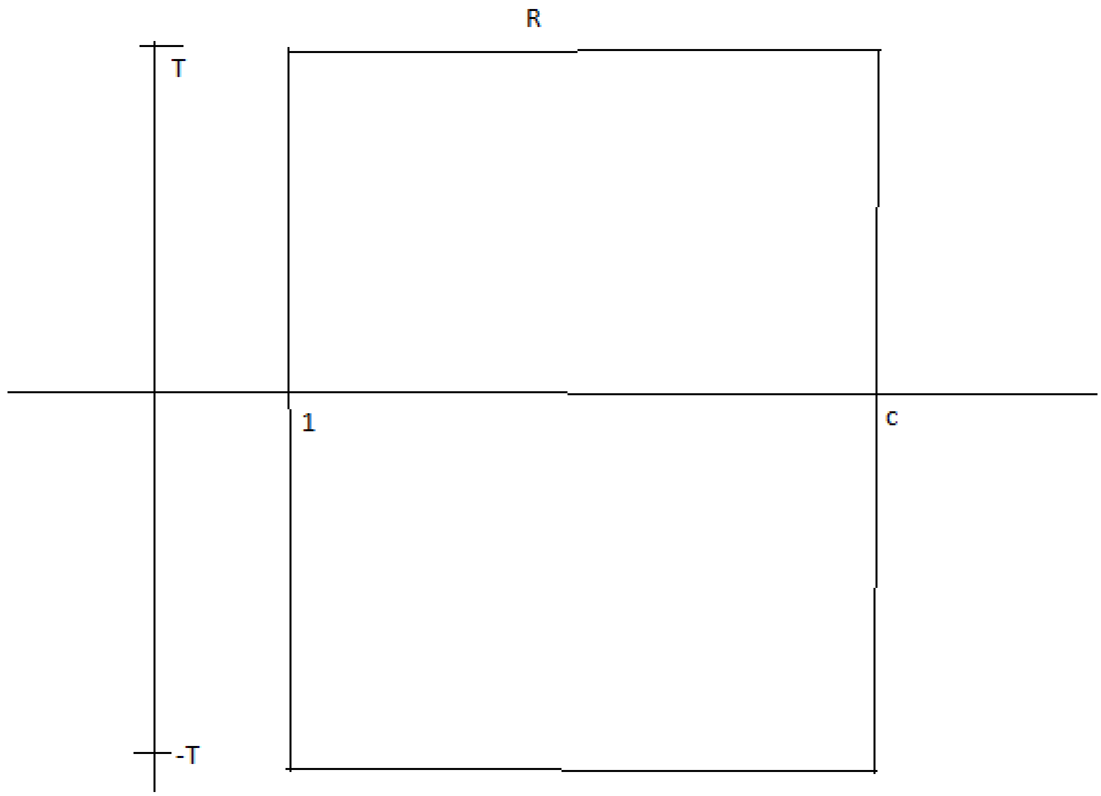
Proof.

Recall that 4.1 was devoted to proving that

$$\frac{\psi_1(x)}{x^2}-\frac{1}{2}\left(1-\frac{1}{x}\right)^2=\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}x^{s-1}h(s)ds,$$

where $h(s)=\frac{1}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}-\frac{1}{s-1}\right)$. This result is true for $c > 1$ and $x \geq 1$. We want to show this also holds for $c = 1$.

Consider the region R below.



Note that this is a simple, closed, positively oriented curve. Since $x^{s-1}h(s)$ is analytic inside and on R, we can use Cauchy's Theorem to see that

$$\int_R x^{s-1}h(s)ds = 0.$$

We can use this fact to help us evaluate parts of this integral. Our goal is to see what happens as $T \to \infty$.

Consider the upper line on R. Note that

$$\begin{aligned} \left|\frac{1}{s(s+1)}\right| &= \left|\frac{1}{(\sigma+it)(\sigma+1+it)}\right| \\ &= \left|\frac{1}{(\sigma+iT)(\sigma_1+iT)}\right| \\ &= \left|\frac{1}{\sigma+iT}\right|\left|\frac{1}{\sigma_1+iT}\right| \\ &\leq \frac{1}{T^2}. \end{aligned} \tag{39}$$

Similarly,

$$\left|\frac{1}{s(s+1)(s-1)}\right| \leq \frac{1}{T^3} \leq \frac{1}{T^2},$$

since $T \geq 1$.

Also, if $T \geq e$, then by section 4.2, there exists a constant $M > 0$, such that

$$\left|\zeta'(s)/\zeta(s)\right| \leq M \ln^9(T),$$

if $\sigma \geq 1$. Thus,

$$|h(s)| \leq \frac{M \ln^9(T)}{T^2}.$$

Using this result, we have that

$$\left|\int_{1+iT}^{c+iT} x^{s-1} h(s) ds\right| \leq \int_1^c x^{c-1} \frac{M \ln^9(T)}{T^2} d\sigma,$$

since here we are only considering the real part of the integrand. Integrating with respect to σ yields,

$$\left|\int_{1+iT}^{c+iT} x^{s-1} h(s) ds\right| \leq M x^{c-1} \frac{\ln^9(T)}{T^2}(c-1).$$

Now letting $T \to \infty$, we get the following:

$$\lim_{T\to\infty} \frac{\ln^9(T)}{T^2} \to 0. \tag{40}$$

So as $T \to \infty$,

$$\int_{1+iT}^{c+iT} x^{s-1} h(s) ds \to 0,$$

and

$$\int_{1-iT}^{c-iT} x^{s-1}h(s)ds \to 0.$$

Since we know the entire integral around R must be 0, and the top and bottom parts are 0, then the vertical integrals must have the same value, and hence cancel out (since we are integrating them in different directions). In other words,

$$\int_{1-\infty i}^{1+\infty i} x^{s-1}h(s)ds = \int_{c-\infty i}^{c+\infty i} x^{s-1}h(s)ds.$$

Recall that the integral along the right side (from $c-\infty i$ to $c+\infty i$) was our resulting integral for $\psi_1(x)/x^2 - \frac{1}{2}\left(1-1/x\right)^2$. So we now see that we can integrate from $(1-\infty i)$ to $(1+\infty i)$ and get the same result. Section 4.2 was devoted to studying the Riemann zeta function near the line of $\sigma = 1$. So our discussion and findings there assure us that we can integrate along the $\sigma = 1$ line without any problems since $\zeta(1+it) \neq 0$, for any $t \in \mathbb{R}$.

So for $\sigma = 1$,

$$\frac{1}{2\pi i}\int_{1-\infty i}^{1+\infty i} x^{s-1}h(s)ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\ln(x)}h(1+it)dt.$$

Note that

$$\int_{-\infty}^{\infty} \left|h(1+it)\right|dt = \int_{-\infty}^{-e} \left|h(1+it)\right|dt + \int_{-e}^{e} \left|h(1+it)\right|dt + \int_{e}^{\infty} \left|h(1+it)\right|dt.$$

By an earlier discussion, we know the integral from e to ∞ converges (it is $\leq \frac{M\ln^9(t)}{t^2}$). The integral from $-e$ to e is finite as well. Using a similar argument as the $\int_e^\infty$ case, $\int_{-\infty}^{-e}$ converges as well. So, $\int_{-\infty}^{\infty} |h(1+it)|dt$ converges. So we can apply the Riemann-Lebesgue Lemma, mentioned at the beginning of this section, to get that the integral tends to 0 as $x \to \infty$.

Therefore,

$$\lim_{x\to\infty} \frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1-\frac{1}{x}\right)^2 = 0.$$

So $\psi_1(x)/x^2 \sim \frac{1}{2}\left(1-1/x\right)^2$. Note that as $x \to \infty$, $\frac{1}{2}\left(1-1/x\right)^2 \to 1/2$.

Therefore $\psi_1(x) \sim x^2/2$.

To finish this proof we use a lemma from [1] which states the following:

Let $A(x) = \sum_{n=1}^{x} a(n)$ and let $A_1(x) = \int_1^x A(t)dt$. Assume that $a(n) \geq 0$ for all n. If $A_1(x) \sim Ax^c$, for $c > 0$ and $A > 0$, then $A(x) \sim cAx^{c-1}$.

So we can differentiate $\psi_1(x)$ and $x^2/2$ to get that $\psi(x) \sim x$.

Recall that from section 3.2, we showed that the above is equivalent to $\pi(x) \sim \frac{x}{\ln(x)}$. Therefore,

$$\lim_{x \to \infty} \pi(x) = \frac{x}{\ln(x)},$$

thus proving the Prime Number Theorem.

$\square$

5 Conclusion

This proof of the Prime Number Theorem had a big influence on the theory of functions of a complex variable. Over the years there have been several other proofs of this theorem, but almost all of them involve some use of complex analysis.

We end this project with a discussion on the Riemann Hypothesis.

The Riemann zeta function, as we have previously defined it, is only defined for a certain portion of the complex plane. Our typical series representation doesn't work for any number less than 0, or equal to 1. What about negative values of s?

Let's first consider $\zeta(-1)$. With our current representation, this would be

$$\zeta(-1) = \sum_{n=1}^{\infty} \frac{1}{n^{-1}} = \sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + ...$$

So $\zeta(-1)$ would be determined by adding up the positive integers, if it were written in this form. A quick Google search will reveal that $\zeta(-1) = -\frac{1}{12}$. Now, from our above formula, this doesn't appear to be correct, so we need a new way to define $\zeta(s)$ for values of $s < 0$.

Note that there is an incorrect way to arrive at this result. The following is motivated from a YouTube video by the user *Numberphile*:

We know that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ...$$

This is from knowledge about geometric series.

Let $S = 1 + 2 + 3 + 4 + ...$

From the series above we get $\frac{d}{dx}(\frac{1}{1-x}) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + ...$ By letting $x = -1$, we see that

$$\frac{1}{4} = 1 - 2 + 3 - 4 + ...$$

So

$$
\begin{aligned}
S - \frac{1}{4} &= (1 + 2 + 3 + 4 + ...) - (1 - 2 + 3 - 4 + ...) \\
&= 4 + 8 + 12 + 16 + ... \\
&= 4(1 + 2 + 3 + 4 + ...) \\
&= 4S
\end{aligned} \tag{41}
$$

So, $S - \frac{1}{4} = 4S$. Therefore, $S = -\frac{1}{12}$.

The interesting thing about this argument is that everything is seemingly legitimate. The problem with this "proof" is that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$ is valid only for $-1 < x < 1$.

There are a few ways to define $\zeta(s)$, for negative values of s. One includes using Bernoulli numbers, but this representation is slightly complicated, so we will examine another. This way involves the functional equation

$$\zeta(s) = \Gamma(1-s)\zeta(1-s)\sin\left(\frac{\pi s}{2}\right)2^s\pi^{s-1}$$

This equation comes from [6]. Note that this new representation is valid for $s < 0$, as we avoid the poles of $\Gamma(z)$ and can use our previous definition of $\zeta(s)$.

From this formula, we have that

$$\zeta(-1) = \Gamma(2)\zeta(2)\sin\left(\frac{-\pi}{2}\right)\frac{1}{2}\frac{1}{\pi^2} = -\frac{1}{12}.$$

Looking at this representation of $\zeta(s)$, we can see that whenever $s = -2k$ for $k \in \mathbb{N}$, we will have a zero. Thus, for natural numbers k, $\zeta(-2k) = 0$. Another result to note is that $\Gamma(z) \neq 0$, for any z, and also 2^s and π^s are never 0. Thus, the only zeros we need to be concerned with are those of the zeta function and sine function, which for the sine we know are at multiples of π.

So one might ask if every zero of $\zeta(s)$ happens at a negative even number? These are called the "simple" zeros of the Riemann Zeta Function. There are actually zeros of the zeta function on the line $\text{Re}(s) = 1/2$. It is unknown if there are infinitely many zeros on this line and if this covers all of the zeros. This problem was posed by Riemann and is called the Riemann Hypothesis. This is one of the Millenium Problems that, if solved, will give the solver a $1,000,000 prize and plenty of notoriety.

References

[1] Apostol, Tom M. *Introduction to Analytic Number Theory* Springer. 1976. Print.

[2] Parveen, Nikhat. *Fibonacci in Nature.* UGA. jwilson.coe.uga.edu

[3] Erdos, P. *On A New Method In Elementary Number Theory Which Leads To An Elementary Proof Of The Prime Number Theorem.* Department of Mathematics Syracuse University. 1949. Print

[4] Boas, Ralph P. *Invitation to Complex Analysis.* MAA Textbooks. 2010. Print.

[5] Thomson, Brian S., Bruckner, Judith B., Bruckner, Andrew M. *Elementary Real Analysis.* Second Edition. Prentice Hall. 2008. Print.

[6] Sondow, Jonathan and Weisstein, Eric W. "Riemann Zeta Function." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/RiemannZetaFunction.html