The Effects of Using 3D Printed Manipulatives in College Trigonometry

by:

Emily Hoopes

Submitted in Partial Fulfillment of the Requirements

for the Degree of

Master of Science

in the

Mathematics

Program

YOUNGSTOWN STATE UNIVERSITY

May, 2018

The Effects of Using 3D Printed Manipulatives in College Trigonometry

Emily Hoopes

I hereby release this thesis to the public. I understand that this **thesis** will be made available from the OhioLINK ETD Center and the Maag Library Circulation Desk for public access. I also authorize the University or other individuals to make copies of this thesis as needed for scholarly research.

Signature:

Emily Hoopes, Student Date

Approvals:

Dr. Anita O'Mellan, Thesis Advisor Date

Dr. Lucy Kerns, Committee Member Date

Dr. Megan List, Committee Member Date

Dr. Salvatore A. Sanders, Dean of Graduate Studies Date

Abstract

Manipulatives are learning tools in mathematics used to make abstract concepts more concrete for students when they are first learning the material. 3D printing is a relatively new technology whose use in education is still being explored. One way that 3D printing can impact manipulatives is by enhancing the perceptual richness of the manipulative, or how the object looks and feels. In this study, we explored the effectiveness of using manipulatives in a college trigonometry class during a lesson on proving trigonometric identities. We wanted to discover if the material used to create the manipulative impacted the level of its effectiveness and usability. The significance value of the independent t-test used to answer this questions was 0.41, so we did not find sufficient evidence to conclude one type of manipulative was more effective than the other. However, we did find that the manipulatives in general tended to help struggling students more than high-achieving. Prior research and implications for teaching are discussed.

Table of Contents

Introduction

As I was teaching my trigonometry class at Youngstown State University, I found that a topic many trigonometry students struggle with is proving trigonometric identities. Completing proofs requires a different form of mathematical thinking from solving equations and using formulas. Since rewriting trigonometric expressions is an essential skill in future calculus classes, teaching trigonometric identities is a subject worth researching. While researching ways of teaching trigonometric proofs, I happened across a YouTube video series of students explaining how to prove the identities using paper manipulatives with the various trigonometric functions (Ostos, 2017). It led to a link with the written instructions given to these students for this project. The original project was to have students create the manipulatives representing the fundamental trigonometric identities using construction paper and then using the pieces to prove an assigned identity. This project, combined with my background in 3D printing, made me wonder if there was a way for me to model it in my own classroom.

After finding a feasible and cost efficient way of modifying the set and printing out enough sets for my classroom, I did so. One modification to the cited example was that students did not make their own pieces due to a greater time restriction in a college environment. There exist a few studies comparing the levels of effectiveness of virtual and concrete manipulatives, where manipulatives are defined as any object that may be used to represent a mathematical concept (Reimer and Moyer, 2005; Hunt, Nipper, & Nash, 2011; Satsangi & Bouck, 2014). I decided to address a variant of this question: would students benefit more from 3D printed manipulatives rather than paper ones?

This question prompted the design of the experiment. Both classes would complete the same guided activity for discovering the manipulatives and trigonometric concepts behind them;

however one class would use the 3D printed manipulatives while the other class would use the paper cutout manipulatives. 3D printing has only become more mainstream in education in the last few years, so there exists very little prior research on the benefits of using 3D printers to create classroom learning tools.

The purpose of this research is therefore two-fold. First, does a discovery-type learning experience with the given mathematical manipulative positively impact student learning at the college level? Secondly, is there a significant difference in student learning between the 3D printed manipulatives and the paper manipulatives?

Review of Literature

Educational Manipulatives

Civilizations have used manipulatives for centuries. One of the first manipulatives was the counting board in the Middle East, which was a tray of sand in which people would draw symbols to tally objects (Hand2Mind, n.d.). The use of manipulatives to teach elementary mathematics was initiated in the early 1900's based on the work of educational pioneers, like Friedrich Froebel and Maria Montessori, who supported the use of hands-on activities to teach (Hand2Mind, n.d.). While manipulatives have been repeatedly shown to be beneficial in primary grades (Suydam, 1986; Sowell, 1989; Clements & McMillen, 1996), the National Council of Mathematics Teachers (NCTM) highly encourages the use of manipulatives at all grade levels, not only in elementary school. Some studies have found that the benefits of using manipulatives can be found at any level of mathematics (Clements & McMillen, 1996; Hand2Mind, n.d.; Sutton & Kreuger, 2002).

Today, mathematical manipulatives are mainly used in the primary grades. The education theory behind this practice involves Piaget's stages of learning in children. The concrete operational age, usually 7-11 years old, is when children are developing their reasoning with concrete objects and cannot grasp abstract concepts as well. As children grow older, their abstract thinking skills improve, which implies that they need concrete representations less. Despite this reasoning, there have been some conflicting studies about whether or not secondary students also benefit from using manipulatives, and several studies showing that manipulatives have some benefits even when used by college students (Gningue, Menil & Fuchs, 2014; Thirey & Wooster, 2013; Kolpas, 2007).

Manipulatives are defined as "physical objects handled by individual students and small groups" (National Council of Supervisors of Mathematics, 2013, par 3). A manipulative may be an object, such as tangrams, specifically designed to represent a concept, or it may be an everyday item, like buttons or seeds. The use of manipulatives in the classroom satisfies many of the NCTM's standards for mathematical practice, including having the students engage in problem-solving, connecting different concepts in mathematics, communicating mathematical ideas effectively, and representing mathematical ideas in a variety of ways.

The idea of manipulatives is strongly linked to the NCTM standard of representing mathematical ideas in multiple ways. While using manipulatives, teachers often use the concrete-representational-abstract model. In the first stage of instruction of this model, the concrete stage, the mathematical concept is introduced by the manipulatives and the students are able to 'play' with them in a carefully structured format. Once they are comfortable with the manipulatives, the students move on to the representational stage, where the concept is represented pictorially instead of using the concrete objects. At this level, students are learning to both visualize and communicate mathematical ideas using drawings. Finally, during the abstract stage students use proper mathematical symbols to express the concept. They demonstrate their learning using the language of mathematics instead of objects. By the end of this three-stage learning process, students have built a schema for the concept, and their understanding has moved from some real-world object to the mathematical idea behind the object.

Several benefits have been found while using manipulatives in education. One important short-term benefit is that they help engage students in the learning process at any level of mathematics. When educators use manipulatives to teach periodically over an extended period of time, students have displayed improved abilities to discuss mathematics and to work in groups (Hand2Mind, n.d.). They also tend to exhibit more ownership in their learning experiences, and they gain more confidence and creativity in their mathematical thinking (Hand2Mind, n.d.) For example, one study found that students who experienced hands-on learning once a week performed 72% of a grade level ahead in mathematics and 40% of a grade level ahead in science (Wenglisky, p. 72).

One study on the effect of manipulatives in the primary grades (Suydam, 1986) involved a teacher introducing the concept of computations and problem-solving to her fourth-grade class using manipulatives. The group who used the manipulatives displayed greater ability at mathematical problem-solving when compared to the students who were not taught with the manipulative. They also scored higher on a retention test of the material.

A meta-analysis (Sowell, 1989) found manipulatives had the most significant impact when used long-term, such as throughout an entire semester or school year. Use of manipulatives over a short amount of time, such as one lesson or a few weeks, was not found to significantly impact student learning; the students in this short-term study performed at the same level as their peers who did not learn the topic with manipulatives. This finding implies that students gradually become more comfortable reasoning with manipulatives. As they practice using these kinds of tools, they improve their ability to make connections between concrete objects and mathematical concepts (Sowell, 1989). Clements and McMillen supported this claim, suggesting teachers should use manipulatives in a variety of topics (1996). They also found that manipulatives, across all grade levels, are best employed when the students are trying to solve difficult problems and the manipulatives serve as an effective tool in solving, an idea also supported by another study in 2008 (Puchner, Taylor, O'Donnell & Fick). Additionally, it is recommended to have students reflect on and justify their reasoning in order to solidify their comprehension (Clements & McMillen, 1996).

Various studies at the secondary level have found similar results. One survey of mathematics teachers in grades 7-12 found that 8% of the surveyed teachers used manipulatives 'often', as in at least once every two weeks, while 85% used them 'sometimes', meaning up to once a month. In algebra, 45% of the teachers used manipulatives to teach some algebra concept, and it is interesting to note that no manipulative was used by more than 40% of the surveyed teachers. When asked why they used manipulatives, 81% of the teachers who did use manipulatives responded that the students seem to achieve higher scores on assessments and 52% said their students enjoyed learning more with manipulatives. It is interesting to note in this study that 61% of teachers who reported a high degree of confidence using manipulatives to teach also said they would like more training in the use of manipulatives. This could be for a variety of reasons, such as being more open to using manipulatives and wanting to learn about other manipulatives or other ways to use the manipulatives they already have.

Kontaş (2016) experimented with using manipulatives while teaching circles and spheres in 7th grade geometry. Two classes of 24 students were included in the study. The pre-test scores of the experimental and control groups were very similar, but the post-test scores of the experimental group were higher than the control group's scores. This analysis showed that the manipulatives were more effective in teaching the 7th grade concept than the traditional lecture method. It was also found that the students were more engaged in the lesson with a more positive attitude towards learning, while the control group showed a more negative attitude towards mathematics after the lesson. (Kontaş, 2016).

One case study in Africa made 3D models of trigonometry problems in order to help students visualize the physical scenario. Specifically, one scenario was about the trajectory of a plane's nose when it is going to land, and the other problem explored the geometric relationship between a cable car and the top of a mountain (Brijlall & Niranjan, 2015). The case study involved a class of ten trigonometry students. The goal was to study the effect of using mathematical manipulatives on students' learning. During the researchers' observations of the students, they noticed students were working together and discussing the mathematics more than they would with an activity sheet. The students also commented that they could imagine the distances and angles given in the problem by seeing a 3D model of it. One student said that the models helped him visualize and understand the problem better. Another student agreed and added that it helped them get a better picture of how equations can be used in different areas of real life. Overall, the researchers found that manipulatives helped learners increase their problem-solving skills, including their logical thinking, and assisted them in internalizing the mathematical concepts.

Other real-world examples of how to apply trigonometry come up without even leaving the mathematics classroom. One teacher in Australia separated the students into groups of three and gave them "a 45-45-90 set square, a drinking straw, some Blu-Tack, and a metre rule" to determine the height of their classroom (Quinlan, 2004). After giving the students some time to brainstorm, he planted the thought of putting the straw on the hypotenuse of the set square.

In the end, the idea is to line it up so the student can view the top of the wall and then measure the height of the table that the tools are set up on and the distance it was from the wall. This Australian teacher found that the project immersed the students in the context of the new concept. It gave them a visual and an experience to look back on that later helped solidify their understanding of triangles.

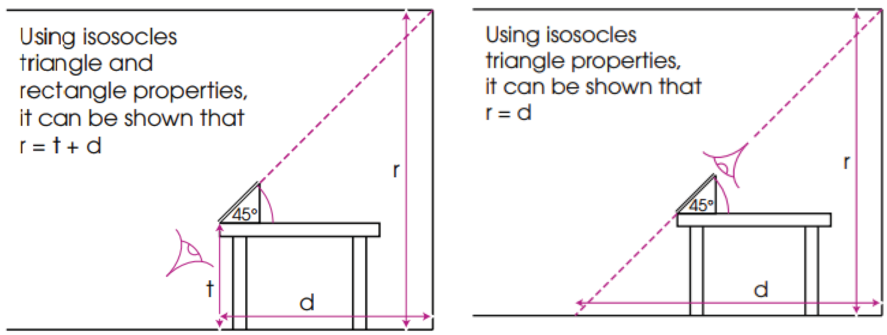


Figure 1: Finding the height of the room using trigonometry

There exists evidence supporting the idea that manipulatives may be especially helpful for students with exceptionalities. One study in particular examined teaching solving one-variable equations using paper plates and dried beans with three 8th grade students with exceptionalities (Allsopp, 1999). After the students used the paper plates and dried beans, they later drew the equations. They drew circles to represent the plates and horizontal or vertical lines to represent the numbers, which helped them transition to the abstract level. After experimenting with the manipulatives and drawing the pictures, the students moved to the abstract stage. At the end of the unit, the students were tested on the concept and showed mastery at the abstract level. The use of manipulatives during the lessons was successful in assisting these students learn the important concepts.

Manipulatives have even been used at the college level with success. For example, two professors, Thirey and Wooster, use manipulatives to teach two major properties of integration

(2013). First, students would take a piece of construction paper, align it in the first quadrant of a coordinate plane, and draw any continuous curve to represent their function. Then they cut along the curve (see figure below). They are now holding a representation of integral of f(t) from a to b. Taking another piece of construction paper, the students draw another function, g(t). They can then manipulate these two pieces to show the sum and difference of the two integrals (see figure below). The students enjoyed the activity, understood the properties the professors wanted to get across, and they felt it was an interesting way to teach mathematics (Thirey & Wooster, 2013).

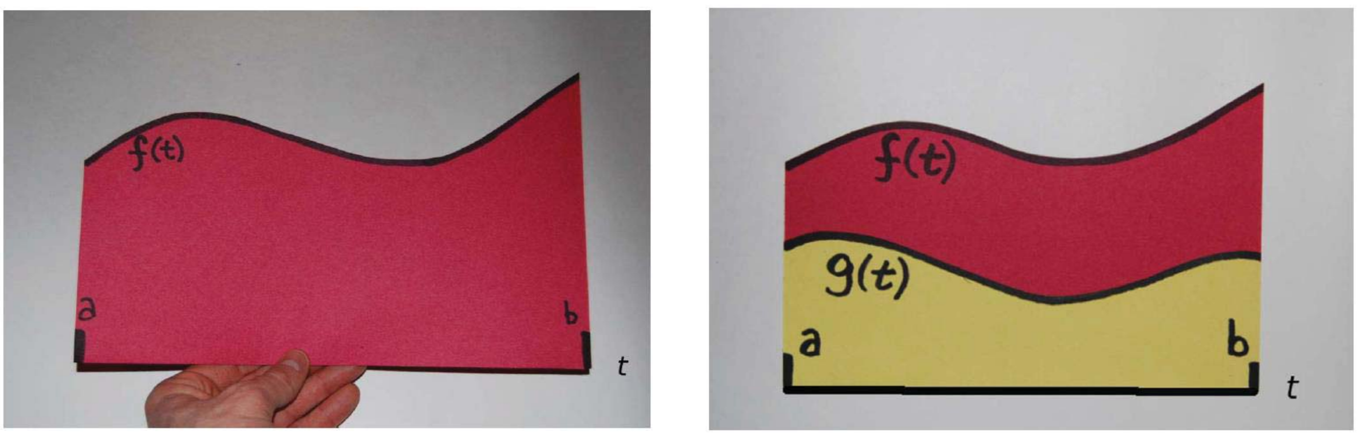


Figure 2: Representations of an integral and of the difference of two integrals.

Another paper at a different university (Sutherland, 2006) also examined how to use manipulatives in a calculus classroom. One topic Sutherland found her students struggling to visualize were solids of revolutions. She would bring in Christmas decorations for visualizing volumes by revolutions, bread for showing volumes by cross-sections, or even washers from the local hardware store to illustrate how the volumes of revolutions relate to real-life. For example, she drew the curve below and asked the students what kind of object they thought would be represented if they revolved it around the x-axis. After the students brainstormed for a minute, she would see looks of understanding and enjoyment when she showed the curve revolved around the x-axis actually represented a Christmas decoration (see figures on next page).

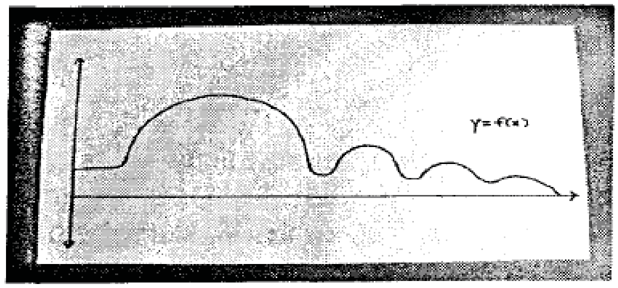


Figure 1. The curve $y = f(x)$.

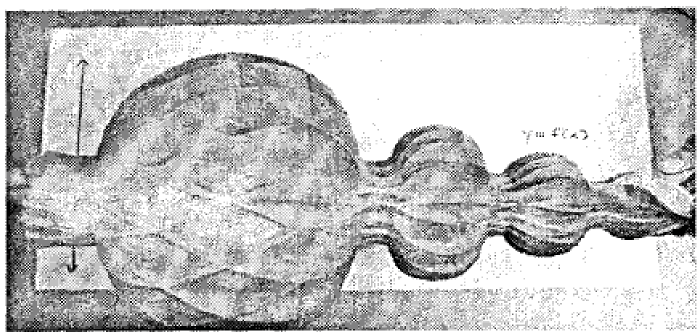


Figure 2. The curve $y = f(x)$ rotated about the x-axis.

Figure 3: Phsyical representation of a volume of revolution.

Sutherland also performs similar hands-on exercises for other topics, such as the surface area of a sphere, drawing cycloids, and finding arc length of cycloids. She has received many positive remarks from the students in her end-of-semester evaluations, describing how the manipulatives helped them become more engaged in the lesson and helped them discover formulas which may be verified using integration techniques (2006).

A professor at another college also found a way to use manipulatives in teaching volumes of revolution and volumes by cross-sections in college calculus (Kolpas, 2007). Their method was to use Computer Assisted Drafting (CAD) to manufacture three-dimensional take-apart models. The students created their own designs, which in this case were two intersecting cylinders. The goal was to find the volume of their intersection. The model is then produced by a

Rapid Prototyping Machine so that students can hold and manipulative the pieces to understand the problem better. The researcher found that students internalized the techniques of finding volume of such shapes when using the manipulative. In these examples, college professors used manipulatives as a way to enrich the students' experiences and gave them a more concrete understanding of the abstract ideas.

Another study on manipulatives focused on algebra topics at the college level (Gningue, Menil, Fuchs, 2014). They wanted to study how the use of virtual manipulatives affected the students' performance in pre-algebra and algebra courses. Virtual manipulatives are defined as "an interactive, Web-based, visual representation of a dynamic object that provides opportunities for constructing mathematical knowledge" (Hunt, Nipper & Nash, 2011, p. 2). The advantage the researchers found with virtual manipulatives was that the students were able to stay on a topic until they mastered it. Since students learn at different rates, this allowed for more flexibility in the class. One disadvantage was that using the manipulatives took more time than the standard lecture format, and students could progress at their own rate, meaning that they did not always reach the higher reasoning questions before the class moved on to the next topic. They found little difference in the performance levels of the classes, but the experimental classes showed more enjoyment during class and displayed an increased confidence in their mathematical skills.

As the previous studies have shown, manipulatives may also impact students' attitudes in mathematics. The use of manipulatives may improve the classroom environment and reduce math anxiety, because it encourages the students to experiment and does not have serious consequences for making mistakes when the topic is first introduced (Cain-Caston, 1996). Marilyn Burns claims that using manipulatives is essential at all levels of mathematics because it makes the material more accessible to all learners. It allows fast learners to be challenged on the concept and is helpful for students with learning disabilities or English language learners (National Council of Supervisors of Mathematics, 2013). Many studies have found that students are more engaged and enjoy the class more through the use of manipulatives (Thirey & Wooster, 2013; Kontaş, 2016; Quinlan, 2004).

In this technological age, many new digital tools have been created to assist students in learning, including virtual manipulatives. While little research exists on the differences between paper and 3D printed manipulatives, several studies have compared the effectiveness of virtual versus concrete manipulatives. One study (Reimer and Moyer, 2005) found that students learned fractions better with virtual manipulatives than the standard paper and pencil lesson. The advantages of virtual manipulatives are that the teacher does not have to worry about having enough materials for every student and virtual manipulatives take less time to work with (Hunt, Nipper & Nash, 2011). Other studies by Olkun in 2003 and by Dorward and Heal in 1999 found that virtual manipulatives an engaging and equally strong as concrete manipulatives.

A meta-analysis of studies using both virtual and physical discovered that teachers may use both types of manipulatives in their classroom, and the order of the types of manipulatives used does matter. They found students were most successful when they used concrete manipulatives to build conceptual understanding, and then solidified their understanding by practicing with virtual manipulatives (Hunt, Nipper & Nash, 2011). According to students who used both virtual and concrete manipulatives, one advantage of the concrete manipulative was that they could see more easily where they went wrong when they made a mistake (Satsangi & Bouck, 2014). However, concrete manipulatives may restrict the types of numbers and problems teachers can show the students, while the virtual manipulatives have more freedom. Also, as students age, they may become less interested in concrete materials, while virtual manipulatives feel less childish and therefore more motivating than concrete manipulatives

(Satsangi & Bouck, 2014). As the students become older and more technologically in tune, it may be time to make our manipulatives more modern. However, one study found that 11th and 12th grade students were more likely to enjoy using concrete manipulatives than 9th and 10th grade students (Mutnansky, 2010).

One possible danger when using either concrete or virtual manipulatives is that students may not make the correct connections between the concrete objects and the mathematical concepts. Thus, it is important to make sure the manipulatives naturally demonstrate the concept being taught. Sometimes students will perform actions with the manipulatives that do not model the desired cognitive activity. For example, in one study, students were using a number line in order to learn how to add two numbers. When adding two numbers, such as 5+4, some students went to the 5 on the number line and counted 1-2-3-4 instead of the desired 6-7-8-9 (Clements & McMillen, 1999). The common mistake with manipulatives is that teachers may assume the mathematical connection to the object is clear, when in actuality it is only clear after they understand the mathematics behind it (Puchner et al., 2008). Another common mistake this study found was that teachers used the manipulatives as an end instead of as a means. In one lesson, students already knew the formula or how to do it the traditional way, and did not enjoy using the manipulative since it was confusing and unnecessary to solve the problem. The students were instructed to use arrays to multiply numbers, but they first found the answer using the traditional algorithm and then arranged the manipulatives in a way they thought represented the correct answer.

An eighth grade class in the same study were supposed to learn how to solve real-life problems with manipulatives. Although the problem presented was challenging and motivated the students to try to use the manipulatives, the students were confused on how to use the manipulatives. While the teachers thought it was clear how the manipulative could help solve the problem, it may have hindered the students who tried to use the manipulative. Successful students in that lesson ended up being the ones who used tables or drawings to find the answers to the problem (Puchner et al., 2008). These possible misconceptions while using manipulatives are why it is important to carefully select manipulative materials and determine the best way to link them to pedagogy.

While choosing manipulatives, one characteristic to take into consideration is their perceptual richness, or the perceived realism and amount of detail possessed by a manipulative. Theoretically, making a manipulative relatable to a real-world object makes the students connect the mathematical concept to something relevant in their life; then they are more motivated to learn the topic. While this may still be true, a study at the fourth and sixth grade level (McNeil, Uttal, Jarvin & Sternberg, 2009) showed that students may actually be hindered by the most perceptually rich manipulatives. The classes used bills and coins to solve problems involving money. One group had bills and coins which looked like real money, and the other group had plain rectangles and circles with the amounts printed plainly in the center. The classes with the realistic manipulatives tended to make the most errors on the post-test. The researchers explained this outcome with the idea that students were distracted by the realism and treated it like play money, instead of using it to learn an algorithm. They found that the transfer and generalization of knowledge may be harmed with perceptually rich objects. This idea is supported by a review of the literature by Pouw and Paas (2014), which found while manipulatives should not be completely decontextualized, too much perceptual richness may impede learning.

It is also interesting to note here that the most well-known manipulatives are rather bland. For example, base-10 blocks, used commonly in elementary classes, are uniformly colored blocks in different sizes. Therefore, objects do not need to be perceptually rich for them to be helpful. Once the students have made the connection to the real-world, it may become redundant information distracting from their learning (McNeil, Uttal, Jarvin & Sternberg, 2009). It draws the students' attention to the features of the manipulative instead of what the object represents mathematically.

The effect of perceptual richness in manipulatives on college students' learning processes has also been researched. Kaminski, Sloutsky, and Heckler (2008) studied how the concreteness of examples help undergraduate students learn. The topic they tested it with was introducing the definition of a group in abstract algebra. In one group, they taught the concept with one example with bland shapes that did not connect to the students' real-world knowledge. The other three groups were introduced with concrete examples, including measuring cups of liquid, pizzas, and tennis balls. Their hypothesis for the experiment was that having multiple concrete examples will help the students recognize other, more abstract, examples of algebra groups. They ended up finding that the students who were taught with the plain symbols scored the highest on the post-test. All groups with the concrete objects scored about the same, so the number of concrete examples did not impact students' learning significantly. A possible explanation behind their finding was that making mathematics very concrete makes it harder for learners to apply it to other situations.

Also at the undergraduate level, Goldstone and Sakamoto (2003) found that students learned ecology principles better while using a bland computer display rather than an attractive one. Thus, the effect of perceptual richness may also be found in virtual manipulatives as well as concrete. While the students enjoyed the visually attractive display, they had more difficulty transferring their knowledge of the given display to another conceptually similar display. The students who used the bland display showed more mastery of the topic when shown a visually dissimilar display. These students may have been less distracted by unnecessary details and hence learned the material more efficiently.

3D printed manipulatives

One kind of manipulative little research has been conducted on are 3D printed manipulatives. This technology is not new, as Chuck Hull invented it in 1980, but it has only become popular in education and the consumer markets in the past few years. To understand why 3D printed manipulatives may be beneficial, let us define this process. 3D printing is a type of additive manufacturing. When an object is additively manufactured, it means that it was constructed by 'adding' the materials together. A simple example is a brick wall. When we build brick walls, we put down a layer of bricks, add a layer of mortar on top, and then so forth until the wall is the desired height. In other words, the base materials come together to form the final product. The most common form of 3D printing, stereolithographic, is a great example of additive manufacturing. These 3D printers melt the material, usually plastic, and then it squeezes the melted plastic out, just like a hot glue gun. When the layer dries, the printer comes back and lays down another layer on top of the first.

Since the 3D printer builds in such layers, the desired product must first be designed in one of the various computer aided-design (CAD) programs. Once it is complete, the computer will slice it into many layers and then feeds the printer the design one layer at a time. The figure below gives a representation of this slicing. The process begins with the complete object, then the computer separates it into equal layers, and finally the printer constructs one layer at a time. Layers may be as thin as 1mm thick, meaning that the objects are able to be quite detailed.

Figure 4: Sliced object

Some prior research has been done on using 3D printers in the classroom (3D Supply Guys, n.d.; Kutch, 2014; Ford & Minshall, 2016), and they have found some benefits. First, the current students are growing up in an increasingly technological world, so it is important to introduce them to different types of technology. Seeing technology used in different ways may inspire some students or at least pique their interest in how to use technology to better the world around them. In this regard, it is important to use 3D printers more than just for the sake of 3D printing. This process is a valuable real-world tool that teachers may use in their own classroom. 3D printing allows for creativity, as one of its strongest attributes is that the files are easily customizable. With manipulatives in particular, a teacher may personalize them to match the school and classroom, or they may modify the manipulatives in any way to best suit their lessons. Whether they want different amounts of objects or think of something to enhance the manipulative, or maybe they have a student with a disability and want to modify it to make the object and the lesson more accessible, teachers would have the freedom to do so with 3D printing.

One way that does not utilize this technology but also achieves the same goal is using paper manipulatives. Teachers may also customize this type and it is a cost-effective way of creating something concrete for the students to learn the mathematical topic with. However, one advantage of 3D printed manipulatives is that they are machine-made. This saves the teacher time and energy since they cut the paper manipulatives themselves while the machine prints out the manipulatives for them. It also cuts down on human error; the 3D printed manipulatives will have very little error in their creation so the pieces will fit together nicely and be the right size.

One disadvantage of 3D printed manipulatives is that they take many hours to print, but once the printer has started the project, the teacher may sit back and work on lesson plans or grading. It simply requires some planning ahead, as it could take several days before the manipulatives are all printed by a machine. There also exists some printer error, where the printing plate may become off balance and the objects do not come out as intended, but this is counterbalanced by the relatively low cost of materials. According to one 3D printer guide, the average cost is $0.02 to $0.08 per cubic centimeter, or $0.33 to $1.32 per cubic inch; educators may purchase a roll of material for $25 and it will last for hundreds of prints (Costs of 3D printing, 2013). Besides saving time and energy, 3D printing manipulatives instead of creating them by hand makes them more durable and easier to hold. Plastics hold up for many years, and they are not as slippery as paper or smaller blocks. Ease of use may allow manipulatives to have a more positive effect on teaching.

History of 3D printing in education

3D printing has grown exponentially in the years since it was invented. It had slowly gained popularity in industry, where the main deterrent was the cost of a 3D printer (Jackson, 2017). In 2009, the company MakerBot produced a line of 3D printers for the everyday consumer and caused the 3D printer to become commercialized (Editors, 2016). To show how quickly the prices have changed, a 3D printer for a consumer still cost about $50,000 until 2011. Today, the price of a single printer is down to nearly $1,800, thanks to their rise in popularity (Jackson, 2017). The real question is how did these printers become so popular and what is the average consumer using it for?

One of the driving forces behind the popularity in 3D printing is called the Maker Movement. This movement started up in 2005 with the launch of *Make*: magazine, which started publishing information on how to complete do-it-yourself projects and created the idea of 'maker faires' (A Brief History, 2015). Maker faires are events where makers (people who have made their own unique projects) show off and share what they have created. President Obama was a supporter of this Maker Movement, especially in education, with his Educate to Innovate campaign (A Brief History, 2015). His campaign emphasized the value of 'making' experiences and improving engagement in science and engineering. The first ever Maker Faire at the White House was in June 2014, and this helped jumpstart the growth of makerspaces and the investment in maker learning. While it is hard to count how many makerspaces exist in schools, libraries, and museums today, according to a poll in 2016, there were over 400 makerspaces in the United States, which is expected to grow tremendously over the next few years (Lou, Peek, 2016).

But why are makerspaces so popular? What benefits do educators see in having makerspaces for their students? The Harvard Graduate School of Education completed a three year study on 3D printing and education to examine the impacts of the Maker Movement. The researchers found that the two main advantages for students included a shift from a consumer to producer mentality and a renewed interest in STEM (Agency by Design, 2015). In the United States, we have seen a decrease in the number of people choosing to major in STEM content areas or enrolling in other higher education programs in STEM, which contrasted the increase in jobs requiring some kind of STEM background (Agency by Design, 2015). Makerspaces are one tool to reverse this trend, as they encourage students to experiment with maker tools, such as 3D printers.

The Use of 3D Printing and 3D Models to Teach Mathematics

The idea of using 3D printing in education is a relatively new one, but people around the world have already thought of great ideas about how to use 3D printers and study their effects on students' learning. Technology is pervasive in today's world, and it is growing at unprecedented rates (Daggett, 2010). That is why schools and educators need to keep up and expose students to the kind of technology and problem-solving skills that come with it in order to prepare them for college and the workplace. In this section, we will look at some recent ideas on how to use 3D modeling and 3D printing technology in the classroom.

One high school teacher in the United States uses 3D printing in their calculus classroom (Kutch, 2014). In order to solidify the students' understanding of optimization problems, the teacher has them construct a container with specific guidelines. Once they have finished their design, they 3D print a prototype. This project exposes students to how calculus may be used in the real world and to the engineering design process of creating a prototype, testing it, and redesigning.

One advantage of 3D printing is that students are able to see their designs come to life, like in the calculus optimization problems. Just as mathematical manipulatives may be used at all levels of education, 3D printers are able to create visualizations of all kinds of mathematical concepts, such as the Fibonacci sequence, lattice structures, and hyperbolic paraboloids (3D Supply Guys, n.d.). For basic arithmetic, it is also possible to 3D print shapes and toys that build on those mathematical concepts. In one high school mathematics class, they 3D printed components of a Rube Goldberg machine. In other classes, 3D printing is being used to represent concurrency of triangles and, of course, as a problem-solving tool in discovery based learning (3D Supply Guys, n.d.).

There are many applications of 3D printing at the middle school level. Researchers have found in 6th grade geometry, the use of 3D printing significantly increased the student's mathematical reflection ability. They designed basic geometric solids in CAD and 3D printed their designs out. They then calculated volume and surface area using the CAD program and the formulas given by the teacher, while using the physical object to compare methods of calculations (Huleihil, 2017). One particular middle school is even having their students develop 3D printed manipulatives for the elementary school in their district. Each class was assigned a grade level and manipulative, and their job was to design it and 3D print prototypes (Teen Leadership Classes, 2017). These students learned both about learning tools in mathematics and design tools used in the real world.

One review studied exactly how 3D printing is being used in numerous places in education, including public schools, universities, and libraries (Ford & Minshall, 2016). A few schools have experimented with using 3D printing to create atomic structure manipulatives in science, and a positive correlation exists between its integration and student learning. Also in a science classroom, a high school in Japan had students create 3D printed police whistles to teach them about frequency of sound waves. The researchers found that the benefits of 3D printing in high schools are strongest when the teachers are enthusiastic, organized and supportive of the use of the technology in their classrooms. While use in high schools is spreading, Ford and Minshall found that the implementation of 3D printing is greatest at the university level, especially in engineering. This rapid prototyping process is used most commonly in creating scientific models and test models for experiments. The most important conclusion this review came to was that teachers need to have a strong knowledge base for 3D printing to be used effectively in the curriculum.

One study in particular looked at the advantages and disadvantages of using 3D printing for students with exceptionalities (Buehler, E., Comrie, N., McDonal, S., Hurst, A., & Hofmann, M., 2016). The study identified three main uses of 3D printing in special education: STEM engagement, creation of instructional aids, and making custom adaptive devices. For example, in a history classroom, the teacher could print a plastic model of a pharaoh's tomb for learners with visual impairments and kinesthetic learners. Specifically, the researchers collaborated with a school's occupational therapists to create a stylus grip that would work for a student at the school. They had tested numerous other grips and stylus-like products, but no product had worked satisfactorily. The designed grip, with its texture and improved grip, were well received by the student. At a school for the blind and visually impaired, where the researchers conducted interviews, they had created geometry manipulatives, hoping to create a strong connection between the students and the mathematical concept in a highly visual subject by instead using a physical object instead of pictures.

Long before 3D printers had become popularized, Rannels, a drafting teacher in Westlawn, Pennsylvania, had considered the advantages of using CAD design to teach geometry (1998). Rannels noticed that her students' had become less comfortable with solid geometric shapes in the years since she had first started teaching. According to Rannels, "Computer Aided Drafting (CAD) systems are designed to draw geometric shapes using points, lines, angles, circles, surfaces, and wireframe solids" (1998). There exist CAD programs that are not too difficult for junior high and high school students to learn and they can help them draw and solve geometry problems. In 1998, Rannels imagined the students printing them out with paper and ink, but in today's world, 3D printers are a clear way for students to see their solid geometric shapes come to life. Students could also model real-life objects and learn about more complex geometries in the process. “Computer technology has made individual memorization skills less important while making individual ability to apply stored information more important than ever.” (Rannels, 1998). While Rannels may not have been imagining 3D printers, designing and testing geometric ideas is certainly a skill set that is emphasized through the use of CAD design and 3D printing.

A 3D printer operates by melting plastic and forming it into objects using the extruder. Now imagine taking the extruder off the machine and using it free hand: this is the idea behind the 3D pen, a spinoff from the 3D printer. The first 3D pen, called the 3Doodler, was invented in 2013, and is useful in making artistic projects (Flaherty, 2013). Margaret Mohr-Schroeder, an associate professor of STEM Education at the University of Kentucky has run numerous mathematics workshops for middle school students using 3D pens to teach concepts. For example, this past summer one camp she organized had students learning about mathematical modeling. First, they made their own polyhedral using the 3D pen, and then the students were allowed to get creative and create their own items with the pen (Mohr-Schroeder, 2017). Afterwards, the students talked about how much they enjoyed getting to be creative have hands-on experiences, and they commented on how much they learned about geometric shapes by using a 3D pen (Mohr-Schroeder, 2017). The beauty of using this kind of 3D printing is that the students get to create their design in real-time and there are no bounds to their creativity as they interact with this modern tool.

In the real-world, 3D printing is used as a rapid prototyping machine, meaning that engineers come up with an idea, use a 3D printer to make a prototype quickly (less than a day), and test it to see if their idea works well. If it does not, the engineer revisits their idea, makes modifications, and then can 3D print another prototype. INVENTOR Cloud, an educational 3D printing company in Youngstown, Ohio, attempts to teach students this engineering process through various hands on projects they have created. There are a variety of projects, including building a bird house, creating the best composter, and designing a musical instrument, and all of them involve the students coming up with an idea, printing and testing it, and making notes on how well their idea worked and what improvements they could make to their design (Michael-Smith, 2013).

The ultimate goal of using manipulatives is to help students understand mathematical concepts easily and comfortably. Ideally, manipulatives will give students an intuitive physical representations of the mathematical process they are supposed to be learning, and then they gradually translate their understanding to the more abstract form of the problems. In the current study, we will examine the effectiveness of a manipulative designed to guide learning on verifying trigonometric identities for a college trigonometry class. We will investigate how the material and method of creation of a manipulative impacts how well students learn the mathematical concepts.

Experimental Methodology

Participants

The data analyzed in this study were from two college trigonometry classes in the Spring 2018 semester at Youngstown State University. Each class had 35 students, but data points were only considered of those students who attended each class day of the learning segment being studied. After these participants were eliminated from the data, one class had 23 valid data points and the second class had 19 data points. Each class was taught by a different instructor, Class A at 10:00 am, and Class B at 1:00 pm. Class A used the 3D printed manipulatives, and Class B used the paper manipulatives. Each type of manipulative had the same number of pieces (Appendices A and B) with the same expressions; the only differing factor was the texture of the manipulatives. The age of the students differed within each class. Both classes had some college freshmen, non-traditional students, and a few high school students who were able to take the class at the college through their high school's program.

Method

The learning segments consisted of two days. A week before the learning segment began, each class completed a pre-test on the material (Appendix D) and a mathematical attitude survey (Appendix E). The students were told to try their best, but neither part would impact their grade in the class. On the first day of the learning segment in each class, the students were split into groups of 3 or 4. Each group was given one set of manipulatives for them to use together. Every piece in a set of manipulatives was the same color, except the '1' piece, which was white. They began by following a worksheet (Appendix C) which guided them through a review of the fundamental identities, starting with the quotient identities of tangent and cotangent and ending with the Pythagorean Identities. This section of the worksheet was designed to refresh each students' memory of the fundamental trigonometric identities and for them to explore how to use the manipulatives.

The manipulatives work by matching the shapes of pieces. For example, placing cosine above sine creates the same shape as cotangent, while placing sine over cosine matches the shape of tangent (see Figure 1). In general, placing one piece over another symbolizes a fraction, or division of two functions, while placing them next to each other symbolizes multiplication. Addition, subtraction and equals signs were demonstrated by leaving a space between the pieces and writing the appropriate symbol in that space.

Figure 5: The sine, cosine, tangent, and cotangent manipulatives.

Once they finished reviewing the fundamental trigonometric identities, the teacher led them through an example of a trigonometric proof using the manipulatives. The students begin by expressing the identity with the pieces. They then pick one side to manipulate. Using the remaining pieces, the students replace one expression at a time with an equivalent expression. They must use their judgement to determine if the replacement gets them closer to proving the identity. Once the students felt more comfortable with the concept of proving, they could simply use the pieces to recall different fundamental identities which would help them in the proof. While the students continued to work in groups while working through the rest of the proofs, the teacher walked around and answered any questions. By the end of the worksheet, it was expected that students would be able to stop using the manipulatives and prove the identity using only paper and pencil. It was a two-day lesson, so on the second day the students worked within their groups again to finish the practice problems and ask any remaining questions. At the end of the second day, the students completed the post-test and retook the mathematical attitude survey.

Quantitative Measures

This study followed a quasi-experimental structure, with a pre-test and a post-test before and after the desired lesson sequence. The students took the pre-test about one week before participating in the lesson, and they took the post-test at the end of the second day of the lesson. The pre- and post-tests had the same exact questions, and they were worth 18 points. Seven of the available points covered the quotient identities and the Pythagorean identities. The remaining 11 points were split between three questions asking the participants to prove identities of varying difficulty using the fundamental identities.

The main research question was whether paper or 3D printed manipulatives impacted student learning more, so to answer this question the independent variable was the type of manipulative used during the participants' class sessions. One class used 3D printed manipulatives, while the second class used paper manipulatives. The dependent variable in the study was the difference in pre- and post-test scores for each participant.

The other main research questions was whether the use of any kind of manipulative helped struggling students learn the material more effectively than high-achieving students. There were 18 possible points on the pre-test. Before they took the pre-test, the students were expected to know the reciprocal and quotient identities, along with the first Pythagorean identity ($\sin^2 x + \cos^2 x = 1$). If they knew these identities and were able to recognize them within an expression, they could have earned up to 7 points. We considered a student who did not display 70% accuracy on these 7 points as low-achieving for the section, and students who achieved a score of 7 or more of the 18 points as high-achieving. Thus, the independent variable for this question was high-achieving versus low achieving, and the dependent variable was the difference between the pre-and post- test scores for each participant.

Since the goal of manipulatives is to make students comfortable enough with the mathematical concepts until they do not need the manipulatives any more, it may be true that students who understand mathematics quickly or have a high confidence in their mathematical ability may view the manipulatives as less useful than the students who have a low mathematical confidence. Therefore, we measured the correlation between the confidence level of the students and how useful they viewed the manipulatives after the lesson. On the mathematical attitude survey, a score of 1 represented low confidence in math while 5 represented high confidence, and on the questions concerning the usefulness of manipulatives, a score of 1 represented that the student did not view manipulatives as useful at all and a score of 5 represented that the student viewed the manipulatives as useful.

Qualitative Measures

The students were also encouraged to leave comments about their mathematical ability or their thoughts on the specific topic and lesson at the end of the survey. From the class which used the 3D printed manipulatives, the comments on the pre-test showed that those students felt low confidence in their mathematical ability or that they did not enjoy learning mathematics. These comments were only from a third of the class, so they do not represent the class's confidence level accurately. A few students commented on how they had never seen this topic before. On the post-test, several students in the class commented on how much they loved working in groups; however, a few students said that they did not feel their group worked well together, which impacted their level of learning. Some students recognized that they needed to spend more time on trigonometric identities and working on these types of problems. One student said that working in his group was 'useful', but the manipulatives were not very useful to him. This student had also marked that he had a high confidence level in mathematics. Another student remarked that his group had not wanted to use the manipulatives, but he felt he would have learned more if his group had been as open to using the manipulatives as him. Several students wrote that they enjoyed the lesson more than the standard kind of lesson and they felt that their peers were able to learn from each other.

In the class with the paper manipulatives, the pre-test also had a few students stating that they felt that math was one of the harder subjects for them to learn. One student in particular said that he was open to different learning methods. A few students commented that this section on trigonometric identities in particular was a difficult topic for them. They had seen the content before and did not understand it very well the first time they had learned it. On the post-test, one student commented that the pieces definitely helped them understand the identities more. Another student stated that he felt he just needs more time in each section, and one other student enjoyed the lesson and wrote “Math is fun!”

Analysis

An independent two sample t-test may be used when the data has one independent categorical variable (paper versus 3d printed) and one continuous dependent variable (difference in test scores). It also assumes normality of the two data sets and we may assume unequal variances. According to the histograms in Figures 2 and 3, the data sets appear normal. We may also use the Shapiro-Wilk test to confirm this conclusion. For the class with 3D printed manipulatives, the p-value was the Shapiro-Wilk test was 0.457, so we do not reject the null hypothesis that the data is normally distributed. For the class with paper manipulatives, the p-value was the Shapiro-Wilk test was 0.900, so we do not reject the null hypothesis that the data is normally distributed. We assumed they had unequal variances, since the samples did not have equal sample variance and the data set was too small to assume equal population variance.

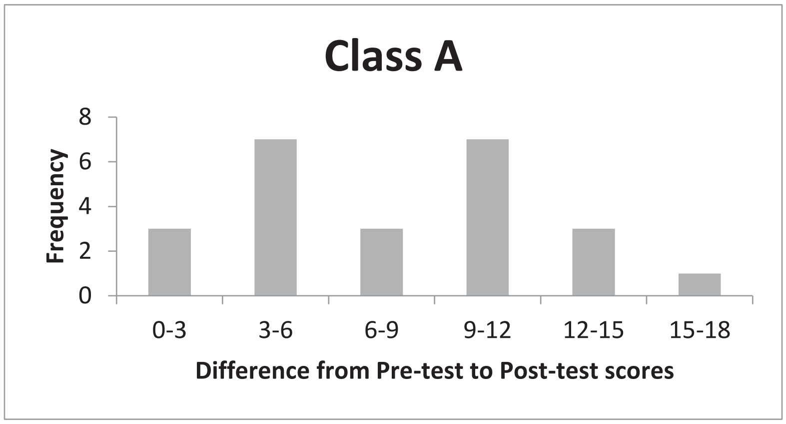


Figure 6: Histogram of differences in scores from pre- to post-test for class with 3D printed manipulatives

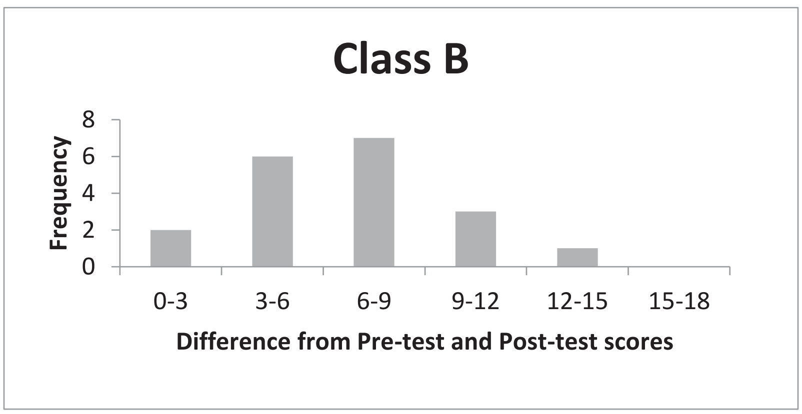


Figure 7: Histogram of differences in scores from pre-to post-test for class with paper manipulatives

For the low-achieving versus the high-achieving students, the Shapiro-Wilk test had a significance value of p=.337, so we fail to reject the null hypothesis that the sample was normally distributed. The high-achieving students sample size was small enough that we will disregard the normality requirement for the t-test.

Results

An independent two sample t-test was conducted on the difference between the post-test and pre-test scores. For Class A, the group who used the 3D printed manipulatives, the mean difference was an improvement of 6.7 points. For Class B, who used the paper manipulatives, the mean difference was an improvement of 5.8 points. While it appears that Class A had the higher average difference, the significance value of the t-test was 0.41, so we fail to reject the null hypothesis that there was no difference between the two groups' scores. We may conclude that there was no significant difference in the effectiveness between the types of manipulatives.

A two independent sample t-test was used to test if there was a statistically significant difference between the two groups' improvements. The test outputted a significance value of p=0.41, so we fail to reject the null hypothesis that there was no difference between the two groups' increases in scores. Data points from each class are listed in the following table.

Table 1: Differences in Pre to Post test scores

Class A (3D Printed Manipulatives)	**Class B (Paper Manipulatives)**
3	9
13	4
4	6
9	10
8	7
4	4
9.5	4.5
3	6
12	7
0	-1
9	9
9	7
3	2
10	12
9	2.5
7.5	4
4	7
9	6
5	4
2	
7	
14	
0	

We also wanted to know if either type of manipulative helped struggling students more than high-achieving students. We reorganized the data from both classes into two groups and ran an independent two-sample t-test. For the struggling students, the average increase from the pre-test to the post-test is 7.14, while the average increase for the high achieving students was 3.95. The significance of the t-test was $p=0.005<0.05$, meaning that we may reject the null hypothesis that there was no difference between the two groups' gains in scores. We may conclude that both types of manipulatives were more helpful for the struggling students than the high achieving students. Individual results are listed in the following table.

Table 2: Low versus high achieving students' scores

Low achieving students			**High achieving students**		
Pre-Test Score	**Post-Test Score**	**Difference**	**Pre-test Score**	**Post-test Score**	**Difference**
2	11	9	11	14	3
3	13	10	7	11	4
3	9	6	7	11	4
4	11	7	7	10	3
1	0	-1	7	11	4
3	12	9	16	18	2
2	9	7	7	14	7
2	4	2	9.5	14	4.5
4	11	7	14	18	4
0	6	6	7	11	4
2	11	9			
4	12	8			
3.5	13	9.5			
4	16	12			
3	3	0			
4.5	12	7.5			
3	8	5			
4	11	7			
4	18	14			
4	4	0			
2	18	16			

For the mathematical attitude survey, the average scores for each class may be found in the table below. In the class with the 3D printed manipulatives, answers in all four categories decreased. Specifically, the average rating of the usefulness of manipulatives decreased from 3.96 to 3.83. In the class who used paper manipulatives, their views of manipulatives became only slightly more positive, increasing from 3.53 to 3.63. Overall, there was not a significant change between the attitudes of the students before and after the lesson. Surveys are also slightly unreliable since they measure subjective responses, which may change depending on the mood of the student that day and not on how the lesson changed their opinion.

Table 3: Mathematical Attitude of Class A (3D printed manipulatives)

Category	**Before Lesson**	**After Lesson**
Confidence in ability in mathematics	4.30	4.04
Confidence in ability in trigonometry	4.04	3.83
Level of enjoyment of mathematics	3.74	3.70
Usefulness of manipulatives	3.96	3.83

Table 4: Mathematical Attitudes of Class B (paper manipulatives)

Category	**Before Lesson**	**After Lesson**
Confidence in ability in mathematics	4.11	4.05
Confidence in ability in trigonometry	3.95	4.00
Level of enjoyment of mathematics	3.47	3.42
Usefulness of manipulatives	3.53	3.63

The hypothesis was that students who had a higher confidence in mathematics would feel the manipulatives were less useful. We found the correlation coefficient between the “confidence in ability to learn mathematics” and “Usefulness of manipulatives” variables using all data points from both classes to be 0.15 after the lesson. There is a very weak positive correlation between the two variables, hinting that students who were more confident in math may have viewed the manipulatives as more helpful, instead of less. The correlation is weak enough that there may be no correlation between the two variables, meaning that students were equally likely to find the manipulatives useful whether or not they felt confident in their mathematical ability.

Conclusions

Manipulatives are only one tool that mathematics teachers may use to help their students learn. In this study, we found that the texture or material of the manipulative did not have a strong impact on its effectiveness. We did find, however, that the use of manipulatives benefited the students who were struggling in the topic significantly more than the students who were high achieving. The high achieving students’ scores increase by an average of 3.95 points out of 18, while the struggling students’ scores increased by an average of 7.14. This could show that the manipulatives helped solidify the struggling students’ understanding of the fundamental identities and also learned how to prove the identities. Finally, we found that the students’ self-confidence in mathematics did not impact whether or not they viewed manipulatives as useful. While their scores did not increase by as much as the struggling students’, the high-achieving students still found some benefits from using the manipulatives to learn the concept, which supports the claim that manipulatives are useful for all learners (National Council of Supervisors of Mathematics, 2013).

Specifically, from the comments made on the survey, many of the students said that they enjoyed the group work and being able to talk through the problems with their peers. Engaging students in learning is an important part of teaching, and it is sometimes easier to do when the students are working in groups and discovering the concepts with their peers, as it was found in the 1996 study by Cain-Caston. According to the previous study, students are less afraid of making mistakes and experience less math anxiety when being introduced a topic through manipulatives. After our learning segment on proving identities with manipulatives, the students said that they found the manipulatives to be helpful in building their memory of the fundamental identities. From these comments and previous research, it may be advantageous to use these manipulatives at the beginning of the chapter to introduce the fundamental identities and let the students discover the identities on their own.

Overall, the scaffolding design of the lesson worked well. The instructors saw the students use the pieces at the beginning of the lesson, and as the students felt more and more comfortable with the fundamental identities, they started simply writing out the steps on their paper. Some students jumped directly to writing on their paper without using the manipulatives at all. Some students commented that they already understood the identities and how to use them, so they did not find the manipulatives helpful. This finding is supported by a previous study found that concrete manipulatives helped students most when they were using the manipulatives to build conceptual understanding (Hunt, Nipper, & Nash, 2011). It is also possible that some students disliked the idea of using manipulatives and just wanted to do the mathematics. A few students commented that they felt the manipulatives would have been more useful if their group members had been more willing to use the manipulatives. From these observations, it may also be beneficial to group the students by willingness to use manipulatives and leave it as a choice as to whether or not they want to use manipulatives or just follow the discovery learning worksheet without manipulatives.

We may have found more differences between the pre- and post-test scores if the manipulatives were used over a long-term period of time. To support this idea, a meta-analysis (Sowell, 1989) found manipulatives had the most significant impact when used long-term, such as throughout an entire semester or school year. Some students commented during class that they did not understand the manipulatives or that they would rather just use pencil and paper. Given time to grow accustomed to this learning technique, the students may have changed their opinion of the manipulatives and grown more open to using this learning technique, making it more effective.

Besides being short-term, another weakness of this study was the small sample size. We saw a small difference between the 3D printed and paper, but it was not big enough to be statistically significant. If we were to conduct the same study with a large enough sample size, we may find that one type was more effective than the other. Another weakness is that two different instructors taught the two classes. The lesson was designed to be identical in both classes, but slight variances between instructors could have caused differences in scores on the post-test.

In practice, the 3D printed manipulatives were easier to handle and move around, but the paper manipulatives may have felt less like toys and more appealing to the college students. This was not studied explicitly, and may be good for future research. In general, we found that teachers may use either type of manipulative in a college classroom and see students learn. An important benefit of these 3D printed manipulatives is that they are thicker and easier for students with disabilities to use than paper manipulatives.

3D printing is a powerful new technology in this century, and educators are still determining its place in education. The trigonometry identity pieces in this experiment allowed students to see 3D printers being used not simply for the sake of 3D printing, but because it creates a solid teaching tool in a timely and efficient manner. If a teacher had more time, it would be a learning experience for the students to create the pieces themselves. The students could even personalize their group's set of pieces, as long as the basic idea or design was maintained, and feel more ownership over their learning since they would be using their own designs to learn the topic. They would have the opportunity to solidify their understanding of the fundamental, and learn about a growing technology at the same time. Creating learning tools for all students is one way 3D printing may be used in the future, and it is a unique way for students to be involved in their own learning.

References

Allsopp, D. H. (1999). Using modeling, manipulatives, and mnemonics with eighth-grade math students. *Teaching Exceptional Children*, 32(2), 74-81.

Brijlall, D., & Niranjan, C. (2015). Using manipulatives to support an embodied approach to learning trigonometry in a South African school: A case study. *Africa Education Review, 12*(3), 361-380.

Buehler, E., Comrie, N., McDonald, S.., Hurst, A.., & Hofmann, M.. (2016). Investigating the implications of 3D printing in special education. *ACM Transactions On Accessible Computing*, *8*(3), doi:10.1145/2870640

Cain-Caston, M. (1996). 'Manipulative Queen'. *JOURNAL OF INSTRUCTIONAL PSYCHOLOGY*, (4). 270.

Clements, D. H. (1999). "Concrete" manipulatives, concrete ideas. *Contemporary Issues in Early Childhood*, 1(1): 45–60. Retrieved from http://journals.sagepub.com/doi/abs/10.2304/ciec.2000.1.1.7

Ford, S. & Minshall, T. (2016). 3D printing in education: a literature review. DOI: 10.13140/RG.2.2.16230.83526

Hand2Mind (n.d.). *Research on the benefits of manipulatives*. Retrieved October 11, 2017, from https://www.hand2mindhan.com/pdf/learning_place/research_math_manips.pdf

Howard, P., Perry, B., & Lindsay, M. (1996). Mathematics and Manipulatives: Views from the Secondary Schools.

Huleihil, M. (2017). 3D printing technology as innovative tool for math and geometry teaching applications, in: 5th Glob. Conf. Mater. Sci. Eng., Taichung City, Taiwan. Doi:10.1088/1757-899X/164/1/012023.

Hunt, A. W., Nipper, K. L., & Nash, L. E. (2011). Virtual vs. Concrete Manipulatives in Mathematics Teacher Education: Is One Type More Effective than the Other?. Current Issues In Middle Level Education, 16(2), 1-6.

Kolpas, S. (2007). Rapid prototyped teaching models for volumes of revolution and volumes by cross-section in college calculus. Glendale, California: Glendale Community College.

Kontas, H. (2016). The Effect of Manipulatives on Mathematics Achievement and Attitudes of Secondary School Students. Journal Of Education And Learning, 5(3), 10-20.

Kutch, A. (2014, January 3). 3D Printing in the Classroom. Retrieved January 12, 2018, from https://kutch3dprint.weebly.com/3d-printing-in-the-classroom

National Council of Supervisors of Mathematics (2013). *Improving Student Achievement in Mathematics by Using Manipulatives with Classroom Instruction.* The National Council of Supervisors of Mathematics Improving Student Achievement Series. Retrieved from http://www.borenson.com/Portals/25/ncsm_positionpaper%20Manipulatives.pdf

Ostos, Damaris (2017, March 9). *Trig Identity Manipulatives*. Retrieved from https://www.youtube.com/watch?v=AUq9Jcs3X_E&t=6s

Puchner, L., Taylor, A., O'Donnell, B., & Fick, K. (2008). Teacher learning and mathematics manipulatives: a collective case study about teacher use of manipulatives in elementary and middle school mathematics lessons. *School Science and Mathematics*, (7), 313.

Quinlan, C. (2004). Sparking interest in trigonometry. Australian Mathematics Teacher, 60(3), 17-20.

Mutnansky, C. (2010). Manipulatives in the secondary mathematics classroom using a traditional algebra text. Bemidji, Minnesota: Bemidji State University.

Reimer, K. & Moyer, P. (2005). Third graders learn about fractions using virtual manipulatives: A classroom study. *Journal of Computers in Mathematics and Science Teaching,* 24(1), 5-25.

Satsangi, R., & Bouck, E. C. (2015). Using Virtual Manipulative Instruction to Teach the Concepts of Area and Perimeter to Secondary Students With Learning Disabilities. Learning Disability Quarterly, 38(3), 174-186.

Sowell, E. (1989). Effects of manipulative materials in mathematics instruction. *Journal for Research in Mathematics Education, Vol 20, No. 5*. The National Council of Teachers of Mathematics. Retrieved from http://www.jstor.org/stable/749423?origin=crossref&seq=1#page_scan_tab_contents

Sutherland, M. (2006). Hands-on calculus. *PRIMUS,* Vol 16, Issue 4.

Sutton, J. & Krueger, A. (2002). EDThoughts: What we know about mathematics teaching and learning. Aurora, CO: Mid-Continent Research for Education and Learning. Retrieved from https://www.gpo.gov/fdsys/pkg/ERIC-ED465514/pdf/ERIC-ED465514.pdf

Suydam, M. (1984). Manipulative materials. *The Arithmetic Teacher, Vol. 31, No. 5, p.27*. National Council of Teachers of Mathematics. Retrieved from http://www.jstor.org/stable/pdf/41190889.pdf?refreqid=excelsior%3A92e7794a7fad54ec40f8304908ade697

Teen Leadership Classes Share Math Manipulatives with Other SISD Schools. (2017, December). Retrieved January 12, 2018, from https://sms.sangerisd.net/apps/news/article/793341

Thirey, B. & Wooster, R. (2013). The touchy-feely integral: Using manipulatives to teach the basic properties of integration. *PRIMUS,* 23(7).

Wenglinsky, H. (2000). How teaching matters: Bringing the classroom back into discussions of teacher quality. Princeton, NJ: Educational Testing Service. Retrieved from https://www.ets.org/research/policy_research_reports/publications/report/2000/idxn

3D Supply Guys, (n.d.). 3D Printers for Mathematics Education. Retrieved February 12, 2018, from https://3dsupplyguys.com/blogs/3d-printing-education/3d-printers-for-mathematics-education3d printers for mathematics education

Appendix A

Trigonometric Identities Manipulatives (3D printed)

Figure 5: 3D printed manipulatives

Appendix B

Trigonometric Identities Manipulatives (paper)

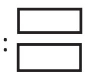
Figure 6: Paper manipulatives

Appendix C

Trigonometric Identities Activity Sheet

<u>Part 1: Simplifying trigonometric functions</u>

Directions: Your group has received a set of trigonometric identity manipulatives. Here are the basic properties of these pieces.

a. If the shapes of the pieces match, then those functions are equal.
b. A fraction is represented by one piece being on top of the other like so: 
c. Multiplication is represented by the pieces being right next to each other.
d. For addition, subtraction, and equal signs, leave a space in between the pieces.

Introductory problems:

1. Find the $\sin\theta, \cos\theta, \tan\theta$, and $\cot\theta$ pieces. By matching the shapes, tell which expression is equal to $\tan\theta$ and which expression is equal to $\cot\theta$.

   a) $\dfrac{\sin\theta}{\cos\theta} =$ b) $\dfrac{\cos\theta}{\sin\theta} =$

2. Using the $\sin\theta$ and $\cos\theta$ pieces, determine what the following are equal to. *Remember, if one piece is on top of another, it represents a fraction, so the usual rules of fractions apply.*

   a) $\cot\theta\tan\theta$

   b) $\dfrac{\tan\theta}{\cot\theta}$

3. Now put away the $\tan\theta$ and $\cot\theta$ pieces, and take out the $1, sec\theta$, and $csc\theta$ pieces. Determine what the following are equivalent to by matching the shapes of the combined pieces to either $sec\theta$ or $\csc\theta$:

   a) $\dfrac{1}{\sin\theta} =$ b) $\dfrac{1}{\cos\theta} =$

4. Now take out the $1 - \sin^2\theta$ piece. Which piece has a similar shape to this one? How many $\sin\theta$ or $\cos\theta$ pieces fit inside of the $1 - \sin^2\theta$ piece?

   a) $1 - \sin^2\theta =$

5. Similarly, look at the $1 - \cos^2\theta$. You may already know what it is equal to, but verify the identity by finding the pieces which make the same shape.

   a) $1 - \cos^2\theta =$

6. From questions 4 and 5, we can conclude that $1 = \cos^2\theta + \sin^2\theta$. This is called our **Pythagorean Identity**. For the following problems, ***simplify using algebra*** to find our other two Pythagorean Identities. (You may use the pieces to remember which functions are equal to each other).

   a) $\frac{1}{\cos^2\theta}(1) = \frac{1}{\cos^2\theta}(\cos^2\theta + \sin^2\theta)$

   b) $\frac{1}{\sin^2\theta}(1) = \frac{1}{\sin^2\theta}(\cos^2\theta + \sin^2\theta)$

7. Now take out the $\sec^2\theta - 1$ and the $\csc^2\theta - 1$ pieces. Using your work from question 6, what are those pieces equal to? Verify using the shapes of the pieces.

   a) $\sec^2\theta - 1 =$

   b) $\csc^2\theta - 1 =$

8. Using the pieces and/or what we have determined in the previous questions, give what the following are equal to:

   a) $\sec\theta\sin\theta$

   b) $\cos\theta\sec\theta$

   c) $\frac{\sin\theta\cos\theta}{\cot\theta}$

Part 2: Proving trigonometric identities

Hints for proving identities:

1. Only work with ONE side. (Do not change anything on the other side once you have started manipulating one side.)
2. Work with what looks like the most complicated side.
3. Try rewriting the functions into expressions of $\sin\theta$ and $\cos\theta$.

Show:

$$\tan\theta\cos\theta + \cot\theta\sin\theta = \sin\theta + \cos\theta$$

1. Find the pieces and place them in the order of the equation above.

2. How can we rewrite $\tan\theta$ and $\cot\theta$?

3. Does anything cancel out? If so, what are we left with?

Identity proven!

Now try the following. Remember, only change **one side**. Feel free to refer to the pieces to remember the trigonometric identities. **SHOW ALL WORK!!**

1. $\frac{\cos\theta}{\cot\theta} + \frac{\sin\theta}{\tan\theta} = \sin\theta + \cos\theta$

2. $\sec^2\theta - 1 = \frac{\sin^2\theta}{1-\sin^2\theta}$

3. $\frac{\cos^2\theta}{1-\sin\theta} = \frac{\cos\theta}{\sec\theta-\tan\theta}$

4. $\frac{\cot\theta-\tan\theta}{\sin\theta\cos\theta} = \csc^2\theta - \sec^2\theta$

Part 3: More Practice Problems

1. $\frac{\csc^2\theta - 1}{1 - \sin^2\theta} = \csc^2\theta$

2. $\tan^2\theta - \sin^2\theta = (\sec^2\theta - 1)(\sin^2\theta)$

3. $\cot^2\theta\,(\sec^2\theta - 1) = 1$

4. $\tan^2\theta\,(\csc^2\theta - 1) = 1$

5. $\csc^2\theta\,(\tan^2\theta + 1) = \frac{1}{\sin^2\theta\cos^2\theta}$

Appendix D

Trigonometric Identities Pre/Post-Test

MATH 1511 Trig Pre-Test

Name:________________ Date:________________

Answer all of the questions in the space provided.

1. Rewrite the following using $cos(\theta)$ and $sin(\theta)$.

   a) $\tan\theta =$ c) $\sec\theta =$

   b) $\cot\theta =$ d) $\csc\theta =$

2. Give what the following are equal to using the Pythagorean identities

   a) $1-\sin^2\theta =$

   b) $\sec^2\theta - 1 =$

   c) $\csc^2\theta - 1 =$

3. Verify the following trigonometric identity. Show all steps.

$$42\csc\theta\sin\theta = 42$$

4. Verify the following trigonometric identity.

$$(1-\cos^2\theta)\sec^2\theta = \tan^2\theta$$

5. Verify the following trigonometric identity

$$(1 + \cos\theta)(1 - \cos\theta) = \sin^2\theta$$

Appendix E

Mathematics Attitude Survey

MATH 1511

Math Attitude Survey

Name: _______________________________ **Date:**
____

Directions: Please answer each question honestly. Rate how much you agree with each statement, where

1 = strongly disagree 2 = somewhat disagree 3 = neutral

4 = somewhat agree 5 = strongly agree

1. I am confident in my ability to learn mathematics. **1 2 3 4 5**

2. I am confident in my ability to learn trigonometry. **1 2 3 4 5**

3. I enjoy learning mathematics. **1 2 3 4 5**

4. I feel manipulatives are useful in learning mathematics. **1 2 3 4 5**

Please write any general comments below:

Appendix F

Letter of Consent

Informed Consent

Dear student:

I am Emily Hoopes, a graduate mathematics student at Youngstown State University. I am conducting a study to investigate the effectiveness of using mathematics manipulatives (a tool that makes an abstract mathematical concept more concrete) in teaching trigonometric identities. In this study, you would participate in a lesson on trigonometric identities during regular class time. During this lesson, you would use the trigonometric identity manipulatives. Before the lesson, you would take a pre-test, and after wards a post-test on trigonometric identities and proofs. These scores would not be used in calculating your grade for the class.

There are no anticipated risks to you from participating in the study. The benefits to you from being in this study include the opportunity to see a teaching technique not used very often in college settings, and exploring a mathematical concept in a concrete way.

Your privacy is important and I will handle all information collected about you in a confidential manner. Once the data collection is complete, your name will be removed from the file that contains your pre and post test scores, and I will report the results of the project in a way that will not identify you. I may publish the results of the study, but your name will not be used in any of the publication.

You do not have to be in this study. If you don't want to, you can say no without losing any benefits that you are entitled to. If you do agree, you can stop participating at any time. If you wish to withdraw, just tell me.

If you have questions about this research project please contact Emily Hoopes at eahoopes@student.ysu.edu. If you have questions about your rights as a participant in a research project, you may contact the Office of Research at YSU (330-941-2377) or at YSUIRB@ysu.edu.

I understand the study described above and have been given a copy of this consent document. I am 18 years of age or older and I agree to participate.

______________________ ______________

Signature of Participant Date

____

Please Print Name Here

Youngstown
STATE UNIVERSITY

One University Plaza, Youngstown, Ohio 44555

Office of Research
330.941.2377
www.ysu.edu

January 31, 2018

Dr. Anita O'Mellan, Principal Investigator
Ms. Emily Hoopes, Co-investigator
Mr. Kyle Gumble, Co-investigator
Department of Mathematics & Statistics
UNIVERSITY

RE: HSRC PROTOCOL NUMBER: 071-2018
TITLE: Use of Mathematical Manipulatives in Teaching Trigonometric Identities

Dear Dr. O'Mellan, Ms. Hoopes and Mr. Gumble:

The Institutional Review Board has reviewed the abovementioned protocol and determined that it is exempt from full committee review based on a DHHS Category 2 exemption.

Any changes in your research activity should be promptly reported to the Institutional Review Board and may not be initiated without IRB approval except where necessary to eliminate hazard to human subjects. Any unanticipated problems involving risks to subjects should also be promptly reported to the IRB.

The IRB would like to extend its best wishes to you in the conduct of this study.

Sincerely,

[redacted]

Mr. Michael Hripko
Associate Vice President for Research
Authorized Institutional Official

MAH:cc

c: Dr. Thomas Wakefield, Chair
Department of Mathematics & Statistics