DYNAMIC STABILITY OF CYLINDRICALLY ORTHOTROPIC CIRCULAR PLATES

by

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ABSTRACT

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THE DIFFERENTIAL EQUATION OF THE DYNAMIC STABILITY OF CYLINDRICALLY ORTHOTROPIC CIRCULAR PLATES IS PRESENTED. THE SOLUTION OF THIS EQUATION IS OBTAINED IN INFINITE SERIES FORM USING THE METHOD OF FROBENIUS. FREQUENCY EQUATIONS FOR SYMMETRICAL AND ASYMMETRICAL VIBRATIONS FOR FIXED, AND SIMPLY SUPPORTED SOLID PLATES ARE PROVIDED.

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NOMENCLATURE

r, θ, z	= cylindrical polar coordinates
$\epsilon_{rr}, \epsilon_{\theta\theta}, \gamma_{r\theta}$	= radial, tangential, and shear strains
$\tau_{rr}, \tau_{\theta\theta}, \tau_{r\theta}$	= radial, tangential, and shear stresses
$M_{rr}, M_{\theta\theta}, M_{r\theta}$	= bending and twisting moments per unit length
$Q_{rz}, Q_{\theta z}$	= shear forces per unit length
u, v, w	= mid-plane displacements in the r, θ, z directions, respectively
E_r, E_θ	= Young's moduli for radial and tangential directions
ν_r, ν_θ	= Poisson's Ratios
G	= shear modulus
D_r, D_θ	= structural rigidities
D_K	= torsional stiffness
h	= thickness of the plate
$q(r, \theta, t)$	= applied load per unit area of plate
a	= radius of the plate
I_r, I_θ	= second moments of area per unit length with respect to the tangential and radial axes respectively
ρ	= mass density of the plate
ω	= frequency of the mode of vibration
C_i	= integration constants
a_i	= constants of Frobenius series
c_i	= indicies of the Frobenius series

$N_{rr}, N_{\theta\theta}, N_{r\theta}$ = in-plane stability forces per unit length

t = time variable

S, T = numerical parameters

$E_r, \nu_\theta = E_\theta, \nu_r$

$D_r = \dfrac{E_r h^3}{12(1-\nu_r \nu_\theta)}$

$D_\theta = \dfrac{E_\theta h^3}{12(1-\nu_r \nu_\theta)}$

$D_k = \dfrac{G h^3}{12}$

$k = D_\theta | D_r$

$\sigma' = D_{r\theta} | D_r$

$R = K r$

$f = \omega | 2\pi$

$K^4 = \rho . h . \omega^2 | D_r$

$\beta^2 = 2n^2\sigma' + k^2$

$n_r^2 = N_{rr} | D_r$.

$n_\theta^2 = N_{\theta\theta} | D_r$.

$\alpha_r^2 = I_r / h$

$\alpha_\theta^2 = I_\theta / h$.

$D_{r\theta} = 2D_k + D_r \nu_\theta = 2D_k + D_\theta \nu_r$

CHAPTER I

INTRODUCTION

A. HISTORICAL REVIEW

The literature contains many analyses of transverse vibrations of elastic orthotropic circular plates from the standpoint of small deflection, thin-plate theory. References [1], [2], [3] deal with the free vibration problem including natural frequencies and mode shapes. The effect of rotary inertia is included in [4], and the effect of an elastic foundation is considered in [5]. The forced vibration problem is presented in [6].

It is the purpose of this thesis to investigate the dynamic stability of cylindrically orthotropic plates including in-plane stability forces and the effects of transverse and rotary inertia.

B. DEFINITION OF A CYLINDRICALLY ORTHOTROPIC MATERIAL

A cylindrically orthotropic material is defined as one for which the elastic constants, as referred to a cylindrical coordinate system, are independent of the magnitude of the radius r and remain invariant under the following coordinate transformations: a rotation about the z axis; a translation along the z axis; a reversal of the z axis.

With these transformations, and the usual assumptions in the development of the theory of bending of thin plates, the number of elastic constants in the cylindrical polar coordinate form of Hooke's Law is reduced to four.

C. STATEMENT OF THE PROBLEM

For clarity of presentation, the features of this thesis are outlined as follows:

1. Solve the equations of motion for the dynamic stability of cylindrically orthotropic circular plates considering transverse and rotary inertia.
2. Determine the frequency equations for symmetrical and asymmetrical vibrations for fixed, and simply supported solid plates.
3. Formulate the complete infinite series solutions of the frequency equations for future programming on the digital computer.

CHAPTER II--THEORY

A. EQUATIONS OF MOTION

The equations of motion for a cylindrically orthotropic circular plate, including the in-plane stability forces and the effects of transverse and rotary inertia, are:

$$\frac{1}{r}\frac{\partial}{\partial r}(r N_{rr}) - \frac{1}{r} N_{\theta\theta} + \frac{1}{r}\frac{\partial N_{r\theta}}{\partial \theta} = \rho h \frac{\partial^2 u}{\partial t^2} \qquad (1)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r N_{r\theta}) + \frac{N_{r\theta}}{r} + \frac{1}{r}\frac{\partial N_{\theta\theta}}{\partial \theta} = \rho h \frac{\partial^2 v}{\partial t^2} \qquad (2)$$

$$\Big[\frac{1}{r}\frac{\partial}{\partial r}(r Q_{rz}) + \frac{1}{r}\frac{\partial Q_{\theta z}}{\partial \theta} + q(r,\theta,t) - \frac{1}{r}\frac{\partial}{\partial r}\left(r N_{rr}\frac{\partial W}{\partial r}\right)$$

$$- \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{N_{\theta\theta}}{r}\frac{\partial W}{\partial \theta}\right) - 2\frac{\partial^2}{\partial r\,\partial \theta}\left(\frac{N_{r\theta}}{r} W\right)\Big] = \rho h \frac{\partial^2 W}{\partial t^2} \qquad (3)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r M_{rr}) - \frac{M_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta} - Q_{rz} = -\rho I_r \frac{\partial^3 W}{\partial r\,\partial t^2} \qquad (4)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r M_{r\theta}) + \frac{M_{r\theta}}{r} + \frac{1}{r}\frac{\partial M_{\theta\theta}}{\partial \theta} - Q_{\theta z} = -\rho \frac{I_\theta}{r}\frac{\partial^3 W}{\partial \theta\,\partial t^2} \qquad (5) \quad *$$

*Novozhilov, Foundations of the Nonlinear Theory of Elasticity, Graylock Press, 1953. Ch. V, § 43, page 156.

Substituting Equations (3) and (4) into (5) and eliminating the shear forces, yields

$$
\begin{aligned}
&\Bigg[\frac{\partial^2 M_{rr}}{\partial r^2} + \frac{2}{r}\,\frac{\partial M_{rr}}{\partial r} - \frac{1}{r}\,\frac{\partial M_{\theta\theta}}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 M_{\theta\theta}}{\partial \theta^2} + \frac{2}{r^2}\,\frac{\partial M_{r\theta}}{\partial \theta} \\
&+ \frac{2}{r}\,\frac{\partial^2 M_{r\theta}}{\partial r \cdot \partial \theta} - \rho \frac{\partial^2}{\partial t^2}\Big[hW - \frac{1}{r}\,\frac{\partial}{\partial r}\Big(r \cdot I_r \cdot \frac{\partial W}{\partial r}\Big) - \frac{1}{r}\,\frac{\partial}{\partial \theta}\Big(\frac{I_\theta}{r}\,\frac{\partial W}{\partial \theta}\Big)\Big] \\
&- \frac{1}{r}\,\frac{\partial}{\partial r}\Big(r N_{rr}\,\frac{\partial W}{\partial r}\Big) - \frac{1}{r}\,\frac{\partial}{\partial \theta}\Big(\frac{N_{\theta\theta}}{r}\,\frac{\partial W}{\partial \theta}\Big) - \frac{1}{r}\,\frac{\partial}{\partial r}\Big(\frac{N_{r\theta}}{r}\,\frac{\partial W}{\partial \theta}\Big) \\
&- \frac{1}{r}\,\frac{\partial}{\partial \theta}\Big(N_{r\theta}\,\frac{\partial W}{\partial r}\Big)\Bigg] = -q(r,\theta,t) \qquad (6)
\end{aligned}
$$

Using the definition of cylindrically orthotropic material, and neglecting the effects of shear stresses $\tau_{rz}, \tau_{\theta z}$, and normal stress τ_{zz}. The generalized Hooke's Law in cylindrical polar coordinates is written as

$$
\begin{bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_r} & -\frac{\nu_r}{E_r} & 0 \\ -\frac{\nu_\theta}{E_\theta} & -\frac{1}{E_\theta} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \tau_{rr} \\ \tau_{\theta\theta} \\ \tau_{r\theta} \end{bmatrix} \qquad (7)
$$

By matrix operations and proper integration of Equations (7), it is readily shown that,

$$M_{rr} = -D_r\left[\frac{\partial^2 W}{\partial r^2} + \nu_\theta\left(\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}\right)\right] \tag{8}$$

$$M_{\theta\theta} = -D_\theta\left[\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} + \nu_r\frac{\partial^2 W}{\partial r^2}\right] \tag{9}$$

$$M_{r\theta} = -2Dk\left[\frac{\partial^2}{\partial r\,\partial\theta}\left(\frac{W}{r}\right)\right] \tag{10}$$

Substituting Equations (8), (9), (10), into Equation (6), and assuming free vibrations (that is, $q = 0$), it follows that:

$$\begin{aligned}
&\Bigg[D_r\frac{\partial^4 W}{\partial r^4} + \frac{2D_r}{r} - \frac{D_\theta}{r^2}\frac{\partial^2 W}{\partial r^2} + \frac{D_\theta}{r^3}\frac{\partial W}{\partial r} + \frac{2D_{r\theta}}{r^2}\frac{\partial^4 W}{\partial r^2\partial\theta^2} \\
&-\frac{2D_{r\theta}}{r^3}\frac{\partial^3 W}{\partial r\,\partial\theta^2} + \frac{2}{r^4}\left(D_{r\theta} + D_\theta\right)\frac{\partial^2 W}{\partial\theta^2} + \frac{D_\theta}{r^4}\frac{\partial^4 W}{\partial\theta^4} \\
&+\rho\frac{\partial^2}{\partial t^2}\left[hW - \frac{1}{r}\frac{\partial}{\partial r}\left(r.I_r.\frac{\partial W}{\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{I_\theta}{r^2}\frac{\partial W}{\partial\theta}\right)\right] \\
&+\frac{1}{r}\frac{\partial}{\partial r}\left(r N_{rr}\frac{\partial W}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{N_{\theta\theta}}{r}\frac{\partial W}{\partial\theta}\right) + 2\frac{\partial^2}{\partial r\,\partial\theta}\left(N_{r\theta}\frac{W}{r}\right)\Bigg] = 0
\end{aligned} \tag{11}$$

Assuming N_{rr}, $N_{\theta\theta}$, $N_{r\theta}$, I_r, and I_θ are independent of r and θ, Equation (11) reduces to

$$\Big\{ D_r \frac{\partial^4 W}{\partial r^4} + \frac{2D_r}{r}\frac{\partial^3 W}{\partial r^3} - \frac{D_\theta}{r^2}\frac{\partial^2 W}{\partial r^2} + \frac{D_\theta}{r^3}\frac{\partial W}{\partial r} + \frac{2D_{r\theta}}{r^2}\frac{\partial^4 W}{\partial r^2 \partial \theta^2}$$

$$-\frac{2D_{r\theta}}{r^3}\frac{\partial^3 W}{\partial r \partial \theta^2} + \frac{2}{r^4}\left(D_{r\theta} + D_\theta\right)\frac{\partial^2 W}{\partial \theta^2} + \frac{D_\theta}{r^4}\frac{\partial^4 W}{\partial \theta^4}$$

$$+\rho\frac{\partial^2}{\partial t^2}\left[hW - \frac{I_r}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right) - \frac{I_\theta}{r^2}\frac{\partial^2 W}{\partial \theta^2}\right] + \frac{N_{rr}}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right)$$

$$+\frac{N_{\theta\theta}}{r^2}\frac{\partial^2 W}{\partial \theta^2} + 2N_{r\theta}\frac{\partial^2}{\partial r \partial \theta}\left(\frac{W}{r}\right)\Big\} = 0 \qquad (12)$$

B. SOLUTION OF THE EQUATION OF FREE VIBRATION

Equation (12) is solved using the method of separation of variables. For the special case of $N_{r\theta} = 0$, space and time variables are separated in the form

$$W(r,\theta,t) = \sum_{n=0}^{\infty} Y(r)\cdot e^{in\theta}\cdot e^{i\omega t} \qquad (13)$$

where ω is the circular frequency and where n = 0,1,2,3,... corresponds to the number of nodal diameters. Substitution of Equation (13) into (12) yields the following ordinary differential equation:

$$\frac{d^4Y}{d\gamma^4} + \frac{2}{\gamma}\frac{d^3Y}{d\gamma^3} - \frac{\beta^2}{\gamma^2}\frac{d^2Y}{d\gamma^2} + \frac{\beta^2}{\gamma^3}\frac{dY}{d\gamma} + \frac{[k^2(n^2-1)^2-\beta^2]Y}{\gamma^4}$$

$$-K^4Y + \frac{[K^4\alpha_\gamma^2 + n_\gamma^2]}{\gamma}\frac{d}{d\gamma}\left(\gamma\frac{dY}{d\gamma}\right) - \frac{[K^4\alpha_\theta^2 + n_\theta^2]n^2Y}{\gamma^2} = 0 \quad (14)$$

For convenience, the change of variable, $R = K\gamma$, is made. It follows then, that

$$\left\{R^4\frac{d^4Y}{dR^4} + 2R^3\frac{d^3Y}{dR^3} - \beta^2R^2\frac{d^2Y}{dR^2} + \beta^2R\frac{dY}{dR} + [k^2(n^2-1)^2-\beta^2]Y\right.$$

$$(15)$$

$$\left.-R^4Y + [k^2\alpha_\gamma^2 + n_\gamma^2/k^2]R^3\frac{d}{dR}\left(R\frac{dW}{dR}\right) - [k^2\alpha_\theta^2 + n_\theta^2/k^2]n^2R^2Y\right\} = 0$$

The latter differential equation is solved in infinite series form using the method of Frobenius. Let

$$Y_{(R)} = \sum_{\lambda=0}^{\infty} a_\lambda R^{\lambda+c} \quad (16)$$

where a_λ are the undetermined coefficients of the series, and c are the indicial roots associated with the differential equation. Substituting Equation (16) into (15) yields the following equation:

$$a_0\left[(c-1)^4-(1+\beta^2)(c-1)^2+k^2(n^2-1)^2\right]R^{c}$$

$$+\,a_1\left[c^4-(1+\beta^2)c^2+k^2(n^2-1)^2\right]R^{c+1}$$

$$+\Big\{a_2\left[(c+1)^4-(1+\beta^2)(c+1)^2+k^2(n^2-1)^2\right]$$

$$+\,a_0\Big[c^2(k^2\alpha_r^2+n_r^2|k^2)-n^2(k^2\alpha_\theta^2+n_\theta^2|k^2)\Big]\Big\}R^{c+2}$$

$$+\Big\{a_3\left[(c+2)^4-(1+\beta^2)(c+2)^2+k^2(n^2-1)^2\right]$$

$$+\,a_1\Big[(1+c)^2(k^2\alpha_r^2+n_r^2|k^2)-n^2(k^2\alpha_\theta^2+n_\theta^2|k^2)\Big]\Big\}R^{c+3}$$

$$+\sum_{\lambda=4}^{\infty}\Big\{a_\lambda\left[(\lambda+c-1)^4-(1+\beta^2)(\lambda+c-1)^2+k^2(n^2-1)^2\right.$$

$$\left.+\,a_{\lambda-2}\Big[(\lambda+c-2)^2(k^2\alpha_r^2+n_r^2|k^2)-n^2(k^2\alpha_\theta^2+n_\theta^2|k^2)\Big]-a_{\lambda-4}\right\}R^{\lambda+c}=0 \qquad (17)$$

Assuming $a_0 \neq 0$, the indicial equation reads:

$$\left[(c-1)^4-(1+\beta^2)(c-1)^2+k^2(n^2-1)^2\right]=0 \qquad (18)$$

The indicial roots, of Equation (18), are:

$$C = 1 \pm \left[\frac{(1+\beta^2)\pm\left[(1+\beta^2)^2-4k^2(n^2-1)^2\right]^{1/2}}{2}\right]^{1/2} \qquad (19)$$

The recurrence formula is:

$$\Big\{a_l\left[(c+l-1)^4-(1+\beta^2)(c+l-1)^2+k^2(n^2-1)^2\right]$$

$$+a_{l-2}\left[(c+l-2)^2(k^2\alpha_r^2+n_r^2/k^2)-n^2(k^2\alpha_\theta^2+n_\theta^2/k^2)\right]-a_{l-4}\Big\}=0$$

$$l \geq 4$$

where

$$\Big\{a_2\left[(c+1)^4-(c+1)^2(1+\beta^2)+k^2(n^2-1)^2\right]$$

$$+a_0\left[c^2(k^2\alpha_r^2+n_r^2/k^2)-n^2(k^2\alpha_\theta^2+n_\theta^2/k^2)\right]\Big\}=0 \qquad (20)$$

The four values of the index c of the indicial equation are written

$$C_1 = 1+T$$

$$C_2 = 1-T$$

$$C_3 = 1+S$$

$$C_4 = 1-S \qquad (21)$$

<u>WHERE</u>

$$S = \left[\frac{(1+\beta^2) + \left[(1+\beta^2) - 4k^2(\eta^2-1)^2\right]^{1/2}}{2}\right]^{1/2}$$

$$T = \left[\frac{(1+\beta^2) - \left[(1+\beta^2) - 4k^2(\eta^2-1)^2\right]^{1/2}}{2}\right]^{1/2} \qquad (22)$$

The following restrictions are placed on the coefficients a_0, a_1, a_2, and a_3 in order to satisfy Equation (17):

$$a_0 \neq 0$$

$a_1 = 0$ <u>PROVIDED</u> $S \neq \frac{1}{2} \,;\, -\frac{1}{2}$

$T \neq \frac{1}{2} \,;\, \frac{1}{2}$

$a_2 \neq 0$ <u>PROVIDED</u> $S \neq 1 \,;\, -1$

$T \neq 1 \,;\, -1$

$a_3 = 0$ <u>PROVIDED</u> $S \neq \frac{2}{3} \,;\, -\frac{2}{3}$

$T \neq \frac{2}{3} \,;\, -\frac{2}{3}$

The recurrence relationship is rewritten in terms of S and T as

$$a_l = \frac{a_{l-4} - \left[(c+l-2)^2\left(k^2\alpha_r^2 + n_r^2/k^2\right) - n^2\left(k^2\alpha_\theta^2 + n_\theta^2/k^2\right)\right]a_{l-2}}{(c+l-1+S)(c+l-1-S)(c+l-1+T)(c+l-1-T)}$$

$$l \geq 4$$

where by the requirements of the coefficients of R^{c+1}, R^{c+2}, and R^{c+3}

$$a_2 = \frac{-a_0\left[c^2\left(k^2\alpha_r^2 + n_r^2/k^2\right) - n^2\left(k^2\alpha_\theta^2 + n_\theta^2/k^2\right)\right]}{(c+1-S)(c+1+S)(c+1-T)(c+1+T)}$$

$$a_1 = a_3 = 0 \qquad (23)$$

CHAPTER III

SOLUTIONS FOR THE CASE OF SYMMETRICAL VIBRATIONS

A. SOLUTION OF THE EQUATION OF MOTION

For symmetrical vibrations the differential equation is independent of variable θ, thus all terms containing n vanish. The parameter β then equals k, and Equation (15) reduces to

$$\Big[R^4 \frac{d^4 Y}{dR^4} + 2R^3 \frac{d^3 Y}{dR^3} - k^2 R^2 \frac{d^2 Y}{dR^2} + k^2 R \frac{dY}{dR}$$

$$+ (k^2 \alpha_r^2 + n_r^2 | k^2) R^3 \frac{d}{dz}\left(R \frac{dw}{dR}\right) - R^4 Y \Big] = 0 \qquad (24)$$

Substituting $\beta = k$ and $n = 0$ into Equations (22) gives $S = 1$ and $T = k$, so that

$$C_1 = 1 + T = 1 + k$$

$$C_2 = 1 - T = 1 - k$$

$$C_3 = 1 + S = 2$$

$$C_4 = 1 - S = 0 \qquad (25)$$

The recurrence formula, Equation (23), then reduces to

$$a_l = \frac{a_{l-4} - [(c+l-2)^2 (K^2\alpha_r^2 + n_r^2 | K^2)]}{(c+l)(c+l-1)(c+l-1+k)(c+l-1-k)} \qquad l \geq 4$$

where

$$a_2 = -\frac{a_0[c^2(K^2\alpha_r^2 + n_r^2 | K^2)]}{c(c+2)(c+1+k)(c+1-k)} \tag{26}$$

and

$$a_1 = a_3 = 0$$

The four solutions corresponding to the indicies $c = 1+k,\ 1-k$, 2, 0, are respectively:

$$C_1 Y_1(r) = a_0 \sum_{J=1}^{\infty} b_J (Kr)^{2J-1+k} \tag{27}$$

where

$$b_1 = 1 \quad ; \quad b_2 = -\frac{[K^2\alpha_r^2 + n_r^2 | K^2]}{4(3+k)}$$

$$b_J = \frac{b_{J-2} - (2J-3+k)^2 (K^2\alpha_r^2 + n_r^2 | K^2) b_{J-1}}{4(J-1)(J-1+k)(2J-1+k)(2J-3+k)} \qquad J \geq 3$$

$$C_2 Y_2(r) = a_0' \sum_{J=1}^{\infty} b_J' (Kr)^{2J-1-k} \tag{28}$$

where

$$b_1' = 1 \quad ; \quad b_2' = -\frac{[K^2\alpha_r^2 - n_r^2 | K^2]}{4(3-k)}$$

$$b_J' = \frac{b_{J-2}' - (2J-3-k)^2 (K^2 \alpha_\gamma^2 + n_\gamma^2 / K^2)\, b_{J-1}'}{4(J-1)(J-1-k)(2J-1-k)(2J-3-k)} \qquad J \geq 3$$

$$C_3 Y_3(\gamma) = a_0'' \sum_{J=1}^{\infty} b_J'' (K\gamma)^{2J} \tag{29}$$

where:

$$b_1'' = 1$$

$$b_2'' = \frac{-[K^2 \alpha_\gamma^2 + n_\gamma^2 / K^2]}{2(3-k)(3+k)}$$

$$b_J'' = \frac{b_{J-2}'' - (2J-2)^2 [K^2 \alpha_\gamma^2 + n_\gamma^2 / K^2]\, b_{J-1}''}{4J(J-1)(2J-1+k)(2J-1-k)} \qquad J \geq 3$$

$$C_4 Y_4(\gamma) = \sum_{J=1}^{\infty} b_J''' (K\gamma)^{2J-2} \tag{30}$$

where

$$b_1''' = 1$$

$$b_2''' = 0$$

$$b_J''' = \frac{b_{J-2}''' - (2(J-2))^2 [K^2 \alpha_\gamma^2 + n_\gamma^2 / K^2]\, b_{J-1}'''}{4(J-1)(J-2)(2J-3+k)(2J-3-k)} \qquad J \geq 3$$

The complete solution of Equation (24) then reads,

$$W(r,t) = \left[C_1 Y_1(r) + C_2 Y_2(r) + C_3 Y_3(r) + C_4 Y_4(r) \right] e^{i\omega t} \tag{31}$$

For the special case of solid plates, $Y_2(r)$ is inadmissible since it yields an infinite slope at $r = 0$. Also for symmetrical vibrations it can be shown that $C_3 = 0$ by using the following argument: For a section of the solid plate of arbitrary radius r, the vertical shear force Q_{rz} must balance the inertial forces, since Newton's Second Law must be satisfied; that is,

$$\int_0^{2\pi} Q_{rz} \cdot r \, d\theta = \int_0^{r} \int_{\theta=0}^{2\pi} \rho h \frac{\partial^2 W}{\partial t^2} \cdot r \, dr . d\theta \tag{32}$$

where

$$W(r,t) = \left[C_1 Y_1(r) + C_3 Y_3(r) + C_4 Y_4(r) \right] . e^{i\omega t} \tag{33}$$

and where

$$Q_{rz} = -\left[D_r \left(\frac{\partial^3 W}{\partial r^3} + \frac{1}{r} \frac{\partial^2 W}{\partial r^2} \right) - \frac{D_\theta}{r^2} \frac{\partial W}{\partial r} - \alpha_r^2 \rho h \frac{\partial^3 W}{\partial r \, \partial t^2} \right] \tag{34}$$

Substituting Equations (27), (29), (30), (33), and (34) into Equation (32), yields

$$C_3 (2)(1 - k^2) = 0 \tag{35}$$

For $k > 0$, and $k \neq 1$, Equation (35) is satisfied by letting $C_3 = 0$. Therefore, the solution for a solid plate becomes:

$$W(r,t) = \left[C_1 Y_1(r) + C_4 Y_4(r) \right] e^{i\omega t} \tag{36}$$

B. FREQUENCY EQUATIONS

1. Clamped Plate

The boundary conditions for a clamped plate are:

$$W(a) = 0 \qquad \frac{dW}{dr}(a) = 0 \tag{37}$$

Inserting these conditions into Equation (36) yields

$$\begin{bmatrix} Y_1(a) & Y_4(a) \\ \frac{dY_1}{dr}(a) & \frac{dY_4}{dr}(a) \end{bmatrix} \begin{bmatrix} C_1 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{38}$$

For nontrivial solutions of the constants C_1 and C_4, the determinant of the 2 x 2 matrix must equal zero. This leads to the frequency equation

$$Y_1(a) \cdot \frac{dY_4}{dr}(a) - Y_4(a) \cdot \frac{dY_1}{dr}(a) = 0 \qquad (39)$$

or, in series representation:

$$\psi(K) = \left[\sum_{J=1}^{\infty} bj\,(Ka)^{2J} \cdot \sum_{J=1}^{\infty} bj''\,(2J-2)\,(Ka)^{2J} \right.$$

$$\left. - \sum_{J=1}^{\infty} bj'''\,(Ka)^{2J} \cdot \sum_{J=1}^{\infty} bj\,(2J-1+k)(Ka)^{2J} \right] = 0 \qquad (40)$$

The roots of Equation (40), $\psi(K)=0$, give the values of K, which when inserted into the 'frequency' equation,

$$f = \frac{\omega}{2\pi} = \frac{K^2}{2\pi}\left[\frac{Dr}{\rho h}\right]^{1/2} \qquad (41)$$

yield values of the natural frequencies in cycles per second for the clamped plate.

2. Simply Supported Plate

For the simply supported plate the boundary conditions are

$$W(a) = 0 \qquad M_{rr}(a) = -Dr\left[\frac{d^2W}{dr^2}(a) + \frac{\nu_\theta}{a}\frac{dW}{dr}(a)\right] = 0 \qquad (42)$$

Conditions (42) together with Equations (36) give

$$\begin{bmatrix} Y_1(a) & Y_4(a) \\ \frac{d^2Y_1(a)}{dr^2} + \frac{\nu_\theta}{a}\frac{dY_1(a)}{dr} & \frac{d^2Y_4(a)}{dr^2} + \frac{\nu_\theta}{a}\frac{dY_4(a)}{dr} \end{bmatrix} \begin{bmatrix} C_1 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (43)$$

which yields the frequency equation in series form as:

$$\sum_{J=1}^{\infty} b_j (Ka)^{2J} \cdot \sum_{J=1}^{\infty} b_j''' \, 2(J-1)(2J-3+\nu_\theta)(Ka)^{2J}$$

$$-\sum_{J=1}^{\infty} b_j''' (Ka)^{2J} \sum_{J=1}^{\infty} b_j (2J-1+k)(2J-2+k+\nu_\theta)(Ka)^{2J} = 0 \qquad (44)$$

The roots of frequency Equation (44) are substituted into Equation (41) and the natural frequencies (in units of cycles/second) are then obtained for a simply supported plate.

CHAPTER IV

SOLUTIONS FOR THE CASE OF ASYMMETRICAL VIBRATIONS

A. EQUATIONS OF MOTION

For asymmetrical vibrations the four values of the index C are given by Equations (21) and (22). The recurrence relations by Equations (23) and $\beta^2 = 2n^2\sigma' + k^2$.

The equations corresponding to the indicies $1+S$, $1-S$, $1+T$, and $1-T$ are respectively:

$$C_1 Y_1(r) = a_0 \sum_{J=1}^{\infty} b_j (Kr)^{2J-1+T} \qquad (45)$$

where

$$b_1 = 1$$

$$b_2 = -\frac{\left[(1+S)^2(K^2\alpha_r^2 + n_r^2/K^2) - n^2(K^2\alpha_\theta^2 + n_\theta^2/K^2)\right]}{4(1+S)(2+S+T)(2+S-T)}$$

$$b_j = b_{j-2} - \frac{\left[(2J-3+S)^2(K^2\alpha_r^2 + n_r^2/K^2) - n^2(K^2\alpha_\theta^2 - n_\theta^2/K^2)\right] b_{j-1}}{4(J-1)(J-1+S)(2J-2+S+T)(2J-2+S-T)}$$

$$J \geq 3$$

$$C_2 Y_2(r) = \sum_{J=1}^{\infty} b_j' (Kr)^{2J-1-S} \qquad (46)$$

where

$$b_1' = 1$$

$$b_2' = \frac{-\left[(1-S)^2(K^2\alpha_r^2 + n_r^2/K^2) - n^2(K^2\alpha_\theta^2 + n_\theta^2/K^2)\right]}{4(1-S)(2-S-T)(2-S+T)}$$

$$b_J' = \frac{b_{J-2}' - \left[(2J-3-S)^2(K^2\alpha_r^2 + n_r^2/K^2) - n^2(K^2\alpha_\theta^2 + n_\theta^2/K^2)\right] b_{J-1}'}{4(J-1)(J-1-S)(2J-2+S-T)(2J-2-S-T)}$$

$$J \geq 3$$

$$C_3 Y_3(r) = \sum_{J=1}^{\infty} b_J'' (Kr)^{2J-1+T} \tag{47}$$

where

$$b_1'' = 1$$

$$b_2'' = \frac{-\left[(1+T)^2(K^2\alpha_r^2 + n_r^2/K^2) - n^2(K^2\alpha_\theta^2 + n_\theta^2/K^2)\right]}{4(1+T)(2+S+T)(2-S+T)}$$

$$b_J'' = \frac{b_{J-2} - \left[(2J+T-3)^2(K^2\alpha_r^2 + n_r^2/K^2) - n^2(K^2\alpha_\theta^2 + n_\theta^2/K^2)\right] b_{J-1}}{4(J-1)(J-1+T)(2J-2+S+T)(2J-2-S-T)}$$

$$J \geq 3$$

$$C_4 Y_4(r) = \sum_{J=1}^{\infty} b_J''' (Kr)^{2J-1-T} \tag{48}$$

where

$$b_1''' = 1$$

$$b_2''' = \frac{-\left[(1-T)^2(K^2\alpha_r^2 + n_r^2/K^2) - n^2(K^2\alpha_\theta^2 + n_\theta^2/K^2)\right]}{4(1-T)(2+S-T)(2-S-T)}$$

$$b_j''' = \frac{b_{j-2}''' - \left[(2J-T-3)^2(K^2\alpha_r^2 + n_r^2/K^2) - n^2(K^2\alpha_\theta^2 + n_\theta^2/K^2)\right] b_{j-1}}{4(J-1)(J-1-T)(2J-2+S+T)(2J-2-S+T)}$$

$$J \geq 3$$

The solution is written

$$W(r,\theta,t) = \left[\sum_n \left(C_1 Y_1(r) + C_2 Y_2(r) + C_3 Y_3(r) + C_4 Y_4(r)\right) e^{in\theta}\right] e^{i\omega t} \quad (49)$$

The solutions $Y_1(r)$, $Y_2(r)$, $Y_3(r)$, and $Y_4(r)$ hold for values of n = 2,3,4,... . For a solid plate the deflection and the slope must be bounded at $r=0$. This condition is satisfied only if $C_2 = C_4 = 0$. For the special case n = 1, $C_1 = C_3 = 1$, that is, the indicial roots are repeated. The solution $Y_3(r)$ is valid but the solution of $Y_4(r)$ must be altered. The new solution $Y_4(r)$ contains logarithmic terms which are unbounded at $r=0$. Therefore $C_2 = C_4 = 0$ must again be zero to satisfy the condition of finite slope and finite deflection at $r=0$, respectively. The solution reduces to

$$W(r,\theta,t) = \left[\sum_n \left(C_1 Y_1(r) + C_3 Y_3(r) \right) e^{in\theta} \right] e^{i\omega t} \qquad (50)$$

B. FREQUENCY EQUATIONS

1. Clamped Plate

The boundary conditions for a clamped plate at $r = a$ are

$$W(a) = 0 \qquad \frac{dW}{dr}(a) = 0 \qquad (51)$$

Inserting these conditions into Equation (50) gives

$$\begin{bmatrix} Y_1(a) & Y_3(a) \\ \frac{dY_1}{dr}(a) & \frac{dY_3}{dr}(a) \end{bmatrix} \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (52)$$

For nontrivial solution of constants C_1 and C_3 the following frequency equation must hold.

$$Y_1(a) \frac{dY_3}{dr}(a) - Y_3(a) \frac{dY_1}{dr}(a) = 0 \qquad (53)$$

or

$$\sum_{J=1}^{\infty} bj(Ka)^{2J} \cdot \sum_{J=1}^{\infty} bj''(2J-1+T)(Ka)^{2J} - \sum_{J=1}^{\infty} bj''(Ka)^{2J} \cdot \sum_{J=1}^{\infty} bj(2J-1+S)(Ka)^{2J} = 0 \quad (54)$$

2. Simply Supported Plate

The boundary conditions for a simply supported plate are:

$$W(a) = 0 \qquad M_{rr}(a) = -D_r\left[\frac{\partial^2 W}{\partial r^2}(a) + \frac{\nabla_\theta}{a}\frac{\partial W(a)}{\partial r} + \frac{1}{a^2}\frac{\partial^2 W(a)}{\partial \theta^2}\right] = 0 \quad (55)$$

but since $W(a) = 0$ then $\frac{\partial^2 W(a)}{\partial \theta^2} = 0$ AND M_{rr}

reduces to

$$M_{rr}(a) = -D_r\left[\frac{d^2W}{dr^2}(a) + \frac{\nabla_\theta}{a}\frac{dW}{dr}(a)\right] = 0 \qquad (56)$$

Using these boundary conditions one obtains

$$\begin{bmatrix} Y_1(a) & Y_3(a) \\ \left[\frac{d^2Y_1(a)}{dr^2} + \frac{\nabla_\theta}{a}\frac{dY_1}{dr}\right] & \left[\frac{d^2Y_3(a)}{dr^2} + \frac{\nabla_\theta}{a}\frac{dY_3}{dr}\right] \end{bmatrix} \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (57)$$

which yields

$$Y_1(a)\left[\frac{d^2 Y_3(a)}{d\gamma^2} + \frac{\nu_\theta}{a}\frac{dY_3}{d\gamma}\right] - Y_3(a)\left[\frac{d^2 Y_1(a)}{d\gamma^2} + \frac{\nu_\theta}{a}\frac{dY_1}{d\gamma}(a)\right] = 0 \quad (58)$$

which in turn yields the frequency equation

$$\sum_{J=1}^{\infty} b_J (Ka)^{2J} \cdot \sum_{J=1}^{\infty} b_J''(2J-1+T)(2J-2+T+\nu_\theta)(Ka)^{2J}$$

$$-\sum_{J=1}^{\infty} b_J''(Ka)^{2J} \cdot \sum_{J=1}^{\infty} b_J(2J-1+S)(2J-2+S+\nu_\theta)(Ka)^{2J} = 0 \quad (59)$$

SUMMARY

The solution of the differential equation for the free vibration of a cylindrically orthotropic circular plate with consideration of in-plane stability forces and the effect of transverse and rotary inertia is determined. The effect of transverse shear and normal is neglected. It is assumed that N_{rr}, $N_{\theta\theta}$, $N_{r\theta}$, I_r, and I_θ are independent of r and θ. The method of separation of variables is used to solve the differential equation for the special case of $N_{r\theta} = 0$. The solution of the differential equation is determined in infinite series form using the method of Frobenius. Frequency equations for symmetrical and asymmetrical vibrations for the special cases of fixed and simply supported solid plates are provided.

CONCLUSION

The solution of differential equation of motion in closed form is obtained by the technique of separation of variables only if $N_{r\theta} = 0$. For a given value of $\eta_r = \left(\frac{N_{rr}}{D_r}\right)^{\frac{1}{2}}$ the frequency equation $\Psi(K) = 0$ is solved for the roots of K, i.e. K_1, K_2, K_3 ... K_m. The dynamic stability criterion states that the instability load is that value of in-plane load at which the natural frequency of vibration of the plate is identically equal to zero. Therefore in order to obtain the critical load one must increment the value of until the value of $K_1 = 0$. This yields the lowest critical load N_{rr}. The higher order critical loads are obtained in the same manner. The value of η_r is incremented beyond the lowest critical buckling load until $K_2 = 0$, $K_3 = 0$, ... , $K_n = 0$.

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