THE ANALYSIS OF A DYNAMIC STABILITY OF A VON MISES

TRUSS INCLUDING THE EFFECT OF TORSIONAL SPRINGS

by

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ABSTRACT

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The purpose of this thesis is to investigate the dynamic stability characteristics of the Von Mises truss including the effect of torsional springs. The snap-through buckling phenomena is investigated for various stiffnesses of the torsional springs. The stability and instability criteria is developed using the phase plane analysis technique. A series of phase plane diagram are drawn. The coordinates of points in the phase plane are determined using the electronic computer.

A comparision of results is made with the Von Mises truss solution where the torsional springs are neglected.

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CHAPTER I

INTRODUCTION

In recent years the phenomena of snap-through buckling is being investigated more extensively.[1,2,3,4,5] The condition is used in a practical sense in a simple oil can. As the pressure on the bottom surface of the can is increased, the surface suddenly snaps through to a new equilibrium position thus forcing oil out of the can. If the pressure is then reduced the surface will snap-through to its original shape. This stability characteristic occurs in shallow-domed roofs, trough-shaped springs, and springs-type electical switches.

The idealized Von Mises truss contains only linear springs in the two sloping members. This condition illustrates the snap-through concept in an extreamely simplified manner. In a shallow spherical shell or in a shallow arch, bending stresses play a major role in the snap-through phe--nomena. A more complete understanding of this condition in these more complicated structure is obtained by adding the torsional springs to the Von Mises truss and investigating the resulting effects.

To investigating the stability of equilibrium configurations, the nonlinear and nonhomogeneous second order differential equations of the motion is formed by the use of Lagrange's equations. The higher order equation of motion is reduced to a family of first order differential equations. Inherent in the reduction of order is the potential energy integral equation is formed by the condition that the sum of the potential plus the kinetic energy remain constant. This equation is used to obtain the phase plane curve or trajectories. This is a geometric image of energy integral equation.

Proper interpretation of the phase plane diagrams yields the conditions of stable and unstable equilibrium configurations. The snap-through phenomena is dramatically observable.

A general outline of the procedure of the thesis is as follows:

1. Formulate the equation of motion for the Von Mises truss for the case of linear spring.

2. Develope the basic mathematical concepts of both static and dynamic equilibrium for the linear spring model.

3. Formulate the equation of motion for the combined linear and torsional spring model.

4. Investigate the mathematical requirements for static and dynamic conditions for stable and unstable equilibrium configurations in the combined spring system.

5. Solve a numerical problem for the linear spring model, and for the combined spring model and interpret the results graphically.

CHAPTER II

MATHEMATICAL ANALYSIS

1. Linear springs analysis:

A load P is applied to two elastic rods which are supported and connected by pin joints at points A, B, and C as shown in Fig. 1.

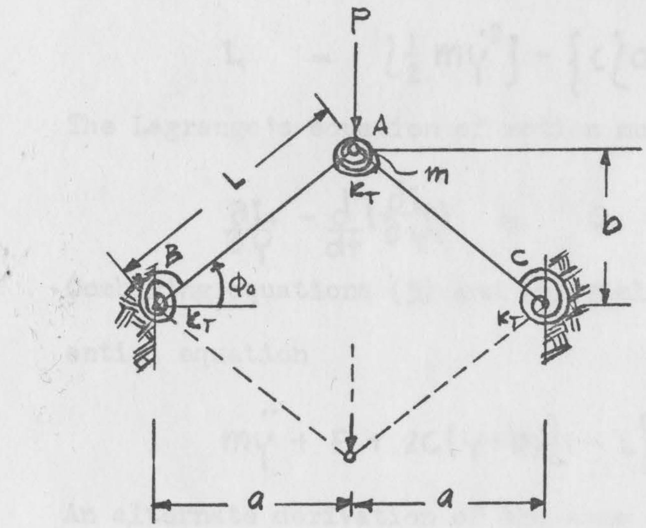


Fig. 1 Static Equilibrium Position.

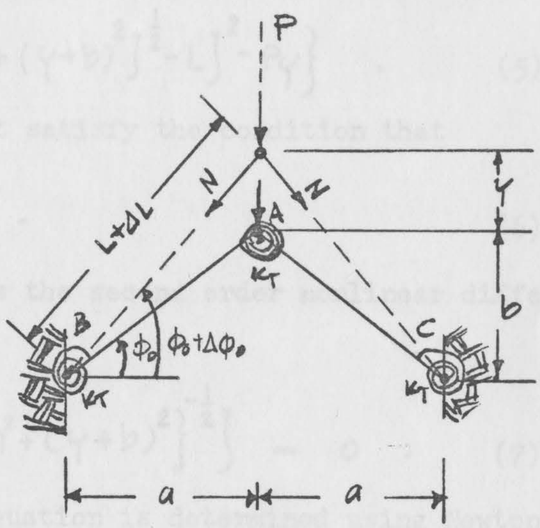


Fig. 2 Deformed Equilibrium Position.

The following assumptions are made for the mathematical model:

1) The entire mass of the system is concentrated at point A.
2) The member AB and AC are linear elastic rods.
3) The angle ϕ_o (see Fig. 1) is measured to the static equilibrium position.
4) $K_T = 0$.

The following relationships hold for the deformed equilibrium position (see Fig. 2)

$$\Delta L = \left[a^2+(y+b)^2\right]^{\frac{1}{2}} - L \tag{1}$$

$$\sin(\phi_o+\Delta\phi_o) = \frac{y+b}{\left[a^2+(y+b)^2\right]^{\frac{1}{2}}} \tag{2}$$

The equation of motion is formulated using Lagrange's equation in the following manner. The Lagrangian defined as the difference between the kinetic and potential energy is expressed in the form

$$L = \left[\tfrac{1}{2} m\dot{y}^2\right] - \left[2\left(\tfrac{1}{2}\right)\frac{AE}{L}\Delta L^2 + Py\right] \quad . \tag{3}$$

Using equation (1), and noting that

$$c = \frac{AE}{L} \quad , \tag{4}$$

Equation (3) is rewritten as

$$L = \left[\tfrac{1}{2} m\dot{y}^2\right] - \left\{c\left[\left[a^2 + (y+b)^2\right]^{\frac{1}{2}} - L\right]^2 - Py\right\} \quad . \tag{5}$$

The Lagrange's equation of motion must satisfy the condition that

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 \quad . \tag{6}$$

Combining equations (5) and (6) yields the second order nonlinear differential equation

$$m\ddot{y} + P + 2c(y+b)\left[1 - L\left\{a^2 + (y+b)^2\right\}^{-\frac{1}{2}}\right] = 0 \quad . \tag{7}$$

An alternate derivation of the same equation is determined using Newton's laws. Where the condition of motion satisfies the equation

$$\Sigma F_y = m a_y \tag{8}$$

Defining N as the compressive force in the rod, and noting that

$$N = \frac{AE}{L}\Delta L \quad , \tag{9}$$

then it follows that

$$m\ddot{y} + P + 2N \, \mathrm{Sin}(\phi_0 + \Delta\phi) = 0 \quad . \tag{10}$$

Substitution of equations (1) and (2) into equation (10) gives the same equation of motion as,

$$m\ddot{y} + P + 2c(y+b)\left[1 - L\left\{a^2 + (y+b)^2\right\}^{\frac{1}{2}}\right] = 0 \quad . \tag{11}$$

For convenience the following parameters are introduced:

$$q_1 = \frac{y+b}{a} \quad , \tag{12}$$

$$\dot{q}_1 = q_2 = \dot{y}/a \quad , \tag{13}$$

$$\ddot{q}_1 = \dot{q}_2 = \ddot{y}/a \quad , \text{ and} \tag{14}$$

$$\alpha = \frac{L}{a} \quad . \tag{15}$$

Substitution of new parameters into equation (7) reduces the second order differential equation into the following pair of first order equations,

$$\dot{q}_1 = q_2 \quad , \tag{16}$$

$$\dot{q}_2 = -\left\{\frac{P}{ma} + \frac{2c}{m} q_1 \left[1 - \alpha\left\{1 + q_1^2\right\}^{-\frac{1}{2}}\right]\right] = -f(q_1) \, , \tag{17}$$

The potential energy function appears in derivative form on the R.H.S. of equation (17), and is determined by noting the integral condition

$$V(q_1) = \int_{q_1=0}^{q_1=q_1} f(q_1) dq_1 \quad . \tag{18}$$

By proper integration it follows that

$$V(q_1) = \frac{P}{ma} q_1 + \frac{c}{m} q_1^2 - 2\frac{c}{m}\alpha(1+q_1^2)^{\frac{1}{2}} - (-2\alpha\frac{c}{m}) \quad . \tag{19}$$

Rearranging terms, there results

$$V(q_1) = \frac{c}{m}\left[\left(\frac{P}{AE}\right)\alpha q_1 + q_1^2 - 2\alpha\left\{(1+q_1^2)^{\frac{1}{2}} - 1\right\}\right] \tag{20}$$

The static stability analysis is completely determined by investigating the potential energy function given in equation (20). The necessary condition for static equlibrium is that

$$\frac{dV(q_1)}{dq_1} = 0 \tag{21}$$

This condition yields the following result:

$$\frac{P_{CR}}{AE} = \frac{2q_1}{\alpha}\left[\alpha(1+q_1^2)^{-\frac{1}{2}} - 1\right] \tag{22}$$

The sufficient condition for equilibrium are that

$$\frac{d^2V}{dq_1^2} < 0 \quad \text{for unstable equilibrium}$$

$$\frac{d^2V}{dq_1^2} > 0 \quad \text{for stable equilibrium}$$

$$\frac{d^2V}{dq_1^2} = 0 \quad \text{for neutral equilibrium (i.e. the boundary between stable and unstable equilibrium states)}$$

The critical value of q_1 is found by the third condition above as

$$q_{1CR} = \pm(\alpha^{\frac{2}{3}} - 1)^{\frac{1}{2}} \quad . \tag{23}$$

Substituting this q_{1CR} into equation (22) gives the associated value of P_{CR} in the form

$$\frac{P_{CR}}{AE} - 2\left(1 - \frac{1}{\alpha^{\frac{2}{3}}}\right)^{\frac{3}{2}} \quad . \tag{24}$$

The concept of snap-through buckling can be interpreted geometrically using the load deflection curve. Referring to equation (7), the inertia term is omitted and the resulting equation is solved for the variable P(y) in the form

$$\frac{P}{AE} = \frac{2}{\alpha}\left(\frac{y+b}{a}\right)\left[\alpha\left\{1 + \left(\frac{y+b}{a}\right)^2\right\}^{-\frac{1}{2}} - 1\right] \quad . \tag{25}$$

The plot of this equation is shown in Figure 3 for increasing values of P and Figure 4 for decreasing values of P. Referring to Figure 3, as P increases the states of equilibrium take the order 1, 2, 3, 4, and 5 as shown. The values of P_{CR1} and Φ_{CR1} at located at point 3. However, since point 3 is a condition of unstable equilibrium the system moves instantaneously to point 4 without an increase in load P. If P is again increased point 5 is reached and the system remains stable for all higher value of P.

If the load on the system is decreased from point 5 a completely different phenomena occurs (see Figure 4). As P decreases the states of equilibrium take the order 6, 7, 8, 9, and 10. Points 6 and

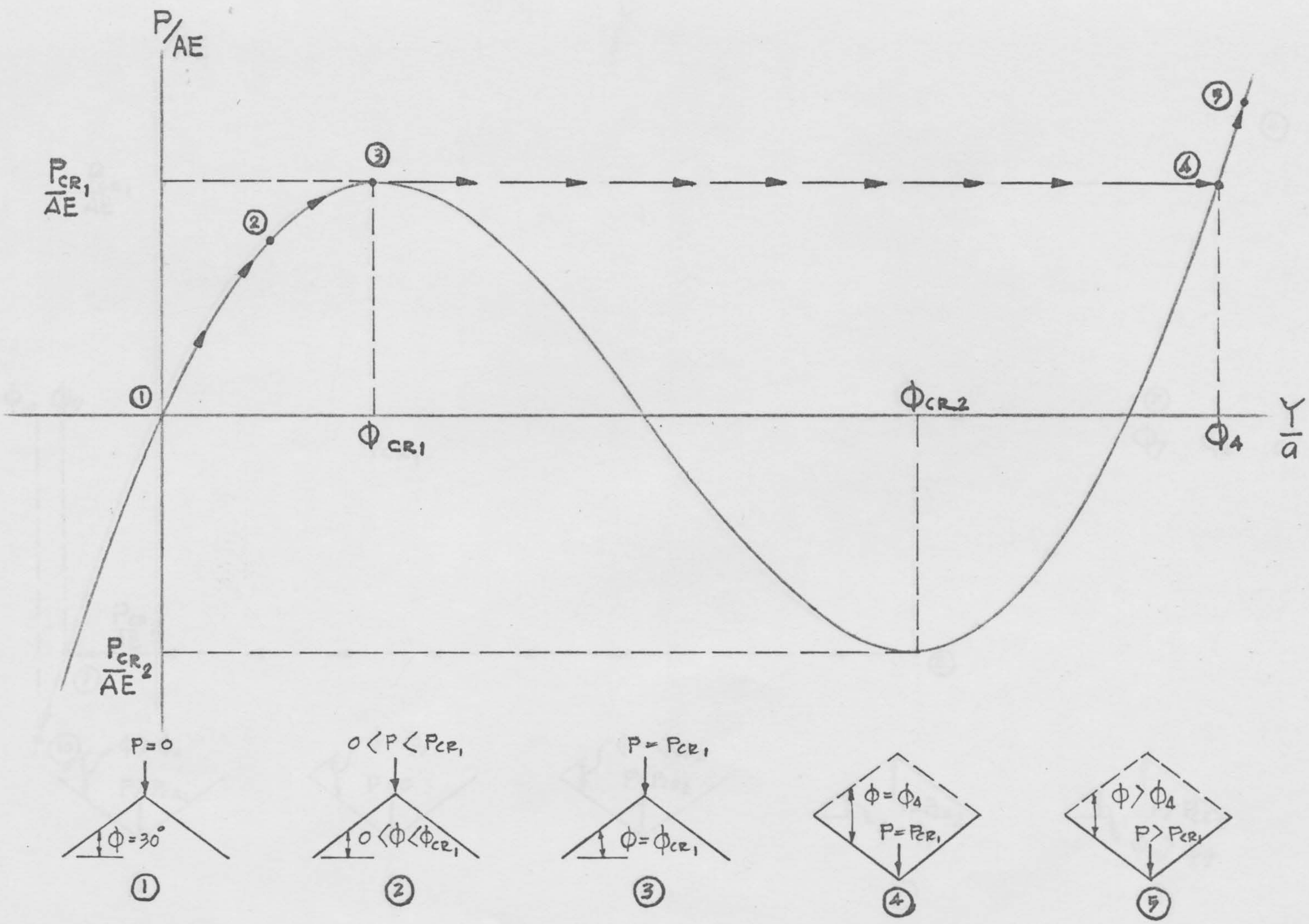


Fig. 3. Load Deflection Curve Shows Snap-Through Buckling By Increasing P from P=0 to P > Pcr.

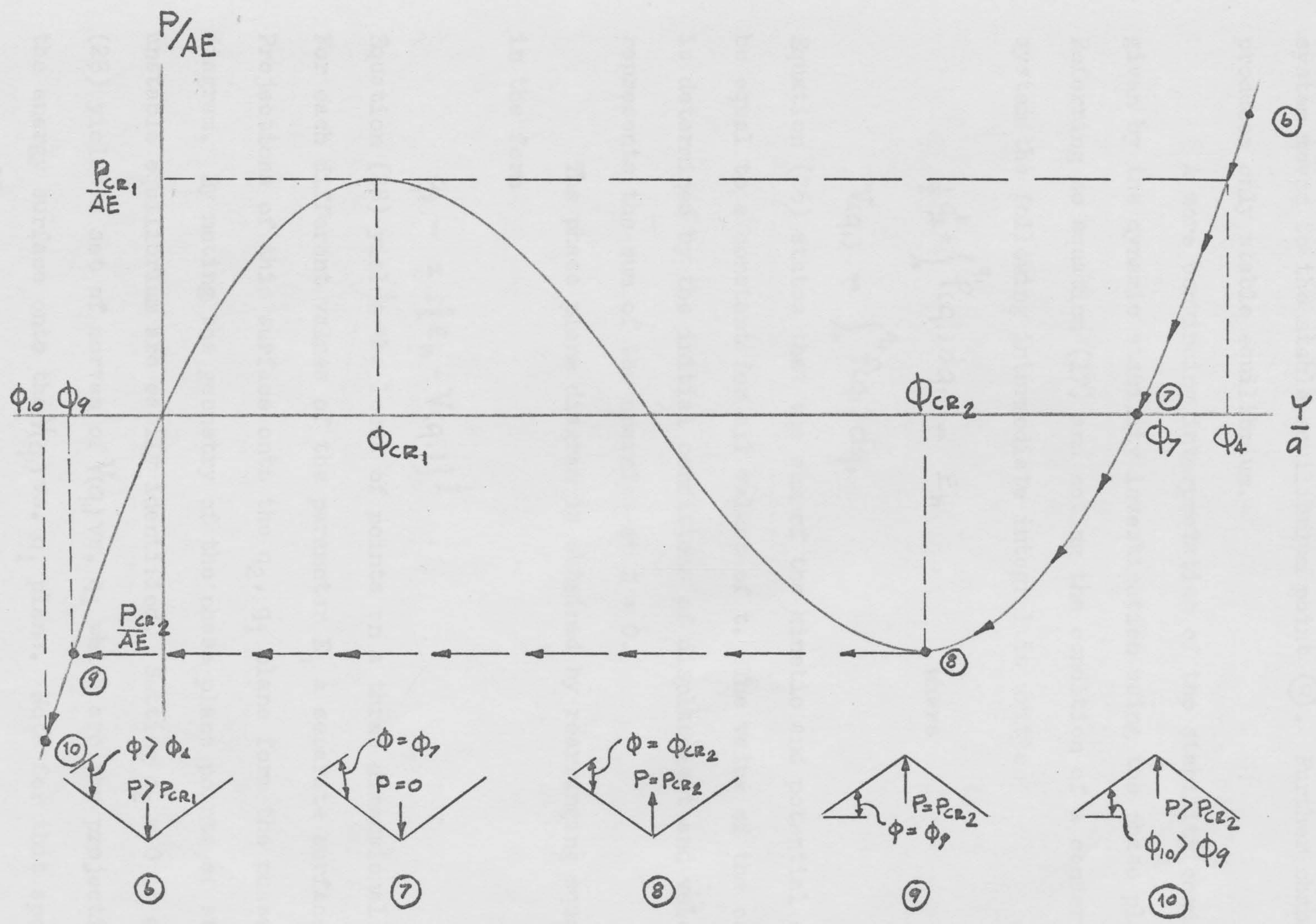


Fig. 4. Load Deflection Curve Shows Snap-Through Buckling By Decreasing P from $P > P_{CR}$ to P negative.

7 are stable points, but point 8 is an unstable point corresponding to P_{CR_2} and ϕ_{CR_2}. Instantaneously snap-through buckling occurs and the system moves to the stable equilibrium point 9. Further change in P produces only stable equilibrium.

A more convincing interpretation of the stability conditions is given by the dynamic stability investigation using the phase plane method. Referring to equation (17) and noting the condition of a conservative system the following intermediate integral is written

$$\frac{1}{2}q_2^2 + \int_0^{q_1} f(q_1)\,dq_1 = E_h \quad , \qquad \text{where} \tag{26}$$

$$V(q_1) = \int_0^{q_1} f(q_1)\,dq_1 \quad . \tag{27}$$

Equation (26) states that the sum of the kinetic and potential energy must be equal to a constant for all values of t. The value of the constant E_h is determined by the initial conditions of displacement and velocity. It represents the sum of the energies at t = 0.

The phase plane diagram is obtained by rearranging equation (26) in the form

$$q_2 = \pm 2\left[E_h - V(q_1)\right]^{\frac{1}{2}} \quad . \tag{28}$$

Equation (28) yields the locus of points on a three dimensional surface. For each different values of the parameter E_h a separate surface exists. Projections of this surface onto the q_2, q_1 plane form the phase plane diagram. By noting the geometry of the phase plane points of stable and unstable equilibrium are easily identified. Setting $q_2 = 0$ in equation (28) yields a set of curves of $V(q_1)$ vs. q_1 which are the projections of the energy surface onto the $V(q_1)$ vs. q_1 plane. Note for this special case only $E_h = V(q_1)$.

2. <u>Combined linear and torsional springs analysis:</u> $(K_T \neq 0)$

For the special case of torsional springs, the following geometic conditions hold:

$$\Delta\phi_0 = \tan^{-1} q_1 - \phi_0 \quad , \tag{29}$$

$$\phi_0 = \tan^{-1} q_0 \quad . \tag{30}$$

Substitution of equation (30) into equation (29) gives

$$\Delta\phi_0 = \tan^{-1} q_1 - \tan q_0 \quad . \tag{31}$$

The Lagrangian of the system is written in the form,

$$L = \{\tfrac{1}{2} m \dot{y}^2\} - [Py + 2(\tfrac{1}{2}\tfrac{AE}{L}\Delta L^2) + 2(4K_T \Delta\phi_0^2)] \quad . \tag{32}$$

By direct substitution of equations (1), (4), and (31) into equation (32) gives,

$$L = \{\tfrac{1}{2} m \dot{y}^2\} - \{Py + C[\{a^2 + (y+b)^2\}^{\frac{1}{2}} - L]^2 - 2K_T\{\tan^{-1} q_1 - \tan^{-1} q_0\}^2\} \quad . \tag{33}$$

The differential equation of motion satisfies the Lagrange's equation

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 \quad . \tag{34}$$

The equation of motions becomes

$$\frac{\ddot{y}}{a} + \frac{P}{ma} + 2\frac{c}{m}\left(\frac{b+y}{a}\right)\left[1 - \frac{L}{a}\left\{1 + \left(\frac{y+b}{a}\right)^2\right\}^{-\frac{1}{2}}\right] + 4\frac{K_T}{ma^2}[\tan^{-1} q_1 - \tan^{-1} q_0]\left(\frac{1}{1+q_1^2}\right) = 0 \quad , \tag{35}$$

Using equations (12) through (15) the differential equation is written in the form

$$\dot{q}_2 = -\left[\frac{P}{ma} + 2\frac{c}{m} q_1\{1 - \alpha(1+q_1^2)^{-\frac{1}{2}}\} + 4\frac{K_T}{ma^2}\{\tan^{-1} q_1 - \tan^{-1} q_0\}(1+q_1^2)^{-1}\right] = -f(q_1) \quad . \tag{36}$$

Also, the potential energy function is rewritten in the form

$$V(q_1) = \int_0^{q_1} f(q_1)\,dq_1 \quad , \quad \text{or} \tag{37}$$

$$V(q_1) = \frac{c}{m}\left\{\left(\frac{P}{AE}\alpha q_1 + q_1^2 - 2\alpha\{(1+q_1^2)^{\frac{1}{2}} - 1\} + 2\frac{K_T}{ca^2}\{(\tan^{-1} q_1)^2 - 2\tan^{-1} q_1 \tan^{-1} q_0\}\right\}\right] \quad . \tag{38}$$

The condition for static stability analysis are

$$\frac{dV(q_1)}{dq_1} = \frac{d^2V(q_1)}{dq_1^2} = 0 \quad . \tag{39}$$

Thus, the critical value of force P becomes

$$\frac{P_{CR}}{AE} = \frac{2q_1}{\alpha}\left\{\alpha(1+q_1^2)^{-\frac{1}{2}}-1\right\} - \frac{4K_T}{\alpha Ca^2}(1+q_1^2)^{-1}\left\{\tan^{-1}q_1 - \tan^{-1}q_0\right\} \quad . \tag{40}$$

The critical value of displacement function must satisfy the equation

$$q_1^2 = \frac{\alpha^{\frac{2}{3}}}{\left[1 + \frac{2K_T}{Ca^2}(1+q_1^2)^{-2}\left\{1+2q_1(\tan^{-1}q_1-\tan^{-1}q_0\right\}\right]^{\frac{2}{3}}} - 1 \quad . \tag{41}$$

In the general case equation (41) cannot be solved directly for q_{1CR}.

For the special case of $K_T = 0$ equations (40) and (41) reduce to equations (23) and (24).

The value of the ratio $\frac{K_T}{Ca^2}$ for the special case where $q_1 = 0$ is obtained from equation (41) in the form

$$\frac{K_T}{Ca^2} = \frac{\alpha - 1}{2} \quad . \tag{42}$$

The corresponding value of $\frac{P_{CR}}{AE}$ is obtained from equation (40) as

$$\frac{P_{CR}}{AE} = \frac{2(\alpha - 1)}{\alpha}\tan^{-1}q_0 \quad . \tag{43}$$

A similar geometric interpretation of the dynamic stability problem using the phase plane diagram as in the case of the linear springs analysis exist for this problem. Equations (26), (27), and (28) hold in this case except the quantity $V(q_1)$ is replaced by the quantity defined by equation (38).

CHAPTER III

NUMERICAL PROBLEM AND GEOMETRIC INTERPRETATIONS

1. Linear springs analysis:

The numerical problem is solved for the special case of the following geometry:

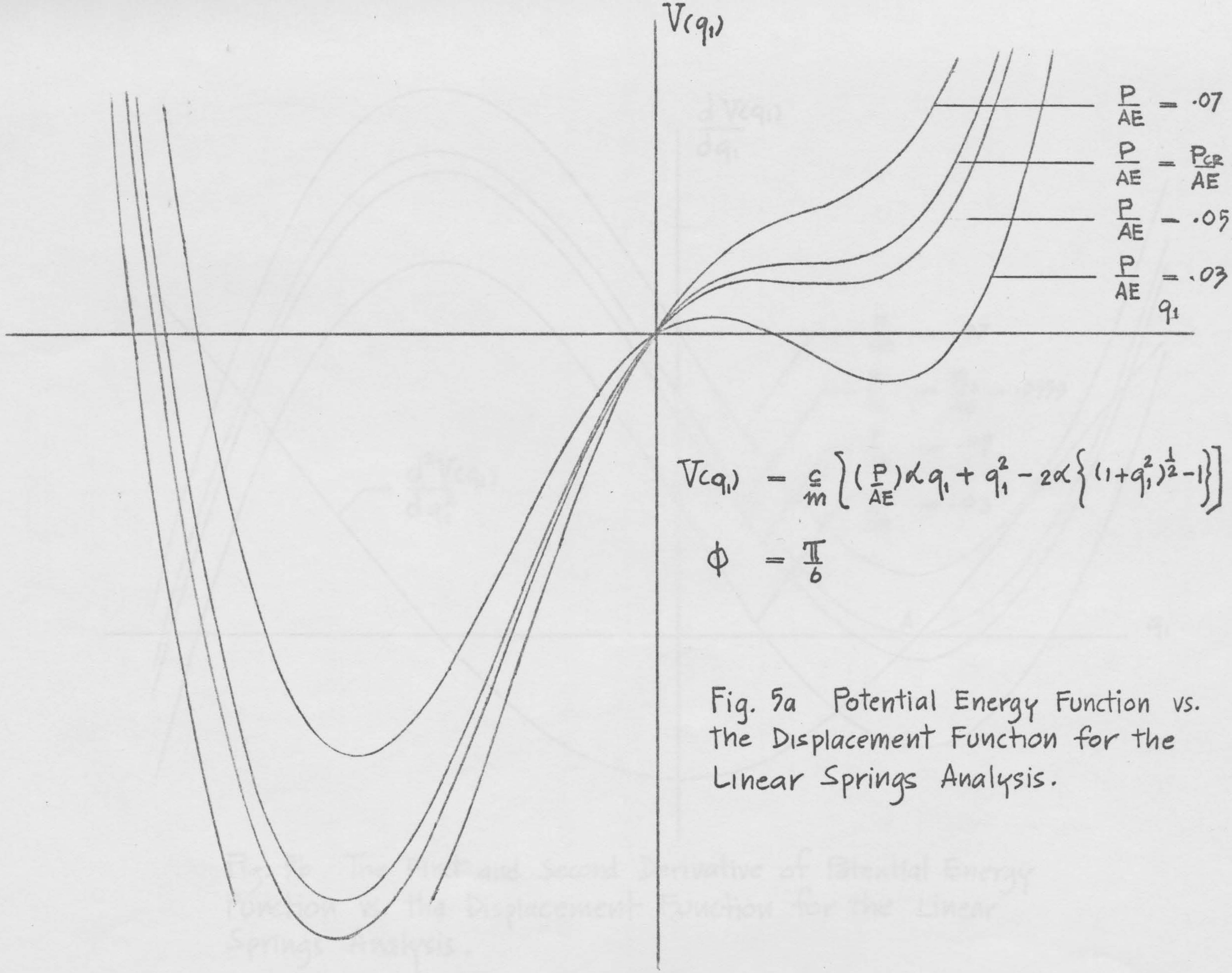
$$\left.\begin{aligned} \Phi_0 &= \pi/6 \quad , \quad \text{and} \\ \alpha &= \frac{2\sqrt{3}}{3} \quad . \end{aligned}\right\} \tag{44}$$

Referring to equations (23) and (24) and using equations (44), we obtain

$$\left.\begin{aligned} q_{1CR} &= \left[\left(\frac{2\sqrt{3}}{3}\right)^{\frac{2}{3}} - 1\right]^{\frac{1}{2}} \approx 0.317 \quad , \quad \text{and} \\ \frac{P_{CR}}{AE} &= 2\left[1 - \frac{1}{\left(\frac{2\sqrt{3}}{3}\right)^{\frac{2}{3}}}\right]^{\frac{3}{2}} \approx 0.0553 \; . \end{aligned}\right\} \tag{45}$$

A geometric interpretation of the static stability are shown in Fig. 5a and 5b. The functions $V(q_1), \frac{dV}{dq_1}(q_1), \frac{d^2V}{dq_1^2}(q_1)$ are plotted vs. the quantity q_1 for different value of $\frac{P}{AE}$. The conditions that $\frac{dV}{dq_1}(q_1) = \frac{d^2V}{dq_1^2}(q_1) = 0$ are shown to exist at point A as shown on Fig. 5b. The corresponding values of q_{1CR} and $\frac{P_{CR}}{AE}$ are shown to be the same as given in equations (45).

The dynamic stability conditions are shown using the phase plane diagram shown in Fig. 5c, 5d, and 5e. Figures 5c and 5d show the phase plane plot for $\frac{P}{AE} < \frac{P_{CR}}{AE}$. For both diagrams two stable points A and B exist and one unstable point C exists. As the $\frac{P}{AE}$ increases the stable point A approaches the unstable point C. These two points remain distinct provided $\frac{P}{AE} < \frac{P_{CR}}{AE}$. Figure 5e illustrates the condition where $\frac{P}{AE} = \frac{P_{CR}}{AE}$. Points A and C coincide and the resulting point is unstable. Only one stable point remains at point B. The mechanism snaps-through to this equilibrium state.



$$V(q_1) = \frac{c}{m}\left[\left(\frac{P}{AE}\right)\alpha q_1 + q_1^2 - 2\alpha\left\{(1+q_1^2)^{\frac{1}{2}} - 1\right\}\right]$$

$$\phi = \frac{\pi}{6}$$

Fig. 5a Potential Energy Function vs. the Displacement Function for the Linear Springs Analysis.

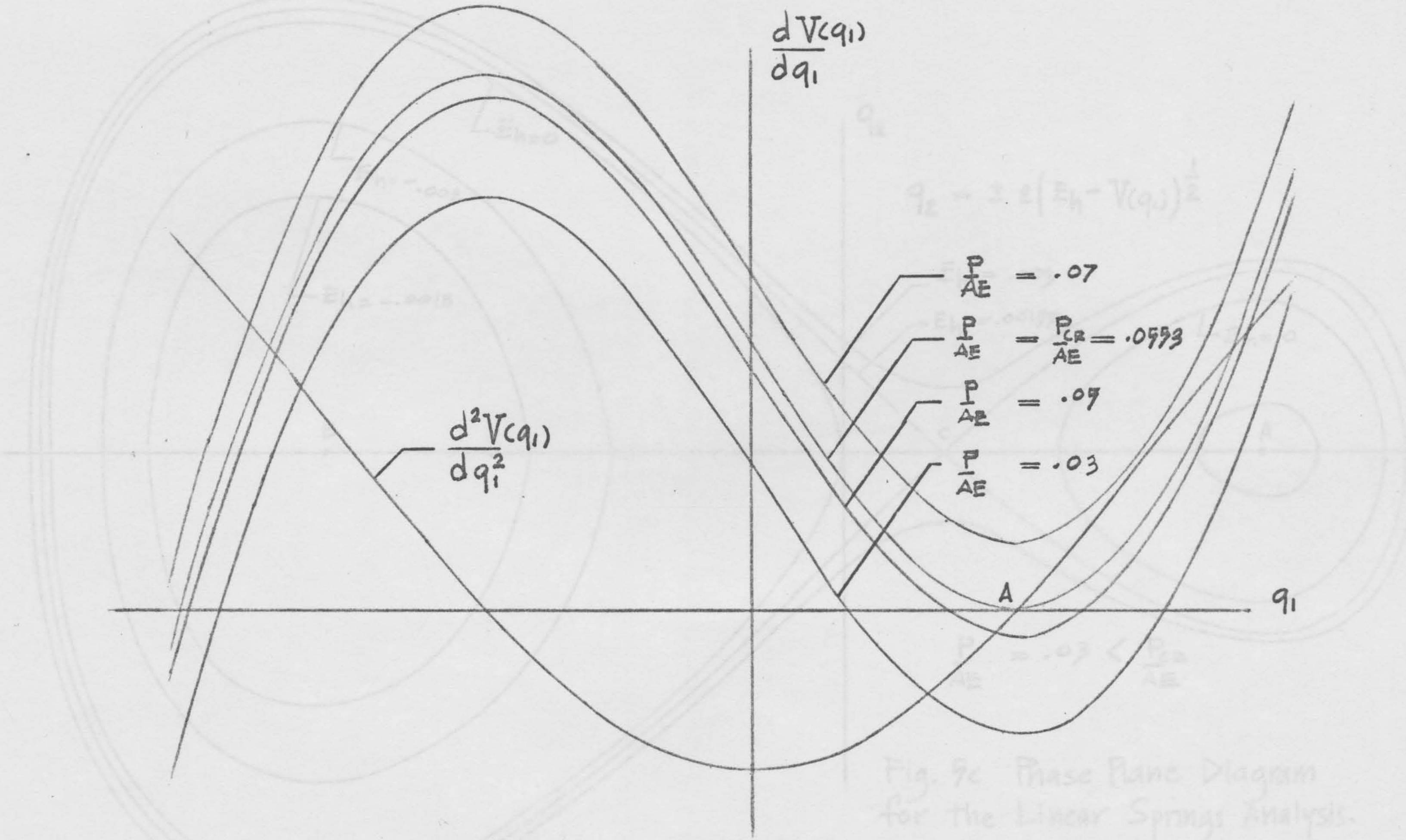


Fig. 5b The First and Second Derivative of Potential Energy Function vs. the Displacement Function for the Linear Springs Analysis.

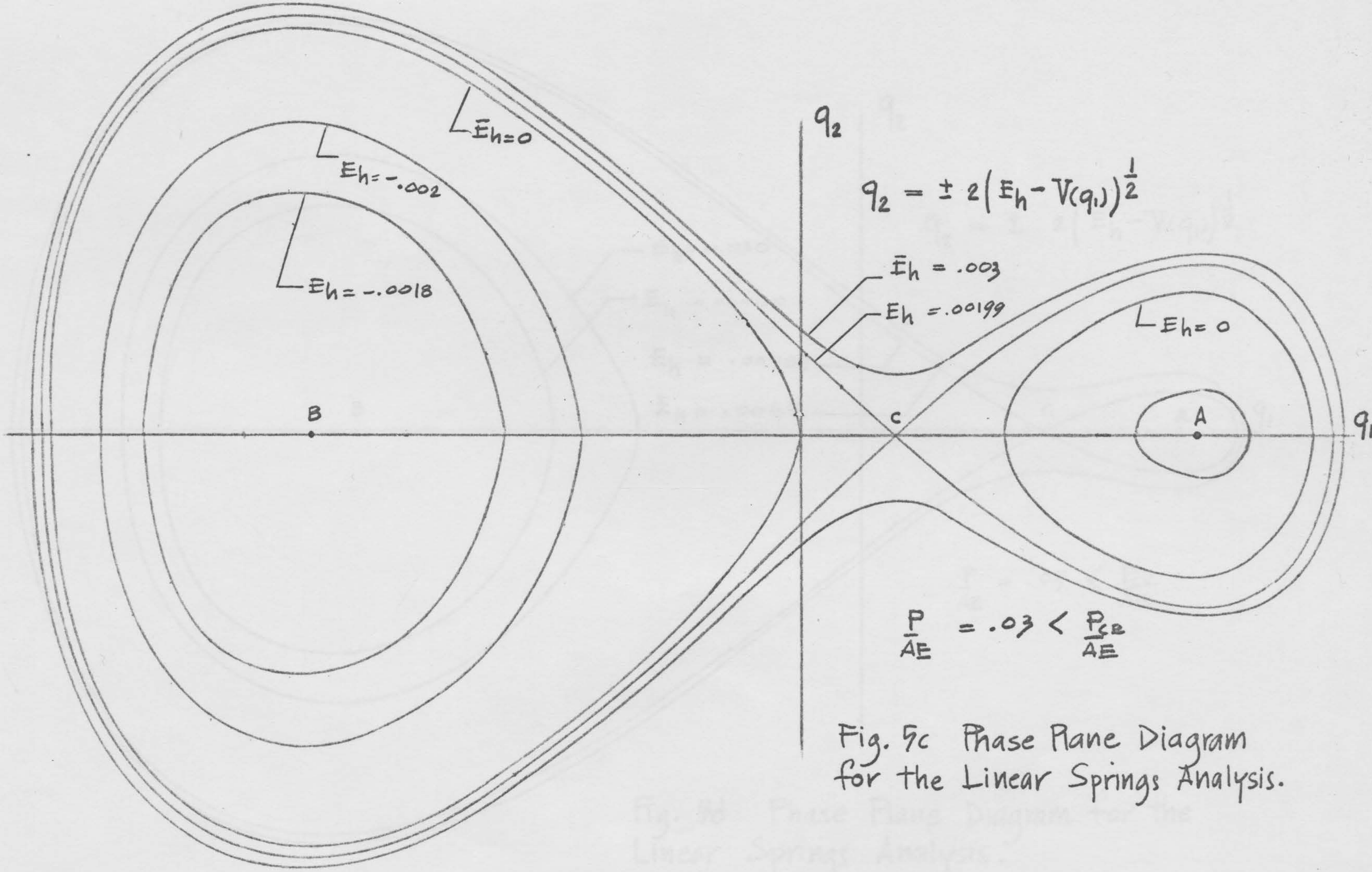


Fig. 5c Phase Plane Diagram for the Linear Springs Analysis.

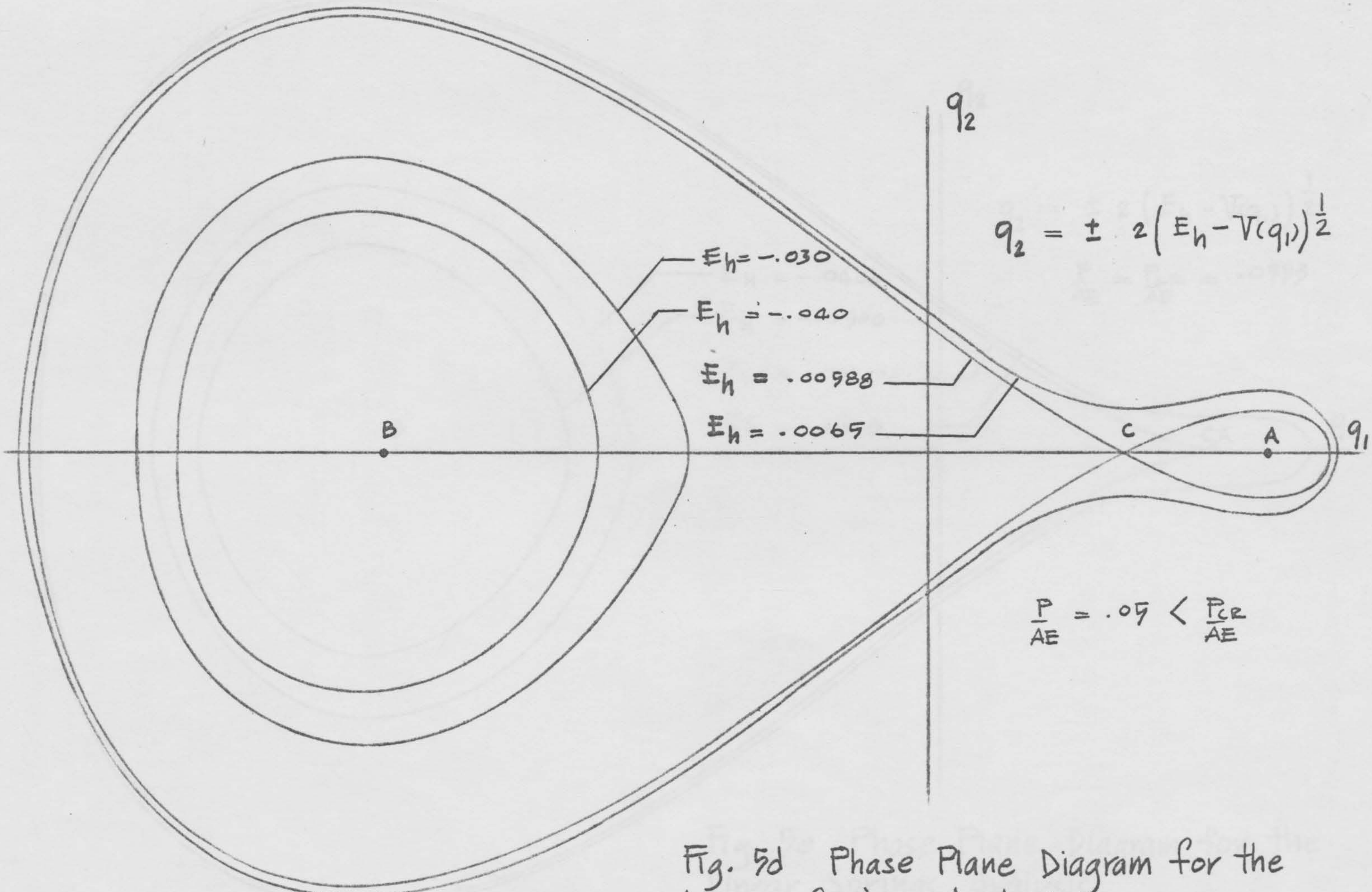


Fig. 5d Phase Plane Diagram for the Linear Springs Analysis.

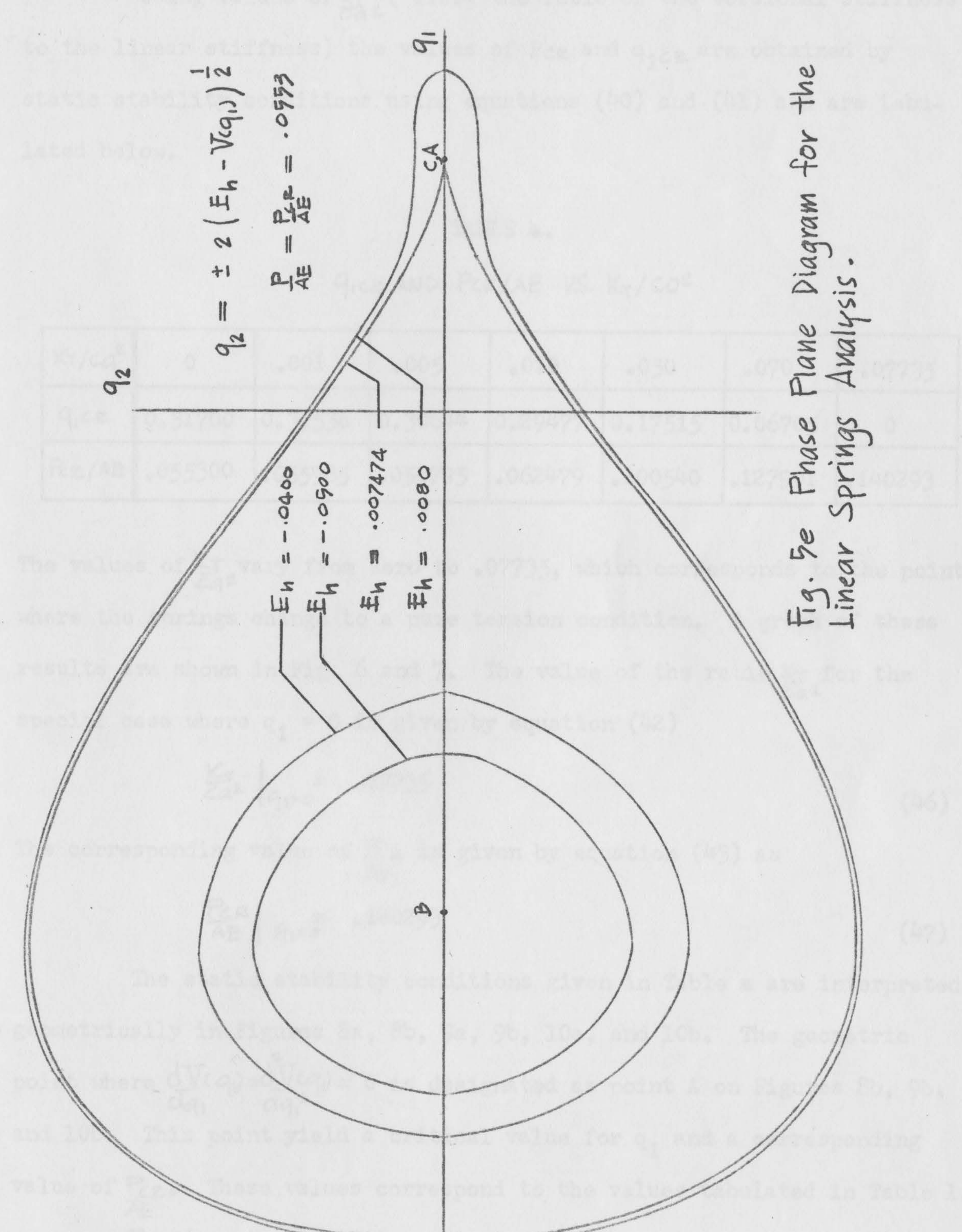


Fig. 5e Phase Plane Diagram for the Linear Springs Analysis.

2. Combined linear and torsional springs analysis: $(K_T \neq 0)$

Using values of $\frac{K_T}{Ca^2}$ (i.e., the ratio of the torsional stiffness to the linear stiffness) the values of P_{CR} and q_{1CR} are obtained by static stability conditions using equations (40) and (41) and are tabulated below.

TABLE a.

q_{1CR} AND P_{CR}/AE VS. K_T/Ca^2

K_T/Ca^2	0	.001	.005	.010	.050	.070	.07735
q_{1CR}	0.31700	0.31536	0.30644	0.29477	0.17515	0.06705	0
P_{CR}/AE	.055300	.055985	.058795	.062479	.100540	.127901	.140293

The values of $\frac{K_T}{Ca^2}$ vary from zero to .07735, which corresponds to the point where the springs change to a pure tension condition. A graph of these results are shown in Fig. 6 and 7. The value of the ratio $\frac{K_T}{Ca^2}$ for the special case where $q_1 = 0$ is given by equation (42)

$$\left.\frac{K_T}{Ca^2}\right|_{q_1=0} = .07735 \tag{46}$$

The corresponding value of $\frac{P_{CR}}{AE}$ is given by equation (43) as

$$\left.\frac{P_{CR}}{AE}\right|_{q_1=0} = .140293 \tag{47}$$

The static stability conditions given in Table a are interpreted geometrically in Figures 8a, 8b, 9a, 9b, 10a, and 10b. The geometric point where $\frac{dV(q_1)}{dq_1} = \frac{d^2V(q_1)}{dq_1^2} = 0$ is designated as point A on Figures 8b, 9b, and 10b. This point yield a critical value for q_1 and a corresponding value of $\frac{P_{CR}}{AE}$. These values correspond to the values tabulated in Table 1.

The dynamic stability criteria is illustrated in the phase plane diagrams in Figures 8c, 8d, 8e, 9c, 9d, 9e, 10c, 10d, and 10e. Figures

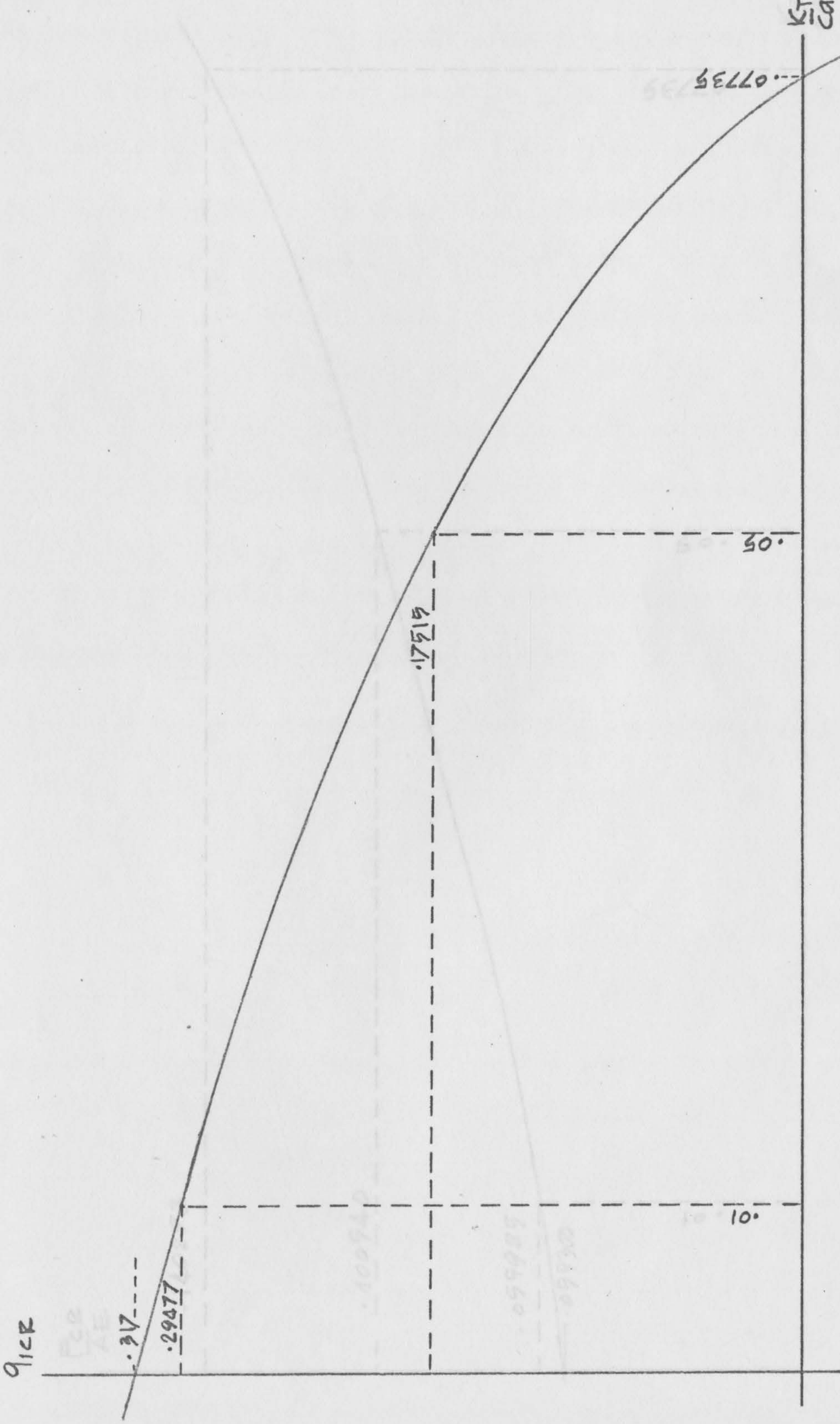


Fig. 6 A Graph of q_{ICE} vs. $\frac{KT}{Ca^2}$

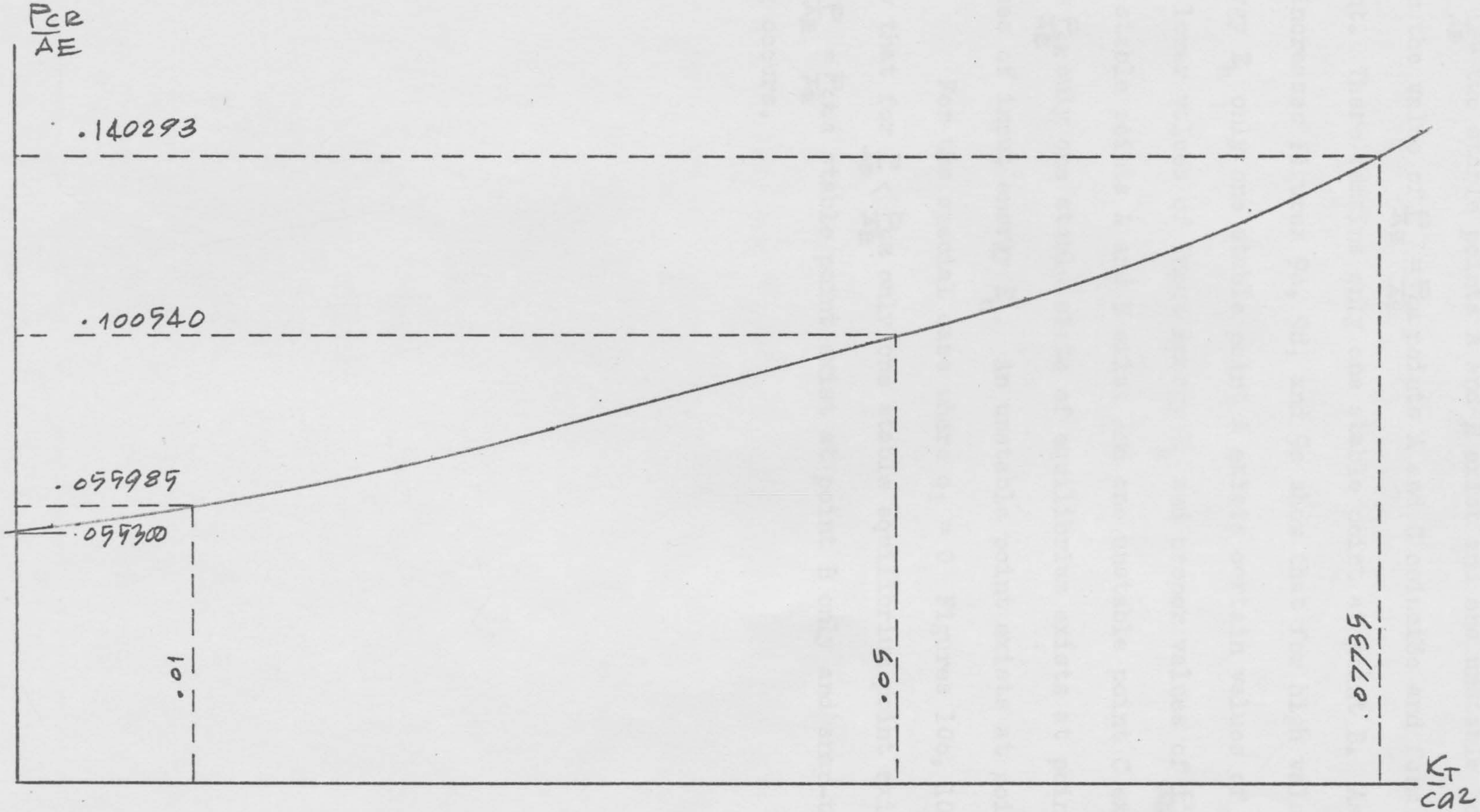
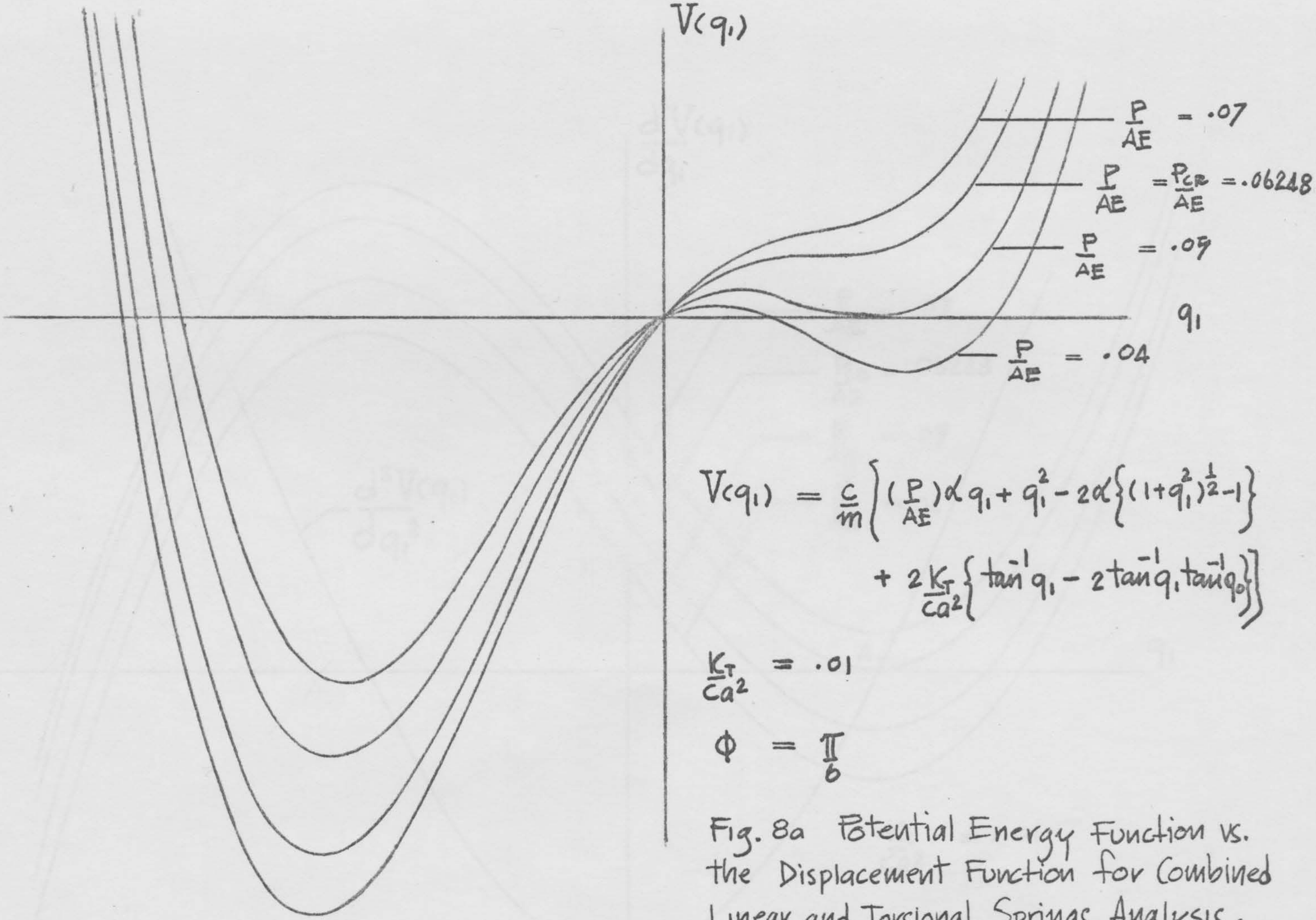


Fig. 7 A Graph of $\frac{P_{CR}}{AE}$ vs. $\frac{K_T}{ca^2}$ corresponding to q_{1CR}.

8c, 8d, and 8e show that for small values of the quantity $\frac{K_T}{Ca^2}$ and for $\frac{P}{AE} < \frac{P_{CR}}{AE}$ two stable points A and B exist and one unstable point C exists. When the value of $\frac{P}{AE} = \frac{P_{CR}}{AE}$ points A and C coincide and form an unstable point. There remains only one stable point at point B. As the value of $\frac{K_T}{Ca^2}$ increases Figures 9c, 9d, and 9e show that for high values of input energy E_h only one stable point A exists certain values of $\frac{P}{AE}$ (see Fig.9c). For lower values of input energy E_h and proper values of $\frac{P}{AE}$ (see Fig.9d) two stable points A and B exist and one unstable point C exists. When $\frac{P}{AE} = \frac{P_{CR}}{AE}$ only one stable state of equilibrium exists at point B for all values of input energy E_h. An unstable point exists at point A.

For the special case where $q_1 = 0$ Figures 10c, 10d, and 10e show that for $\frac{P}{AE} < \frac{P_{CR}}{AE}$ only one stable equilibrium point exist at point A. For $\frac{P}{AE} = \frac{P_{CR}}{AE}$ a stable point exist at point B only and snap-through buckling occurs.



$$V(q_1) = \frac{c}{m}\left\{ \left(\frac{P}{AE}\right)\alpha q_1 + q_1^2 - 2\alpha\left\{(1+q_1^2)^{\frac{1}{2}} - 1\right\} \right.$$

$$\left. + 2\frac{K_T}{Ca^2}\left\{ \tan^{-1} q_1 - 2\tan^{-1} q_1 \tan^{-1} q_0 \right\}\right\}$$

$$\frac{K_T}{Ca^2} = .01$$

$$\phi = \frac{\pi}{6}$$

Fig. 8a Potential Energy Function vs. the Displacement Function for Combined Linear and Torsional Springs Analysis.

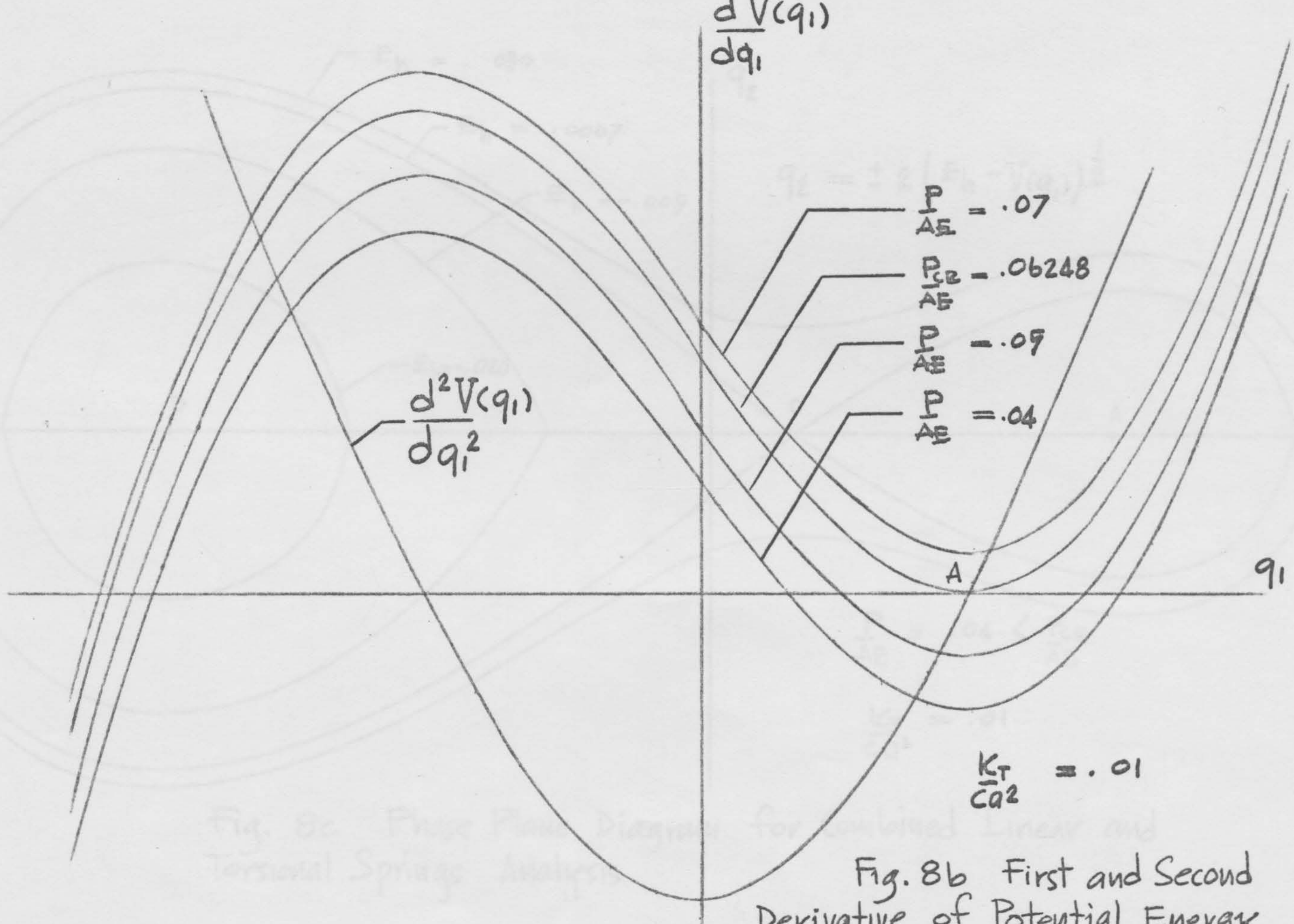
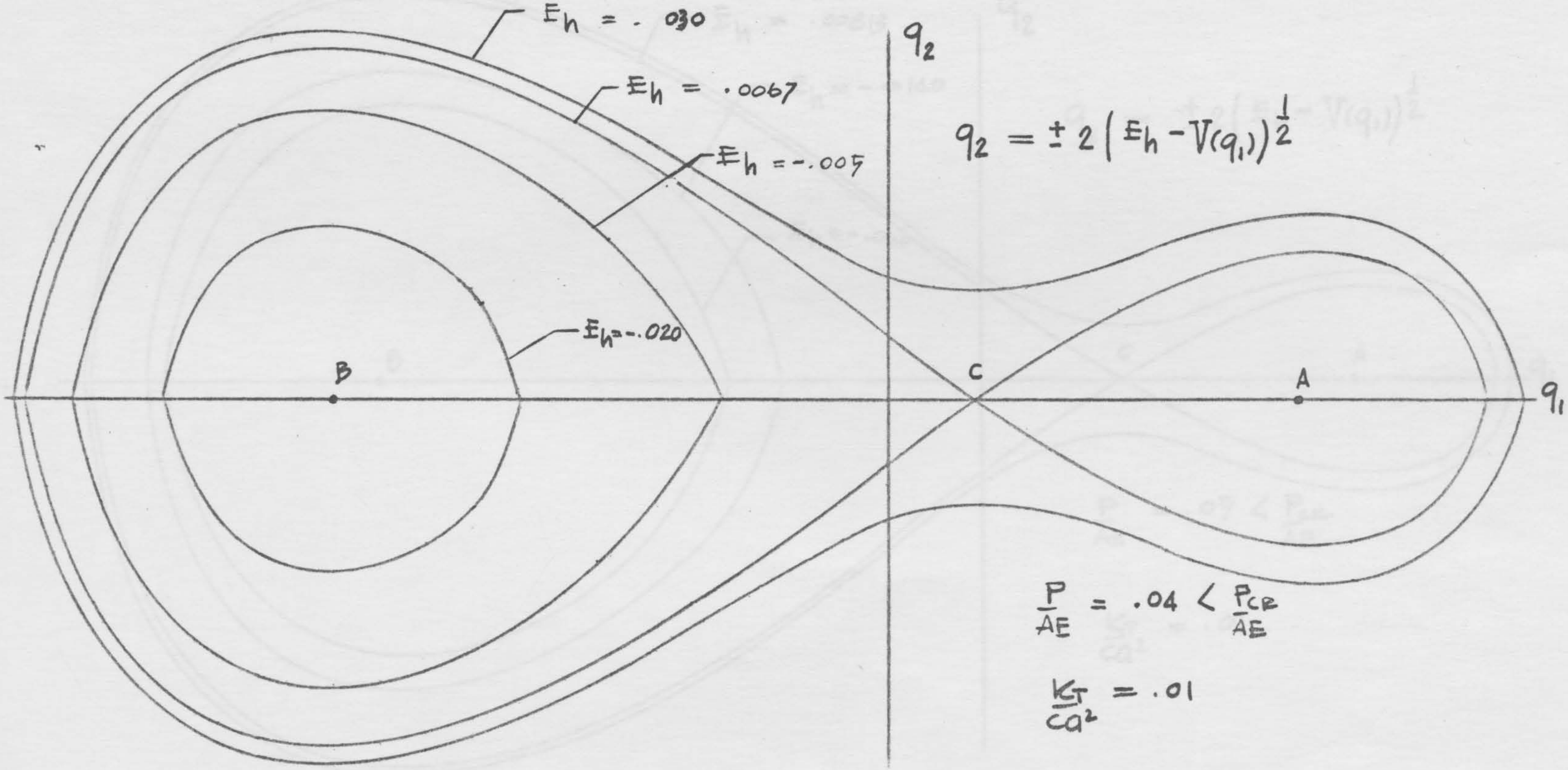


Fig. 8b First and Second Derivative of Potential Energy for Combined Linear and Torsional Springs Analysis.



$$q_2 = \pm 2\left(E_h - V(q_1)\right)^{\frac{1}{2}}$$

$$\frac{P}{AE} = .04 < \frac{P_{CR}}{AE}$$

$$\frac{K_T}{ca^2} = .01$$

Fig. 8c Phase Plane Diagram for Combined Linear and Torsional Springs Analysis.

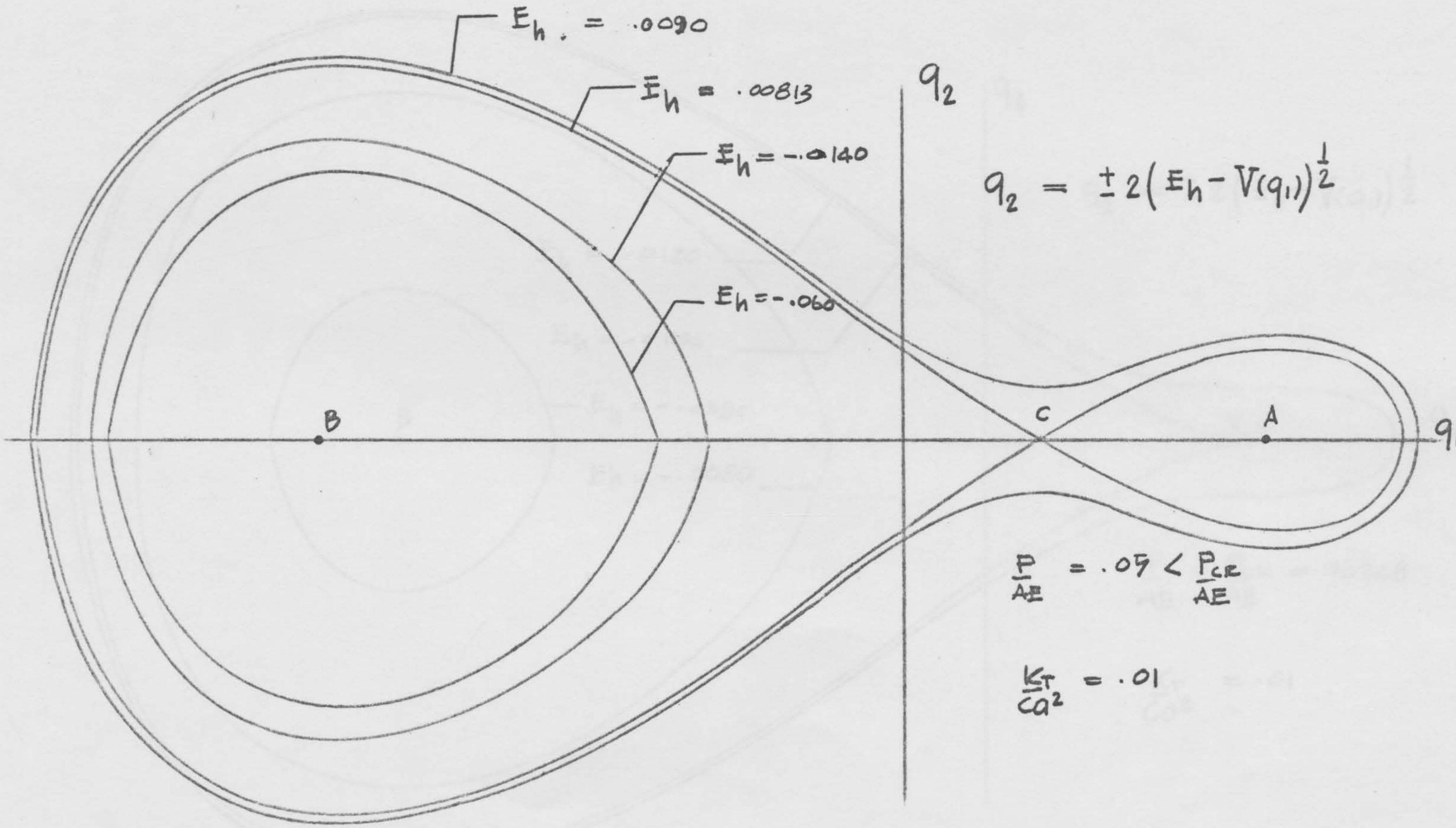


Fig. 8d Phase Plane Diagram for Combined Linear and Torsional Springs Analysis.

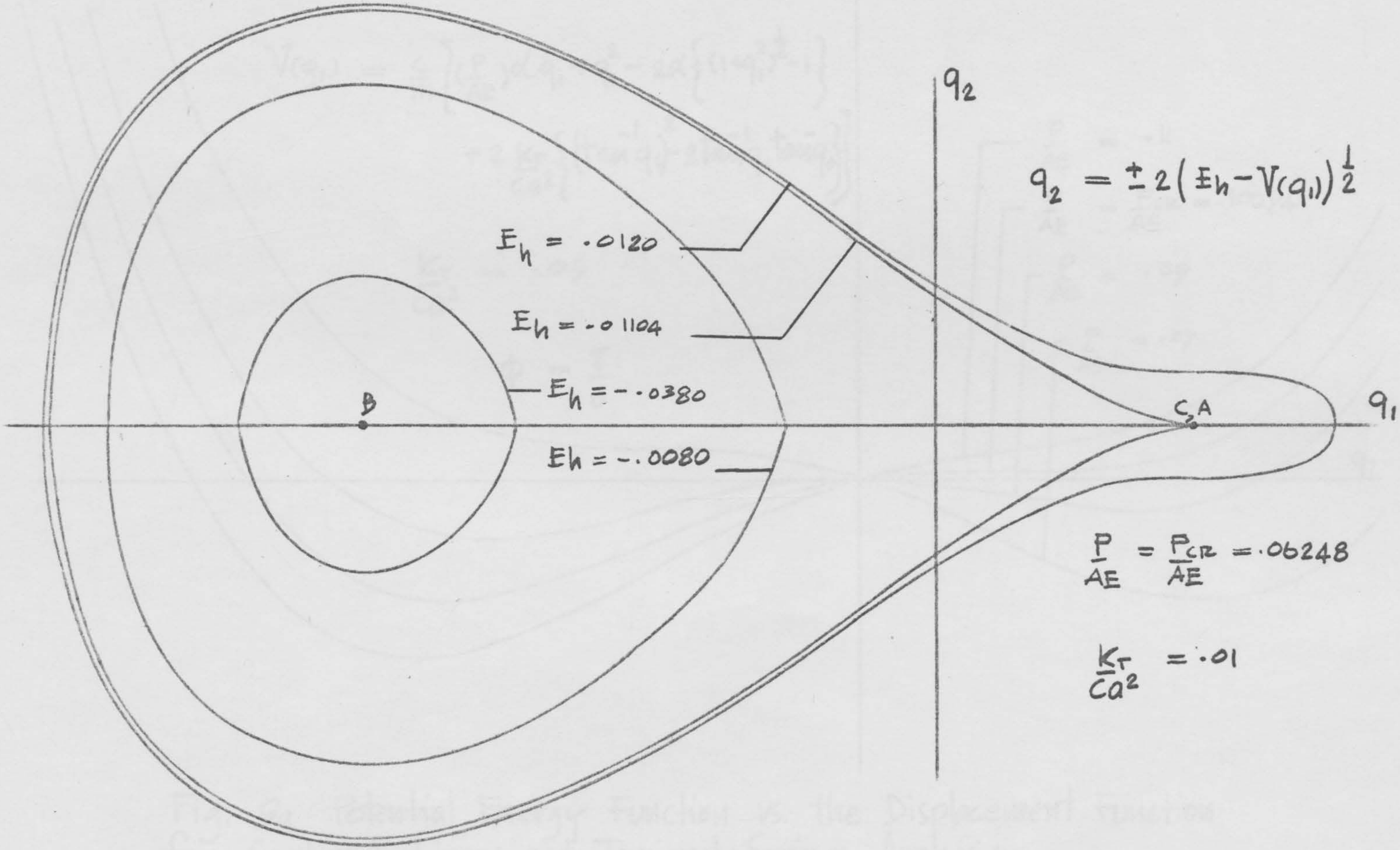


Fig. 8e Phase Plane Diagram for Combined Linear and Torsional Springs Analysis.

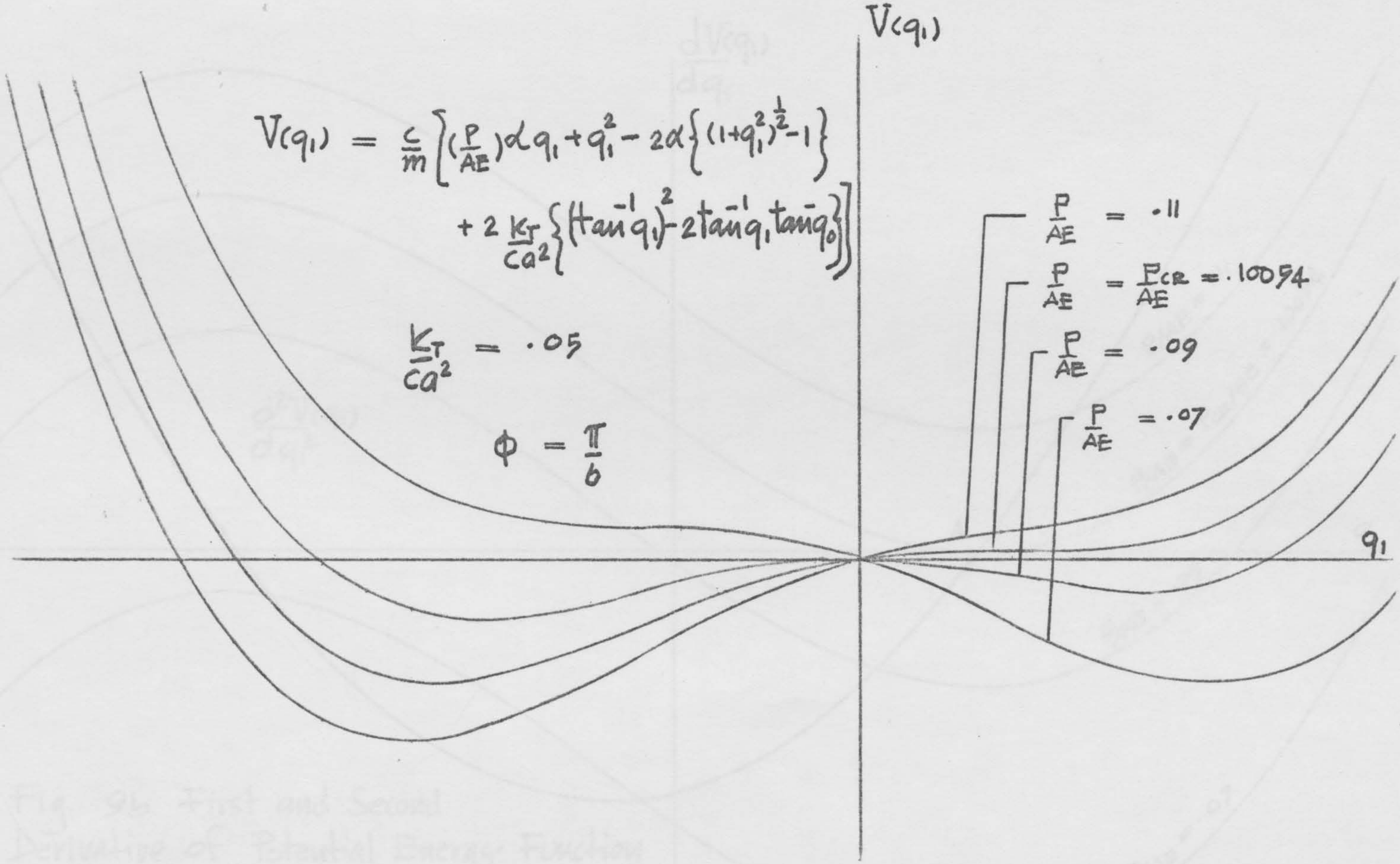


Fig. 9a Potential Energy Function vs. the Displacement Function for Combined Linear and Torsional Springs Analysis.

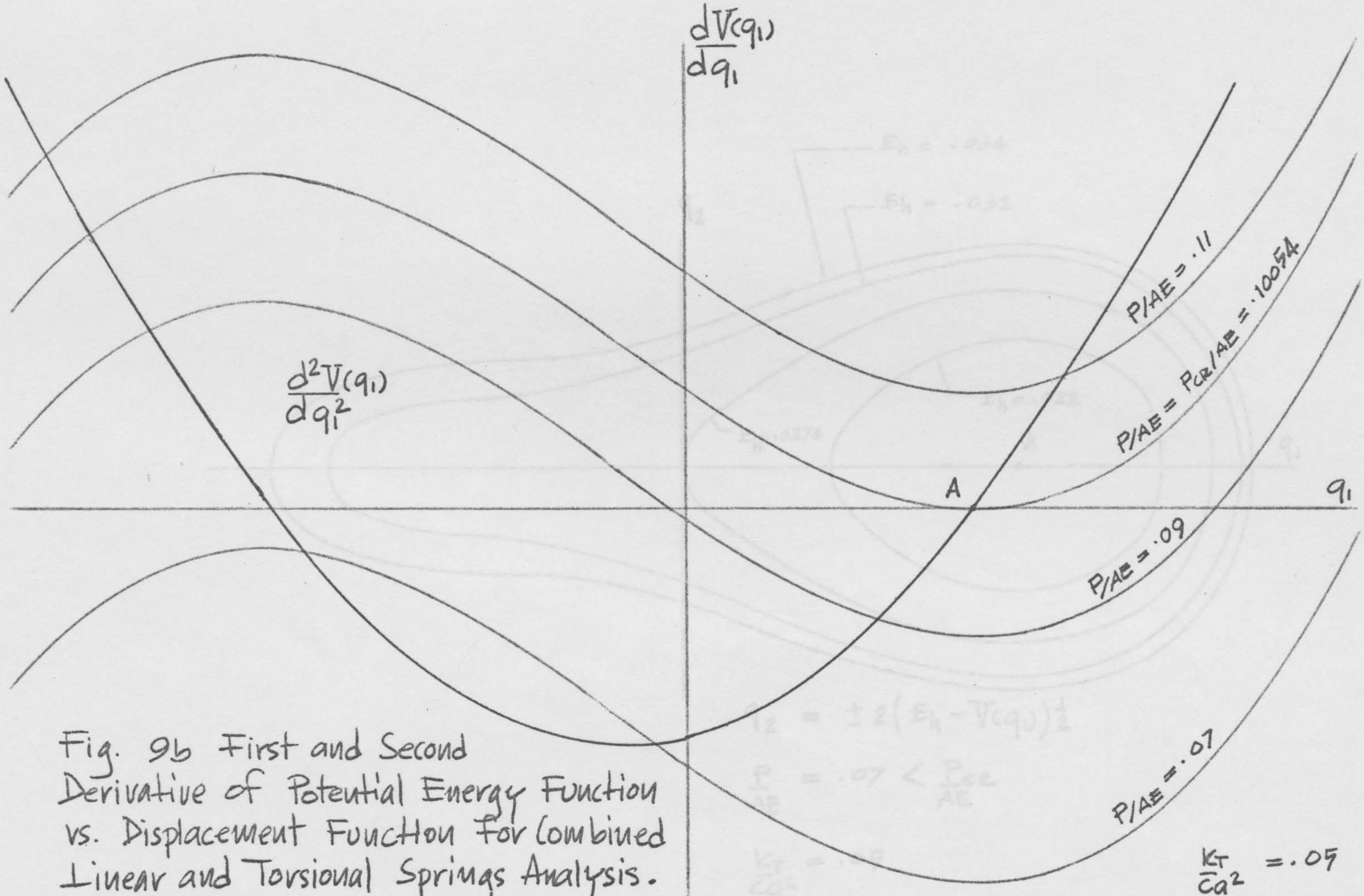
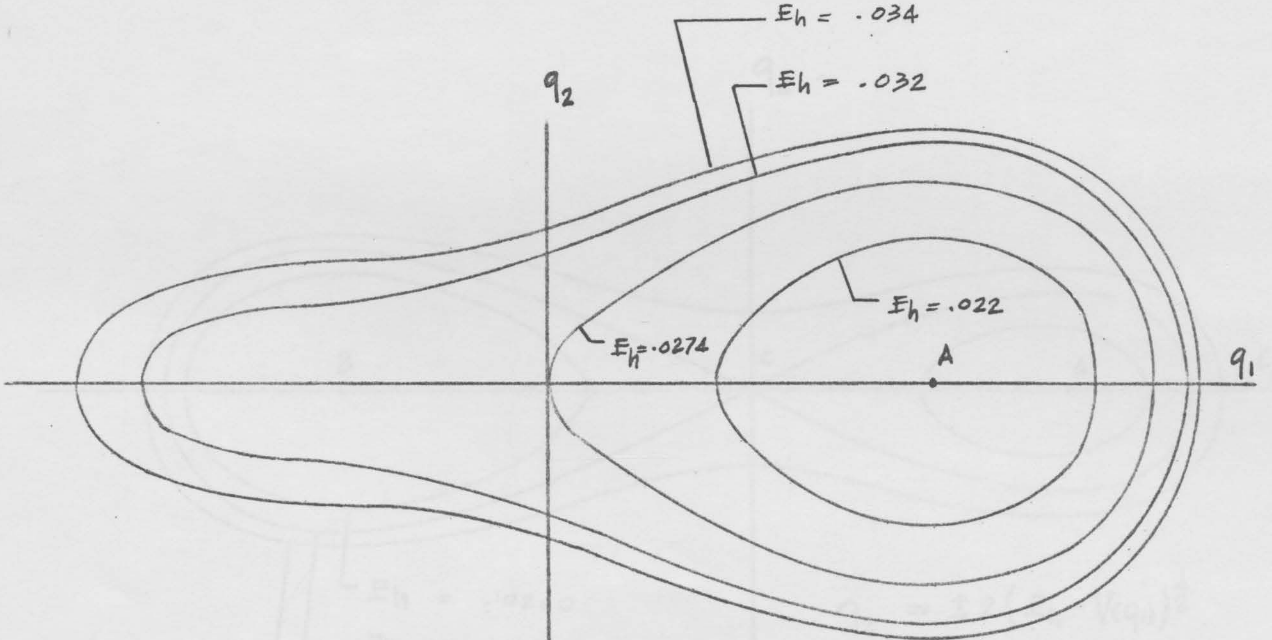


Fig. 9b First and Second Derivative of Potential Energy Function vs. Displacement Function for Combined Linear and Torsional Springs Analysis.



$$q_2 = \pm 2\left(E_h - V(q_1)\right)^{\frac{1}{2}}$$

$$\frac{P}{AE} = .07 < \frac{P_{CR}}{AE}$$

$$\frac{K_T}{Ca^2} = .05$$

Fig. 9c Phase Plane Diagram for Combined Linear and Torsional Springs Analysis.

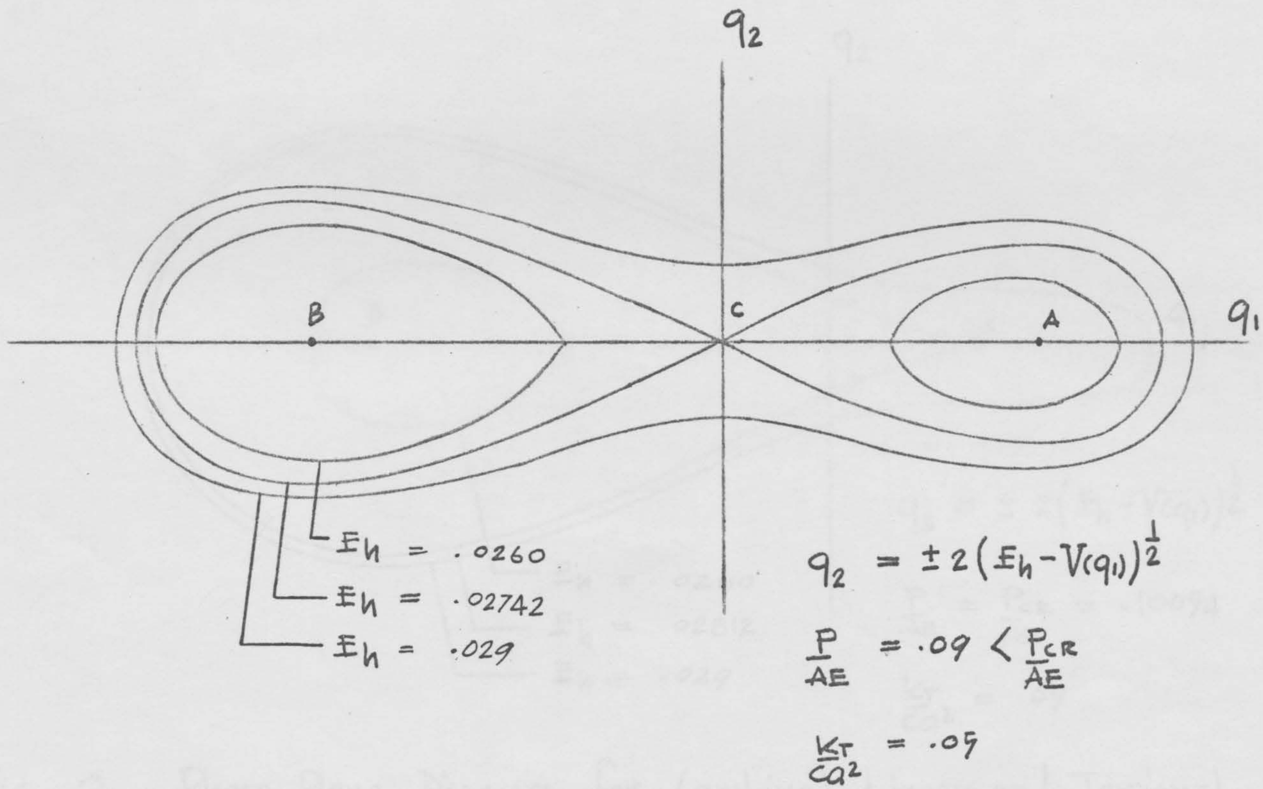
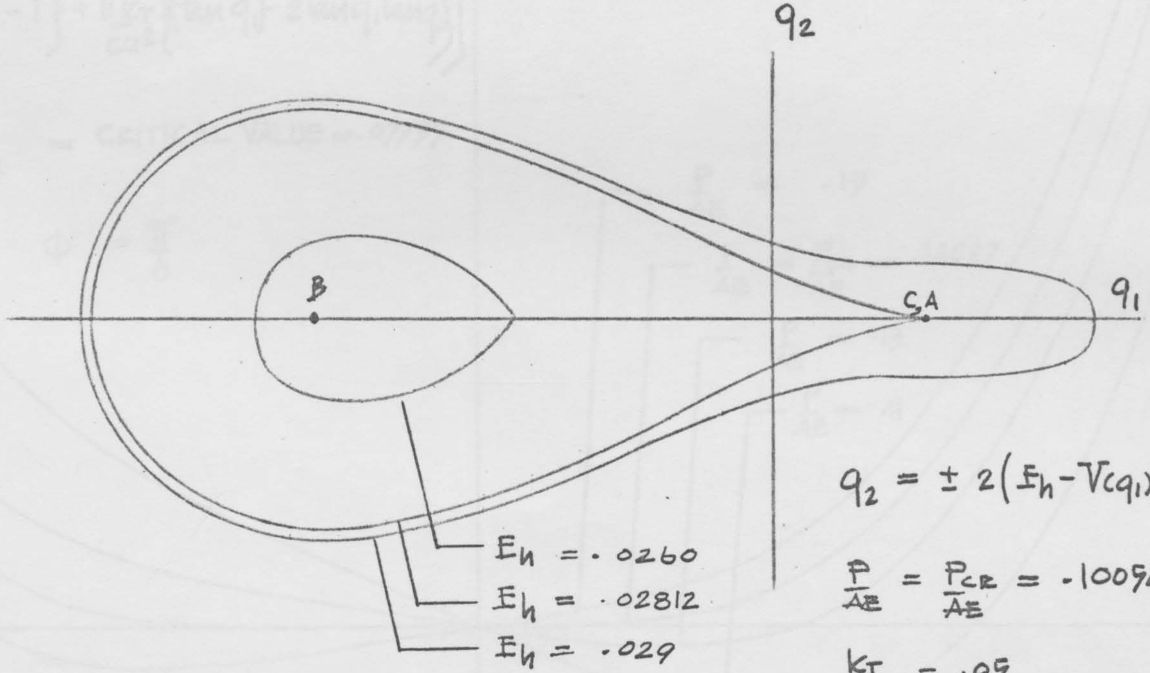


Fig. 9d Phase Plane Diagram for Combined Linear and Torsional Springs Analysis.



$$q_2 = \pm 2(E_h - V(q_1))^{\frac{1}{2}}$$

$$\frac{P}{AE} = \frac{P_{CR}}{AE} = .10094$$

$$\frac{K_T}{Ca^2} = .09$$

Fig. 9e Phase Plane Diagram for Combined Linear and Torsional Springs Analysis.

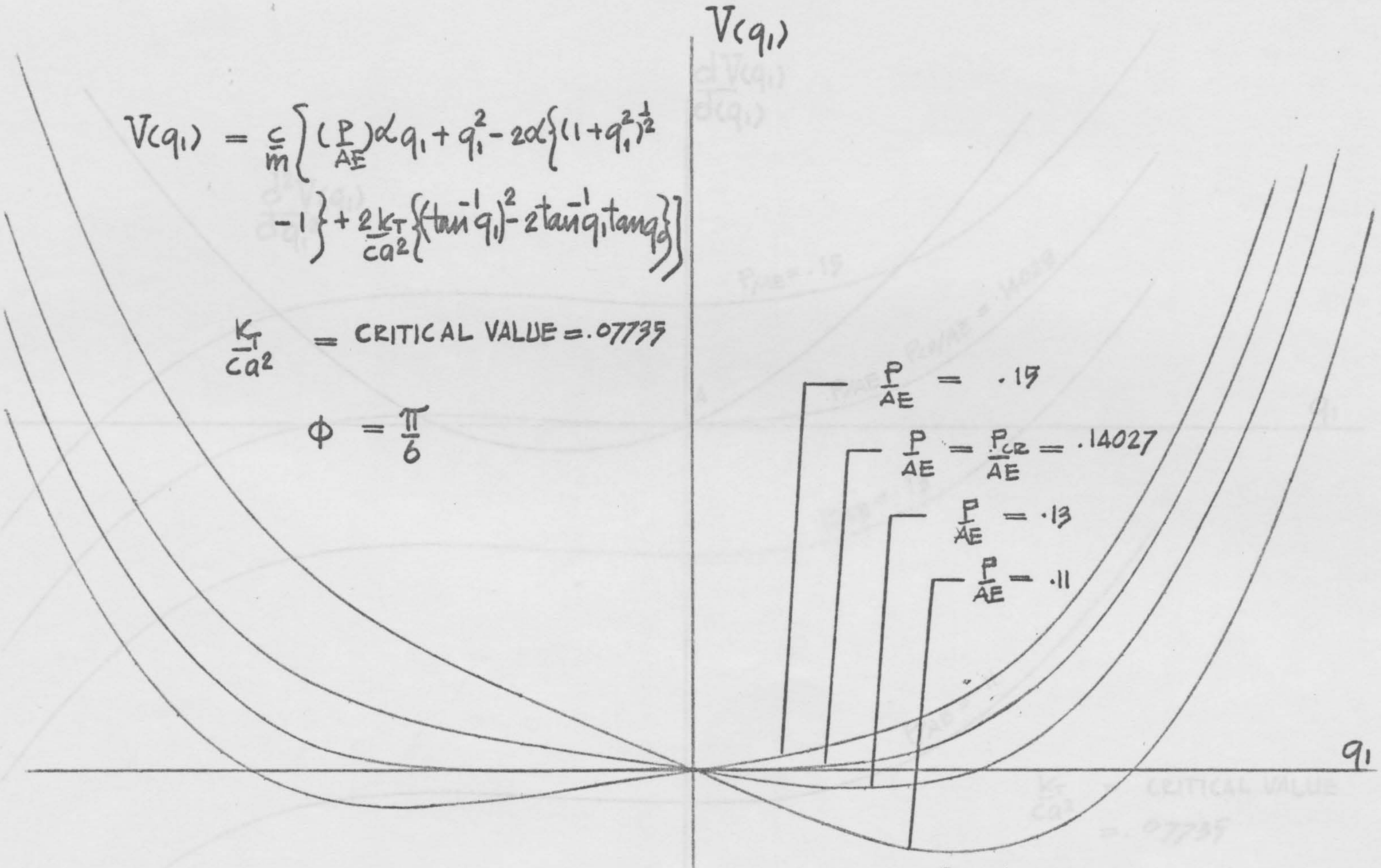


Fig. 10a Potential Energy Function vs. the Displacement Function for Combined Linear and Torsional Springs Analysis.

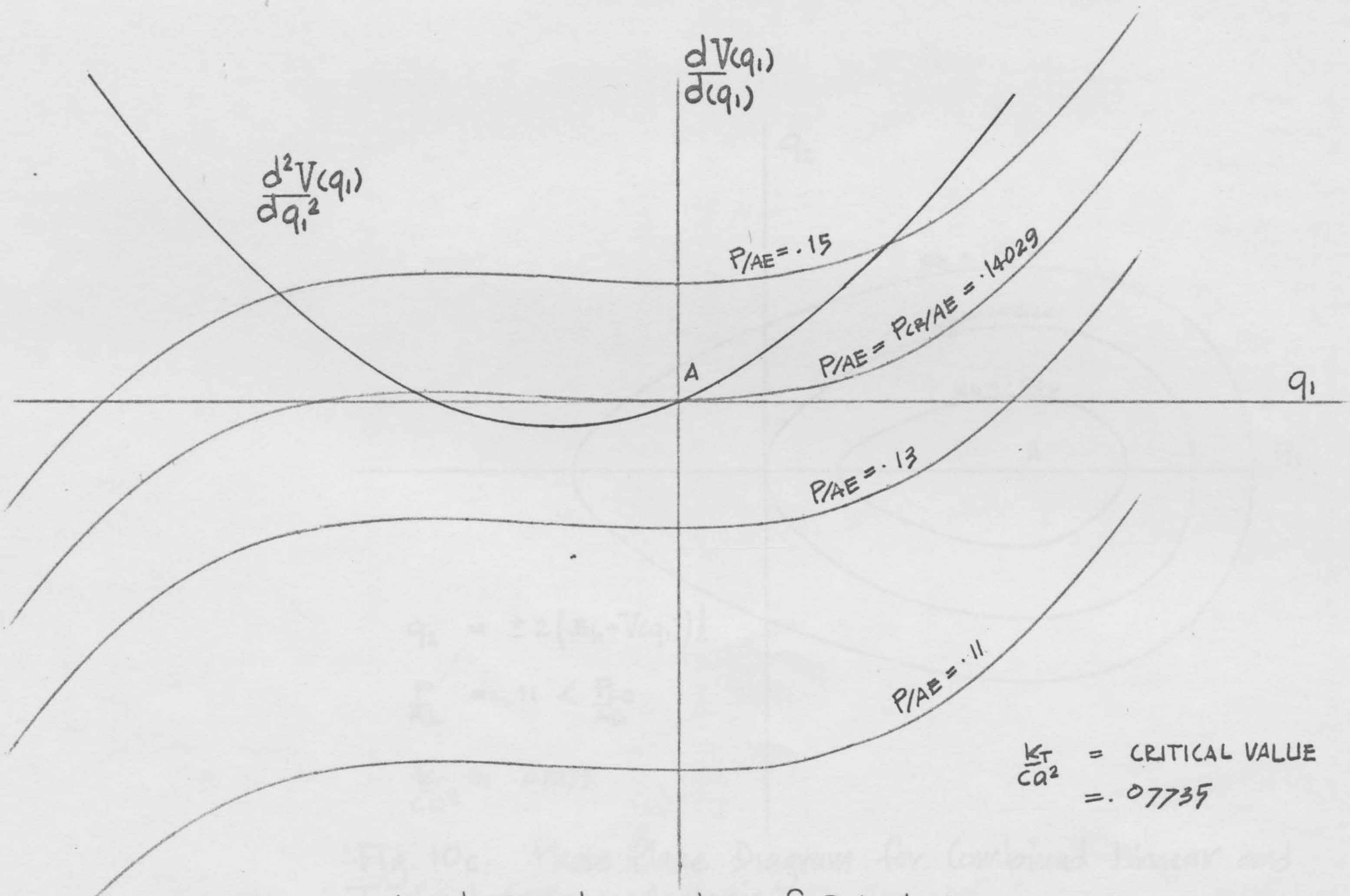
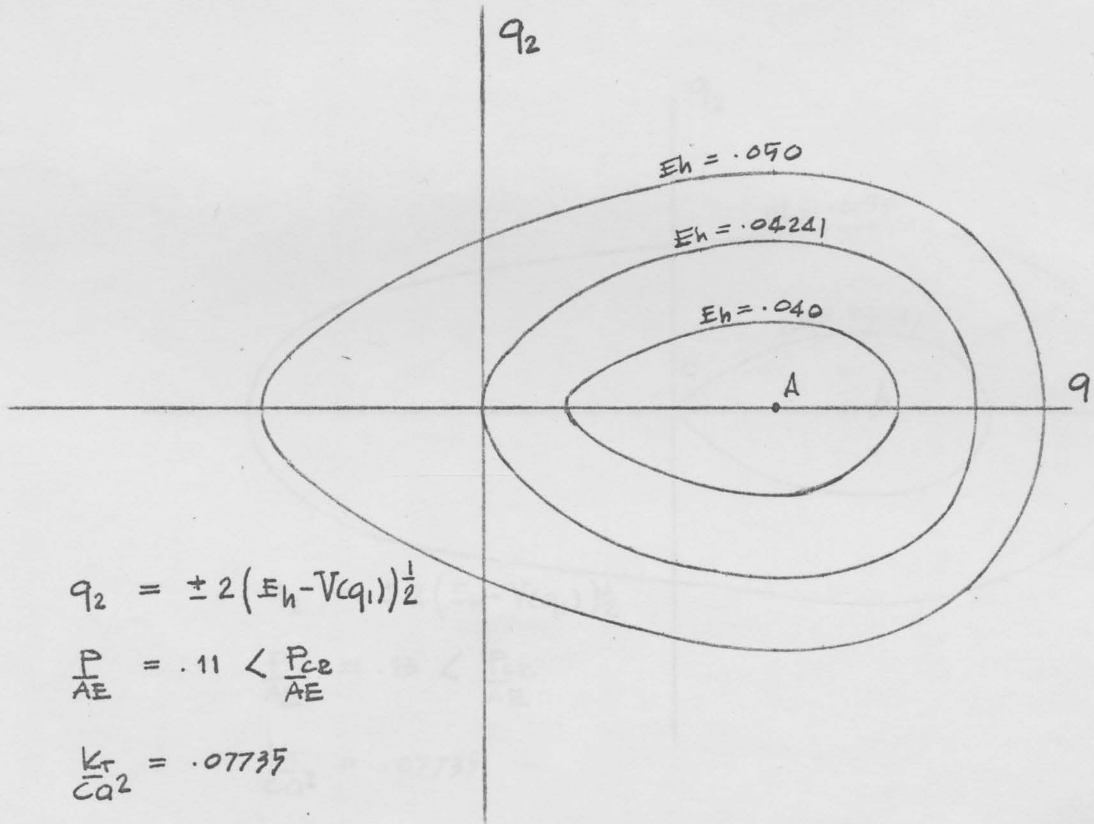


Fig. 10b First and Second Derivative of Potential Energy Function for Combined Linear and Torsional Springs Analysis.

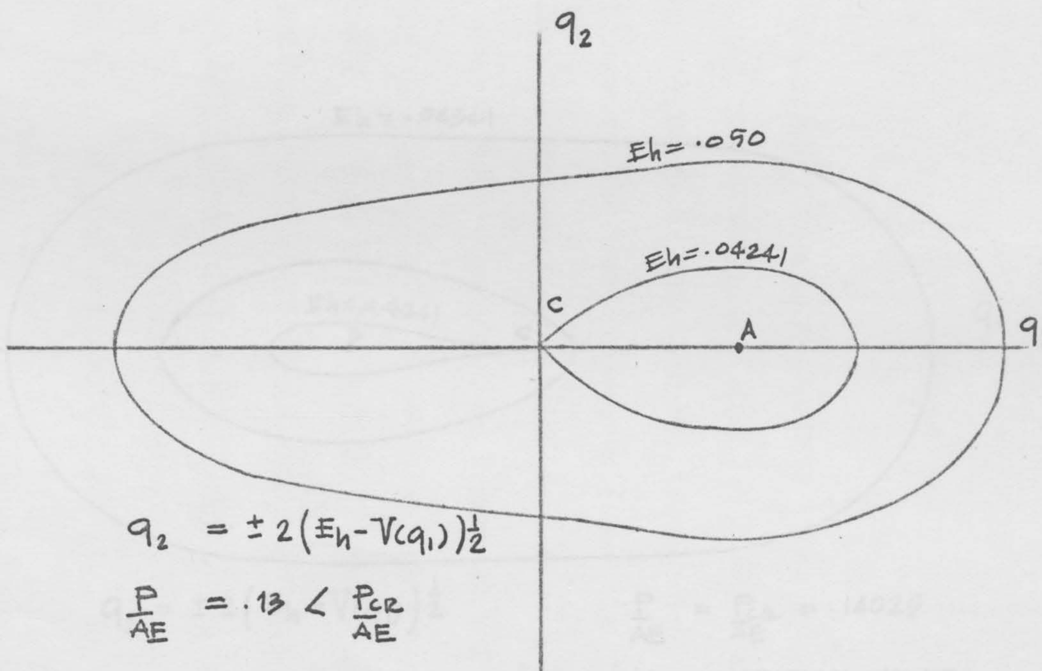


$$q_2 = \pm 2\left(E_h - V(q_1)\right)^{\frac{1}{2}}$$

$$\frac{P}{AE} = .11 < \frac{P_{cr}}{AE}$$

$$\frac{K_T}{Ca^2} = .07735$$

Fig. 10c Phase Plane Diagram for Combined Linear and Torsional Springs Analysis.



$$q_2 = \pm 2\left(E_h - V(q_1)\right)^{\frac{1}{2}}$$

$$\frac{P}{AE} = .13 < \frac{P_{CR}}{AE}$$

$$\frac{k_T}{Ca^2} = .07735$$

Fig. 10d Phase Plane Diagram for Combined Linear and Torsional Springs Analysis.

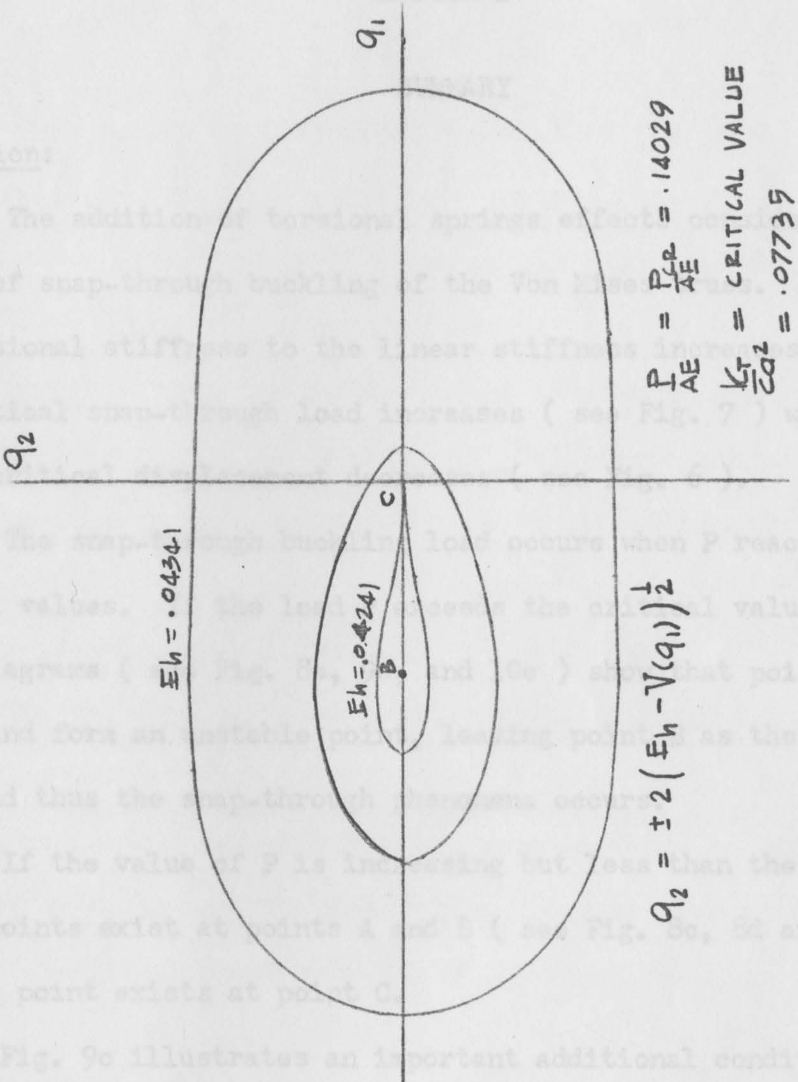


Fig. 10e Phase Plane Diagram for Combined Linear and Torsional Springs Analysis.

CHAPTER IV

SUMMARY

Discussion:

The addition of torsional springs effects considerably the phenomena of snap-through buckling of the Von Mises truss. As the value of the torsional stiffness to the linear stiffness increases the value of the critical snap-through load increases (see Fig. 7) while the value of the critical displacement decreases (see Fig. 6).

The snap-through buckling load occurs when P reaches a set of critical values. If the load P exceeds the critical values, the phase plane diagrams (see Fig. 8e, 9e, and 10e) show that points A and C coincide and form an unstable point, leaving point B as the only stable point and thus the snap-through phenomena occurs.

If the value of P is increasing but less than the critical value, stable points exist at points A and B (see Fig. 8c, 8d and 9d) and an unstable point exists at point C.

Fig. 9c illustrates an important additional condition which results due to the torsional springs. If the initial input of energy at time t = 0 (defined by the quantity E_h) is large only one possible equilibrium configuration exists. This stable equilibrium configuration is defined by point A. For this condition the snap-through phenomena never takes place.

Fig. 10a, 10b, 10c, 10d, and 10e illustrate the stability conditions which occur for the special case when the two bars are in the horizontal position at the instant snap-through occurs. As P is increased point A moves to the left; when $P = P_{cR}$ point A is at the origin and is an unstable point. The system snaps through to point B the stable point from the horizontal position.

Conclusion:

The problem of the snap-through stability of the Von Mises truss is investigated from both a static stability and a dynamic stability point of view. The mathematical theory together the necessary geometric interpretation is completely formulated so that the analysis of the more complex problem, that is, the additional of torsional springs, has the solid base upon which it may be formulated. This procedure proved invaluable in the approach to the solution of the torsional springs problem. It is also important as a building block approach for a reader not familar with this phenomena of instability.

The combination of the static stability approach and the dynamic stability approach leads to a more complete understanding of this complicate concept. Both method yield the same results, however the dynamic stability criteria and the phase plane diagrams give a more satisfying and complete visual interpretation of the results.

APPENDIX

APPENDIX

TABLE 1. Table of $\frac{P}{AE}$ and $\frac{y}{a}$. (See Fig. 3, and 4)

y/a	P/AE	y/a	P/AE	y/a	P/AE
-0.50	-0.40014	0.30	0.05413	1.10	-0.02114
-0.48	-0.37828	0.32	0.05271	1.12	-0.01400
-0.46	-0.35681	0.34	0.05076	1.14	-0.00617
-0.44	-0.33575	0.36	0.04831	1.16	0.00233
-0.42	-0.31509	0.38	0.04540	1.18	0.01149
-0.40	-0.29487	0.40	0.04206	1.20	0.02133
-0.38	-0.27508	0.42	0.03833	1.22	0.03183
-0.36	-0.25574	0.44	0.03424	1.24	0.04298
-0.34	-0.23683	0.46	0.02983	1.26	0.05477
-0.32	-0.21848	0.48	0.02516	1.28	0.06719
-0.30	-0.20058	0.50	0.02025	1.30	0.08022
-0.28	-0.18318	0.52	0.01517	1.32	0.09387
-0.26	-0.16631	0.54	0.00994	1.34	0.10810
-0.24	-0.14996	0.56	0.00463	1.36	0.12292
-0.22	-0.13416	0.58	-0.00071	1.38	0.13831
-0.20	-0.11892	0.60	-0.00606	1.40	0.15425
-0.18	-0.10426	0.62	-0.01135	1.42	0.17074
-0.16	-0.09018	0.64	-0.01654	1.44	0.18775
-0.14	-0.07669	0.66	-0.02159	1.46	0.20528
-0.12	-0.06382	0.68	-0.02643	1.48	0.22332
-0.10	-0.05147	0.70	-0.03104	1.50	0.24183
-0.08	-0.03995	0.72	-0.03536	1.52	0.26083
-0.06	-0.02879	0.74	-0.03936	1.54	0.28029
-0.04	-0.01864	0.76	-0.04300	1.56	0.30019
-0.02	-0.00898	0.78	-0.04622	1.58	0.32053
0	0	0.80	-0.04901	1.60	0.34129
0.02	0.00833	0.82	-0.05133	1.62	0.36246
0.04	0.01597	0.84	-0.05314	1.64	0.38403
0.06	0.02293	0.86	-0.05442	1.66	0.40599
0.08	0.02920	0.88	-0.05514	1.68	0.42832
0.10	0.03478	0.90	-0.05527	1.70	0.45101
0.12	0.03968	0.92	-0.05481	1.72	0.47405
0.14	0.04390	0.94	-0.05371	1.74	0.49743
0.16	0.04743	0.96	-0.05198	1.76	0.52114
0.18	0.05030	0.98	-0.04960	1.78	0.54517
0.20	0.05251	1.00	-0.04655	1.80	0.56591
0.22	0.05407	1.02	-0.04282	1.82	0.59414
0.24	0.05499	1.04	-0.03844	1.84	0.61907
0.26	0.05530	1.06	-0.03336	1.86	0.64427
0.28	0.05500	1.08	-0.02759	1.88	0.66975

TABLE 2. Table of $\frac{K_T}{ca^2}$, q_{1CR} and $\frac{P_{CR}}{AE}$ (See Fig. 6 and 7)

$\frac{K_T}{ca^2}$	q_{1CR}	$\frac{P_{CR}}{AE}$
-0.01101	0.34000	0.0482119
-0.00128	0.32000	0.0544342
0	0.31700	0.0553004
0.00766	0.30000	0.0607315
0.01000	0.29447	0.0624794
0.01586	0.28000	0.0670521
0.02339	0.26000	0.0733675
0.03030	0.24000	0.0796416
0.03664	0.22000	0.0858449
0.04776	0.18000	0.0979054
0.05000	0.17155	0.1005387
0.05262	0.16000	0.1037146
0.05705	0.14000	0.1093360
0.06107	0.12000	0.1147339
0.06469	0.10000	0.1198618
0.06795	0.08000	0.1247191
0.07084	0.06000	0.1292372
0.07337	0.04000	0.1333756
0.07554	0.02000	0.1370803
0.07735	0	0.1402977
0.07878	-0.02000	0.1429387

TABLE 3. Table of q_1, $V(q_1)$, $\frac{dV(q_1)}{dq_1}$, $\frac{d^2V(q_1)}{dq_1^2}$ and $|q_2|$ For $\frac{K_T}{ca^2} = 0$.

$\frac{P}{AE} = .03 < \frac{P_{CR}}{AE}$. (See Fig. 5a, 5b and 5c)						
q_1	$V(q_1)$	$\frac{dV(q_1)}{dq_1}$	$\frac{d^2V(q_1)}{dq_1^2}$	$\lvert q_2 \rvert$		
				$E_h = 0$	$E_h = .00199$	$E_h = .0030$
-0.90	-0.01877	-0.22044	0.52580	0.27391	0.28808	0.29501
-0.80	-0.03579	-0.12268	0.45019	0.37836	0.38874	0.39390
-0.70	-0.04383	-0.04100	0.36511	0.41872	0.42812	0.43281
-0.60	-0.04458	0.02282	0.27194	0.42230	0.43162	0.43627
-0.50	-0.03990	0.06744	0.17375	0.39954	0.40938	0.41428
-0.40	-0.03175	0.09233	0.07575	0.35640	0.36740	0.37286
-0.30	-0.02207	0.09824	-0.01469	0.39716	0.31026	0.31670
-0.20	-0.01266	0.08755	-0.08873	0.22405	0.24209	0.25030
-0.10	-0.00498	0.06443	-0.13760	0.14116	0.16699	0.17868
0	0	0.03464	-0.15470	0.00103	0.08921	0.10954
0.10	0.00195	0.00484	-0.13759	0.08826	0.01308	0.06489
0.20	0.00119	-0.01827	-0.08872	0.09611	0.05643	0.08500
0.30	-0.00129	-0.02896	-0.01467	0.07185	0.11456	0.13101
0.40	-0.00404	-0.02305	0.07578	0.12719	0.15536	0.16786
0.50	-0.00527	0.00185	0.17378	0.14515	0.17038	0.18185
0.60	-0.00302	0.04647	0.27196	0.10982	0.14149	0.15511
$\frac{P}{AE} = .05 \frac{P_{CR}}{AE}$. (See Fig. 5a, 5b and 5d)						
				$E_h = -.030$	$E_h = .00788$	$E_h = .0065$
-0.90	-0.03955	-0.19735	0.52580	0.19536	0.42627	0.42915
-0.80	-0.05427	-0.09958	0.45019	0.31154	0.49051	0.49301
-0.70	-0.06000	-0.01790	0.36511	0.34639	0.51335	0.51574
-0.60	-0.05844	0.04592	0.27194	0.33728	0.50725	0.50967
-0.50	-0.05145	0.09053	0.17375	0.29295	0.47891	0.48147
-0.40	-0.04099	0.11543	0.07575	0.20970	0.43303	0.43586
-0.30	-0.02900	0.12134	-0.01469	0.06314	0.37357	0.37685
-0.20	-0.01728	0.11064	-0.08873	0.22556	0.30440	0.30842
-0.10	-0.00729	0.08753	-0.13760	0.30139	0.22957	0.23487
0	0	0.05773	-0.15470	0.34641	0.15342	0.16124
0.10	0.00426	0.02794	-0.13759	0.37017	0.08070	0.09472
0.20	0.00581	0.00482	-0.08872	0.37849	0.01699	0.05243
0.30	0.00564	-0.00587	-0.01467	0.37756	0.03146	0.05873
0.40	0.00519	0.00005	0.07578	0.37520	0.05259	0.07229
0.50	0.00628	0.024940	0.17378	0.38095	0.03975	0.02966
0.60	0.01084	0.06957	0.27196	0.40419	0.14081	0.13178

TABLE 3. Table of q_1, $V(q_1)$, $\frac{dV(q_1)}{dq_1}$, $\frac{d^2V(q_1)}{dq_1^2}$ and $|q_2|$ For $\frac{K_T}{ca^2} = 0$

(Continued)

q_1	$V(q_1)$	$\frac{dV(q_1)}{dq_1}$	$\frac{d^2V(q_1)}{dq_1^2}$	$\lvert q_2 \rvert$		
$\frac{P}{AE} = .0933 = \frac{P_{CB}}{AE}$. (See Fig. 5a, 5b and 5e)						
				$Eh = -.040$	$Eh = .00747$	$Eh = .0080$
-0.90	-0.04506	-0.19123	0.52580	0.14212	0.45836	0.46065
-0.80	-0.05818	-0.09347	0.45019	0.27684	0.51627	0.51831
-0.70	-0.06428	-0.01178	0.36511	0.31165	0.53574	0.53770
-0.60	-0.06211	0.05204	0.27194	0.29740	0.52758	0.52957
-0.50	-0.05451	0.09665	0.17375	0.24095	0.49795	0.50006
-0.40	-0.04344	0.12155	0.07575	0.11732	0.45129	0.45361
-0.30	-0.03083	0.12746	-0.01469	0.19142	0.31948	0.39415
-0.20	-0.01850	0.11676	-0.08873	0.29323	0.32236	0.32560
-0.10	-0.00790	0.09365	-0.13760	0.35831	0.24801	0.25221
0	0	0.06385	-0.15470	0.40000	0.17290	0.17888
0.10	0.00487	0.03406	-0.13759	0.42365	0.10208	0.11191
0.20	0.00704	0.01094	-0.08872	0.43376	0.04181	0.06207
0.30	0.00747	0.00025	-0.01467	0.43577	0.00129	0.04589
0.40	0.00764	0.00617	0.07578	0.43654	0.02589	0.03787
0.50	0.00934	0.03106	0.17378	0.44425	0.08639	0.07321
0.60	0.01451	0.07569	0.27196	0.46696	0.16870	0.16141
$\frac{P}{AE} = .07 > \frac{P_{CB}}{AE}$ (See Fig. 5a and 5b)						
-0.90	-0.06033	-0.17425	0.52580			
-0.80	-0.07274	-0.07649	0.45019			
-0.70	-0.07616	0.00519	0.36511			
-0.60	-0.07230	0.06901	0.27194			
-0.50	-0.06300	0.11363	0.17375			
-0.40	-0.05023	0.13852	0.07575			
-0.30	-0.03593	0.14443	-0.01469			
-0.20	-0.02190	0.13374	-0.08873			
-0.10	-0.00960	0.11062	-0.13760			
0	0	0.08083	-0.15470			
0.10	0.00657	0.05103	-0.13759			
0.20	0.01043	0.02792	-0.08872			
0.30	0.01257	0.01723	-0.01467			
0.40	0.01433	0.02314	0.07578			
0.50	0.01783	0.04804	0.17378			
0.60	0.02470	0.09266	0.27196			

TABLE 4. Table of q_1, $V(q_1)$, $\frac{dV(q_1)}{dq_1}$, $\frac{d^2V(q_1)}{dq_1^2}$ and $|q_2|$ For $\frac{K_T}{ca^2} = .01$

q_1	$V(q_1)$	$\frac{dV(q_1)}{dq_1}$	$\frac{d^2V(q_1)}{dq_1^2}$	$\lvert q_2 \rvert$		
$\frac{P}{AE} = .04 < \frac{P_{CR}}{AE}$ (See Fig. 8a, 8b and 8c)						
				$E_h = -.0050$	$E_h = .0067$	$E_h = .030$
-0.90	-0.00306	-0.23667	0.51881	0.17232	0.13079	0.33212
-0.80	-0.02179	-0.14037	0.44338	0.21264	0.30334	0.43037
-0.70	-0.03166	-0.05991	0.35982	0.29106	0.36265	0.47404
-0.60	-0.03435	0.00307	0.26895	0.30899	0.37720	0.48526
-0.50	-0.03167	0.04739	0.17392	0.29113	0.36271	0.47408
-0.40	-0.02551	0.07270	0.07987	0.24516	0.32696	0.44733
-0.30	-0.01773	0.07988	-0.00608	0.17035	0.27535	0.41112
-0.20	-0.01006	0.07137	-0.07557	0.04124	0.21237	0.37189
-0.10	-0.00385	0.05130	-0.12043	0.16290	0.14236	0.33685
0	0	0.02524	-0.13470	0.20477	0.06977	0.31316
0.10	0.00122	-0.00040	-0.11632	0.21629	0.00449	0.30532
0.20	0.00015	-0.01927	-0.06782	0.20622	0.06539	0.31221
0.30	-0.00223	-0.02593	0.00450	0.18166	0.11748	0.32711
0.40	-0.00449	-0.01643	0.09233	0.15474	0.15118	0.34067
0.50	-0.00490	0.01148	0.18733	0.14938	0.15647	0.34305
0.60	-0.00156	0.05851	0.28255	0.18888	0.10548	0.32300
$\frac{P}{AE} = .05 < \frac{P_{CR}}{AE}$ (See Fig. 8a, 8b and 8d)						
				$E_h = -.0140$	$E_h = .00813$	$E_h = .0090$
-0.90	-0.01345	-0.22512	0.51881	0.15532	0.23576	0.26053
-0.80	-0.03102	-0.12882	0.44338	0.21487	0.36700	0.37171
-0.70	-0.04074	-0.04836	0.35982	0.28469	0.41179	0.41599
-0.60	-0.04128	0.01462	0.26895	0.29528	0.41918	0.42331
-0.50	-0.03754	0.05894	0.17392	0.26805	0.40046	0.40478
-0.40	-0.03013	0.08425	0.07987	0.20635	0.36208	0.36685
-0.30	-0.02120	0.09143	-0.00608	0.08291	0.30886	0.31444
-0.20	-0.01236	0.08291	-0.07557	0.16871	0.24506	0.25206
-0.10	-0.00500	0.06284	-0.12043	0.24066	0.17494	0.18462
0	0	0.03679	-0.13470	0.27916	0.10289	0.11861
0.10	0.00237	0.01115	-0.11632	0.29563	0.03349	0.06783
0.20	0.00246	-0.00772	-0.06782	0.29625	0.02752	0.06510
0.30	0.00124	-0.01438	0.00450	0.28785	0.07526	0.09563
0.40	0.00012	-0.00489	0.09233	0.28003	0.10050	0.11654
0.50	0.00087	0.02302	0.18733	0.28532	0.08432	0.10291
0.60	0.00437	0.07006	0.28255	0.31526	0.10424	0.08595

TABLE 4. Table of q_1, $V(q_1)$, $\frac{dV(q_1)}{dq_1}$, $\frac{d^2V(q_1)}{dq_1^2}$ and $|q_2|$ For $\frac{K_T}{ca^2} = .01$

(Continued)

q_1	$V(q_1)$	$\frac{dV(q_1)}{dq_1}$	$\frac{d^2V(q_1)}{dq_1^2}$	q_2		
				$E_h = -.0080$	$E_h = .01104$	$E_h = .0120$
$\frac{P}{AE} = .06248 = \frac{P_{cr}}{AE}$ (See Fig. 8a, 8b and 8e)						
-0.90	-0.02642	-0.21071	0.51881	0.22749	0.35765	0.36298
-0.80	-0.04255	-0.11441	0.44338	0.34100	0.43868	0.44304
-0.70	-0.04983	-0.03395	0.35982	0.38131	0.47070	0.47476
-0.60	-0.04992	0.02903	0.26895	0.38181	0.47110	0.47516
-0.50	-0.04465	0.07335	0.17392	0.35309	0.44814	0.45241
-0.40	-0.03589	0.09866	0.07987	0.29939	0.40718	0.41187
-0.30	-0.02552	0.10584	-0.00608	0.21947	0.35260	0.35800
-0.20	-0.01525	0.09732	-0.07557	0.08405	0.28849	0.29507
-0.10	-0.00644	0.07725	-0.12043	0.16779	0.21911	0.22770
0	0	0.05120	-0.13470	0.23223	0.14909	0.16145
0.10	0.00381	0.02556	-0.11632	0.26299	0.08365	0.10410
0.20	0.00534	0.00669	-0.06782	0.27439	0.02948	0.06862
0.30	0.00556	0.00003	0.00450	0.27595	0.00340	0.06206
0.40	0.00589	0.00952	0.09233	0.27834	0.03628	0.05204
0.50	0.00808	0.03743	0.18733	0.29365	0.10035	0.07894
0.60	0.01401	0.08447	0.28255	0.33162	0.18388	0.17313
$\frac{P}{AE} = .07 > \frac{P_{cr}}{AE}$. (See fig. 8a and 8b)						
-0.90	-0.03423	-0.20203	0.51881			
-0.80	-0.04950	-0.10573	0.44338			
-0.70	-0.05591	-0.02527	0.35982			
-0.60	-0.05513	0.03771	0.26895			
-0.50	-0.04899	0.08203	0.17392			
-0.40	-0.03936	0.10734	0.07987			
-0.30	-0.02813	0.11452	-0.00608			
-0.20	-0.01698	0.10601	-0.07557			
-0.10	-0.00731	0.08594	-0.12043			
0	0	0.05988	-0.13470			
0.10	0.00468	0.03425	-0.11632			
0.20	0.00708	0.01537	-0.06782			
0.30	0.00816	0.00871	0.00450			
0.40	0.00936	0.01821	0.09233			
0.50	0.01242	0.04612	0.18733			
0.60	0.01922	0.09315	0.28255			

TABLE 5. Table of q_1, $V(q_1)$, $\frac{dV}{dq_1}(q_1)$, $\frac{d^2V}{dq_1^2}(q_1)$ and $|q_2|$ For $\frac{K_T}{ca^2} = .05$

$\frac{P}{AE} = .07 < \frac{P_{CR}}{AE}$				(See Fig. 9a, 9b and 9c)		
q_1	$V(q_1)$	$\frac{dV(q_1)}{dq_1}$	$\frac{d^2V(q_1)}{dq_1^2}$	$\lvert q_2 \rvert$ $E_h = .0274$	$E_h = .032$	$E_h = .034$
-0.90	0.07001	-0.31308	0.48730	0.54965	0.52956	0.50410
-0.80	0.04362	-0.22263	0.41609	0.44209	0.41685	0.38399
-0.70	0.02527	-0.14707	0.33863	0.34932	0.31677	0.27208
-0.60	0.01268	-0.08746	0.25697	0.27507	0.23234	0.16632
-0.50	0.00722	-0.04433	0.17457	0.22328	0.16785	0.04306
-0.40	0.00427	-0.01736	0.09632	0.19504	0.12790	0.09980
-0.30	0.00326	-0.00512	0.02832	0.18437	0.11097	0.11835
-0.20	0.00284	-0.00491	-0.02294	0.17982	0.10322	0.12517
-0.10	0.00201	-0.01280	-0.05179	0.17025	0.08547	0.13789
0	0	-0.02389	-0.05470	0.14718	0.00430	0.16229
0.10	-0.00270	-0.03291	-0.03125	0.10072	0.10740	0.19456
0.20	-0.00617	-0.04381	0.01580	0.06093	0.15935	0.22740
0.30	-0.00929	-0.02536	0.08122	0.12723	0.19459	0.25335
0.40	-0.01076	-0.00153	0.15860	0.14859	0.20919	0.26742
0.50	-0.00905	0.03845	0.24161	0.12350	0.19218	0.25150
0.60	-0.00251	0.09513	0.32494	0.10442	0.10381	0.19261
$\frac{P}{AE} = .09 < \frac{P_{CR}}{AE}$				(See Fig. 9a, 9b and 9d) $E_h = .026$	$E_h = .02742$	$E_h = .029$
-0.90	0.04932	-0.28999	0.48730	0.45053	0.44418	0.43701
-0.80	0.02496	-0.19954	0.41609	0.32488	0.31602	0.30586
-0.70	0.00892	-0.12397	0.33863	0.20338	0.18890	0.17136
-0.60	-0.00036	-0.06437	0.25697	0.06509	0.03799	0.08811
-0.50	-0.00650	-0.02124	0.17457	0.11105	0.13421	0.15599
-0.40	-0.00515	0.00573	0.09632	0.12210	0.14349	0.16404
-0.30	-0.00385	0.01797	0.02832	0.09858	0.12409	0.14737
-0.20	-0.00195	0.01818	-0.02294	0.04626	0.08843	0.11891
-0.10	-0.00048	0.01030	-0.05179	0.06123	0.04395	0.09084
0	0	-0.00080	-0.05470	0.07525	0.00416	0.07961
0.10	-0.00057	-0.00982	-0.03125	0.05817	0.04792	0.09282
0.20	-0.00173	-0.01172	0.01580	0.03515	0.08316	0.11505
0.30	-0.00254	-0.00227	0.08122	0.06688	0.10076	0.12835
0.40	-0.00170	0.02157	0.15860	0.03357	0.08250	0.11457
0.50	0.00232	0.06154	0.24161	0.12222	0.09621	0.05419
0.60	0.01117	0.11823	0.32494	0.22434	0.21130	0.19578

TABLE 5. Table of q_1, $V(q_1)$, $\frac{dV(q_1)}{dq_1}$, $\frac{d^2V(q_1)}{dq_1^2}$ and $|q_2|$ For $\frac{K_T}{ca^2} = .05$

(Continued)

q_1	$V(q_1)$	$\frac{dV(q_1)}{dq_1}$	$\frac{d^2V(q_1)}{dq_1^2}$	$\lvert q_2 \rvert$ $Eh = .026$	$\lvert q_2 \rvert$ $Eh = .02812$	$\lvert q_2 \rvert$ $Eh = .029$
$\frac{P}{AE} = .10054 = \frac{P_{CR}}{AE}$ (See Fig. 9a, 9b and 9e)						
-0.90	0.03837	-0.27782	0.48730	0.39895	0.38818	0.38362
-0.80	0.01523	-0.18737	0.41609	0.25807	0.24109	0.23367
-0.70	0.00040	-0.11180	0.33863	0.08536	0.03454	0.06865
-0.60	-0.00766	-0.05220	0.25697	0.15802	0.18290	0.19228
-0.50	-0.01059	-0.00907	0.17457	0.19150	0.21249	0.22062
-0.40	-0.01002	0.01790	0.09632	0.18542	0.20703	0.21536
-0.30	-0.00750	0.03014	0.02832	0.15595	0.18111	0.19058
-0.20	-0.00439	0.03035	-0.02294	0.10898	0.14268	0.15452
-0.10	-0.00170	0.02247	-0.05179	0.03345	0.09797	0.11454
0	0	0.01137	-0.05470	0.07525	0.05307	0.07960
0.10	0.00064	0.00235	-0.03125	0.09084	0.01509	0.06122
0.20	0.00071	0.00045	0.01580	0.09220	0.00460	0.05915
0.30	0.00111	0.00990	0.08122	0.10066	0.04064	0.04322
0.40	0.00316	0.03374	0.15860	0.13545	0.09933	0.07966
0.50	0.00840	0.07371	0.24161	0.19819	0.17549	0.16516
0.60	0.01847	0.13040	0.32494	0.28203	0.26657	0.25988
$\frac{P}{AE} = .11 > \frac{P_{CR}}{AE}$ (See Fig. 9a and 9b)						
-0.90	0.02854	-0.26690	0.48730			
-0.80	0.00649	-0.17644	0.41609			
-0.70	-0.00725	-0.10088	0.33863			
-0.60	-0.01422	-0.04127	0.25697			
-0.50	-0.01605	0.00186	0.17457			
-0.40	-0.01439	0.02883	0.09632			
-0.30	-0.01078	0.04107	0.02832			
-0.20	-0.00637	0.04127	-0.02294			
-0.10	-0.00279	0.03339	-0.05179			
0	0	0.02230	-0.05470			
0.10	0.00174	0.01328	-0.03125			
0.20	0.00289	0.01253	0.01580			
0.30	0.00439	0.02534	0.08122			
0.40	0.00753	0.04466	0.15860			
0.50	0.01386	0.08464	0.24161			
0.60	0.02502	0.14132	0.32494			

TABLE 6. Table of q_1, $V(q_1)$, $\frac{dV(q_1)}{dq_1}$, $\frac{d^2V(q_1)}{dq_1^2}$ and $|q_2|$ For $\frac{K_T}{ca^2} = .07735$

$\frac{P}{AE} = .11 < \frac{P_{CR.}}{AE}$				(See Fig. 10a, 10b and 10c)		
q_1	$V(q_1)$	$\frac{dV(q_1)}{dq_1}$	$\frac{d^2V(q_1)}{dq_1^2}$	$\lvert q_2\rvert$		
				$E_h = .040$	$E_h = .04241$	$E_h = .050$
-0.90	0.09989	-0.34284	0.46623	0.63971	0.63213	0.60674
-0.80	0.07005	-0.25638	0.39743	0.53837	0.52934	0.49985
-0.70	0.04814	-0.18417	0.32414	0.44970	0.43885	0.40277
-0.60	0.03272	-0.12687	0.24879	0.37486	0.36177	0.31705
-0.50	0.02227	-0.08455	0.17502	0.31422	0.29848	0.24235
-0.40	0.01534	-0.05644	0.10757	0.26643	0.24768	0.17603
-0.30	0.01057	-0.04074	0.05185	0.22786	0.20562	0.10918
-0.20	0.00687	-0.03457	0.01305	0.19267	0.16578	0.05365
-0.10	0.00347	-0.03412	-0.00485	0.15331	0.11775	0.12843
0	0	-0.03499	0	0.09822	0.00256	0.17422
0.10	-0.00342	-0.03264	0.02692	0.06370	0.11704	0.20990
0.20	-0.00628	-0.02294	0.07297	0.12441	0.15849	0.23554
0.30	-0.00765	-0.00247	0.13367	0.14481	0.17496	0.24692
0.40	-0.00633	0.03117	0.20390	0.12528	0.15917	0.23600
0.50	-0.00093	0.07939	0.27872	0.07689	0.06106	0.18643
0.60	0.01004	0.14268	0.35391	0.22320	0.20044	0.09908
$\frac{P}{AE} = .13 < \frac{P_{CR.}}{AE}$				(See Fig. 10a, 10b and 10d)		
				$E_h = .04241$	$E_h = .050$	
-0.90	0.07912	-0.31974	0.46623	0.56254	0.53488	
-0.80	0.05158	-0.23329	0.39743	0.45421	0.41946	
-0.70	0.03198	-0.16107	0.32414	0.35767	0.31235	
-0.60	0.01986	-0.10377	0.24879	0.27469	0.21236	
-0.50	0.01073	-0.06145	0.17502	0.20713	0.11200	
-0.40	0.00610	-0.03334	0.10757	0.15619	0.07722	
-0.30	0.00464	-0.01764	0.05185	0.12070	0.12567	
-0.20	0.00225	-0.01147	0.01305	0.09491	0.14612	
-0.10	0.00116	-0.01103	-0.00485	0.06803	0.16041	
0	0	-0.01189	0	0.00277	0.17422	
0.10	0.00111	-0.00955	0.02692	0.06678	0.18660	
0.20	-0.00166	-0.00016	0.07297	0.08150	0.19236	
0.30	-0.00072	0.02062	0.13367	0.05381	0.18236	
0.40	0.00290	0.05426	0.20390	0.10778	0.13691	
0.50	0.01061	0.10248	0.27872	0.20606	0.11001	
0.60	0.02390	0.16577	0.35391	0.30920	0.25543	

TABLE 6. Table of q_1, $V(q_1)$, $\frac{dV}{dq_1}(q_1)$, $\frac{d^2V}{dq_1^2}(q_1)$ and $|q_2|$ For $\frac{K_T}{ca^2} = .07735$

(Continued)

$\frac{P}{AE} = .14029 = \frac{P_{CR}}{AE}$ (See Fig. 10a, 10b and 10e)					
q_1	$V(q_1)$	$\frac{dV(q_1)}{dq_1}$	$\frac{d^2V(q_1)}{dq_1^2}$	$\lvert q_2 \rvert$ $Eh = .04241$	$\lvert q_2 \rvert$ $Eh = .04341$
-0.90	0.06842	-0.30786	0.46623	0.51931	0.52314
-0.80	0.04207	-0.22141	0.39743	0.40532	0.41023
-0.70	0.02366	-0.14919	0.32414	0.30109	0.30766
-0.60	0.01174	-0.09189	0.24879	0.20722	0.21666
-0.50	0.00479	-0.04957	0.17502	0.12305	0.13835
-0.40	0.00135	-0.02146	0.10757	0.03724	0.07339
-0.30	0.00008	-0.00576	0.05185	0.06074	0.01761
-0.20	-0.00012	0.00041	0.01305	0.06706	0.02230
-0.10	-0.00003	0.00086	-0.00485	0.06422	0.01112
0	0	-0.00001	0	0.06318	0.00287
0.10	0.00007	0.00233	0.02692	0.06087	0.01716
0.20	0.00072	0.01204	0.07297	0.03371	0.05351
0.30	0.00284	0.03250	0.13367	0.08581	0.10660
0.40	0.00765	0.06614	0.20390	0.16318	0.17501
0.50	0.01656	0.11437	0.27872	0.24945	0.25734
0.60	0.03103	0.17765	0.35391	0.34659	0.35231
$\frac{P}{AE} = .15 > \frac{P_{CR}}{AE}$ (See Fig. 10a and 10b)					
-0.90	0.05833	-0.29665	0.46623		
-0.80	0.03310	-0.21019	0.39743		
-0.70	0.01581	-0.13798	0.32414		
-0.60	0.00601	-0.08068	0.24879		
-0.50	-0.00082	-0.03836	0.17502		
-0.40	-0.00314	-0.01025	0.10757		
-0.30	-0.00329	0.00545	0.05185		
-0.20	-0.00237	0.01162	0.01305		
-0.10	-0.00115	0.01207	0.00485		
0	0	0.01120	0		
0.10	0.00120	0.01355	0.02692		
0.20	0.00296	0.02325	0.07297		
0.30	0.00621	0.04371	0.13367		
0.40	0.01214	0.07735	0.20390		
0.50	0.02216	0.12558	0.27872		
0.60	0.03776	0.18886	0.35391		

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