BASIC CONCEPTS OF ELASTIC STABILITY

by

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ABSTRACT

BASIC CONCEPTS OF ELASTIC STABILITY

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The purpose of this thesis is to explain from a pedagogical point of view the basic, underlying concepts of the theory of elastic stability. Two simple mathematical models consisting of a rigid rod combined with a linear and a torsional springs are investigated using the criteria of static and dynamic stability.

The basic laws of elastic stability are interpreted mathematically and geometrically. The mathematical solutions for both stable and unstable equilibrium states are supplemented by the necessary graphs which permit a visual interpretation of the mathematical results.

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I wish to take this opportunity to thank everyone who helped in the completion of this thesis. I especially would like to thank Dr. Paul X. Bellini, my advisor, who helped me immensely in this work. Also, I would like to thank Dr. Gilbert Williamson and Mr. John Ritter for their cooperation.

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LIST OF SYMBOLS

SYMBOL	DEFINITION	UNITS OR REFERENCE
θ	Angular displacement	radians
$\dot{\theta}$	Angular velocity	rad/sec
Kt	Torsional stiffness factor	lb in/rad
K	Linear stiffness factor	lb/in
Eo	Sum of the kinetic energy plus the potential energy	See Eq. (9)
α	Kt/L	See Eq. (10)
β	Kt/KLo	See Eq. (38)
Lo	Length of the rigid rod	feet

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CHAPTER I

INTRODUCTION

Object:

Elastic stability of engineering structures has been a subject of extensive analysis in recent years. The conditions governing the stability of structural elements such as beams, columns, plates, and shells have been formulated. The mathematical concepts involved in the theory are somewhat complex and difficult to comprehend, especially by someone new to the field. It is the purpose of this thesis to develop the underlying concepts of elastic stability and present them in a unified and simple manner, from a mathematical point of view, and from a pedagogical point view.

Two extremely simple mathematical models are chosen to illustrate the principles. The models are made up of rigid rods and linear and torsional springs. The stability loads are assumed to be of conservative type. The two models considered, are an idealized form of a contilever-type beam column. Thus, the resulting stability criteria has a direct relation to this actual physical problem. The mathematical model is simplified, so that a new reader may easily comprehend the basic underlying principles.

Approach to the problem:

Two different approaches are taken to each problem.

1. Static stability analysis.

2. Dynamic stability analysis.

A comparison of the two different analysis is investigated.

1. Static stability analysis

The criteria for static stability is based on the potential energy function of the system designated as V. If the function must be at least twice differentiable in (a,b) then a necessary condition for existence of an equilibrium state at θo for which $\theta_1 < \theta o < \theta_2$ is that $dV/d\theta\ (\theta_1) = o$. This condition is not sufficient to guarantee that θo is a state of stable equilbrium. A sufficient condition for a stable equilibrium state or an unstable equilibrium state is determined by investigating the sign of the function $d^2V/d\theta^2\ (\theta o)$.

If $d^2V/d\theta^2(\theta o) < 0$, this corresponds to a maximum point and thus an unstable state of equilibrium exists.

If $d^2V/d\theta^2(\theta o) > 0$, this signifies a minimum point exists and hence, a stable point of equililbrium.

Finally, if $d^2V/d\theta^2\ (\theta o) = o$, the point is neither a stable point nor an unstable point. This condition determines the critical positions of system separating stable and unstable zones of equilibrium. It is designated a neutral equilibrium.

2. Dynamic stability analysis.

Dynamic stability analysis involves a function of more than one variable. The extremums of a multivariable function must be investigated. For a function at more than one variable a necessary condition for an extremum is $fx = fy = fz = \ldots = fn = o$ at P, that is the total derivative of the function must vanish. The sufficiency conditions are more complicated. In the case of two variables, the function f(x,y) yields a maximum at point A if, in addition to $fx = fy = o$ at A, $fxx < o$ and $(fxxfyy - fxy^2) > o$. It results a minimum when $fxx > o$ and $(fxxfyy - fxy^2) > o$. For the proper function f point A corresponds to a state of unstable and stable point respectively. If $(fxxfyy - fxy^2) < o$, at point A the function is neither maximum nor minimum at the point, a saddle point occurs which is an unstable equilibrium.

Fundamental to dynamic stability analysis is the formulation of the differential equation of motion, which is usually of the nonlinear type even for the most simplified stability problems. The equation of motion is reduced in order into a set of first order differential equation which have inherent in them a form of the potential energy of the system. Secondly, an intermediate energy integral equation is formed which exists for conservative force system. The equation states the condition that the sum of the kinetic and potential energies remain constant for all values of time t. For a one degree of freedom system, a function of the variables results which is plotted as a surface in three space, the variables being displacement, velocity, and potential energy. Projections of this surface onto the displacement-velocity plane define the phase plane. A specified level of potential yields a single continous trace or curve on the phase plane. A geometric interpretation of the phase plane plots yields the criteria for stable and unstable equilibrium configurations.

CHAPTER II

THE MATHEMATICAL ANALYSIS OF ONE DEGREE OF FREEDOM SYSTEM.

The following assumptions are made for the idealized mathematical model (See Figure 1):

1. The bar is assumed as a rigid body.
2. The spring is assumed to have a linear moment - rotation relationship of the form $m = k_T \theta$.
3. The force P acts in the vertical direction for all values of the angular rotation θ, and is thus a conservative force.
4. The weight of the bar is neglected.
5. The total mass of the bar m is assumed as a point mass located at the point of application load P.

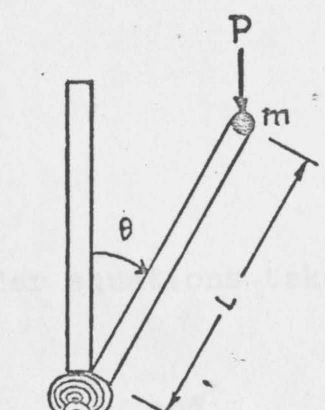

Figure 1

The Lagrangian of the system defined as the difference between the kinetic and potential energy is given as,

$$L(\dot{\theta},\theta,P) = \frac{m}{2}L^2\dot{\theta}^2 - \left\{\frac{K_T}{2}\theta^2 - PL(1-\cos\theta)\right\}. \tag{1}$$

The differential Equation of motion must satisfy the following Lagrange equation.

$$\frac{\partial L}{\partial \theta} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 . \tag{2}$$

Combining equation (1) and (2), the following nonlinear second order differential equation of motion is obtained,

$$mL^2\ddot{\theta} + K_T\theta - PL\sin\theta = 0. \tag{3}$$

For convenience equation (3) is reduced to a pair of first order differential equation using the following substitutions,

$$\left.\begin{aligned} \theta &= \theta_1 , \\ \dot{\theta} &= \dot{\theta}_1 = \theta_2 , \\ \ddot{\theta} &= \ddot{\theta}_1 = \dot{\theta}_2 . \end{aligned}\right\} \tag{4}$$

Thus, the two first order equations take the form

$$\dot{\theta}_1 = \theta_2 , \qquad \text{and,} \tag{5}$$

$$\dot{\theta}_2 = \frac{(-1)}{mL^2}(K_T\theta_1 - PL\sin\theta_1) . \tag{6}$$

Denoting the right hand side of equation (6) as $-f(\theta_1)$, then,

$$\int_{\theta_1=0}^{\theta_1=\theta} f(\theta_1)\, d\theta_1 = \frac{V(\theta_1, P)}{mL^2} = V^*(\theta_1, P), \tag{7}$$

where V (θ,P) is defined as the total potential energy as defined by the second term on the R.H.S. of equation (1). Performing the integration of equation (7) yields the result

$$V^*(\theta_1, P) = \frac{1}{mL^2}\left(K_T \frac{\theta_1^2}{2} - PL\left(1 - \cos\theta_1\right)\right). \tag{8}$$

For a conservative force field, the following intermediate integral holds

$$\frac{1}{2} mL^2 \theta_2^2 + \int_{\theta=0}^{\theta=\theta_1} f(\theta_1)\, d\theta_1 = E_o(\theta_1, P) = \text{constant} \tag{9}$$

That is, the sum of the kinetic energy plus the potential energy remains constant. The parameter $E_o(\theta_1, P)$ represents this energy sum evaluated at the initial starting time t = o. Solving equation (9) for the function θ_2, yields

$$\theta_2 = \pm\sqrt{2}\left[\frac{E_o(\theta_1, P)}{mL^2} - \frac{1}{mL^2}\left(\frac{K_T}{L}\frac{\theta_1^2}{2} - P\left(1 - \cos\theta_1\right)\right)\right]^{\frac{1}{2}}. \tag{10}$$

Equation (10) gives relationship between angular velocity, the angular displacement, the stability load, and the initial energy of the elastic system at any time t.

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The analysis of this problem is divided into two parts:

1. Static stability analysis.
2. Dynamic stability analysis.

1. <u>Static Stability</u>

The static stability analysis is based upon the behavior of the potential energy function V defined by equations (7) and (8). The mathematical conditions for stability and instability are defined as follows:

1. $\frac{dV}{d\theta} = 0$ is a necessary condition for equilibrium.
2. If $\frac{d^2V}{d\theta^2} > 0$, the equilibrium condition is stable.
3. If $\frac{d^2V}{d\theta^2} < 0$, the equilibrium condition is unstable.
4. If $\frac{d^2V}{d\theta^2} = 0$, the equilibrium condition is neutral.

and the critical values of deformation and loading are determined.

For the problem in question, then

$$mL^2 V^* = V = \frac{K_T}{L} \cdot \frac{\theta_1^2}{2} - P(1-\cos\theta_1) . \qquad (11)$$

The necessary condition for equilibrium takes the form,

$$\frac{dV}{d\theta_1} = 0 = \frac{K_T}{L}\theta_1 - P\sin\theta_1 \text{ , or} \qquad (12)$$

$$P = \frac{K_T}{L}\frac{\theta}{\sin\theta} . \qquad (13)$$

Neutral equilibrium yields the condition,

$$\frac{dV^2}{d\theta_1^2} \doteq 0 = \frac{K_T}{L} - p\cos\theta_1 \qquad (14)$$

Combining equations (13) and (14) and eliminating the parameter P yields the conditions for θ_{cr}, the critical value of 0 for instability, as

$$\tan\theta = \theta, \qquad \text{where} \qquad 0 \leq \theta \leq \pi \qquad (15)$$

The corresponding value P_{cr} satisfying the condition θ_{cr} = o is given by equation (13) as

$$P_{CR} = K_T/L \qquad (16)$$

LOAD VS. ANGULAR DISPLACEMENT

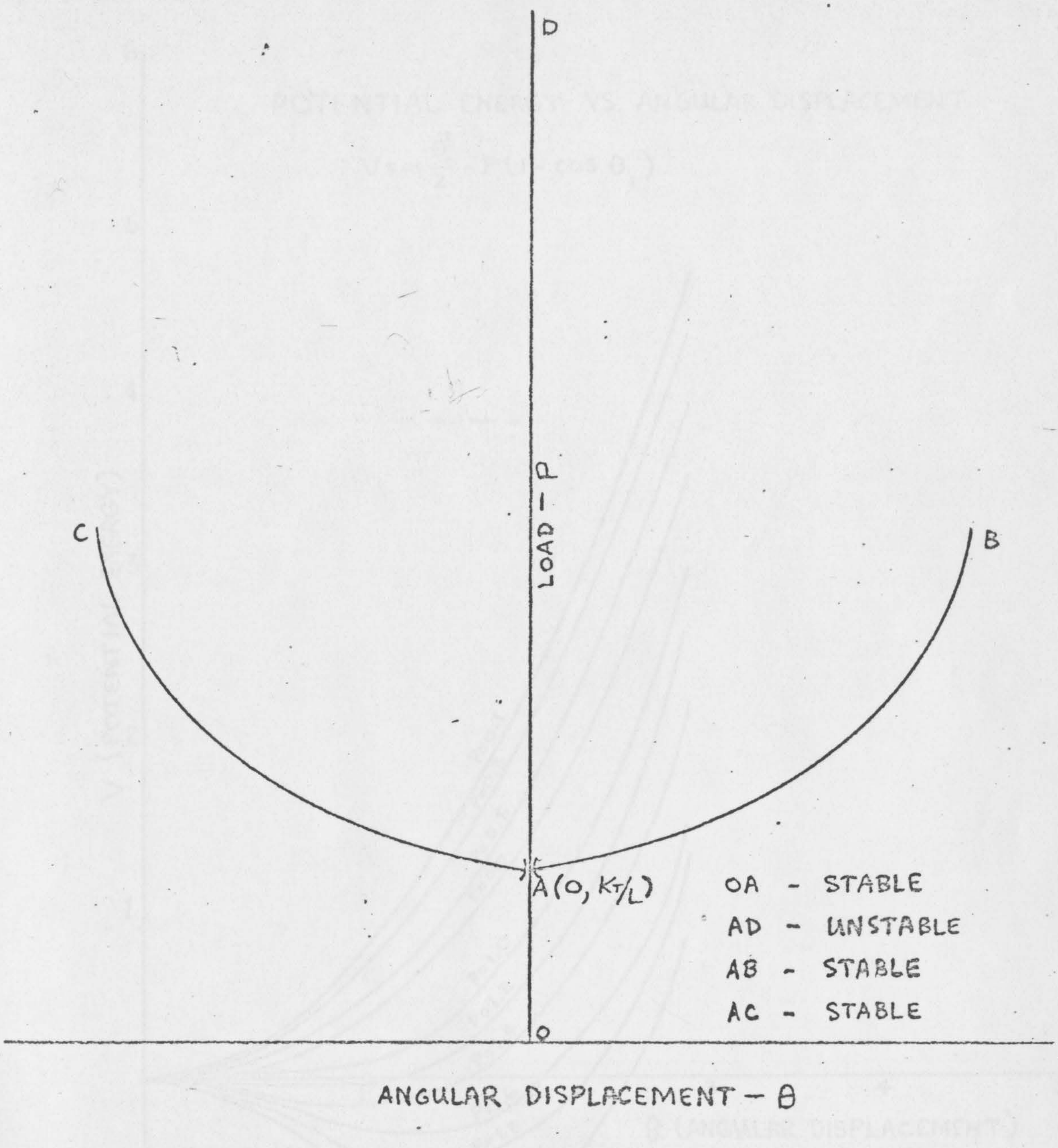


Figure 2

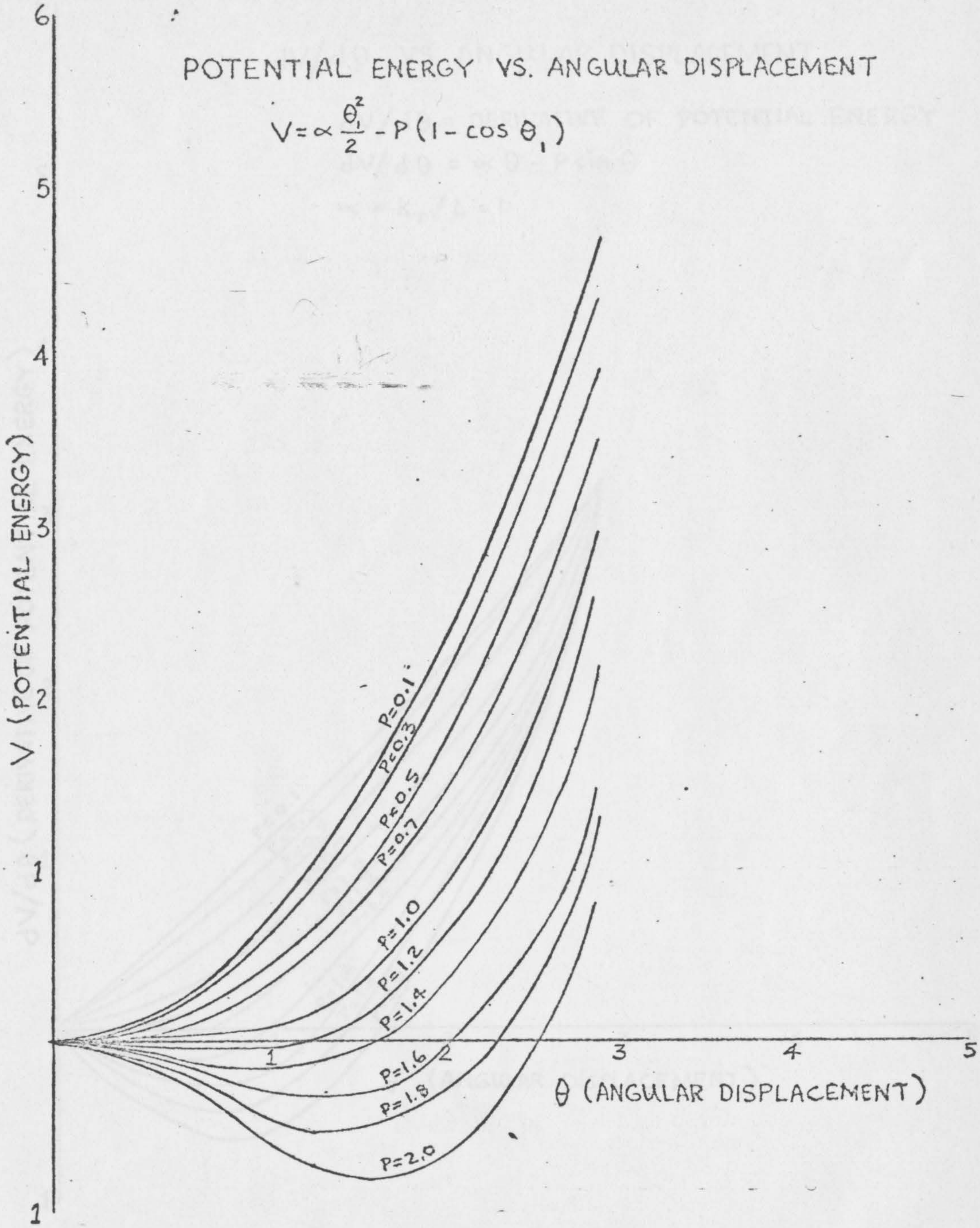

POTENTIAL ENERGY VS. ANGULAR DISPLACEMENT
$V = \alpha \frac{\theta_1^2}{2} - P(1 - \cos\theta_1)$
V (POTENTIAL ENERGY)
θ (ANGULAR DISPLACEMENT)
P=0.1
P=0.3
P=0.5
P=0.7
P=1.0
P=1.2
P=1.4
P=1.6
P=1.8
P=2.0
6
5
4
3
2
1
1
1
2
3
4
5

Figure 3

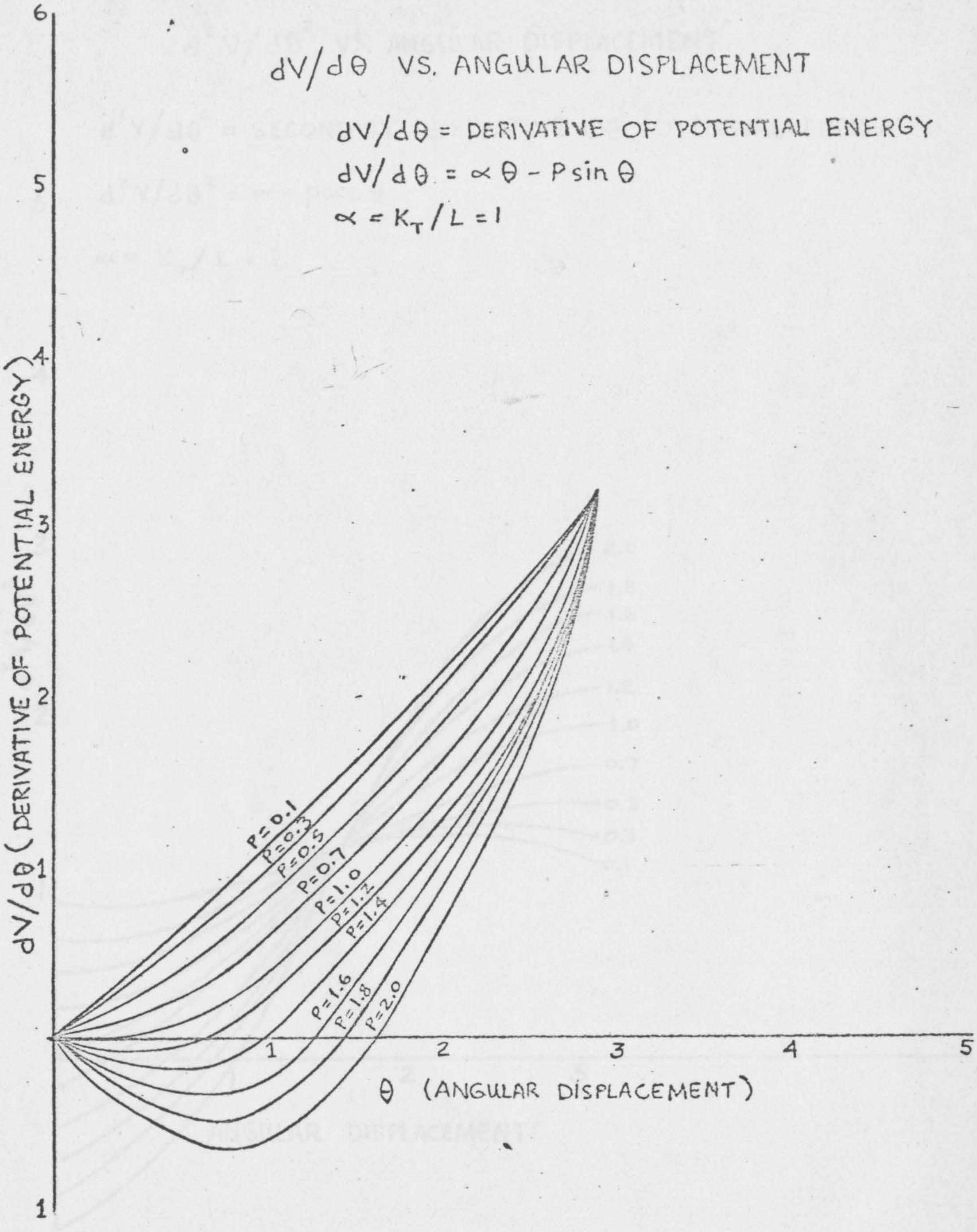
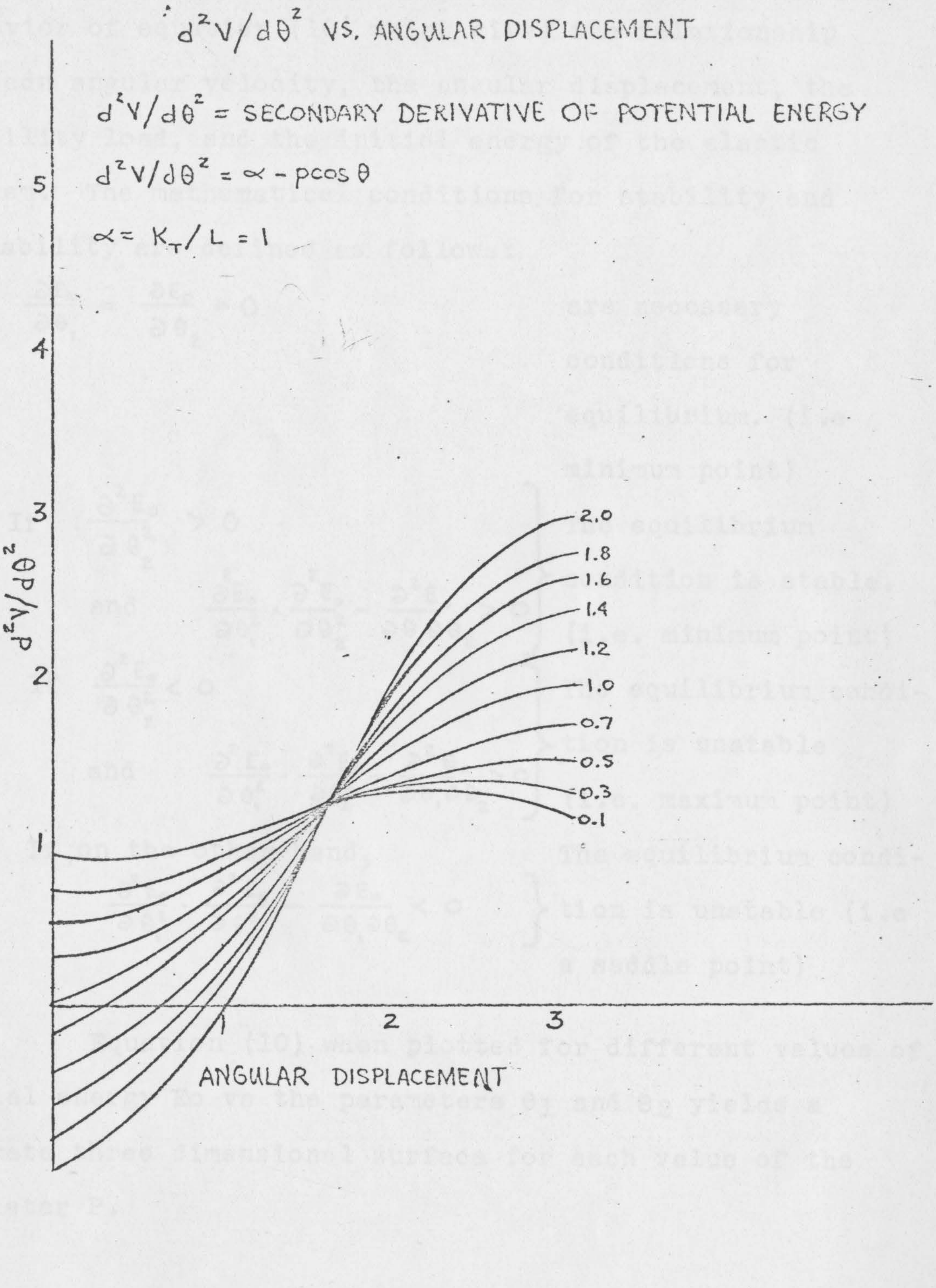

dV/dθ VS. ANGULAR DISPLACEMENT
dV/dθ = DERIVATIVE OF POTENTIAL ENERGY
dV/dθ = ∝θ - P sin θ
∝ = K_T / L = 1
dV/dθ (DERIVATIVE OF POTENTIAL ENERGY)
θ (ANGULAR DISPLACEMENT)
P=0.1
P=0.3
P=0.5
P=0.7
P=1.0
P=1.2
P=1.4
P=1.6
P=1.8
P=2.0
6
5
4
3
2
1
1
1
2
3
4
5

Figure 4



$d^2V/d\theta^2$ VS. ANGULAR DISPLACEMENT
$d^2V/d\theta^2$ = SECONDARY DERIVATIVE OF POTENTIAL ENERGY
$d^2V/d\theta^2 = \alpha - p\cos\theta$
$\alpha = K_T/L = 1$
$d^2V/d\theta^2$
ANGULAR DISPLACEMENT
1
2
3
4
5
2.0
1.8
1.6
1.4
1.2
1.0
0.7
0.5
0.3
0.1

Figure 5

2. Dynamic Stability Analysis

The dynamic stability analysis is based upon the behavior of equation (10) which gives the relationship between angular velocity, the angular displacement, the stability load, and the initial energy of the elastic system. The mathematical conditions for stability and instability are defined as follows:

1. $\frac{\partial E_o}{\partial \theta_1} = \frac{\partial E_o}{\partial \theta_2} = 0$ are necessary conditions for equilibrium. (i.e minimum point)

2. If $\frac{\partial^2 E_o}{\partial \theta_2^2} > 0$ and $\frac{\partial^2 E_o}{\partial \theta_1^2} \cdot \frac{\partial^2 E_o}{\partial \theta_2^2} - \frac{\partial^2 E_o}{\partial \theta_1 \partial \theta_2} > 0$ } The equilibrium condition is stable. (i.e. minimum point)

3. If $\frac{\partial^2 E_o}{\partial \theta_2^2} < 0$ and $\frac{\partial^2 E_o}{\partial \theta_1^2} \cdot \frac{\partial^2 E_o}{\partial \theta_2^2} - \frac{\partial^2 E_o}{\partial \theta_1 \partial \theta_2} > 0$ } The equilibrium condition is unstable (i.e. maximum point)

4. If, on the other hand, $\frac{\partial^2 E_o}{\partial \theta_1^2} \cdot \frac{\partial^2 E_o}{\partial \theta_2^2} - \frac{\partial E_o}{\partial \theta_1 \partial \theta_2} < 0$ } The equilibrium condition is unstable (i.e a saddle point)

Equation (10) when plotted for different values of initial energy Eo vs the parameters θ_1 and θ_2 yields a separate three dimensional surface for each value of the parameter P.

Equation (10) is rewritten in the following form,

$$E_o(\theta_1, \theta_2, P) = \frac{mL^2}{2}\dot{\theta}_2^2 + \frac{K_T}{2}\theta_1^2 - PL(1-\cos\theta_1). \tag{17}$$

The necessary conditions of dynamic equilibrium are written,

$$\frac{\partial E_o}{\partial \theta_1} = 0 = \frac{K_T}{L}\theta_1 - P\sin\theta_1, \text{ and} \tag{18}$$

$$\frac{\partial E}{\partial \theta_2} = 0 = mL^2\,\theta_2. \tag{19}$$

Equations (18) and (19) are reduced to the form,

$$P = \frac{K_T}{L}\cdot\frac{\theta_1}{\sin\theta_1}, \tag{20}$$

and

$$\theta_2 = 0. \tag{21}$$

the sufficiency conditions are investigated by noting following restrictions,

$$\left.\begin{aligned} \frac{\partial^2 E_o}{\partial \theta_1} &= \frac{K_T}{L} - P\cos\theta_1, \\ \frac{\partial E_o}{\partial \theta_2^2} &= mL^2, \text{ and} \\ \therefore \frac{\partial^2 E}{\partial \theta_1 \partial \theta_2} &= 0 \end{aligned}\right\} \tag{22}$$

Using equation (17) and expanding $\cos\theta_1$ as a Maclaurin's series we obtain

$$E_o(\theta_1, \theta_2, P) = \frac{mL^2}{2}\theta_2 + \frac{1}{2}K_T\theta_1^2 - PL\left[1-\left(1-\frac{\theta_1^2}{2}+\frac{\theta_1^4}{24}-\dots\right)\right]. \quad (23)$$

The necessary conditions of dynamic equilibrium are

$$\frac{\partial E_o}{\partial \theta_1} = 0 = K_T\theta_1 - PL\left(\theta_1 - \frac{\theta_1^3}{6}\right), \quad \text{and} \quad (24)$$

$$\frac{\partial E_o}{\partial \theta_2} = 0 = mL^2\theta_2. \quad (25)$$

The solutions of equations are $\theta_2 = o$, and either

$$\theta_1 = 0 \quad \text{or} \quad \theta_1 = \pm\sqrt{6\left(1-\frac{K_T}{PL}\right)} \quad (26)$$

By constructing the discriminant

$$\frac{\partial^2 E_o}{\partial^2 \theta_1^2}\frac{\partial^2 E_o}{\partial \theta_2^2} - \frac{\partial^2 E_o}{\partial \theta_1 \partial \theta_2} \quad (27)$$

Sustituting equation (23) into equation (27) yields

$$mL^3\left[\frac{K_T}{L} - P\left(1-\frac{\theta_1^2}{2}\right)\right], \text{ where } L \neq 0. \quad (28)$$

Evaluating equation (28) at $\theta_1 = o$ and noting that by equation (20) $P_{cr} = kt/l$, it follows that for $P > P_{cr}$ the evaluation is negative. Hence, the origin is a saddle point and is an unstable state of equilibrium.

For $P < Pcr = kt/l$ and $\theta_1 = 0$ evaluation of equation (28) is positive, also $\partial^2 E_o/\partial\theta_1^2 > 0$ and thus origin is a minimum point and a state of stable equilibrium. For the condition $\theta_1 = \pm\sqrt{6(1-K_T/PL)}$ the discriminant reduces to the value $2mpL^3(1-K_T/PL)$ which is positive for $P > kt/l$ and thus a minimum point. Hence, it is a point of stable equilibrium.

Summary:

		Discriminant		$\partial^2 E_o/\partial\theta_1^2$	Equilibrium
$\theta_2 = 0$	$\theta_1 = 0$	$P < K_T/L$	positive	positive	stable
		$P > K_T/L$	negative	-	unstable
	$\theta_1 = \sqrt{6(1-\frac{K_T}{PL})}$	$P > K_T/L$	positive	positive	stable

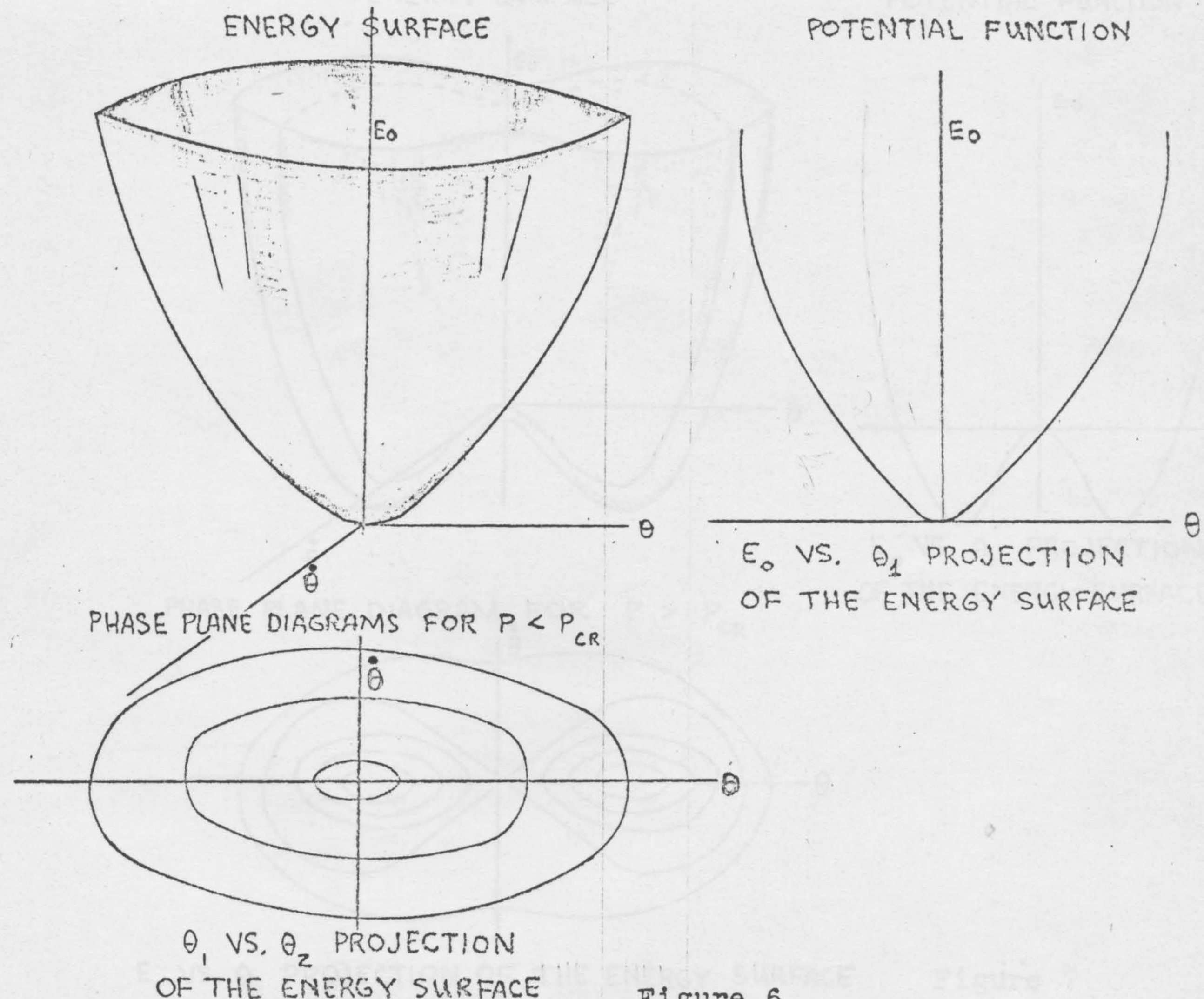
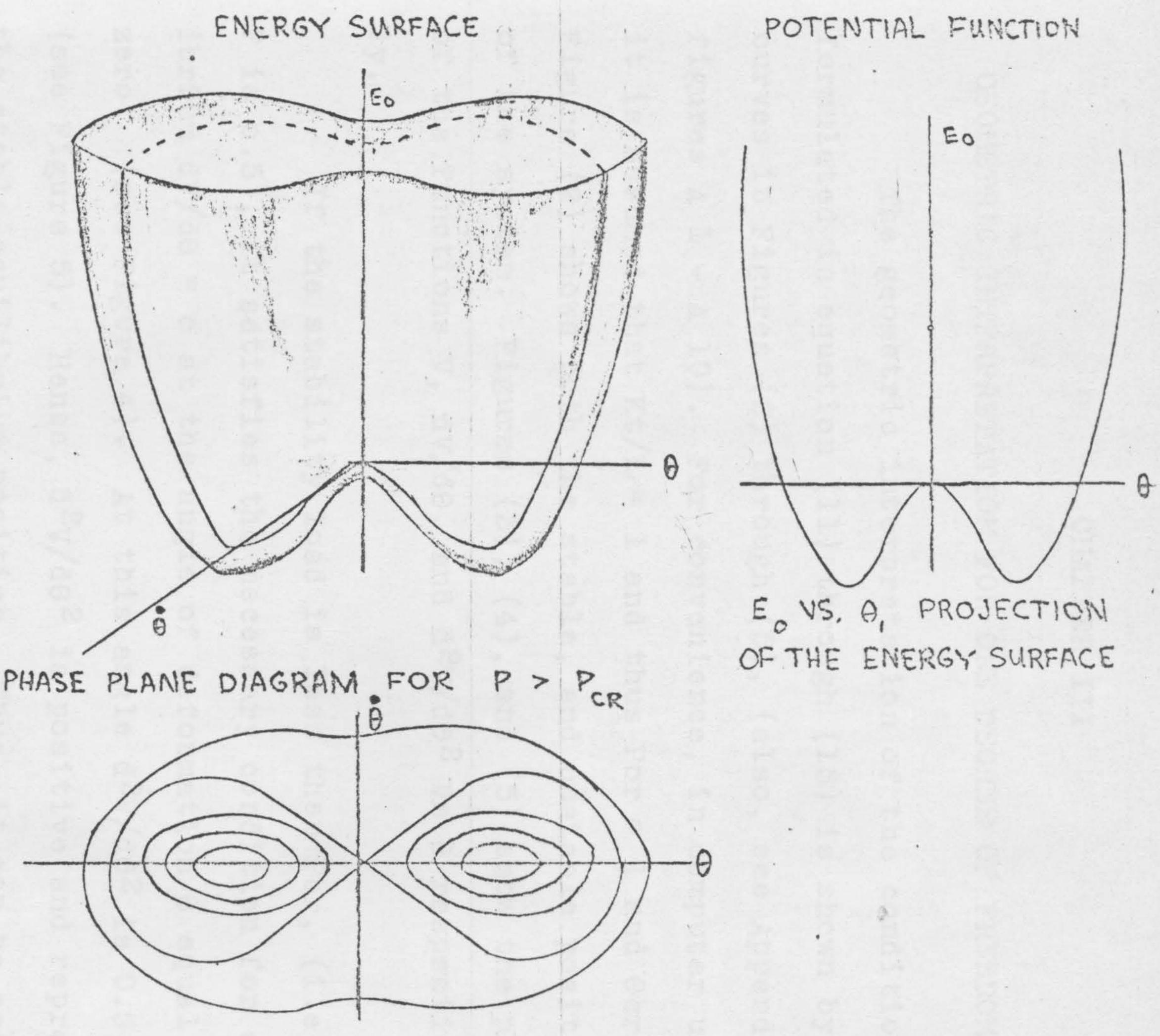

ENERGY SURFACE
E_0
θ
$\dot{\theta}$
POTENTIAL FUNCTION
E_0
θ
E_0 VS. θ_1 PROJECTION
OF THE ENERGY SURFACE
PHASE PLANE DIAGRAMS FOR $P < P_{CR}$
$\dot{\theta}$
θ
θ_1 VS. θ_2 PROJECTION
OF THE ENERGY SURFACE

Figure 6



E_1 VS. θ_2 PROJECTION OF THE ENERGY SURFACE

Figure 7

CHAPTER III

GEOMETRIC INTERPRETATION FOR ONE DEGREE OF FREEDOM SYSTEM

The geometric interpretation of the conditions formulated in equation (11) through (15) is shown by the curves in Figures (2) through (5), (also, see Appendix figures A 1 - A 10). For convenience, in computer usage, it is assumed that $Kt/L = 1$ and thus $Pcr = 1$ and $\theta cr = 0$. Figure (2) shows both the stable, and unstable positions of the system. Figures (3), (4), and (5) show the plots of the functions V, $dV/d\theta$, and $d^2V/d\theta^2$ Vs θ respectively.

If the stability load is less than Pcr, (i.e if P is o.5), it satisfies the necessary condition for equilibrium $dV/d\theta = o$ at the angle of deformation θ equal to zero. (see Figure 4). At this angle $d^2V/d\theta^2$ is 0.5. (see Figure 5). Hence, $d^2V/d\theta^2$ is positive and represents the stable equilibrium position. Thus, it can be seen from Figures (3), (4) and (5) that line segment oA represents the stable equilibrium condition of the elastic system in question.

If the stability load is equal to Pcr, (i.e if P is exactly equal to 1), it satisfies the necessary condition for equilibrium $dV/d\theta$ o at the angle of deformation θ equal to zero. At this angle $d^2V/d\theta^2$ for P = 1 is also zero. Hence, point A represents the neutral equilibrium condition of elastic system in consideration. As this point separates the stable and unstable conditions of equilibrium, it is known as the "bifurcation point".

If the stability load is greater than Pcr, (i.e if P is 1.6), it satisfies the necessary condition for equilibrium $dV/d\theta = o$ at two values of angle of deformation: (1) $\theta = o$ and (2) $\theta = 1.6$ radians. At $\theta = o$, and P = 1.6, $d^2V/d\theta^2 = -0.6$, thus $d^2V/d\theta^2$ is negative and represents the unstable equilibrium position. At $\theta = 1.6$ radians, $d^2V/d\theta^2 = 1.05$, thus $d^2V/d\theta^2$ is positive and represents the stable equilibrium position. Referring to Figure (2) it follows that the locus of points on the lines oA, AB, and AC are stable positions of equilibrium. The locus of points on the line AD are unstable positions of equilibrium. Finally, point A represents the "bifurcation point".

These results imply that the elastic system considered is loaded with a force which is less than Pcr, and given a slight rotation, from the origin the system is stable and vibrates with small oscillations about the origin (i.e. $\theta = o$). If the stability load is gradually increased and the condition P = Pcr holds, a slight rotation of the system from the origin will produce no oscillations about the origin. The system remains in this deformed position in a motionless state called neutral equilibrium such that $P = Pcr$. A small rotation from the point $\theta = o$ will cause large unstable oscillations. However, a slight rotation about the point $\theta = \theta_2$ (i.e θ satisfying equation (26)) the system will vibrate with small oscillations about this point which indicates a stable equilibrium condition.

Figures (6) and (7) show the behavior of the model from the dynamic stability point of view. These two figures show a three dimensional surface. The Figures (A 1) through (A 10) (see Appendix) show the plots of the phase plane diagram for angular displacement vs angular velocity. Figures (A 1 through (A 4) are drawn for the stability load $P = Pcr$. Figure (A 5) is drawn for the stability load $P = Pcr$. Figures (A 6) through (A 10) are drawn for stability load $P = Pcr$.

If applied force $P = Pcr$ the system is characterized by the energy surface shown in Figure (6) which is formed from equation (10). The projection of this surface on to the θ_1 and θ_2 plane produce the phase plane diagram. The three dimensional surface possesses the geometric properties of an elliptic parabolid. It is characterized by a minimum point at the origin indicating the origin, a stable point of equilibrium for $P = Pcr$. The phase plane diagram is made up of system of concentric ellipsi. This set of curves show that for all values of time and for small values of initial conditions θ_1 and θ_2 the origin is stable.

Figure (7) illustrates the three dimensional energy surface for the system when applied force $P = P_{cr}$. A saddle point occurs at the origin. Hence, origin is unstable. Two minimum points occur at the left and at the right of the origin, indicating two stable points for $\theta > \theta = o$. The phase plane diagram shows a set of concentric ellipsi for these two minimum points. The saddle point at $\theta = o$ is indicated on the phase plane plot on a closed curve which takes the approximate shape of lemniscate with principal points at the origin. This geometric configuration indicates an unstable point at $\theta = o$.

CHAPTER IV

MATHEMATICAL ANALYSIS OF TWO DEGREE OF FREEDOM SYSTEM

The mathematical model given in problem 1 is changed by adding a linear spring as shown in Figures (8 a) and (8 b).

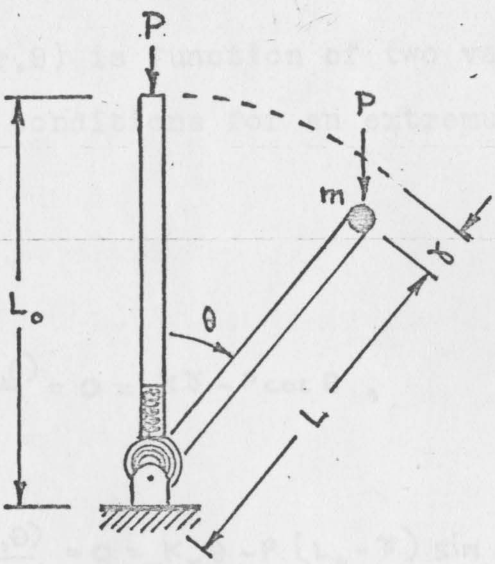


Figure 8 a

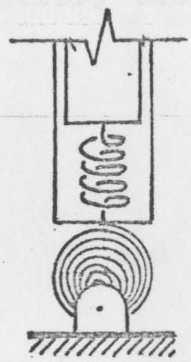


Figure 8 b

The strain energy of deformation for this system is given by the equation

$$U_o = \frac{1}{2} K \gamma^2 + \frac{1}{2} K_T \theta^2 \tag{29}$$

The potential energy of the external forces is given by the equation

$$W = P\left[L_o(1-\cos\theta) + \gamma\cos\theta\right] \tag{30}$$

The total potential energy in the system is written in the form

$$V(\gamma,\theta) = U_o - W = \frac{1}{2}K\gamma^2 + \frac{1}{2}K_T\theta^2 - P\left[L_o(1-\cos\theta) + \gamma\cos\theta\right], \quad (31)$$

where

$$0 \leq \theta \leq \pi \quad (32)$$

Since V (r,θ) is function of two variables, the necessary conditions for an extremum satisfy the equations

$$\frac{\partial V(\gamma,\theta)}{\partial \gamma} = 0 = K\gamma - P\cos\theta \,, \qquad \text{and} \quad (33)$$

$$\frac{\partial V(\gamma,\theta)}{\partial \theta} = 0 = K_T\theta - P(L_o - \gamma)\sin\theta \,. \quad (34)$$

Equation (33) is solved for the parameter γ , yielding

$$\gamma = P\cos\theta / K \quad (35)$$

Equation (34) is solved for the value of P in the form

$$P = \left[K_T\theta / (L_o - \gamma)\sin\theta \right] \quad (36)$$

Substituting the value of r given in equation (35) into equation (36) give,

$$P^{*}_{1,2} = \frac{P}{(KL_o/2)} = \frac{1}{\cos\theta}\left[1 \pm \left(1-\beta\frac{\theta}{\tan\theta}\right)^{1/2}\right] \qquad (37)$$

where $\beta = \dfrac{4K_T}{KL_o}$, and (38)

$$0 < \frac{\theta}{\tan\theta} < 1 \quad . \qquad (39)$$

The previous equations may be rearranged and solved for the parameter r in the form

$$\gamma_{1,2} = \frac{\gamma}{(L/2)} = \left[1 \pm \left(1-\beta\frac{\theta}{\tan\theta}\right)^{1/2}\right] \qquad (40)$$

The displacement r, corresponds to the force P_1, and the displacement r_2 to the load P_2.

Substituting the value of r given in equation (35) into equation (31) gives

$$V(\theta) = \frac{K_T\theta^2}{2} - \frac{P^2}{2K}\cos^2\theta - PL_o(1-\cos\theta) \qquad (41)$$

The potential energy function is reduced to a function in one variable. The condition $dV(\theta)/d\theta = o$ yields equation (37) given above. The sufficiency conditions of equilibrium are determined by the equation

$$\frac{d^2V(\theta)}{d\theta^2} = \frac{4}{KL_0^2}\frac{d^2V(\theta)}{d\theta^2} = \beta + \overset{*}{P}{}^2 \cos 2\theta - 2\overset{*}{P} \cos \theta \qquad (42)$$

For the stable equilibrium equation (42) must be positive, for unstable equilibrium, negative, and for neutral equilibrium, zero.

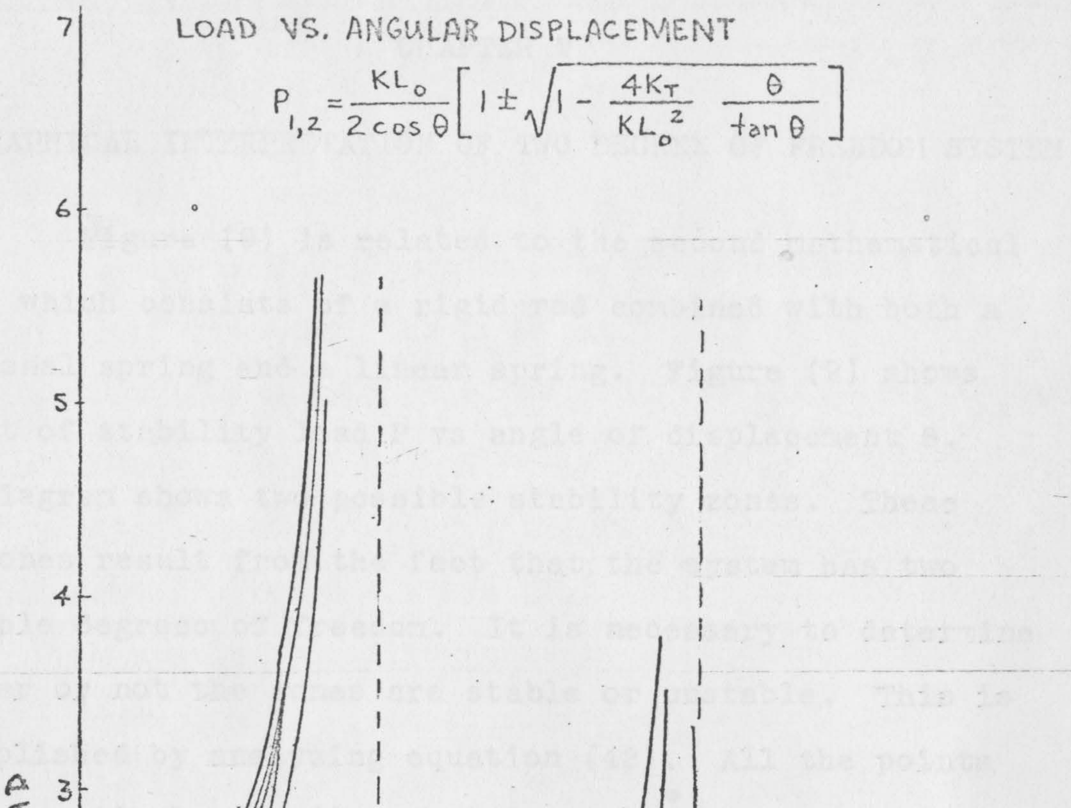

LOAD VS. ANGULAR DISPLACEMENT
$P_{1,2} = \frac{KL_o}{2\cos\theta}\left[1 \pm \sqrt{1 - \frac{4K_T}{KL_o^2}\ \frac{\theta}{\tan\theta}}\right]$
LOAD
1
2
3
4
5
6
7
ANGULAR DISPLACEMENT
1
π/2
2
3
π
β=0..
β=1.0
β=1.0
β=0.8
β=0.4
β=0.1
β=0

Figure 9

CHAPTER V

GRAPHICAL INTERPRETATION OF TWO DEGREE OF FREEDOM SYSTEM

Figure (9) is related to the second mathematical model which consists of a rigid rod combined with both a torsional spring and a linear spring. Figure (9) shows a plot of stability load P vs angle of displacement θ. The diagram shows two possible stability zones. These two zones result from the fact that the system has two possible degrees of freedom. It is necessary to determine whether or not the zones are stable or unstable. This is accomplished by analyzing equation (42). All the points on the vertical axis (i.e. $\theta = 0$) for $0 < \bar{P} \leq 2$ are "bifurcation points". Figures (A 11 through A 16) show that for the lower stability region including the boundary curves $\beta = 0$ and $\beta = 1$, the function $d^2V(\theta)/d\theta^2 > 0$ for all values of θ. Thus all points lying in and on the boundary of the lower stability zones are stable. Figures (A 11 through A 16) also show that for the upper region including the boundary curves $\beta = 0$ and $\beta = 1$, the function $d^2V(\theta)/d\theta^2 < 0$ for all values of θ. Thus all points lying in and on the boundary of the lower stability zones are unstable.

CHAPTER VI

CONCLUSION

The mathematical analysis for the static and dynamic stability criteria of the one degree of freedom system yields the similar results for the critical stability loading conditions. The dynamic stability criteria for the system yields a more atenable interpretation of the stability characteristic of the system.

The static stability analysis of the two degree of freedom system yields a set of algebric equations which are readily solvable. The dynamic stability criteria is somewhat more cumbersome to apply and interpret geometrically since the simplest two degree of freedom problem yields a function in five variables, which cannot be plotted in usual 3 space.

A variety of additional problems may be investigated by using this thesis as a basis.

1. The conservative vertical force P in the one degree of freedom problem may be replaced by a force which rotates as the rod rotates, (i.e. "the follower force"). The rotation of the rod and the rotation of the force should be taken as independent variables.

2. The analysis of a two degree of freedom system subjected to "the follower force" can be considered.

APPENDIX

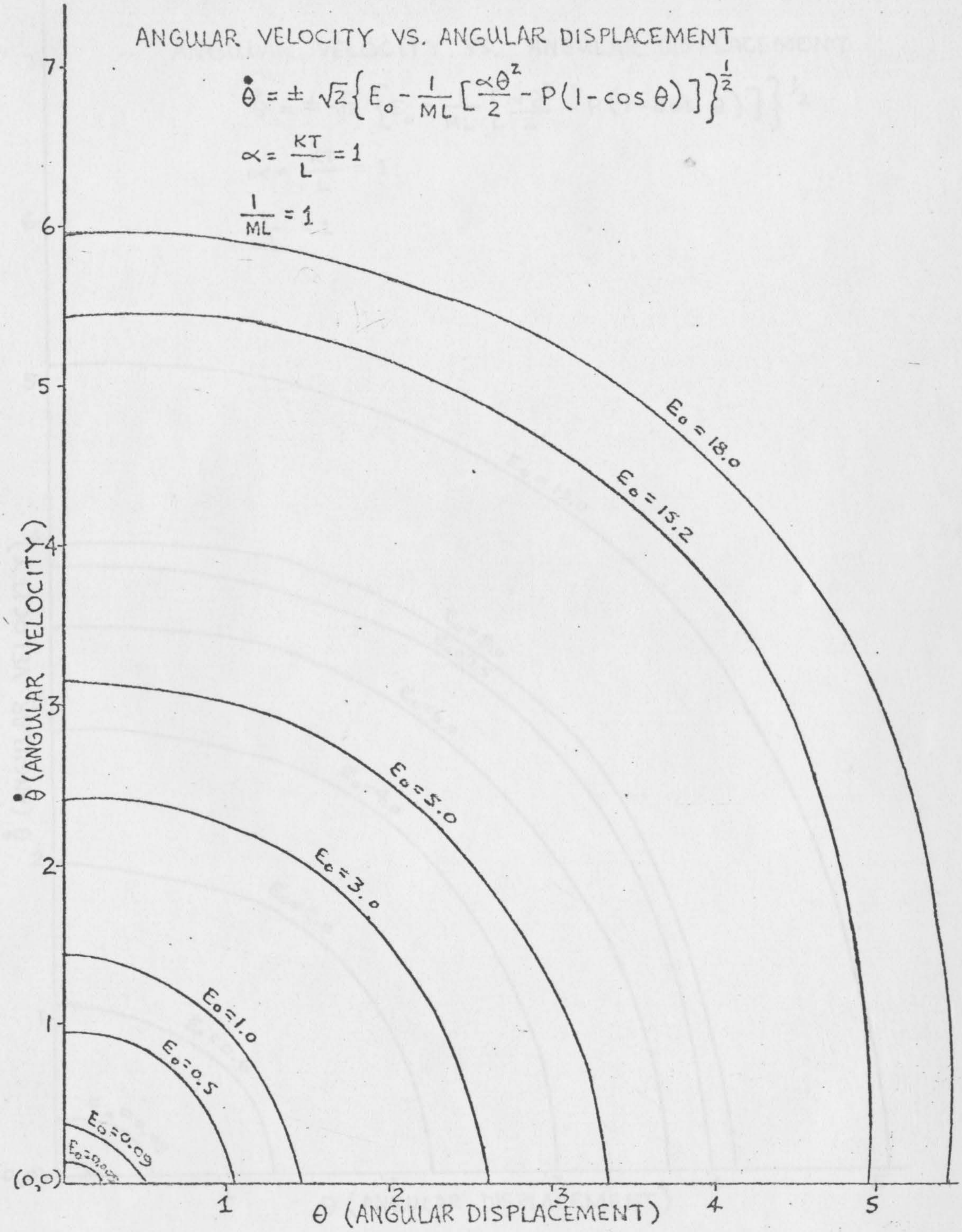


Figure A 1 PHASE PLANE PLOT FOR P = 0.2

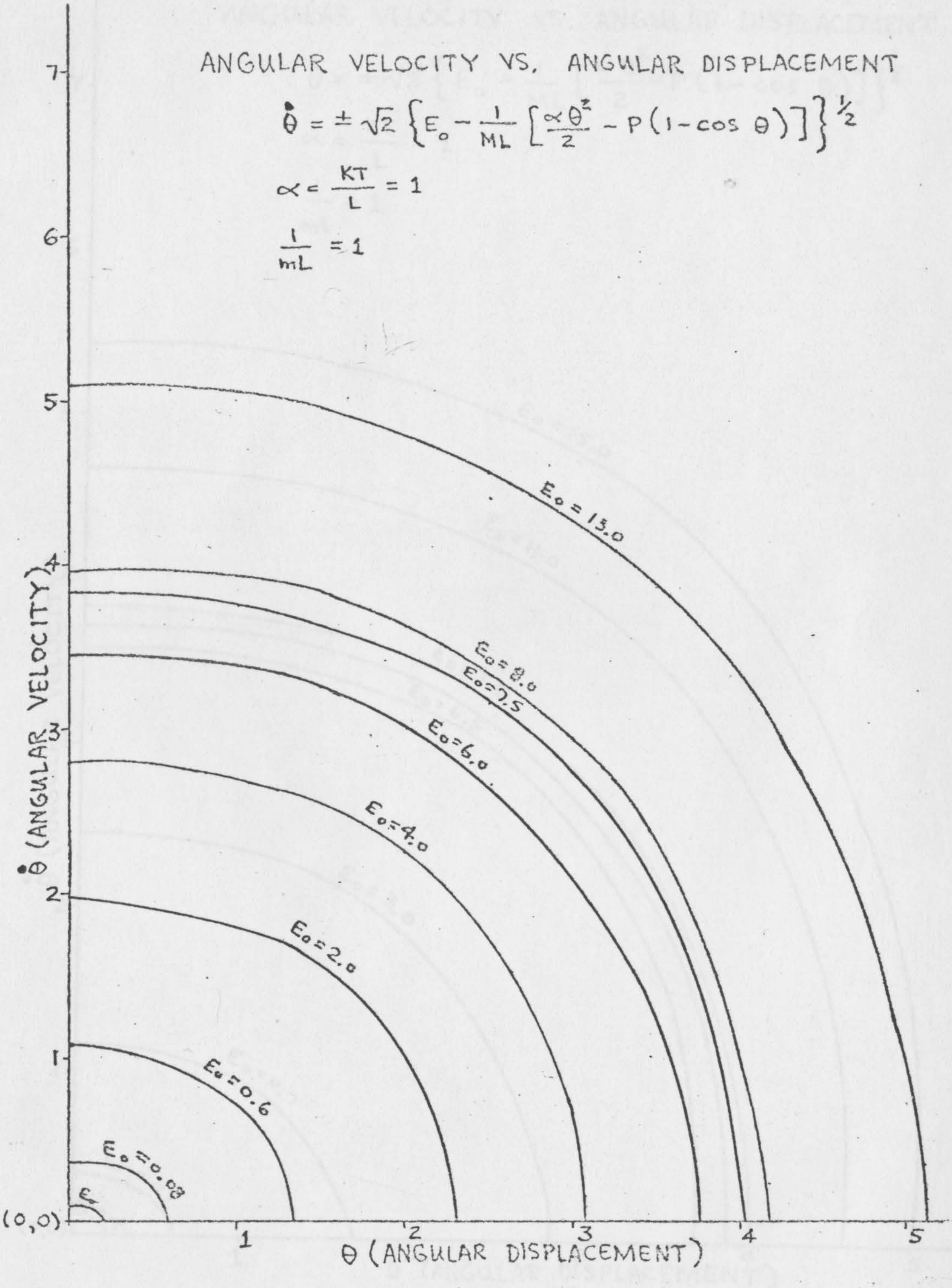


Figure A 2 PHASE PLANE PLOT FOR P = 0.4

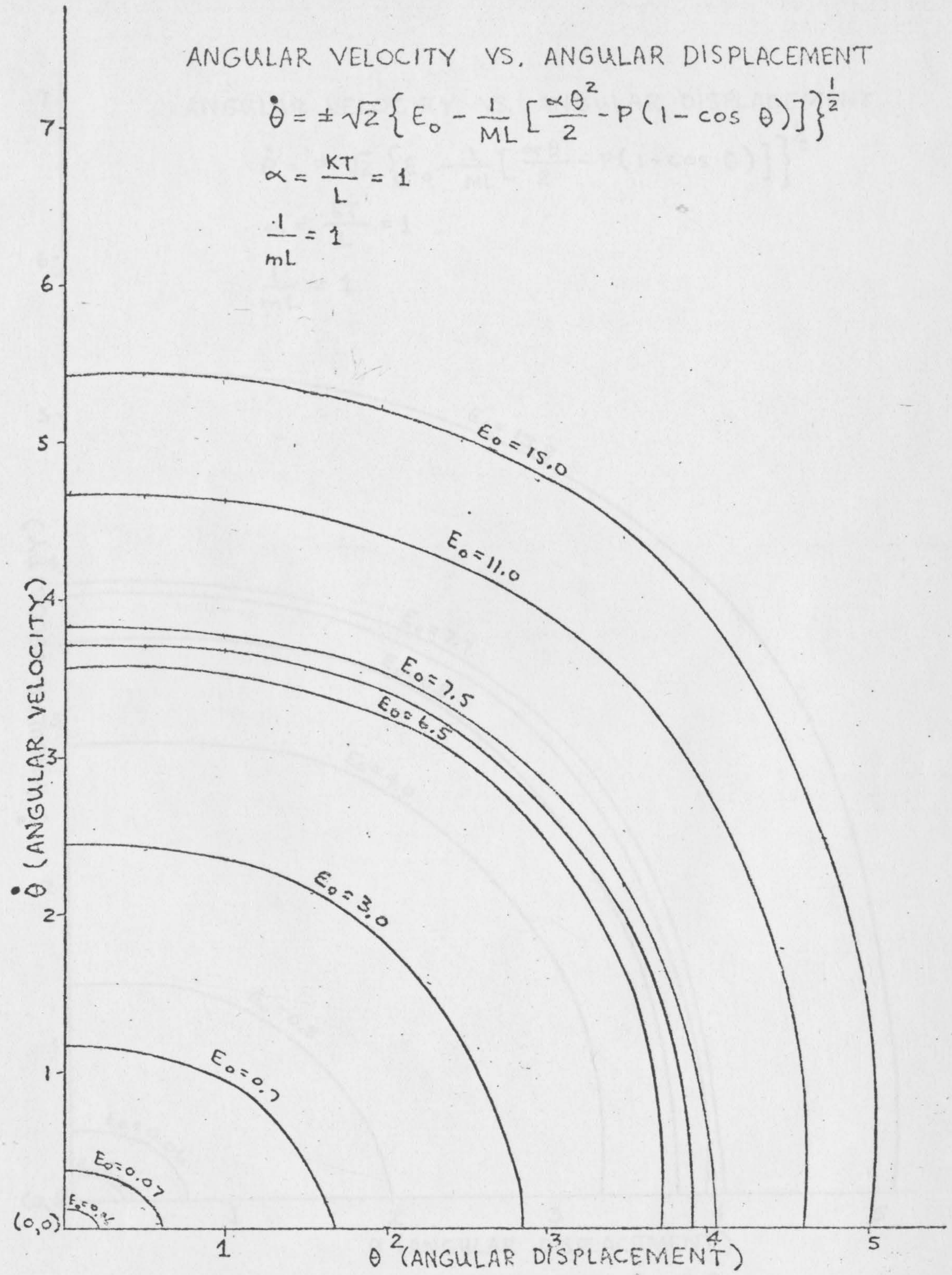


Figure A 3 PHASE PLANE PLOT FOR P = 0.6

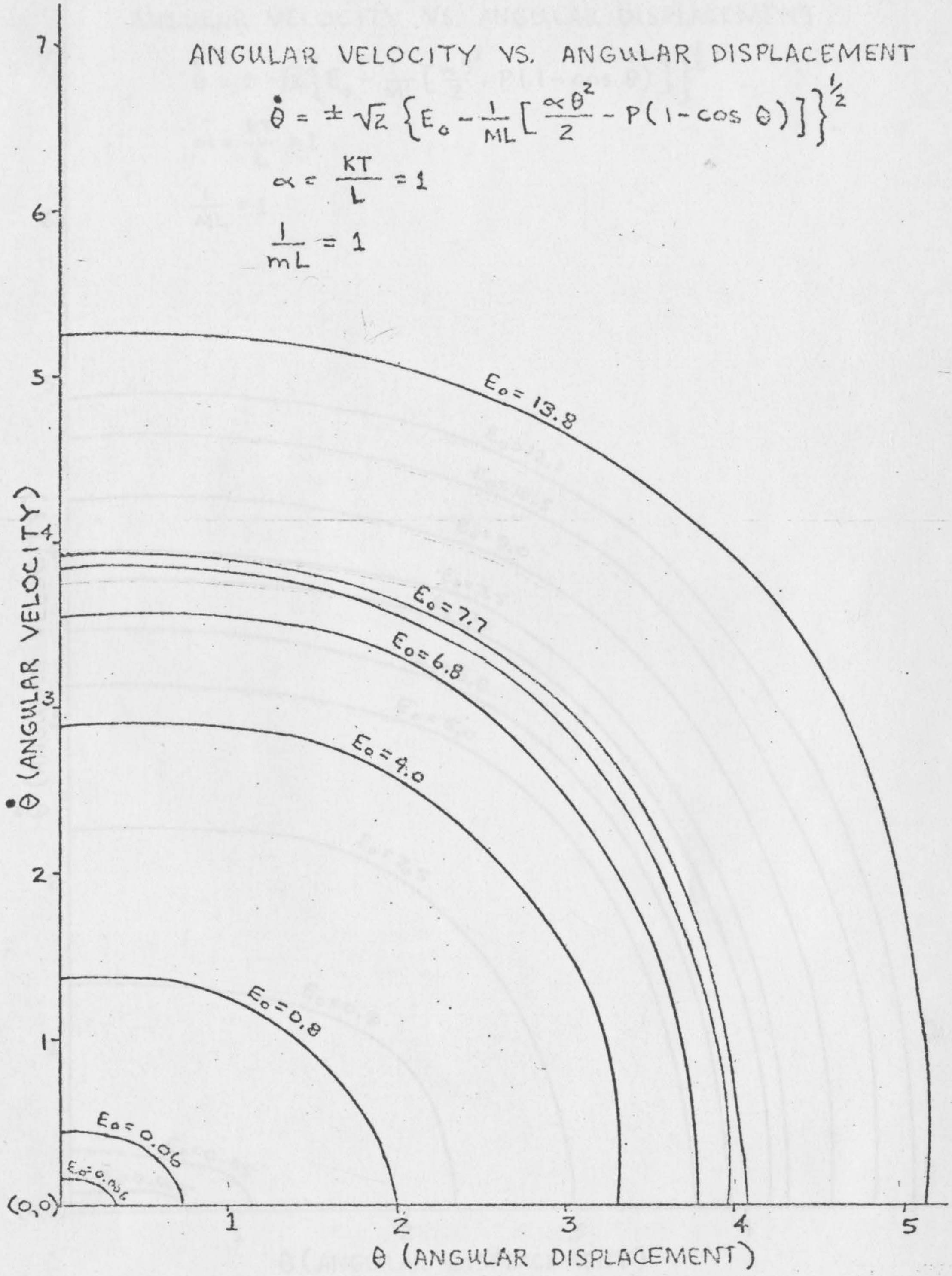


Figure A 4 PHASE PLANE PLOT FOR P = 0.8

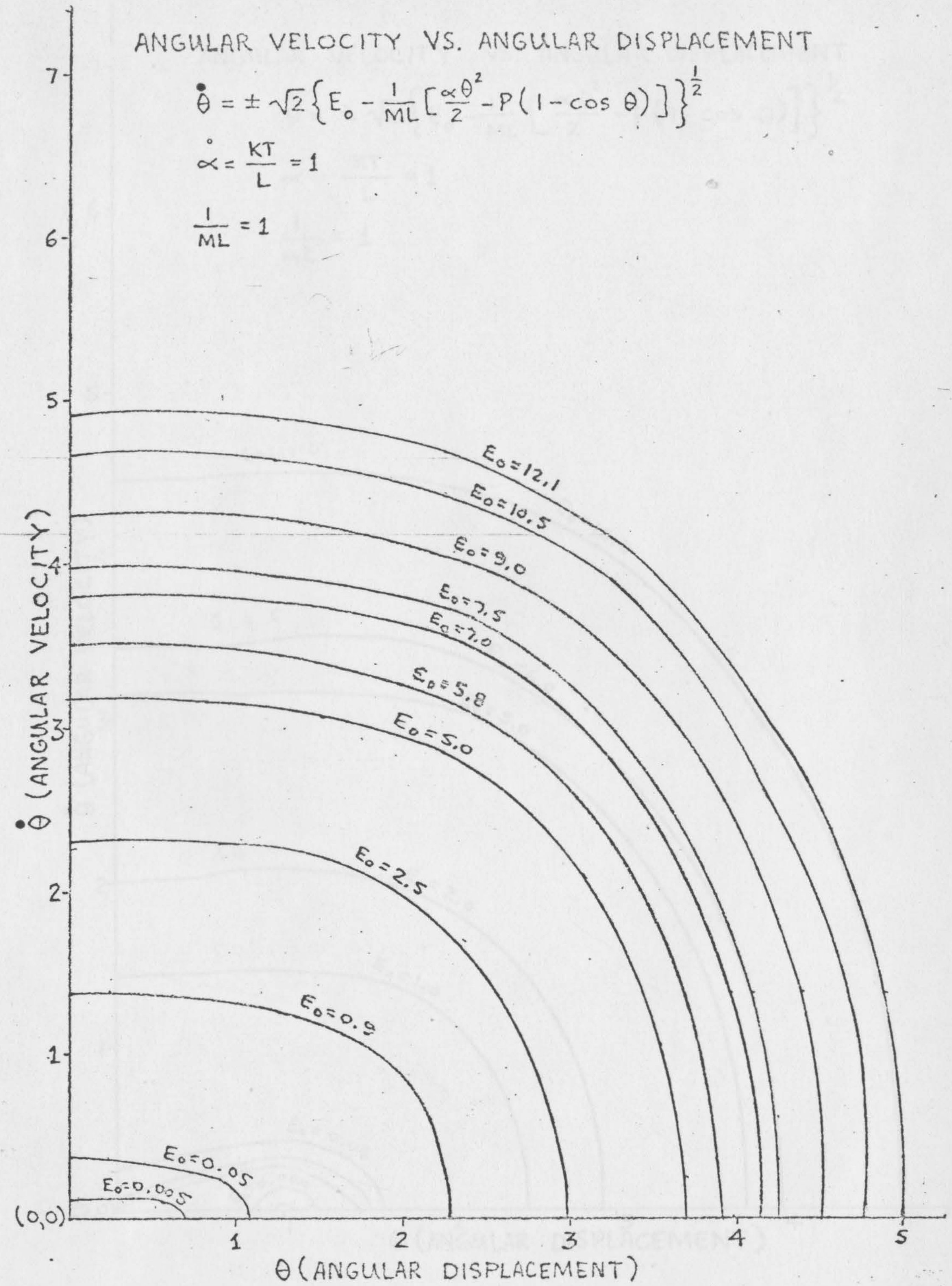


Figure A 5 PHASE PLANE PLOT FOR P = 1

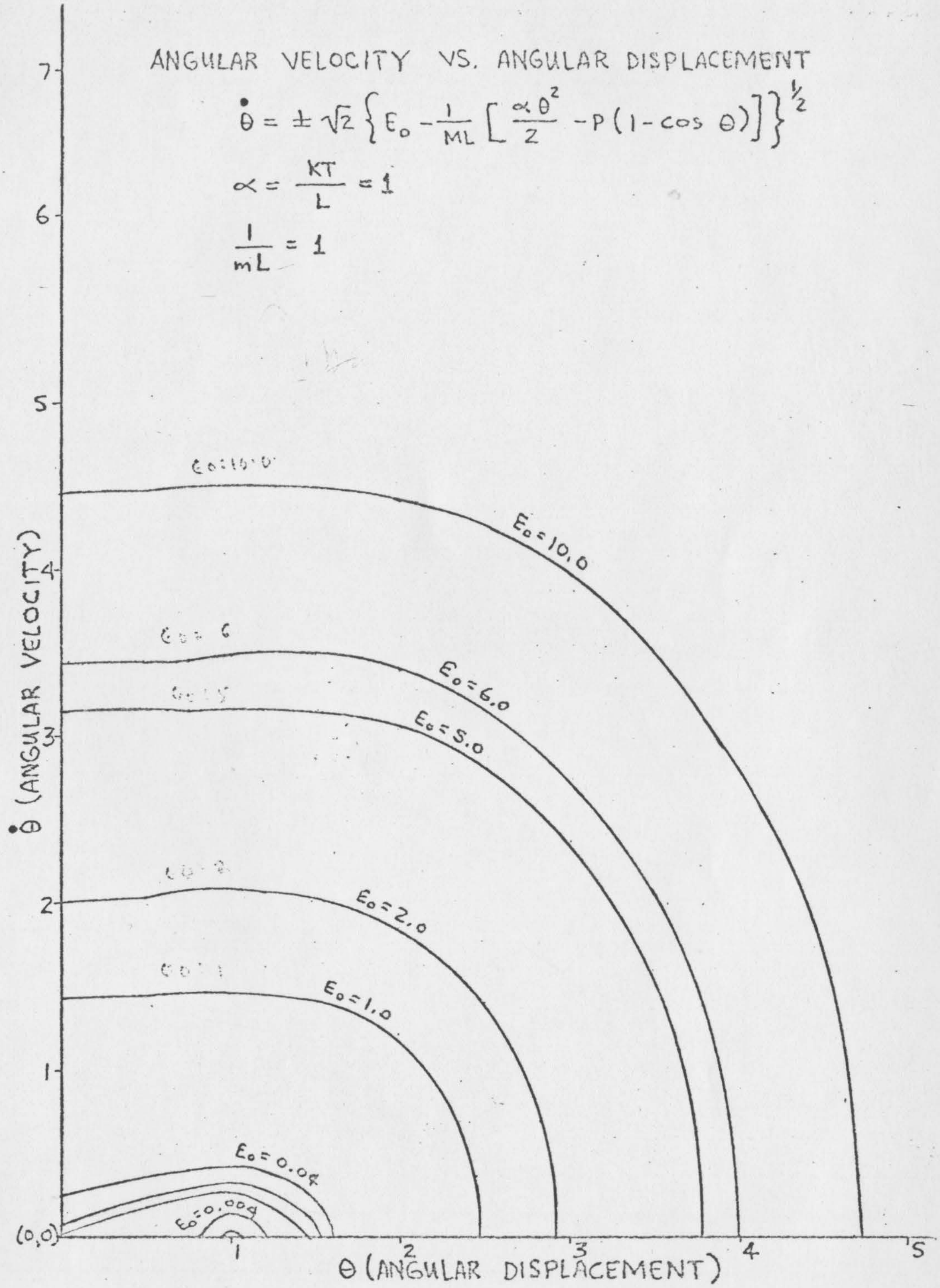


Figure A 6 PHASE PLANE PLOT FOR P = 1.2

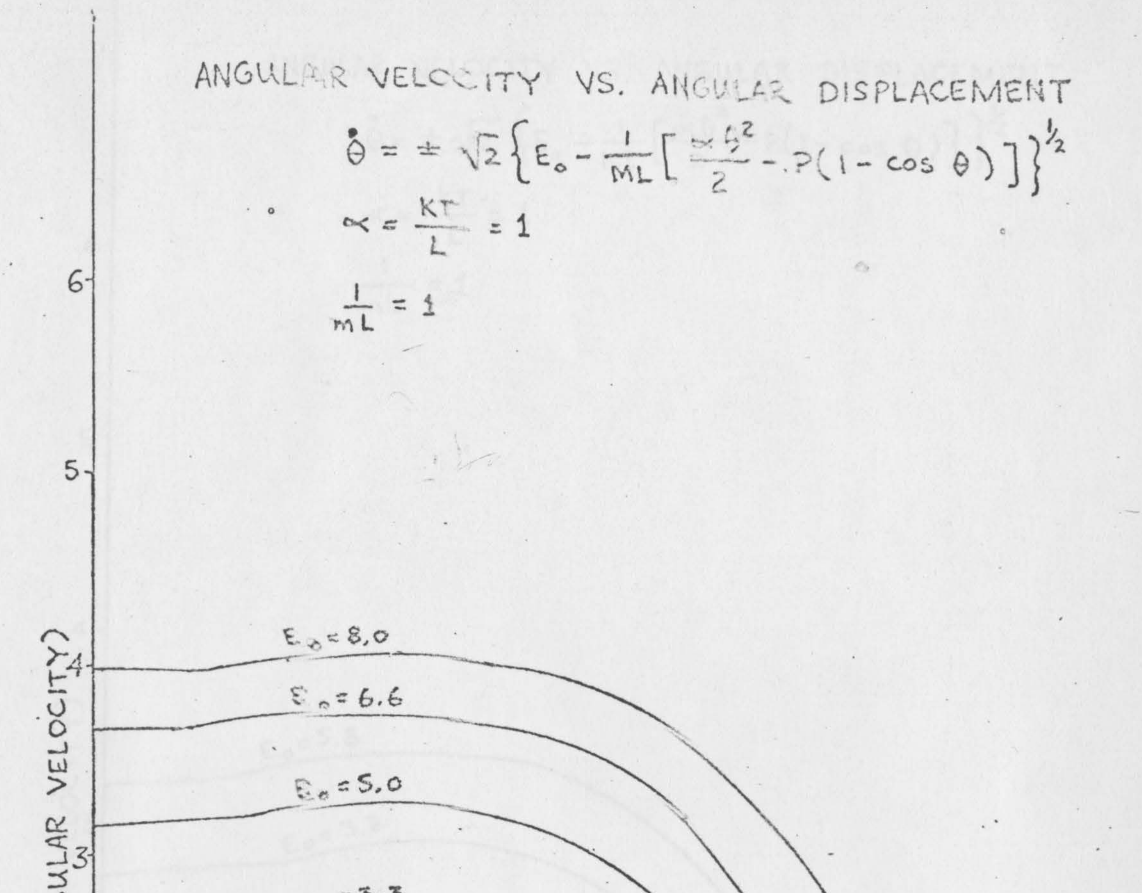


Figure A 8 PHASE PLANE PLOT FOR P = 1.6

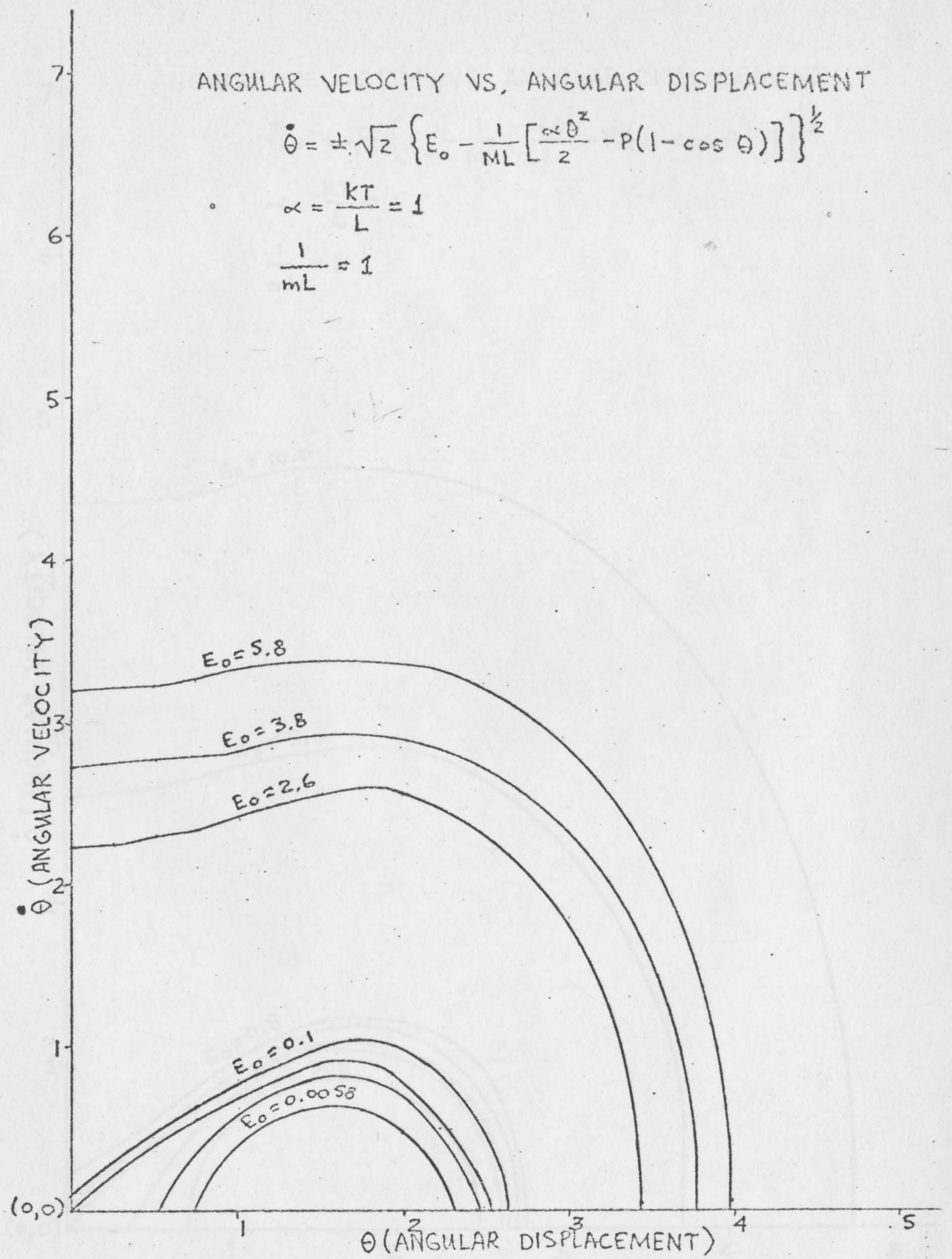


Figure A 9 PHASE PLANE PLOT FOR P = 1.8

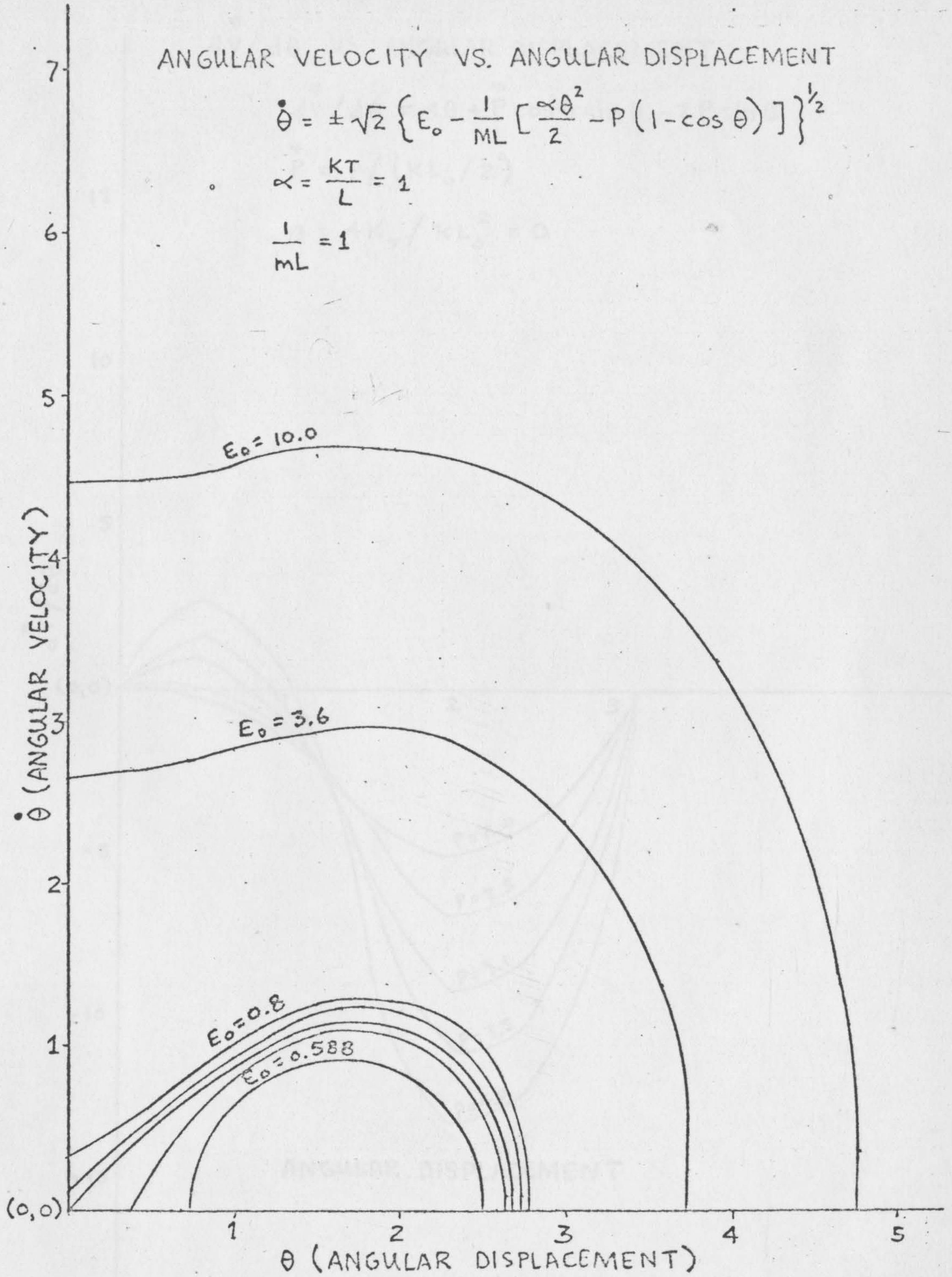


Figure A 10 PHASE PLANE PLOT FOR P = 2.0

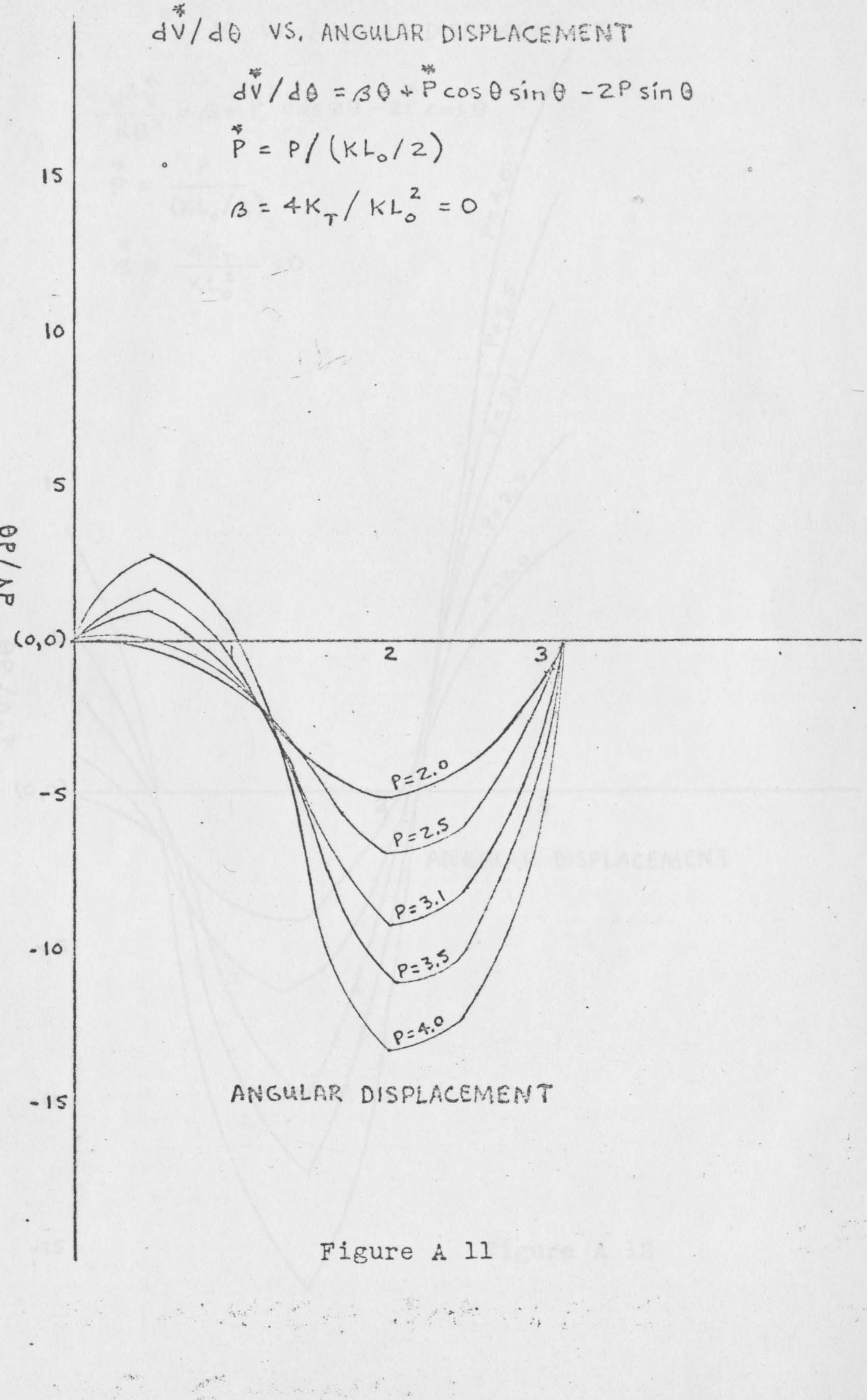
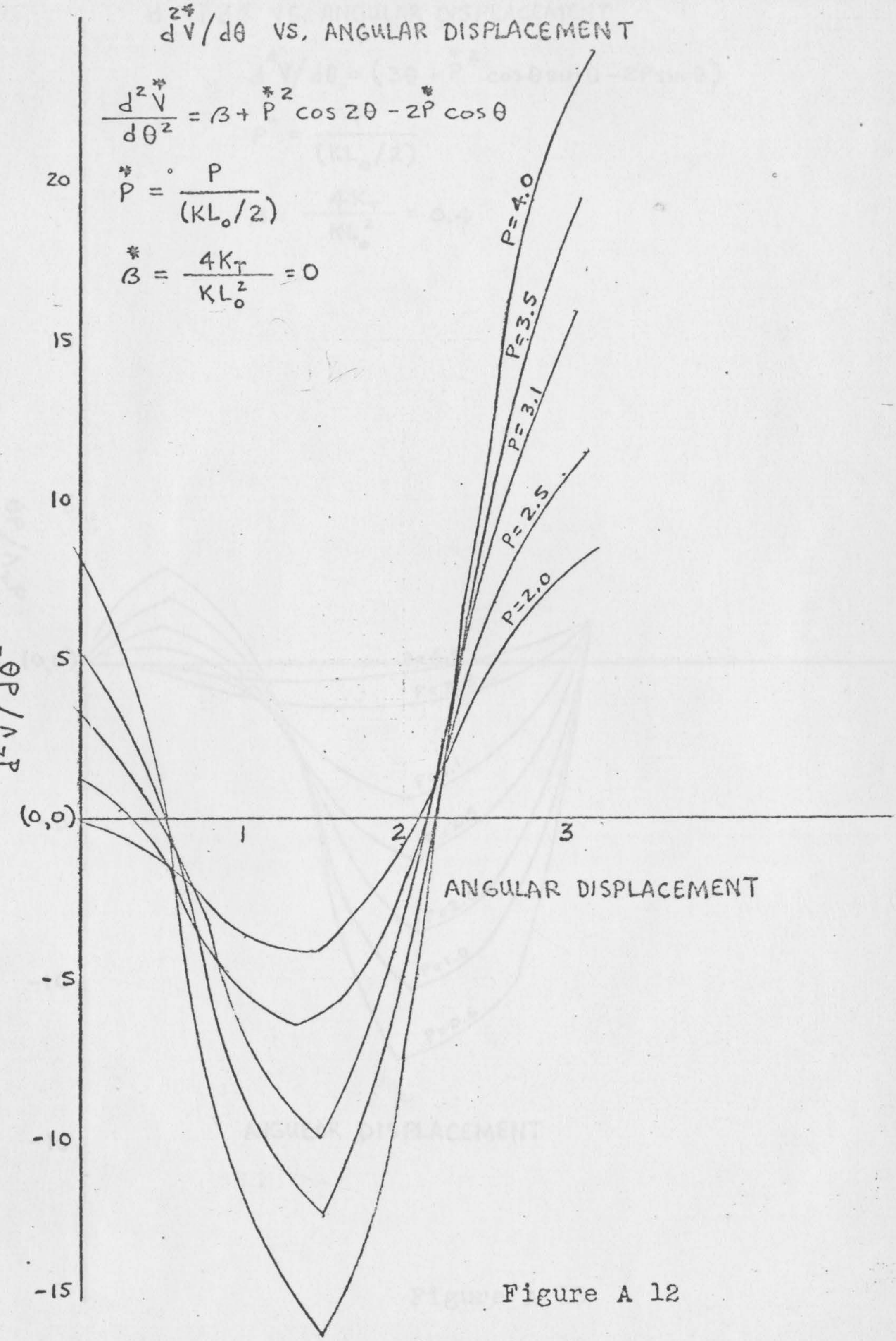

$d\overset{*}{V}/d\theta$ VS. ANGULAR DISPLACEMENT
$d\overset{*}{V}/d\theta = \beta\theta + \overset{*}{P}\cos\theta\sin\theta - 2P\sin\theta$
$\overset{*}{P} = P/(KL_o/2)$
$\beta = 4K_T/KL_o^2 = 0$
15
10
5
(0,0)
-5
-10
-15
1
2
3
$d\overset{*}{V}/d\theta$
P=2.0
P=2.5
P=3.1
P=3.5
P=4.0
ANGULAR DISPLACEMENT

Figure A 11



d2*V/dθ VS. ANGULAR DISPLACEMENT
$\frac{d^2 \overset{*}{V}}{d\theta^2} = \beta + \overset{*}{P}^2 \cos 2\theta - 2\overset{*}{P} \cos\theta$
$\overset{*}{P} = \frac{P}{(KL_o/2)}$
$\overset{*}{\beta} = \frac{4K_T}{KL_o^2} = 0$
P=4.0
P=3.5
P=3.1
P=2.5
P=2.0
20
15
10
5
(0,0)
-5
-10
-15
1
2
3
ANGULAR DISPLACEMENT
d2*V / dθ2

Figure A 12

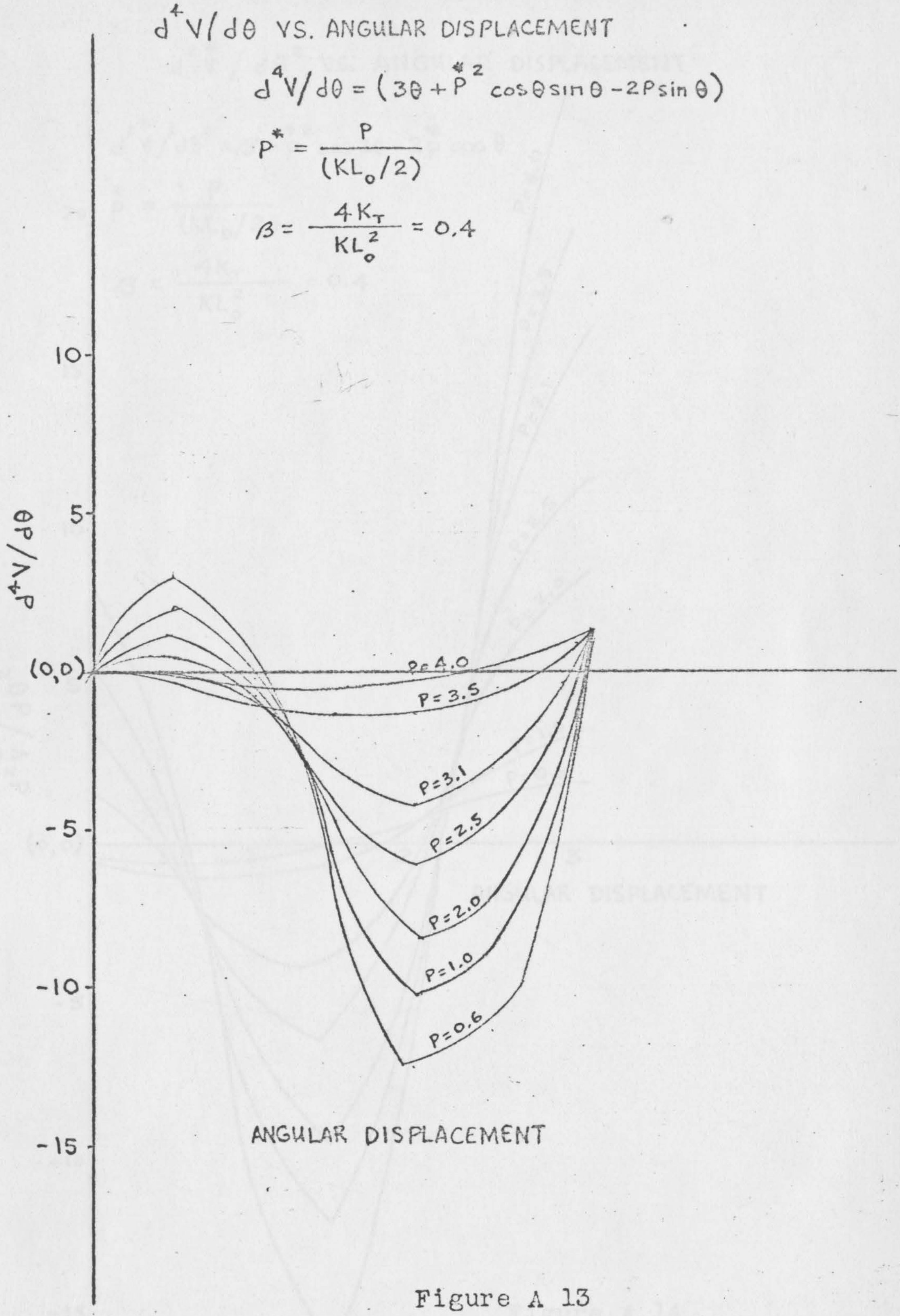

d⁴V/dθ VS. ANGULAR DISPLACEMENT
$d^4V/d\theta = (3\theta + P^{*2} \cos\theta \sin\theta - 2P\sin\theta)$
$P^* = \frac{P}{(KL_o/2)}$
$\beta = \frac{4K_T}{KL_o^2} = 0.4$
10
5
(0,0)
-5
-10
-15
d⁴V/dθ
P=4.0
P=3.5
P=3.1
P=2.5
P=2.0
P=1.0
P=0.6
ANGULAR DISPLACEMENT

Figure A 13

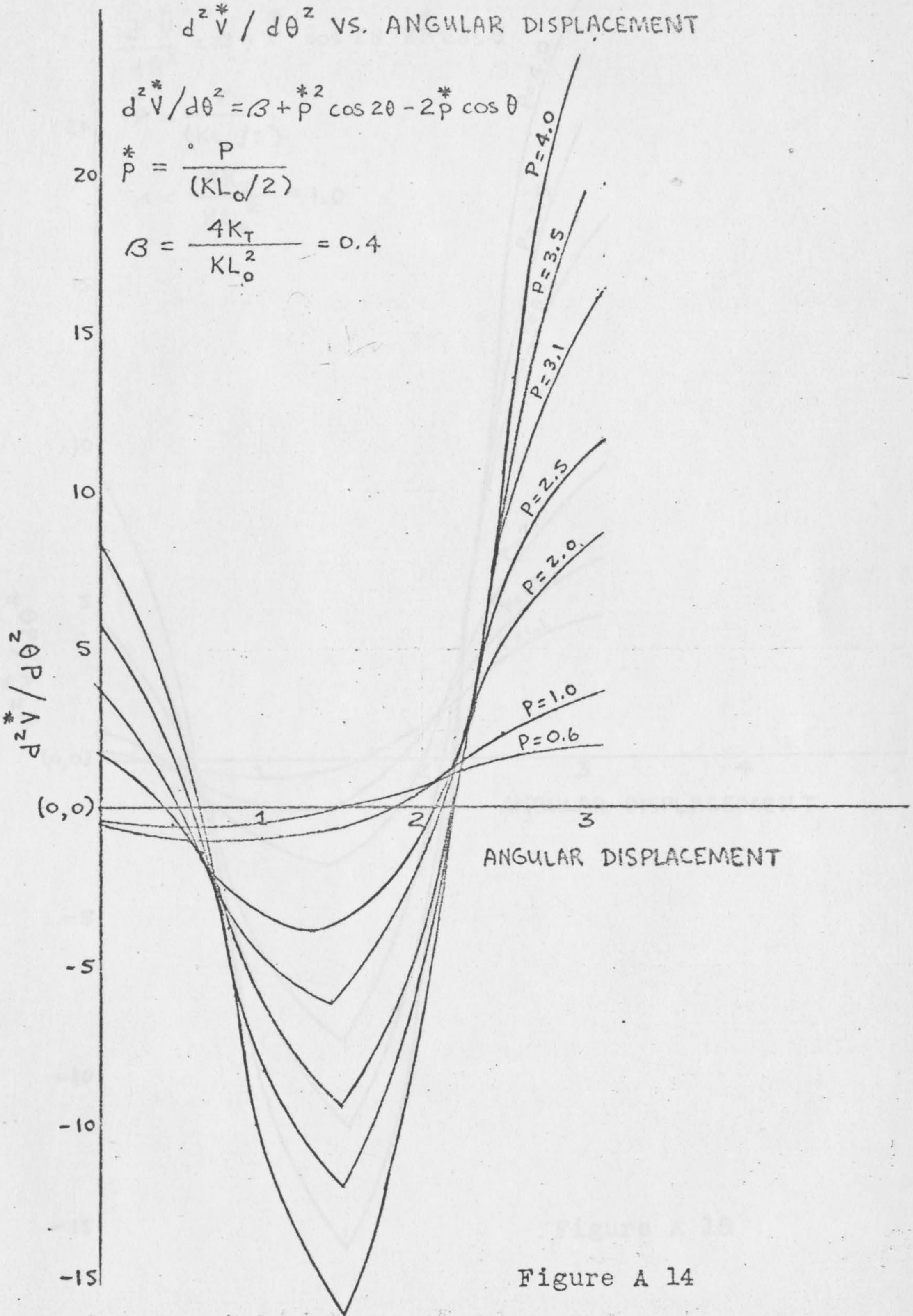
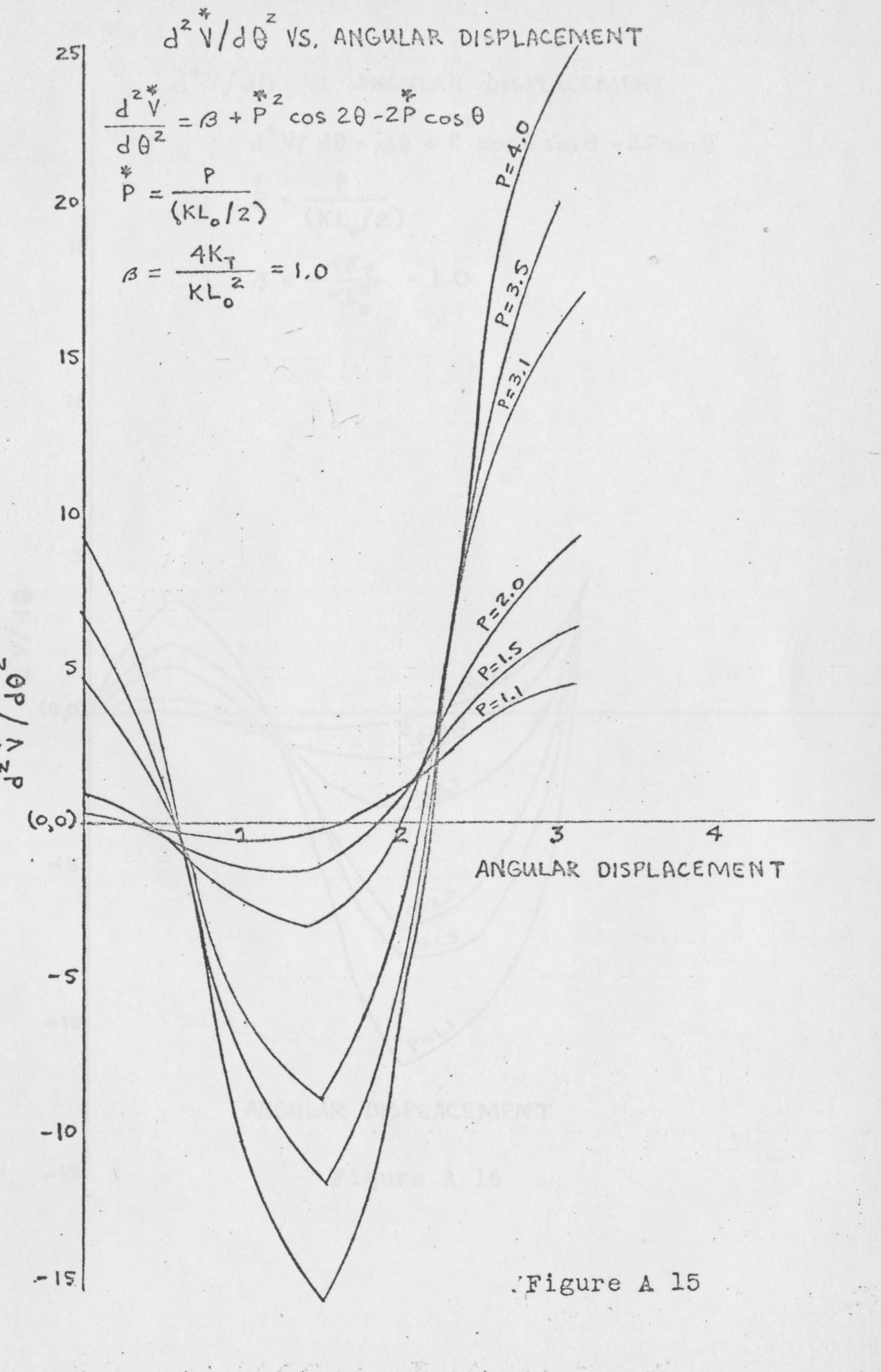

$d^2\overset{*}{V} / d\theta^2$ VS. ANGULAR DISPLACEMENT
$d^2\overset{*}{V}/d\theta^2 = \beta + \overset{*}{P}^2 \cos 2\theta - 2\overset{*}{P}\cos\theta$
$\overset{*}{P} = \frac{P}{(KL_o/2)}$
$\beta = \frac{4K_T}{KL_o^2} = 0.4$
P=4.0
P=3.5
P=3.1
P=2.5
P=2.0
P=1.0
P=0.6
20
15
10
5
(0,0)
-5
-10
-15
1
2
3
ANGULAR DISPLACEMENT
$d^2\overset{*}{V}/d\theta^2$

Figure A 14



$d^2\overset{*}{V}/d\theta^2$ VS. ANGULAR DISPLACEMENT
$\frac{d^2\overset{*}{V}}{d\theta^2} = \beta + \overset{*}{P}^2 \cos 2\theta - 2\overset{*}{P}\cos\theta$
$\overset{*}{P} = \frac{P}{(KL_0/2)}$
$\beta = \frac{4K_T}{KL_0^2} = 1.0$
P=4.0
P=3.5
P=3.1
P=2.0
P=1.5
P=1.1
25
20
15
10
5
(0,0)
-5
-10
-15
1
2
3
4
ANGULAR DISPLACEMENT
$d^2\overset{*}{V}/d\theta^2$

Figure A 15

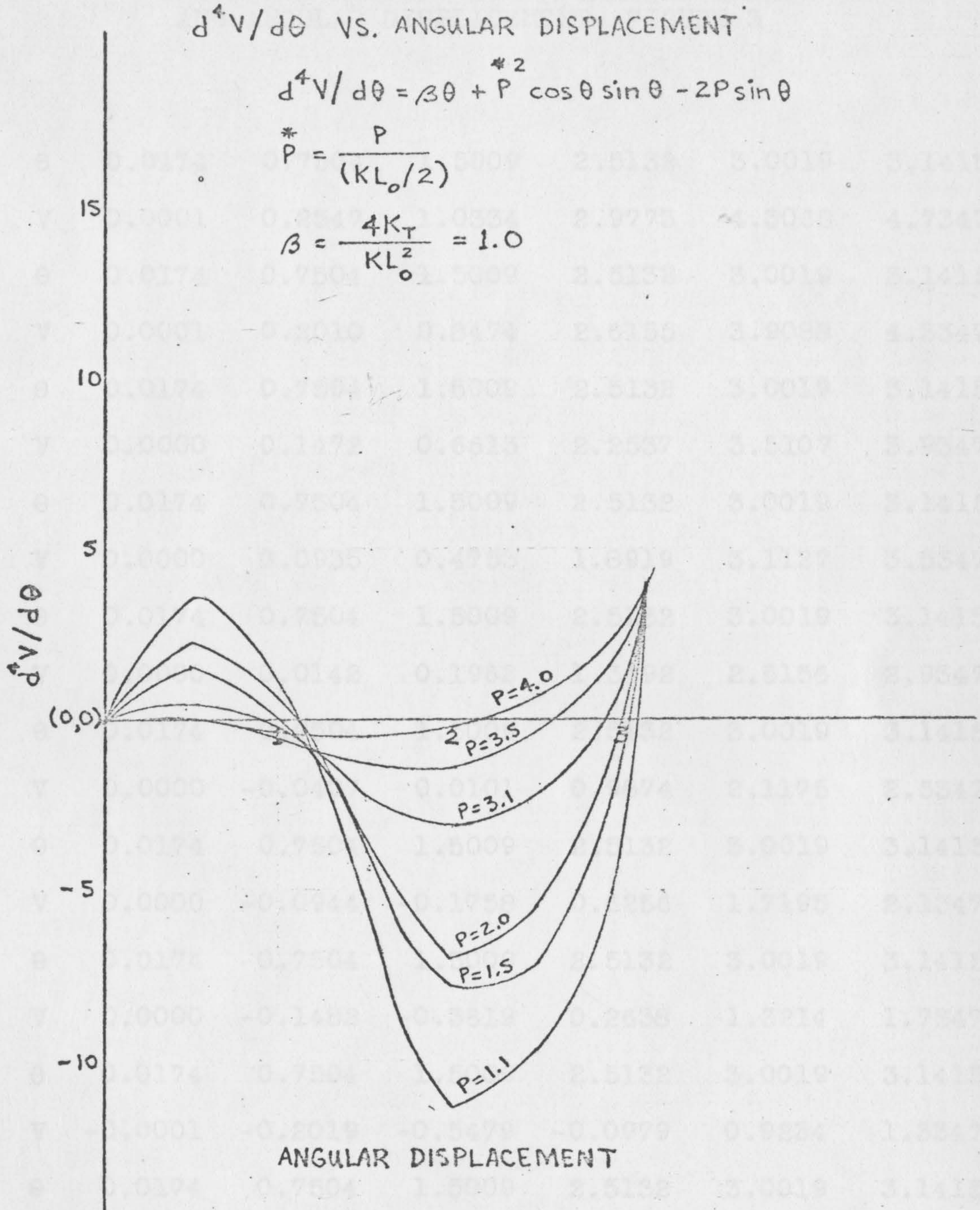

d⁴V/dθ VS. ANGULAR DISPLACEMENT
d⁴V/dθ = βθ + P*² cos θ sin θ - 2P sin θ
P* = P / (KL₀/2)
β = 4K_T / KL₀² = 1.0
15
10
5
-5
-10
-15
(0,0)
d⁴V/dθ
P=4.0
P=3.5
P=3.1
P=2.0
P=1.5
P=1.1
ANGULAR DISPLACEMENT

Figure A 16

TABLE A 1

TABULATED RESULTS FOR POTENTIAL ENERGY AND ANGULAR DISPLACEMENT. FIGURE 3

P							
0.1	θ	0.0174	0.7504	1.5009	2.5132	3.0019	3.1415
	V	0.0001	0.2547	1.0334	2.9773	4.3068	4.7347
0.3	θ	0.0174	0.7504	1.5009	2.5132	3.0019	3.1415
	V	0.0001	0.2010	0.8474	2.6155	3.9088	4.3347
0.5	θ	0.0174	0.7504	1.5009	2.5132	3.0019	3.1415
	V	0.0000	0.1472	0.6613	2.2537	3.5107	3.9347
0.7	θ	0.0174	0.7504	1.5009	2.5132	3.0019	3.1415
	V	0.0000	0.0935	0.4753	1.8919	3.1127	3.5347
1.0	θ	0.0174	0.7504	1.5009	2.5132	3.0019	3.1415
	V	0.0000	0.0142	0.1962	1.3492	2.5156	2.9347
1.2	θ	0.0174	0.7504	1.5009	2.5132	3.0019	3.1415
	V	0.0000	-0.0407	0.0101	0.9874	2.1175	2.5347
1.4	θ	0.0174	0.7504	1.5009	2.5132	3.0019	3.1415
	V	0.0000	-0.0944	-0.1758	0.6256	1.7195	2.1347
1.6	θ	0.0174	0.7504	1.5009	2.5132	3.0019	3.1415
	V	0.0000	-0.1482	-0.3619	0.2638	1.3214	1.7347
1.8	θ	0.0174	0.7504	1.5009	2.5132	3.0019	3.1415
	V	-0.0001	-0.2019	-0.5479	-0.0979	0.9234	1.3347
2.0	θ	0.0174	0.7504	1.5009	2.5132	3.0019	3.1415
	V	-0.0001	-0.2556	-0.7340	-0.4597	0.5253	0.9347

TABLE A 2

TABULATED RESULTS FOR dV/dθ
AND ANGULAR DISPLACEMENT. FIGURE 4

P							
0.1	θ	0.0174	0.7504	1.5009	2.5132	3.0019	3.1415
	dV/dθ	0.0157	0.6822	1.4012	2.4544	2.9880	3.1415
0.3	θ	0.0174	0.7504	1.5009	2.5132	3.0019	3.1415
	dV/dθ	0.0122	0.5458	1.2017	2.3369	2.9602	3.1415
0.5	θ	0.0174	0.7504	1.5009	2.5132	3.0019	3.1415
	dV/dθ	0.0087	0.4094	1.0021	2.2193	2.9323	3.1415
0.7	θ	0.0174	0.7504	1.5009	2.5132	3.0019	3.1415
	dV/dθ	0.0052	0.2730	0.8026	2.1018	2.9045	3.1415
1.0	θ	0.0174	0.7504	1.5009	2.5132	3.0019	3.1415
	dV/dθ	0.0000	0.0684	0.5034	1.9254	2.8627	3.1415
1.2	θ	0.0174	0.7504	1.5009	2.5132	3.0019	3.1415
	dV/dθ	-0.0034	-0.0679	0.3039	1.8079	2.8349	3.1415
1.4	θ	0.0174	0.7504	1.5009	2.5132	3.0019	3.1415
	dV/dθ	-0.0069	-0.1725	0.1043	1.6903	2.8071	3.1415
1.6	θ	0.0174	0.7504	1.5009	2.5132	3.0019	3.1415
	dV/dθ	-0.0209	-0.3407	-0.0951	1.5728	2.7792	3.1415
1.8	θ	0.0174	0.7504	1.5009	2.5132	3.0019	3.1415
	dV/dθ	-0.0139	-0.4771	-0.2946	1.4552	2.7516	3.1415
2.0	θ	0.0174	0.7504	1.5009	2.5132	3.0019	3.1415
	dV/dθ	-0.0174	-0.6135	-0.4941	1.3376	2.7236	3.1415

TABLE A 3

TABULATED RESULTS FOR $d^2V/d\theta^2$ AND ANGULAR DISPLACEMENT. FIGURE 5

P						
0.1	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2V/d\theta^2$	0.9000	0.9268	0.9930	1.0809	1.0999
0.3	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2V/d\theta^2$	0.7000	0.7805	0.9790	1.2427	1.2999
0.5	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2V/d\theta^2$	0.5000	0.6343	0.9651	1.4046	1.5000
0.7	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2V/d\theta^2$	0.3000	0.4880	0.9511	1.5663	1.6999
1.0	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2V/d\theta^2$	0.0001	0.2686	0.9302	1.8090	2.0000
1.2	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2V/d\theta^2$	-0.1998	0.1223	0.9162	1.9708	2.1999
1.4	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2V/d\theta^2$	-0.3997	-0.0238	0.9023	2.1326	2.3999
1.6	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2V/d\theta^2$	-0.5997	-0.1701	0.8883	2.2944	2.5999
1.8	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2V/d\theta^2$	-0.7997	-0.3164	0.8744	2.4562	2.7999
2.0	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2V/d\theta^2$	-0.9997	-0.4627	0.8604	2.6180	3.0000

TABLE A 4

TABULATED RESULTS FOR ANGULAR DISPLACEMENT AND ANGULAR VELOCITY FOR P = 0.2. PHASE PLANE PLOT FIGURE A 1

Eo							
18	θ	0.0174	0.7504	1.5009	3.5081	4.5029	5.4800
	$\dot{\theta}$	5.9991	5.9613	5.8414	4.9461	4.0250	0.4230
15.2	θ	0.0174	0.7504	1.5009	3.5081	4.5029	5.4800
	$\dot{\theta}$	5.4773	5.4350	5.3072	4.2973	3.1944	0.2960
5	θ	0.0174	0.7504	1.5009	3.2812	-	-
	$\dot{\theta}$	3.1620	3.0896	2.8492	0.1722	-	-
3	θ	0.0174	0.7504	1.5009	2.5830	-	-
	$\dot{\theta}$	2.4494	2.3546	2.0295	0.2586	-	-
1.0	θ	0.0174	0.7504	1.5009	-	-	-
	$\dot{\theta}$	1.4141	1.2420	0.0968	-	-	-
0.5	θ	0.0174	0.7504	1.0995	-	-	-
	$\dot{\theta}$	0.9882	0.7371	0.0968	-	-	-
0.09	θ	0.0174	0.2443	0.4104	-	-	-
	$\dot{\theta}$	0.4235	0.3638	0.2250	-	-	-
0.009	θ	0.0174	0.0872	0.1396	-	-	-
	$\dot{\theta}$	0.1333	0.0961	0.0485	-	-	-

TABLE A 5

TABULATED RESULTS FOR ANGULAR DISPLACEMENT AND ANGULAR VELOCITY FOR P = 0.4. PHASE PLANE PLOT FIGURE A 2

Eo							
13	θ	0.0174	0.7504	1.5009	3.5081	4.5029	5.1310
	$\dot{\theta}$	5.0982	5.0644	4.9483	3.9033	2.5862	0.3800
8	θ	0.0174	0.7504	1.5009	3.5081	4.1364	-
	$\dot{\theta}$	3.9991	3.9564	3.8060	2.2891	0.3548	-
7.5	θ	0.0174	0.7504	1.5009	3.5081	4.0142	-
	$\dot{\theta}$	3.8726	3.8274	3.6671	2.0594	0.2200	-
6	θ	0.0174	0.7504	1.5009	3.5081	3.6651	-
	$\dot{\theta}$	3.4641	3.4135	3.2396	1.1131	0.2438	-
4	θ	0.0174	0.7504	1.5009	3.0019	3.0892	-
	$\dot{\theta}$	2.8283	2.7661	2.5477	0.7618	0.2357	-
2	θ	0.0174	0.7504	1.5009	-	-	-
	$\dot{\theta}$	1.9990	1.9106	1.5783	-	-	-
0.6	θ	0.0174	0.7504	1.0122	-	-	-
	$\dot{\theta}$	1.0952	0.9226	0.7423	-	-	-
0.08	θ	0.0174	0.2443	0.4537	-	-	-
	$\dot{\theta}$	0.3997	0.3522	0.1875	-	-	-
0.008	θ	0.0174	0.0523	0.1221	-	-	-
	$\dot{\theta}$	0.1252	0.1194	0.0838	-	-	-

TABLE A 6

TABULATED RESULTS FOR ANGULAR DISPLACEMENT AND ANGULAR VELOCITY FOR P = 0.6. PHASE PLANE PLOT FIGURE A 3

Eo							
15	θ	0.0174	0.7504	1.5009	3.5081	4.5029	5.0610
	$\dot{\theta}$	5.4410	5.4182	5.3352	4.4286	3.7824	0.3722
11	θ	0.0174	0.7504	1.5009	3.5081	4.5029	4.7996
	$\dot{\theta}$	4.6901	4.6642	4.5678	3.4661	1.7819	0.242
7.5	θ	0.0174	0.7504	1.5009	3.5081	4.0840	-
	$\dot{\theta}$	3.8473	3.8151	3.6960	2.1936	0.1603	-
7	θ	0.0174	0.7504	1.5009	3.5081	3.9968	-
	$\dot{\theta}$	3.7416	3.7044	3.5861	2.0032	0.1130	-
6.5	θ	0.0174	0.7504	1.5009	3.5081	3.8746	-
	$\dot{\theta}$	3.6050	3.5713	3.4441	2.5751	0.2814	-
3.0	θ	0.0174	0.7504	1.5009	2.8797	-	-
	$\dot{\theta}$	2.4494	2.3998	2.2052	0.2567	-	-
0.7	θ	0.0174	0.7504	1.5009	-	-	-
	$\dot{\theta}$	1.1813	1.0766	0.5130	-	-	-
0.007	θ	0.0174	0.5061	0.5759	-	-	-
	$\dot{\theta}$	0.3748	0.2042	0.0431	-	-	-
0.006	θ	0.0174	0.1047	0.1745	-	-	-
	$\dot{\theta}$	0.1173	0.0981	0.0425	-	-	-

TABLE A 7

TABULATED RESULTS FOR ANGULAR DISPLACEMENT AND ANGULAR VELOCITY FOR P = 0.8. PHASE PLANE PLOT FIGURE A 4

Eo							
13.8	θ	0.0174	0.7504	1.5009	3.5081	4.5029	5.131
	$\dot{\theta}$	5.2382	5.2256	5.1638	4.3451	3.2383	0.4441
7.7	θ	0.0174	0.7504	1.5009	3.5081	3.9793	-
	$\dot{\theta}$	3.9315	3.9253	3.8270	2.2852	0.1362	-
7.4	θ	0.0174	0.7504	1.5009	3.5081	4.0142	4.0666
	$\dot{\theta}$	3.8413	3.8236	3.7381	2.1873	0.7171	0.1592
6.8	θ	0.0174	0.7504	1.5009	3.5081	3.8397	-
	$\dot{\theta}$	3.5643	3.5443	3.4511	1.6690	0.2863	-
4	θ	0.0174	0.7504	1.5009	3.3335	-	-
	$\dot{\theta}$	2.8213	2.8043	2.6893	0.2400	-	-
0.8	θ	0.0174	0.7504	1.5009	-	-	-
	$\dot{\theta}$	1.2642	1.2113	0.9142	-	-	-
0.06	θ	0.0174	0.5061	0.7155	-	-	-
	$\dot{\theta}$	0.3461	0.2615	0.0201	-	-	-
0.006	θ	0.0174	0.1221	0.2792	-	-	-
	$\dot{\theta}$	0.1091	0.0940	0.0391	-	-	-

TABLE A 8

TABULATED RESULTS FOR ANGULAR DISPLACEMENT AND ANGULAR VELOCITY FOR P = Pcr. PHASE PLANE PLOT FIGURE A 5

Eo							
12.1	θ	0.0174	0.7504	1.5009	3.5081	4.5029	4.9567
	$\dot{\theta}$	4.9161	4.9160	4.8754	3.9811	2.4200	0.4833
10.5	θ	0.0174	0.7504	1.5009	3.5081	4.5029	4.7647
	$\dot{\theta}$	4.6821	4.6792	4.6292	3.6448	1.8115	0.4340
9	θ	0.0174	0.7504	1.5009	3.5081	4.5029	-
	$\dot{\theta}$	4.3421	4.3392	4.2946	3.1912	0.3738	-
7.5	θ	0.0174	0.7504	1.5009	3.5081	4.2236	-
	$\dot{\theta}$	3.9721	3.9695	3.9213	2.6611	0.3150	-
7.0	θ	0.0174	0.7505	1.5009	3.5081	0.3900	-
	$\dot{\theta}$	3.8400	3.8383	3.6882	2.3588	0.3900	-
5.8	θ	0.0174	0.7504	1.5009	3.5081	3.8746	-
	$\dot{\theta}$	3.5053	3.5021	3.4476	1.8771	0.2710	-
5	θ	0.0174	1.0122	2.0071	3.7000	-	-
	$\dot{\theta}$	3.2622	3.2488	3.0692	0.0734	-	-
2.5	θ	0.0174	1.0122	2.0071	-	-	-
	$\dot{\theta}$	2.3360	2.3170	2.0536	-	-	-
0.9	θ	0.0174	1.0122	2.0071	-	-	-
	$\dot{\theta}$	1.4411	1.4315	1.1865	-	-	-
0.05	θ	0.0174	0.7504	1.0122	-	-	-
	$\dot{\theta}$	0.4162	0.3225	0.2040	-	-	-
0.005	θ	0.1501	0.2617	0.5061	-	-	-
	$\dot{\theta}$	0.1501	0.0984	0.0673	-	-	-

TABLE A 9

TABULATED RESULTS FOR ANGULAR DISPLACEMENT AND ANGULAR VELOCITY FOR P = 1.2. PHASE PLANE PLOT FIGURE A 6

Eo							
10	θ	0.0174	0.7504	1.5009	3.5081	4.5029	
	$\dot{\theta}$	4.4720	4.4813	4.4691	3.5113	1.1696	0.4398
6	θ	0.0174	0.7504	1.5009	3.5081	3.96	-
	$\dot{\theta}$	3.4649	3.4753	3.4611	2.0815	0.0113	-
5	θ	0.0174	0.7504	1.5009	3.5081	3.769	-
	$\dot{\theta}$	3.1622	3.1754	3.1590	1.5270	0.3591	-
2	θ	0.0174	0.7504	1.5009	2.9496	-	-
	$\dot{\theta}$	2.0000	2.0202	1.9949	0.2360	-	-
1	θ	0.0174	0.7504	1.5009	2.5132	-	-
	$\dot{\theta}$	1.4142	1.4427	1.4069	0.1584	-	-
0.04	θ	0.0174	0.5061	0.7504	1.5707	-	-
	$\dot{\theta}$	0.2821	0.3536	0.4010	0.1126	-	-
0.004	θ	0.0174	0.5061	0.7504	1.4660	-	-
	$\dot{\theta}$	0.0896	0.2298	0.2999	0.0801	-	-
0.0002	θ	0.0174	0.5061	0.7504	1.4486	-	-
	$\dot{\theta}$	0.0161	0.2115	0.2858	0.0953	-	-
-0.044	θ	0.8377	0.9250	1.0122	1.2042	-	-
	$\dot{\theta}$	0.0036	0.1562	0.2021	0.0020	-	-
-0.048	θ	0.9250	1.0122	1.1170	-	-	-
	$\dot{\theta}$	0.0014	0.1308	0.0883	-	-	-

TABLE A 10

TABULATED RESULTS FOR ANGULAR DISPLACEMENT AND ANGULAR VELOCITY FOR P = 1.4. PHASE PLANE PLOT FIGURE A 7

Eo							
7	θ	0.0174	0.7504	1.5009	3.5081	4.0142	4.2411
	$\dot{\theta}$	3.7413	3.7668	3.7884	2.6692	1.5760	0.2898
5	θ	0.0174	0.7504	1.5009	3.5081	3.8571	-
	$\dot{\theta}$	3.1488	3.1726	3.2548	1.7521	0.1684	-
3	θ	0.0174	0.7504	1.5009	3.0019	3.3859	-
	$\dot{\theta}$	2.4493	2.4876	2.5203	1.6001	0.2783	-
1	θ	0.0174	0.7504	1.5009	2.5132	2.7052	-
	$\dot{\theta}$	1.4142	1.4795	1.5335	0.8652	0.1386	-
0.03	θ	0.0174	0.7504	1.5335	2.0071	-	-
	$\dot{\theta}$	0.2452	0.4981	0.6413	0.1216	-	-
0.003	θ	0.0174	0.5061	0.7506	0.9722	-	-
	$\dot{\theta}$	0.0781	0.3176	0.4414	0.1010	-	-
0.0002	θ	0.0174	0.5061	0.7506	0.9722	-	-
	$\dot{\theta}$	0.0223	0.3186	0.4350	0.0695	-	-
-0.06	θ	0.5759	0.7506	0.0122	1.8151	-	-
	$\dot{\theta}$	0.0025	0.2208	0.4046	0.0036	-	-
-0.056	θ	0.8028	1.0122	1.2566	0.7106	-	-
	$\dot{\theta}$	0.0061	0.2536	0.3510	0.0019	-	-

TABLE A 11

TABULATED RESULTS FOR ANGULAR DISPLACEMENT AND ANGULAR VELOCITY FOR P = 1.6. PHASE PLANE PLOT FIGURE A 8

Eo							
8	θ	0.0174	0.7504	1.5009	3.5081	4.0142	4.4680
	$\dot{\theta}$	4.0000	4.0363	4.0891	3.1438	2.2676	0.1030
6.6	θ	0.0174	0.7504	1.5009	3.5081	4.0142	4.2586
	$\dot{\theta}$	3.6873	3.7276	3.7849	2.7350	1.6563	0.2598
5	θ	0.0174	0.7504	1.5009	3.5081	3.9269	-
	$\dot{\theta}$	3.1620	3.2088	3.2747	1.9699	0.2036	-
3.3	θ	0.0174	0.7504	1.5009	3.5081	3.5779	-
	$\dot{\theta}$	2.6070	2.6631	2.7428	0.8249	0.3146	-
3	θ	0.0174	0.7504	1.5009	3.0019	3.4906	-
	$\dot{\theta}$	2.4495	2.5092	2.5930	1.8322	0.1494	-
1	θ	0.0174	0.7504	1.5009	2.5132	2.5830	-
	$\dot{\theta}$	1.4144	1.5153	1.6509	1.2139	0.2826	-
0.006	θ	0.0174	0.7504	1.5009	2.3212	-	-
	$\dot{\theta}$	0.2042	0.5803	0.8736	0.1846	-	-
0.002	θ	0.0174	0.7504	1.5009	2.3038	-	-
	$\dot{\theta}$	0.0648	0.5831	0.8531	0.1933	-	-
-0.08	θ	0.0174	0.7504	1.5009	2.2514	-	-
	$\dot{\theta}$	0.0031	0.0453	0.8593	0.1517	-	-
-0.056	θ	0.62	0.75	1.5	2.15	-	-
	$\dot{\theta}$	0.0013	0.2568	0.8290	0.1536	-	-
-0.0584	θ	0.7	1.0	1.5	-	-	-
	$\dot{\theta}$	0.0010	0.3761	0.5696	-	-	-

TABLE A 12

TABULATED RESULTS FOR ANGULAR DISPLACEMENT AND ANGULAR VELOCITY FOR P = 1.8. PHASE PLAN PLOT FIGURE A 9

Eo							
5.2	θ	0.0174	0.7504	1.5009	3.5081	4.0142	4.0317
	$\dot{\theta}$	3.2241	3.2868	3.3903	2.2486	0.4460	0.1044
3.8	θ	0.0174	0.7504	1.5009	3.5081	3.7524	-
	$\dot{\theta}$	2.7568	2.8291	2.9488	1.5013	0.2608	-
2.6	θ	0.0174	0.7504	1.5009	3.0019	3.4732	-
	$\dot{\theta}$	2.2801	2.3674	2.5098	1.8316	0.3751	-
0.01	θ	0.0174	0.7504	1.5009	2.5132	2.5830	-
	$\dot{\theta}$	0.1420	0.6511	1.0564	0.4643	0.0253	-
0.0004	θ	0.0174	0.7504	1.5009	2.5132	2.5656	-
	$\dot{\theta}$	0.0321	0.6364	1.0472	0.4433	0.1936	-
-0.056	θ	0.5061	0.7504	1.5009	2.2689	-	-
	$\dot{\theta}$	0.0034	0.4128	0.9341	0.0068	-	-
-0.058	θ	0.7504	1.5009	2.0071	-	-	-
	$\dot{\theta}$	0.0021	0.6532	0.5018	-	-	-

TABLE A 13

TABULATED RESULTS FOR ANGULAR DISPLACEMENT AND ANGULAR VELOCITY FOR P = 2.0. PHASE PLANE PLOT FIGURE A 10

Eo							
10	θ	0.0174	0.7504	1.5009	3.5008	4.5029	4.8345
	$\dot{\theta}$	4.4726	4.5281	4.6338	3.9275	2.1376	0.3732
3.6	θ	0.0174	0.7504	1.5009	3.5008	3.7873	-
	$\dot{\theta}$	2.6831	2.7764	2.9440	1.6203	0.2246	-
0.08	θ	0.0174	0.7504	1.5009	2.5008	2.8009	-
	$\dot{\theta}$	0.4001	0.8193	1.2758	1.0390	0.2147	-
0.008	θ	0.0174	0.7504	1.5009	2.5008	2.7750	-
	$\dot{\theta}$	0.1275	0.7261	1.2118	0.9674	0.2729	-
0.001	θ	0.0174	0.7504	1.5009	2.5008	2.7576	-
	$\dot{\theta}$	0.0223	0.7158	1.2116	0.9590	0.1864	-
-0.56	θ	0.4014	1.0129	1.5009	2.5132	2.7227	-
	$\dot{\theta}$	0.0024	0.8013	1.1006	0.7481	0.0015	-
-0.588	θ	0.6632	0.0129	1.5009	2.5132	2.6179	-
	$\dot{\theta}$	0.0015	0.5538	0.9163	0.3315	0.0020	-

TABLE A 14

TABULATED RESULTS FOR , P, dV, d^2V AND ANGULAR DISPLACEMENT. FIGURE 9

β	P	dV	d^2V	Comment
0.0	.00001	2.6179	0.0001	stable
	.0001	3.1066	0.0019	stable
	2.01	0.1047	-0.0259	unstable
	2.1	0.3141	-0.4266	unstable
	2.2	0.4363	-0.8766	unstable
	2.3	0.5235	-1.3387	unstable
	2.4	0.5934	-1.8216	unstable
	2.5	0.6283	-2.1137	unstable
	2.6	0.6806	-2.6356	unstable
	2.7	0.7330	-3.2509	unstable
	2.8	0.7853	-3.9597	unstable
	3.0	0.8377	-4.9555	unstable
	3.1	0.8726	-5.6403	unstable
	3.2	0.8901	-6.1566	unstable
	3.3	0.9750	-6.1566	unstable
	3.4	0.9474	-7.5691	unstable
	3.5	0.9599	-8.2047	unstable
	3.6	0.9948	-9.1926	unstable
	3.7	1.0122	-9.9226	unstable
	3.8	1.0297	-10.6934	unstable
	3.9	1.0471	-0.0011	unstable
	4.0	1.0646	-0.0012	unstable

TABLE A 14 continued

β	P	dV	d^2V	Comment
0.2	1.7	- 3.0543	- 6.4331	not defined stable
	1.8	- 3.0545	- 6.9996	not defined stable
	1.9	- 3.0717	- 7.5656	not defined stable
	2.0	- 3.089	- 8.1726	not defined stable
	2.1	- 3.071	- 8.7568	not defined stable
	2.2	0.5235 3.0892	-1.1905 9.4074	unstable stable
	2.3	0.5759 3.0892	-1.5062 10.0547	unstable stable
	2.4	0.6456 3.0892	-2.0457 10.7218	unstable stable
	2.5	0.6981 3.0892	-2.5449 11.4089	unstable stable
	2.6	0.7330 3.0892	-2.9577 12.1158	unstable stable
	2.7	0.7679 3.0892	-3.4300 12.8926	unstable stable
0.4	0.8	- 2.6354	- 2.3184	not defined stable
	1.7	- 2.9670	- 6.4640	not defined stable
	1.8	0.1745 2.9670	-0.1007 6.9899	unstable stable
	1.9	0.3490 2.9845	-0.405 7.5865	unstable stable
	2.0	0.4537 3.0019	-0.7325 8.2061	unstable stable

TABLE A 14 continued

β	P	dV	d^2V	Comment
0.4	2.1	0.5410	-1.1293	unstable
		3.0019	8.7982	stable
	2.5	0.7504	-2.8207	unstable
		3.0368	11.4860	stable
	3.1	0.9250	-5.9801	unstable
		3.0717	16.1013	stable
	3.2	0.9424	-6.7731	unstable
		3.0717	16.9751	stable
	3.3	0.9599	-7.1101	unstable
		3.0717	17.7679	stable
	3.4	0.9738	-7.7329	unstable
		3.0892	18.6873	stable
	3.5	0.9948	-8.6794	unstable
		3.0892	19.5732	stable
	3.6	1.0122	-9.0966	unstable
		3.0892	20.4163	stable
	3.7	1.0297	-9.8383	unstable
		3.0892	21.4048	stable
	3.8	1.0646	-10.9365	unstable
		3.0892	22.3504	stable
	3.9	1.0821	-11.7671	unstable
		3.0892	23.3151	stable
	4.0	1.0995	-12.6364	unstable
		3.0892	24.3013	stable
0.6	1.7	0.2617	-0.1813	unstable
		2.8623	6.3191	stable
	1.8	0.4014	-0.4621	unstable
		2.8972	6.9538	stable
	1.9	0.4886	-0.7365	unstable
		2.9146	7.547	stable
	2.0	0.5585	-1.0285	unstable
		2.9321	8.1369	stable

TABLE A 14 continued

β	P	dV	d^2V	Comment
0.6	2.1	0.6283	-1.3138	unstable
		2.9496	8.7245	stable
	2.2	0.6806	-1.8131	unstable
		2.9670	9.4812	stable
	2.3	0.7155	-2.1354	unstable
		2.9845	10.1744	stable
	2.4	0.7679	-2.1354	unstable
		2.9845	10.1744	stable
	2.5	0.8203	-3.2459	unstable
		2.9845	11.4825	stable
	2.6	0.8203	-3.4179	unstable
		3.0019	12.2475	stable
	2.7	0.8552	-3.9372	unstable
		3.0019	12.9550	stable
	3.1	0.9599	-6.2429	unstable
		3.0368	16.1660	stable
	3.2	1.0122	-7.2803	unstable
		3.0368	16.9811	stable
	3.3	0.9948	-7.4239	unstable
		3.0543	17.8994	stable
	3.4	1.0122	-8.0710	unstable
		3.0543	18.7584	stable
	3.5	1.0297	-8.7562	unstable
		3.0543	19.6372	stable
	3.6	1.0471	-9.4799	unstable
		3.0543	20.5356	stable
0.8	1.1	-	-	not defined
		2.4783	2.8263	stable
	1.2	-	-	not defined
		2.5481	3.3291	stable
	1.3	-	-	not defined
		2.6179	3.8966	stable

TABLE A 14 continued

β	P	dV	d^2V	Comment
0.8	1.4	-	-	not defined
		2.6703	4.446	stable
	1.5	0.2094	-0.0789	unstable
		2.6832	4.8813	stable
	1.6	0.3665	-0.2850	unstable
		2.7401	5.5239	stable
	1.7	0.4712	-0.5307	unstable
		2.7923	6.2088	stable
	1.8	0.5410	-0.7647	unstable
		2.8099	6.9032	stable
	2.01	0.6632	-1.3906	unstable
		2.8448	8.0904	stable
	2.1	0.7155	-1.7560	unstable
		2.8797	8.7690	stable
	2.2	0.7504	-2.0803	unstable
		2.8972	0.3427	stable
	3.1	0.9948	-6.4854	unstable
		3.0019	16.7710	stable
	3.2	1.0122	-7.0803	unstable
		3.0019	16.9810	stable
	3.3	1.0297	-7.7117	unstable
		3.0194	17.9173	stable
	3.4	1.0296	-7.8710	unstable
		3.0194	18.7651	stable
	3.5	1.0479	-8.824	unstable
		3.0193	19.3266	stable
	3.6	1.0645	-9.9031	unstable
		3.0192	20.5898	stable
0.9	1.1	-	-	not defined
		2.3736	2.5247	stable
	1.2	-	-	not defined
		2.4609	2.9885	stable

TABLE A 14 continued

β	P	dV	d^2V	Comment
0.9	1.3	-	-	not defined
		2.5481	3.6885	stable
	1.4	-	-	not defined
		2.6179	4.3048	stable
	1.5	-	-	not defined
		2.6522	4.8070	stable
	1.6	-	-	not defined
		2.6878	5.3522	stable
	1.7	0.5410	-0.6575	unstable
		2.7401	6.0372	stable
	1.8	0.6108	-0.94-7	unstable
		2.7576	6.5685	stable
	1.9	0.6632	-1.2213	unstable
		2.8099	7.3376	stable
	2.0	0.7155	-1.5621	unstable
		2.8774	7.9402	stable
0.98	1.1	-	-	not defined
		2.3212	2.3959	stable
	1.2	0.1765	-0.0303	unstable
		2.3910	2.8356	stable
	1.3	0.3141	-0.1255	unstable
		2.4783	3.4376	stable
	1.4	0.4014	-0.2358	unstable
		2.5481	4.0355	stable
	1.5	0.4886	-0.4106	unstable
		3.6005	4.6078	stable
	1.6	0.5585	-0.6152	unstable
		2.6529	5.2369	stable
	1.7	0.6108	-0.8166	unstable
		2.6703	5.7081	stable
	1.8	0.6632	-1.0730	unstable
		2.7227	6.4367	stable

TABLE A 14 continued

β	P	dV	d^2V	Comment
0.98	1.9	0.6981	-1.3060	unstable
		2.7576	7.1000	stable
	2.0	0.7504	-1.5589	unstable
		2.9670	7.8264	stable
0.99	1.1	0.1570	-0.0231	unstable
		2.2863	2.2739	stable
	1.2	0.2617	-0.0721	unstable
		2.3910	2.8546	stable
	1.3	0.3490	-0.0864	unstable
		2.4783	3.4566	stable
	1.4	0.4363	-0.2787	unstable
		2.5370	3.9629	stable
	1.5	0.5061	-0.4375	unstable
		2.5830	4.5294	stable
	1.6	0.5585	-0.5925	unstable
		2.6529	5.2559	stable
	1.7	0.6783	-0.8585	unstable
		2.6878	5.8341	stable
	1.8	0.6806	-1.1250	unstable
		2.7227	6.4557	stable
	1.9	0.7145	-1.4519	unstable
		2.7750	7.485	stable
1.0	1.0	-	-	not defined
		2.1467	1.6285	stable
	1.1	0.1398	-0.0154	unstable
		2.2863	2.2749	stable
	1.2	0.2617	-0.0711	unstable
		2.3910	2.8556	stable
	1.3	0.3665	-0.1713	unstable
		2.4783	3.4576	stable
	1.4	0.4363	-0.2777	unstable
		2.5307	3.9633	stable

TABLE A 14 continued

β	P	dV	d^2V	Comment
1.0	1.5	0.5061	-0.4315	unstable
		2.6000	4.6278	stable
	1.6	0.5759	-0.6424	unstable
		2.6529	5.2569	stable
	1.7	0.6457	-0.9187	unstable
		2.6878	5.8351	stable
	1.8	0.7185	-0.6326	unstable
		2.7221	6.4567	stable
	1.9	0.7330	-1.4466	unstable
		2.2401	7.0056	stable
	2.0	0.7504	-1.6463	unstable
		2.7750	7.0060	stable
	2.1	0.7853	-1.9698	unstable
		2.8099	8.4462	stable
	2.2	0.8203	-2.3384	unstable
		2.8274	9.1002	stable
	2.3	0.8552	-2.7560	unstable
		2.8448	9.7845	stable
	2.5	0.9075	-3.5903	unstable
		2.8992	11.3698	stable
	3.0	1.0122	-6.1248	unstable
		2.9696	15.2344	stable
	3.5	1.0821	-9.1364	unstable
		3.0019	19.7073	stable
	4.0	1.0646	-12.9532	unstable
		3.0019	24.3023	stable

TABLE A 15

TABULATED RESULTS FOR LOAD AND ANGULAR DISPLACEMENT. FIGURE 9

P						
0.0001	θ	0.0174	0.7504	1.5009	3.0019	3.1415
	P_1	0.00005	0.00005	0.00007	0.00107	0.00890
	θ	0.0174	0.5061	0.7504	1.2042	1.5533
	P_2	2.0002	2.2866	2.7345	5.5807	114.589
.1	θ	0.0174	0.7504	1.5009	3.0019	3.1415
	P_1	0.0513	0.0561	0.0754	0.7784	3.3475
	θ	0.0174	0.5061	0.7504	1.2042	1.5533
	P_2	1.9489	2.2332	2.6784	5.5155	114.511
.6	θ	0.0174	0.7504	1.5009	3.0019	3.1415
	P_1	0.2254	0.24139	0.3041	2.1098	7.5209
	θ	0.0174	0.5061	0.7504	1.2042	1.5533
	P_2	1.7748	2.0542	2.4932	5.3096	114.278
0.8	θ	0.0174	0.7504	1.5009	3.0019	3.1415
	P_1	0.5527	0.5513	0.6150	3.2849	11.0088
	θ	0.0174	0.5061	0.7504	1.2042	1.5533
	P_2	1.4475	1.7369	2.1833	5.0055	113.964
1.0	θ	0.0174	0.7504	1.5009	3.0019	3.1415
	P_1	0.9669	0.7619	0.7723	3.7629	12.4103
	θ	0.0174	0.5061	0.7504	1.2042	1.5533
	P_2	1.0033	1.4821	1.9726	4.8374	113.808

TABLE A 16

TABULATED RESULTS FOR $d\overset{*}{V}/d\theta$ AND ANGULAR DISPLACEMENT. FIGURE A 11

P						
2.0	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d\overset{*}{V}/d\theta$	0.0000	-0.7328	-3.7119	-4.2532	-0.0000
2.5	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d\overset{*}{V}/d\theta$	0.0217	-0.2926	-4.5538	-5.9109	-0.0000
3.1	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d\overset{*}{V}/d\theta$	0.0594	0.5649	-5.5161	-8.2141	-0.0000
3.5	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d\overset{*}{V}/d\theta$	0.0915	1.3336	-6.1304	-9.9397	-0.0000
4.0	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d\overset{*}{V}/d\theta$	0.1395	2.5245	-6.8671	12.3107	-0.0000

TABLE A 17

TABULATED RESULTS FOR $d^2\overset{*}{V}/d\theta^2$ AND ANGULAR DISPLACEMENT. FIGURE A 12

P						
2.0	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2\overset{*}{V}/d\theta^2$	-0.0018	-2.6463	-4.2401	4.4721	8.0000
2,5	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2\overset{*}{V}/d\theta^2$	1.2469	-3.2207	-6.5379	5.9764	11.2500
3.1	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2\overset{*}{V}/d\theta^2$	3.4050	-3.8640	-9.9489	7.9855	15.8099
3.5	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2\overset{*}{V}/d\theta^2$	5.2436	-4.2649	-12.6190	9.4485	19.2500
4.0	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2\overset{*}{V}/d\theta^2$	7.9914	-4.7347	-16.4023	11.4163	24.0000

TABLE A 18

TABULATED RESULTS FOR $dV^*/d\theta$ VS ANGULAR DISPLACEMENT. FIGURE A 13

P						
0.6	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$dV^*/d\theta$	-0.0076	-0.3386	-0.5716	0.1287	1.2565
1.0	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$dV^*/d\theta$	-0.0104	-0.5650	-1.3251	-0.6457	1.2566
2.0	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$dV^*/d\theta$	0.0069	-0.4326	-3.1115	-3.2479	1.2566
2.5	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$dV^*/d\theta$	0.0287	-0.0430	-3.9525	-4.9056	1.2566
3.1	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$dV^*/d\theta$	0.0664	0.8650	-4.9157	-7.2087	1.2565
3.5	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$dV^*/d\theta$	0.0985	1.6362	-5.5390	-8.9344	1.2565
4.0	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$dV^*/d\theta$	0.1465	2.82.47	-6.2667	-11.3054	1.2565

TABLE A 19

TABULATED RESULTS FOR $d^2\overset{*}{V}/d\theta^2$ VS ANGULAR DISPLACEMENT. FIGURE A 14

P						
0.6	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2\overset{*}{V}/d\theta^2$	-0.0044	-0.4525	-4.0206	1.4820	1.9599
1.0	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2\overset{*}{V}/d\theta^2$	-0.6003	-0.9929	-0.7297	2.3270	3.4000
2.0	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2\overset{*}{V}/d\theta^2$	0.3981	-2.2463	-3.8878	4.8721	8.4000
2.5	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2\overset{*}{V}/d\theta^2$	1.6469	-2.8207	-6.1908	6.3764	11.6500
3.1	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2\overset{*}{V}/d\theta^2$	3.8050	-3.4640	-9.5489	8.3855	16.2099
3.5	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2\overset{*}{V}/d\theta^2$	5.6436	-3.8649	-12.2190	9.8485	19.6499
4.0	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2\overset{*}{V}/d\theta^2$	8.3658	-4.3347	-16.0023	11.8163	24.3999

TABLE A 20

TABULATED RESULTS FOR $d\overset{*}{V}/d\theta$ AND ANGULAR DISPLACEMENT. FIGURE A 15

P						
1.1	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d\overset{*}{V}/d\theta$	0.0001	-0.1463	-0.6094	0.6447	3.1415
1.5	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d\overset{*}{V}/d\theta$	0.0043	-0.1732	-1.3351	-3.2002	3.1415
2.0	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d\overset{*}{V}/d\theta$	0.0174	0.0176	-2.2109	-1.7399	3.1415
3.1	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d\overset{*}{V}/d\theta$	0.0769	1.3153	-4.0151	-5.7008	3.1415
3.5	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d\overset{*}{V}/d\theta$	0.1090	2.0865	-4.6295	-7.4264	3.1415
4.0	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d\overset{*}{V}/d\theta$	0.1570	3.2750	-5.3661	-9.7974	3.1415

TABLE A 21

TABULATED RESULTS FOR $d^2\overset{*}{V}/d\theta^2$ AND ANGULAR DISPLACEMENT. FIGURE A 16

P						
1.1	θ	0.0174	0.07504	1.5009	2.5132	3.1415
	$d^2\overset{*}{V}/d\theta^2$	0.0095	-0.5245	-0.3516	3.1537	4.4099
1.5	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2\overset{*}{V}/d\theta^2$	0.2490	-0.1037	-0.1437	4.1223	6.2500
2.0	θ	0.0174	0.7504	1.5009	2.4132	3.1415
	$d^2\overset{*}{V}/d\theta^2$	0.09981	-1.6465	-3.2401	5.4721	9.0000
3.1	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2\overset{*}{V}/d\theta^2$	4.4050	-2.6029	-8.9489	8.9855	16.8099
3.5	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2\overset{*}{V}/d\theta^2$	6.2436	-3.2649	-11.6190	10.4483	20.2500
4.0	θ	0.0174	0.7504	1.5009	2.5132	3.1415
	$d^2\overset{*}{V}/d\theta^2$	8.9914	-3.7347	-15.4023	12.4163	25.0000

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