ELASTIC STABILITY

OF

THIN CYLINDRICAL SHELLS

BY

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ABSTRACT

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The purpose of this thesis is the analysis of forced vibration of thin cylindrical shells including effects of transverse shear, rotary inertia, and in-plane stability forces.

The solution of the free vibration problem is formulated for the usual classical boundary conditions. The orthogonality conditions of the free vibration mode shapes are obtained. The forced and free boundary conditions are determined as an inherent part of the orthogonality conditions.

The analysis includes both symmetric and asymmetric motion of the shell. The forced vibration is solved in Duhamel integral form which allows for the application of any arbitrary static or dynamic surface loading.

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LIST OF SYMBOLS

SYMBOLS	DEFINITIONS
a =	Mean radius of the shell
$a_{mn}(t)$ =	An arbitrary function of time
$A_{mn}, B_{mn}, C_{mn}, D_{mn}$ =	Numerical coefficients
D =	Flexural rigidity of the shell, $[D = Eh^3/12(1-\mu^2)]$.
E =	Modulus of elasticity
G =	Shear modulus, $[G = E/2(1+\mu)]$
h =	Thickness of the shell
L =	Length of the shell
m, n, p, q =	An integers
$M_x, M_\theta, M_{x\theta}, M_{\theta x}$ =	Bending and twisting moments
$N_x, N_\theta, N_{x\theta}, N_{\theta x}$ =	In-plane axial and in-plane shear forces
Q_x, Q_θ =	Shearing forces in shell
p_x, p_θ, p_z =	Axial, tangential and normal components of surface stress traction respectively
m_x, m_θ =	Surface bending moment traction in X-Z and θ-Z planes respectively
U, V, W =	Axial, tangential and normal components of displacement of the middle surface of the shell
Φ, Ψ =	Angle of rotation of normal to middle surface in X-Z and θ-Z planes respectively
x, θ, z =	Cylindrical coordinates

t = Time variable

μ = Poisson's ratio

ρ = Mass density per unit volume

Ω = Natural frequency of free vibration, when in-plane forces are included

ω = Natural frequency of free vibration, when in-plane forces are neglected

λ = Scalar parameter

$\beta = \rho \frac{(1-\mu^2)}{E}$

$K = \beta \Omega^2_{mn}$ when in-plane forces are included

$\bar{K} = \beta \omega^2_{mn}$ when in-plane forces are neglected

$S = \frac{5(1-\mu)}{12}$

$\alpha = \frac{h^2}{12a^2}$

$\eta = \frac{\lambda}{L}.$

$p = \frac{m\pi a}{L}$

$\bar{N}_x = \frac{(1-\mu^2)}{Eh} N_x$

LIST OF FIGURES

CHAPTER-I

INTRODUCTION

The elastic stability problem of the thin cylindrical shells is analysed using a nonlinear theory of the stability for thin elastic shells as formulated by Archer[1]. This theory includes the effects of transverse shear, rotary inertia, and in-plane stability forces.

Kraus[3] presents the theory of free and forced vibration of cylindrical shells on basis of Donnell type analysis, neglecting the effect of shear, rotary inertia, and stability forces.

Harrmann and Armenakes[2] consider the linear theory for thin elastic cylindrical shells which includes the effects of transverse shear, rotary inertia as a special case of Flugge theory for shells, but neglects the effect of in-plane stability forces.

The object of this thesis is to formulate the mathematical solutions for free and forced vibration of the cylindrical shells together with the orthogonality conditions of the mode shapes, and the associated free and natural boundary conditions.

CHAPTER-II

METHOD OF ANALYSIS

The analysis of the forced vibration of thin cylindrical shells including effects of in-plane forces is carried out according to the following assumptions:

1. The thickness of the thin elastic shells is assumed uniform and small when compared with radius of curvature, that is, terms of order higher than $\alpha = \frac{h^2}{12a^2}$ are dropped relative to unity.
2. Lines which are normal to the middle surface before deformation do not remain normal to the middle surface after deformation (i.e. the effect of shear deformation is accounted for).
3. Linear elastic stress-strain relationships are assumed to hold.
4. The in-plane force N_θ is negligible in comparison to applied axial force N_x .
5. All nonlinear terms are omitted from the equations of motion except stability term ($N_x W_{xx}$).

2.1 EQUATIONS OF MOTION:

The equations of motion for forced vibration of the thin cylindrical shells including the effects of transverse shear, rotary inertia, and in-plane stability forces are formulated by Archer[1] in orthogonal curvilinear coordinates. If these equations are transformed in to cylindrical polar coordinates, and using the notations of figures 2.1a and 2.1b, the following equations of motion are obtained.

$$\left.\begin{aligned}
&N_{x,x} + \tfrac{1}{a} N_{\theta x,\theta} + P_x = \rho h (U_{tt} + \alpha a \Phi_{tt}), \\
&\tfrac{1}{a} N_{\theta,\theta} + N_{x\theta,x} + \tfrac{1}{a} Q_\theta + P_\theta = \rho h (V_{tt} + \alpha a \Psi_{tt}), \\
&Q_{x,x} + \tfrac{1}{a} Q_{\theta,\theta} - \tfrac{1}{a^2} N_\theta + N_x W_{xx} + \tfrac{1}{a^2} N_\theta W_{\theta\theta} + P_z \\
&+ N_{x,x} W_x + \tfrac{1}{a^2} N_{\theta,\theta} W_\theta + \tfrac{1}{a^2}(1+\alpha)\left[(N_{x\theta} W_\theta)_x + (N_{x\theta} W_x)_\theta\right] \\
&- \tfrac{1}{a^2}\left[(M_{x\theta} W_\theta)_x + (M_{x\theta} W_x)_\theta\right] = \rho h W_{tt}, \\
&M_{x,x} + \tfrac{1}{a} M_{\theta x,\theta} - Q_x + m_x = \rho \tfrac{h^3}{12}\left(\tfrac{1}{a} U_{tt} + \Phi_{tt}\right),
\end{aligned}\right] \quad (1)$$

and

$$\tfrac{1}{a} M_{\theta,\theta} + M_{x\theta,x} - Q_\theta + m_\theta = \rho \tfrac{h^3}{12}\left(\tfrac{1}{a} V_{tt} + \Psi_{tt}\right).$$

Neglecting the products of the stability forces with derivatives of displacements (except the term $N_x W_{xx}$), one obtains from equation (1):

$$N_{x,x} + \tfrac{1}{a} N_{\theta x,\theta} + P_x = \rho h (U_{tt} + \alpha a \Phi_{tt}),$$

$$\frac{1}{a}N_{\theta,\theta} + N_{x\theta,x} + \frac{1}{a}Q_\theta + p_\theta = \rho h(V_{tt} + \alpha a \Psi_{tt}),$$

$$Q_{x,x} + \frac{1}{a}Q_{\theta,\theta} - \frac{1}{a^2}N_\theta + N_x W_{xx} + p_z = \rho h W_{tt},$$

$$M_{x,x} + \frac{1}{a}M_{\theta x,\theta} - Q_x + m_x = \rho \frac{h^3}{12}(\frac{1}{a}U_{tt} + \Phi_{tt}),$$

and

$$\frac{1}{a}M_{\theta,\theta} + M_{x\theta,x} - Q_\theta + m_\theta = \rho \frac{h^3}{12}(\frac{1}{a}V_{tt} + \Psi_{tt}), \tag{2}$$

where

$$Q_x = \frac{5Gh}{6}[W_x + \Phi], \quad Q_\theta = \frac{5Gh}{6}[\frac{1}{a}(V + W_\theta) + \Psi],$$

$$N_x = \frac{12D}{h^2}[U_x + \frac{\mu}{a}(V_\theta + W) + \alpha a \Phi_x],$$

$$N_\theta = \frac{12D}{h^2}[\frac{1}{a}(V_\theta + W) + \mu U_x - \alpha \Psi_\theta],$$

$$N_{x\theta} = Gh[V_x + \frac{1}{a}U_\theta + \alpha a \Psi_x],$$

$$N_{\theta x} = Gh[V_x + \frac{1}{a}U_\theta - \alpha \Phi_\theta],$$

$$M_x = D[\Phi_x + \frac{1}{a}(U_x + \mu \Psi_\theta)],$$

$$M_\theta = D[\mu \Phi_x - \frac{1}{a}\{\frac{1}{a}(V_\theta + W) - \Psi_\theta\}],$$

$$M_{x\theta} = (\frac{1-\mu}{2})D[\frac{1}{a}(V_x + \Phi_\theta) + \Psi_x],$$

and

$$M_{\theta x} = (\frac{1-\mu}{2})D[\frac{1}{a}(U_\theta + \Phi_\theta) + \Psi_x], \tag{3}$$

where X and θ subscripts on U, V, W, Φ and Ψ denotes differentiation with respect to X and θ respectively;

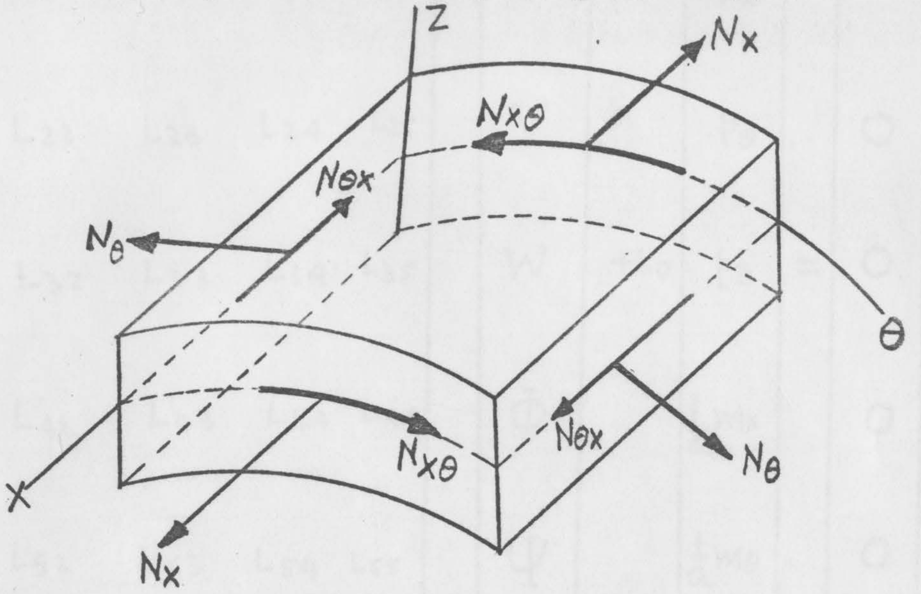


Fig. 2.1a

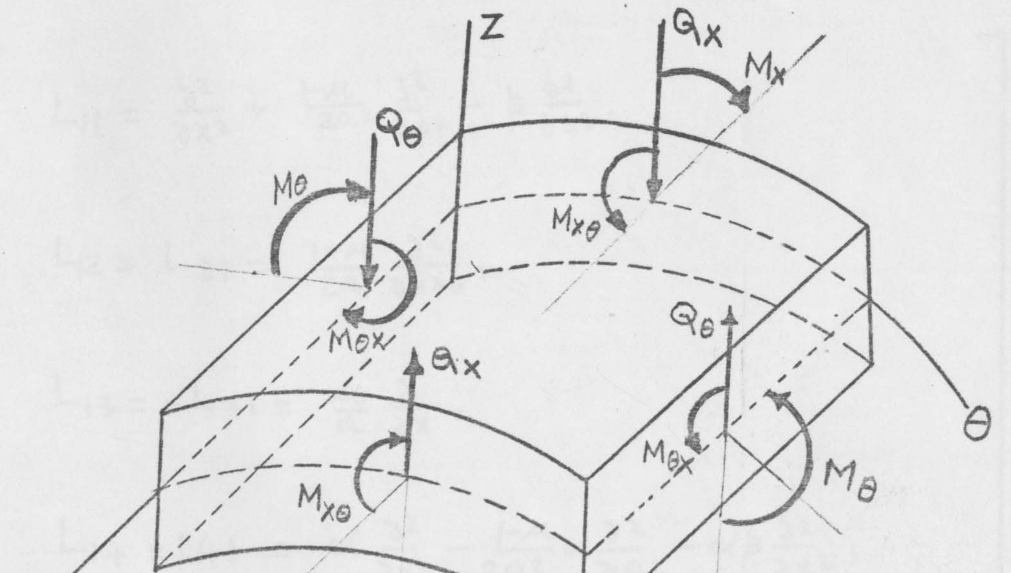


Fig. 2.1b

Substitution of equation (3) in to equation (2), yields the following five homogeneous partial differential equations in matrix form:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25} \\ L_{31} & L_{32} & L_{33} & L_{34} & L_{35} \\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45} \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} \end{bmatrix} \begin{bmatrix} U \\ V \\ W \\ \Phi \\ \Psi \end{bmatrix} + L_0 \begin{bmatrix} p_x \\ p_\theta \\ p_z \\ \frac{1}{a} m_x \\ \frac{1}{a} m_\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad -(4)$$

where

$$L_{11} = \frac{\partial^2}{\partial x^2} + \frac{1-\mu}{2a^2} \cdot \frac{\partial^2}{\partial \theta^2} - \beta \frac{\partial^2}{\partial t^2},$$

$$L_{12} = L_{21} = \frac{1+\mu}{2a} \frac{\partial^2}{\partial x \partial \theta},$$

$$L_{13} = -L_{31} = \frac{\mu}{a} \frac{\partial}{\partial x},$$

$$L_{14} = L_{41} = \alpha \frac{\partial^2}{\partial x^2} - \frac{1-\mu}{2a^2} \alpha \frac{\partial^2}{\partial \theta^2} - \alpha\beta \frac{\partial^2}{\partial t^2},$$

$$L_{15} = L_{51} = 0,$$

$$L_{22} = \frac{1-\mu}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{a^2} \frac{\partial^2}{\partial \theta^2} - \frac{s}{a^2} - \beta \frac{\partial^2}{\partial t^2},$$

$$
\left.
\begin{aligned}
L_{23} &= -L_{32} = \frac{1}{a^2}(1+s)\frac{\partial}{\partial \theta}, \\
L_{24} &= L_{42} = 0, \\
L_{25} &= L_{52} = \frac{1-\mu}{2}\alpha\frac{\partial^2}{\partial x^2} - \frac{1}{a^2}\alpha\frac{\partial^2}{\partial \theta^2} + \frac{s}{a^2} - \alpha\beta\frac{\partial^2}{\partial t^2}, \\
L_{33} &= (\bar{N}_x + s)\frac{\partial^2}{\partial x^2} + \frac{s}{a^2}\frac{\partial^2}{\partial \theta^2} - \frac{1}{a^2} - \beta\frac{\partial^2}{\partial t^2}, \\
L_{34} &= -L_{43} = \frac{s}{a}\frac{\partial}{\partial x}, \\
L_{35} &= L_{53} = \frac{1}{a^2}(\alpha+s)\frac{\partial}{\partial \theta}, \\
L_{44} &= \alpha\frac{\partial^2}{\partial x^2} + \frac{1-\mu}{2a^2}\alpha\frac{\partial^2}{\partial \theta^2} - \frac{s}{a^2} - \alpha\beta\frac{\partial^2}{\partial t^2}, \\
L_{45} &= +L_{54} = \frac{1+\mu}{2a}\alpha\frac{\partial^2}{\partial x \partial \theta}, \\
L_{55} &= \frac{1-\mu}{2}\alpha\frac{\partial^2}{\partial x^2} + \frac{\alpha}{a^2}\frac{\partial^2}{\partial \theta^2} - \frac{s}{a^2} - \alpha\beta\frac{\partial^2}{\partial t^2},
\end{aligned}
\right\} \quad (5)
$$

and

$$L_0 = \frac{1-\mu^2}{Eha}.$$

2.2 FREE VIBRATION ANALYSIS:

For free vibration, the eigenvalues and eigenfunctions are found by setting

$$\left.\begin{aligned}
& p_x(x,\theta,t) = 0, \\
& p_\theta(x,\theta,t) = 0, \\
& p_z(x,\theta,t) = 0, \\
& m_x(x,\theta,t) = 0, \\
\text{and} \quad & m_\theta(x,\theta,t) = 0,
\end{aligned}\right\} \quad (6)$$

and by defining the following free vibration form as :

$$\left.\begin{aligned}
U_{mn}(x,\theta,t) &= \hat{u}_{mn}(x,\theta)\, e^{i\Omega_{mn}t}, \\
V_{mn}(x,\theta,t) &= \hat{V}_{mn}(x,\theta)\, e^{i\Omega_{mn}t}, \\
W_{mn}(x,\theta,t) &= \hat{W}_{mn}(x,\theta)\, e^{i\Omega_{mn}t}, \\
\Phi_{mn}(x,\theta,t) &= \hat{\phi}_{mn}(x,\theta)\, e^{i\Omega_{mn}t},
\end{aligned}\right\} \quad (7)$$

and

$$\Psi_{mn}(x,\theta,t) = \hat{\psi}_{mn}(x,\theta)\, e^{i\Omega_{mn}t},$$

where

m = Longitudinal mode index,
  = 1,2,3,...,

n = Circumferential mode index,
  = 0,1,2,...,

and also,

$$\hat{U}_{mn}(x,\theta) = U_{mn}(x)\,\mathrm{Cos}\,n\theta,$$

$$\hat{V}_{mn}(x,\theta) = V_{mn}(x)\,\mathrm{Sin}\,n\theta,$$

$$\hat{W}_{mn}(x,\theta) = W_{mn}(x)\,\mathrm{Cos}\,n\theta, \qquad (8)$$

$$\hat{\phi}_{mn}(x,\theta) = \phi_{mn}(x)\,\mathrm{Cos}\,n\theta,$$

and

$$\hat{\psi}_{mn}(x,\theta) = \psi_{mn}(x)\,\mathrm{Sin}\,n\theta.$$

Substitution of equation (5), (7), and (8) in to equation (4), yields a set of five simultaneous, linear, ordinary, total, differential equations with constant coefficients given in matrix form as:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix} \begin{bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ \phi_{mn} \\ \psi_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad -(9)$$

where

$$A_{11} = \frac{\partial^2}{\partial x^2} - \left(\frac{1-\mu}{2a^2} n^2 - K \right),$$

$$A_{12} = -A_{21} = \frac{1+\mu}{2a} n \frac{\partial}{\partial x},$$

$$A_{13} = -A_{31} = \frac{\mu}{a} \frac{\partial}{\partial x},$$

$$A_{14} = A_{41} = \alpha \frac{\partial^2}{\partial x^2} + \left(\alpha \frac{1-\mu}{2a^2} n^2 + \alpha K \right),$$

$$A_{15} = A_{51} = 0,$$

$$A_{22} = \frac{1-\mu}{2} \frac{\partial^2}{\partial x^2} - \left(\frac{n^2}{a^2} + \frac{S}{a^2} - K \right),$$

$$
\left.
\begin{aligned}
A_{23} &= A_{32} = -\frac{1}{a^2}(1+S)n, \\
A_{24} &= A_{42} = 0, \\
A_{25} &= A_{52} = \frac{1-\mu}{2}\alpha\frac{\partial^2}{\partial x^2} + \left(\frac{\alpha n^2}{a^2} + \frac{S}{a^2} + \alpha K\right), \\
A_{33} &= (\bar{N}_x + S)\frac{\partial^2}{\partial x^2} - \left(\frac{S}{a^2}n^2 + \frac{1}{a^2} - K\right), \\
A_{34} &= -A_{43} = \frac{S}{a}\frac{\partial}{\partial x}, \\
A_{35} &= A_{53} = \frac{1}{a^2}(\alpha + S)n, \\
A_{44} &= \alpha\frac{\partial^2}{\partial x^2} - \left(\frac{1-\mu}{2a^2}\alpha n^2 + \frac{S}{a^2} - \alpha K\right), \\
A_{45} &= -A_{54} = \frac{1+\mu}{2a}\alpha n\frac{\partial}{\partial x},
\end{aligned}
\right\} \quad (10)
$$

and

$$A_{55} = \frac{1-\mu}{2}\alpha\frac{\partial^2}{\partial x^2} - \left(\alpha\frac{n^2}{a^2} + \frac{S}{a^2} - \alpha K\right).$$

For specific set of boundary conditions, equations (9) are satisfied by five fold infinity of eigenvalues Ω^2_{mn} for each combination of m and n. The group of five frequency equations may be obtained by setting specific mode shapes in to equations (9), and equating determinant of the coefficients of the modal constants to zero. The resulting equation is of order five in Ω^2_{mn}. The modal constants associated with each frequency are then obtained up to common, constant multiple of one another by substituting the values of Ω^2_{mn} in to equations (9).

For a specific set of boundary conditions, the mode shape functions are determined by an additional separation of variables in the following form:

$$
\left.
\begin{aligned}
U_{mn}(x) &= A_{mn}\, e^{qx}, \\
V_{mn}(x) &= B_{mn}\, e^{qx}, \\
W_{mn}(x) &= C_{mn}\, e^{qx}, \\
\phi_{mn}(x) &= D_{mn}\, e^{qx}, \\
\text{and} \quad & \\
\psi_{mn}(x) &= E_{mn}\, e^{qx}, \\
\text{where} \quad & \\
q &= \frac{\lambda}{L}\,.
\end{aligned}
\right] \quad (11)
$$

Substituting equation (11) in to equation (9), yields a set of five linear algebraic simultaneous equations which take the following matrix form:

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} \\ B_{21} & B_{22} & B_{23} & B_{24} & B_{25} \\ B_{31} & B_{32} & B_{33} & B_{34} & B_{35} \\ B_{41} & B_{42} & B_{43} & B_{44} & B_{45} \\ B_{51} & B_{52} & B_{53} & B_{54} & B_{55} \end{bmatrix} \begin{bmatrix} A_{mn} \\ B_{mn} \\ C_{mn} \\ D_{mn} \\ E_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad -(12)$$

where

$$B_{11} = q^2 - \frac{1-\mu}{2a^2} n^2 + K,$$

$$B_{12} = -B_{21} = \frac{1+\mu}{2a} qn,$$

$$B_{13} = -B_{31} = \frac{\mu}{a} q,$$

$$B_{14} = B_{41} = \alpha q^2 + \frac{1-\mu}{2a^2} \alpha n^2 + \alpha K,$$

$$B_{15} = B_{51} = 0,$$

$$B_{22} = \frac{1-\mu}{2} q^2 - \frac{n^2}{a^2} - \frac{s}{a^2} + K,$$

$$
\left.
\begin{aligned}
B_{23} &= B_{32} = -\frac{1}{a^2}(1+s)n, \\
B_{24} &= B_{42} = 0, \\
B_{25} &= B_{52} = \frac{1-\mu}{2}\alpha q^2 + \frac{\alpha}{a^2}n^2 + \frac{s}{a^2} + \alpha K, \\
B_{33} &= (\bar{N}x+s)q^2 - \frac{s}{a^2}n^2 - \frac{1}{a^2} + K, \\
B_{34} &= -B_{43} = \frac{s}{a}q, \\
B_{35} &= B_{53} = \frac{1}{a^2}(\alpha+s)n, \\
B_{44} &= \alpha q^2 - \frac{1-\mu}{2a^2}\alpha n^2 - \frac{s}{a^2} + \alpha K, \\
B_{45} &= -B_{54} = \frac{1+\mu}{2a}\alpha n q,
\end{aligned}
\right\} \quad (13)
$$

and

$$
B_{55} = \frac{1-\mu}{2}\alpha q^2 - \alpha\frac{n^2}{a^2} - \frac{s}{a^2} + \alpha K.
$$

For the nontrivial solution for constants A_{mn}, B_{mn}, C_{mn}, D_{mn}, E_{mn}, the determinant of coefficients in equation (12) must to zero. The algebraic expansion of this condition (given in appendix A), yield a tenth order equation in parameter $\frac{\lambda}{L}$ which may be solved for the ten roots of $\frac{\lambda_i}{L}$ ($i = 1,2,3,\ldots,10.$). These roots when substituted back to equation (11), yields the result :

$$
\left.
\begin{aligned}
U_{mn}(x) &= \sum_{i=1}^{10} A_{imn}\, e^{q_i x}, \\
V_{mn}(x) &= \sum_{i=1}^{10} B_{imn}\, e^{q_i x}, \\
W_{mn}(x) &= \sum_{i=1}^{10} C_{imn}\, e^{q_i x}, \\
\phi_{mn}(x) &= \sum_{i=1}^{10} D_{imn}\, e^{q_i x}, \\
\text{and} \quad & \\
\psi_{mn}(x) &= \sum_{i=1}^{10} E_{imn}\, e^{q_i x}, \\
\text{where} \quad & \\
q_i &= \frac{\lambda_i}{L} .
\end{aligned}
\right] \quad (14)
$$

A combination of ten natural and forced boundary conditions, five on each edge, are applied to equations (14). This result yields the mode shapes of free vibration.

2.3 ORTHOGONALITY CONDITIONS:

The orthogonality conditions on eigenfunctions or mode shapes are obtained by using free vibration form of equation (2). Considering equation (7), it follows from the form of equations (2) through (5), that the bending moments, twisting moments, and shear forces are also harmonic in time. Therefore, applying the subscripts mn to each moment, shear force, and displacement term of the free vibration form of equation (2), and multiplying these resulting equations by $\hat{U}_{pq}$, $\hat{V}_{pq}$, $\hat{W}_{pq}$, $\hat{\phi}_{pq}$, and $\hat{\psi}_{pq}$ respectively, and integrating these equations over the surface area of the cylinder, yields after proper algebraic summation a single equation. If the subscripts of this equation are interchanged and these two equations are subtracted, one obtains

$$
\begin{aligned}
(\Omega_{mn}^2-\Omega_{pq}^2)\int_0^{2\pi}\!\!\int_0^L \Big[& \rho h a(\hat{U}_{mn}\hat{U}_{pq}+\hat{V}_{mn}\hat{V}_{pq}+\hat{W}_{mn}\hat{W}_{pq}) \\
& +\rho \frac{h^3}{12} a(\hat{\phi}_{mn}\hat{\phi}_{pq}+\hat{\psi}_{mn}\hat{\psi}_{pq})+\rho\frac{h^3}{12}(\hat{\phi}_{mn}\hat{U}_{pq}+\hat{\phi}_{pq}\hat{U}_{mn}+\hat{\psi}_{mn}\hat{V}_{pq}+\hat{\psi}_{pq}\hat{V}_{mn})\Big]a\,dx\,d\theta \\
=\int_0^{2\pi}\Big|\, a\Big[& N_{xpq}\hat{U}_{mn}-N_{xmn}\hat{U}_{pq}+M_{xpq}\hat{\phi}_{mn}-M_{xmn}\hat{\phi}_{pq} \\
& +M_{x\theta pq}\hat{\psi}_{mn}-M_{x\theta mn}\hat{\psi}_{pq}+N_{x\theta pq}\hat{V}_{mn} \\
& -N_{x\theta mn}\hat{V}_{pq}+(Q_{xpq}+N_{xpq}\hat{W}_{xpq})\hat{W}_{mn}
\end{aligned}
$$

$$- (Q_{xmn} + N_{xmn}\hat{W}_{xmn})\hat{W}_{pq} \Big] \Big|_{x=0}^{x=L} a\,d\theta$$

$$+\int_0^{2\pi}\int_0^{L} \Big[aN_{xmn}\hat{U}_{pq} - aN_{xpq}\hat{U}_{mn} + N_{x\theta mn}\hat{U}_{pq}$$

$$- N_{x\theta pq}\hat{U}_{mn} + aM_{xmn}\hat{\phi}_{xpq} - aM_{xpq}\hat{\phi}_{xmn}$$

$$+ M_{\theta xmn}\hat{\phi}_{\theta pq} - M_{\theta xpq}\hat{\phi}_{\theta mn} + N_{\theta mn}\hat{V}_{\theta pq}$$

$$- N_{\theta pq}\hat{V}_{\theta mn} + aN_{x\theta mn}\hat{V}_{xpq} - aN_{x\theta pq}\hat{V}_{xmn}$$

$$+ M_{\theta mn}\hat{\Psi}_{\theta pq} - M_{\theta pq}\hat{\Psi}_{\theta mn} + aM_{x\theta mn}\hat{\Psi}_{xpq} \qquad (15)$$

$$- aM_{x\theta pq}\hat{\Psi}_{xmn} + N_{\theta mn}\hat{W}_{pq} - N_{\theta pq}\hat{W}_{mn}$$

$$- N_{\theta pq}\hat{W}_{\theta mn}\hat{W}_{\theta pq} - N_{\theta mn}\hat{W}_{\theta pq}\hat{W}_{\theta mn} + aQ_{xmn}\hat{\phi}_{pq}$$

$$- aQ_{xpq}\hat{\phi}_{mn} + Q_{\theta pq}\hat{V}_{mn} - Q_{\theta mn}\hat{V}_{pq}$$

$$+ aQ_{\theta mn}\hat{\Psi}_{pq} - aQ_{\theta pq}\hat{\Psi}_{mn} + aQ_{xmn}\hat{W}_{xpq}$$

$$- aQ_{xpq}\hat{W}_{xmn} + Q_{\theta mn}\hat{W}_{\theta pq} - Q_{\theta pq}\hat{W}_{\theta mn} \Big] a\,dx\,d\theta .$$

$$+\int_0^L \Big| \Big[a N_{x\theta pq} \hat{U}_{mn} - a N_{x\theta mn} \hat{U}_{pq} + a M_{x\theta pq} \hat{\phi}_{mn} - a M_{x\theta mn} \hat{\phi}_{pq} + a N_{\theta pq} \hat{V}_{mn} - a N_{\theta mn} \hat{V}_{pq} + a M_{\theta pq} \hat{\psi}_{mn} - a M_{\theta mn} \hat{\psi}_{pq} + a Q_{\theta pq} \hat{W}_{mn} - a Q_{\theta mn} \hat{W}_{pq} + N_{\theta pq} \hat{W}_{mn} \hat{\omega}_{\theta pq} - N_{\theta mn} \hat{W}_{pq} \hat{\omega}_{\theta mn} \Big] \Big|_{\theta=0}^{\theta=2\pi} dx.$$

The last term on right hand side of equation (15) is equal to zero by direct substitution of the limits shown. The sum of second integral is also equal to zero by noting the free vibration form of equations (2) and (3). This reduces the form of equation (15) to the following:

$$\left.\begin{aligned} &(\Omega^2_{mn} - \Omega^2_{pq}) \int_0^{2\pi}\int_0^{L} \Big[\rho h a (\hat{U}_{mn}\hat{U}_{pq} + \hat{V}_{mn}\hat{V}_{pq} + \hat{W}_{mn}\hat{W}_{pq}) \\ &+ \rho \frac{h^3}{12} a (\hat{\phi}_{mn}\hat{\phi}_{pq} + \hat{\psi}_{mn}\hat{\psi}_{pq}) + \rho \frac{h^3}{12} (\hat{\phi}_{mn}\hat{U}_{pq} + \hat{\phi}_{pq}\hat{U}_{mn} + \hat{\psi}_{mn}\hat{V}_{pq} + \hat{\psi}_{pq}\hat{V}_{mn}) \Big] a\,dx\,d\theta \\ &= \int_0^{2\pi} \Big| a \Big[N_{xpq}\hat{U}_{mn} - N_{xmn}\hat{U}_{pq} + M_{xpq}\hat{\phi}_{mn} - M_{xmn}\hat{\phi}_{pq} \\ &+ M_{x\theta pq}\hat{\psi}_{mn} - M_{x\theta mn}\hat{\psi}_{pq} + N_{x\theta pq}\hat{V}_{mn} - N_{x\theta mn}\hat{V}_{pq} \\ &+ (Q_{xpq} + N_{xpq}\hat{\omega}_{xpq})\hat{W}_{mn} - (Q_{xmn} + N_{xmn}\hat{\omega}_{xmn})\hat{W}_{pq} \Big] \Big|_{x=0}^{x=L} a\,d\theta. \end{aligned}\right\} \quad (16)$$

The integral on right hand side of equation (16) contains terms which form the natural and forced boundary conditions.

The right hand side of equation (16) is identically equal to zero, if the following conditions hold at $X=0$ and $X=L$:

$$
\left.
\begin{array}{lll}
\text{Either} & a N_x(x,\theta) = 0 & \text{or} \quad \hat{u}(x,\theta)=0, \\
\text{Either} & a N_{x\theta}(x,\theta) = 0 & \text{or} \quad \hat{v}(x,\theta)=0, \\
\text{Either} & a Q_x(x,\theta) + a N_x(x,\theta)\hat{W}_x(x,\theta) = 0 & \text{or} \quad \hat{w}(x,\theta)=0, \\
\text{Either} & a M_x(x,\theta) = 0 & \text{or} \quad \hat{\phi}(x,\theta)=0, \\
\text{and} & & \\
\text{Either} & a M_{x\theta}(x,\theta) = 0 & \text{or} \quad \hat{\psi}(x,\theta)=0.
\end{array}
\right\} \quad (17)
$$

The following conditions hold for the usual boundary conditions at $X=0$ and $X=L$:

<u>Simple Support:</u>

$$\hat{u}(x,\theta)=0,$$

$$a N_{x\theta}(x,\theta)=0,$$

$$\hat{w}(x,\theta)=0,$$

$$aM_x(x,\theta)=0,$$

and

$$aM_{x\theta}(x,\theta)=0.$$

Clamped:

$$\hat{u}(x,\theta)=0,$$

$$\hat{v}(x,\theta)=0,$$

$$\hat{w}(x,\theta)=0, \qquad -(18)$$

$$\hat{\phi}(x,\theta)=0,$$

and

$$\hat{\psi}(x,\theta)=0.$$

Free:

$$\hat{u}(x,\theta)=0,$$

$$aN_{x\theta}(x,\theta)=0,$$

or $$a[Q_x(x,\theta)+N_x(x,\theta)\hat{W}_x(x,\theta)]=0,$$

$$aM_x(x,\theta)=0,$$

or $$aM_{x\theta}(x,\theta)=0$$

ing to the boundary conditions given in

case , the right hand side of equation (16) is

, or

$$\rho h a(\hat{U}_{mn}\hat{U}_{pq}+\hat{V}_{mn}\hat{V}_{pq}+\hat{W}_{mn}\hat{W}_{pq})$$
$$+\hat{\psi}_{mn}\hat{\psi}_{pq})+\rho\frac{h^3}{12}(\hat{\Phi}_{mn}\hat{U}_{pq}+\hat{\Phi}_{pq}\hat{U}_{mn}+\hat{\psi}_{mn}\hat{V}_{pq}$$
$$+\hat{\psi}_{pq}\hat{V}_{mn})]\,a\,dx\,d\theta=0 \qquad (19)$$

he case where $mn \neq pq$ (i.e. $\Omega_{mn}^2 \neq \Omega_{pq}^2$),

$$+\hat{V}_{mn}\hat{V}_{pq}+\hat{W}_{mn}\hat{W}_{pq})+\rho\frac{h^3}{12}a(\hat{\Phi}_{mn}\hat{\Phi}_{pq}+\hat{\Psi}_{mn}\hat{\Psi}_{pq})$$
$$+\hat{\Phi}_{pq}\hat{U}_{mn}+\hat{\Psi}_{mn}\hat{V}_{pq}+\hat{\Psi}_{pq}\hat{V}_{mn})]\,a\,dx\,d\theta=0 \qquad (20)$$

2.4 FORCED VIBRATION ANALYSIS:

The solution of the forced vibration problem is assumed in the form of a double infinite series as follow:

$$
\left.
\begin{aligned}
U_{mn}(x,\theta,t) &= \sum_m \sum_n \hat{U}_{mn}(x,\theta)\, a_{mn}(t), \\
V_{mn}(x,\theta,t) &= \sum_m \sum_n \hat{V}_{mn}(x,\theta)\, a_{mn}(t), \\
W_{mn}(x,\theta,t) &= \sum_m \sum_n \hat{W}_{mn}(x,\theta)\, a_{mn}(t), \\
\Phi_{mn}(x,\theta,t) &= \sum_m \sum_n \hat{\phi}_{mn}(x,\theta)\, a_{mn}(t),
\end{aligned}
\right]
$$

and

$$
\Psi_{mn}(x,\theta,t) = \sum_m \sum_n \hat{\psi}_{mn}(x,\theta)\, a_{mn}(t), \qquad (21)
$$

where

$a_{mn}(t)$ is an arbitrary function of time to be determined.

Noting equation (21), it follows from equations (1) through (4) that the bending moments, twisting moments, and shear forces are similar in the form of equations (21). Substituting equations (21) in to equations (1) through (4) and utilization of the free vibration form of equation (1), gives the following results:

$$\left.\begin{aligned}
&\sum_m \sum_n \left[\rho h a \left(\hat{U}_{mn} + \alpha a \hat{\phi}_{mn} \right) \left(\ddot{a}_{mn}(t) + \Omega_{mn}^2 a_{mn}(t) \right) \right] = p_x(x,\theta,t), \\
&\sum_m \sum_n \left[\rho h a \left(\hat{V}_{mn} + \alpha a \hat{\psi}_{mn} \right) \left(\ddot{a}_{mn}(t) + \Omega_{mn}^2 a_{mn}(t) \right) \right] = p_\theta(x,\theta,t), \\
&\sum_m \sum_n \left[\rho h a \, \hat{W}_{mn} \left(\ddot{a}_{mn}(t) + \Omega_{mn}^2 a_{mn}(t) \right) \right] = p_z(x,\theta,t), \\
&\sum_m \sum_n \left[\rho \frac{h^3 a}{12} \left(\frac{1}{a} \hat{U}_{mn} + \hat{\phi}_{mn} \right) \left(\ddot{a}_{mn}(t) + \Omega_{mn}^2 a_{mn}(t) \right) \right] = M_x(x,\theta,t), \\
&\text{and} \\
&\sum_m \sum_n \left[\rho \frac{h^3}{12} a \left(\frac{1}{a} \hat{V}_{mn} + \hat{\psi}_{mn} \right) \left(\ddot{a}_{mn}(t) + \Omega_{mn}^2 a_{mn}(t) \right) \right] = M_\theta(x,\theta,t).
\end{aligned}\right] \text{—(22)}$$

Multiplying equations (22) by $\hat{U}_{pq}$, $\hat{V}_{pq}$, $\hat{W}_{pq}$, $\hat{\phi}_{pq}$, and $\hat{\psi}_{pq}$ respectively, integrating each equation over the surface area of the cylinder, adding and making use of orthogonality conditions, yields

$$\ddot{a}_{mn}(t) + \Omega_{mn}^2 a_{mn}(t) = \frac{1}{I_{mn}(x,\theta)} \int_0^{2\pi} \int_0^{L} \left[E_{mn}(x,\theta,t) \right] a \, dx \, d\theta , \qquad \text{—(23)}$$

where

$$\left.\begin{aligned}
I_{mn}(x,\theta) = \int_0^{2\pi} \int_0^{L} \Big[&\rho h a \left(\hat{U}_{mn}^2 + \hat{V}_{mn}^2 + \hat{W}_{mn}^2 \right) \\
&+ \rho \frac{h^3 a}{12} \left(\hat{\phi}_{mn}^2 + \hat{\psi}_{mn}^2 \right) + 2\rho \frac{h^3}{12} \left(\hat{\phi}_{mn} \hat{U}_{mn} + \hat{\psi}_{mn} \hat{V}_{mn} \right) \Big] a \, dx \, d\theta ,
\end{aligned}\right] \text{—(24)}$$

and

$$E_{mn}(x,\theta,t) = p_x(x,\theta,t)\,\hat{u}_{mn}(x,\theta) + p_\theta(x,\theta,t)\,\hat{v}_{mn}(x,\theta) + p_z(x,\theta,t)\,\hat{w}_{mn}(x,\theta) + m_x(x,\theta,t)\,\hat{\phi}_{mn}(x,\theta) + m_\theta(x,\theta,t)\,\hat{\psi}_{mn}(x,\theta). \tag{25}$$

The integral on the right hand side of equation (23) represents the work-done by the external loads in the mn^{th} mode. The solution of equation (23) is determined by using the Lagrange variation of parameter method, and takes the form:

$$a_{mn}(t) = C_{mn}\cos(\Omega_{mn}t) + D_{mn}\sin(\Omega_{mn}t) + \frac{1}{\Omega_{mn} I_{mn}(x,\theta)} \int_{\tau=0}^{\tau=t} \left[\int_0^{2\pi}\int_0^{L} \{E_{mn}(x,\theta,t)\}\, a\,dx\,d\theta \right] \sin\Omega_{mn}(t-\tau)\,d\tau. \tag{26}$$

The arbitrary constants C_{mn} and D_{mn} are determined by noting the conditions of linear and angular displacement, and linear and angular velocity defined by equation (21) at time $t = 0$. Using Leibtniz's rule and noting the orthogonality conditions defined by equation (20), there results

$$a_{mn}(0) = \frac{J_{mn}(x,\theta,0)}{I_{mn}(x,\theta)}, \tag{27}$$

and

$$\dot{a}_{mn}(0) = \frac{\dot{J}_{mn}(x,\theta,0)}{I_{mn}(x,\theta)}, \tag{28}$$

where

$$\begin{aligned} J_{mn}(x,\theta,0) = \int_0^{2\pi}\int_0^{L} \Big[& \rho h a\{u_{mn}\hat{u}(x,\theta,0) + v_{mn}\hat{v}(x,\theta,0) + \\ & \hat{w}_{mn}\hat{w}(x,\theta,0)\} + \rho\frac{h^3}{12}a\{\phi_{mn}\hat{\phi}(x,\theta,0) + \psi_{mn}\hat{\psi}(x,\theta,0)\} \\ & + \rho\frac{h^3}{12}\{\hat{\phi}_{mn}\hat{u}(x,\theta,0) + \hat{u}_{mn}\hat{\phi}(x,\theta,0) + \hat{\psi}_{mn}\hat{v}(x,\theta,0) + \hat{v}_{mn}\hat{\psi}(x,\theta,0)\}\Big] a\,dx\,d\theta , \end{aligned} \tag{29}$$

and

$$\begin{aligned} \dot{J}_{mn}(x,\theta,0) = \int_0^{2\pi}\int_0^{L} \Big[& \rho h a\{\hat{u}_{mn}\dot{\hat{u}}(x,\theta,0) + \hat{u}_{mn}\dot{\hat{v}}(x,\theta,0) + \\ & \hat{w}_{mn}\dot{\hat{w}}(x,\theta,0)\} + \rho\frac{h^3}{12}a\{\hat{\phi}_{mn}\dot{\hat{\phi}}(x,\theta,0) + \hat{\psi}_{mn}\dot{\hat{\psi}}(x,\theta,0)\} \\ & + \rho\frac{h^3}{12}\{\hat{\phi}_{mn}\dot{\hat{u}}(x,\theta,0) + \hat{u}_{mn}\dot{\hat{\phi}}(x,\theta,0) + \hat{\psi}_{mn}\dot{\hat{v}}(x,\theta,0) + \hat{v}_{mn}\dot{\hat{\psi}}(x,\theta,0)\}\Big] a\,dx\,d\theta . \end{aligned} \tag{30}$$

The dot refers to differentiation with respect to time. Using equations (27) and (28), the solution for the parameters C_{mn} and D_{mn} are given as:

$$C_{mn} = a_{mn}(0), \tag{31}$$

and

$$D_{mn} = \frac{1}{\Omega_{mn}} \dot{a}_{mn}(0). \tag{32}$$

The complete solution to the differential equation (23) is written as:

$$\begin{aligned} a_{mn}(t) = {} & a_{mn}(0) \operatorname{Cos}(\Omega_{mn} t) + \frac{1}{\Omega_{mn}} \dot{a}_{mn}(0) \operatorname{Sin}(\Omega_{mn} t) \\ & + \frac{1}{\Omega_{mn} I_{mn}(x,\theta)} \int_{\tau=0}^{\tau=t} \left[\int_0^{2\pi} \int_0^{L} \left\{ E_{mn}(x,\theta,t) \right\} a\, dx\, d\theta \right] \operatorname{Sin} \Omega_{mn}(t-\tau)\, d\tau. \end{aligned} \tag{33}$$

The last integral is called the Duhamel integral, and the parameters $a_{mn}(0)$ and $\dot{a}_{mn}(0)$ are given by equations (27) and (28) respectively.

2.5 ILLUSTRATIVE EXAMPLE:

Considering the case of simple support at each end of the cylindrical shell, and the solution to the equation (9) is assumed in the form:

$$
\left.\begin{aligned}
U_{mn}(x) &= A_{mn} \cos \frac{m\pi}{L} x, \\
V_{mn}(x) &= B_{mn} \sin \frac{m\pi}{L} x, \\
W_{mn}(x) &= C_{mn} \sin \frac{m\pi}{L} x, \\
\Phi_{mn}(x) &= D_{mn} \cos \frac{m\pi}{L} x,
\end{aligned}\right] \quad (34)
$$

and

$$
\Psi_{mn}(x) = E_{mn} \sin \frac{m\pi}{L} x .
$$

Substituting equations (34) in to equation (9) and using notation

$$
\left.\frac{m\pi a}{L} = p,\right] \quad (35)
$$

the following five equations are obtained in the matrix form:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} \end{bmatrix} \begin{bmatrix} A_{mn} \\ B_{mn} \\ C_{mn} \\ D_{mn} \\ E_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad -(36)$$

where

$$C_{11} = -\frac{p^2}{a^2} - \frac{1-\mu}{2a^2} n^2 + K,$$

$$C_{12} = C_{21} = \frac{1+\mu}{2a^2} np,$$

$$C_{13} = C_{31} = \frac{\mu}{a^2} p,$$

$$C_{14} = C_{41} = -\alpha\frac{p^2}{a^2} + \frac{1-\mu}{2a^2}\alpha n^2 + \alpha K,$$

$$C_{15} = C_{51} = 0,$$

$$C_{22} = -\frac{1-\mu}{2a^2} p^2 - \frac{n^2}{a^2} - \frac{s}{a^2} + K,$$

$$\left.\begin{aligned}
C_{23} &= C_{32} = -\frac{1}{a^2}(1+s)n, \\
C_{24} &= C_{42} = 0, \\
C_{25} &= C_{52} = -\frac{1-\mu}{2a^2}\alpha p^2 + \frac{\alpha n^2}{a^2} + \frac{s}{a^2} + \alpha K, \\
C_{33} &= -(\bar{N}_x + s)\frac{p^2}{a^2} - \frac{s}{a^2}n^2 - \frac{1}{a^2} + K, \\
C_{34} &= C_{43} = -\frac{s}{a^2}p, \\
C_{35} &= C_{53} = \frac{1}{a^2}(1+\alpha)n, \\
C_{44} &= -\frac{\alpha}{a^2}p^2 - \alpha\frac{1-\mu}{2a^2}n^2 - \frac{s}{a^2} + \alpha K, \\
C_{45} &= C_{54} = \frac{1+\mu}{2a^2}\alpha p n,
\end{aligned}\right\} \quad (37)$$

and

$$C_{55} = -\frac{1-\mu}{2a^2}\alpha p^2 - \alpha\frac{n^2}{a^2} - \frac{s}{a^2} + \alpha K.$$

For the non-trivial solution, the determinant of coefficients in equation (36) must vanish, The algebraic expansion of this condition (given in appendix B), yields fifth order equation in Ω^2_{mn}. For each combination of m and n , five eigenvalues for Ω^2_{mn} can be found, corrosponding to five natural frequencies associated with the mode shapes of the free vibration.

Considering symmetric motions only (i.e. $n = 0$), the fifth order equation reduces to a cubic equation and a quadratic equation given respectively as,

$$K^3 - d_2 K^2 + d_1 K - d_0 = 0, \qquad (38)$$

and

$$C_2 K^2 - C_2 K + C_0 = 0, \qquad (39)$$

where

$$d_2 = \frac{1}{\alpha a^2}\left\{ \alpha p^2 (2 + \bar{N}_x) + s(1 + \alpha p^2)\right\},$$

$$d_1 = \frac{1}{\alpha a^4}\left\{ \alpha p^2 [p^2 (1 + 2\bar{N}_x) + 2 - \mu^2] + s[p^2\{1 + 2\alpha(p^2 - \mu) + \bar{N}_x]\right\},$$

$$d_0 = \frac{p^2}{\alpha a^6}\left\{ (\alpha p^2 + s)(1 - \mu^2) - \alpha s p^2 (2\mu - p^2) + \bar{N}_x(\alpha p^2 + s)p^2\right\},$$

$$C_2 = \alpha a^4,$$

$$C_1 = a^2 \left\{\alpha(1 - \mu) p^2 + s(1 + 3\alpha)\right\}, \qquad (40)$$

and

$$C_0 = \left(\frac{1-\mu}{2}\right) p^2 \left\{ \alpha\left(\frac{1-\mu}{2}\right) p^2 + s(1 + 3\alpha)\right\}.$$

The roots of equations (38) and (39), in view of requirement that the natural frequency of free vibration must be real quantities, the latter equation will possess five, real, positive, unequal roots, which can be expressed as:

$$\left.\begin{aligned}
K_1 &= \frac{1}{a^2}\left[\left(\frac{1-\mu}{2}\right)p^2\right], \\
K_2 &= \frac{1}{a^2}\left[\left(\frac{1-\mu}{2}\right)p^2 + \frac{5}{\alpha}(1+3\alpha)\right], \\
K_3 &= 2R^{\frac{1}{3}}\,\mathrm{Cos}\left(\frac{\theta}{3}\right) + \frac{d_2}{3}, \\
K_4 &= 2R^{1/3}\,\mathrm{Cos}\left(\frac{\theta+2\pi}{3}\right) + \frac{d_2}{3}, \\
&\text{and} \\
K_5 &= 2R^{1/3}\,\mathrm{Cos}\left(\frac{\theta+4\pi}{3}\right) + \frac{d_2}{3},
\end{aligned}\right] \tag{41}$$

where

$$\left.\begin{aligned}
R &= \left[-\frac{1}{27}\left(d_1 - \frac{1}{3}d_2^2\right)^3\right]^{\frac{1}{2}}, \\
&\text{and} \\
\theta &= \mathrm{Cos}^{-1}\left[\frac{1}{2R}\left(d_0 - \frac{1}{3}d_1 d_2 + \frac{2}{27}d_2^3\right)\right].
\end{aligned}\right] \tag{42}$$

Equation (41) includes the effect of in-plane stability forces. If these forces are neglected, the five roots of natural frequency equations are written as:

$$\left.\begin{aligned}
\overline{K}_1 &= \frac{1}{a^2}\left[\left(\frac{1-\mu}{2}\right)p^2\right],\\
\overline{K}_2 &= \frac{1}{a^2}\left[\left(\frac{1-\mu}{2}\right)p^2 + \frac{S}{\alpha}(1+3\alpha)\right],\\
\overline{K}_3 &= 2r^{1/3}\cos\left(\frac{\hat{\theta}}{3}\right) + \frac{e_2}{3},\\
\overline{K}_4 &= 2r^{1/3}\cos\left(\frac{\hat{\theta}+2\pi}{3}\right) + \frac{e_2}{3},\\
&\text{and}\\
\overline{K}_5 &= 2r^{1/3}\cos\left(\frac{\hat{\theta}+4\pi}{3}\right) + \frac{e_2}{3},
\end{aligned}\right] \tag{43}$$

where

$$\left.\begin{aligned}
r &= \left[-\frac{1}{27}\left(e_1 - \frac{1}{3}e_2^2\right)^3\right]^{\frac{1}{2}},\\
\hat{\theta} &= \cos^{-1}\left[\frac{1}{2r}\left(e_0 - \frac{1}{3}e_1e_2 + \frac{2}{27}e_2^3\right)\right],\\
e_2 &= \frac{1}{\alpha a^2}\left\{2\alpha p^2 + S(1+\alpha p^2)\right\},\\
e_1 &= \frac{1}{\alpha a^4}\left\{\alpha p^2(p^2+2-\mu^2) + S\left[p^2\left\{1+2\alpha(p^2-\mu)\right\}\right]\right\},\\
&\text{and}\\
e_0 &= \frac{p^2}{\alpha a^6}\left\{(\alpha p^2+S)(1-\mu^2) - \alpha S p^2(2\mu-p^2)\right\}.
\end{aligned}\right] \tag{44}$$

Solving for in-plane stability force, setting natural frequency equal zero in to equation (38), where $n = 0$ (i.e. for symmetric motion only), gives

$$\bar{N}_x = N_x \frac{(1-\mu^2)}{Eh} = -\left[\frac{1-\mu^2}{p^2} + \alpha \frac{p^2 - 2\mu}{1 + \frac{\alpha}{s} p^2}\right]. \quad -(45)$$

One obtained the smallest value of equation (45) using condition $\frac{dN_x}{dp}$ equal to zero, which yields

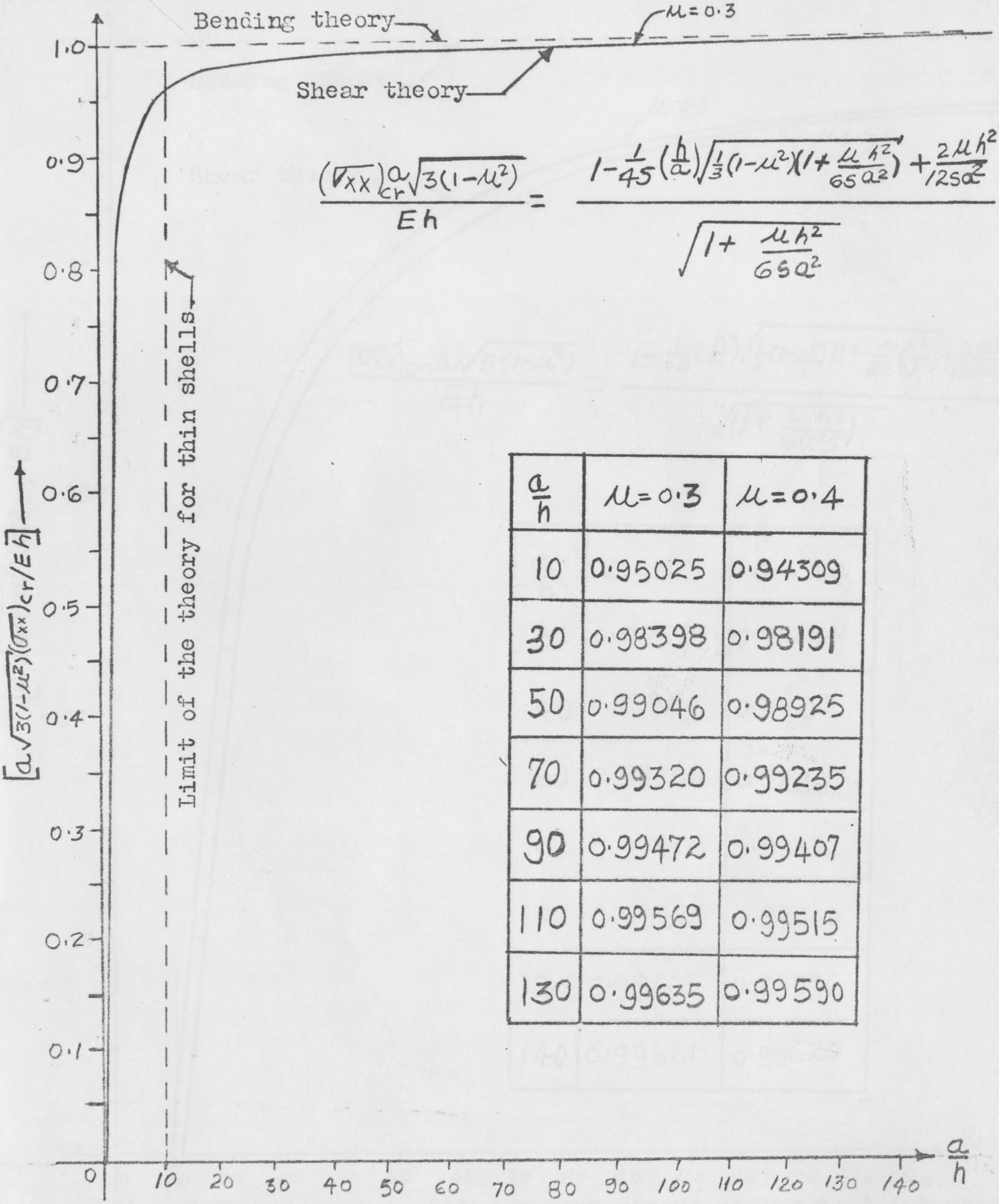
$$p^2 = \frac{2a}{h}\left[\frac{\sqrt{3(1-\mu^2)}}{\sqrt{1 + \frac{h^2}{6a^2 s}\mu} - \frac{h}{2as}\sqrt{\frac{1}{3}(1-\mu^2)}}\right]. \quad -(46)$$

Substitution of equation (46) in to equation (45), gives the critical buckling load as

$$(N_x)_{cr} = -\frac{Eh^2}{a}\left[\frac{1 - \frac{1}{4a}\left[\left(\frac{h}{s} + \frac{2\mu}{E}\right)\sqrt{\frac{1}{3}(1-\mu^2)\left(1 + \frac{h^2}{6a^2 s}\mu\right)} - \frac{2h^2\mu}{3as}\right]}{\sqrt{3(1-\mu^2)\left(1 + \frac{h^2}{6a^2 s}\mu\right)}}\right], \quad -(47)$$

from which the critical buckling stress is given as

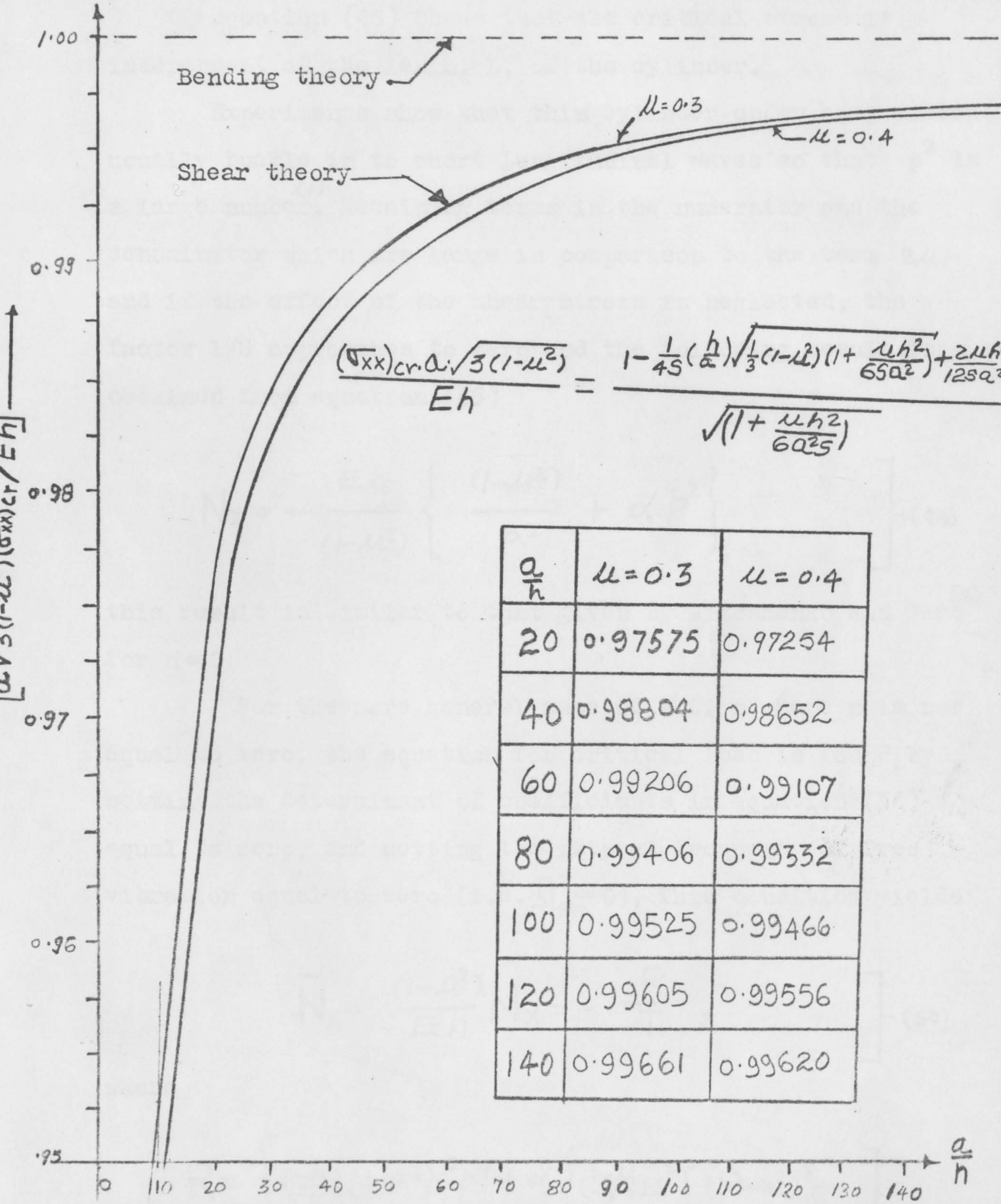
$$(\sigma_{xx})_{cr} = \frac{(N_x)_{cr}}{h} = -\frac{Eh}{a}\left[\frac{1 - \frac{1}{4a}\left[\left(\frac{h}{s} + \frac{2\mu}{E}\right)\sqrt{\frac{1}{3}(1-\mu^2)\left(1 + \frac{h^2\mu}{6a^2 s}\right)} - \frac{2h^2\mu}{3as}\right]}{\sqrt{3(1-\mu^2)\left(1 + \frac{h^2}{6a^2 s}\mu\right)}}\right]. \quad -(48)$$



$$\frac{(\sigma_{xx})_{cr}\, a\sqrt{3(1-\mu^2)}}{Eh} = \frac{1-\frac{1}{45}\left(\frac{h}{a}\right)\sqrt{\frac{1}{3}(1-\mu^2)\left(1+\frac{\mu h^2}{65a^2}\right)}+\frac{2\mu h^2}{125a^2}}{\sqrt{1+\frac{\mu h^2}{65a^2}}}$$

$\frac{a}{h}$	$\mu=0.3$	$\mu=0.4$
10	0.95025	0.94309
30	0.98398	0.98191
50	0.99046	0.98925
70	0.99320	0.99235
90	0.99472	0.99407
110	0.99569	0.99515
130	0.99635	0.99590

Fig. 2.5a critical buckling stress versus radius to thickness ratio.



$$\frac{(\sigma_{xx})_{cr}\cdot a\cdot\sqrt{3(1-\mu^2)}}{Eh} = \frac{1-\frac{1}{45}\left(\frac{h}{a}\right)\sqrt{\frac{1}{3}(1-\mu^2)\left(1+\frac{\mu h^2}{65a^2}\right)}+\frac{2\mu h^2}{125a^2}}{\sqrt{\left(1+\frac{\mu h^2}{6a^2 5}\right)}}$$

$\frac{a}{h}$	$\mu = 0.3$	$\mu = 0.4$
20	0.97575	0.97254
40	0.98804	0.98652
60	0.99206	0.99107
80	0.99406	0.99332
100	0.99525	0.99466
120	0.99605	0.99556
140	0.99661	0.99620

Fig. 2.5b critical buckling stress versus radius to thickness ratio.

Equation (48) shows that the critical stress is independent of the lengh, L, of the cylinder.

Experiments show that thin cylinder under compression usually buckle in to short longitudinal waves so that p^2 is a large number[4]. Retaining terms in the numerator and the denominator which are large in comparison to the term 2μ, and if the effect of the shear stress is neglected, the factor 1/S approaches to zero and the following result is obtained from equation (45)

$$N_x = -\frac{Eh}{(1-\mu^2)}\left\{\frac{(1-\mu^2)}{p^2} + \alpha p^2\right\}, \tag{49}$$

this result is similar to that given by Timoshenko and Gere[4] for $n = 0$.

For the more general case of motion where n is not equal to zero, the equation for critical load is found by setting the determinant of coefficients in equation (36) equal to zero, and setting the natural frequency of free vibration equal to zero (i.e. $\Omega_{mn}^2 = 0$), This condition yields

$$\bar{N}_x = \frac{(1-\mu^2)}{Eh}N_x = -\frac{R}{T}, \tag{50}$$

where

$$R = \Big[(\tfrac{1-\mu}{2})(1-\mu^2)s^2p^4 + \alpha\Big\{(\tfrac{1-\mu}{2})s^2\big[(p^2+n^2)^4 -$$

$$\left\{ 2\mu + \tfrac{1}{4}(3+\mu)(1-\mu)n^2 \right\} p^6 + \tfrac{1}{2}(1+\mu)\left\{ \tfrac{1}{2}(3+\mu)n^2 - 2\mu \right\} n^2 p^5$$

$$+ \left\{ \left[\tfrac{1}{2}(1-\mu^2)n^2 + (1-\mu)\mu \right] + 3(1-\mu^2) \right\} n^2 p^4 - \left\{ 1 + \tfrac{1}{2}(1+\mu)n^2 \right\} n p^3$$

$$+ \left\{ \left[1 + \tfrac{1}{4}(1+\mu)^2 \right] n^2 + (1-\mu)(1+n^2) + 6n^4 \right\} n^2 p^2 - \tfrac{1}{2}(1-\mu) n^5 p$$

$$+ \left\{ 2(1-\mu^2) + \tfrac{1}{2}(3+\mu) n + \tfrac{3}{2}(1-\mu) n^2 \right\} n^4 \Big]$$

$$+ \tfrac{1}{2}(1-\mu) S \Big[(1-\mu^2) p^6 + \tfrac{1}{4}(1+\mu)(1-3\mu) n^2 p^5$$

$$+ (1-\mu^2) \left\{ 2 + \tfrac{1}{4}(1-\mu)(3-\mu) \right\} n^2 p^4 - \left[1 + \tfrac{1}{2}(1-\mu)n^2 \right] n p^3$$

$$+ \left\{ \tfrac{1}{4}(1-\mu)(3-\mu) - \mu^2 \right\} n^2 p^2 - \tfrac{1}{2}(1-\mu) n^5 p + \tfrac{1}{2}(3-\mu) n^6 \Big] \qquad (51)$$

$$+ S \Big[(1-\mu^2) n^2 p^4 + \tfrac{1}{2}(1+\mu) \left\{ \tfrac{1}{2}(1-2\mu)(1-\mu) n - 1 \right\} n^3 p^3$$

$$+ (1-\mu^2) n^4 p^2 \Big] + S^2 \Big[n^2 p^6 - \tfrac{1}{8}(1+\mu)^3 n^4 p^5$$

$$+ \left\{ 2(1-\mu) + \tfrac{1}{2}(1+\mu)\left[(1+\mu)\mu + n^2 \right] \right\} n^2 p^4 + \left\{ \tfrac{1}{2}(1-2\mu)(1-\mu) n - 1 \right\} \times$$

$$\tfrac{1}{2}(1+\mu) n^3 p^3 + \left\{ 1 + \mu^2 - (1+\mu) n^2 + \left[n^2 - \{ 2 + (1+\mu)\mu \} n^2 \right\} n^2 p^2 \Big]$$

$$+ S^3 \Big[\tfrac{5}{2}(1-\mu) n^2 p^4 - \tfrac{1}{2}(1-\mu)\mu n^4 p^3 + \left\{ 3 \left[\tfrac{1-\mu}{2} \left(\tfrac{1-\mu}{2} + \tfrac{1+\mu}{2} n^2 \right) - n^2 \right. \right.$$

$$
\left.\begin{aligned}
&+ \; n^2\left[\tfrac{1}{2}(1-\mu)(5-\mu) - (1-\mu)n\right]\Big\} n^2p^2 + \left\{1 - \tfrac{1-\mu}{2}(1+3n^2)\right\}\tfrac{1-\mu}{2}n^4\Big] \\
&- s^4\left[2p^2 + \left(\tfrac{1-\mu}{2}\right)n^2\right]n^2\Big\} \\
&+ \Big\{ s^2\Big[-\tfrac{1}{2}(1-\mu)n^2p^4 + \left\{1+\tfrac{1}{4}(1+\mu)^2 n\right\}np^3 \\
&\quad + \left\{(1+\mu)(1-2\mu) - \tfrac{1}{4}(1-\mu)^2 n^2\right\}n^2p^2 - \tfrac{1}{2}(p+n)(1-\mu)n^3\Big] \\
&+ s^3\Big[\tfrac{1}{4}(1+\mu)^2 n^2p^5 + \left\{\tfrac{1}{4}(1+\mu)^2 - \tfrac{1}{2}(1-\mu)\right\}n^2p^4 + \left\{1+\tfrac{1}{8}(1+\mu)^3 n^3\right\}np^3 \\
&\quad + \tfrac{1}{2}(1+\mu)^2 n^4p^2 - \tfrac{1}{2}(1-\mu)n^3p\Big] - \tfrac{s^4}{2}\left[(p^2+n^2)(1-\mu)n^2\right]\Big\},
\end{aligned}\right]
$$

and

$$
\left.\begin{aligned}
T = &\tfrac{1}{2}(1-\mu)sp^2\Big[s(p^2+n^2)^2 + \alpha\Big\{p^2\left[p^4 + \left\{\tfrac{1}{2}(7-3\mu)\right\}n^2p^2\right. \\
&+ \tfrac{1}{2}(1+\mu)n^2p^3 + (1-\mu)\left\{n^2+\tfrac{1}{2}(1-\mu)\right\}n^2 + \tfrac{1}{2}(1-\mu)n^6 \\
&\left. + s\left[3p^2 + \left\{n^2(2p^2-1) + \tfrac{1}{2}(1-\mu)\right\}\right]n^2\Big\}\right].
\end{aligned}\right] \text{—(52)}
$$

Assuming a condition similar to that used in equation (49) where p^2 is assumed large, equation (50) reduces to the following simplified form:

$$
\bar{N}_x = \frac{1-\mu^2}{Eh}N_x = -\frac{\hat{R}}{\hat{T}}\,, \qquad \text{—(53)}
$$

where

$$
\begin{aligned}
\hat{R} = & \left\{ \frac{(1-\mu^2)\,p^2}{(p^2+n^2)^2} + \alpha \frac{(p^2+n^2)^2}{p^2} \right\} \\
& - \frac{2\alpha p^4}{(1-\mu)(p^2+n^2)^2 s^2} \left\{ \frac{1}{2}(1-\mu)s^2 \left[2\mu + \frac{1}{2}(1-\mu)(3+\mu)n^2 \right] - S\left[1-\mu^2+5n^2 \right] \right\} \\
& - \frac{2p^2 n^2}{(1-\mu)(p^2+n^2)^2 s^2} \left\{ s^2 \left[\frac{1}{2}(1-\mu) - S\left\{ \frac{1}{4}(1+\mu)^2(p+1) - \frac{1}{2}(1-\mu) \right\} \right] \right\},
\end{aligned}
\quad (54)
$$

and

$$
\hat{T} = \left\{ 1 + \frac{2\alpha p^3}{(1-\mu)(p^2+n^2)^2 s^2} \left[p + \frac{1}{4}(1+\mu^2)\,n^2 \right] \right\}. \quad (55)
$$

The critical value of the stability force may be determined by setting $\frac{d\bar{N}_x}{dp}$ equal to zero, and solving for the parameter p which yields, when substituted back in to the general equation (53), the critical buckling load.

Noting the degree of the parameter n in the numerator and the denominator of equation (53), it follows that as the factor n increases the critical buckling load increases. For $n = 0$, equation (50) and (53) coincide with equation (45), which was obtained for the symmetric motion only.

CHAPTER-III

SUMMARY

The equations of motion for the forced vibration of thin elastic cylindrical shells with consideration of transverse shear, rotary inertia, and in-plane stability forces have been investigated by using normal mode theory. Orthogonality conditions are determined for free vibration together with associated boundary conditions, five on each edge.

The application of resulting equations is made for simply supported end conditions and roots of the natural frequency equations are formulated in the term of in-plane stability forces. The fifth order equation for natural frequency of free vibration is solved algebrically for the case of symmetric motion only (i.e. $n = 0$).

Forced vibrations solution is formulated in the Duhamel integral form which allows for the application of any arbitrary surface load, static or dynamic. As well as any initial conditions on displacement and velocity.

The determination of the natural frequencies of free vibration from the fifth order equation is an extremely tedious algebraic operation, and is not carried out numerically in this thesis.

Only algebraic solutions of the problem presented in the thesis are obtained, additional numerical work on the problem could possibly carried out by using digital computer.

CONCLUSIONS

The critical buckling stress of a cylindrical shell as defined by equation (48) is valid for the special case of thin shells only. The theory used to derive this equation applies only when the quantity $\frac{h^2}{12a^2}$ is the same order as unity, and when higher order forms of this ratio (i.e. $\frac{h^3}{a^3}$, $\frac{h^4}{a^4}$, ...) are negligible in comparison to unity. This condition restricts the radius-to-thickness ratio a/h to be equal to or greater than 10. Referring to figures 2.5a and 2.5b, the inclusion of the effect of shear stress reduces the critical buckling stress by approximately 5%, for a/h=10, when compared with the theory given by Timoshenko and Gere.[4] As the ratio a/h increases, the reduction in the critcal buckling stress due to shear stress becomes less significant, that is, for the condition a/h=48 the critical buckling stress is only reduced by 1%.

Referring to equation (48), as Poisson's ratio, μ, increases, the critical buckling stress decreases. Increasing the value of μ from 0.3 to 0.4 decreases the value of the critical buckling stress by approximately 0.7% in the range of a/h=10. This decrease becomes insignificant for the ratio a/h greater than 50.

In restricting the solution to symmetric motions only, equation (48) for the critical buckling stress applies only for the special case of short cylinders which usually buckle in large number of short longitudinal waves.

For long cylinders, equation (48) does not apply because the critical buckling stress usually occurs for the condition of antisymmetric motions (i.e. $n \neq 0$) and for the case where total number of longitudinal waves is small (i.e. p^2 is small). Equation (50) applies to this condition and can be used to obtain the critical buckling stress for the case of a long cylinder.

APPENDICES

APPENDIX-A

The determinant of the equation (12) is given in the following form:

$$|D| = \begin{vmatrix} B_{11} & B_{12} & B_{13} & B_{14} & 0 \\ -B_{12} & B_{22} & B_{23} & 0 & B_{25} \\ -B_{13} & B_{23} & B_{33} & B_{34} & B_{35} \\ B_{14} & 0 & -B_{34} & B_{44} & B_{45} \\ 0 & B_{25} & B_{35} & -B_{45} & B_{55} \end{vmatrix} = 0 \quad \text{(A-1)}$$

$$\begin{aligned} = \Big\{ & (B_{22}B_{55} - B_{25}^2)\big[B_{33}(B_{11}B_{44} - B_{14}^2) + B_{34}(2B_{13}B_{14} + B_{11}B_{34}) \\ & + B_{13}^2 B_{44}\big]\Big\} + \Big\{(B_{44}B_{55} + B_{45}^2)\big[B_{12}(B_{12}B_{33} - B_{13}B_{23}) \\ & - B_{23}(B_{12}B_{13} + B_{11}B_{23})\big] + B_{45}\big[B_{45}(B_{11}B_{22}B_{23} + B_{13}^2 B_{22}) \\ & + B_{35}(B_{11}B_{22}B_{34} + 2B_{13}B_{14}B_{22})\big] \\ & + (B_{34}B_{45} - B_{35}B_{44})\big[B_{35}(B_{11}B_{22} - B_{12}^2) - B_{25}(B_{12}B_{13} + B_{11}B_{23})\big] \end{aligned} \quad \text{(A-2)}$$

$$+2B_{14}B_{25}\left[B_{12}(B_{33}B_{45}+B_{34}B_{35})-B_{13}B_{23}B_{45}\right]$$

$$\left.-B_{14}^{2}\left[B_{35}(B_{23}B_{25}-B_{22}B_{35})-B_{23}(B_{23}B_{55}-B_{25}B_{35})\right]\right\}=0$$

Considering symmetric motions only (i.e. $n=0$), the quantities in the second bracket of equation (A-2) equal to zero, and the equation reduces to the form:

$$(B_{22}B_{55}-B_{25}^{2})\left[B_{33}(B_{11}B_{44}-B_{14}^{2})+B_{34}(2B_{13}B_{14}+B_{11}B_{34})+B_{13}^{2}B_{44}\right]=0 \quad \text{-(A-3)}$$

which gives

$$(B_{22}B_{55}-B_{25}^{2})=0\,, \quad \text{-(A-4)}$$

and

$$\left[B_{33}(B_{11}B_{44}-B_{14}^{2})+B_{34}(2B_{13}B_{14}+B_{34}B_{11})+B_{13}^{2}B_{44}\right]=0. \quad \text{-(A-5)}$$

APPENDIX-B

The determinant of the equation (36) is given in the following form:

$$
|D| = \begin{vmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & 0 \\
C_{12} & C_{22} & C_{23} & 0 & C_{25} \\
C_{13} & C_{23} & C_{33} & C_{34} & C_{35} \\
C_{14} & 0 & C_{34} & C_{44} & C_{45} \\
0 & C_{25} & C_{35} & C_{45} & C_{55}
\end{vmatrix} = 0 \quad -(B\text{-}1)
$$

$$
\left.\begin{aligned}
= & \Big\{(C_{22}C_{55} - C_{25}^2)\big[C_{33}(C_{11}C_{44} - C_{14}^2) + C_{34}(2C_{13}C_{14} - C_{11}C_{34}) - C_{13}^2C_{44}\big]\Big\} \\
& + \Big\{(C_{35}C_{45} - C_{34}C_{55})\big[C_{12}(2C_{13}C_{23} - C_{12}C_{33}) - C_{11}C_{23}^2\big] \\
& + C_{12}(2C_{14}C_{23} + C_{12}C_{34})(C_{35}C_{45} - C_{34}C_{55}) \\
& + (C_{34}C_{45} - C_{35}C_{44})\big[C_{35}(C_{11}C_{22} - C_{12}^2) + 2C_{25}(C_{12}C_{13} - C_{11}C_{23})\big]
\end{aligned}\right] \quad -(B\text{-}2)
$$

$$+C_{14}^2\left[C_{23}^2 C_{55} - C_{35}\left(2C_{25}C_{23} - C_{22}C_{35}\right)\right]$$

$$\left.+ 2C_{25}\left[C_{12}C_{14}\left(C_{34}C_{35} - C_{33}C_{45}\right) + C_{13}C_{14}C_{23}C_{45}\right]\right\} = 0.$$

Quantities in second bracket of equation (B-2) vanish when considering symmetric motions of free vibration (i.e. $n = 0$), the equation (B-2) reduces to the equations (38) and (39).

REFERENCES

1. Archer, R. R., "A Nonlinear Shear Deformation Theory for Thin Elastic Shells," Sponsored in part by grant No. GK-1856 from the National Science Foundation.
2. Herrmann, G., and Armenakas, A. E., "Dynamic Behaviour of Cylindrical Shells under Initial Stress," Proceeding of Applied Mechanics Vol. I, June, 1962. pp. 203-213.
3. Kraus, H., "Thin Elastic Shells," John Wiley and Sons, Inc, New York.
4. Timoshenko, S., and Gere, J. M., "Theory of Elastic Stability," McGraw-Hill Book Company, New York.