DYNAMIC STIFFNESS MATRIX OF A BEAM-COLUMN

USING POWER SERIES EXPANSION TECHNIQUES

by

Chuchart Laohasiripunya

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Dean   Date

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ABSTRACT

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Chuchart Laohasiripunya

Master of Science in Engineering

Youngstown State University, Year 1976.

The purpose of this thesis is to obtain a power series expansion of the general stiffness matrix for a beam-column element. The general stiffness matrix for a beam-column element is derived from the Bernoulli-Euler differential equation with the inclusion of the axial force. The terms of the general stiffness matrix containing hyperbolic and harmonic functions are expanded in power series form. The resulting matrix series allows for an efficient procedure for computer operations.

The matrix series expansion produces an elastic stiffness matrix, a consistent mass matrix, and a geometric matrix associated with axial force. Higher terms in the series are retained to the order of the fourth power in natural frequency and axial force. All coupling terms of lower order between these parameters are retained.

A numerical problem is included as an explanation of the derived equations.

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LIST OF NOTATIONS

SYMBOL	DEFINITION
A	Area of cross-section
A_1, A_2, A_3, A_4	Constants of integration
a	$= u^2$
b	$= v^2$
B	$= \dfrac{EI}{2p_1p_2 - 2p_1p_2Cc + (p_1^2 - p_2^2)Ss}$
C	$= \cosh p_1L$
c	$= \cos p_2L$
D	Denominator of elements of general stiffness matrix
E	Modulus of elasticity
I	$= Ar^2$, Cross-section moment of inertia
k^2	$= \dfrac{P}{EI}$
L	Length of beam-column element
M_{AB}, M_{BA}	Bending moment at the ends of beam-column element
N_{ij}	Numerator of elements of general stiffness matrix
P	Axial force
P_{cr}	Critical buckling load
p_1	$= \left[-\dfrac{k^2}{2} + \sqrt{\left(\dfrac{k^2}{2}\right)^2 + \lambda^4}\right]^{\frac{1}{2}}$
p_2	$= \left[+\dfrac{k^2}{2} \sqrt{\left(\dfrac{k^2}{2}\right)^2 + \lambda^4}\right]^{\frac{1}{2}}$

SYMBOL	DEFINITION
r	= Radius of gyration
S_{ij}	Element of general stiffness matrix
$\hat{S}_{ij}$	$= \frac{L}{EI} S_{ij}$
S	$= \sinh p_1 L$
s	$= \sin p_2 L$
t	Time variable
u	$= p_1 L$
v	$= p_2 L$
V_{AB}, V_{BA}	Shear forces at the ends of beam element
x, y	Co-ordinates along the axes of a deflected beam-column
$Y(x)$	Mode shape
$\hat{P}$	$= \frac{P}{AE}$
$\hat{P}_{cr}$	$= \frac{P_{cr}}{AE}$
$\hat{s}$	$= \frac{L}{r}$
Ω	Natural frequency of beam-column
$\hat{\Omega}^2$	$= \frac{L^2 \Omega^2}{c^2}$
ω	Natural frequency of beam
$\hat{\omega}^2$	$= \frac{L^2 \omega^2}{c^2}$

SYMBOL	DEFINITION
λ^4	$= \frac{\rho A \ell^2}{EI}$
δ_{AB} , δ_{BA}	Transverse deflection at the ends of beam-column
θ_{AB} , θ_{BA}	Angular deflection at the ends of beam-column
$[A_1]$	Second order coupling matrix
$[A_2]$	Third order coupling matrix
$[G_o]$	First order geometrical matrix
$[G_1]$	Second order geometrical matrix
$[G_2]$	Third order geometrical matrix
$[G_3]$	Fourth order geometrical matrix
$[K]$	Elastic stiffness matrix
$[M_o]$	First order mass matrix
$[M_1]$	Second order mass matrix
$[S]$	General stiffness matrix

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CHAPTER I

INTRODUCTION

Using the Bernoulli-Euler equations of a beam-column, the components of the structural stiffness matrix are composed of terms containing hyperbolic and trigonometric functions. This form of the stiffness matrix is inefficient in the usual computer operations currently in use. As a result, a number of authors have developed techniques to circumvent this problem. The stiffness matrix has been separated into a number of parts: the bending stiffness matrix of a beam, the geometric stiffness matrix expressing the effect of axial force, and a consistent mass matrix defining the effect of inertia forces. All three of the above matrices contain numerical components which allows for efficient computer operations. In determining the numerical forms of these matrices, the components of the geometric stiffness matrix and the mass matrix are in error since they are approximation to that given by the exact Bernoulli-Euler theory.

Rubenstein[(6)*] derived the required stiffness mass, and geometric matrices utilizing static displacement functions for the beam-column element. Henshell[(1)] used the 'exact' equation in obtaining the dynamic stiffness coefficients

* Number in parenthesis refers to literature cited in the Bibliography.

(i.e., mass matrix) for a beam element. Wang[7] used the 'exact' equation in deriving the geometric stiffness matrix for a beam-column element. Przemieniecki[5] and Paz[3] derived second order coefficients for the mass matrix by using alternate formulations.

Paz[4] has expanded the terms of the exact general stiffness matrix for a beam-column element into a power series expansion. He obtained the first and second order terms of the geometric stiffness and the mass matrices.

In this thesis, terms beyond the second order effect are obtained and comparisons of these higher order effects are investigated among the different approximations. The following steps are taken to establish the results:

1. The general stiffness matrix is obtained based on the exact differential equation.

2. The hyperbolic and trigonometric components are expanded using power series techniques.

3. Terms of the power series are retained up to and including the order of fourth power in natural frequency, the fourth power in axial force, and similar lower order combinations between the two.

4. The higher order effects are investigated in a numerical problem for the special case of simply-supported boundary conditions.

CHAPTER II

FORMULATION OF THE STIFFNESS MATRIX

2.1 Solution of the Differential Equation

The dynamic stiffness matrix of a general beam-column is derived using the exact equation of motion, if the effects of rotary inertia and shear deformation are neglected, the differential equation of motion for a uniform beam-column is given by the Bernoulli-Euler equation as

$$\frac{\partial^4 y}{\partial x^4} + \frac{P}{EI}\frac{\partial^2 y}{\partial x^2} + \frac{\rho A}{EI}\frac{\partial^2 y}{\partial t^2} = 0 \qquad (2\text{-}1)$$

with X & Y - the co-ordinates axes of the beam as shown in Fig. I.

P - the axial forces

EI - the flexural stiffness

ρA - the mass per unit length

t - the time variable

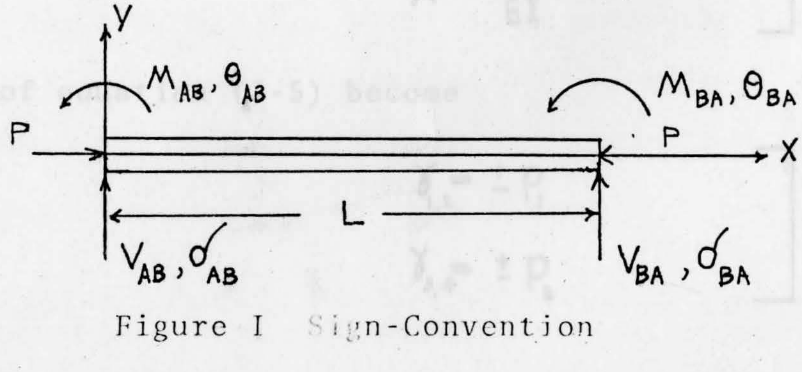


Figure I Sign-Convention

If the beam is not subjected to external forces, the assumed solution of equation (2-1) is a harmonic function of time

$$y(x,t) = \bar{y}(x) \sin \Omega t, \tag{2-2}$$

where Ω - natural frequency

$\bar{y}(x)$ - the corresponding mode shape.

By substituting equation (2-2) into (2-1) results in the ordinary differential equation

$$\frac{d^4 \bar{y}}{dx^4} + \frac{P}{EI}\frac{d^2 \bar{y}}{dx^2} - \frac{\rho A \Omega^2}{EI}\bar{y} = 0 \, . \tag{2-3}$$

Assuming the solution of the differential equation (2-3) is

$$\bar{y} = A_m e^{\gamma x} \, , \tag{2-4}$$

it follows that

$$\gamma^4 + k\gamma^2 - \lambda^4 = 0 \, , \tag{2-5}$$

where

$$\left.\begin{aligned} k^2 &= \frac{P}{EI} \\ \lambda^4 &= \frac{\rho A \Omega^2}{EI} \, . \end{aligned}\right] \tag{2-6}$$

The roots of equation (2-5) become

$$\left.\begin{aligned} \gamma_{1,2} &= \pm p_1 \\ \gamma_{3,4} &= \pm p_2 \end{aligned}\right] \tag{2-7a}$$

where

$$p_1 = \left[-\frac{k^2}{2} + \sqrt{\left(\frac{k^2}{2}\right)^2 + \lambda^4} \right]^{\frac{1}{2}} \qquad (2\text{-}7b)$$

$$p_2 = \left[+\frac{k^2}{2} + \sqrt{\left(\frac{k^2}{2}\right)^2 + \lambda^4} \right]^{\frac{1}{2}} . \qquad (2\text{-}7c)$$

The general solution of the differential equation (2-3) is

$$\bar{y}(x) = A_1 \sin p_2 x + A_2 \cos p_2 x + A_3 \sinh p_1 x + A_4 \cosh p_1 x \, , \qquad (2\text{-}8)$$

where A_1, A_2, A_3 and A_4 are the constants.

2.2 Boundary Conditions

The eight boundary conditions associated with the beam-column are

$$\left.\begin{array}{ll} \bar{y}(0) = \delta_{AB} & \bar{y}(L) = \delta_{BA} \\[2ex] \dfrac{d\bar{y}(0)}{dx} = \theta_{AB} & \dfrac{d\bar{y}(L)}{dx} = \theta_{BA} \end{array}\right] \text{and} \qquad (2\text{-}9a)$$

$$\left.\begin{array}{ll} \dfrac{d^3\bar{y}(0)}{dx^3} = \dfrac{V_{AB}}{EI} - \dfrac{P\theta_{AB}}{EI} & \dfrac{d^3\bar{y}(L)}{dx^3} = -\dfrac{V_{BA}}{EI} - \dfrac{P\theta_{BA}}{EI} \\[2ex] \dfrac{d^2\bar{y}(0)}{dx^2} = -\dfrac{M_{AB}}{EI} & \dfrac{d^2\bar{y}(L)}{dx^2} = \dfrac{M_{BA}}{EI} \end{array}\right] , \qquad (2\text{-}9b)$$

where δ_{AB}, δ_{BA} and θ_{AB} , θ_{BA} are respectively the transverse displacements and angular rotations at the ends of the beam-column. V_{AB}, V_{BA} and M_{AB}, M_{BA} are corresponding shear forces and moments at the boundaries (see Figure I).

Substituting the boundary conditions given by equation (2-9a) and (2-9b) into equation (2-8) yields respectively the matrix forms

$$\begin{Bmatrix} \delta_{AB} \\ \theta_{AB} \\ \delta_{BA} \\ \theta_{BA} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ p_2 & 0 & p_1 & 0 \\ s & c & S & C \\ p_2 c & -p_2 s & p_1 C & p_1 S \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} \qquad (2\text{-}10)$$

and

$$\begin{Bmatrix} \frac{V_{AB}}{EI} - \frac{P}{EI}\theta_{AB} \\ -\frac{M_{AB}}{EI} \\ -\frac{V_{BA}}{EI} - \frac{P}{EI}\theta_{BA} \\ \frac{M_{BA}}{EI} \end{Bmatrix} = \begin{bmatrix} -p_2^3 & 0 & p_1^3 & 0 \\ 0 & -p_2^2 & 0 & p_1^2 \\ -p_2^3 c & p_2^3 s & p_1^3 C & p_1^3 S \\ -p_2^2 s & -p_2^2 c & p_1^2 S & p_1^2 C \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix}, \qquad (2\text{-}11)$$

where

$$s = \sin p_2 L$$

$$c = \cos p_2 L$$

$$S = \sinh p_1 L$$

$$C = \cosh p_1 L$$

2.3 Stiffness Matrix

Performing the inverse operation on equation (2-10) gives

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{Bmatrix} \delta_{AB} \\ \theta_{AB} \\ \delta_{BA} \\ \theta_{BA} \end{Bmatrix} \qquad (2\text{-}12)$$

where

$$a_{11} = -p_1 p_2 Cs - p_1^2 Sc$$

$$a_{12} = p_1 - p_1 Cc - p_2 Ss$$

$$a_{13} = p_1^2 S + p_1 p_2 s$$

$$a_{14} = -p_1 C + p_1 c$$

$$a_{21} = -p_1 p_2 (S^2 + C^2) - p_1 p_2 Cc - p_1^2 Ss$$

$$a_{22} = p_1 Cs - p_2 Sc$$

$$a_{23} = -p_1 p_2 C + p_1 p_2 c$$

$$a_{24} = p_2 S - p_1 s \qquad (2\text{-}13)$$

$$a_{31} = p_1 p_2 Sc + p_2^2 Cs$$

$$a_{32} = -p_2 Cc + p_2 (s^2 - c^2) - p_1 Ss$$

$$a_{33} = -p_2^2 s - p_1 p_2 S$$

$$
\left.
\begin{aligned}
a_{34} &= -p_2 c + p_2 C \\
a_{41} &= -p_1 p_2 Cc + p_1 p_2 (s^2 - c^2) + p_2^2 Ss \\
a_{42} &= p_2 Sc - p_1 Cs \\
a_{43} &= -p_1 p_2 c + p_1 p_2 C \\
a_{44} &= p_1 s - p_2 S \;.
\end{aligned}
\right]
$$

Substituting equation (2-12) into equation (2-11), the final form of the general stiffness matrix relating end harmonic forces and moments to displacements and rotations is

$$
\begin{Bmatrix} V_{AB} \\ M_{AB} \\ V_{BA} \\ M_{BA} \end{Bmatrix} = \begin{bmatrix} S_{11} & & \text{Symmetric} & \\ S_{21} & S_{22} & & \\ S_{31} & S_{32} & S_{33} & \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} \begin{Bmatrix} \delta_{AB} \\ \theta_{AB} \\ \delta_{BA} \\ \theta_{BA} \end{Bmatrix}, \tag{2-14}
$$

where

$$S_{11} = S_{33} = B\left[(p_1^2 p_2^3 + p_1^4 p_2) Sc + (p_1 p_2^4 + p_1^3 p_2^2) Cs\right] \tag{2-15}$$

$$S_{21} = -S_{43} = B\left[(p_1 p_2^3 - p_1^3 p_2) + (p_1^3 p_2 - p_1 p_2^3) Cc + 2 p_1^2 p_2^2 Ss\right] \tag{2-16}$$

$$S_{22} = S_{44} = B\left[(p_1 p_2^2 + p_1^3) Cs - (p_1^2 p_2 + p_2^3) Sc\right] \tag{2-17}$$

$$S_{32} = -S_{41} = B\left[(p_1 p_2^3 + p_1^3 p_2)(c - C)\right] \tag{2-18}$$

$$S_{31} = B\left[(-p_1^2 p_2^3 - p_1^4 p_2) S - (p_1^3 p_2^2 + p_1 p_2^4) s\right] \tag{2-19}$$

$$S_{42} = B\left[(p_1^2 p_2 + p_2^3)S - (p_1 p_2^2 + p_1^3)s\right] \tag{2-20}$$

and

$$B = \frac{EI}{2p_1p_2 - 2p_1p_2Cc + (p_1^2 - p_2^2)Ss}, \tag{2-21}$$

subjected to the condition that

$$2p_1p_2 - 2p_1p_2Cc + (p_1^2 - p_2^2)Ss \neq 0 . \tag{2-22}$$

It is convenient to introduce the following change in notation

$$\left.\begin{aligned} u &= p_1L, \text{ and} \\ v &= p_2L . \end{aligned}\right] \tag{2-23}$$

Substituting equation (2-23) in equation (2-15), (2-16) (2-17), (2-18), (2-19) and (2-20), the elements of the general stiffness matrix of equation (2-14) become

$$S_{11} = S_{33} = \frac{\frac{EI}{L^5}(uv)\left[uv^2(Sc) + u^3(Sc) + v^3(Cs) - u^2v(Cs)\right]}{\frac{(uv)}{L^2}\left[-2(Cc-1) + \frac{(u^2-v^2)}{uv}Ss\right]} \tag{2-24}$$

$$S_{21} = -S_{43} = \frac{\frac{EI}{L^4}(uv)\left[(u^2-v^2)(Cc-1) + 2uvSs\right]}{\frac{(uv)}{L^2}\left[-2(Cc-1) + \frac{(u^2-v^2)}{uv}Ss\right]} \tag{2-25}$$

$$S_{22} = S_{44} = \frac{\frac{EI}{L^3}\left[uv^2(Cs) + u^3(Cs) - v^3(Sc) - u^2v(Sc)\right]}{\frac{(uv)}{L^2}\left[-2(Cc-1) + \frac{(u^2-v^2)}{uv}Ss\right]} \tag{2-26}$$

$$S_{32} = -S_{41} = \frac{\frac{EI}{L^4}(uv)\left[v^2(c) - v^2(C) + u^2(c) - u^2(C)\right]}{\frac{(uv)}{L^2}\left[-2(Cc-1) + \frac{(u^2-v^2)}{uv}Ss\right]} \tag{2-27}$$

$$S_{31} = \frac{\frac{EI}{L^5}(uv)\left[-uv^2S - u^3S - u^2vS - v^3S\right]}{\frac{(uv)}{L^2}\left[-2(Cc-1) + \frac{(u^2-v^2)}{uv}Ss\right]} \tag{2-28}$$

$$S_{42} = \frac{\frac{EI}{L^3}\left[u^2vS + v^3S - uv^2S - u^3S\right]}{\frac{(uv)}{L^2}\left[-2(Cc-1) + \frac{(u^2-v^2)}{uv}Ss\right]} \tag{2-29}$$

CHAPTER III

POWER SERIES EXPANSION

3.1 Power Series Expansion in Functions u, v

The known power series expansions of trigonometric and hyberbolic functions about the points u=o, v=o are

$$\cosh u = 1 + \frac{u^2}{2!} + \frac{u^4}{4!} + \frac{u^6}{6!} + \frac{u^8}{8!} + \frac{u^{10}}{10!} + \frac{u^{12}}{12!} + \frac{u^{14}}{14!} + \frac{u^{16}}{16!} + \cdots\cdots \tag{3-1a}$$

$$\sinh u = u + \frac{u^3}{3!} + \frac{u^5}{5!} + \frac{u^7}{7!} + \frac{u^9}{9!} + \frac{u^{11}}{11!} + \frac{u^{13}}{13!} + \frac{u^{15}}{15!} + \frac{u^{17}}{17!} + \cdots\cdots \tag{3-1b}$$

$$\cos v = 1 - \frac{v^2}{2!} + \frac{v^4}{4!} - \frac{v^6}{6!} + \frac{v^8}{8!} - \frac{v^{10}}{10!} + \frac{v^{12}}{12!} - \frac{v^{14}}{14!} + \frac{v^{16}}{16!} + \cdots\cdots \tag{3-1c}$$

$$\sin v = v - \frac{v^3}{3!} + \frac{v^5}{5!} - \frac{v^7}{7!} + \frac{v^9}{9!} - \frac{v^{11}}{11!} + \frac{v^{13}}{13!} - \frac{v^{15}}{15!} + \frac{v^{17}}{17!} + \cdots\cdots \tag{3-1d}$$

Substituting equation (3-1a), (3-1b), (3-1c) and (3-1d) into equation (2-24), (2-25), (2-26), (2-27), (2-28) and (2-29), the denominator of these terms becomes

$$D = \frac{uv}{L^2}\Big[\, u^4\Big(\frac{1}{12} + \frac{u^2}{180} + \frac{u^4}{6,720} + \frac{u^6}{453,600} + \frac{u^8}{47,900,160} + \frac{u^{10}}{7,264,857,600} + \cdots\cdots\Big)$$

$$+ v^2\Big(\frac{u^2}{6} + \frac{u^4}{180} - \frac{u^6}{5,040} - \frac{u^8}{90,720} - \frac{u^{10}}{4,790,016} - \frac{u^{12}}{444,787,200} + \cdots\cdots\Big)$$

$$+ v^4\Big(\frac{1}{12} - \frac{u^2}{180} - \frac{u^4}{1,440} - \frac{u^6}{75,600} + \frac{u^8}{217,720,800} - \frac{u^{10}}{239,500,800} + \cdots\cdots\Big)$$

$$+ V^{6}\left(-\frac{1}{180} - \frac{u^{2}}{5{,}040} + \frac{u^{4}}{75{,}600} + \frac{u^{6}}{1{,}814{,}400} + \frac{u^{8}}{152{,}409{,}600} + \cdots\cdots\cdots \right)$$

$$+ V^{8}\left(\frac{1}{6{,}720} + \frac{u^{2}}{90{,}720} + \frac{u^{4}}{21{,}772{,}800} - \frac{u^{6}}{15{,}240{,}960} + \cdots\cdots\cdots \right)$$

$$\left. + V^{10}\left(-\frac{1}{453{,}600} - \frac{u^{2}}{4{,}790{,}016} - \frac{u^{4}}{239{,}500{,}800} + \cdots\cdots\cdots \right) \right] .$$

(3-2)

The numerator of these terms take the form

$$N_{11} = N_{33} = \frac{EI}{L^5}(uv)\Bigg[\left(u^{4} + \frac{u^{4}v^{2}}{6} - \frac{u^{4}v^{4}}{12} + \frac{u^{4}v^{6}}{360} + \frac{u^{4}v^{8}}{24{,}192} - \frac{11\,u^{4}v^{10}}{3{,}628{,}800} + \frac{u^{4}v^{12}}{17{,}107{,}200} + \cdots \right)$$

$$+ \left(2u^{2}v^{2} - \frac{u^{2}v^{4}}{6} - \frac{u^{2}v^{6}}{30} + 13\frac{u^{2}v^{8}}{5{,}040} - \frac{13\,u^{2}v^{10}}{181{,}440} + \frac{43\,u^{2}v^{12}}{97{,}297{,}200} + \cdots\cdots \right)$$

$$+ \left(v^{4} - \frac{v^{6}}{6} + \frac{v^{8}}{120} - \frac{v^{10}}{5{,}040} + \frac{v^{12}}{362{,}880} + \frac{v^{14}}{39{,}916{,}800} + \cdots\cdots\cdots \right)$$

$$+ \left(\frac{u^{6}}{6} - \frac{u^{6}v^{2}}{30} - \frac{u^{6}v^{4}}{360} + \frac{u^{6}v^{6}}{4{,}320} - \frac{u^{6}v^{8}}{241{,}920} + \cdots\cdots\cdots \right)$$

$$+ \left(\frac{u^{8}}{120} - \frac{13\,u^{8}v^{2}}{5{,}040} + \frac{u^{8}v^{4}}{24{,}192} + \frac{u^{8}v^{6}}{24{,}192} + \cdots\cdots\cdots \right)$$

$$+ \left(\frac{u^{10}}{5{,}040} - \frac{13\,u^{10}v^{2}}{181{,}440} + \frac{11\,u^{10}v^{4}}{3{,}628{,}800} + \cdots\cdots\cdots \right)$$

$$+ \left(\frac{u^{12}}{362{,}880} - \frac{43\,u^{12}v^{2}}{39{,}916{,}800} + \cdots\cdots\cdots \right)$$

$$\left. + \left(\frac{u^{14}}{39{,}916{,}800} + \cdots\cdots\cdots \right) \right] ,$$

(3-3a)

$$
\begin{aligned}
N_{21} = -N_{43} = \frac{EI}{L^4}(uv) \Bigg[& \left(\frac{u^4}{2} + \frac{u^6}{24} + \frac{u^8}{720} + \frac{u^{10}}{40,320} + \frac{u^{12}}{3,628,800} + \frac{u^{14}}{479,001,600} + \cdots \right) \\
& + \left(u^2v^2 + \frac{u^4v^2}{24} - \frac{u^6v^2}{180} - \frac{13u^8v^2}{40,320} - \frac{13u^{10}v^2}{1,814,400} - \frac{43u^{12}v^2}{479,001,600} + \cdots \right) \\
& + \left(\frac{v^4}{2} - \frac{u^2v^4}{24} - \frac{u^4v^4}{72} - \frac{u^6v^4}{2,880} + \frac{u^8v^4}{241,920} + \frac{11u^{10}v^4}{43,545,600} + \cdots \right) \\
& + \left(\frac{-v^6}{24} - \frac{u^2v^6}{180} + \frac{u^4v^6}{2,880} + \frac{u^6v^6}{43,200} + \frac{u^8v^6}{2,903,040} + \cdots \right) \\
& + \left(\frac{v^8}{720} + \frac{13u^2v^8}{40,320} + \frac{u^4v^8}{241,920} - \frac{u^6v^8}{2,903,040} + \cdots \right) \\
& + \left(\frac{-v^{10}}{40320} - \frac{13u^2v^{10}}{1,814,400} - \frac{11u^4v^{10}}{43,545,600} \cdots \right) \Bigg],
\end{aligned}
\tag{3-3b}
$$

$$
\begin{aligned}
N_{22} = N_{44} = \frac{EI}{L^3}(uv) \Bigg[& \left(\frac{v^4}{3} - \frac{v^6}{30} + \frac{v^8}{840} - \frac{v^{10}}{45,360} + \frac{v^{12}}{3,991,680} + \cdots \right) \\
& + \left(\frac{2u^2v^2}{3} - \frac{u^2v^4}{30} - \frac{u^2v^6}{630} + \frac{u^2v^8}{9,072} - \frac{u^2v^{10}}{399,168} + \cdots \right) \\
& + \left(\frac{u^4}{3} + \frac{u^4v^2}{30} - \frac{u^4v^4}{180} + \frac{u^4v^6}{7,560} + \frac{u^4v^8}{1,814,400} + \cdots \right) \\
& + \left(\frac{u^6}{30} - \frac{u^6v^2}{630} - \frac{u^6v^4}{7,560} + \frac{u^6v^6}{151,200} + \cdots \right) \\
& + \left(\frac{u^8}{840} - \frac{u^8v^2}{9,072} + \frac{u^8v^4}{1,814,400} + \cdots \right) \\
& + \left(\frac{u^{10}}{45,360} - \frac{u^{10}v^2}{399,168} + \cdots \right) \\
& + \left(\frac{u^{12}}{3,991,680} + \cdots \right) \Bigg],
\end{aligned}
\tag{3-3c}
$$

$$N_{32} = -N_{41} - \frac{EI}{L^4}(uv)\Bigg[\left(-\frac{v^4}{2} + \frac{v^6}{24} - \frac{v^8}{720} + \frac{v^{10}}{40{,}320} - \frac{v^{12}}{3{,}628{,}800} + \frac{v^{14}}{479{,}001{,}600} + \cdots\right)$$

$$+\left(-u^2v^2 + \frac{u^2v^4}{24} - \frac{u^2v^6}{720} + \frac{u^2v^8}{40{,}320} - \frac{u^2v^{10}}{3{,}628{,}800} + \frac{u^2v^{12}}{479{,}001{,}600} + \cdots\right)$$

$$+\left(-\frac{u^4}{2} - \frac{u^4v^2}{24} + \cdots\cdots\right)$$

$$+\left(-\frac{u^6}{24} - \frac{u^6v^2}{720} - \frac{u^8v^2}{40{,}320} - \frac{u^{10}v^2}{3{,}628{,}800} + \frac{u^{12}v^2}{479{,}001{,}600}\cdots\cdots\right)$$

$$+\left(-\frac{u^8}{720} - \frac{u^{10}}{40{,}320} - \frac{u^{12}}{3{,}628{,}800} - \frac{u^{14}}{479{,}001{,}600}\cdots\cdots\right)\Bigg],$$

(3-3d)

$$N_{31} = \frac{EI}{L^5}(uv)\Bigg[\left(-v^4 + \frac{v^6}{6} - \frac{v^8}{120} + \frac{v^{10}}{5{,}040} - \frac{v^{12}}{362{,}880} + \frac{v^{14}}{39{,}916{,}800}\cdots\cdots\right)$$

$$+\left(-2u^2v^2 + \frac{u^2v^4}{6} - \frac{u^2v^6}{120} + \frac{u^2v^8}{5{,}040} - \frac{u^2v^{10}}{362{,}880} + \frac{u^2v^{12}}{39{,}916{,}800}\cdots\cdots\right)$$

$$+\left(-u^4 - \frac{u^6}{6} - \frac{u^8}{120} - \frac{u^{10}}{5{,}040} - \frac{u^{12}}{362{,}880} - \frac{u^{14}}{39{,}916{,}800}\cdots\cdots\right)$$

$$+\left(-\frac{u^4v^2}{6} - \frac{u^6v^2}{120} - \frac{u^8v^2}{5{,}040} - \frac{u^{10}v^2}{362{,}880} - \frac{u^{12}v^2}{39{,}916{,}800}\cdots\cdots\right)\Bigg],$$

(3-3e)

$$N_{42} = \frac{EI}{L^3}(uv)\Bigg[\left(\frac{v^4}{6} - \frac{v^6}{120} + \frac{v^8}{5{,}040} - \frac{v^{10}}{362{,}880} + \frac{v^{12}}{39{,}916{,}800} - \frac{v^{14}}{6{,}227{,}020{,}800} + \cdots\right)$$

$$+\left(\frac{u^2v^2}{3} - \frac{u^2v^4}{120} + \frac{u^2v^6}{5{,}040} - \frac{u^2v^8}{362{,}880} + \frac{u^2v^{10}}{39{,}916{,}800} - \frac{u^2v^{12}}{6{,}227{,}020{,}800} + \cdots\right)$$

$$+\left(\frac{u^4v^2}{120} + \frac{u^6v^2}{5{,}040} + \frac{u^8v^2}{362{,}880} + \frac{u^{10}v^2}{39{,}916{,}800} + \frac{u^{12}v^2}{6{,}227{,}020{,}800} + \cdots\right)$$

$$+\left(\frac{u^4}{6}+\frac{u^6}{120}+\frac{u^8}{5{,}040}+\frac{u^{10}}{362{,}880}+\frac{u^{12}}{39{,}916{,}800}+\frac{u^{14}}{6{,}227{,}020{,}800}+\cdots\right)\Bigg] .$$

(3-3f)

3.2 Power Series Expansion in the Functions a, b

Since the series expansion in equation (3-2), (3-3a) (3-3b), (3-3c), (3-3d), (3-3e), and (3-3f) are in even functions of variables u and v, it is efficient to introduce the simpler notation of the form

$$\left.\begin{aligned} a &= u^2 , \text{ and} \\ b &= v^2 . \end{aligned}\right] \qquad (3\text{-}4)$$

Referring to equation (2-7b) and (2-7c), the variables a, b are related to the parameters k^2 and λ^4 as follows:

$$\left.\begin{aligned} a &= \left(-\frac{k^2}{2}+\hat{r}\right)L^2 \\ b &= \left(\frac{k^2}{2}+\hat{r}\right)L^2 \end{aligned}\right] \qquad (3\text{-}5a)$$

where $$\hat{r} = \sqrt{\left(\frac{k^2}{2}\right)^2+\lambda^4} . \qquad (3\text{-}5b)$$

Substituting equation (3-4) into equation (3-2), and rearranging terms in increasing powers of a and b, the equation (3-2) becomes

$$D = \frac{\sqrt{ab}}{L^2}(a+b)^2\left[\frac{1}{12}+\frac{a}{180}-\frac{b}{180}+\frac{a^2}{6{,}720}-\frac{ab}{2{,}016}+\frac{b^2}{6{,}720}+\frac{a^3}{453{,}600}-\frac{7a^2b}{453{,}600}+\frac{7ab^2}{453{,}600}\right.$$
$$\left.-\frac{b^3}{453{,}600}+\frac{a^4}{47{,}900{,}160}+\frac{126a^2b^2}{23{,}9{,}500{,}800}-\frac{12ab^3}{47{,}900{,}160}+\frac{b^4}{47{,}900{,}160}+\cdots\right]$$

(3-6)

Similarly, the numerators of equation (3-3a), (3-3b), (3-3c), (3-3d), (3-3e) and (3-3f) are respectively,

$$N_{11} = N_{33} = \frac{EI}{L^5}\sqrt{ab}\,(a+b)^2\left[1 + \frac{a}{6} - \frac{b}{6} + \frac{a^2}{120} - \frac{ab}{20} + \frac{b^2}{120} + \frac{a^3}{5{,}040} - \frac{15a^2b}{5{,}040} + \frac{15ab^2}{5{,}040}\right.$$
$$\left. - \frac{b^3}{5{,}040} + \frac{a^4}{362{,}880} - \frac{28a^3b}{362{,}880} + \frac{70a^2b^2}{362{,}880} - \frac{28ab^3}{362{,}880} + \frac{b^4}{362{,}880} + \cdots\right], \tag{3-7a}$$

$$N_{21} = -N_{43} = \frac{EI}{L^4}\sqrt{ab}\,(a+b)^2\left[\frac{1}{2} + \frac{a}{24} - \frac{b}{24} + \frac{a^2}{720} - \frac{6ab}{720} + \frac{b^2}{720} + \frac{a^3}{40{,}320} - \frac{15a^2b}{40{,}320} + \frac{15ab^2}{40{,}320}\right.$$
$$\left. - \frac{b^3}{40{,}320} + \frac{a^4}{3{,}628{,}800} - \frac{28a^3b}{3{,}628{,}800} + \frac{70a^2b^2}{3{,}628{,}800} - \frac{28ab^3}{3{,}628{,}800} + \frac{b^4}{3{,}628{,}800} + \cdots\right], \tag{3-7b}$$

$$N_{22} = N_{44} = \frac{EI}{L^3}\sqrt{ab}\,(a+b)^2\left[\frac{1}{3} + \frac{a}{30} - \frac{b}{30} + \frac{a^2}{840} - \frac{ab}{252} + \frac{b^2}{840} + \frac{a^3}{45{,}360} - \frac{7a^2b}{45{,}360}\right.$$
$$+ \frac{7ab^2}{45{,}360} - \frac{b^3}{45{,}360} + \frac{a^4}{3{,}991{,}680} - \frac{12a^3b}{3{,}991{,}680} + \frac{126a^2b^2}{19{,}958{,}400}$$
$$\left. - \frac{12ab^3}{3{,}991{,}680} + \frac{b^4}{3{,}991{,}680} + \cdots\right], \tag{3-7c}$$

$$N_{32} = -N_{41} = \frac{EI}{L^4}\sqrt{ab}\,(a+b)^2\left[-\frac{1}{2} - \frac{a}{24} + \frac{b}{24} - \frac{a^2}{720} + \frac{ab}{720} - \frac{b^2}{720} - \frac{a^3}{40{,}320} + \frac{a^2b}{40{,}320}\right.$$
$$- \frac{ab^2}{40{,}320} + \frac{b^3}{40{,}320} - \frac{a^4}{3{,}628{,}800} + \frac{a^3b}{3{,}628{,}800} - \frac{a^2b^2}{3{,}628{,}800}$$
$$\left. + \frac{ab^3}{3{,}628{,}800} - \frac{b^4}{3{,}628{,}800} + \cdots\right], \tag{3-7d}$$

$$N_{31} = \frac{EI}{L^5}\sqrt{ab}\,(a+b)^2\Bigg[-1-\frac{a}{6}+\frac{b}{6}-\frac{a^2}{120}+\frac{ab}{120}-\frac{b^2}{120}-\frac{a^3}{5{,}040}+\frac{a^2b}{5{,}040}-\frac{ab^2}{5{,}040}$$
$$+\frac{b^3}{5{,}040}-\frac{a^4}{362{,}880}+\frac{a^3b}{362{,}880}-\frac{a^2b^2}{362{,}880}+\frac{ab^3}{362{,}880}-\frac{b^4}{362{,}880}+\cdots\Bigg],$$

(3-7e)

$$N_{42} = \frac{EI}{L^3}\sqrt{ab}\,(a+b)^2\Bigg[\frac{1}{6}+\frac{a}{120}-\frac{b}{120}+\frac{a^2}{5{,}040}-\frac{ab}{5{,}040}+\frac{b^2}{5{,}040}+\frac{a^3}{362{,}880}-\frac{a^2b}{362{,}880}$$
$$+\frac{ab^2}{362{,}880}-\frac{b^3}{362{,}880}+\frac{a^4}{39{,}916{,}800}-\frac{a^3b}{39{,}916{,}800}+\frac{a^2b^2}{39{,}916{,}800}$$
$$-\frac{ab^3}{39{,}916{,}800}+\frac{b^4}{39{,}916{,}800}+\cdots\cdots\Bigg].$$

(3-7f)

The series expansion of the coefficient S_{11}, S_{21}, S_{22}, S_{31}, S_{32}, and S_{42} in equation (2-24), (2-25), (2-26), (2-27), (2-28) and (2-29) are determined by performing the division of the numerator terms of equations (3-7a) through (3-7f) by the denominator form given in equation (3-6).

After considerable amount of algebraic operations, the following series terms are obtained:

$$S_{11}=S_{33}=\frac{EI}{L^3}\Bigg[12+\frac{6a}{5}-\frac{6b}{5}-\frac{a^2}{700}-\frac{129ab}{350}-\frac{b^2}{700}+\frac{a^3}{63{,}000}+\frac{17a^2b}{63{,}000}-\frac{17ab^2}{63{,}000}$$
$$-\frac{b^3}{63{,}000}-\frac{111a^4}{582{,}120{,}000}-\frac{17a^3b}{5{,}390{,}000}-\frac{34{,}751a^2b^2}{97{,}020{,}000}$$
$$-\frac{17ab^3}{5{,}390{,}000}-\frac{111b^4}{582{,}120{,}000}+\cdots\cdots\Bigg],$$

(3-8a)

$$S_{21} = -S_{43} = \frac{EI}{L^2}\left[6 + \frac{a}{10} - \frac{b}{10} - \frac{a^2}{1{,}400} - \frac{107\,ab}{2{,}100} - \frac{b^2}{1{,}400} + \frac{a^3}{126{,}000} + \frac{97\,a^2b}{126{,}000} - \frac{97\,ab^2}{126{,}000}\right.$$

$$- \frac{b^3}{126{,}000} - \frac{111\,a^4}{1{,}164{,}240{,}000} - \frac{1{,}693\,a^3b}{97{,}020{,}000} - \frac{24{,}173\,a^2b^2}{562{,}120{,}000} - \frac{1{,}693\,ab^3}{97{,}020{,}000}$$

$$\left. - \frac{111\,b^4}{1{,}164{,}240{,}000} + \cdots\cdots\right], \tag{3-8b}$$

$$S_{22} = S_{44} = \frac{EI}{L}\left[4 + \frac{2a}{15} - \frac{2b}{15} - \frac{11a^2}{6{,}300} - \frac{19ab}{3{,}150} - \frac{11b^2}{6{,}300} + \frac{a^3}{27{,}000} + \frac{13\,a^2b}{63{,}000} - \frac{13\,ab^2}{63{,}000}\right.$$

$$- \frac{b^3}{27{,}000} - \frac{5{,}599\,a^4}{6{,}403{,}320{,}000} - \frac{109\,a^3b}{16{,}170{,}000} - \frac{901\,a^2b^2}{873{,}180{,}000} - \frac{109\,ab^3}{16{,}170{,}000}$$

$$\left. - \frac{901\,a^2b^2}{873{,}180{,}000} - \frac{109\,ab^3}{16{,}170{,}000} - \frac{5{,}599\,b^4}{6{,}403{,}320{,}000} + \cdots\cdots\right], \tag{3-8c}$$

$$S_{32} = -S_{41} = \frac{EI}{L^2}\left[-6 - \frac{a}{10} + \frac{b}{10} + \frac{a^2}{1{,}400} - \frac{17ab}{525} + \frac{b^2}{1{,}400} - \frac{a^3}{126{,}000} + \frac{91\,a^2b}{147{,}000} - \frac{91\,ab^2}{147{,}000}\right.$$

$$+ \frac{b^3}{126{,}000} + \frac{111\,a^4}{1{,}164{,}240{,}000} - \frac{2{,}273\,a^3b}{145{,}530{,}000} - \frac{499\,a^2b^2}{12{,}127{,}500} - \frac{2{,}273\,ab^3}{145{,}530{,}000}$$

$$\left. + \frac{111\,b^4}{1{,}164{,}240{,}000} + \cdots\cdots\right], \tag{3-8d}$$

$$S_{31} = \frac{EI}{L^3}\left[-12 - \frac{6a}{5} + \frac{6b}{5} + \frac{a^2}{700} - \frac{23ab}{175} + \frac{b^2}{700} - \frac{a^3}{63{,}000} - \frac{17a^2b}{63{,}000} + \frac{17\,ab^2}{63{,}000} + \frac{b^3}{63{,}000}\right.$$

$$\left. + \frac{111a^4}{582{,}120{,}000} + \frac{17a^3b}{5{,}390{,}000} - \frac{2{,}039a^2b^2}{6{,}063{,}750} + \frac{17ab^3}{5{,}390{,}000} + \frac{111\,a^4}{582{,}120{,}000} + \cdots\cdots\right], \tag{3-8e}$$

$$S_{42} = \frac{EI}{L}\Bigg[\, 2 - \frac{a}{30} + \frac{b}{30} + \frac{13a^2}{12{,}600} + \frac{24ab}{4{,}725} + \frac{13b^2}{12{,}600} - \frac{11a^3}{378{,}000} - \frac{a^2b}{5{,}250} + \frac{ab^2}{5{,}250}$$

$$+ \frac{11b^3}{378{,}000} + \frac{29{,}931a^4}{38{,}419{,}920{,}000} + \frac{1{,}298a^3b}{200{,}103{,}750} + \frac{704a^2b^2}{600{,}311{,}250} + \frac{1{,}298ab^3}{200{,}103{,}750}$$

$$+ \frac{29{,}931b^4}{38{,}419{,}920{,}000} + \cdots\cdots \Bigg].$$

(3-8f)

3.3 Powers Series Expansion in Basic Parameters

Substituting equation (3-5a) and (3-5b) into equation (3-8a), (3-8b), (3-8c), (3-8d), (3-8e) and (3-8f), and by noting $k^2 = \frac{P}{EI}$ and $\lambda^4 = \frac{\rho A \Omega^2}{EI}$, the coefficients S_{11}, S_{21}, S_{22}, S_{31}, S_{32}, and S_{42} become

$$S_{11} = S_{33} = 12\frac{EI}{L^3} - \frac{6(P)}{5L} - \frac{156L(\rho A)(\Omega^2)}{420} - \frac{L^3(\rho A)(P\Omega^2)}{3150EI} - \frac{L(P^2)}{700EI}$$

$$- \frac{59L^5(\rho A)^2(\Omega^4)}{161{,}700\,EI} - \frac{L^3(P^3)}{63{,}000(EI)^2} - \frac{111L^5(P^4)}{582{,}120{,}000} - \frac{19L^5(\rho A)(P^2\Omega^2)}{4{,}851{,}000(EI)^2} + \cdots\cdots$$

(3-9a)

$$S_{21} = -S_{43} = \frac{6EI}{L^2} - \frac{(P)}{10} - \frac{11L^2(\rho A)(\Omega^2)}{210} - \frac{L^4(\rho A)(P\Omega^2)}{1{,}260} - \frac{L^2(P^2)}{1{,}400EI} - \frac{223L^6(\rho A)^2(\Omega^4)}{2{,}910{,}600\,EI}$$

$$- \frac{L^4(P^3)}{126{,}000(EI)^2} - \frac{111L^6(P^4)}{1{,}164{,}240{,}000(EI)^3} - \frac{173L^6(\rho A)(P^2\Omega^2)}{9{,}702{,}000\,(EI)^2} + \cdots\cdots$$

(3-9b)

$$S_{22} = S_{44} = \frac{4EI}{L} - \frac{2L(P)}{15} - \frac{L^3(\rho A)(\Omega^2)}{105} - \frac{L^5(\rho A)(P\Omega^2)}{3{,}150\,EI} - \frac{11L^3(P^2)}{6{,}300EI} - \frac{71L^7(\rho A)^2(\Omega^4)}{4{,}365{,}900(EI)}$$

$$-\frac{L^5(P^3)}{27,000(EI)^3} - \frac{509L^7(P^4)}{582,120,000(EI)^3} - \frac{149L^7(\rho A)(P^2\Omega^2)}{14,553,000(EI)^2} + \cdots$$

(3-9c)

$$S_{32} = -S_{41} = -\frac{6EI}{L^2} - \left(\frac{1}{10}\right)(P) - \frac{13L^2(\rho A)(\Omega^2)}{420} - \frac{L^4}{1,680}\frac{(\rho A)(P\Omega^2)}{EI} - \left(\frac{-1}{1,400\,EI}\right)L^2(P^2)$$

$$-\frac{1,681\,L^6(\rho A)^2(\Omega^4)}{23,284,800\,EI} - \left(\frac{-1}{126,000}\right)\frac{L^4(P^3)}{(EI)^2} - \left(\frac{-111}{1,164,240,000}\right)\frac{L^6(P^4)}{(EI)^3}$$

$$-\frac{887\,L^6(\rho A)(P^2\Omega^2)}{58,212,000\,(EI)^2} + \cdots$$

(3-9d)

$$S_{31} = -\frac{12EI}{L^3} - \left(\frac{-6}{5L}\right)(P) - \frac{54L(\rho A)(\Omega^2)}{420} - \left(\frac{-1}{3150}\right)\frac{(L^3)(\rho A)(P\Omega^2)}{EI} - \left(\frac{-1}{700\,EI}\right)L(P^2)$$

$$-\frac{1,279\,L^5(\rho A)^2(\Omega^4)}{3,880,800\,(EI)} - \left(\frac{-1}{63,000}\right)\frac{L^3(P^3)}{(EI)^2} - \left(\frac{-37}{194,040,000}\right)\frac{L^5(P^4)}{(EI)^3}$$

$$-\left(\frac{-19}{4,851,000}\right)\frac{L^5(\rho A)(P^2\Omega^2)}{(EI)^2} + \cdots$$

(3-9e)

$$S_{42} = \frac{2EI}{L} - \left(\frac{-1}{30}\right)L(P) - \left(\frac{-3}{420}\right)L^3(\rho A)(\Omega^2) - \left(\frac{-1}{3,600}\right)\frac{L^5(\rho A)(P\Omega^2)}{EI}$$

$$-\left(\frac{-13}{12,600}\right)\frac{L^3(P^2)}{EI} - \left(\frac{-1,097}{69,854,400}\right)\frac{L^7(\rho A)^2(\Omega^4)}{EI} - \left(\frac{-11}{378,000}\right)\frac{L^5(P^3)}{(EI)^2}$$

$$-\left(\frac{-907}{1,164,240,000}\right)\frac{L^7(P^4)}{(EI)^3} - \left(\frac{-559}{58,212,000}\right)\frac{L^7(\rho A)(P^2\Omega^2)}{(EI)^2} + \cdots$$

(3-9f)

3.4 Power Series Expansion in Matrix Form

The power series expansion of the general stiffness matrix for beam-column element of equation (2-14) is written as

$$[S] = [K] - [G_0]P - [M_0]\Omega^2 - [A_1]P\Omega^2 - [G_1]P^2 - [M_1]\Omega^4 - [A_2]P^2\Omega^2 - [G_2]P^3$$

$$-[G_3]P^4 \ldots\ldots\ldots , \tag{3-10}$$

where $[K]$, $[G_0]$, $[M_0]$, $[A_1]$, $[G_1]$, $[M_1]$, $[A_2]$, $[G_2]$ and $[G_3]$ are defined as

$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & & \text{symmetric} & \\ 6L & 4L^2 & & \\ -12 & -6L & 12 & \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$[G_0] = \begin{bmatrix} \frac{6}{5L} & & \text{symmetric} & \\ \frac{1}{10} & \frac{2L}{15} & & \\ -\frac{6}{5L} & -\frac{1}{10} & \frac{6}{5L} & \\ \frac{1}{10} & -\frac{L}{30} & -\frac{1}{10} & \frac{2L}{5} \end{bmatrix}$$

$$
\left[M_0\right] = \frac{\rho A L}{420}\begin{bmatrix} 156 & \text{symmetric} & & \\ 22L & 4L^2 & & \\ 54 & 13L & 156 & \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}
$$

$$
\left[A_1\right] = \frac{\rho A L^3}{EI}\begin{bmatrix} \frac{1}{3{,}150} & \text{symmetric} & & \\ \frac{L}{1{,}260} & \frac{L^2}{3{,}150} & & \\ -\frac{1}{3{,}150} & \frac{L}{1{,}680} & \frac{1}{3{,}150} & \\ -\frac{L}{1{,}680} & -\frac{L^2}{3{,}600} & -\frac{L}{1{,}260} & \frac{L^2}{3{,}150} \end{bmatrix} \qquad (3\text{-}11)
$$

$$
\left[G_1\right] = \frac{1}{EI}\begin{bmatrix} \frac{L}{700} & \text{symmetric} & & \\ \frac{L^2}{1{,}400} & \frac{11L^3}{6{,}300} & & \\ -\frac{L}{700} & -\frac{L^2}{1{,}400} & \frac{L}{700} & \\ \frac{L^2}{1{,}400} & -\frac{13L^3}{12{,}600} & -\frac{L^2}{1{,}400} & \frac{11L^3}{6{,}300} \end{bmatrix}
$$

$$[M_1] = \frac{(\rho A)^2 L^5}{100\,EI}\begin{bmatrix} \frac{59}{1{,}617} & & & \\ \frac{223L}{29{,}106} & \frac{71L^2}{43{,}659} & & \\ \frac{1{,}279}{38{,}809} & \frac{1681L}{232{,}848} & \frac{59}{1{,}617} & \\ -\frac{1{,}681L}{232{,}848} & -\frac{1{,}097L^2}{698{,}544} & -\frac{223L}{29{,}106} & \frac{71L^2}{43{,}659} \end{bmatrix}$$

$$[A_2] = \frac{\rho A L^5}{1000\,(EI)^2}\begin{bmatrix} \frac{19}{4{,}851} & & & \\ \frac{173L}{9{,}702} & \frac{149L^2}{14{,}553} & & \\ -\frac{19}{4{,}851} & \frac{887L}{58{,}212} & \frac{19}{4{,}851} & \\ -\frac{887L}{58{,}212} & -\frac{559L^2}{58{,}212} & -\frac{173L}{9{,}702} & \frac{149L^2}{14{,}553} \end{bmatrix}$$

$$[G_2] = \frac{L^3}{(EI)^2}\begin{bmatrix} \frac{1}{63{,}000} & & & \\ \frac{L}{126{,}000} & \frac{L^2}{27{,}000} & & \\ -\frac{1}{63{,}000} & -\frac{L}{126{,}000} & \frac{1}{63{,}000} & \\ \frac{L}{126{,}000} & -\frac{11L^2}{378{,}000} & -\frac{L}{126{,}000} & \frac{L^2}{27{,}000} \end{bmatrix}$$

$$[G_3] = \frac{L^5}{1{,}000(EI)^3}\begin{bmatrix} \frac{111}{582{,}120} & & \text{symmetric} & \\ \frac{111L}{1{,}164{,}240} & \frac{509L^2}{582{,}120} & & \\ -\frac{37}{194{,}040} & -\frac{111L}{1{,}164{,}240} & \frac{111}{582{,}120} & \\ \frac{111L}{1{,}164{,}240} & -\frac{907L^2}{1{,}164{,}240} & -\frac{111L}{1{,}164{,}240} & \frac{509L^2}{582{,}120} \end{bmatrix}.$$

CHAPTER IV

NUMERICAL EXAMPLE

4.1 Simply-Supported Beam-Column

Consider a beam-column AB that is simply-supported at the ends, as shown in Figure II. The beam is assumed to have the length L, a Young's Modulus E, and a second moment of area I.

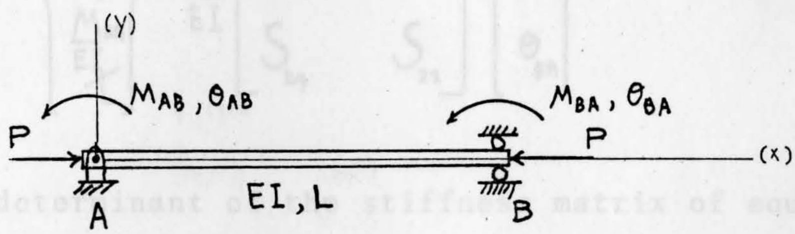


Figure II Simply-Supported Beam-Column

Referring to Figure I, the four boundary conditions are

$$\left.\begin{aligned} V_{AB} &\neq 0 \\ V_{BA} &\neq 0 \\ \delta_{AB} &= 0 \text{ , and} \\ \delta_{BA} &= 0 \text{ .} \end{aligned}\right] \tag{4-1}$$

Substituting the boundary conditions given by equation (4-1) into equation (2-14) yields the matrix form

$$\begin{Bmatrix} M_{AB} \\ M_{BA} \end{Bmatrix} = \begin{bmatrix} S_{22} & S_{24} \\ S_{42} & S_{44} \end{bmatrix} \begin{Bmatrix} \theta_{AB} \\ \theta_{BA} \end{Bmatrix}, \tag{4-2}$$

where $S_{22} = S_{44}$ and $S_{24} = S_{42}$.

Dividing equation (4-2) by $\frac{EI}{L}$, the stiffness matrix of equation (4-2) takes the non-dimensional form

$$\begin{Bmatrix} \frac{M_{AB}}{EI/L} \\ \frac{M_{BA}}{EI/L} \end{Bmatrix} = \frac{L}{EI} \begin{bmatrix} S_{22} & S_{24} \\ S_{24} & S_{22} \end{bmatrix} \begin{Bmatrix} \theta_{AB} \\ \theta_{BA} \end{Bmatrix} \tag{4-3}$$

Setting the determinant of the stiffness matrix of equation (4-3) equal to zero, and using the following definitions,

$$\begin{aligned} I &= Ar^2 \\ \hat{P} &= \frac{P}{AE} \\ c^2 &= \frac{E}{\rho} \\ \hat{\Omega} &= \frac{L^2 \Omega^2}{c^2}, \text{ and} \\ \hat{S} &= \frac{L}{r}, \end{aligned} \tag{4-4}$$

yields

$$\hat{S}_{22}^2 - \hat{S}_{24}^2 = 0 \,, \tag{4-5}$$

where

$$\hat{S}_{22} = \frac{L}{EI} S_{22} = 2 - \frac{2}{3.5}\hat{p}\hat{s}^2 - \frac{1}{3.5.7}\hat{\Omega}^2\hat{s}^2 - \frac{1}{2.3^2.5^2.7}\hat{\Omega}^2\hat{p}\hat{s}^4 - \frac{11}{2^2.3^2.5^2.7}\hat{p}^2\hat{s}^4 - \frac{71}{2^2.3^4.5^2.7^2.11}\hat{\Omega}^4\hat{s}^4$$

$$- \frac{149}{2^3.3^3.5^3.7^2.11}\hat{\Omega}^2\hat{p}^2\hat{s}^6 - \frac{7}{2^3.3^3.5^3.7}\hat{p}^3\hat{s}^6 - \frac{509}{2^6.3^3.5^4.7^2.11}\hat{p}^4\hat{s}^8 - \cdots\cdots \tag{4-6a}$$

$$\hat{S}_{24} = \frac{L}{EI} S_{24} = 2 + \frac{1}{2.3.5}\hat{p}\hat{s}^2 + \frac{1}{2^2.5.7}\hat{\Omega}^2\hat{s}^2 + \frac{1}{2^4.3^2.5^2}\hat{\Omega}^2\hat{p}\hat{s}^4 + \frac{13}{2^3.3^2.5^2.7}\hat{p}^2\hat{s}^4$$

$$+ \frac{1097}{2^6.3^4.5^2.7^2.11}\hat{\Omega}^4\hat{s}^4 + \frac{559}{2^5.3^3.5^3.7^2.11}\hat{\Omega}^2\hat{p}^2\hat{s}^6 + \frac{11}{2^4.3^3.5^3.7}\hat{p}^3\hat{s}^6 + \frac{907}{2^7.3^3.5^4.7^2.11}\hat{p}^4\hat{s}^8 + \cdots\cdots \tag{4-6b}$$

Two possible solutions of equation (4-5) are

$$\hat{S}_{22} = \hat{S}_{24}\text{, and} \tag{4-7a}$$

$$\hat{S}_{22} = -\hat{S}_{24}. \tag{4-7b}$$

Substituting equation (4-6a) and (4-6b) into equation (4-7a) and (4-7b) yields respectively,

$$2 - \frac{1}{2.3}\hat{p}\hat{s}^2 - \frac{1}{2^2.3.5}\hat{\Omega}^2\hat{s}^2 - \frac{1}{2^4.3.5.7}\hat{\Omega}^2\hat{p}\hat{s}^4 - \frac{1}{2^3.3^2.5}\hat{p}^2\hat{s}^4 - \frac{29}{2^6.3^4.5^2.7}\hat{\Omega}^4\hat{s}^4$$

$$- \frac{1}{2^5.3^2.5^2.7}\hat{\Omega}^2\hat{p}^2\hat{s}^6 - \frac{1}{2^4.3^3.5.7}\hat{p}^3\hat{s}^6 - \frac{1}{2^7.3^3.5^2.7}\hat{p}^4\hat{s}^8 = 0 \tag{4-8a}$$

and

$$6-\frac{1}{2.5}\hat{p}\hat{s}^2-\frac{1}{2^2.3.5.7}\hat{\Omega}^2\hat{s}^2-\frac{1}{2^4.3^2.5^2.7}\hat{\Omega}^2\hat{p}\hat{s}^4-\frac{1}{2^3.5^2.7}\hat{p}^2\hat{s}^4-\frac{13}{2^6.3^3.5^2.7^2.11}\hat{\Omega}^4\hat{s}^4$$

$$-\frac{37}{2^5.3^3.5^3.7^2.11}\hat{\Omega}^2\hat{p}^2\hat{s}^6-\frac{1}{2^4.3^2.5^3.7}\hat{p}^3\hat{s}^6-\frac{37}{2^7.3^2.5^4.7^2.11}\hat{p}^4\hat{s}^8=0.$$

(4-8b)

4.2 Case I - Beam Vibration

If the simply-supported beam is not subjected to axial forces (i.e. P=o), equation (4-8a) and equation (4-8b) reduce respectively as follows:

$$(\hat{\omega}^2\hat{s}^2)^2+\frac{2^4.3^3.5.7}{29}(\hat{\omega}^2\hat{s}^2)-\frac{2^7.3^4.5^2.7}{29}=0$$

(4-9a)

and
$$(\hat{\omega}^2\hat{s}^2)^2+\frac{2^4.3^2.5.7.11}{13}(\hat{\omega}^2\hat{s}^2)-\frac{2^7.3^4.5^2.7^2.11}{13}=0.$$

(4-9b)

Two possible solutions of equation (4-9a) are

$$(\hat{\omega}\hat{s}^2)=100.59925\text{, and}$$

(4-10)

$$(\hat{\omega}\hat{s}^2)=-621.97156\ .$$

The latter value has no physical significance.

Similarly, two possible solutions of equation (4-9b) are

$$(\hat{\omega}\hat{s}^2)=1778.39133\text{, and}$$

(4-11)

$$(\hat{\omega}\hat{s}^2)=-6043.006717\ ,$$

where the negative root is neglected.

4.3 Case II Static Beam-Column

If the simply-supported beam-column is not subjected to dynamic forces (i.e. $\Omega = 0$), equation (4-8a) and equation (4-8b) respectively reduce to the form

$$(\hat{p}\hat{s}^2)^4 + 2^3.5(\hat{p}\hat{s}^2)^3 + 2^4.3.5.7(\hat{p}\hat{s}^2)^2 + 2^6.3^2.5^2.7(\hat{p}\hat{s}^2) - 2^8.3^3.5^2.7 = 0$$

and (4-12a)

$$37(\hat{p}\hat{s}^2)^4 + 2^3.5.7.11(\hat{p}\hat{s}^2)^3 + 2^4.3^2.5^2.7.11(\hat{p}\hat{s}^2)^2 - 2^6.3^2.5^3.7^2.11(\hat{p}\hat{s}^2) - 2^8.3^3.5^4.7^2.11 = 0 .$$

(4-12b)

Two possible solutions of equation (4-12a) are as

$$(\hat{p}\hat{s}^2)_{CR} = 9.89069 \text{ , and}$$

(4-13)

$$(\hat{p}\hat{s}^2)_{CR} = -52.69116 .$$

Similarly, two possible solutions of equation (4-12b) are

$$(\hat{p}\hat{s}^2)_{CR} = 40.475974 \text{, and}$$

(4-14)

$$(p\hat{s}^2)_{CR} = -124.019150 .$$

The two negative values in the above equations are neglected.

4.4 Case III Beam-Column Vibration

When the simply-supported beam-column is subjected to both axial forces and dynamic forces, equation (4-8a) and equation (4-8b) take the forms respectively as

$$(\hat{\Omega}\hat{S})^4 + \frac{18}{29}\left[840 + 30(\hat{P}\hat{S}^2) + (\hat{P}\hat{S}^2)^2\right](\hat{\Omega}\hat{S})^2 + \frac{3}{58}\left[(\hat{P}\hat{S}^2)^4 + 40(\hat{P}\hat{S}^2)^3 + 1680(\hat{P}\hat{S}^2)^2 + 10800(\hat{P}\hat{S}^2) - 1209600\right] = 0\,, \tag{4-15}$$

where $$\hat{P}\hat{S}^2 = \hat{P}_{CR}\left(\frac{P}{P_{CR}}\right)\hat{S}^2 = 9.89069\left(\frac{P}{P_{CR}}\right) \tag{4-16}$$

and $$(\hat{\Omega}\hat{S})^4 + \frac{2}{65}\left[138600 + 2310(\hat{P}\hat{S}^2) + 37(\hat{P}\hat{S}^2)^2\right](\hat{\Omega}S)^2 + \frac{3}{650}\left[37(\hat{P}\hat{S}^2)^4 + 3080(\hat{P}\hat{S}^2)^3 + 277200(\hat{P}\hat{S}^2)^2 + 38808000(\hat{P}\hat{S}^2) - 2328480000\right] = 0 \tag{4-17}$$

where $$\hat{P}\hat{S}^2 = \hat{P}_{CR}\left(\frac{P}{P_{CR}}\right)\hat{S}^2 \quad 40.475974\left(\frac{P}{P_{CR}}\right). \tag{4-18}$$

Substituting equation (4-16) and (4-18) into equation (4-15) and (4-17) gives respectively

$$(\hat{\Omega}\hat{S})^4 + \frac{1}{29}\left[15120 + 5340.97152\left(\frac{P}{P_{CR}}\right) + 1760.86276\left(\frac{P}{P_{CR}}\right)^2\right](\hat{\Omega}\hat{S})^2$$

$$+ \frac{1}{58}\left[28709.60809\left(\frac{P}{P_{CR}}\right)^4 + 116107.62810\left(\frac{P}{P_{CR}}\right)^3 + 493041.57390\left(\frac{P}{P_{CR}}\right)^2\right.$$

$$\left. + 2990944.05100\left(\frac{P}{P_{CR}}\right) - 3628800\right] = 0$$

(4-19a)

and

$$(\hat{\Omega}\hat{S})^4 + \frac{1}{65}\left[277200 + 186998.99990\left(\frac{P}{P_{CR}}\right) + 121234.55090\left(\frac{P}{P_{CR}}\right)^2\right](\hat{\Omega}\hat{S})^2$$

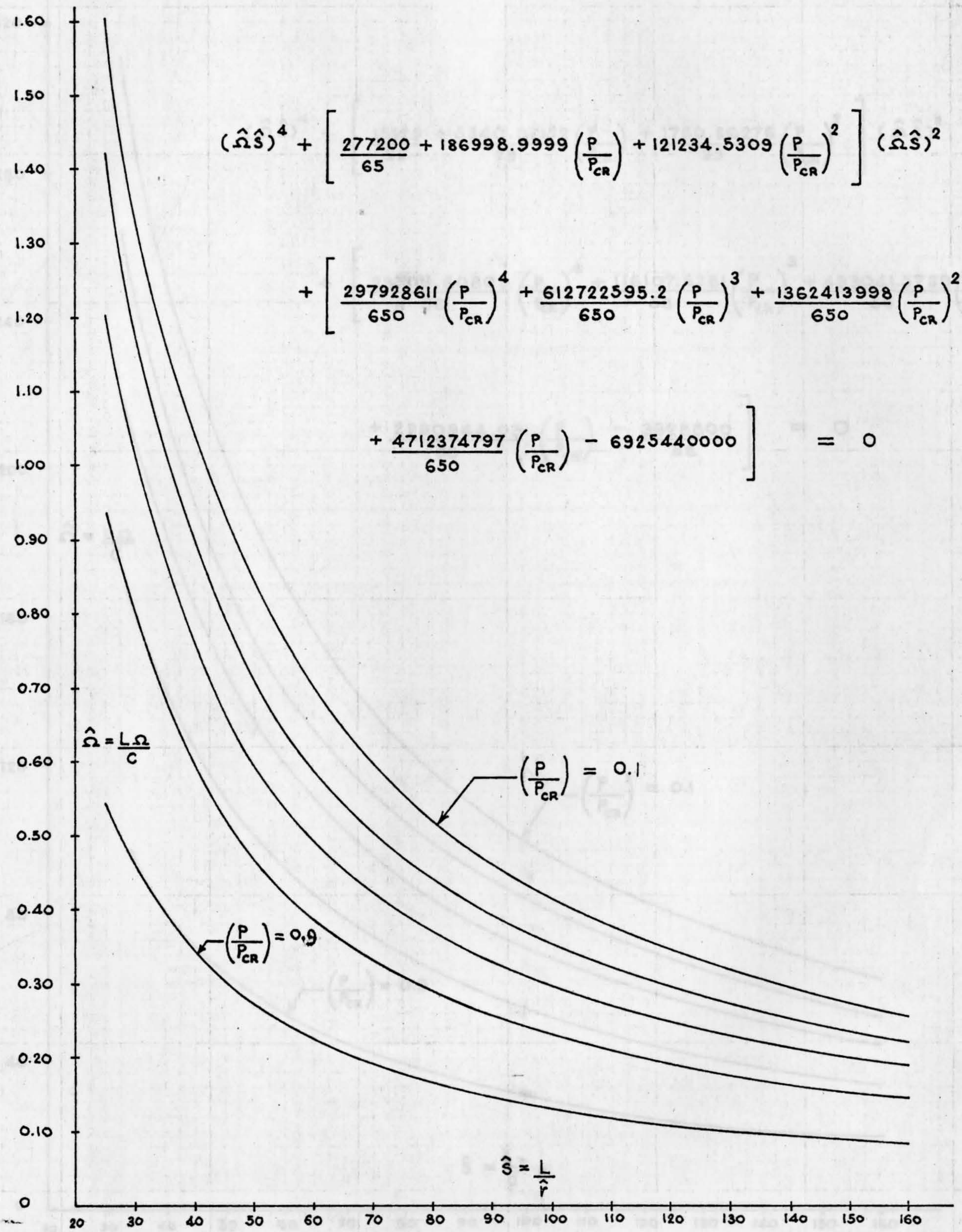
$$\frac{1}{650}\left[897428611\left(\frac{P}{P_{CR}}\right)^4 + 612722595.2\left(\frac{P}{P_{CR}}\right)^3 + 1362413998\left(\frac{P}{P_{CR}}\right)^2\right.$$

$$\left. + 4712374797\left(\frac{P}{P_{CR}}\right) - 6985440000\right] = 0$$

(4-19b)

The solutions of equation (4-19a) and (4-19b) are dependent on the value of $\frac{P}{P_{CR}}$ where $0 \leq \frac{P}{P_{CR}} \leq 1$.

The numerical computation of the natural frequencies of the simply-supported beam-column for $\hat{\Omega}$ vs. $\hat{S}$ are shown in the Table I and II with graphical results shown in Figures I and II.

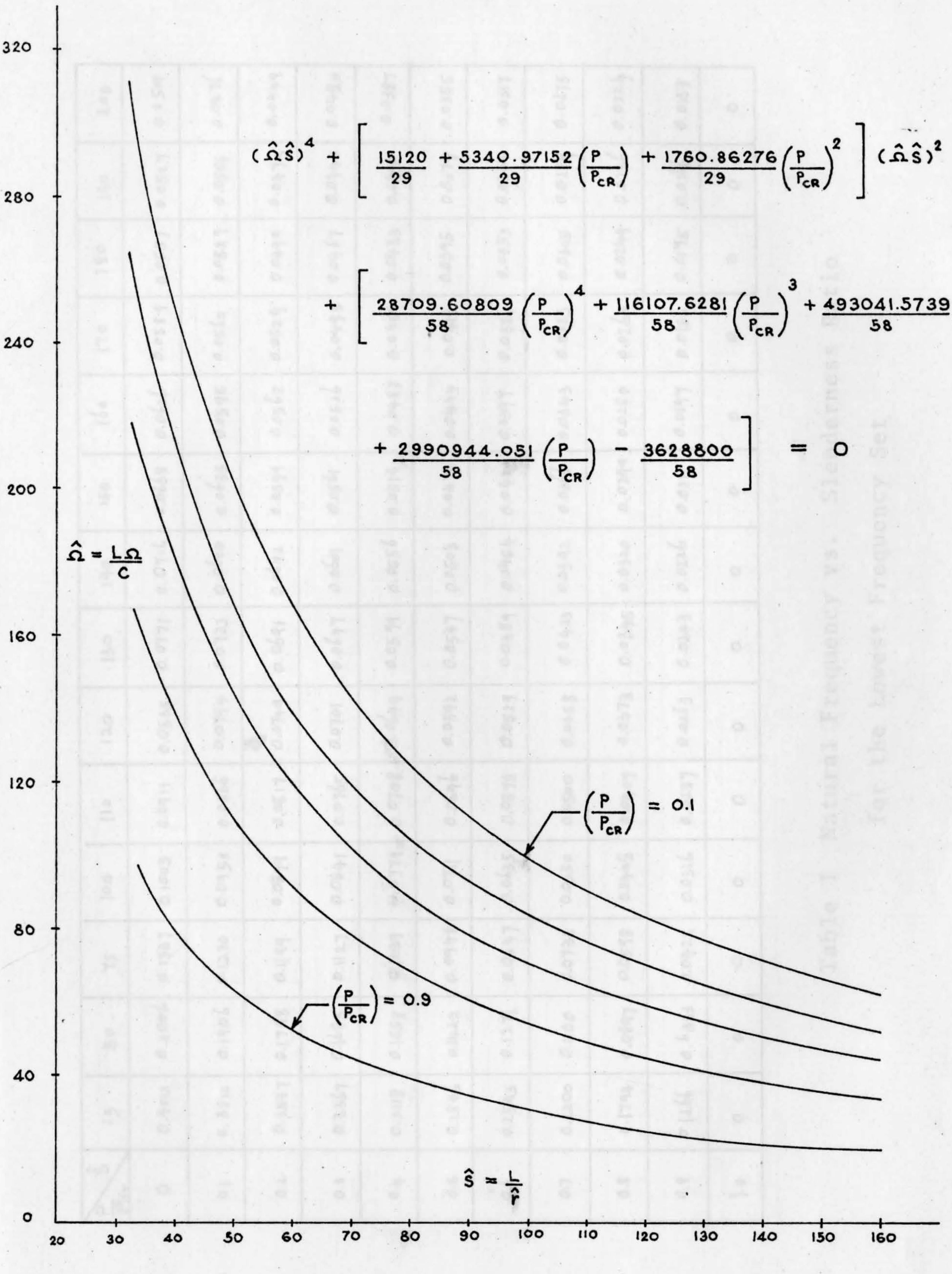


$$(\hat{\Omega}\hat{S})^4 + \left[\frac{277200}{65} + 186998.9999\left(\frac{P}{P_{CR}}\right) + 121234.5309\left(\frac{P}{P_{CR}}\right)^2\right](\hat{\Omega}\hat{S})^2$$

$$+\left[\frac{297928611}{650}\left(\frac{P}{P_{CR}}\right)^4 + \frac{612722595.2}{650}\left(\frac{P}{P_{CR}}\right)^3 + \frac{1362413998}{650}\left(\frac{P}{P_{CR}}\right)^2\right.$$

$$\left. + \frac{4712374797}{650}\left(\frac{P}{P_{CR}}\right) - 6925440000\right] = 0$$

Figure II Natural Frequency vs. Slenderness Ratio for the Higher Frequency Set



$$(\hat{\Omega}\hat{S})^4 + \left[\frac{15120}{29} + \frac{5340.97152}{29}\left(\frac{P}{P_{CR}}\right) + \frac{1760.86276}{29}\left(\frac{P}{P_{CR}}\right)^2\right](\hat{\Omega}\hat{S})^2$$

$$+ \left[\frac{28709.60809}{58}\left(\frac{P}{P_{CR}}\right)^4 + \frac{116107.6281}{58}\left(\frac{P}{P_{CR}}\right)^3 + \frac{493041.5739}{58}\left(\frac{P}{P_{CR}}\right)^2\right.$$

$$\left. + \frac{2990944.051}{58}\left(\frac{P}{P_{CR}}\right) - \frac{3628800}{58}\right] = 0$$

Figure I Natural Frequency vs. Slenderness Ratio for the Lowest Frequency Set

$\frac{P}{P_{cr}}$ \ $\hat{S}$	25	50	75	100	110	120	130	140	150	160	170	180	190	200
0	0.4012	0.2006	0.1337	0.1003	0.0911	0.0835	0.0771	0.0716	0.0668	0.0626	0.0589	0.0557	0.0527	0.0501
0.1	0.3812	0.1906	0.1270	0.0953	0.0866	0.0794	0.0733	0.0680	0.0635	0.0595	0.0560	0.0529	0.0501	0.0476
0.2	0.3597	0.1798	0.1199	0.0899	0.0817	0.0749	0.0691	0.0642	0.0599	0.0562	0.0529	0.0499	0.0473	0.0449
0.3	0.3367	0.1683	0.1127	0.0841	0.0765	0.0701	0.0647	0.0601	0.0561	0.0526	0.0495	0.0467	0.0443	0.0420
0.4	0.3118	0.1558	0.1039	0.0779	0.0708	0.0649	0.0599	0.0556	0.0519	0.0487	0.0458	0.0433	0.0410	0.0387
0.5	0.2845	0.1422	0.0948	0.0711	0.0646	0.0592	0.0547	0.0508	0.0474	0.0444	0.0418	0.0395	0.0374	0.0355
0.6	0.2543	0.1271	0.0847	0.0635	0.0578	0.0529	0.0489	0.0454	0.0423	0.0397	0.0374	0.0353	0.0334	0.0317
0.7	0.2200	0.1100	0.0733	0.0550	0.0500	0.0458	0.0423	0.0392	0.0366	0.0343	0.0323	0.0305	0.0289	0.0275
0.8	0.1794	0.0897	0.0598	0.0448	0.0407	0.0373	0.0345	0.0320	0.0299	0.0280	0.0263	0.0249	0.0236	0.0224
0.9	0.1266	0.6333	0.0422	0.0316	0.0287	0.0263	0.0243	0.0226	0.0211	0.0197	0.0186	0.0175	0.0166	0.0158
1.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table I Natural Frequency vs. Slenderness Ratio for the Lowest Frequency Set

$\frac{P}{P_{cr}}$ \ $\hat{S}$	25	50	75	100	110	120	130	140	150	160	170	180	190	200
0	1.6868	0.8434	0.5623	0.4217	0.3834	0.3514	0.3244	0.3012	0.2811	0.2636	0.2481	0.2343	0.2219	0.2109
0.1	1.6081	0.8041	0.5360	0.4020	0.3655	0.3350	0.3093	0.2872	0.2680	0.2513	0.2365	0.2234	0.2116	0.2010
0.2	1.5215	0.7608	0.5072	0.3804	0.3458	0.3169	0.2926	0.2717	0.2536	0.2377	0.2238	0.2113	0.2002	0.1902
0.3	1.4266	0.7133	0.4755	0.3567	0.3242	0.2972	0.2744	0.2549	0.2378	0.2229	0.2098	0.1981	0.1877	0.1783
0.4	1.3224	0.6612	0.4408	0.3306	0.3005	0.2755	0.2543	0.2361	0.2204	0.2066	0.1945	0.1837	0.1740	0.1653
0.5	1.2074	0.6037	0.4025	0.3019	0.2744	0.2518	0.2322	0.2156	0.2012	0.1887	0.1776	0.1677	0.1589	0.1509
0.6	1.0792	0.5396	0.3599	0.2698	0.2453	0.2248	0.2075	0.1927	0.1798	0.1686	0.1587	0.1498	0.1420	0.1349
0.7	0.9333	0.4663	0.3111	0.2333	0.2120	0.1944	0.1795	0.1667	0.1555	0.1458	0.1372	0.1296	0.1228	0.1166
0.8	0.7604	0.3802	0.2535	0.1901	0.1728	0.1584	0.1462	0.1358	0.1267	0.1188	0.1118	0.1056	0.1006	0.0951
0.9	0.5364	0.2682	0.1788	0.1341	0.1219	0.1117	0.1032	0.0958	0.0894	0.0838	0.0789	0.0745	0.0706	0.0671
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table II Natural Frequency vs. Slenderness Ratio

for the Higher Frequency Set

4.5 Summary of Numerical Results

4.5a Case I - Beam Vibration

The natural frequency of the vibrating beam with simple supports is given by the exact classical solution as

$$\omega_n^2 = \left(\frac{n\pi}{L}\right)^4 \frac{EI}{\rho A} \qquad n \quad 1, 2, 3\ldots \tag{4-20}$$

or in the usual form as

$$(\hat{\omega}_n)^2 = \left(\frac{n\pi}{\S_2}\right)^4 \qquad n \quad 1, 2, 3\ldots \tag{4-21}$$

The following table is constructed for comparison purposes.

Type of Solution		$\hat{\omega}_n\hat{S}$		% Diff.	
		$\hat{\omega}_1\hat{S}$	$\hat{\omega}_2\hat{S}$	$\hat{\omega}_1\hat{S}$	$\hat{\omega}_2 S$
Exact Solution		9.8696	39.4784	0	0
Series	Two Terms	10.9545	50.1996	10.99	27.16
Form	Three Terms	10.0295	42.1709	1.62	6.82

Table III Comparison of Beam Frequencies

4.5b Case II - Beam Column

The critical buckling load of the beam-column with simple supports is given by the exact classical solution as

$$P_{cr_n} = \frac{(n\pi)^2 EI}{L^2} \qquad n \quad 1, 2, 3 \ldots \qquad (4\text{-}22)$$

or in the usual form as

$$\hat{P}_n = \frac{(n\pi)^2}{\hat{s}^2} \qquad n \quad 1, 2, 3 \ldots \qquad (4\text{-}23)$$

The following table is constructed for comparison purposes.

Type of Solution		$\hat{P}_n \hat{s}^2$		% Diff.	
		$\hat{P}_1 \hat{s}^2$	$\hat{P}_2 \hat{s}^2$	$\hat{P}_1 \hat{s}^2$	$\hat{P}_2 \hat{s}^2$
Exact Solution		9.8696	39.4784	0	0
Series Form	Two Terms	12.0000	60.0000	21.58	51.98
	Three Terms	10.2490	45.3256	3.84	14.81
	Four Terms	9.9562	41.7619	0.88	5.78
	Five Terms	9.8906	40.4759	0.21	2.53

Table IV Comparison of Beam-Column Critical Buckling Loads

4.5c Case III Beam-Column Vibration

The natural frequency of the vibrating beam-column with simple supports is given by the exact classical solution as

$$\Omega_n^2 = \frac{(n\pi)^4}{L^4}\frac{EI}{\rho A}\left(1-\frac{P}{P_{CR_n}}\right) \qquad n=1,2,3\ldots \qquad (4\text{-}24)$$

or in the usual form

$$(\hat{\Omega}_n)^2 = \frac{(n\pi)^4}{\hat{S}^2}\left(1-\frac{P}{P_{CR_n}}\right) \qquad n=1,2,3\ldots \qquad (4\text{-}25)$$

where $P_{CR_n} = \frac{(n\pi)^2 EI}{L^2}$ with the lowest critical buckling load P_{1CR} (i.e. n=1) used in the natural frequency equation.

CHAPTER V

DISCUSSION AND CONCLUSIONS

5.1 Discussion

For the special case of vibrating beam, Table III shows that as the number of terms in the power series increases, the series is convergent to the exact numerical value. Taking two terms of series expansion gives a value of the lowest frequency form ($\hat{\omega}_1\hat{s}$) as approximately 11.0% different than the exact value, and the value of the second lowest frequency form ($\hat{\omega}_2\hat{s}$) as approximately 27.0% different than the exact value. Three terms of series gives the value of the lowest frequency form ($\hat{\omega}_1\hat{s}$) as approximately 1.6% different than the exact value, and the value of the second lowest frequency form ($\hat{\omega}_2\hat{s}$) as approximately 6.8% different than the exact value.

For the special case of beam-column, Table IV shows that as the number of power series increases, the series is convergent to the exact value. Two terms of series gives the value of the lowest critical buckling form ($\hat{P}_1\hat{s}$) approximately 21.5% different from the exact value and the second lowest critical buckling form ($\hat{P}_2\hat{s}$) as approximately 51.9% different from the exact value. Three terms of series gives the value of the lowest critical buckling form ($\hat{P}_1\hat{s}$) as approximately 3.8% different from the exact value and the value of the second lowest critical buckling form ($\hat{P}_2\hat{s}$) as approximately

14.8% different from the exact value. Four terms of series gives the value of the lowest critical buckling form $(\hat{P}_1\hat{s})$ as approximately 0.8% different from the exact value and the value of the second lowest critical buckling form $(\hat{P}_2\hat{s})$ as approximately 5.7% different from the exact value. Five terms of series give the value of the lowest critical buckling form $(\hat{P}_1\hat{s})$ as approximately 0.2% different than the exact value and the value of the second lowest critical buckling form $(\hat{P}_2\hat{s})$ as approximately 2.5% different from the exact value.

For the special case of vibrating beam-column, Table I and II show that the power series expansion is convergent. However, the series containing the first six terms of the series are efficiently convergent. This condition holds for all value of $\frac{P}{Pcr}$. Table I and II also show that the series containing terms up to the nine terms of the series are not efficiently convergent. It is convergent only for the value of $\frac{P}{Pcr}$ close to unity.

5.2 Conclusions

Taking the power series expansion of the dynamic stiffness matrix for beam vibration in the three term matrix form

$$\left[K_d\right] = \left[K\right] - \left[M_o\right]\Omega^2 - \left[M_1\right]\Omega^4 - \ldots.$$

yields a convergent series with solutions extremely accurate in comparison with the exact solution, in the order of 2% of the exact value.

Taking power series expansion for the static beam-column in the five term form

$$\left[K_s\right] = \left[K\right] - \left[G_o\right] P - \left[G_1\right] P^2 - \left[G_2\right] P^3 - \left[G_3\right] P^4 - \ldots$$

yields approximate solution within less than 1% of the exact value.

Taking power series expansion of the vibrating beam-column in nine term matrix form as

$$\left[S\right] = \left[K\right] - \left[G_o\right] P - \left[M_o\right]\Omega^2 - \left[A_1\right]P\Omega^2 - \left[G_1\right] P^2 - \left[M_1\right]\Omega^4$$
$$\left[G_2\right] P^3 - \left[G_3\right] P^4 - \left[A_2\right] P^2\Omega^2 - \ldots.$$

yields numerical solution more accurate than the six term matrix form only for the special case when $\frac{P}{Pcr}$ is close to unity.

To avoid the situation of less accurate numerical values for small values of $\frac{P}{Pcr}$, it is suggested that additional terms of the matrix expansion be determined. This would involve a ten term series expansion in the form

$$\left[S\right] = \Big[\left[K\right] - \left[G_o\right] P - \left[G_1\right] P^2 - \left[G_2\right] P^3\Big]$$
$$- \Big[\left[M_o\right] + \left[A_1\right] P + \left[A_2\right] P^2\Big]\Omega^2$$
$$- \Big[\left[M_1\right] + \left[A_3\right] P\Big]\Omega^4 - \left[M_2\right]\Omega^6$$

The latter two matrices $\left[A_3\right]$ and $\left[M_2\right]$ must be determined by numerical investigation of the terms $(a^5 + \cdots\cdots\cdots + b^5)$ and

$(a^6+\cdots\cdots\cdots+b^6)$ respectively.

It is recommended that this numerical task be undertaken as a future topic of a Master's Thesis to extend the analysis.

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