THE STATIC BUCKLING OF SLANTED COLUMNS

by

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SYMBOL	DEFINITION
A	Area of cross-section
a,b,c,d	Scalar parameters
E	Modulus of elasticity
F	Acting force
f	Dimensionless acting force
L	Length of column
P	External force
R	Slenderness ratio
$\hat{V}, V, v$	Functionals
U,u	Axial displacements
W,w	Lateral displacements
x,y	Axial and Lateral coordinates
e_x	Strain in x-direction
α	Angle between center line of column on vertical line
β	Eigenvalue
β_0	β for $\alpha = 0$
γ	Nomalized reaction at constraint
δ	Head displacement
η	= X/L
$\hat{\lambda}, \lambda, \rho$	Lagrangian multipliers or reactions at constraint

SYMBOL	DEFINITION
τ	Nomalized acting force
ϕ	Dimensionless shear forces
ψ	Dimensionless axial forces
k_t	Stiffness of Spring

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CHAPTER I

1.1 INTRODUCTION

The stability of the simple truss subjected to the single force P, shown in Fig. 1 was investigated by Mises (1) as a basic model to illustrate some of the buckling phenomena in structures. In this model, known as the Mises truss, the buckling phenomenon is induced by axial thrust only, the bending effect is neglected.

Herein, the effects of both bending and axial thrust are taken into consideration. The structural response of such a truss subjected to a vertical force at the central joint is analyzed as a slanted column, as shown in Fig. 2a.

The following two constraints are imposed on the slanted column: (1) The top end B displaces only in the vertical direction; and (2) The lower end A has a zero displacement in all directions.

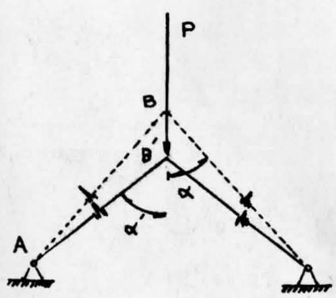


Figure 1. Mises Truss

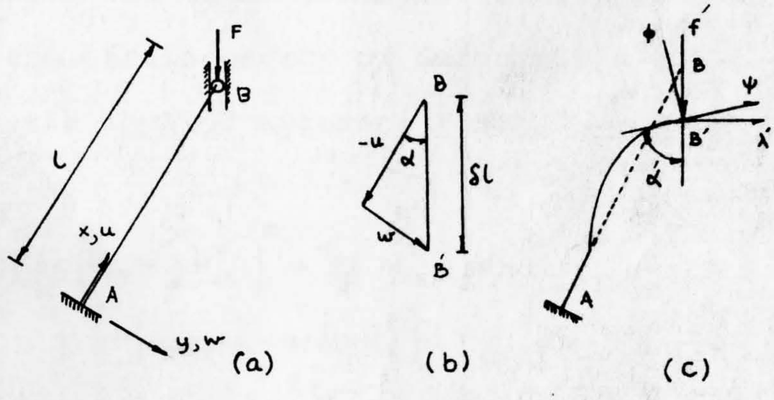


Figure 2. Slanted Column

1.2 Large Deflection Theory for Slanted Column

Consider an elastic and prismatic slanted column, AB, of length L (Fig. (2a)) subjected to a vertical force, F. Let x and y be the axial and lateral coordinates with the origin located at the centroid of the cross-section at the lower end, A. Let u and w be the axial and lateral displacements, respectively, and let α ($0 \leq \alpha \leq \frac{\pi}{2}$) be the angle between the center line of the column and the vertical line. End A. has no displacement while the upper end, B, may have a displacement only in the vertical direction. Thus, end B has a geometrical constraint that the total displacement in the horizontal direction must vanish which is expressed in equation form as

$$u \sin\alpha + w \cos\alpha = 0 \quad , \text{ at } x = L \qquad (1)$$

If the effects of bending strain and axial strain are considered the longitudinal strain component is defined as

$$e_x = u_{,x} + \frac{1}{2} w_{,x}^2 - y\, w_{,xx} \qquad (2)$$

in which a subscript preceded by a comma represents the appropriate derivative. The total potential energy of the system consists of the strain energy of deformation and the potential energy of the applied external force F, and is written as

$$V = \int_0^L \left[\frac{EA}{2}\left(u_{,x} + \frac{1}{2} w_{,x}^2\right)^2 + \frac{EI}{2} w_{,xx}^2 \right] dx - \left[F\left(-u\cos\alpha + w\sin\alpha\right) \right] \qquad (3)$$

The total potential energy function is constrained to satisfy equation (1) by utilizing the Lagrange multiplier technique in the form

$$V = \int_0^L \left[EA/2 \left(u_{,x} + 1/2\, w_{,x}^2 \right)^2 + EI/2\, w_{,xx}^2 \right] dx$$
$$- \left[F \left(-u\cos\alpha + w\sin\alpha \right) \right] - \hat{\lambda} \left(u\sin\alpha + w\cos\alpha \right) \qquad (4)$$

where $\hat{\lambda}$ is an arbitrary multiplier to be determined, A = the cross-sectional area, E = Young's modulas, I = second moment of area, and F = applied vertical force at the movable end.

Noting the following dimensionless parameters:

$$\eta = x/L \qquad (5a)$$
$$U = u/L \qquad (5b)$$
$$W = w/L \qquad (5c)$$
$$\lambda = \hat{\lambda}/AE \qquad (5d)$$
$$f = F/AE \qquad (5e)$$
$$R^2 = AL^2/I \qquad (5f)$$

then, equation (4) is rewritten in dimensionless form as

$$V/AEL = 1/2 \int_0^1 \left(U_{,\eta} + 1/2\, W_{,\eta}^2 \right)^2 d\eta + 1/2R^2 \int_0^1 W_{,\eta\eta}^2\, d\eta$$
$$+ \left[f\cos\alpha\, U - f\sin\alpha\, W - \lambda \left(U\sin\alpha + W\cos\alpha \right) \right]_{\eta=1} \qquad (6)$$

where the quantity

$$\psi = U_{,\eta} + 1/2\, W_{,\eta}^2 \qquad (7)$$

is the axial strain produced by the **dimensionless axial force.** For equilibrium the first variation of the Equation (6) must be zero. Performing the individual variations on the parameters

U and W yields respectively

$$\psi = -(f\cos\alpha - \lambda\sin\alpha) \qquad (7a)$$

$$W_{,\eta\eta\eta\eta} + \beta^2 W_{,\eta\eta} = 0 \quad ; \text{ for } 0 < \eta < 1 \qquad (7b)$$

together with the following boundary conditions:

	either	or	
@ $\eta = 0$	$W_{,\eta} = 0$	$W_{,\eta\eta} = 0$	(8a,b)
	$W = 0$	$W_{,\eta\eta\eta} + \beta^2 W_{,\eta} = 0$	(9a,b)
	$u = 0$	$\beta^2 = 0$	(10a,b)
@ $\eta = 1$	$W_{,\eta} = 0$	$W_{,\eta\eta} = 0$	(11a,b)
	$W = 0$	$W_{,\eta\eta\eta} + \beta^2 W_{,\eta} = \phi R^2$	(12a,b)
	$u = 0$	$\beta^2/R^2 + (f\cos\alpha - \lambda\sin\alpha) = 0$	(13a,b)

in which

$$\phi = -(f\sin\alpha + \lambda\cos\alpha) \qquad (14)$$

where the function ϕ is the shear force in the column.

Integration of Equation (7) yields the relationship between axial deformation, bending deformation, and the parameter ψ as

$$U(\eta) = \psi\eta - 1/2\int_0^\eta W_{,\eta}^2\, d\eta \qquad (15)$$

Also, the dimensionless form of the constraint equation, which must be satisfied, is obtained from Equation (1) as

$$U \sin\alpha + W \cos\alpha = 0 \quad ; \text{ at } \eta = 0 \tag{16}$$

The general solution of Equation (7b) is written as

$$W(\eta) = a \sin\beta\eta + b \cos\beta\eta + c\eta + d \tag{17}$$

Applying the boundary condition $U = 0$ @ $\eta = 0$ implies that parameter β is nonzero. At $\eta = 1$, since β is nonzero it follows that

$$\beta^2 = -\psi R^2 \tag{18a}$$

Applying the boundary conditions on the function W, $W(0) = 0$ and $W(1) = 0$ $\left(\text{i.e. } W_{,\eta\eta\eta} + \beta^2 W_{,\eta} = \phi R^2 \right)$ yields a partial solution of Equation (17a)as

$$W(\eta) = b(\cos\beta\eta - 1) + c \sin\beta\eta - \phi\eta/\psi \tag{18b}$$

Taking the function ψ as the independent parameter in the problem, the three constants, b, c, and ϕ in Equation (18b) are to be determined by the two additional boundary at $\eta = 0$ and $\eta = 1$ on the function W, together the constraint Equation (16).

Solving for f and λ from Equations (7a) and (14) and normalizing them with respect to the Euler buckling load for the column at $\alpha = 0$, one obtains

$$\tau = fR^2/\beta_0^2 = -R^2/\beta_0^2 \left(\psi \cos\alpha + \phi \sin\alpha \right) \tag{19a}$$

$$\gamma = \lambda R^2/\beta_0^2 = R^2/\beta_0^2 \left(\psi \sin\alpha - \phi \cos\alpha \right) \tag{19b}$$

The coefficient, β_0, depends on the end constraints of the particular column under consideration. For $\alpha = 0$, $f = -\psi$, $\phi = -\lambda$, and $\beta^2 = \beta_0^2$ which reduce Equation (18a) to the form $\beta_0^2 = fR^2$. In dimensional form the latter equation is written as $F_{cr} = \beta_0^2 EI/L^2$ which is the standard form of the Euler buckling equation (2,3). For any angle $0 \angle \alpha \angle \pi/2$ equations (17), (18), and (19a) are valid. For the special case $\alpha = \pi/2$ the equations are valid for another class of problems, namely the bending of beams.

The head shortening of the column (Fig. 2b) is

$$\delta = -U\cos\alpha + W\sin\alpha \quad , \text{ at } \eta = 1$$

The general solution problem consist of formulation a plot of $\P$ (related to the applied force) vs. δ (the head shortening) Equations (19a) and (20) form a pair of parametric Equations where ψ is the parametric variable.

CHAPTER II

Critical Buckling Load of Vertical Columns

2.1 Buckling Loads of a Vertical Column with a Spring Support at One End, The Other End Simply Supported.

Consider a vertical column subjected to the loads P as shown in Fig. 3

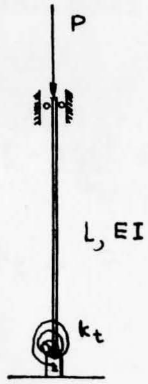


Figure 3. Vertical Column with a Spring Support and a Simple Support.

Boundary conditions for spring support at one end and a simple support at the other end are

@ $\eta = 0$ $\quad W(\eta) = 0$ (21a)

$M(\eta) = -k_t W'(\eta)$ (21b)

@ $\eta = L$ $\quad W(\eta) = 0$ (21c)

$M(\eta) = 0$ (21d)

The general solution of the static displacement Equation (7b), becomes

Substitution of the four boundary conditions of Equation (21 a,b,c,d) into Equation (22) gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \cos\beta l & \sin\beta l & l & 1 \\ EI\beta^2\cos\beta l & EI\beta^2\sin\beta l & 0 & 0 \\ \frac{EI}{k_t}\beta^2 & \beta & 1 & 0 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

For nontrivial solutions of these four constants the determinant of the coefficient matrix is set equal to zero giving

$$EI\beta^2\left(\sin\beta l - \beta^2 l\cos\beta l + \frac{EI\beta^3 l}{k_t}\sin\beta l\right) = 0 \qquad (23a)$$

For the special case $k_t = 0$, it follows that

$$\sin\beta l = 0$$
$$\beta l = n\pi \qquad n \quad 1, 2, 3 \ldots\ldots \qquad (23b)$$

For the special case $k_t = \infty$, one obtains

$$\sin\beta l - \beta l\cos\beta l = 0 \qquad \text{or}$$
$$\tan\beta l = \beta l \qquad (23c)$$

Using numerical techniques and noting Fig. 4 with $y_1 = \tan\beta l$, $y_2 = \beta l$, and $x = \beta l$,

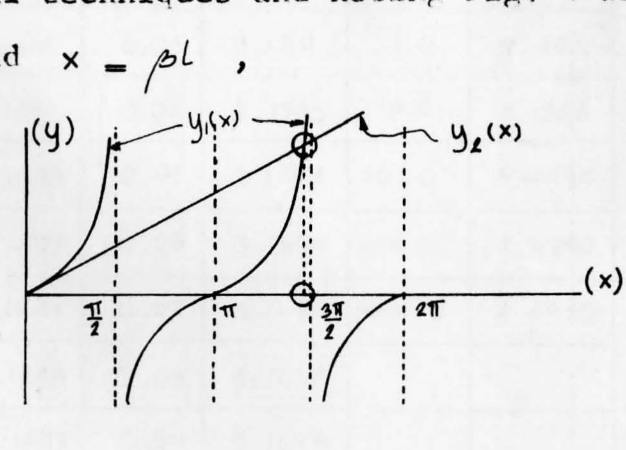


Figure 4. Geometric Plot of the Equation $\tan\beta l = \beta l$

The lowest root of this Equation (23c) is $\beta l = 4.4935$.

The general solution of Equation (23a) satisfies the equation

$$\frac{k_t l}{EI} \sin\beta l + \beta l(\beta l \sin\beta l - \cos\beta l) = 0 \qquad (24)$$

The solution of Equation (24) is obtained by letting

$$Z = RK \sin B + (B \sin B - \cos B) B$$

with $\beta l = B$ and $\frac{k_t l}{EI} = RK$

and using the computer program Number 1 given in Appendix I. For particular values of $\frac{k_t l}{EI}$. The associated values of $\beta_o l$, are listed in Table 1 and a plot of the results is shown in Fig. 5.

TABLE 1. Critical buckling load for a vertical column with spring support at one end, and simply supported at the other.

$\frac{k_t l}{EI}$	$\beta_o l$	$\frac{k_t l}{EI}$	$\beta_o l$	$\frac{k_t l}{EI}$	$\beta_o l$
0.000	3.1416	0.02	3.1480	0.1	3.1727
0.001	3.1419	0.03	3.1511	1.0	3.4057
0.005	3.1431	0.04	3.1543	10.0	4.1324
0.006	3.1435	0.05	3.1574	100.0	4.4494
0.007	3.1438	0.06	3.1605	1000.0	4.4890
0.008	3.1441	0.07	3.1636	10000.0	4.4935
0.009	3.1444	0.08	3.1667		
0.01	3.1448	0.09	3.1698		

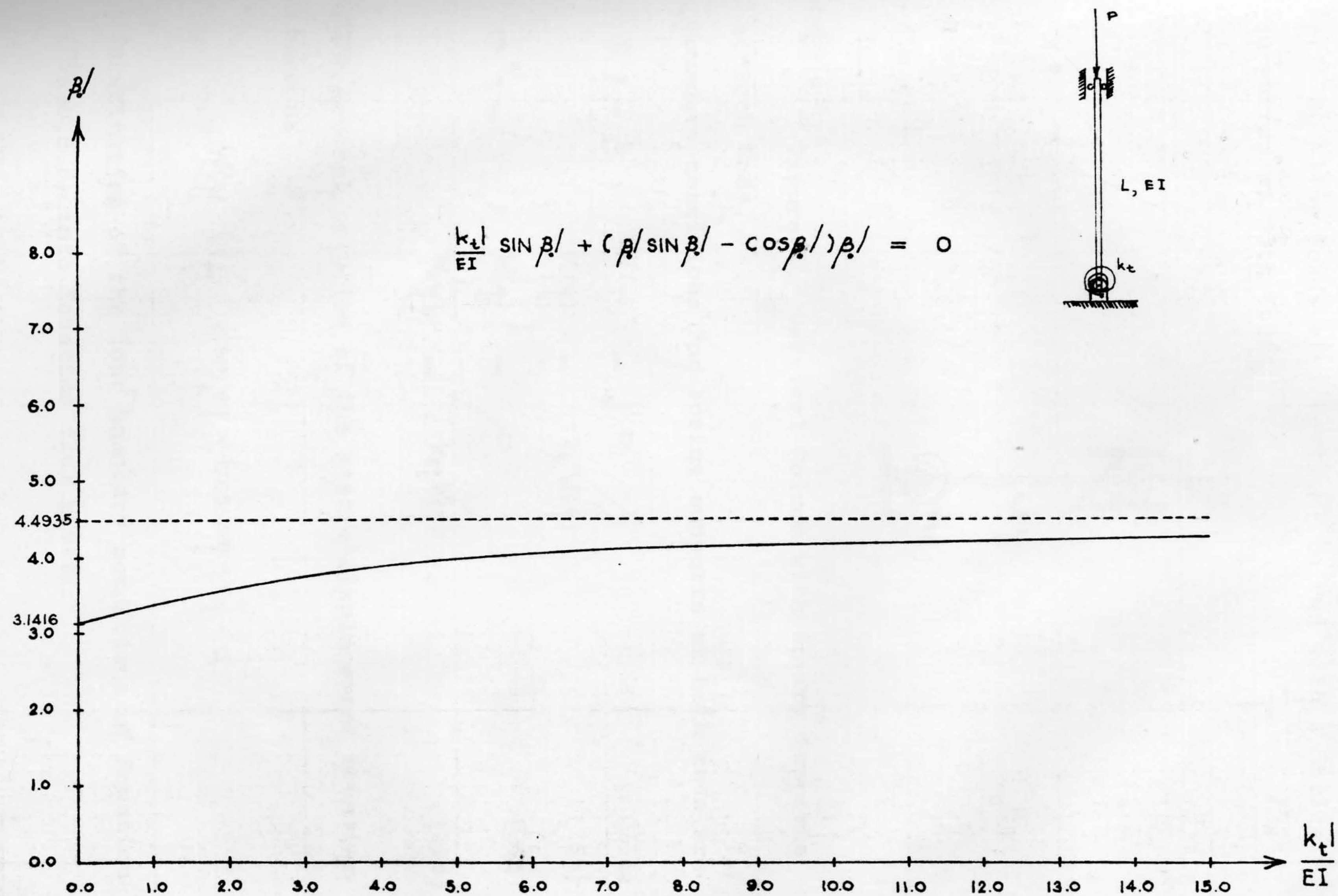


Figure 5. Curve of βl vs. $\frac{k_t l}{EI}$ for Vertical Column with a Spring Support at One End, The Other End Simply Support.

2.2 Buckling Loads of a Vertical Column with a Spring Supported at Both Ends.

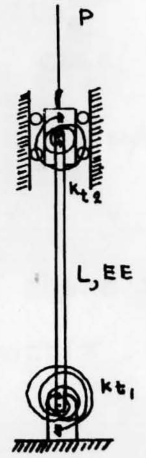


Figure 6. Vertical Column with Spring Supports at Both Ends.

Boundary conditions for spring supports at both ends are

$$@\ \eta = 0 \qquad W(\eta) = 0 \tag{25a}$$

$$M(\eta) = -k_{t_1} W'(\eta) \tag{25b}$$

$$@\ \eta = l \qquad W(\eta) = 0 \tag{25c}$$

$$M(\eta) = k_{t_2} W'(\eta) \tag{25d}$$

The general solution of the static displacement equation becomes

$$W(\eta) = a \sin \beta\eta + b \cos \beta\eta + c\eta + d \tag{26}$$

Substitution of the four boundary conditions of Equation (25a,b,c,d) into Equation (26) gives

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \beta & \frac{EI}{kt_1}\beta^2 & 1 & 0 \\ \sin\beta l & \cos\beta l & l & 1 \\ \left(-\frac{EI}{kt_2}\beta^2\sin\beta l + \beta\cos\beta l\right) & -\frac{EI}{kt_2}\beta^2\cos\beta l - \beta\sin\beta l & 1 & 0 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

For nontrivial solutions of these four constants the determinant of the coefficient matrix is set equal to zero giving

$$\beta + \left(\frac{EI}{kt_2}\beta^2\sin\beta l - \beta\cos\beta l\right)\left(1 + \frac{EI}{kt_1}\beta^2 l - \cos\beta l\right)$$
$$+ \left(\frac{EI}{kt_2}\beta^2\cos\beta l + \beta\sin\beta l\right)\left(\sin\beta l - \beta l\right)$$
$$+ \left(\frac{EI}{kt_1}\beta^2\sin\beta l - \beta\cos\beta l\right) = 0$$

For the special case $kt_1 = \infty$ and $kt_2 = 0$, it follows that

$$\tan\beta l = \beta l \tag{27b}$$

which yields the same values as Equation (23c).
For the special case $kt_1 = \infty$ and $kt_2 = \infty$, one obtains

$$2(\cos\beta l - 1) + \beta l\sin\beta l = 0 \tag{27c}$$

The lowest root of this Equation (27c) is $\beta_o l = 6.2832$.

The general solution of Equation (27a) satisfies the equation with $\frac{kt_1}{EI} = \infty$,

$$\left(\sin\beta l-\beta l\cos\beta l\right)\beta l-\frac{k_{t_2} l}{EI}\left[\beta l\sin\beta l-2\left(1-\cos\beta l\right)\right]=0 \qquad (28)$$

The solution of Equation (28) is obtained by letting

Z = (sinB - BcosB)B - RK(BsinB - 2(1 -cosB))

which βl = B and $\frac{k_{t_2} l}{EI}$ = RK

and using the computer program Number 2 given in Appendix I. For particular values of $\frac{k_{t_2} l}{EI}$ the corresponding values of $\beta_o l$ are listed in Table 2 and a plot of the results is shown in Figure 7.

TABLE 2. Critical buckling load for a vertical column with spring support at one end, and the other end fixed.

$\frac{k_{t_2} l}{EI}$	$\beta_o l$	$\frac{k_{t_2} l}{EI}$	$\beta_o l$	$\frac{k_{t_2} l}{EI}$	$\beta_o l$
0.00	4.4935	0.80	4.7118	7.00	5.6005
0.10	4.5274	0.90	4.7664	8.00	5.6617
0.20	4.5605	1.00	4.7925	9.00	5.7136
0.30	4.5926	2.00	5.0181	10.00	5.7578
0.40	4.6237	3.00	5.1921	100.00	6.2210
0.50	4.6539	4.00	5.3282	1000.00	6.2769
0.60	4.6833	5.00	5.4382	10000.00	6.2825
0.70	4.7118	6.00	5.5271	1000000.00	6.2832

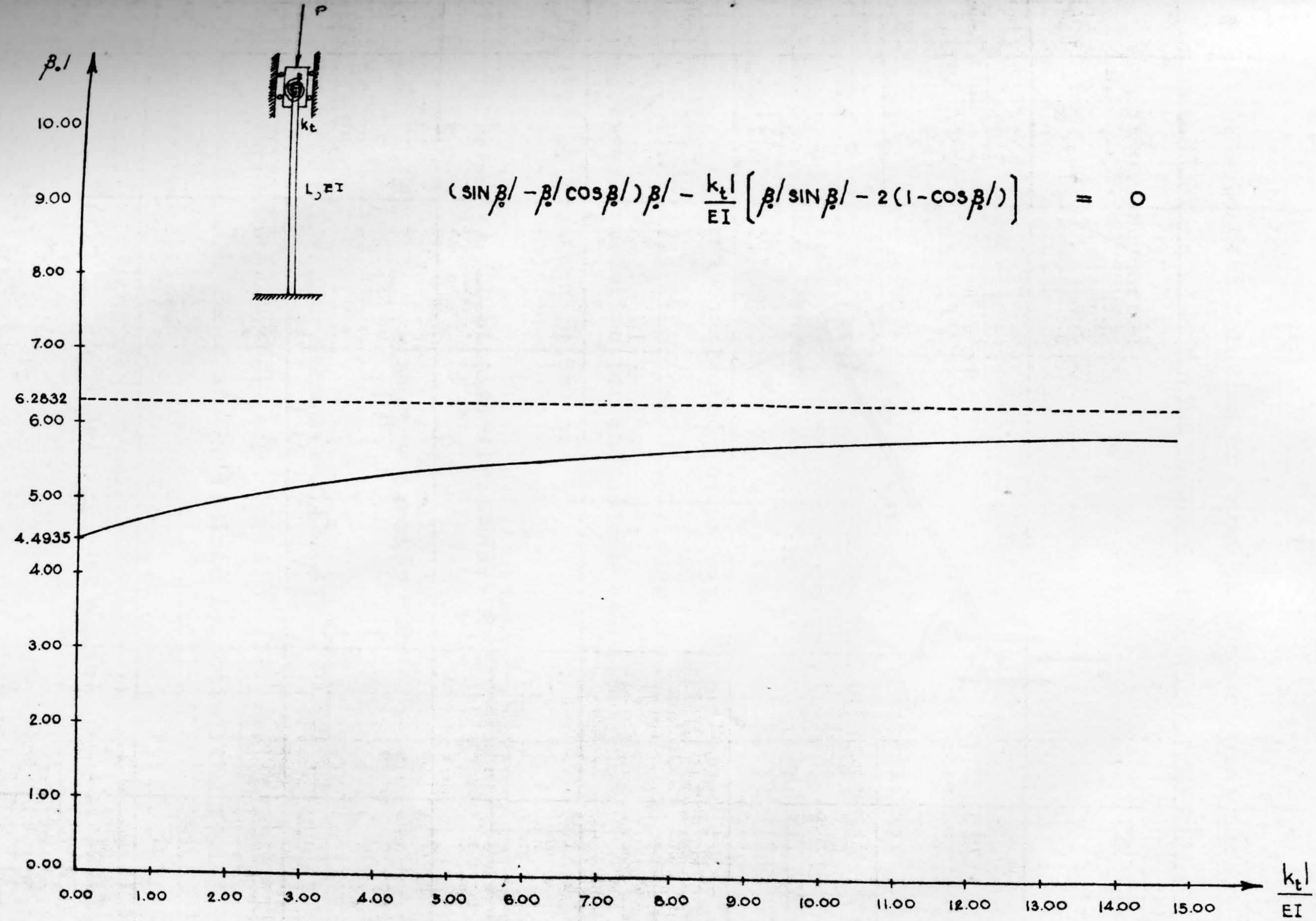


Figure 7. Curve of $\beta_o l$ vs. $\frac{k_t l}{EI}$ for a Vertical Column with a Spring Support at One End, The Other End Fixed.

CHAPTER III

3.1 The General Solution of Force-Head Shortening for Slanted Column with Spring Support at One End, The Other End Simply Supported.

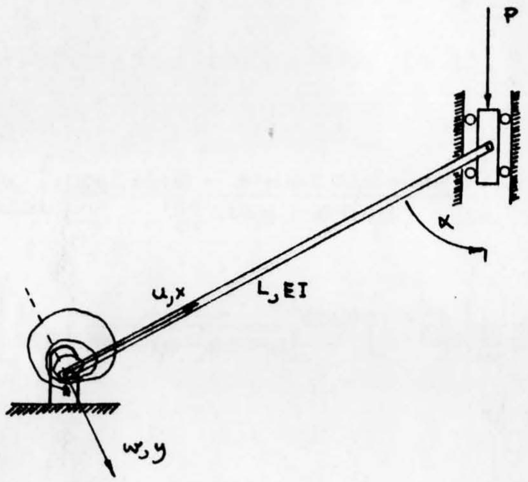


Figure 8. Slanted Column with Spring Support at One End, The Other End Simply Supported.

The general solution for the column takes the form given by Equation (18b) as

$$W(\eta) = b(\cos\beta\eta - 1) + c\sin\beta\eta - \frac{\phi}{\psi}\eta$$

with
$$W_{,\eta} = -\beta b\sin\beta\eta + c\beta\cos\beta\eta - \frac{\phi}{\psi}$$

and
$$W_{,\eta\eta} = -\beta^2 b\cos\beta\eta - c\beta^2\sin\beta\eta$$

The constants b and c are determined by noting the two boundary conditions

$$M(0) = -k_t W_{,\eta}(0) = -EI\, W_{,\eta\eta}(0) \qquad (29a)$$

and
$$M(1) = 0 \qquad (29b)$$

These become

$$b = k_t/EI\left[-c/\beta + \phi/\psi\beta^2\right] \quad (30a)$$

and

$$c = \left(\phi/\psi\right) k_t/EI\beta \cos\beta\left[\frac{1}{\frac{k_t}{EI}\cos\beta - \beta\sin\beta}\right] \quad (30b)$$

The lateral displacements takes the form

$$W(\eta) = \left(\frac{\phi}{\psi}\right)\frac{k_t}{EI\beta}\left[\frac{\cos\beta\sin\beta - \sin\beta\cos\beta + \sin\beta}{\frac{k_t}{EI}\cos\beta - \beta\sin\beta}\right] - \left(\frac{\phi}{\psi}\right)\eta \quad (31a)$$

with $W_{,\eta} = \left(\frac{\phi}{\psi}\right)\left[\frac{k_t}{EI\beta}\left(\frac{\beta\cos\beta}{\frac{k_t}{EI}\cos\beta - \beta\sin\beta}\right)\cos\beta\eta + \frac{k_t}{EI\beta}\left(\frac{\beta\sin\beta}{\frac{k_t}{EI}\cos\beta - \beta\sin\beta}\right)\sin\beta\eta - 1\right]$

Noting

$$\hat{A} = \left(\frac{k_t}{EI\beta}\right)\left(\frac{\beta\sin\beta}{\frac{k_t}{EI}\cos\beta - \beta\sin\beta}\right) \quad (32a)$$

$$\hat{B} = \left(\frac{k_t}{EI\beta}\right)\left(\frac{\beta\cos\beta}{\frac{k_t}{EI}\cos\beta - \beta\sin\beta}\right) \quad (32b)$$

It follows from Equation (31b) that

$$W_{,\eta} = \left(\frac{\phi}{\psi}\right)\left(\hat{B}\cos\beta\eta + \hat{A}\sin\beta\eta - 1\right) \quad (33a)$$

$$W_{,\eta}^2 = \left(\frac{\phi}{\psi}\right)^2\left(\hat{A}^2\sin^2\beta\eta + 2\hat{A}\hat{B}\cos\beta\eta\sin\beta\eta - 2\hat{A}\sin\beta\eta + \hat{B}^2\cos^2\beta\eta - 2\hat{B}\cos\beta\eta + 1\right) \quad (33b)$$

Noting Equation (15), the axial displacement becomes

$$U(\eta) = \psi\eta - \left(\frac{1}{4\beta}\right)\left(\frac{\phi}{\psi}\right)^2\left[\beta\hat{A}^2\eta - \hat{A}^2\sin\beta\eta\cos\beta\eta + 2\hat{A}\hat{B}\sin^2\beta\eta + 4\hat{A}\cos\beta\eta + \hat{B}^2\beta\eta + \hat{B}^2\sin\beta\eta\cos\beta\eta - 4\hat{A} - 4\hat{B}\sin\beta\eta + 2\beta\eta\right] \quad (34)$$

Recalling the characteristic Equation (16), it follows that

$$\psi - \left(\frac{1}{4\beta}\right)\left(\frac{\phi}{\psi}\right)^2\Big[\hat{A}^2\beta - \hat{A}^2\sin\beta\cos\beta + 2\hat{A}\hat{B}\sin^2\beta + 4\hat{A}\cos\beta - 4\hat{A} + \hat{B}^2\beta$$

$$+ \hat{B}^2\sin\beta\cos\beta - 4\hat{B}\sin\beta + 2\beta\Big]$$

$$+\left(\frac{\phi}{\psi}\right)\left[\left(\frac{\sin\beta}{\frac{k_t}{EI}\cos\beta - \beta\sin\beta}\right)\frac{k_t}{EI\beta} - 1\right]\cot\alpha = 0 \qquad (34)$$

Letting

$$S = \left(\frac{1}{4\beta}\right)\Big[\hat{A}^2\beta - \hat{A}^2\sin\beta\cos\beta + 2\hat{A}\hat{B}\sin^2\beta + 4\hat{A}\cos\beta - 4\hat{A} + \hat{B}^2\beta + \hat{B}^2\sin\beta\cos\beta - 4\hat{B}\sin\beta + 2\beta\Big] \qquad (35a)$$

$$T = \left[\left(\frac{-\sin\beta}{\frac{k_t}{EI}\cos\beta - \beta\sin\beta}\right)\frac{k_t}{EI\beta} + 1\right] \qquad (35b)$$

one obtains

$$\psi - S\left(\frac{\phi}{\psi}\right)^2 - \left(\frac{\phi}{\psi}\right)T\cot\alpha = 0 \qquad (36a)$$

Using the quadratic equation, it follows that

$$\phi = \frac{1}{2S}\psi\left[-T\cot\alpha \pm \left(T^2\cot^2\alpha + 4S\psi\right)^{\frac{1}{2}}\right] \qquad (36b)$$

Numerically it is found that for $T < 0$, the minus sign in front of the radical in Equation (36b) must be used, while $T > 0$ the positive sign is necessary.

3.2 The Solution of Force-Head Shortening for Slanted Column with Two End Simply Supported.

Letting $k_t = 0$, Equations 32a and 32b become

$$T_2 = 1 \qquad (37a)$$

$$S_2 = \frac{1}{2} \tag{37b}$$

Notting Equation (20), the head displacement becomes

$$\delta = \left[-\psi - \frac{1}{4\beta}\left(\frac{\phi}{\psi}\right)^2 \left\{ \hat{A}^2\beta - \hat{A}^2\sin\beta\cos\beta + 2\hat{A}\hat{B}\sin^2\beta + 4\hat{A}\cos\beta + \beta\hat{B}^2 + \hat{B}^2\sin\beta\cos\beta - 4\hat{A} - 4\hat{B}\sin\beta + 2\beta \right\} \right]\cos\alpha + \left[\frac{k_t}{EI\beta}\left(\frac{\sin\beta}{\frac{k_t}{EI}\cos\beta - \beta\sin\beta} \right) - 1 \right]\left(\frac{\phi}{\psi}\right)\sin\alpha \tag{38a}$$

Noting Equations (35a) and (35b), the head displacement parameter is written

$$\delta = \left[-\psi + \left(\frac{\phi}{\psi}\right)^2 S \right]\cos\alpha - T\left(\frac{\phi}{\psi}\right)\sin\alpha \tag{38b}$$

The corresponding load parameter is given by Equation (19a) as

$$\tau = -\frac{R^2}{\beta_0^2}\left(\psi\cos\alpha + \phi\sin\alpha\right) \tag{38c}$$

The Load-Displacement diagram is obtained by plotting τ vs. δ given by the latter two equations. The range of the parametric scalar quantity ψ is predetermined by noting the conditions that

$$@\ \tau = \delta = 0 \qquad \psi = 0 \tag{39a}$$

$$@\ \tau = 0 \qquad \psi = (S - T)\cot^2\alpha \tag{39b}$$

The curve τ versus δ shown in Figure 9 for $\alpha = 82.5^\circ$ and R = 50 is obtained from program Number 4 in Appendix I. The range of the parameter ψ is determined utilizing program Number 3 in Appendix I.

3.3 The Solution of Force-Head Shortening for Slanted Column With Lower End Fixed and Upper End Simply Supported.

Letting $k_t = \infty$, Equations (32a) and (32b) yields the following equations

$$T_3 = \frac{1}{\beta}\left(\beta - \tan\beta\right) \tag{40a}$$

$$S_3 = \left(\frac{1}{4}\right)\left[3\left(1 - \frac{\tan\beta}{\beta}\right) + \tan^2\beta\right] \tag{40b}$$

The corresponding head-shortening parameter is given by Equation (38b) as

$$\delta = \left[-\psi + \left(\frac{\phi}{\psi}\right)^2 S\right]\cos\alpha - T\left(\frac{\phi}{\psi}\right)\sin\alpha \tag{41a}$$

The following equations are also applicable for the present case

$$\tau = \frac{-R^2}{\beta_0^2}\left(\psi\cos\alpha + \phi\sin\alpha\right) \tag{41b}$$

$$@\ \tau = \delta = 0 \qquad \psi = 0 \tag{42a}$$

$$@\ \tau = 0 \qquad \psi = \left(S - T\right)\cot^2\alpha \tag{42b}$$

The curve τ versus δ shown in Figure 9 for $\alpha = 82.5^\circ$ and R = 50 is obtained from program Number 6 in Appendix I. The range of the parameter ψ is determined utilizing program Number 5 in Appendix I.

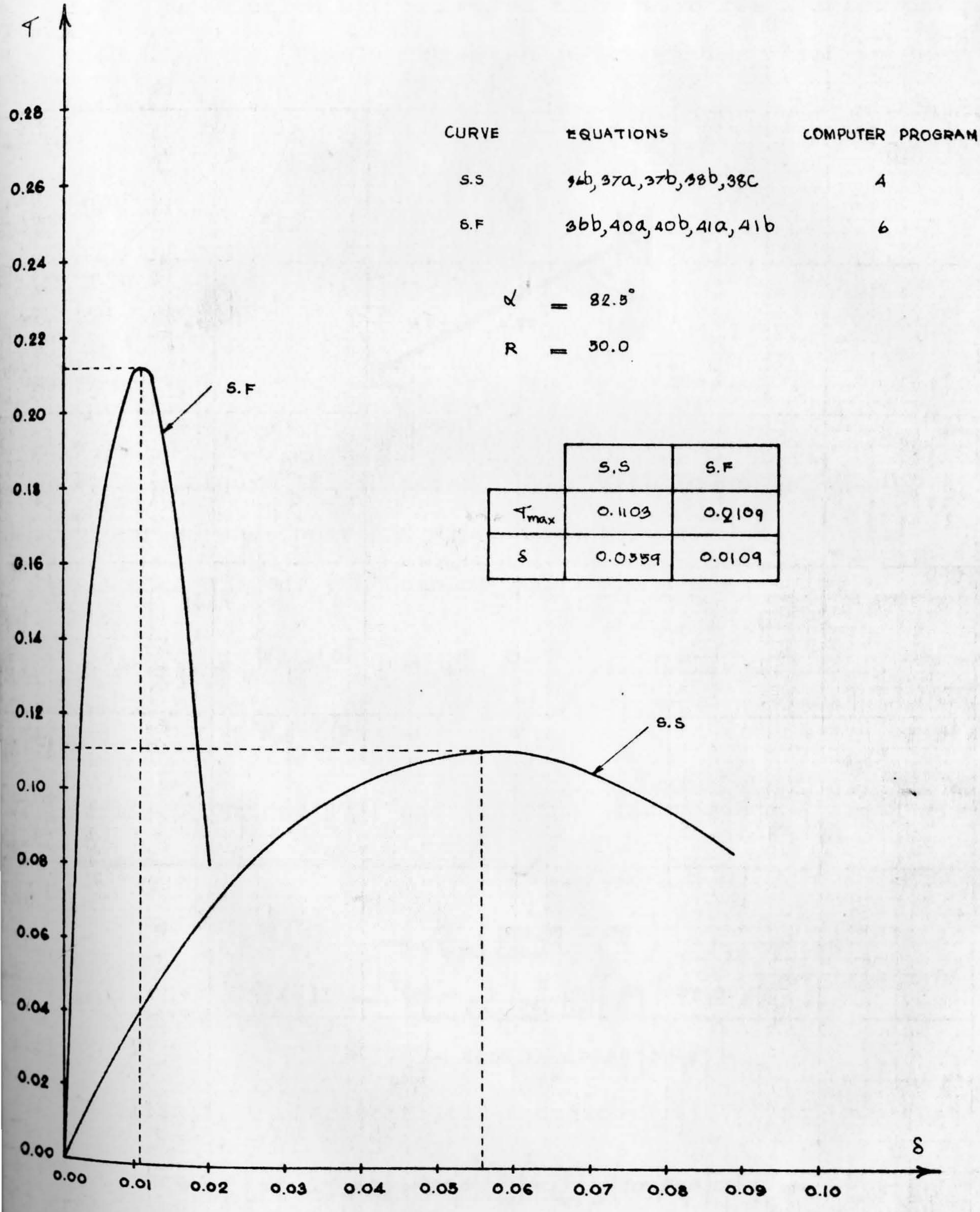


Figure 9. Curve of τ vs. δ for The Slanted Column with various end condition.

3.4 The Solution of Force-Head Shortening for Slanted Column with Lower End Simply Supported and Upper End Fixed in Rotation.

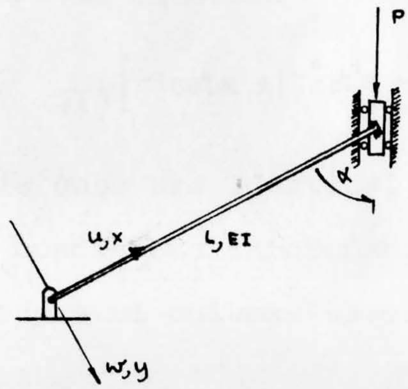


Figure 10. Slanted Column with Lower End Simply Supported and Upper End Fixed in Rotation.

For this column, the boundary conditions are

$$W_{,\eta\eta}(0) = 0 \tag{43a}$$

and

$$W_{,\eta}(1) = 0 \tag{43b}$$

Noting Equations (7b) and (15) the transverse and axial displacements become, respectively,

$$W(\eta) = \frac{\phi}{\psi\beta}\left(\frac{\sin\beta\eta}{\cos\beta} - \beta\eta\right) \tag{44}$$

and

$$U(\eta) = \psi\eta - \frac{1}{4\beta\cos^2\beta}\left(\frac{\phi}{\psi}\right)^2\Big[\beta\eta\left(1+2\cos^2\beta\right) - \left(4\cos\beta - \cos\beta\eta\right)\sin\beta\eta\Big] \tag{45}$$

Recalling the characteristic Equation (16), it follows that

$$\psi - \frac{1}{4}\left[3\left(1-\frac{\tan\beta}{\beta}\right)+\tan^2\beta\right]\left(\frac{\phi}{\psi}\right)^2 + \left(\frac{\phi}{\psi}\right)\frac{1}{\beta}\left(\tan\beta-\beta\right)\cot\alpha = 0$$

Letting

$$T_4 = \frac{1}{\beta}\left(\beta - \tan\beta\right) \tag{46a}$$

$$S_4 = \frac{1}{4}\left[3\left(1-\frac{\tan\beta}{\beta}\right)+\tan^2\beta\right] \tag{46b}$$

one obtains

$$\psi - S\left(\frac{\phi}{\psi}\right)^2 - \left(\frac{\phi}{\psi}\right) T\cot\alpha = 0 ,$$

or using the quadratic equation

$$\phi = \frac{1}{2S}\psi\left[-T\cot\alpha \pm \left(T^2\cot^2\alpha + 4S\psi\right)^{\frac{1}{2}}\right] \tag{47}$$

Equations for this case are identical to the latter case with the end boundary conditions interchanged.

Thus, these two types of columns have the same curve τ versus δ curve.

3.5 The Solution of Force-Head Shortening for Slanted Column with Both Ends Fixed.

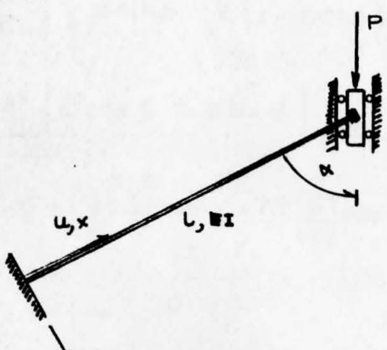


Figure 11. Slanted Column with Both Ends Fixed.

For slanted columns with both the lower and the upper end fixed in rotation, Equations (8a) and (11a) are applicable with

$$W_{,\eta}(0) = 0$$

$$W_{,\eta}(1) = 0$$

The resulting displacements are

$$W(\eta) = \frac{\phi}{\beta\eta}\left[\sin\beta\eta - \beta\eta + \left\{\frac{1-\cos\beta}{\sin\beta\eta}\left(1-\cos\beta\eta\right)\right\}\right] \quad (48)$$

$$U(\eta) = \psi\eta - \frac{1}{8\beta}\left(\frac{\phi}{\psi}\right)^2\left\{\frac{1-\cos\beta}{\sin\beta}\left[\frac{1-\cos\beta}{\sin\beta}\left(2\beta\eta - \sin 2\beta\eta\right)\right.\right.$$
$$\left. -6 + 8\cos\beta\eta - 2\cos 2\beta\eta\right]$$
$$\left. +6\beta\eta - 8\sin\beta\eta + \sin 2\beta\eta\right\} \quad (49)$$

Equations (19a), (36b), (38b), (39a) and (39b) are also applicable for this case except

$$S_5 = \frac{1}{4\beta\sin^2\beta}\left[\left(1-\cos\beta\right)\left(\beta - \beta\cos\beta - 3\sin\beta\right)\right]$$
$$-3\sin\beta\left(\cos 2\beta - \cos\beta - \beta\sin\beta\right) \quad (50a)$$

$$T_5 = \frac{1}{\beta\sin\beta}\left[\beta\sin\beta - 2\left(1-\cos\beta\right)\right] \quad (50b)$$

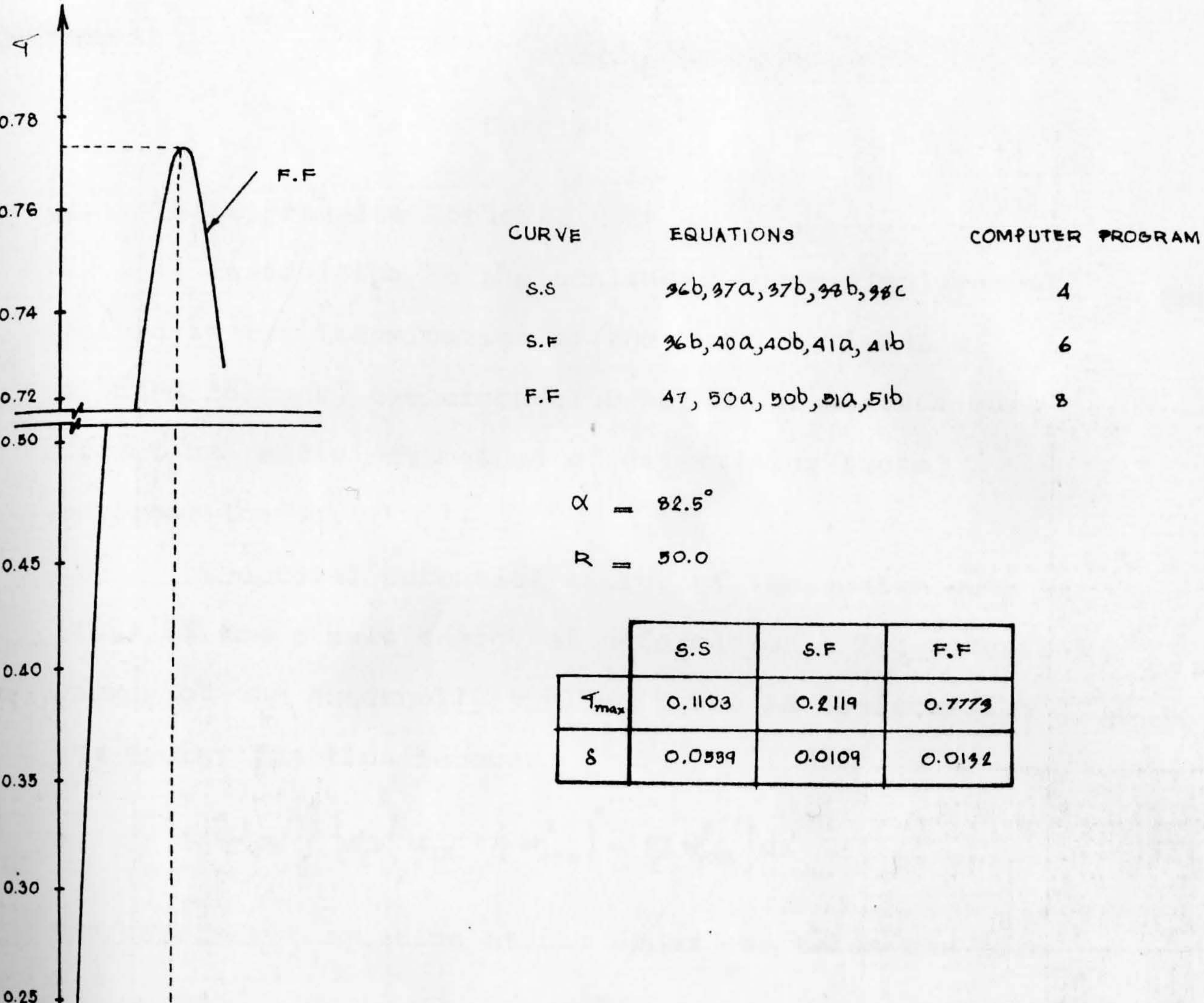
$$\tau = \frac{-R^2}{\beta_0^2}\left(\psi\cos\alpha + \phi\sin\alpha\right) \quad (51a)$$

$$\delta = \left[-\psi + \left(\frac{\phi}{\psi}\right)^2 S\right]\cos\alpha - T\left(\frac{\phi}{\psi}\right)\sin\alpha \quad (51b)$$

$$@\ \tau = \delta = 0 \qquad \psi = 0 \quad (52a)$$

$$@\ \tau = 0 \qquad \psi = \left(S - T\right)\cot^2\alpha \quad (52b)$$

The curve τ versus δ shown in Figure 12 for $\alpha = 82.5^\circ$ and R = 50 is obtained from program Number 7 in Appendix I. The range of the parameter ψ is determined utilizing program Number 8 in Appendix I.



	S.S	S.F	F.F
τ_{max}	0.1103	0.2119	0.7779
δ	0.0559	0.0109	0.0132

Figure 12. Curve of τ vs. δ for The Slanted Column with Various End Condition.

CHAPTER IV

4.1 The Eigenvalue Method

In addition to the nonlinear force-displacement method in the dtermination of the buckling loads of slanted columns, the eigenvalue method presents a more direct and efficient method of determining these critical loads.

The total potential energy of the system consists of the strain energy of deformation. The potential energy of the externally applied force is neglected. The energy function becomes

$$\hat{V} = \int_0^l \left[\frac{EA}{2}\left(u_{,x} + \frac{1}{2}w_{,xx}^2\right)^2 + \frac{EI}{2}w_{,xx}^2\right]dx$$

The constraint equation at the upper end takes the form

$$@\ x = l \qquad u\sin\alpha + w\cos\alpha = 0$$

The dimensionless form of the total energy, utilizing the Lagrange multiplier technique, is written

$$\hat{V}_d = AEl\left\{\frac{1}{2}\int_0^1\left[u_{,\eta}^2 - \psi w_{,\eta}^2\right]d\eta + \frac{1}{2R^2}\int_0^1 w_{,\eta\eta}^2\, dx - \rho\left[u(1)\sin\alpha + w(1)\cos\alpha\right]\right\} \tag{53}$$

where the nondimensional notation given in Equation (5a, b, c, d, e, f) is utilized. The parameter ρ is the Lagrange multiplier constant.

The variational operations on Equation (53) yield the equations

$$W_{,\eta\eta\eta\eta} + \beta W_{,\eta\eta} = 0 \quad ; \text{ for } 0 \leq \eta \leq 1 \tag{54a}$$

and
$$u \sin\alpha + w \cos\alpha = 0 \quad ; \text{ at } \eta = 1 \tag{54b}$$

Equation (54a) in solution form is written

$$W(\eta) = a\cos\beta\eta + b\sin\beta\eta + c\eta + d \tag{55}$$

Equation (55) contains four constants a, b, c, and d, which together with parameter ρ (included in the boundary conditions) yields five constants which must be investigated. Four boundary conditions on the function (two at each end) as given by Equations (8,9,11 and 12) (with $\phi = -\rho\cos\alpha$) yield four homogeneous equations. These four equations together with Equation (54b) produce a family of five homogeneous equations from which the variable β relating the critical buckling load and the slope of the column is determinable.

4.2 Eigenvalues for a Slanted Column with Two Ends Simply Supported

The boundary conditions for simple supports at both end are

$$W(0) = 0 \tag{56a}$$

$$W_{,\eta\eta}(0) = 0 \tag{56b}$$

$$W_{,\eta\eta}(1) = 0 \tag{56c}$$

$$W_{,\eta\eta\eta}(1) + \beta^2 W_{,\eta}(1) + \rho R^2 \cos\alpha = 0 \tag{56d}$$

with $\quad u(1)\sin\alpha + w(1)\cos\alpha = 0$ (56e)

Noting Equation (55) and Equations (56a, b, c, d, e) one obtains the five equations

$$a + d = 0$$

$$a\beta^2 = 0$$

$$a\beta^3\sin\beta - b\beta^3\cos\beta - \beta^3 a\sin\beta + \beta^3 b\cos\beta\,\eta + c\beta^3 + \rho R^2\cos\alpha = 0$$

$$\rho\sin^2\alpha + a\cos\alpha\cos\beta + b\cos\alpha\sin\beta + c\beta\cos\alpha + d\cos\alpha = 0$$

with a matrix form of these equations written as

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \cos\beta & \sin\beta & 0 & 0 & 0 \\ 0 & 0 & \beta^3 & 0 & R^2\cos\alpha \\ \cos\beta\cos\alpha & \cos\alpha\sin\beta & \beta\cos\alpha & \cos\alpha & \sin^2\alpha \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \\ d \\ \rho \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Investigating the possibility of curved forms of equilibrium, we observe that the only way to have a nontrivial solution of these five equations is to have the determinant of the coefficients equal to zero. This determinant is

$$\begin{vmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \cos\beta & \sin\beta & 0 & 0 & 0 \\ 0 & 0 & \beta^3 & 0 & R^2\cos\alpha \\ \cos\beta\cos\alpha & \cos\alpha\cos\beta & \beta\cos\alpha & \cos\alpha & \sin^2\alpha \end{vmatrix} = 0$$

which yields

$$\sin\beta\left(\beta^3\sin^2\alpha - \beta R^2\cos^2\alpha\right) = 0 \qquad (58)$$

Solutions of the latter Equation take the form

$$\sin\beta = 0 \qquad (59a)$$

and $$R\cot\alpha - \beta = 0 \qquad (59b)$$

The roots of Equation (59a) represent the critical buckling loads for the special case of a vertical column (i.e. $\alpha = 0$). The roots of Equation (59b) yield the critical buckling loads in term of the variable α . The numerical solution of the latter equation are shown in Table 3. Computer program Number 9 used to determine these values is contained in Appendix I.

TABLE 3. Eigenvalue for a slanted column with two end simply supported.

α	β	α	β	α	β	α	β
0.0	3.1415	45.0	3.1415	86.0	3.1415	88.0	1.7400
15.0	3.1415	60.0	3.1415	86.5	3.0581	89.0	0.8727
30.0	3.1415	75.0	3.1415	87.0	2.6200	90.0	0.0000

4.3 Eigenvalues for a Slanted Column with Spring Supports at Both Ends.

Boundary conditions for spring supports at both ends are

$$@\ \eta = 0 \qquad W(\eta) = 0 \tag{60a}$$

$$W_{,\eta\eta}(\eta) = \frac{kt_1}{EI}\, y'(\eta) \tag{60b}$$

$$@\ \eta = 1 \qquad W_{,\eta\eta}(\eta) = \frac{-kt_2}{EI}(y') \tag{60c}$$

$$W_{,\eta\eta\eta}(\eta) + \beta^2 W_{,\eta}(\eta) + \rho R^2 \cos\alpha = 0 \tag{60d}$$

with

$$u(1)\sin\alpha + w(1)\cos\alpha = 0 \tag{60e}$$

Eqution (54a) in solution form is written

$$W(\eta) = a\cos\beta\eta + b\sin\beta\eta + c\eta + d \tag{61}$$

Substitution of the five boundary conditions of Equation (60a,b,c,d,e) into Equation (61) gives

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ \beta & \frac{kt_1}{EI} & \frac{kt_1}{EI} & 0 & 0 \\ (\beta\cos\beta + \frac{kt_2}{EI}\sin\beta) & (\beta\sin\beta - \frac{kt_2}{EI}\cos\beta) & -\frac{kt_2}{EI} & 0 & 0 \\ 0 & 0 & \beta^3 & 0 & R^2\cos\alpha \\ \cos\beta\cos\alpha & \cos\beta\sin\beta & \beta\cos\alpha & \cos\alpha & \sin^2\alpha \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \\ d \\ \rho \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

For nontrivial solutions of these five constants the determinant of the coefficient matrix is set equal to zero giving

$$\left\{R^2\left(\frac{kt_1}{EI}\right)\left(\frac{kt_2}{EI}\right)\cos^2\alpha\right\}\cos^2\beta - \left\{2\left(\frac{kt_1}{EI}\right)\left(\frac{kt_2}{EI}\right)R^2\cos^2\alpha - \beta^4\left(\frac{kt_2}{EI}\right)\sin^2\alpha\right.$$

$$\left. + \beta^2R^2\left(\frac{kt_2}{EI}\right)\cos^2\alpha - \beta^4\left(\frac{kt_1}{EI}\right)\sin^2\alpha + \beta^2R^2\left(\frac{kt_1}{EI}\right)\cos^2\alpha\right\}\cos\beta$$

$$+\left\{\left(\frac{kt_1}{EI}\right)\beta R^2\cos^2\alpha - \beta^5\sin^2\alpha + \beta^3R^2\cos^2\alpha + \beta R^2\left(\frac{kt_2}{EI}\right)\cos^2\alpha\right.$$

$$\left. + \beta^3\left(\frac{kt_1}{EI}\right)\left(\frac{kt_2}{EI}\right)\sin^2\alpha - \beta R^2\left(\frac{kt_1}{EI}\right)\cos^2\alpha + R^2\left(\frac{kt_1}{EI}\right)\left(\frac{kt_2}{EI}\right)\cos^2\alpha\sin\beta\right\}\sin\beta$$

$$+\left(\frac{kt_1}{EI}\right)\left(\frac{kt_2}{EI}\right)R^2\cos^2\alpha \quad = \quad 0$$

For the special case of a fixed support at the lower end and a simple support at the upper end, that is,

$$\left(\beta-\tan\beta\right)\frac{\cos^2\alpha}{\beta^3} - \frac{\sin^2\alpha}{R^2} \quad = \quad 0 \qquad (62b)$$

For the special case of both ends fixed, that is, $\frac{kt_1}{EI} = \infty$, one obtains

$$\frac{\beta^3}{R^2}\sin^2\alpha\sin\beta + \cos^2\alpha\cos^2\beta - 2\cos^2\alpha\cos\beta - \beta\cos^2\alpha\sin\beta$$

$$+\cos^2\alpha\sin^2\beta + \cos^2\alpha \quad = \quad 0 \qquad (62c)$$

The roots of Equations (62b, c) yield the critical buckling loads in term of the variable α . The numerical solution of the latter equation are shown in Table 4 and Table 5. Computer program 10 and program 11 are used to determine these values is contained in Appendix I

TABLE 4. Eigenvalues of a slanted column fixed at lower ends and upper end simply supported for various inclination angles.

α	β	α	β	α	β	α	β
0.0	4.4935	77.0	4.4545	85.0	3.8535	89.5	1.6150
15.0	4.4933	78.0	4.4463	86.0	3.3728	89.6	1.6030
30.0	4.4928	79.0	4.4350	86.5	3.0828	89.7	1.5849
45.0	4.4916	80.0	4.4185	87.0	2.7883	89.8	1.5750
60.0	4.4879	81.0	4.3970	88.0	2.2150	89.9	0.8149
75.0	4.4656	83.0	4.2855	88.5	1.9600	90.0	0.0000
76.0	4.4610	84.0	4.1454	89.0	1.7556		

TABLE 5. Eigenvalues of slanted column fixed at both ends for various inclination angles.

α	β	α	β	α	β	α	β
0.0	6.2831	80.0	6.2831	87.0	3.9219	89.8	3.1483
15.0	6.2831	82.0	6.2830	88.0	3.5115	89.9	3.1383
30.0	6.2831	83.0	6.1869	89.0	3.2383	89.95	0.5483
45.0	6.2831	84.0	5.5865	89.5	3.1682	90.00	0.0000
60.0	6.2831	85.0	4.9915	89.6	3.1543		
75.0	6.2831	86.0	4.4272	89.7	3.1483		

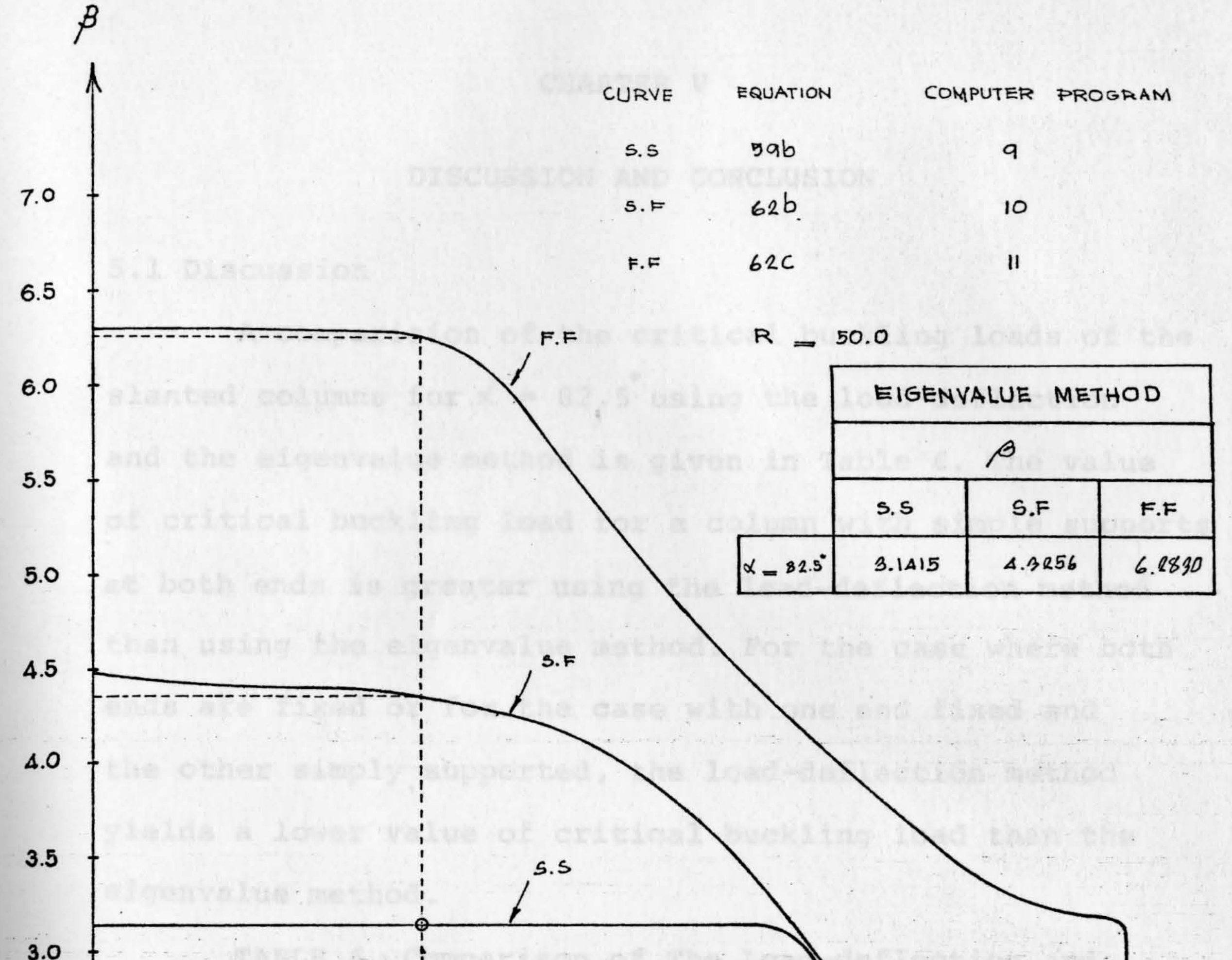


Figure 13. Plots of Critical Buckling Load versus

Inclination Angles.

CHAPTER V

DISCUSSION AND CONCLUSION

5.1 Discussion

A comparision of the critical buckling loads of the slanted columns for $\alpha = 82.5^\circ$ using the load-deflection and the eigenvalue method is given in Table 6. The value of critical buckling load for a column with simple supports at both ends is greater using the load-deflection method than using the eigenvalue method. For the case where both ends are fixed or for the case with one end fixed and the other simply supported, the load-deflection method yields a lower value of critical buckling load than the eigenvalue method.

TABLE 6. Comparison of The Load-deflection and Eigenvalue Methods.

β at $\alpha = 82.5^\circ$, $R = 50$	S.S	S.F	F.F
From Load-deflection method	3.76649	1.59059	3.84772
From Eigenvalue method	3.1416	4.3256	6.2892

This condition may be explained by noting Figure 14. For the case of simple supports at both ends the critical buckling load remains independent of the angle α up to approximately 86°. The two methods differ by only 23% which is a reasonable difference since the first method is nonlinear and the second linear. For the fixed-fixed and fixed-simply supported case the value of critical buckling load is more critically effected by a change in α for α greater than 80°. In neither method is the change in α accounted for directly. Thus, there is no restriction that one method is more accurate than the other.

5.2 Conclusion

This thesis presents the solution of a class of large deflection problems of slanted columns. The buckling loads determined from the load-deflection solutions are not the same as those obtained by eigenvalue method. The latter, nevertheless, provides a simplified method for determining the critical buckling loads for a given column. In fact, the eigenvalues are practically applicable for column with small angle (i.e. $0 < \alpha < \frac{2\pi}{3}$) and large slenderness ratio (i.e. $R \geq 50$).

The comparison of the load-deflection method and eigenvalue method shows that the effects of the change in angle α must be taken into consideration. It is a basic principle of elastic stability that the equations of equilibrium are derived in a deformed state. If the original state is used as the reference as it was done earlier, Equation (19a) yields the solution. If the deformed state is used as the reference, the angle α, at the load point becomes α' where

$$\alpha' = \alpha + \theta = \alpha + \tan^{-1}\left(\frac{w}{1-u}\right), \quad \text{at } \eta = 1$$

The potential energy is written in dimensionless form for the load-deflection method as

$$\frac{V}{AEL} = \frac{1}{2}\int_0^1 \left[U_{,\eta} + \frac{1}{2}W_{,\eta}^2 \right]^2 d\eta + \frac{1}{2R^2}\int_0^1 W_{,\eta\eta}^2 d\eta + \left[f\left(\cos\alpha' U - \sin\alpha' W\right) - \lambda\left(U\sin\alpha' + W\cos\alpha'\right)_{\eta=1} \right]$$

The corresponding values of τ and γ denoted by τ' and γ' are obtained from Figure (14) as

$$\tau' = \frac{fR^2}{\beta_0^2} = \frac{-R^2}{\beta_0^2}\left(\psi\cos\alpha' + \phi\sin\alpha'\right) \quad (63a)$$

$$\gamma' = \frac{\lambda' R^2}{\beta_0^2} = \frac{R^2}{\beta_0^2}\left(\psi\sin\alpha' - \phi\cos\alpha'\right) \quad (63b)$$

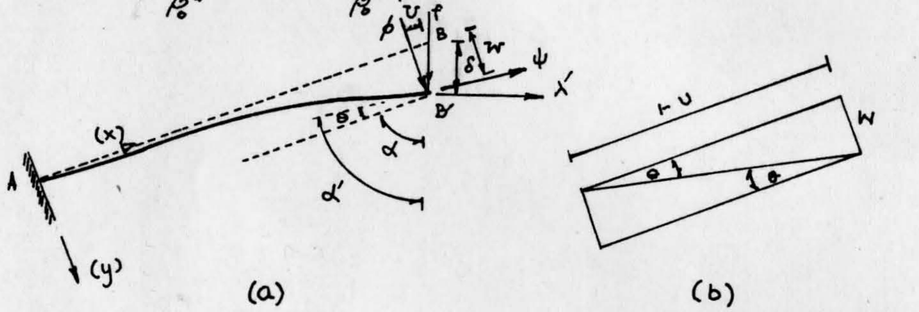


Figure 14. Correction Factor for the Angular Change.

The potential energy fuction for the eigenvalue method is written as

$$\hat{V}_d = AEL\left\{ \frac{1}{2}\int_0^1 \left[u_{,\eta}^2 - \psi w_{,\eta}^2 \right] d\eta + \frac{1}{2R^2}\int_0^1 w_{,\eta\eta}^2 d\eta - \rho\left[u(1)\sin\alpha' + w(1)\cos\alpha' \right] \right\} \quad (64a)$$

with $\alpha' = \alpha + \tan^{-1}\left(\frac{w}{1-U}\right)$. Equation (54a) is satisfied. Equation (54b) becomes

$$u\sin\alpha' + w\cos\alpha' = 0 \quad , \text{at } \eta = 1 \quad (64b)$$

with $$\phi = -\rho\cos\alpha' \quad (64c)$$

Equations (63) and (64) define a new dual set of equations which may be utilized to perform an extended comparison between the two methods.

APPENDIX I

Computer Program Number 1

Solution of the equation

$$\frac{K_t L}{EI} \sin \beta L + \left(\beta L \sin \beta L - \cos \beta L \right) \beta L = 0$$

```
1       DOUBLE PRECISION RK,Z1,B,Z,B1
        RK=0.1
10      Z1=10.0
        DO 50 I=31416,62830
        B=I/10000.0
        Z=RK*(DSIN(B))+(B*DSIN(B)-DCOS(B))*B
        IF (Z.LT.Z1) GO TO 20
        GO TO 50
20      IF (Z.GE.0) GO TO 30
        GO TO 50
30      Z1=Z
        B1=B
50      CONTINUE
        WRITE (6,100) RK,B1,Z1
100     FORMAT (3F15.5)
        RK=RK+0.1
        IF (RK.LE.200) GO TO 10
        STOP
        END
```

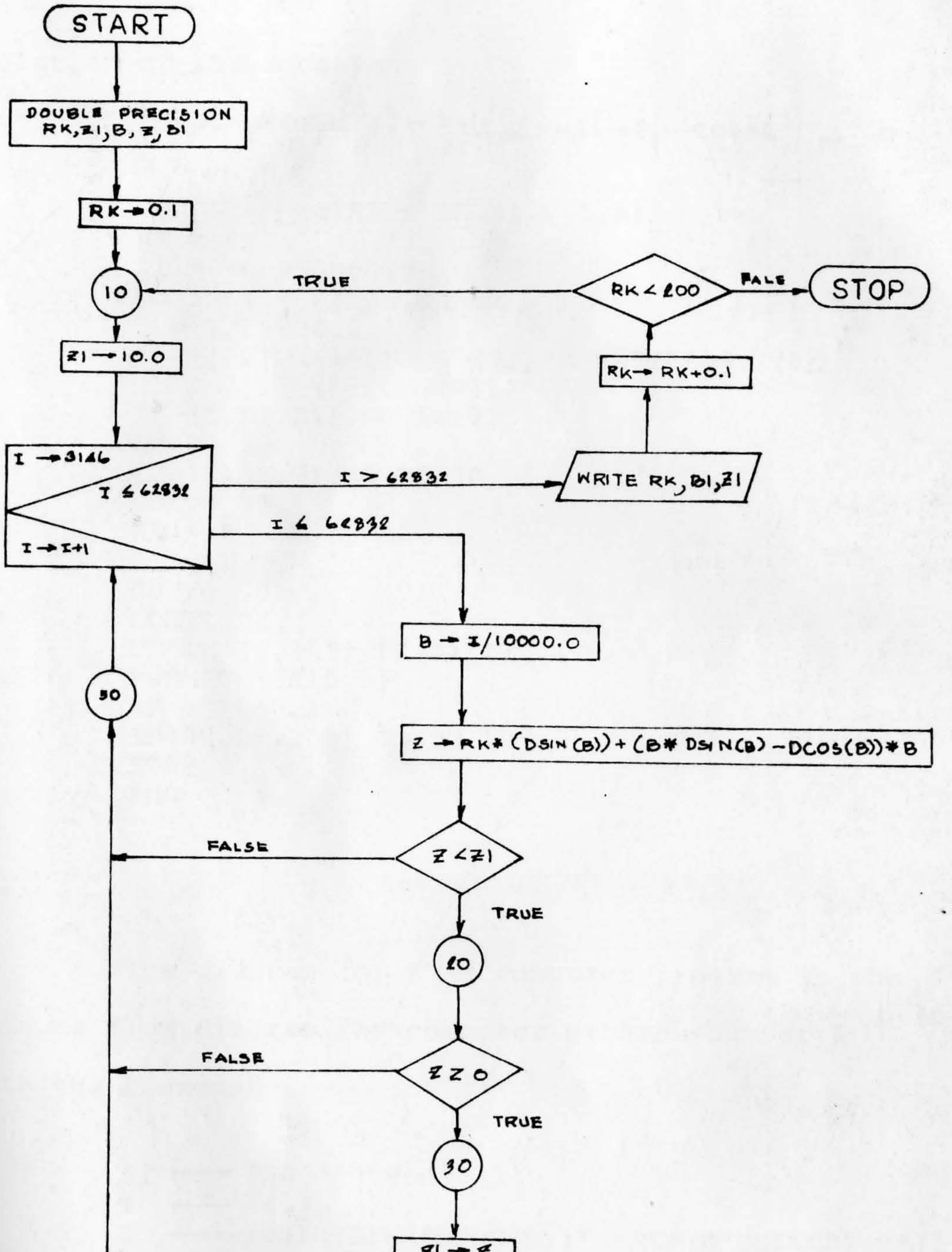


Figure A1. Flow diagram for computer program Number 1.

Computer Program Number 2

Solution of the equation

$$\left(\sin\beta l - \beta l\cos\beta l\right)\beta l - \frac{k_t l}{EI}\left(\beta l\sin\beta l - 2\left(1-\cos\beta l\right)\right) = 0$$

```
1         DOUBLE PRECISION RK,Z1,B,Z,B1
          RK = 0.1
10        Z1 = 400000000.0
          DO 50 I = 44935,62832
          B = I/10000.0
          Z = (DSIN(B)-B*DCOS(B))*B-RK*(B*DSIN(B)
               -2*(1-DCOS(B)))
          IF (Z.LT.Z1) GO TO 20
          GO TO 50
20        IF (Z.GE.O) GO TO 30
          GO TO 50
30        Z1 = Z
          B1 = B
          GO TO 50
50        CONTINUE
          WRITE (6,100) RK,B1,Z1
100       FORMAT (3F15.5)
          RK = RK+0.1
          IF(RK.LE.200) GO TO 10
          STOP
          END
```

Flow diagram for this computer program is the same as flow diagram for computer program Number 1 with the changes

```
Z1 ——→ 400000000.0
I  ——→ 44935
Z  ——→ (DSIN(B)-B*DCOS(B))*B-RK*(B*DSINCB)-Z*(1-DCOS(B)))
```

Computer Program Number 3

Solution of the equation

$$\psi - (S-T)\cot^2\alpha = 0$$

```
1         DOUBLE PRECISION X,ALPHA,XINCR,B,T,S,Z1
          X = -0.00000001
          ALPHA = 1.4398966
          XINCR = -0.001
          DO 10 I = 1,10
          DCOSA = DCOS(ALPHA)
          DSINA = DSIN(ALPHA)
          DCOTA = DCOSA/DSINA
          T = 1.0
          S = 0.5
          Z1 = X - ((DCOTA**2)*(S-T))
          WRITE (6,100) ALPHA,X,Z1
10        x = X+XINCR
100       FORMAT (3F15.7)
          STOP
          END
```

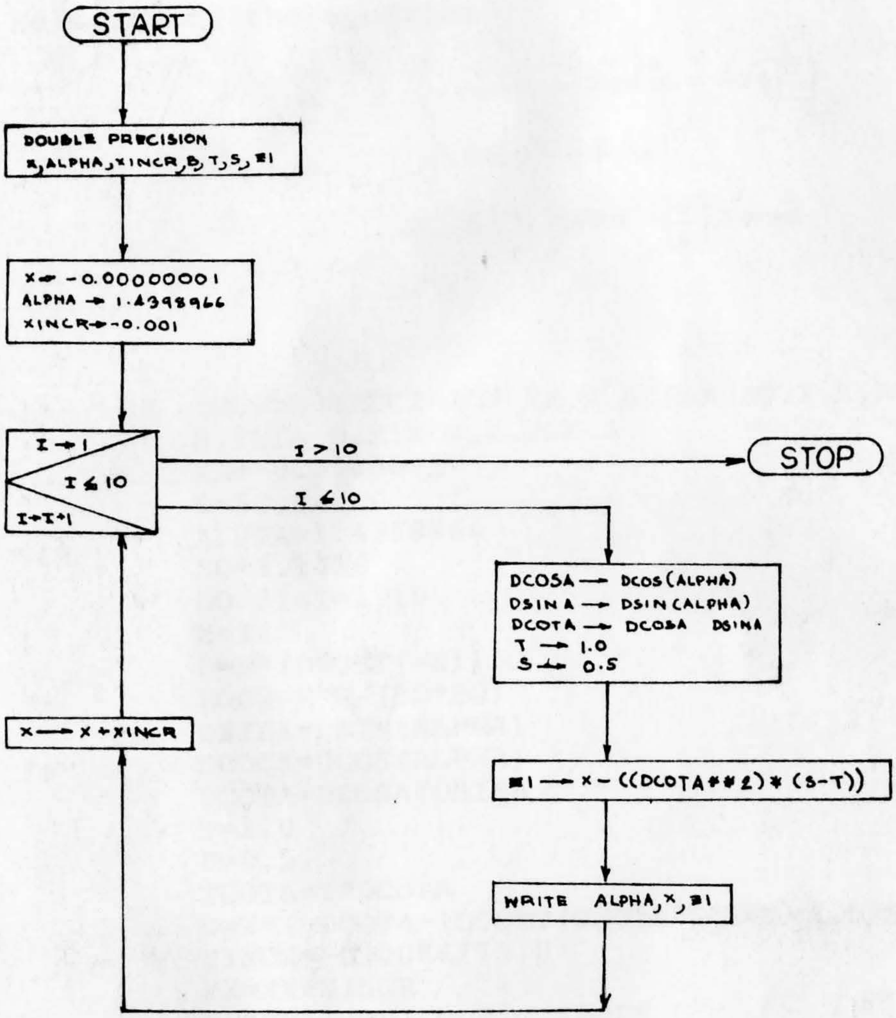


Figure A2. Flow diagram for computer program Number 3.

Computer Program Number 4

Solution of the equation

$$\phi = \frac{\psi}{2S}\left[-T\cot\alpha + \left(T^2\cot^2\alpha + 4S\psi\right)^{\frac{1}{2}}\right]$$

$$\varsigma = \frac{-R^2}{\beta^2}\left(\psi\cos\alpha + \phi\sin\alpha\right)$$

$$\delta = \left[-\psi + S\left(\frac{\phi}{\psi}\right)^2\right]\cos\alpha - \left(\frac{\phi}{\psi}\right)T\sin\alpha$$

$$\beta^2 = -R^2\psi$$

```
1       DOUBLE PRECISION XX,R,ALPHA,BO,X,B,ROB2,T,
        S,TOTA,O,XINCR,Y,OOX,Z
        XX=-0.0000001
        R=50.0
        ALPHA=1.4398966
        BO=3.1416
        DO 21 I=1,10
        X=XX
        B=R*(DSQRT(-X))
        ROB2=R*R/(BO*BO)
        DSINA=DSIN(ALPHA)
        DCOSA=DCOS(ALPHA)
        DCOTA=DCOSA/DSINA
        T=1.0
        S=0.5
        TCOTA=T*DCOTA
        O=X*(-TCOTA+(DSQRT(TCOTA*TCOTA+4.0*S*X)))/(2.0*S)
        XINCR=-0.0086/10.0
        XX=XX+XINCR
        WRITE(6,19) T,S,O,XINCR
19      FORMAT (4F15.10)
        Y=-ROB2*(X*DCOSA+O*DSINA)
        OOX=O/X
        Z=((-X+S*OOX*OOX)*DCOSA)-(T*OOX*DSINA)
21      WRITE (6,23) BO,Y,Z
23      FORMAT (F12.4,2F30.15)
        STOP
        END
```

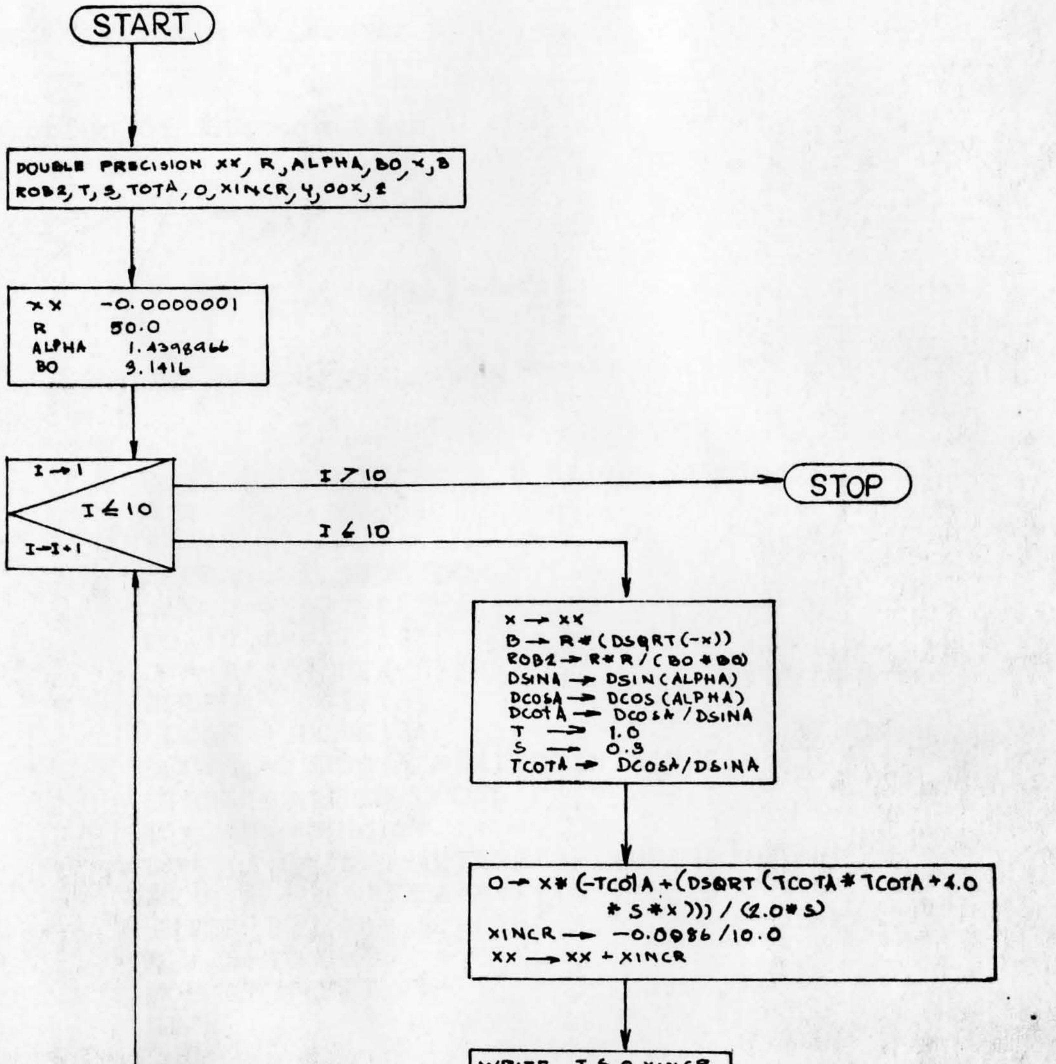


Figure A3. Flow diagram for coputer program Number 4.

Computer Program Number 5

Solution of the equation

$$T = \frac{1}{\beta}\left(\beta - \tan\beta\right)$$

$$S = \frac{1}{4}\left[3\left(1 - \frac{\tan\beta}{\beta}\right) + \tan^2\beta\right]$$

$$\psi - \left(S - T\right)\cot^2\alpha = 0$$

```
1         DOUBLE PRECISION X,R,ALPHA,XINCR,B,T,S,Z1
          X = -0.00000001
          R = 50.0
          ALPHA = 1.4398966
          XINCR = -0.001
          DO 10 I = 1,10
          B = R*(DSQRT(-X))
          DSINB = DSIN(B)
          DCOSB = DCOS(B)
          DCOTA = DCOS(ALPHA)/DSIN(ALPHA)
          DTANB = DSINB/DCOSB
          T = (B-DTANB)/B
          S = (3.0*(1.0-(DTANB/B))+DTANB*DTANB)/4.0
          Z1 = X - ((DCOTA**2)*(S-T))
          WRITE (6,100) R,ALPHA,X,Z1
10        X = X+XINCR
100       FORMAT (4F15.7)
          STOP
          END
```

Flow diagram for this computer program is the same as flow diagram for computer program Number 3 with the changes

```
T → (B-DTANB)/B
S → (3.0*(1.0-(DTANB/B))+DTANB*DTANB)/4.0
B → R*(DSQRT(-X))
```

Computer Program Number 6

Solution of the equation

$$T = \frac{1}{\beta}\left(\beta - \tan\beta\right)$$

$$S = \frac{1}{4}\left[3\left(1 - \frac{\tan\beta}{\beta}\right) + \tan^2\beta\right]$$

$$\phi = \frac{\psi}{2S}\left[-T\cot\alpha \pm \left(T^2\cot^2\alpha + 4S\psi\right)^{\frac{1}{2}}\right]$$

$$\tau = -\frac{R^2}{\beta^2}\left(\psi\cos\alpha + \phi\sin\alpha\right)$$

$$\delta = \left[-\psi + S\left(\frac{\phi}{\psi}\right)^2\right]\cos\alpha - T\left(\frac{\phi}{\psi}\right)\sin\alpha$$

```
1     DOUBLE PRECISION XX,R,ALPHA,BO,X,B,ROB2,T,
      S,TCOTA,O,XINCR,Y,OOX,Z
      XX = -0.0000001
      R = 50.0
      ALPHA=1.4398966
      BO=4.4934
      DO 21 I=1,10
      X=XX
      B=R*(DSQRT(-X))
      ROB2=R*R/(BO*BO)
      DSINA=DSIN(ALPHA)
      DCOSA=DCOS(ALPHA)
      DCOTA=DCOSA/DSINA
      DSINB=DSIN(B)
      DCOSB=DCOS(B)
      DTANB=DSINB/DCOSB
      T=(B-DTANB)/B
      S=(3.0*(1.0-(DTANB/B))+DTANB*DTANB)/4.0
      TCOTA=T*DCOTA
      O=X*(-TCOTA-(DSQRT(TCOTA*TCOTA+4.0*S*X)))/(2.0*S)
      XINCR=-0.0023/10.0
      XX=XX+XINCR
      WRITE (6,19) T,S,O,XINCR,X
19    FORMAT (5F15.10)
      Y=-ROB2*(X*DCOSA+O*DSINA)
      OOX=O/X
21    Z=((-X+S*OOX*OOX)*DCOSA)-(T*OOX*DSINA)
23    WRITE (6,23) BO,Y,Z
      FORMAT (F10.7,F12.4,2F30.15)
      STOP
      END
```

Flow diagram for this computer program is the same as flow diagram for computer program Number 4 with the changes

```
T     ───→  (B-DTANB)/B
S     ───→  (3.0*(1.0-(DTANB/B))+DTANB*DTANB)/4.0
BO    ───→  4.4934
XINCR ───→  -0.0023/10.0
```

Computer Program Number 7

Solution of the equation

$$T = \frac{1}{\beta \sin\beta}\left[\beta \sin\beta - 2\left(1-\cos\beta\right)\right]$$

$$S = \frac{1}{4\beta \sin^2\beta}\left[\left(1-\cos\beta\right)\left(\beta - \beta\cos\beta - 3\sin\beta\right) - 3\sin\beta\left(\cos 2\beta - \cos\beta - \beta\sin\beta\right)\right]$$

$$\psi - \left(S-T\right)\cot^2\alpha = 0$$

```
1         DOUBLE PRECISION X,R,ALPHA,XINCR,B,T,S,Z1
          X=-0.00000001
          R=50.0
          ALPHA=1.4398966
          XINCR=-0.001
          DO 10 I=1,10
          B=R*(DSQRT(-X))
          DSINB=DSIN(B)
          DCOSB=DCOS(B)
          DCOTA=DCOS(ALPHA)/DSIN(ALPHA)
          DTANB=DSINB/DCOSB
          T=(B*DSINB-2*(1.0-DCOSB))/(B*DSINB)
          S=((1.0-DCOSB)*(B-B*DCOSB-3.0*DSINB)-3.0*
            DSINB*(DCOS(2*B)-DCOSB-B*DSINB))/(4.0*B*
            DSINB*DSINB)
          Z1=X-((DCOTA**2)*(S-T))
          WRITE (6,100) R,ALPHA,X,Z1
10        X=X+XINCR
100       FORMAT (4F15.7)
          STOP
          End
```

Flow diagram for this computer program is the same as flow diagram for computer program Number 3 with the changes

```
T → (B*DSINB-2*(1.0-DCOSB))/(B*DSINB)
S → ((1.0-DCOSB)*(B-B*DCOSB-3.0*DSINB)
      -3.0*DSINB*(DCOS(2*B)-DCOSB-B*DSINB))
      /(4.0*B*DSINB*DSINB)
```

Computer Program Number 8

Solution of the equation

$$T = \frac{1}{\beta \sin\beta}\left[\beta \sin\beta - 2\left(1-\cos\beta\right)\right]$$

$$S = \frac{1}{4\beta \sin^2\beta}\left[\left(1-\cos\beta\right)\left(\beta-\beta\cos\beta-3\sin\beta\right)-3\sin\beta\left(\cos 2\beta-\cos\beta-\beta\sin\beta\right)\right]$$

$$\phi = \frac{\psi}{2S}\left[-T\cot\alpha \pm \left(T^2\cot^2\alpha + 4S\psi\right)^{\frac{1}{2}}\right]$$

$$\tau = \frac{-R^2}{\beta_0^2}\left(\psi\cos\alpha + \phi\sin\alpha\right)$$

$$\delta = \left[-\psi + S\left(\frac{\phi}{\psi}\right)^2\right]\cos\alpha - T\left(\frac{\phi}{\psi}\right)\sin\alpha$$

```
1     DOUBLE PRECISION XX,R,ALPHA,BO,X,B,ROB2,
      T,S,TCOTA,O,XINCR,Y,OOX,Z
      XX=-0.0000001
      R=50.0
      ALPHA=1.4398966
      BO=6.2835
      DO 21 I=1,10
      X=XX
      B=R*(DSQRT(-X))
      ROB2=R*R/(BO*BO)
      DSINA=DSIN(ALPHA)
      DCOSA=DCOS(ALPHA)
      DCOTA=DCOSA/DSINA
      DSINB=DSIN(B)
      DCOSB=DCOS(B)
      DTANB=DSINB/DCOSB
      T=(B*DSINB-2.0*(1.0-DCOSB))/(B*DSINB)
      S=((1.0-DCOSB)*(B-B*DCOSB-3.0*DSINB)-3.0*
         DSINB*(DCOS(2.0*B)-DCOSB-B*DSINB))/(4.0*B
         *DSINB*DSINB)
      TCOTA=T*DCOTA
      O=X*(-TCOTA-(DSQRT(TCOTA*TCOTA+4.0*S*X)))/(2.0*S)
      XINCR=-0.0068/10.0
      XX=XX+XINCR
      WRITE (6,19) T,S,O,XINCR,X
19    FORMAT (5F15.10)
      Y=-ROB2*(X*DCOSA+O*DSINA)
      OOX=O/X
21    Z=((-X+S*OOX*OOX)*DCOSA)-(T*OOX*DSINA)
      WRITE (6,23) BO,Y,Z
23    FORMAT (F10.7,2F30.15)
      STOP
      END
```

Flow diagram for this computer program is the same as flow diagram for computer program Number 4 with the changes

```
T      ——→ (B*DSINB-2.0*(1.0-DCOSB))/CB*DSINB)
S      ——→ ((1.0-DCOSB)*(B-B*DCOSB-3.0*DSINB)
               -3.0*DSINB*(DCOS(2.0*B)-DCOSB-B*DSINB))
               /(4.0*B*DSINB*DSINB)
BO     ——→ 6.2835
XINCR  ——→ -0.0068/10.0
```

Computer Program Number 9

Solution of the equation

$$\beta - R\cot\alpha = 0$$

```
1         DOUBLE PRECISION R,ALPHA,B,Z,BINCR
          R=50.0
          ALPHA=1.5620696
          B=0.436699
          DO 10 I=1,10
          BINCR=0.0001
          DSINB=DSIN(B)
          DCOSB=DCOS(B)
          DSINA=DSIN(ALPHA)
          DCOSA=DCOS(ALPHA)
          DCOTA=DCOSA/DSINA
          Z=B-R*DCOTA
          WRITE (6,100) R,ALPHA,B,Z
10        B=B-BINCR
100       FORMAT (4F20.7)
          STOP
          END
```

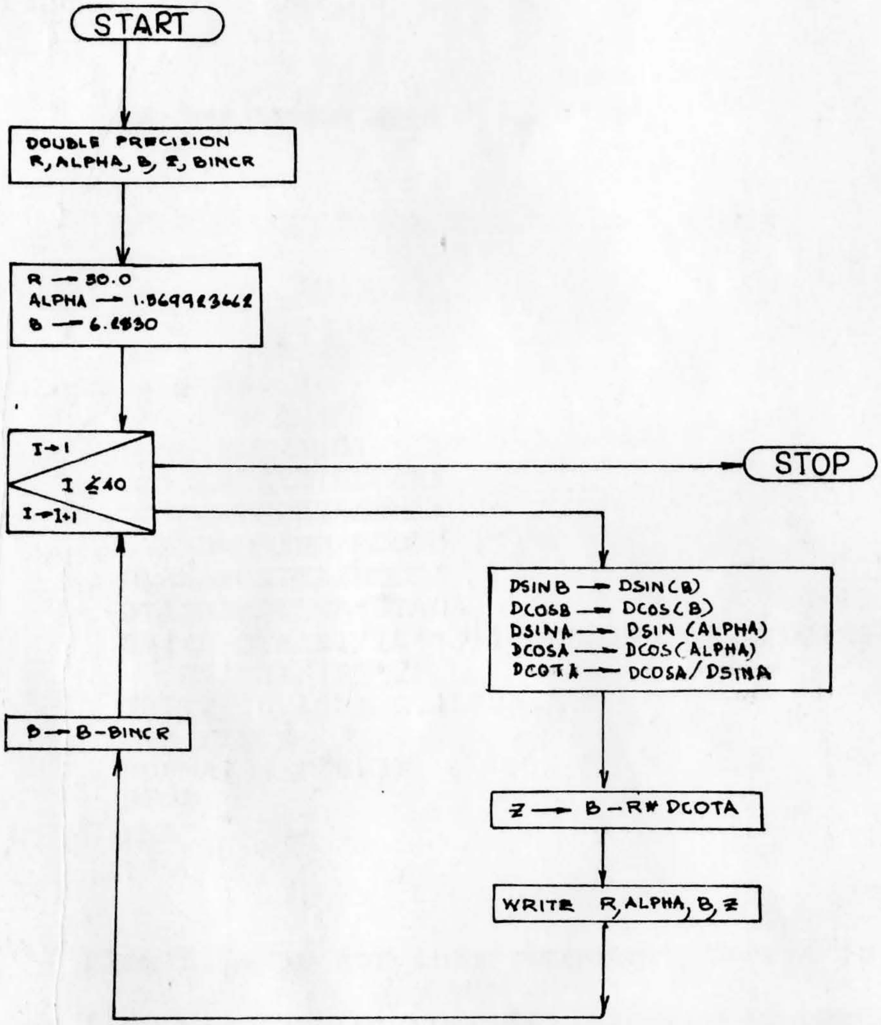


Figure A4. Flow diagram for computer program Number 9.

Computer Program Number 10

Solution of the equation

$$\left(\frac{\beta - \tan\beta}{\beta^3}\right)\cos^2\alpha - \frac{\sin^2\alpha}{R^2} = 0$$

```
1         DOUBLE PRECISION R,ALPHA,B,Z,BINCR
          R=50.0
          ALPHA=1.43986633
          B=4.3556
          DO 10 I=1,50
          BINCR=0.01
          DSINB=DSIN(B)
          DCOSB=DCOS(B)
          DSINA=DSIN(ALPHA)
          DCOSA=DCOS(ALPHA)
          DTANB=DSINB/DCOSB
          DTANA=DSINA/DCOSA
          DTANA2=DTANA*DTANA
          Z=((B-DTANB)/(B**3))*DCOSA*DCOSA-(DSINA*
            DSINA)/(R**2)
          WRITE (6,100) R,ALPHA,B,Z
10        B=B-BINCR
100       FORMAT (4F20.7)
          STOP
          END
```

Flow diagram for this computer program is the same as flow diagram for computer program Number 9 with the changes

```
Z →((B-DTANB)/(B**3))*DCOSA*DCOSA
     -(DSINA*DSINA)/(R**2)
```

Computer Program Number 11

Solution of the equation

$$\frac{\beta^3}{R^2}\sin^2\alpha\sin\beta + \cos^2\alpha\cos^2\beta - 2\cos^2\alpha\cos\beta - \beta\cos^2\alpha\sin\beta + \cos^2\alpha\sin^2\beta + \cos^2\alpha = 0$$

```
1         DOUBLE PRECISION R,ALPHA,B,Z,BINCR
          R=50.0
          ALPHA=1.569923662
          B=6.2830
          DO 10 I=1.40
          BINCR=0.01
          DCOSB=DCOS(B)
          DSINA=DSIN(ALPHA)
          DCOSA=DCOS(ALPHA)
          DTANB=DSINB/DCOSB
          DTANA=DSINA/DCOSA
          DTANA2=DTANA*DTANA
          Z=((DSINA*DSINA)*DSINB)*((B**3)/(R**2))+((DCOSA
             **2)*(DCOSB**2))-2.0*((DCOSA**2)*DCOSB-B*
             ((DCOSA**2)*DSINB)+((DCOSA**2)*(DSINB
             **2))+(DCOSA**2)
          WRITE (6,100) R,ALPHA,B,Z
10        B=B-BINCR
100       FORMAT (4F20.7)
          STOP
          END
```

Flow diagram for this computer program is the same as flow diagram for computer program Number 9 with the changes

```
Z ──►((DSINA*DSINA)(DSINB)*((B**3)/(R**2))
     +((DCOSA**2)*(DCOSB**2))-2.0*((DCOSA**2)
     *DCOSB)-B*((DCOSA**2)*DSINB)+((DCOSA**2)
     *(DSINB**2))+(DCOSA**2)
```

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