CRITICAL BUCKLING LOADS

FOR HYDRAULIC ACTUATING CYLINDERS

by

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ABSTRACT

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The objective of this thesis is to formulate and verify buckling equations that are applicable for various types of hydraulic actuating cylinders. Particular attention was focused on the variation of these equations for their application to both fluid filled plunger rod and fluid unfilled plunger rod. The equations were formulated based on the moment equations and the differential equations of the simple beam theory. The analysis was made with a consideration of the effect of various types of assumptions that were included in the study. Because of the complexity of the solutions, the computer programming was necessary. The actual dimensions were taken into the solutions and the characteristic result of each solution was plotted for comparison.

To verify the analytical approach and results obtained, experiments were performed using an unfilled plunger rod. Satisfactory experimental agreement was obtained.

ACKNOWLEDGEMENTS

The author wishes to express his sincere gratitude and appreciation to his advisor, Dr. Frank J. Tarantine, whose generous help, encouragement and guidance throughout this thesis was invaluable.

I am deeply grateful to my dearly beloved parents, Mr. and Mrs. Decha-umphai, for their encouragement and inspiration for higher education.

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LIST OF NOTATIONS

SYMBOL	DEFINITION
A_C	Internal cross-sectional area of the cylinder
A_R	Internal cross-sectional area of the rod
A_S	Cross-sectional area of the rod (see figure 9(b))
A_T	Total cross-sectional area of the rod
A,B,C,D,E,F	Constants of integration
E_1	Modulus of elasticity of the rod
E_2	Modulus of elasticity of the cylinder
I_1	Moment of inertia of the rod cross-sectional area
I_2	Moment of inertia of the cylinder cross-sectional area
k	Axial load factor ($k_1^2 = P/E_1 I_1$)
L	Length of the hydraulic cylinder
L_1	Length of the rod portion
L_2	Length of the cylinder portion
P	Vertical applied load
P_{CR}	Critical buckling load
p	Internal fluid pressure
r	Center distance
x,y	Rectangular coordinates
θ	Deflection angle at the pinned end of the rod
α	Initial deflection angle between the cylinder and rod

SYMBOL	DEFINITION
β	Deflection angle of the rod at the connecting point
ϕ	Deflection angle of the cylinder at the connecting point
δ	Deflection at the pinned end of the rod
δ_2	Deflection of the cylinder at the connecting point

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CHAPTER I

INTRODUCTION

The essential components of a hydraulic actuating cylinder are the main cylinder, plunger rod and piston. There are many types of application, two of the most widely used are fluid filled plunger rod and fluid unfilled plunger rod as shown by the drawing in figure 1(a) and 1(b), respectively.

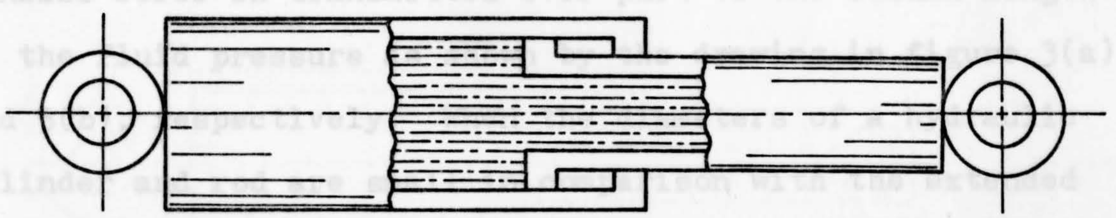


Figure 1(a) Fluid Filled Plunger Rod

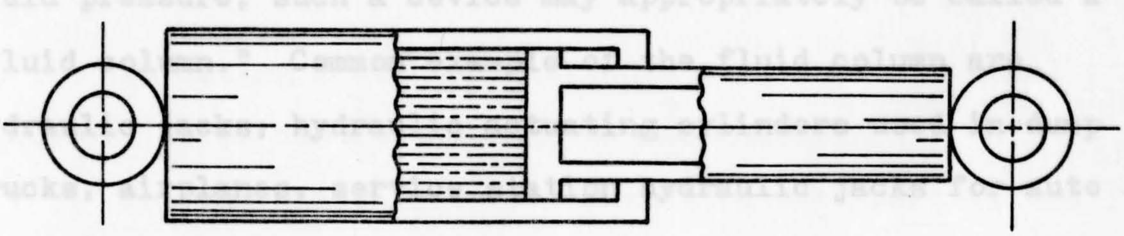


Figure 1(b) Fluid Unfilled Plunger Rod

Other possible application is the telescopic type which consists of more than one plunger rod connected in series as shown by the drawing in figure 2.

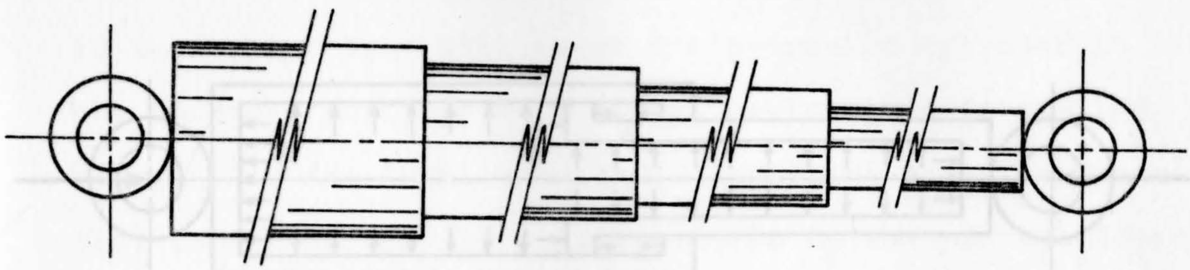


Figure 2 Telescopic Type Cylinder

As the load is applied, the compressive force or the tensile force is transmitted over part of the column length by the fluid pressure as shown by the drawing in figure 3(a) and 3(b), respectively. When the diameters of a hydraulic cylinder and rod are small in comparison with the extended length of the cylinder and rod, failure under compressive load is likely to be by column buckling. Since the compressive force is transmitted over a part of the cylinder by fluid pressure, such a device may appropriately be called a "fluid column." Common example of the fluid column are hydraulic jacks; hydraulic actuating cylinders used in dump trucks, airplanes, service station hydraulic jacks for auto repairing purposes, etc.

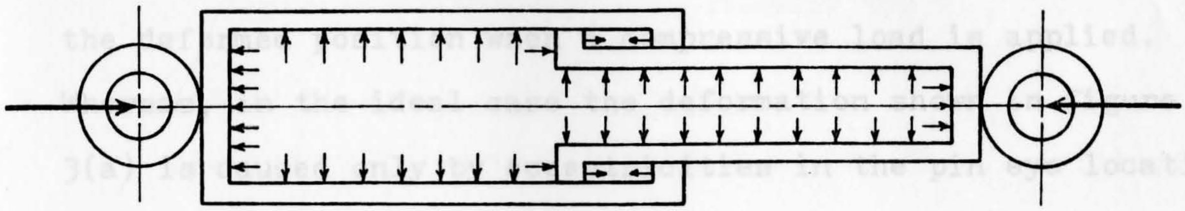


Figure 3(a) Hydraulic Actuating Cylinder Under Compressive Load.

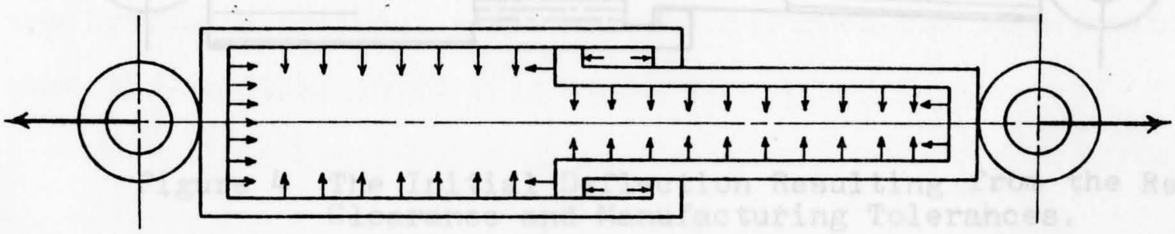


Figure 3(b) Hydraulic Actuating Cylinder Under Tensile Load.

In making a column buckling analysis of a hydraulic cylinder, three conditions must be recognized:

1. The axial load is transmitted over part of the length of the column by fluid pressure.

2. The column is stepped; that is its length is divided into two parts of different moments of inertia.

3. These two parts are joined as shown by the drawing in figure 4, and ther is the initial deflection resulting from the required clearance and manufacturing tolerances at this point that will cause the hydraulic cylinder to take the deformed position when a compressive load is applied. Whereas, in the ideal case the deformation shown in figure 3(a) is caused only by eccentricities in the pin eye locations.

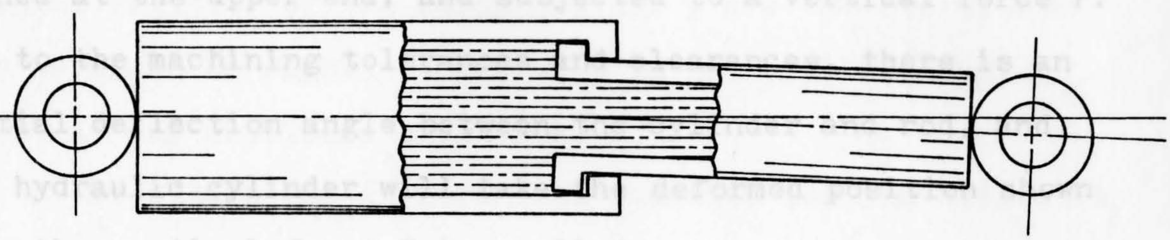


Figure 4 The Initial Deflection Resulting from the Required Clearance and Manufacturing Tolerances.

In practice, there is an initial deflection angle between the cylinder and rod for the reason mentioned above, the deflection will increase gradually with increasing compressive loads. When the loads approach the critical point, the deflection becomes increasingly greater and finally reaches the point where the column buckles. The compressive load at this point is called the critical buckling load. For any compressive load applied greater than the critical buckling load, the applied bending moment will always be greater than the resisting moment and the deflection will increase until failure results.

CHAPTER II

FORMULATION OF THE BUCKLING EQUATION FOR THE FLUID UNFILLED PLUNGER ROD

2.1 Free Body Diagrams and Moment Equations

The hydraulic actuating cylinder with fluid unfilled plunger rod as shown by the drawing in figure 5 is shown in a vertical position with cylinder end fixed, free and pinned at the upper end, and subjected to a vertical force P. Due to the machining tolerances and clearances, there is an initial deflection angle between the cylinder and rod, and the hydraulic cylinder will take the deformed position shown when the vertical force P is applied.

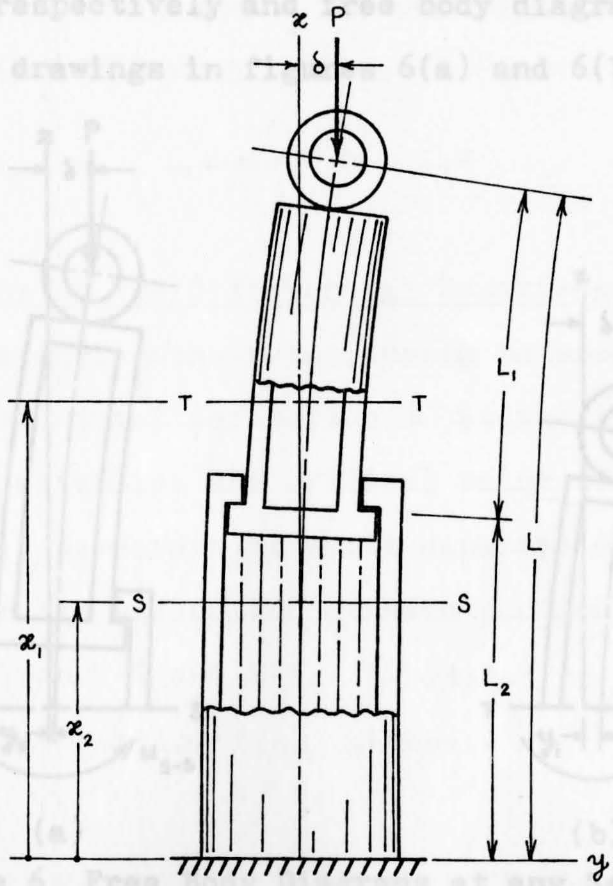


Figure 5 Hydraulic Cylinder Subjected to a vertical force P

Assuming that the shearing deformations and shortening compression of the hydraulic cylinder are negligible compared to those caused by the bending moment, the double-integration equation based on simple beam theory can be applied to obtain the column's deflection: i.e.,

$$EI\frac{d^2y}{dx^2} = M(x) \tag{2-1}$$

where E is the modulus of elasticity, I is the moment of inertia of area, the quantity EI is generally defined as the flexural rigidity of the column, x and y are the coordinate axes as shown in figure 5, and M(x) is the applied bending moment at any arbitrary section x.

To obtain the differential equation for the deflection of the cylinder, the cylinder and rod are cut at sections S-S and T-T respectively and free body diagrams are drawn as shown by the drawings in figures 6(a) and 6(b), respectively.

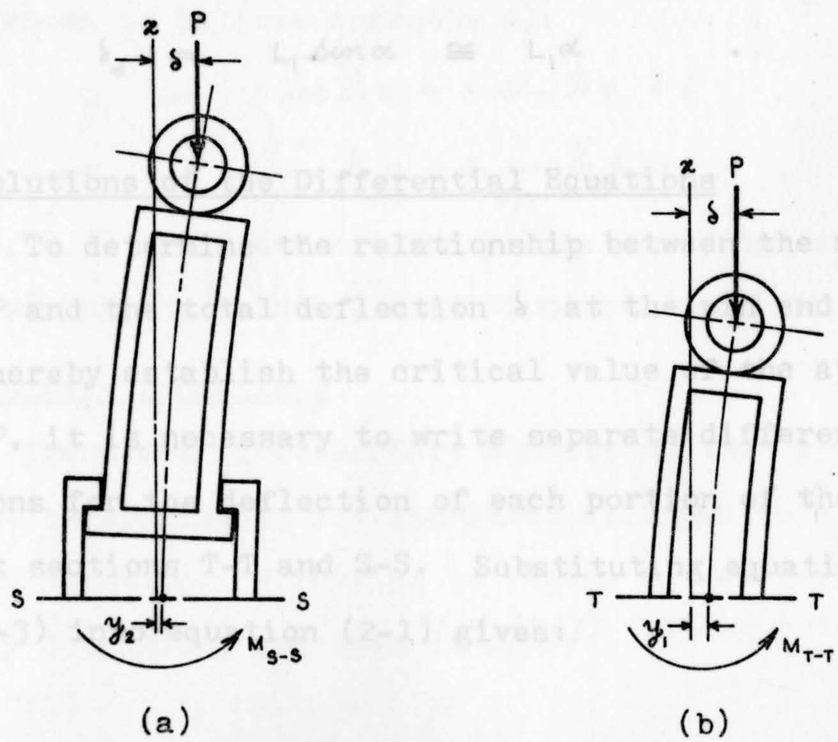


Figure 6 Free Body Diagrams at any Section on the Cylinder and Rod Portion.

When the coordinate axes are taken as indicated in figure 5, the bending moments at the arbitrary cross sections T-T and S-S are

$$M_{T-T} \quad = \quad P(\delta - y_1) \quad , \tag{2-2}$$

and

$$M_{S-S} \quad = \quad P(\delta - y_2) \quad , \tag{2-3}$$

where δ is the total deflection at the pin end of the rod consisting of the deflection δ_α (tolerances and clearances), and the deflection δ_P caused by the applied force P, both of which are measured at the pin end of the rod; i.e.,

$$\delta \quad = \quad \delta_\alpha + \delta_P \quad . \tag{2-4}$$

Assuming α is the small initial deflection angle between the cylinder and rod, and L_1 is the length of the rod portion (see figure 5), then,

$$\delta_\alpha \quad = \quad L_1 \sin\alpha \quad \cong \quad L_1\alpha \quad . \tag{2-5}$$

2.2 Solutions of the Differential Equations

To determine the relationship between the applied force P and the total deflection δ at the pin end of the rod and, thereby establish the critical value of the applied force P, it is necessary to write separate differential equations for the deflection of each portion of the column; e.g. at sections T-T and S-S. Substituting equations (2-2) and (2-3) into equation (2-1) gives:

$$E_1I_1 \frac{d^2y_1}{dx_1^2} = P(\delta - y_1) , \qquad (2\text{-}6)$$

and

$$E_2I_2 \frac{d^2y_2}{dx_2^2} = P(\delta - y_2) , \qquad (2\text{-}7)$$

where E_1I_1 and E_2I_2 are the flexural rigidity of the rod and cylinder, respectively.

Using the notations:

$$k_1^2 = \frac{P}{E_1I_1} \quad \text{and} \quad k_2^2 = \frac{P}{E_2I_2} , \qquad (2\text{-}8)$$

Equations (2-6) and (2-7) become:

$$\frac{d^2y_1}{dx_1^2} + k_1^2 y_1 = k_1^2 \delta , \qquad (2\text{-}9)$$

$$\frac{d^2y_2}{dx_2^2} + k_2^2 y_2 = k_2^2 \delta . \qquad (2\text{-}10)$$

Equations (2-9) and (2-10) are second order linear differential equations with constant coefficients whose general solutions can be shown to be (see Appendix A):

$$y_1 = A \cos k_1x_1 + B \sin k_1x_1 + \delta , \qquad (2\text{-}11)$$

$$y_2 = C \cos k_2x_2 + D \sin k_2x_2 + \delta , \qquad (2\text{-}12)$$

where A, B, C and D are arbitrary constants that are determined by satisfying the boundary conditions of the problem.

2.3 Boundary Conditions

There are five boundary conditions that are needed and must be applied to the hydraulic cylinder to obtain the required solution. Referring to figure 5, the following boundary conditions are seen to apply:

(1) $y_2(0) = 0$

(2) $\dot{y}_2(0) = 0$

(3) $y_1(L) = \delta$

(4) $y_1(L_2) = y_2(L_2)$

(5) At the connection point between the cylinder and rod ($X_1 = X_2 = L_2$), the deflection angle on the rod is equal to the summation of the deflection angle on the cylinder and the initial deflection angle as shown by the drawing in figure 7.

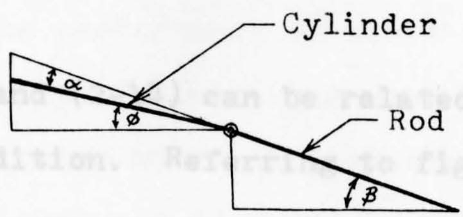


Figure 7 The Change in Angle at the Connecting Point Between the Cylinger and Rod.

Applying the first two boundary conditions into equation (2-12) gives the constant values:

$$C = -\delta \quad \text{and} \quad D = 0 \quad ,$$

and equation (2-12) becomes:

$$y_2 = \delta\,(1 - \cos k_2 x_2) \quad . \qquad (2\text{-}13)$$

Applying the second two boundary conditions into equation (2-11) gives the constant values:

$$A = \frac{-\delta \cos k_2 L_2 \sin k_1 L}{\sin k_1 L_1} ,$$

$$B = \frac{\delta \cos k_2 L_2 \cos k_1 L}{\sin k_1 L_1} .$$

Substituting for A and B as given above into equation (2-11) gives the following:

$$y_1 = \frac{-\delta \cos k_2 L_2 \sin k_1 L}{\sin k_1 L_1} \cos k_1 x_1 + \frac{\delta \cos k_2 L_2 \cos k_1 L}{\sin k_1 L_1} \sin k_1 x_1 + \delta . \tag{2-14}$$

Equations (2-13) and (2-14) can be related by applying the last boundary condition. Referring to figure 7, since

$$\hat{\beta} = \hat{\phi} + \hat{\alpha} ,$$

it follows that

$$\tan \hat{\beta} = \tan(\hat{\phi} + \hat{\alpha}) = \frac{\tan \hat{\phi} + \tan \hat{\alpha}}{1 - \tan \hat{\phi} \tan \hat{\alpha}} ,$$

or

$$\frac{dy_1}{dx_1} = \frac{\dfrac{dy_2}{dx_2} + \dfrac{\delta_\alpha}{L_1}}{1 - \dfrac{dy_2}{dx_2} \cdot \dfrac{\delta_\alpha}{L_1}} , \tag{2-15}$$

where $\frac{dy_1}{dx_1}$ and $\frac{dy_2}{dx_2}$ are obtained by differentiating equations (2-14) and (2-13), respectively;

$$\left.\begin{aligned} \frac{dy_1}{dx_1} &= \frac{\delta k_1 \cos k_2 L_2 \sin k_1 L \sin k_1 L_2}{\sin k_1 L_1} + \frac{\delta k_1 \cos k_2 L_2 \cos k_1 L \cos k_1 L_2}{\sin k_1 L_1} \\ \frac{dy_2}{dx_2} &= \delta k_2 \sin k_2 L_2 \end{aligned}\right] \tag{2-16}$$

Simplifying the equation obtained by substituting equation (2-16) into equation (2-15), and applying equation (2-4) yields the actual deflection at the pin end of the rod caused by the applied force P;

$$\delta_P = \left[\left[-L_1 (k_2 \sin k_2 L_2 - k_1 \cos k_2 L_2 \cot k_1 L_1) - \left\{ L_1^2 (k_2 \sin k_2 L_2 - k_1 \cos k_2 L_2 \cot k_1 L_1)^2 - (4 \delta_\alpha^2 k_1 k_2 \sin k_2 L_2 \cos k_2 L_2 \cot k_1 L_1) \right\}^{1/2} \right] \Big/ (2 \delta_\alpha k_1 k_2 \sin k_2 L_2 \cos k_2 L_2 \cot k_1 L_1) \right] - \delta_\alpha \, . \tag{2-17}$$

To show the performance of the derived formula (equation (2-17)), actual dimensions were taken from a presently designed and fabricated hydraulic actuating cylinder with a fluid unfilled plunger rod as shown in figure 18. The equation was evaluated by using the various arbitrary values of initial deflection angle. A computer program was written for convenience in making the required calculations and is shown on page 52 as computer program no. 1. The characteristic results of each initial deflection angle are shown in figure 19, which is a plot of load vs deflection for the various trial calculations that were made using program no. 1.

CHAPTER III

FORMULATION OF THE BUCKLING EQUATION FOR THE FLUID FILLED PLUNGER ROD

3.1 Analysis of the Major Effect in the Fluid Filled Plunger Rod

3.1.1 Free Body Diagrams and Moment Equations.

The hydraulic actuating cylinder with fluid filled plunger rod, as shown by the drawing in figure 8, is in a vertical position with cylinder end fixed, free and pinned at the upper end, and subjected to a vertical force P.

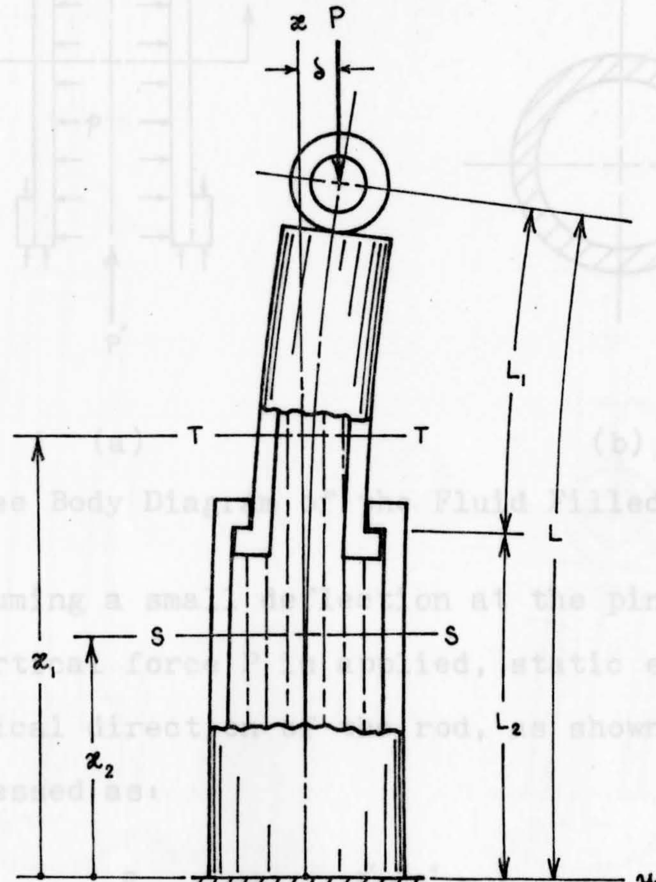
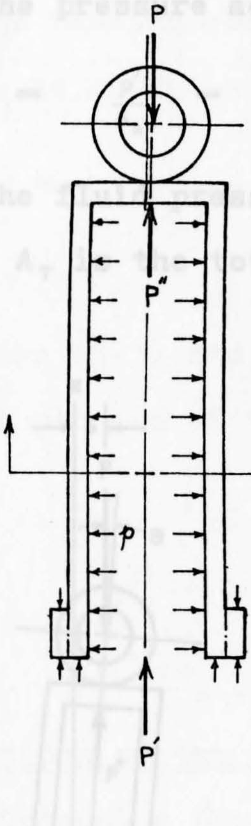


Figure 8 Hydraulic Cylinder with Fluid Filled Plunger Rod Subjected to a Vertical Force P.

To analyze the bending characteristics and the performance of the hydraulic cylinder shown in figure 8 to resist buckling, the free body diagram of the rod and its cross-sectional area are drawn as shown in figures 9(a) and 9(b), respectively.



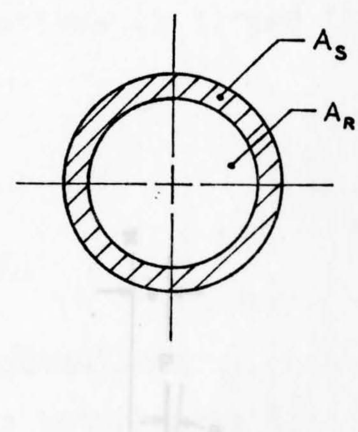


Figure 9 Free Body Diagram of the Fluid Filled Plunger Rod.

Assuming a small deflection at the pin end of the rod when the vertical force P is applied, static equilibrium in the vertical direction of the rod, as shown in figure 9(a), can be expressed as:

$$P \quad = \quad P' + P'' \quad , \qquad (3\text{-}1)$$

where P' and P'' are the summation of the force due to fluid pressure acting on the cross-sectional area of the rod A_S, and the internal area at the rod end A_R, as shown in figure 9(b), respectively. Both forces P' and P'' can be determined in term of the applied force P which depends on the area over which the pressure acts:

$$\frac{P}{A_T} = \frac{P'}{A_S} = \frac{P''}{A_R} = p \quad , \tag{3-2}$$

where p is the fluid pressure corresponding to the applied force P, and A_T is the total cross-sectional area of the rod.

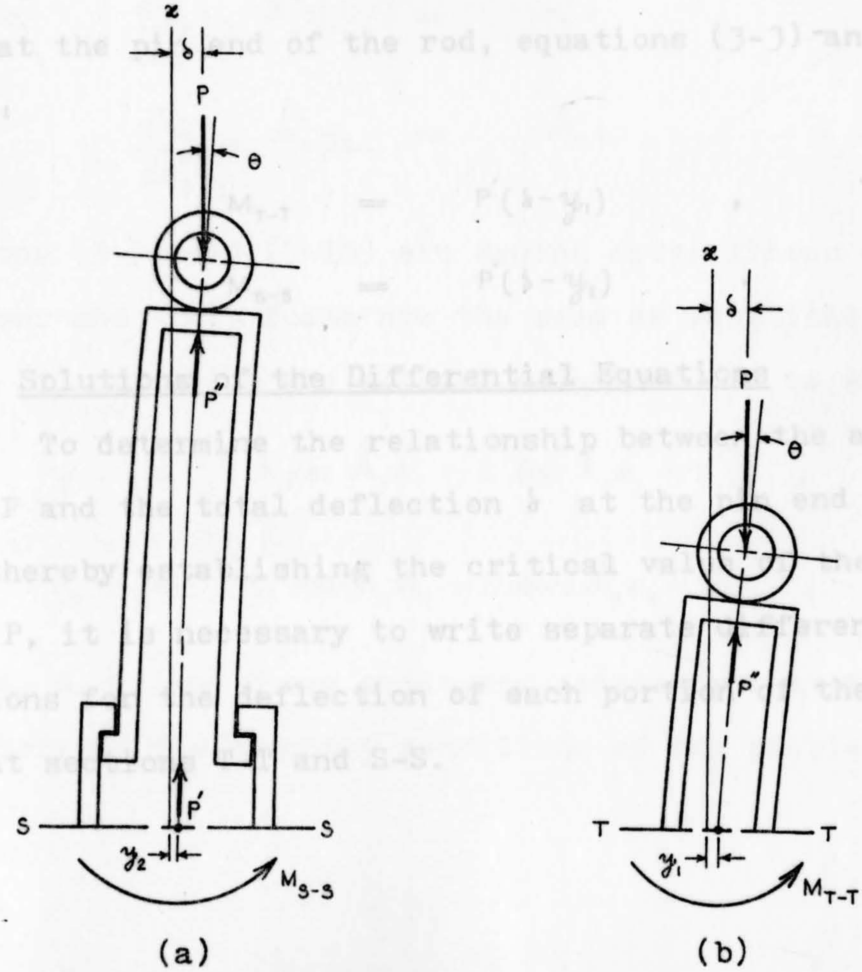


Figure 10 Free Body Diagrams of the Fluid Filled Plunger Rod at any Section on the Cylinder and Rod Portion.

To obtain the differential equation for the deflection of the cylinder, the cylinder and rod are cut at arbitrary sections S-S and T-T and free body diagrams are drawn as shown by the sketches in figures 10(a) and 10(b), respectively.

When the coordinate axes are taken as indicated in figure 8, the bending moments at the arbitrary cross sections T-T and S-S, as shown in figure 10, are:

$$M_{T-T} \quad = \quad (P - P'' \cos\theta)(\delta - y_1) \quad , \qquad (3\text{-}3)$$

and

$$M_{S-S} \quad = \quad (P - P'' \cos\theta)(\delta - y_2) \quad . \qquad (3\text{-}4)$$

Using equation (3-1) and by assuming θ is a small deflection angle at the pin end of the rod, equations (3-3) and (3-4) become:

$$M_{T-T} \quad = \quad P'(\delta - y_1) \quad , \qquad (3\text{-}5)$$

and

$$M_{S-S} \quad = \quad P'(\delta - y_2) \quad . \qquad (3\text{-}6)$$

3.1.2 Solutions of the Differential Equations

To determine the relationship between the applied force P and the total deflection δ at the pin end of the rod, thereby establishing the critical value of the applied force P, it is necessary to write separate differential equations for the deflection of each portion of the column; e.g. at sections T-T and S-S.

Substituting equations (3-5) and (3-6) into equation (2-1) gives:

$$E_1I_1\frac{d^2y_1}{dx_1^2} = P'(\delta - y_1) , \qquad (3\text{-}7)$$

and

$$E_2I_2\frac{d^2y_2}{dx_2^2} = P'(\delta - y_2) , \qquad (3\text{-}8)$$

respectively.

Using the notations:

$$k_3^2 = \frac{P'}{E_1I_1} \text{ and } k_5^2 = \frac{P'}{E_2I_2} ,$$

equations (3-7) and (3-8) become:

$$\frac{d^2y_1}{dx_1^2} + k_3^2 y_1 = k_3^2\delta , \qquad (3\text{-}9)$$

$$\frac{d^2y_2}{dx_2^2} + k_5^2 y_2 = k_5^2\delta . \qquad (3\text{-}10)$$

Equations (3-9) and (3-10) are second order linear differential equations and their forms are the same as equations (2-9) and (2-10) for which the general solutions can be shown to be:

$$y_1 = A\cos k_3x_1 + B\sin k_3x_1 + \delta , \qquad (3\text{-}11)$$

$$y_2 = C\cos k_5x_2 + D\sin k_5x_2 + \delta , \qquad (3\text{-}12)$$

where A, B, C, and D are arbitrary constants that are determined by satisfying the boundary conditions of the problems.

3.1.3 Formulation of the Buckling Equation

Referring to figure 8, the same five boundary conditions as described in section 2.3 can be applied to the hydraulic cylinder with fluid filled plunger rod. Since the general solutions of the differential equations obtained, given by equations (3-11) and (3-12), are of the same form as equations (2-11) and (2-12) of the previous case (chapter II), the actual deflection at the pin end of the rod caused by the applied force P can be determined as:

$$\delta_P = \left[\left[-L_1(k_5 \sin k_5 L_2 - k_3 \cos k_5 L_2 \cot k_3 L_1) - \left\{ L_1^2 (k_5 \sin k_5 L_2 - k_3 \cos k_5 L_2 \cot k_3 L_1)^2 - (4 \delta_\alpha^2 k_3 k_5 \sin k_5 L_2 \cos k_5 L_2 \cot k_3 L_1) \right\}^{1/2} \right] \Big/ (2 \delta_\alpha k_3 k_5 \sin k_5 L_2 \cos k_5 L_2 \cot k_3 L_1) \right] - \delta_\alpha \quad . \qquad (3\text{-}13)$$

The performance of the derived formula (equation (3-13)) was illustrated by using the same actual dimensions of the hydraulic actuating cylinder as the previous case. The equation was evaluated by using various arbitrary values of initial deflection angle. The computer program of this equation is shown on page 53 as computer program no. 2. The characteristic results, i.e. plots of load vs deflection, for the various initial deflection angles, are shown in figure 20.

3.2 Analysis Concerning the Moment Effect of the Horizontal Force Component at Rod End

3.2.1 Free Body Diagrams and Moment Equations.

Considering the F. B. D. of the hydraulic cylinder with a deformed position δ at the pin end of the rod when the vertical force P is applied, the resisting force P'' caused by the fluid pressue at the rod end area A_R remains acting in a direction perpendicular to the internal surface of the rod end as shown by the drawing in figure 11.

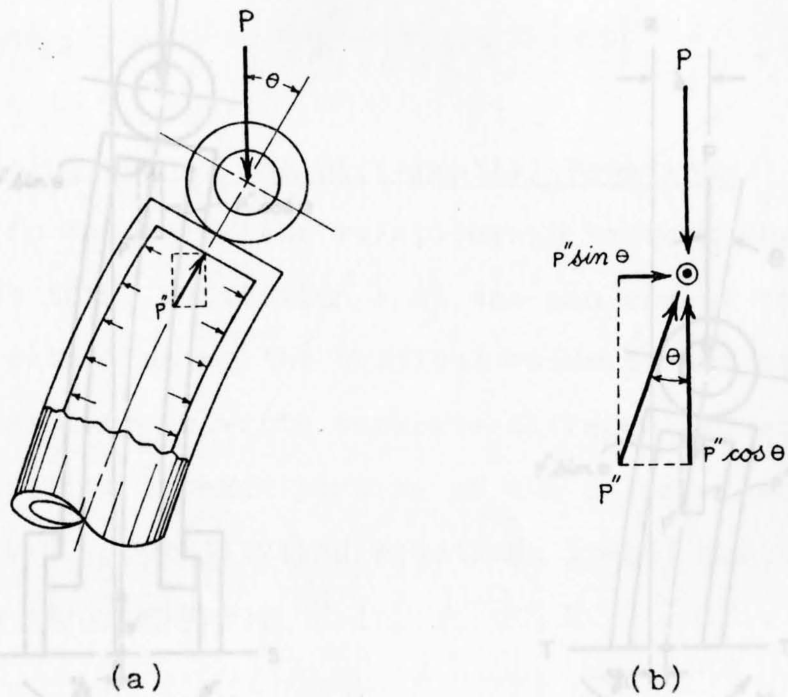


Figure 11 Moment Effect of Force Component at Rod End.

Although the force component $P'' \sin\theta$ is small according to the small deflection angle (θ) at the pin end of the rod, as shown in figure 11 (b), the moment effect produced by this force must be considered due to the long length of the hydraulic cylinder.

In this case, considering the moment effect produced by the force component $P'' \sin\theta$, the free body diagrams of the cylinder and rod cut at sections S-S and T-T are drawn as shown in figures 12(a) and 12(b), respectively.

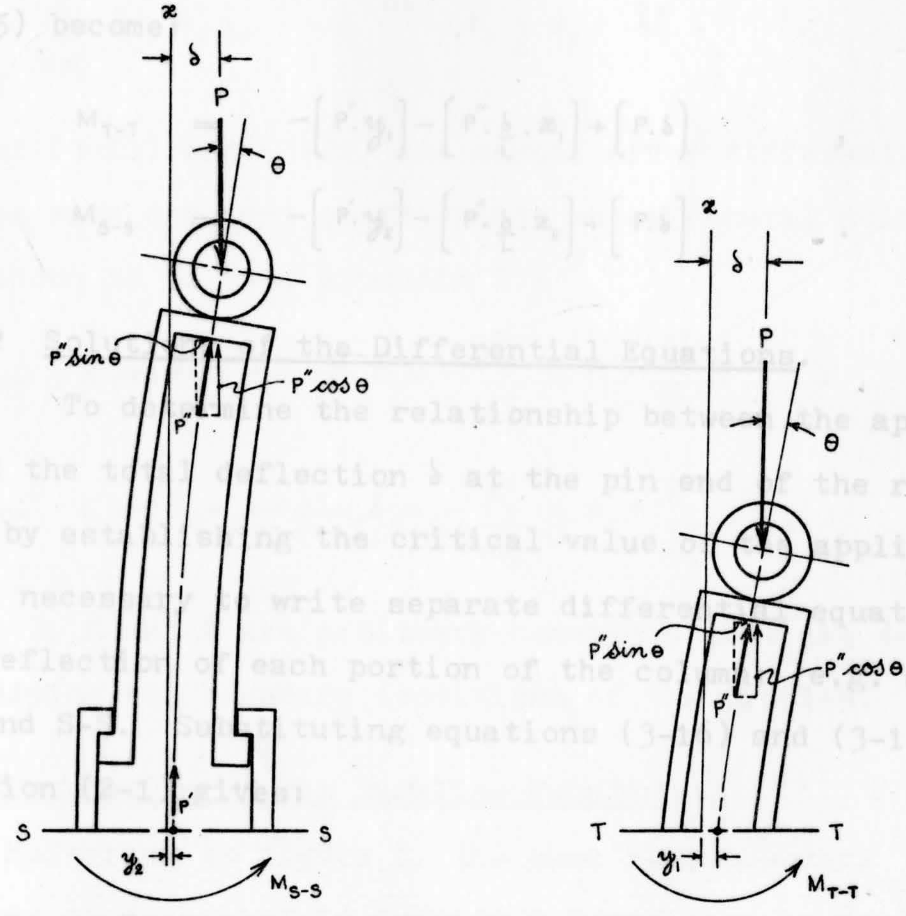


Figure 12 Free Body Diagrams of the Fluid Filled Plunger Rod at any Section on the Cylinder and Rod Portion.

When the coordinate axes are taken as indicated in figure 8, the bending moments at the arbitrary cross sections T-T and S-S as shown in figure 12 are:

$$M_{T-T} \;=\; \left[(P-P''\cos\theta)(\delta-y_1)\right]+\left[(P''\sin\theta)(L-x_1)\right], \qquad (3\text{-}14)$$

$$M_{S-S} \;=\; \left[(P-P''\cos\theta)(\delta-y_2)\right]+\left[(P''\sin\theta)(L-x_2)\right] \;. \qquad (3\text{-}15)$$

Assuming θ is a small deflection angle at the pin end of the rod ($\theta \cong \frac{\delta}{L}$) and using equation (3-1), equations (3-14) and (3-15) become:

$$M_{T-T} \;=\; -\left[P'.y_1\right]-\left[P''.\frac{\delta}{L}.x_1\right]+\left[P.\delta\right] \;, \qquad (3\text{-}16)$$

and $$M_{S-S} \;=\; -\left[P'.y_2\right]-\left[P''.\frac{\delta}{L}.x_2\right]+\left[P.\delta\right] \;. \qquad (3\text{-}17)$$

3.2.2 <u>Solutions of the Differential Equations.</u>

To determine the relationship between the applied force P and the total deflection δ at the pin end of the rod, and thereby establishing the critical value of the applied force P, it is necessary to write separate differential equations for the deflection of each portion of the column; e.g. at sections T-T and S-S. Substituting equations (3-16) and (3-17) into equation (2-1) gives:

$$E_1I_1\frac{d^2y_1}{dx_1^2} \;=\; -\left[P'.y_1\right]-\left[P''.\frac{\delta}{L}.x_1\right]+\left[P.\delta\right] \;, \qquad (3\text{-}18)$$

and $$E_2I_2\frac{d^2y_2}{dx_2^2} \;=\; -\left[P'.y_2\right]-\left[P''.\frac{\delta}{L}.x_2\right]+\left[P.\delta\right] \;. \qquad (3\text{-}19)$$

Using the notations:

$$\left.\begin{aligned} k_1^2 &= \frac{P}{E_1I_1}, & k_2^2 &= \frac{P}{E_2I_2}, & k_3^2 &= \frac{P'}{E_1I_1}, \\ k_4^2 &= \frac{P''}{E_1I_1}, & k_5^2 &= \frac{P'}{E_2I_2}, & k_6^2 &= \frac{P''}{E_2I_2}, \end{aligned}\right] \quad (3\text{-}20)$$

equations (3-18) and (3-19) become:

$$\frac{d^2y_1}{dx_1^2} + k_3^2 y_1 = -k_4^2 . \frac{\delta}{L} . x_1 + k_1^2 . \delta , \qquad (3\text{-}21)$$

and

$$\frac{d^2y_2}{dx_2^2} + k_5^2 y_2 = -k_6^2 . \frac{\delta}{L} . x_2 + k_2^2 . \delta . \qquad (3\text{-}22)$$

Equations (3-21) and (3-22) are second order differential equations with constant coefficients whose general solutions can be shown to be (see Appendix B):

$$y_1 = A \cos k_3 x_1 + B \sin k_3 x_1 - \frac{k_4^2}{k_3^2} . \frac{\delta}{L} . x_1 + \frac{k_1^2}{k_3^2} . \delta , \qquad (3\text{-}23)$$

$$y_2 = E \cos k_5 x_2 + F \sin k_5 x_2 - \frac{k_6^2}{k_5^2} . \frac{\delta}{L} . x_2 + \frac{k_2^2}{k_5^2} . \delta , \qquad (3\text{-}24)$$

where A, B, E and F are arbitrary constants that are determined by satisfying the boundary conditions of the problem.

3.2.3 Formulation of the Buckling Equation.

Referring to figure 8, the same five boundary conditions as described in Section 2.3 can also be applied to the hydraulic cylinder with fluid filled plunger rod. Substituting the first two boundary conditions into equation (3-24) gives the following constant values:

$$E = -\frac{k_2^2}{k_5^2} . \delta \quad \text{and} \quad F = \frac{k_6^2}{k_5^3} . \frac{\delta}{L} .$$

Substituting the above values for E and F into equation (3-24) gives:

$$y_2 = -\frac{k_2^2}{k_5^2}.\delta.\cos k_5 x_2 + \frac{k_6^2}{k_5^3}.\frac{\delta}{L}.\sin k_5 x_2 - \frac{k_6^2}{k_5^2}.\frac{\delta}{L}.x_2 + \frac{k_2^2}{k_5^2}.\delta \quad . \tag{3-25}$$

Applying the second two boundary conditions to equation (3-23) gives the following values for constants A and B:

$$\left.\begin{aligned} A &= -\delta.\frac{\sin k_3 L}{\sin k_3 L_1}\left[\frac{k_2^2}{k_5^2}\cos k_5 L_2 - \frac{k_6^2}{k_5^3.L}\sin k_5 L_2\right] , \\ B &= \delta.\frac{\cos k_3 L}{\sin k_3 L_1}\left[\frac{k_2^2}{k_5^2}\cos k_5 L_2 - \frac{k_6^2}{k_5^3.L}\sin k_5 L_2\right] . \end{aligned}\right] \tag{3-26}$$

Substituting for A and B as given in equation (3-26) into equation (3-23) gives the following:

$$\begin{aligned} y_1 = &-\left\{\delta.\frac{\sin k_3 L}{\sin k_3 L_1}\left(\frac{k_2^2}{k_5^2}\cos k_5 L_2 - \frac{k_6^2}{k_5^3.L}\sin k_5 L_2\right)\right\}\cos k_3 x_1 \\ &+\left\{\delta.\frac{\cos k_3 L}{\sin k_3 L_1}\left(\frac{k_2^2}{k_5^2}\cos k_5 L_2 - \frac{k_6^2}{k_5^3.L}\sin k_5 L_2\right)\right\}\sin k_3 x_1 \\ &- \frac{k_4^2}{k_3^2}.\frac{\delta}{L}.x_1 + \frac{k_1^2}{k_3^2}.\delta \quad . \end{aligned} \tag{3-27}$$

Equations (3-25) and (3-27) can be related by applying the last boundary condition. Referring to equation (2-15),

$$\frac{dy_1}{dx_1} = \frac{\dfrac{dy_2}{dx_2} + \dfrac{\delta_\alpha}{L_1}}{1 - \dfrac{dy_2}{dx_2}.\dfrac{\delta_\alpha}{L_1}} \tag{2-15}$$

where $\frac{dy_1}{dx_1}$ and $\frac{dy_2}{dx_2}$ are obtained in this case by differentiating equations (3-27) and (3-25), respectively; i.e.,

$$\left.\begin{aligned} \frac{dy_1}{dx_1} &= -k_3 A \sin k_3 L_2 + k_3 B \cos k_3 L_2 - \frac{k_4^2}{k_3^2} \cdot \frac{\delta}{L} \,, \\ \frac{dy_2}{dx_2} &= \frac{k_2^2}{k_5} \cdot \delta \cdot \sin k_5 L_2 + \frac{k_6^2}{k_5^2} \cdot \frac{\delta}{L} \cdot \left(\cos k_5 L_2 - 1 \right) \,. \end{aligned}\right] \qquad (3\text{-}28)$$

By simplifying the equation obtained by substituting equation (3-28) into equation (2-15), and by applying equation (2-4), the actual deflection δ_P at the pin end of the rod can be related to the applied force P through the following equation:

$$\begin{aligned} & -k_3 A \sin k_3 L_2 + k_3 B \cos k_3 L_2 - \frac{k_4^2}{k_3^2 . L}(\delta_P + \delta_\alpha) - \frac{k_2^2}{k_5}(\delta_P + \delta_\alpha) \sin k_5 L_2 \\ & - \frac{k_6^2}{k_5^2 . L}(\delta_P + \delta_\alpha)(\cos k_5 L_2 - 1) - \frac{\delta_\alpha}{L_1}\left[\left\{ -k_3 A \sin k_3 L_2 + k_3 B \cos k_3 L_2 \right.\right. \\ & \left. - \frac{k_4^2}{k_3^2 . L}(\delta_P + \delta_\alpha) \right\} \left\{ \frac{k_2^2}{k_5}(\delta_P + \delta_\alpha) \sin k_5 L_2 + \frac{k_6^2}{k_5^2 . L}(\delta_P + \delta_\alpha)(\cos k_5 L_2 - 1) \right\} \\ & \left. + 1 \right] = 0 \end{aligned} \qquad (3\text{-}29)$$

where the constants A and B have the values given by equations (3-26).

The derived formula (equation (3-29)) can be solved by a trial and error method, and a computer program is very helpful in making the necessary calculations. The alphabetic letters used in the derived formula, (3-29), correspond to those used in the computer program for convenience and clarity as shown on page 54 as computer program no. 3. The same actual dimensions of the hydraulic actuating cylinder used in the previous case were taken to show the performance of the derived formula. The equation was evaluated by using various arbitrary values of initial deflection angle and the characteristic results are shown by a plot of load vs deflection in figure 21. The calculations required for the results given by the plots of figure 21 were obtained from program no. 3.

3.3 Analysis Concerning the Moment Effect of Fluid Pressure on the Column Wall.

3.3.1 Free Body Diagrams and Moment Equations.

When a vertical force P is applied at the pin end of the rod, the hydraulic cylinder will take a deformed position as shown in figure 8. The fluid pressure is of course normal to the tube's inner surface at all points.

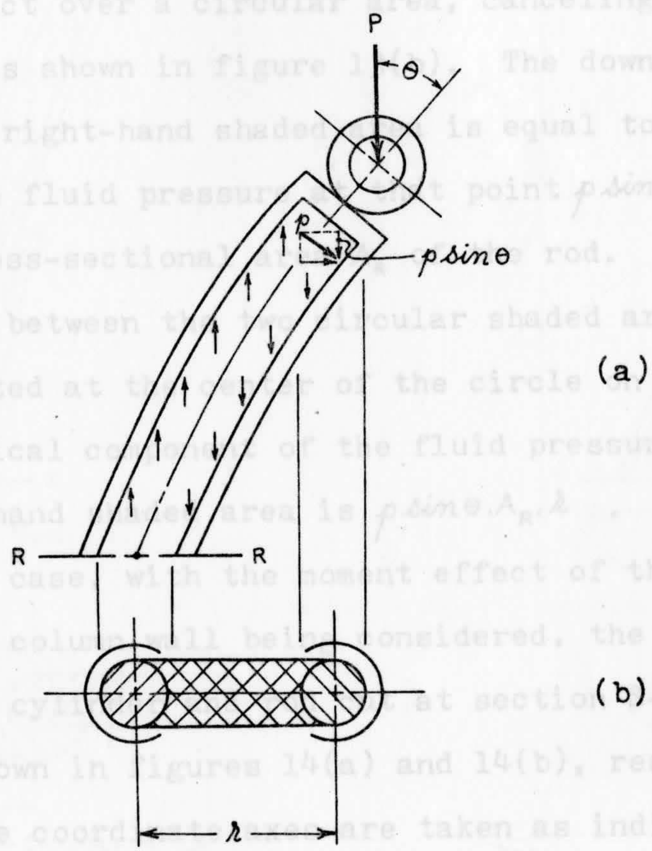


Figure 13 Rod Portion of Hydraulic Cylinder in Deflected Position.

In the hollow rod section of the cylinder, as shown by the drawing in figure 13, the horizontal component of the fluid pressure balances itself out and is not shown. However, the vertical component of the fluid pressure, as shown in the hollow section, figure 13, produces a bending moment. Figure 13(b) shows the projection on the horizontal plane of the chosen cross-section R-R, of the areas upon which the vertical component of the fluid pressure acts. The vertical force is seen to be upward in the left-hand shaded area. The moment of the vertical force system is easily found by assuming that the upward and downward vertical pressures each act over a circular area, canceling out the co-shaded area as shown in figure 13(b). The downward force on the circular right-hand shaded area is equal to the component of the fluid pressure at that point $p \sin\theta$ times the internal cross-sectional area A_R of the rod. Since the center distance between the two circular shaded areas is λ , the moment exerted at the center of the circle on section R-R by the vertical component of the fluid pressure on the circular right-hand shaded area is $p \sin\theta . A_R . \lambda$.

In this case, with the moment effect of the fluid pressure on the column wall being considered, the free body diagrams of the cylinder and rod cut at section S-S and T-T are drawn as shown in figures 14(a) and 14(b), respectively.

When the coordinate axes are taken as indicated in figure 8, the bending moment produced by the fluid pressure acting on the rod wall at an arbitrary cross section T-T as shown in figure 14(b) is:

$$M_{T-T} = (p \sin\theta . A_R)(\delta - y_1) \quad . \qquad (3\text{-}30)$$

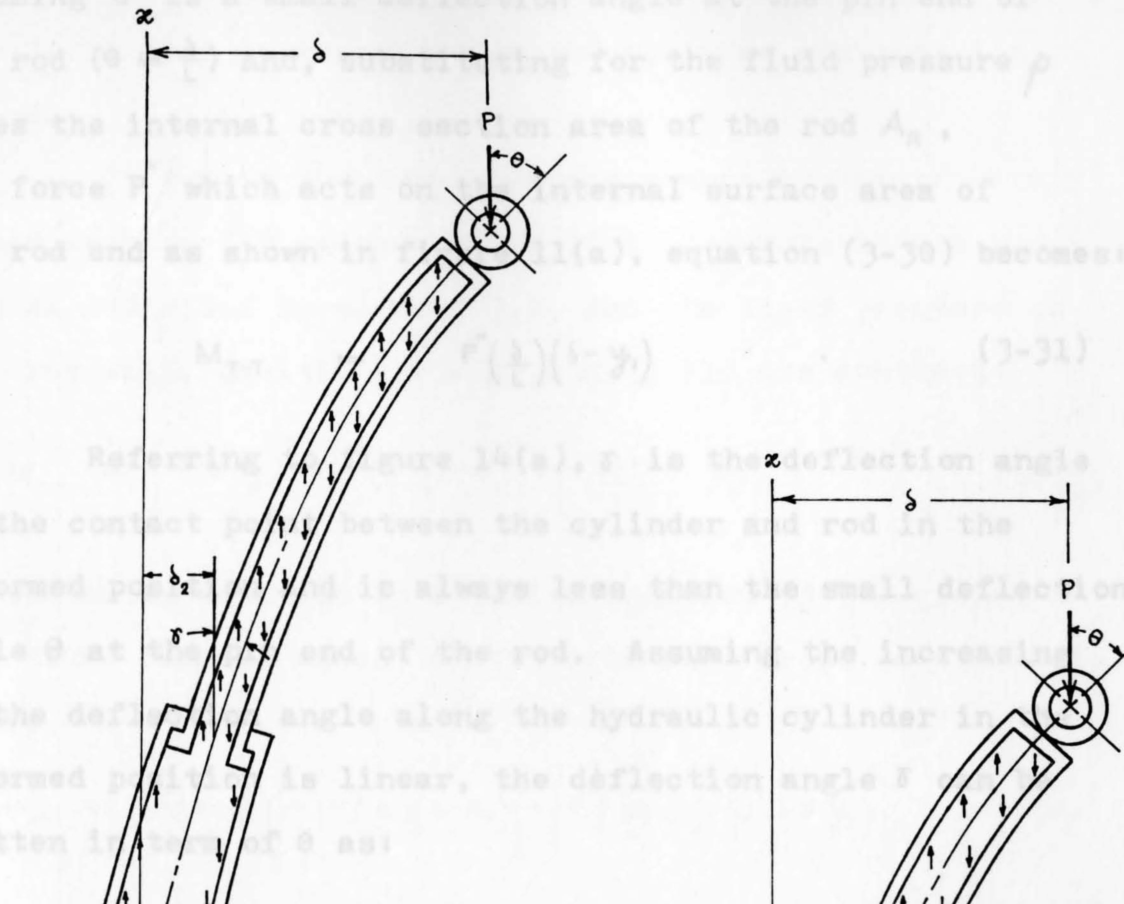


Figure 14 Free Body Diagrams of Fluid Filled Plunger Rod at any Arbitrary Section with Fluid Pressure Acting on the Column Wall.

Assuming θ is a small deflection angle at the pin end of the rod ($\theta \cong \frac{\delta}{L}$) and, substituting for the fluid pressure p times the internal cross section area of the rod A_R, the force P'' which acts on the internal surface area of the rod end as shown in figure 11(a), equation (3-30) becomes:

$$M_{T-T} = P''\left(\frac{\delta}{L}\right)\left(\delta - y_1\right) \quad . \qquad (3\text{-}31)$$

Referring to figure 14(a), γ is the deflection angle at the contact point between the cylinder and rod in the deformed position and is always less than the small deflection angle θ at the pin end of the rod. Assuming the increasing of the deflection angle along the hydraulic cylinder in the deformed position is linear, the deflection angle γ can be written in term of θ as:

$$\gamma = \frac{L_2}{L} . \theta \quad . \qquad (3\text{-}32)$$

Then, the bending moment produced by the fluid pressure acting on the column wall at any arbitrary cross section S-S as shown in figure 14(a) is:

$$M_{S-S} = \left[\left\{ p \sin\gamma (A_C - A_R)\right\}(\delta_2 - y_2)\right] + \left[(p \sin\theta . A_R)(\delta - y_2)\right] \quad , \qquad (3\text{-}33)$$

where A_c is the internal cross-sectional area of the cylinder and δ_2 is the deflection at the cylinder end.

By simplifying the equation obtained (equation (3-33)) and applying equation (3-32) gives:

$$M_{S-S} = \left[P'''\left(\frac{L_2}{L} . \frac{\delta}{L}\right)\left(\frac{L_2}{L} . \delta - y_2\right)\right] + \left[P'' . \frac{\delta}{L} . (\delta - y_2)\right] \quad , \qquad (3\text{-}34)$$

where P''' represents the result of the fluid pressure p times the difference in area between the cylinder and rod; i.e., $A_C - A_R$.

To obtain the total bending moment at the arbitrary cross section T-T caused by the fluid pressure at the rod end as described in section 3.2, and the fluid pressure on the rod wall, equations (3-14) and (3-31) are combined:

$$M_{T-T} = \left[P'(\delta - y_1)\right] + \left[P''\left(\frac{\delta}{L}\right)(L - x_1)\right] + \left[P''\left(\frac{\delta}{L}\right)(\delta - y_1)\right] \quad . \qquad (3\text{-}35)$$

By combining equation (3-15) with equation (3-34) yields the total bending moment at the arbitrary cross section S-S; i.e.,

$$M_{S-S} = \left[P'(\delta - y_2)\right] + \left[P''\left(\frac{\delta}{L}\right)(L - x_2)\right] + \left[P'''\left(\frac{L_2}{L}\cdot\frac{\delta}{L}\right)\left(\frac{L_2}{L}\cdot\delta - y_2\right)\right]$$
$$+ \left[P''\cdot\frac{\delta}{L}\cdot(\delta - y_2)\right] \quad . \qquad (3\text{-}36)$$

3.3.2 Solutions of the Differential Equations.

To determine the relationship between the applied force P and the total deflection δ at the pin end of the rod, and, thereby establish the critical value of the applied force P, it is necessary to write separate differential equations for the deflection for each portion of the column; e.g. at sections T-T and S-S.

Substituting equations (3-35) and (3-36) into equation (2-1) and simplifying the terms on the right-hand side of the equations gives:

$$E_1 I_1 \frac{d^2 y_1}{dx_1^2} = -\left[\left(P' + P''\cdot\frac{\delta}{L}\right) y_1\right] - \left[P''\cdot\frac{\delta}{L}\cdot x_1\right] + \left[\left(P + P''\cdot\frac{\delta}{L}\right)\cdot\delta\right] \quad , \qquad (3\text{-}37)$$

and

$$E_2 I_2 \frac{d^2 y_2}{dx_2^2} = -\left[\left(P' + P''\cdot\frac{\delta}{L} + P'''\cdot\frac{L_2\delta}{L^2}\right) y_2\right] - \left[P''\cdot\frac{\delta}{L}\cdot x_2\right] + \left[\left(P + P''\cdot\frac{\delta}{L} + P'''\frac{L_2^2\cdot\delta}{L^3}\right)\cdot\delta\right] \quad . \qquad (3\text{-}38)$$

Using the notations:

$$\left.\begin{aligned} k_1^2 &= \frac{P}{E_1 I_1} , \quad k_2^2 = \frac{P}{E_2 I_2} , \quad k_3^2 = \frac{P'}{E_1 I_1} , \\ k_4^2 &= \frac{P''}{E_1 I_1} , \quad k_5^2 = \frac{P'}{E_2 I_2} , \quad k_6^2 = \frac{P''}{E_2 I_2} , \\ \text{and} \quad k_7^2 &= \frac{P'''}{E_2 I_2} , \end{aligned}\right] \qquad (3\text{-}39)$$

equations (3-37) and (3-38) become:

$$\frac{d^2 y_1}{dx_1^2} + \left[\left(k_3^2 + k_4^2\cdot\frac{\delta}{L}\right) y_1\right] = -\left[k_4^2\cdot\frac{\delta}{L}\cdot x_1\right] + \left[\left(k_1^2 + k_4^2\cdot\frac{\delta}{L}\right)\cdot\delta\right] \quad , \qquad (3\text{-}40)$$

and

$$\frac{d^2 y_2}{dx_2^2} + \left[\left(k_5^2 + k_6^2\cdot\frac{\delta}{L} + k_7^2\cdot\frac{L_2\delta}{L^2}\right) y_2\right] = -\left[k_6^2\cdot\frac{\delta}{L}\cdot x_2\right] + \left[\left(k_2^2 + k_6^2\cdot\frac{\delta}{L} + k_7^2\cdot\frac{L_2^2\delta}{L^3}\right)\cdot\delta\right] \quad . \qquad (3\text{-}41)$$

Equations (3-40) and (3-41) are second order differential equations with constant coefficients whose general solutions can be shown to be (see Appendix C):

$$y_1 = A\cos k_8 x_1 + B\sin k_8 x_1 - \frac{k_4^2}{k_8^2}\cdot\frac{\delta}{L}\cdot x_1 + \left(\frac{k_1^2}{k_8^2} + \frac{k_4^2}{k_8^2}\cdot\frac{\delta}{L}\right)\cdot\delta \quad , \qquad (3\text{-}42)$$

$$y_2 = E\cos k_9 x_2 + F\sin k_9 x_2 - \frac{k_6^2}{k_9^2}\cdot\frac{\delta}{L}\cdot x_2 + \left(\frac{k_2^2}{k_9^2} + \frac{k_6^2}{k_9^2}\cdot\frac{\delta}{L} + \frac{k_7^2}{k_9^2}\cdot\frac{L_2^2\delta}{L^3}\right)\cdot\delta \quad , \qquad (3\text{-}43)$$

where;

$$\left.\begin{aligned} k_8^2 &= k_3^2 + k_4^2\cdot\frac{\delta}{L} \, , \\ k_9^2 &= k_5^2 + k_6^2\cdot\frac{\delta}{L} + k_7^2\cdot\frac{L_2\delta}{L^2} \, , \end{aligned}\right] \qquad (3\text{-}44)$$

and A, B, E, F are arbitrary constants that are determined by satisfying the boundary conditions of the problem..

3.3.3 Formulation of the Buckling Equation.

Referring to figure 8, the same five boundary conditions as described in section 2.3 can be applied to the hydraulic cylinder with fluid filled plunger rod. Applying the first two boundary conditions to equation (3-43) gives the constant values:

$$E = -\left(\frac{k_2^2}{k_9^2} + \frac{k_6^2}{k_9^2}\cdot\frac{\delta}{L} + \frac{k_7^2}{k_9^2}\cdot\frac{L_2^2\delta}{L^3}\right)\cdot\delta \quad ,$$

and

$$F = \frac{k_6^2}{k_9^3}\cdot\frac{\delta}{L} \quad ,$$

and substituting these values into equation (3-43), the following is obtained:

$$y_2 = -\left(\frac{k_2^2}{k_9^2}+\frac{k_6^2}{k_9^2}\cdot\frac{\delta}{L}+\frac{k_7^2}{k_9^2}\cdot\frac{L_2^2\delta}{L^3}\right)\cdot\delta\cdot\cos k_9 x_2+\frac{k_6^2}{k_9^3}\cdot\frac{\delta}{L}\cdot\sin k_9 x_2$$

$$-\frac{k_6^2}{k_9^2}\cdot\frac{\delta}{L}\cdot x_2+\left(\frac{k_2^2}{k_9^2}+\frac{k_6^2}{k_9^2}\cdot\frac{\delta}{L}+\frac{k_7^2}{k_9^2}\cdot\frac{L_2^2\delta}{L^3}\right)\cdot\delta \qquad . \qquad (3\text{-}45)$$

Applying the second two boundary conditions to equation (3-42) gives the constant values:

$$\left.\begin{aligned}
B = \frac{\cos k_8 L}{\sin k_8 L_1}\Bigg[&\left(\frac{k_2^2}{k_9^2}+\frac{k_6^2}{k_9^2}\cdot\frac{\delta}{L}+\frac{k_7^2}{k_9^2}\cdot\frac{L_2^2\delta}{L^3}\right)\cdot\delta\cdot\cos k_9 L_2\\
&-\frac{k_6^2}{k_9^3}\cdot\frac{\delta}{L}\cdot\sin k_9 L_2+\frac{k_6^2}{k_9^2}\cdot\frac{\delta}{L}\cdot L_2-\left(\frac{k_2^2}{k_9^2}+\frac{k_6^2}{k_9^2}\cdot\frac{\delta}{L}+\frac{k_7^2}{k_9^2}\cdot\frac{L_2^2\delta}{L^3}\right)\cdot\delta\\
&+\frac{\cos k_8 L_2}{\cos k_8 L}\left(\delta-\frac{k_3^2}{k_8^2}\cdot\delta-\frac{k_4^2}{k_8^2}\cdot\frac{\delta^2}{L}\right)-\frac{k_4^2}{k_8^2}\cdot\frac{\delta}{L}\cdot L_2\\
&+\frac{k_1^2}{k_8^2}\cdot\delta+\frac{k_4^2}{k_8^2}\cdot\frac{\delta^2}{L}\Bigg]\\
\text{and } A = \frac{1}{\cos k_8 L}\Bigg[&\delta - B\sin k_8 L-\frac{k_3^2}{k_8^2}\cdot\delta-\frac{k_4^2}{k_8^2}\cdot\frac{\delta^2}{L}\Bigg] \quad ,
\end{aligned}\right] \qquad (3\text{-}46)$$

or equation (3-42) can be written as:

$$y_1 = A\cos k_8 x_1 + B\sin k_8 x_1 - \frac{k_4^2}{k_8^2}\cdot\frac{\delta}{L}\cdot x_1$$

$$+\left(\frac{k_1^2}{k_8^2}+\frac{k_4^2}{k_8^2}\cdot\frac{\delta}{L}\right)\cdot\delta \quad , \qquad (3\text{-}47)$$

where A and B are the constant values expressed by equations (3-46).

Equations (3-45) and (3-47) can be related by applying the last boundary condition. Referring to equation (2-15):

$$\frac{dy_1}{dx_1} = \frac{\dfrac{dy_2}{dx_2} + \dfrac{\delta_\alpha}{L_1}}{1 - \dfrac{dy_2}{dx_2} \cdot \dfrac{\delta_\alpha}{L_1}} \quad , \qquad (2\text{-}15)$$

where $\frac{dy_1}{dx_1}$ and $\frac{dy_2}{dx_2}$ are obtained by differentiating equations (3-47) and (3-45), respectively;

$$\left.\begin{aligned} \frac{dy_1}{dx_1} &= -k_8 A \sin k_8 L_2 + k_8 B \cos k_8 L_2 - \frac{k_4^2}{k_8^2} \cdot \frac{\delta}{L}, \\ \frac{dy_2}{dx_2} &= \left(\frac{k_2^2}{k_9} + \frac{k_6^2}{k_9} \cdot \frac{\delta}{L} + \frac{k_7^2}{k_9} \cdot \frac{L_2^2 \delta}{L^3} \right) \cdot \delta \cdot \sin k_9 L_2 \\ &\quad + \frac{k_6^2}{k_9^2} \cdot \frac{\delta}{L} \left(\cos k_9 L_2 - 1 \right) \end{aligned}\right] \qquad (3\text{-}48)$$

By simplifying the equation obtained by substituting equation (3-48) into equation (2-15), and applying equation (2-4), yields the actual deflection δ_P at the pin end of the rod as related to the applied force P; i.e.:

$$\begin{aligned} & -k_8 A \sin k_8 L_2 + k_8 B \cos k_8 L_2 - \frac{k_4^2}{k_8^2 . L}(\delta_\alpha + \delta_P) - \left[\frac{k_2^2}{k_9} + \frac{k_6^2}{k_9 . L}(\delta_\alpha + \delta_P) \right. \\ & \left. + \frac{k_7^2}{k_9} \cdot \frac{L_2^2}{L^3}(\delta_\alpha + \delta_P) \right] (\delta_\alpha + \delta_P) \sin k_9 L_2 - \frac{k_6^2}{k_9^2 . L}(\delta_\alpha + \delta_P)(\cos k_9 L_2 - 1) \\ & - \frac{\delta_\alpha}{L_1} \left[\left\{ -k_8 A \sin k_8 L_2 + k_8 B \cos k_8 L_2 - \frac{k_4^2}{k_8^2 . L}(\delta_\alpha + \delta_P) \right\} \left\{ \left(\frac{k_2^2}{k_9} \right. \right. \right. \\ & + \left. \frac{k_6^2}{k_9 . L}(\delta_\alpha + \delta_P) + \frac{k_7^2}{k_9} \cdot \frac{L_2^2}{L^3}(\delta_\alpha + \delta_P) \right)(\delta_\alpha + \delta_P) \sin k_9 L_2 \\ & \left. \left. + \frac{k_6^2}{k_9^2 . L}(\delta_\alpha + \delta_P)(\cos k_9 L_2 - 1) \right\} + 1 \right] \quad = \quad 0 \qquad (3\text{-}49) \end{aligned}$$

where A and B are the constant values expressed in equation (3-46).

The derived formula (equation (3-49)) can be solved by a trial and error method, and a computer program is very helpful in making the necessary calculations required for a solution. The alphabetic letters used in the derived formula, (3-49), correspond to those used in the computer program for convenience and clarity as shown on page 55 as computer program no. 4. The same actual dimensions of the hydraulic actuating cylinder used in the previous case were taken to show the performance of the derived formula. The equation was evaluated by using various arbitrary values of initial deflection angle and the characteristic results are shown by a plot of load vs deflection in figure 22. The calculations required for the results given by the plots of figure 22 were obtained from program no. 4.

CHAPTER IV

EXPERIMENTAL STUDY

4.1 Experimental Procedure

An experimental study of the hydraulic actuating cylinder behavior was conducted at the Cylinder Division of the Commercial Shearing Inc. plant by using a fluid unfilled plunger rod with both ends pinned. The important dimensions of the hydraulic test cylinder are shown by the drawing in figure 15. The pinned end of the cylinder

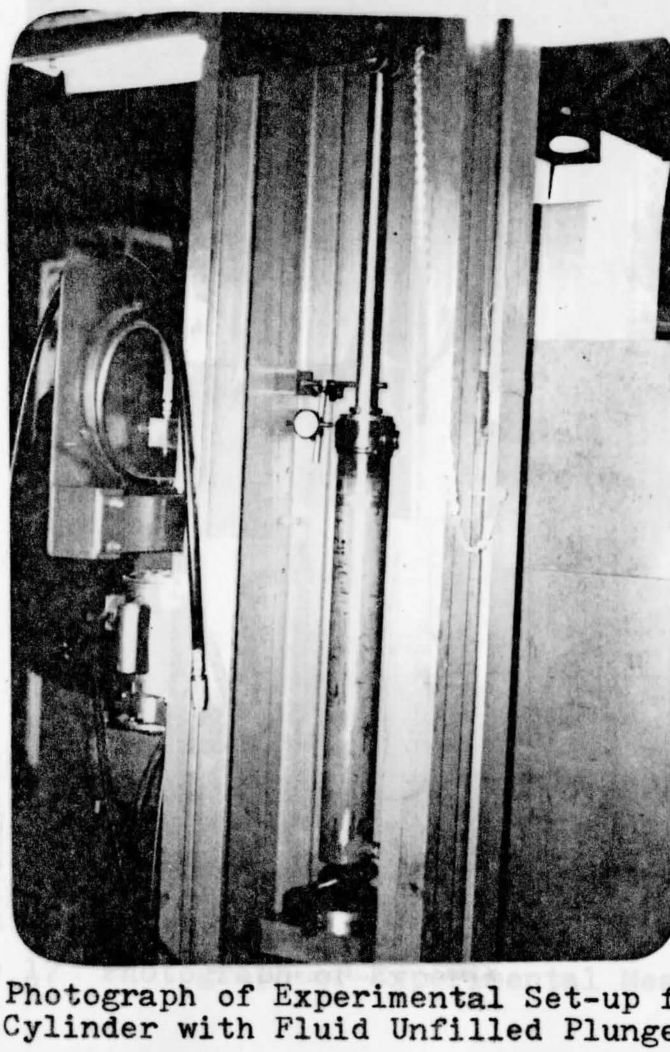


Figure 16 Photograph of Experimental Set-up for Hydraulic Cylinder with Fluid Unfilled Plunger Rod.

(bottom pin) was connected by a 1.0 inch bolt to a horizontal circular steel base, 9.0 inches diameter, which was capable of moving in a vertical direction by pumping fluid through a special fitting and line connected to a hydraulic pump. The upper end of the hydraulic test cylinder was connected by a 1.0 inch bolt to a pineye that was fixed to a steel column. The cylinder portion was filled with a fluid to keep the hydraulic test cylinder extended to its full stroke. A dial gauge was placed at the wall close to the connecting line

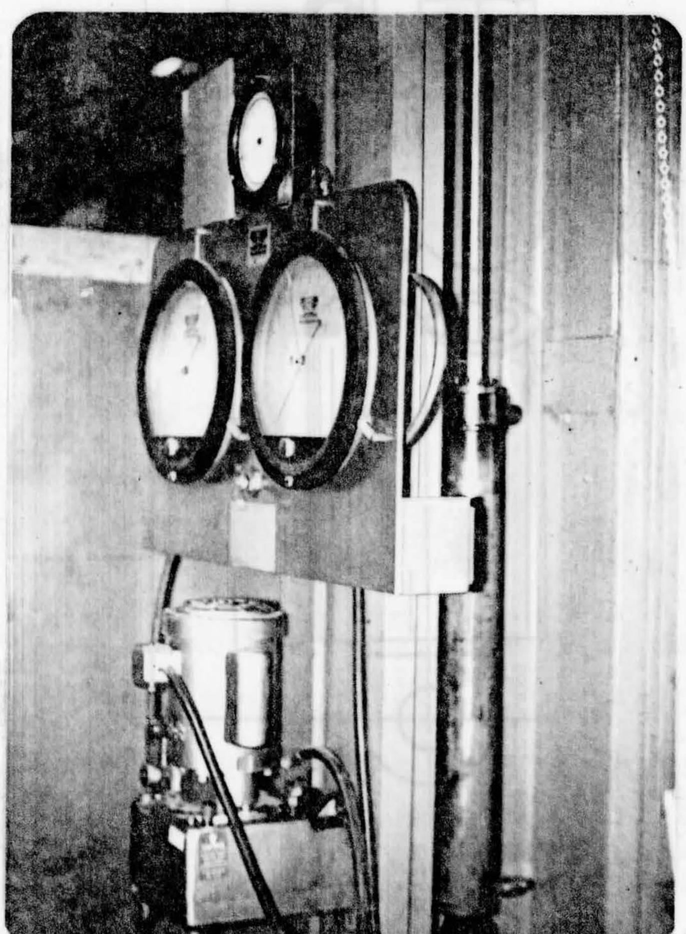


Figure 17 Photograph of Experimental Measurements.

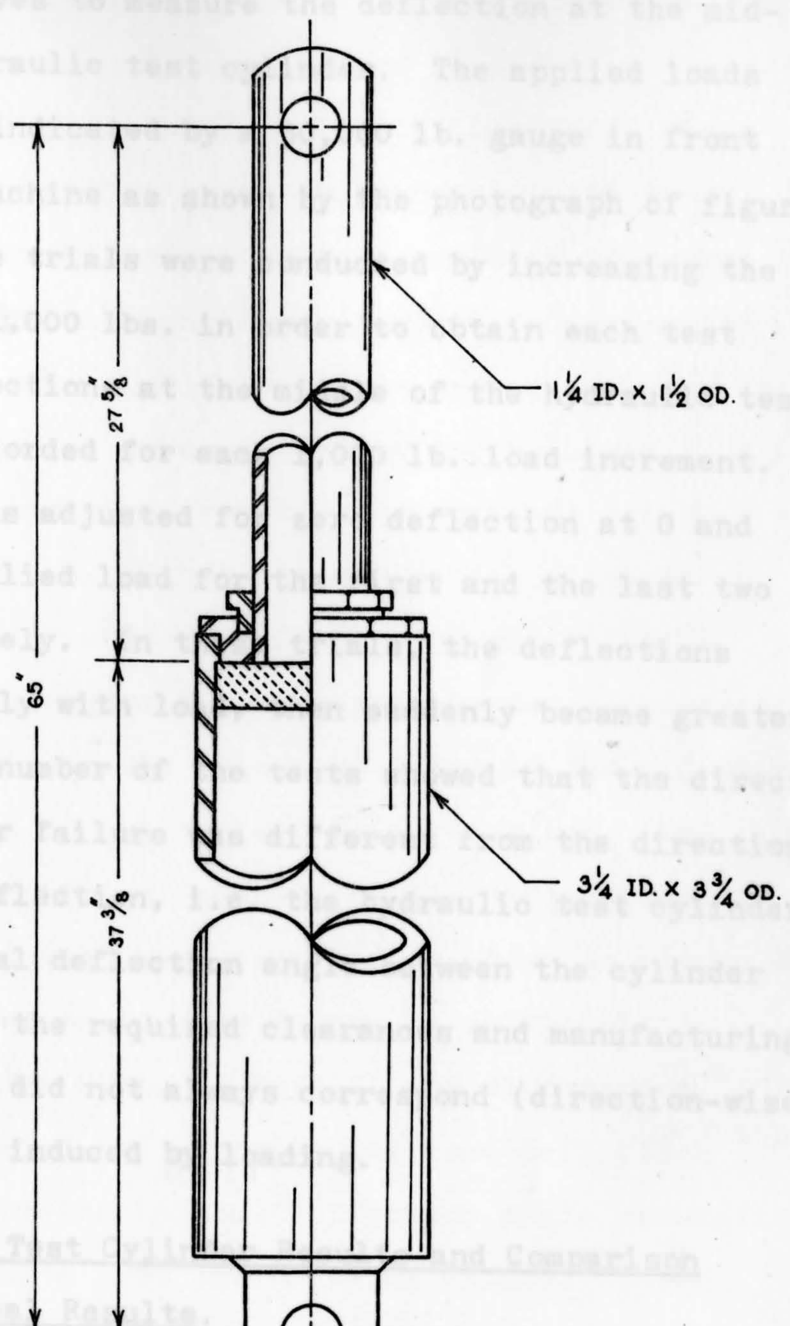


Figure 15 Dimensions of the Testing Hydraulic Cylinder with Fluid Unfilled Plunger Rod.

line of the pineyes to measure the deflection at the mid-point of the hydraulic test cylinder. The applied loads were accurately indicated by a 30,000 lb. gauge in front of the testing machine as shown by the photograph of figure 16.

The three trials were conducted by increasing the applied load by 1,000 lbs. in order to obtain each test point. The deflections at the middle of the hydraulic test cylinder were recorded for each 1,000 lb. load increment. The dial gauge was adjusted for zero deflection at 0 and 2,000 lbs. of applied load for the first and the last two trials, respectively. In these trials, the deflections increased gradually with load, then suddenly became greater near failure. A number of the tests showed that the direction of deflection near failure was different from the direction of the initial deflection, i.e. the hydraulic test cylinder had a small initial deflection angle between the cylinder and rod resulting the required clearances and manufacturing tolerances and it did not always correspond (direction-wise) to the deflection induced by loading.

4.2 Experimental Test Cylinder Results and Comparison with Analytical Results.

The relations between the applied loads and deflections at the mid-point of the test cylinder for three experimental trials are shown in table 1. The load-deflection curves were plotted using the data on table 1 as shown in figure 24. The formulation of the buckling equation for the hydraulic cylinder with fluid unfilled plunger rod (equation (2-17)) was used to calculate analytical values in order to compare the derived formula with the experimental results. Since the clearances and manufacturing tolerances of the test cylinder and rod were found to be very small, 0.10 degree of the initial deflection angle was assumed in performing this calculation. The double precision method was used in the computer program in order to obtain the required precision. The calculated results were obtained using computer program no. 5. Since the deflection obtained from the calculated results are for the pinned end of the rod as shown in figure 8, it reasonable for comparison purposes to approximate the values obtained at the mid-point as being equivalent to twice their measured value. The calculated and experimental results of the deflections at the various applied load were compared as shown in figure 24 and the agreement shown in this figure is sufficient to verify the analytical method. Most of the differences between the calculated and experimental load-deflection curves occurred from the initial deflection angle assumption. The calculated results from the previous chapter showed that the load-deflection curve varied with the given initial deflection angle and, it is difficult, from a practical standpoint, to specify the exact value of the initial deflection angle between the cylinder and rod.

CHAPTER V

CONCLUSIONS AND DISCUSSION

5.1 Conclusions.

There are two ways in which a hydraulic actuating cylinder can be constructed, either with a fluid-filled plunger rod or a fluid-unfilled plunger rod. In the analysis of the two types, the double-integration method based on simple beam theory was applied, and many different effects were considered particularly in the case of the fluid-filled plunger rod. The main effect of the internal pressure for the fluid-filled plunger rod case was introduced in section 3.1. The effect of the small force component at the rod end which produced a significant moment was added to the bending equation as described in section 3.2. In the last section 3.3, the moment effects of all the previous sections were combined with the moment effect produced by the fluid pressure acting on the column wall in the deflected position in order to obtain the complete final result. Since the final formulations of the buckling equations obtained for each case were complicated, computer programs were written for convenience in performing the required calculations.

The initial portion of the deflection-curves for any given initial deflection angle, α , which amount to the plotting of the applied load vs the deflection at the pinned end of the rod, tends to increase in slope as the initial deflection angle decreases.(refer to figures 19 to 22). If the initial deflection angle were assumed to be zero, each formulation gives the characteristic result that there is no deflection at the pinned end of the rod as the applied load is increased until it reaches the critical buckling load value at which point the deflection suddenly increases to infinity (buckles).

In the analysis, the deflection angle at the pinned end of the rod was assumed to be small, i.e. $\sin\theta \cong \theta$ and $\cos\theta \cong 1$, which gives fairly accurate results (within 0.4%), for deflection angles not exceeding five degrees. In practice, the deflection angle at pinned end of the rod should not exceed this limit and this coupled with the mathematical simplifications resulting from the assumption, justify its use.

5.2 Discussion

For the given actual dimensions of a representative cylinder taken from one presently designed and fabricated as shown in figure 18, the analytical results are compared for the fluid-filled plunger rod and fluid-unfilled plunger rod for each case as described in sections 2.1, 3.1, 3.2 and 3.3. Each characteristic result, i.e. a plot of the applied load vs deflection at the pinned end of the rod, were determined and plotted for an initial deflection angle of three degrees as shown in figure 23. Here, it is shown that the fluid-filled plunger rod, in the first case of the analysis (line no. 2), has a buckling load of almost twice that of the fluid-unfilled plunger rod (line no.1) for the same given dimensions. When the effect of the bending moment produced by the small component of force at the rod end was considered, the capability of the hydraulic cylinder to support the applied load was reduced about 20% (line no. 3) from the previous case which neglected this effect (compare line no. 2 with line no. 3). When the bending moment produced by the fluid pressure acting on the column wall at the deflected position was included in the analysis, the capability of the hydraulic cylinder to support the applied load was reduced again but to a lesser extent than what occurred for the previous effect (compare line no. 3 with line no. 4).

The final result of the fluid-filled plunger rod which includes all the moments (line no. 4, figure 23) is seen to have a higher buckling load than that for the fluid unfilled plunger rod (line no. 1). Furthermore, it should be noted that in the typical cases discussed in sections 3.1 and 3.2, as shown by the lines no. 2 and no. 3 in figure 23, the procedure of neglecting some forces that are destabilizing leads to an unsafe rather than an over designed cylinder.

5.3 Comment of Further Work

The hydraulic actuating cylinder with both pinned ends can be analysed by applying the double integration method based on the simple beam theory. Some boundary conditions as described on section 2.3 must be changed in order to satisfy the new condition. The analysis of many different effects that occur in both types of the fluid filled and unfilled plunger rod can follow the method described in chapters II and III.

The results of the experimental study described in chapter IV are compared with the theoretical values obtained for the fluid unfilled plunger rod. An experimental study using a fluid filled plunger rod can be conducted, and compared to theoretical results obtained by using equation (3-49) and computer program no. 4.

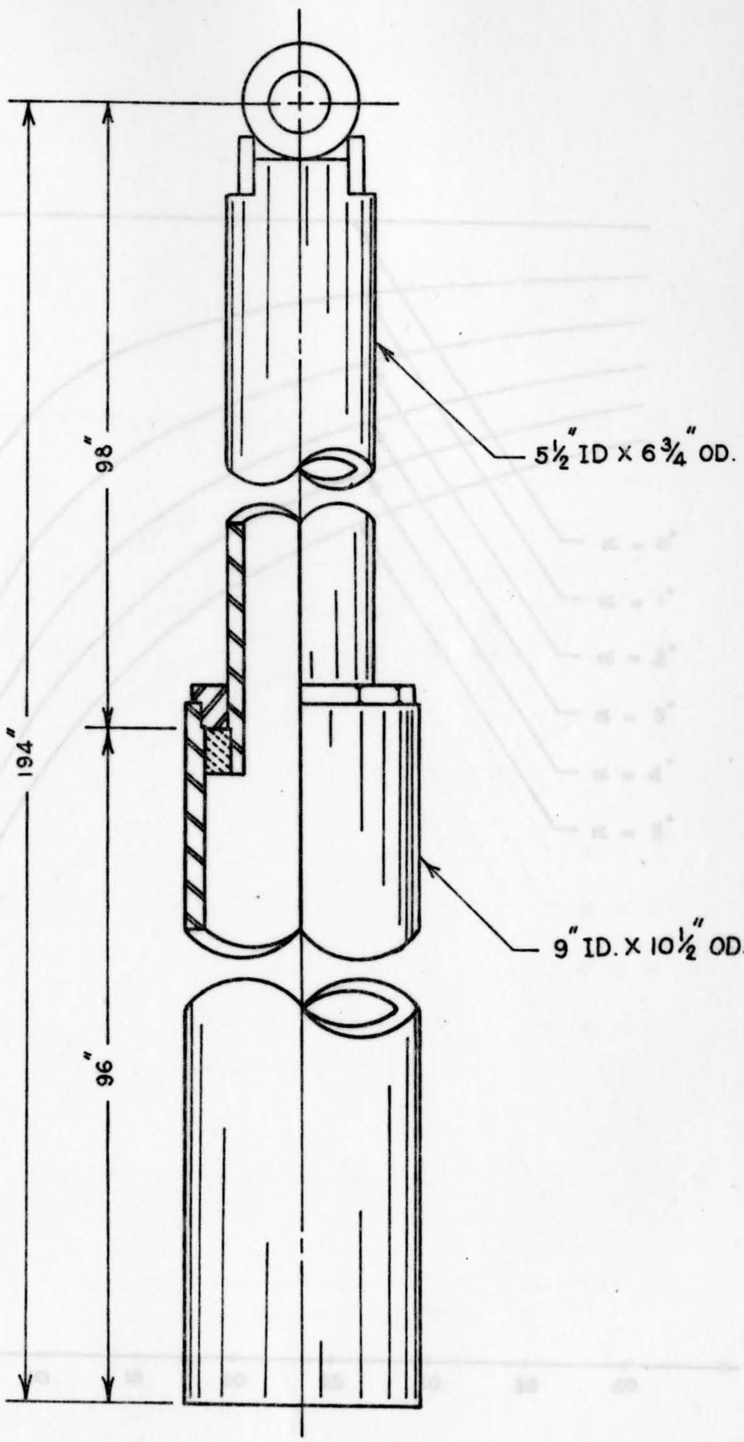


Figure 18 Dimensions of the Hydraulic Cylinder for the Theoretical Calculation

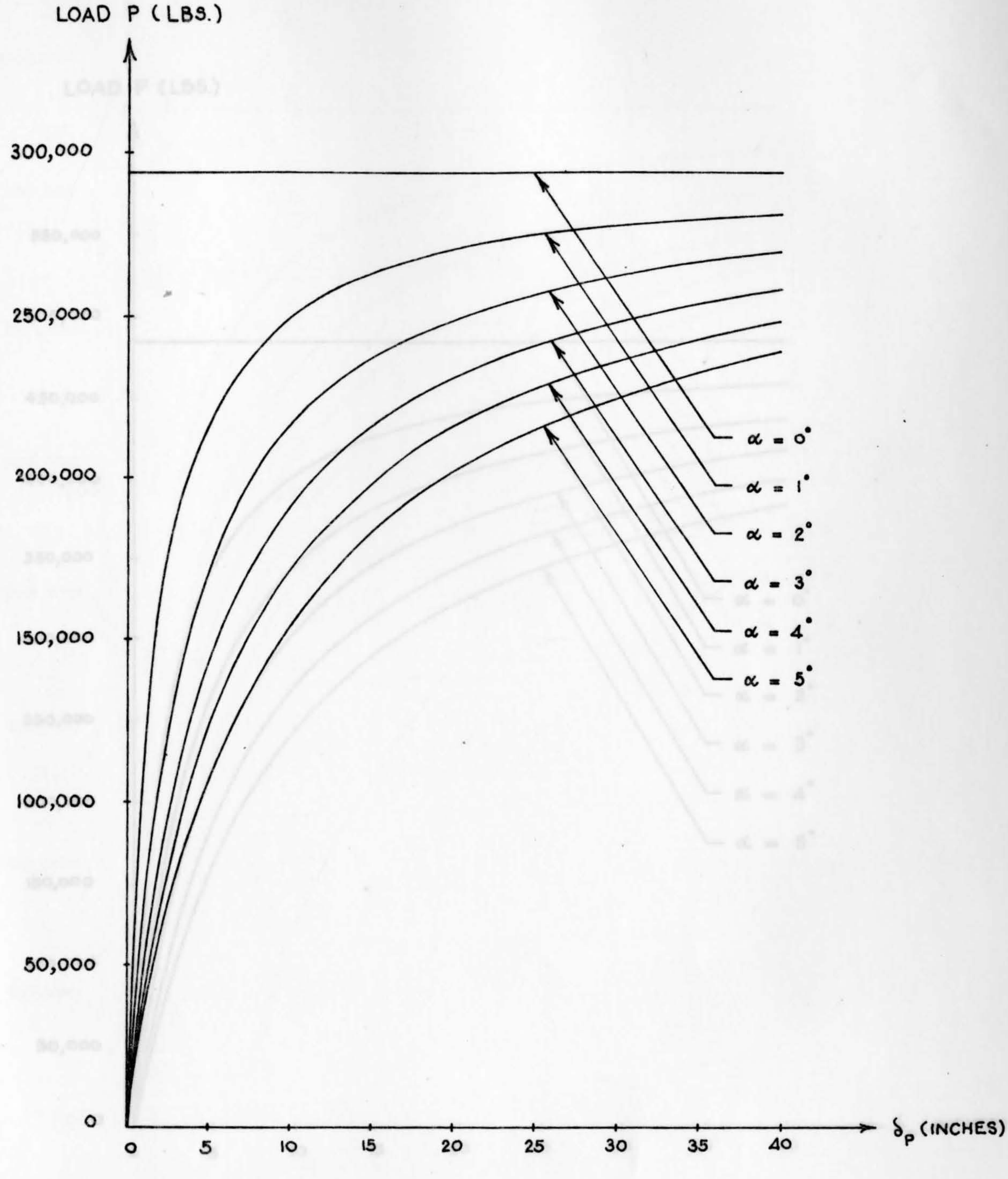


Figure 19 Load-Deflection Curves for the Fluid Unfilled Plunger Rod.

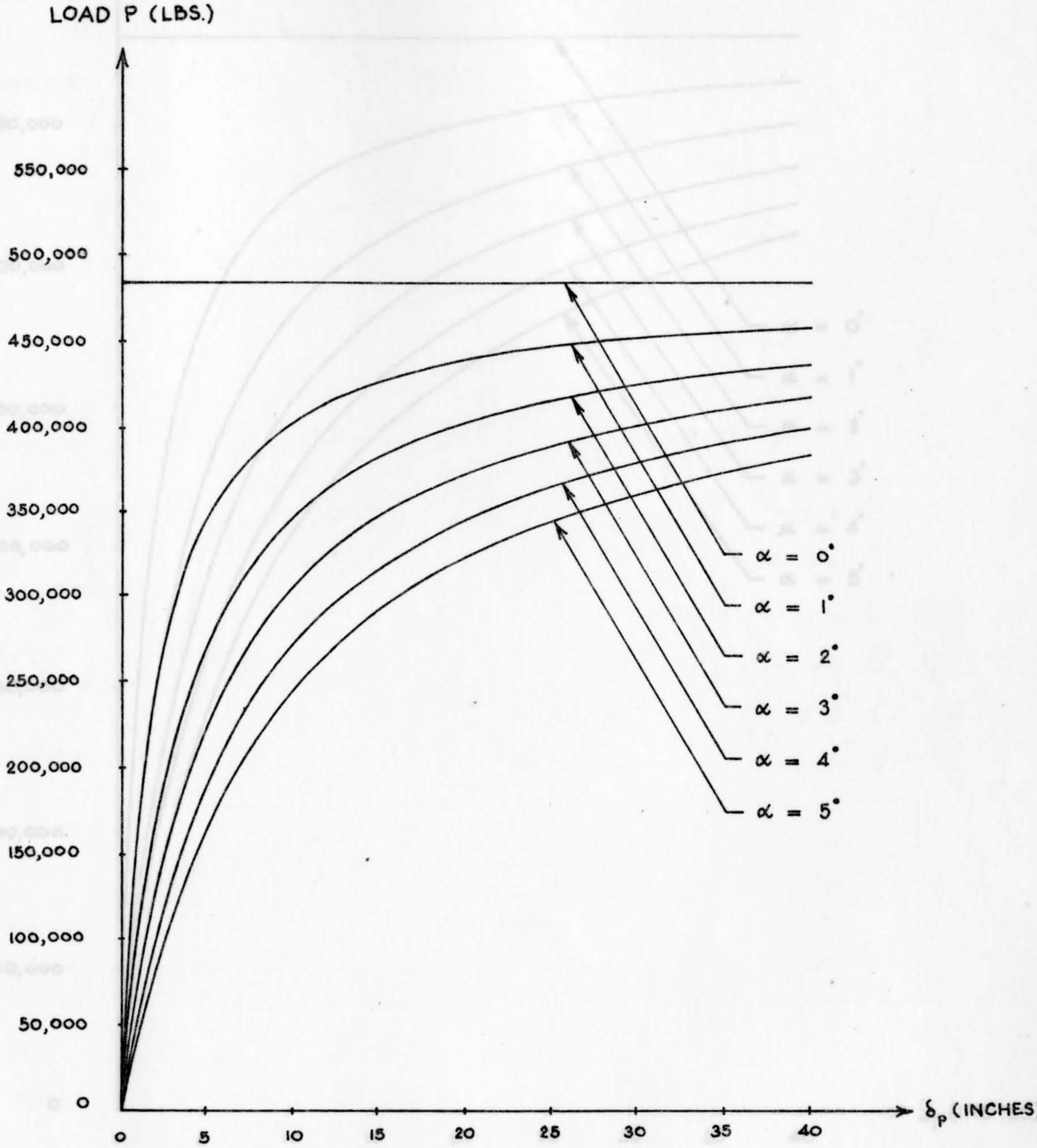


Figure 20 Load-Deflection Curves for the Fluid Filled Plunger Rod when the Main Effect of the Fluid Pressure was considered.

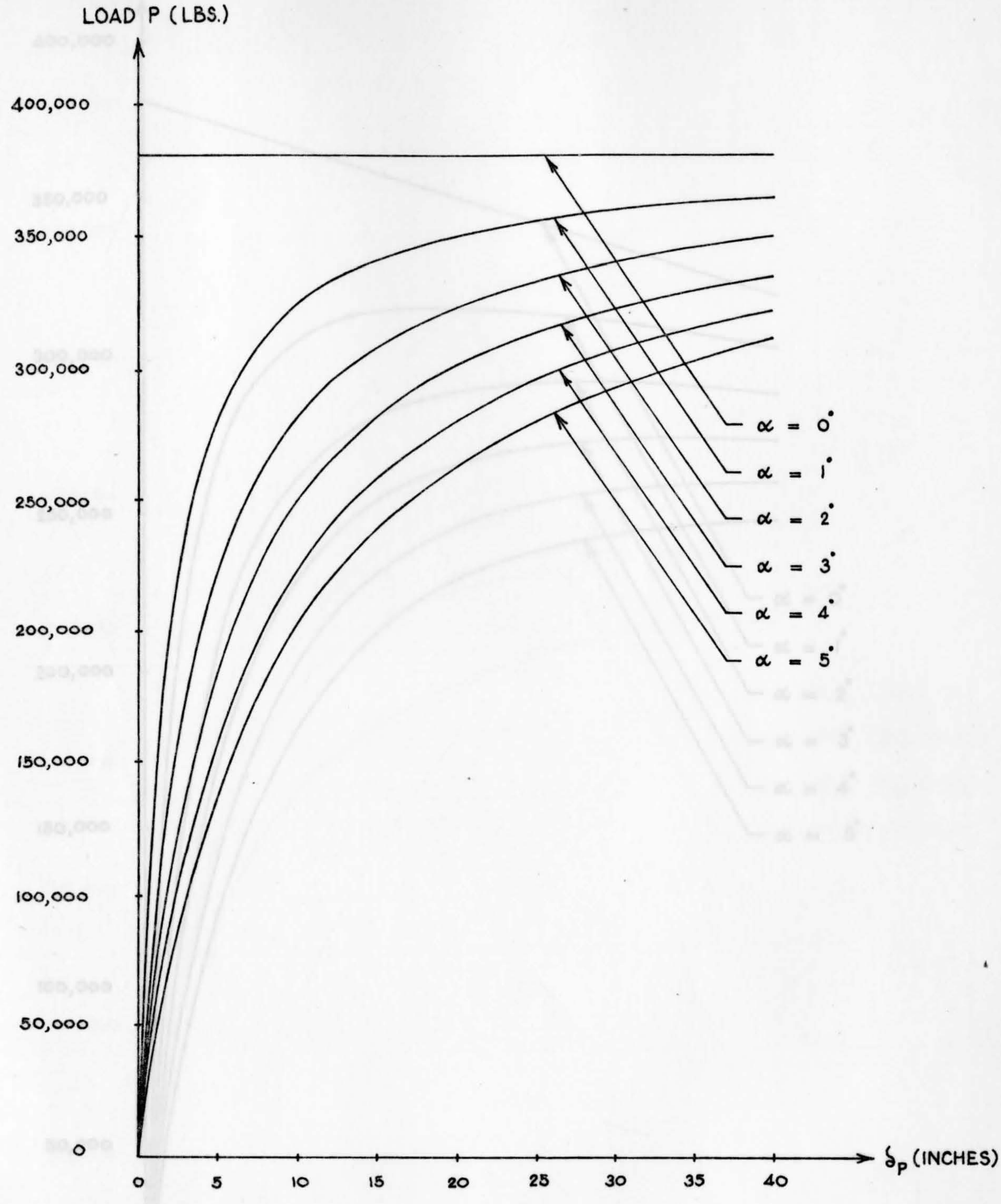


Figure 21 Load-Deflection Curves for the Fluid Filled Plunger Rod when the Bending Moment Produced by Force P sinθ was considered.

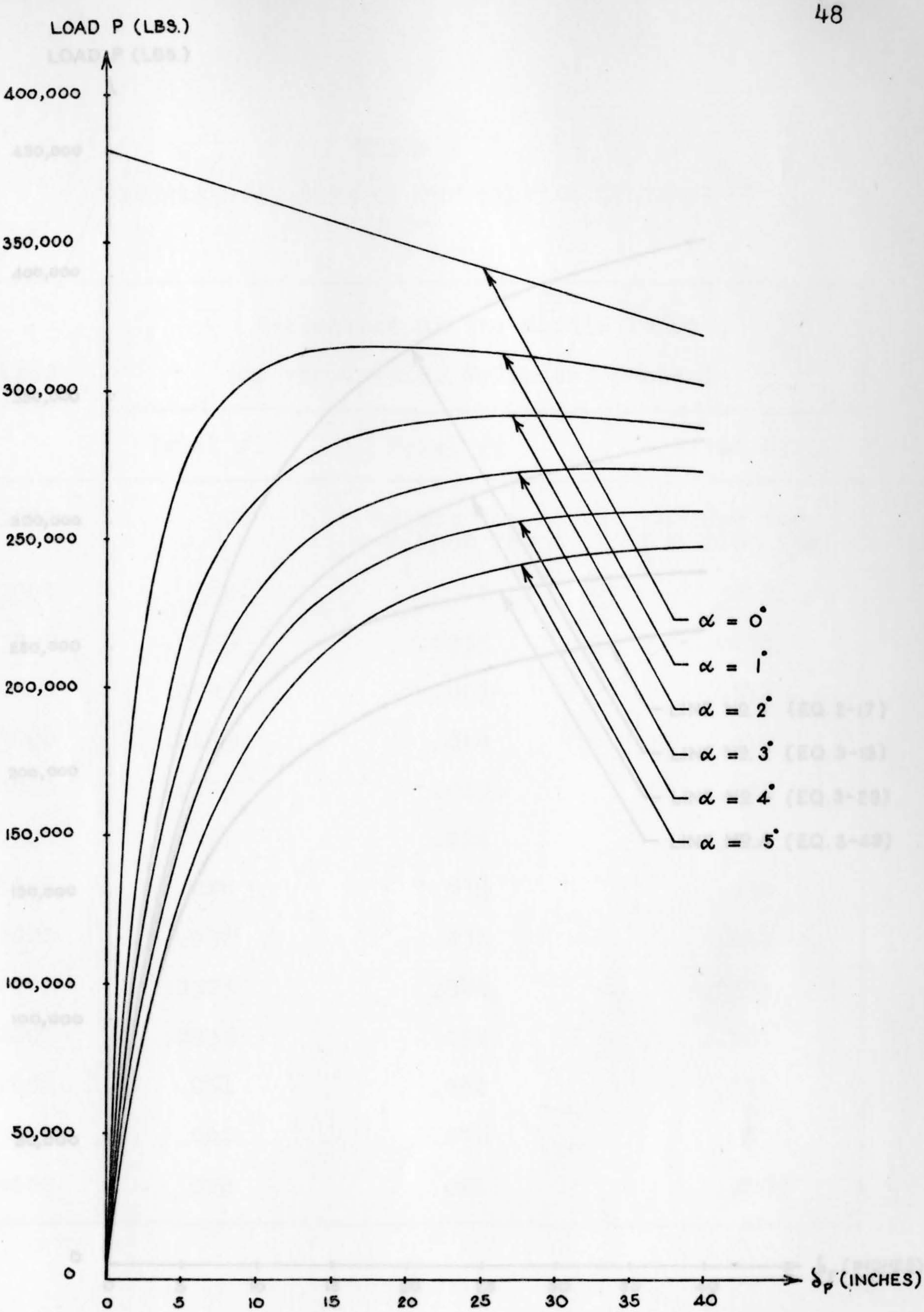


Figure 22 Load-Deflection Curves for the Fluid Filled Plunger Rod at the Final Result.

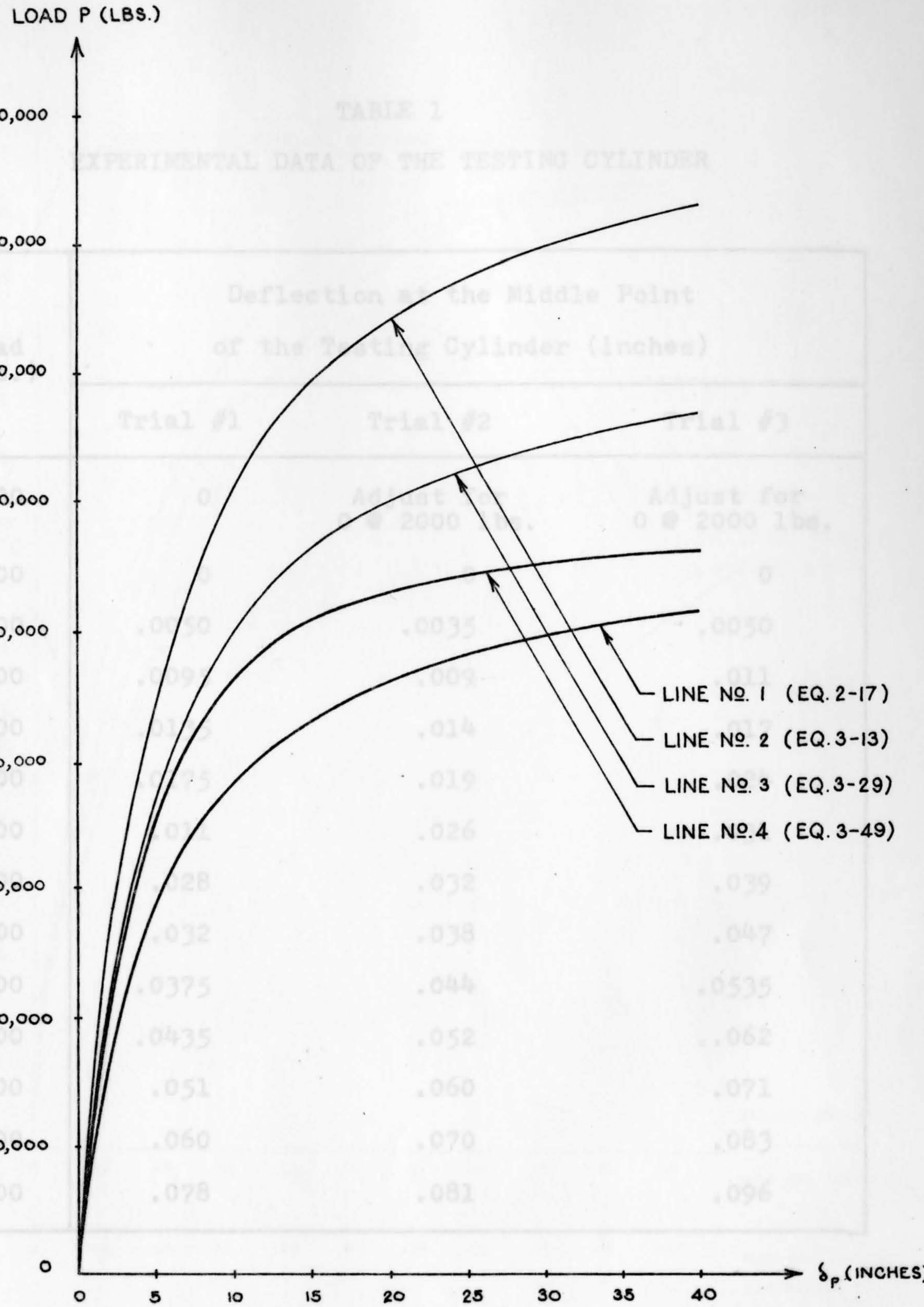


Figure 23 Theoretical Results Comparison at Three Degrees of Initial Deflection Angle.

TABLE 1

EXPERIMENTAL DATA OF THE TESTING CYLINDER

Load (lbs.)	Deflection at the Middle Point of the Testing Cylinder (inches)		
	Trial #1	Trial #2	Trial #3
1000	0	Adjust for 0 @ 2000 lbs.	Adjust for 0 @ 2000 lbs.
2000	0	0	0
3000	.0050	.0035	.0050
4000	.0095	.009	.011
5000	.0135	.014	.017
6000	.0175	.019	.024
7000	.011	.026	.031
8000	.028	.032	.039
9000	.032	.038	.047
10000	.0375	.044	.0535
11000	.0435	.052	.062
12000	.051	.060	.071
13000	.060	.070	.083
14000	.078	.081	.096

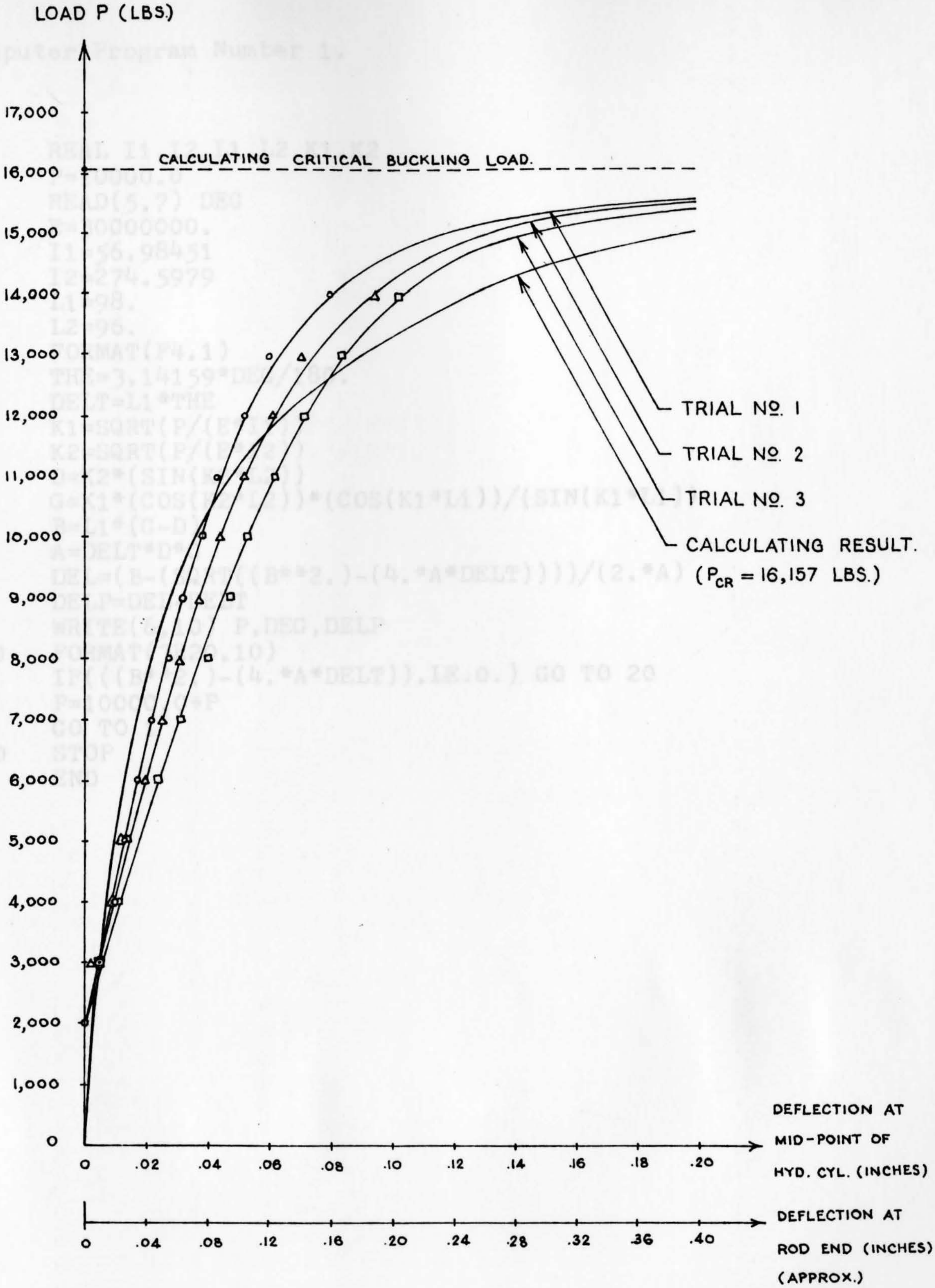


Figure 24 Experimental and Analytical Results Comparison

Computer Program Number 1.

```
      REAL I1,I2,L1,L2,K1,K2
      P=10000.0
      READ(5,7) DEG
1     E=30000000.
      I1=56.98451
      I2=274.5979
      L1=98.
      L2=96.
7     FORMAT(F4.1)
      THE=3.14159*DEG/180.
      DELT=L1*THE
      K1=SQRT(P/(E*I1))
      K2=SQRT(P/(E*I2))
      D=K2*(SIN(K2*L2))
      G=K1*(COS(K2*L2))*(COS(K1*L1))/(SIN(K1*L1))
      B=L1*(G-D)
      A=DELT*D*G
      DEL=(B-(SQRT((B**2.)-(4.*A*DELT))))/(2.*A)
      DELP=DEL-DELT
5     WRITE(6,10) P,DEG,DELP
10    FORMAT(3E20.10)
      IF(((B**2.)-(4.*A*DELT)).LE.0.) GO TO 20
      P=10000.0+P
      GO TO 1
20    STOP
      END
```

Computer Program Number 2.

```
      REAL I1,I2,L1,L2,K3,K5
      P=10000.0
      READ(5,7) DEG
1     E=30000000.
      I1=56.98451
      I2=274.5979
      L1=98.
      L2=96.
7     FORMAT(F4.1)
      THE=3.14159*DEG/180.
      DELT=L1*THE
      K3=SQRT((0.33608*P)/(E*I1))
      K5=SQRT((0.33608*P)/(E*I2))
      D=K5*(SIN(K5*L2))
      G=K3*(COS(K5*L2))*(COS(K3*L1))/(SIN(K3*L1))
      B=L1*(G-D)
      A=DELT*D*G
      DEL=(B-(SQRT((B**2.)-(4.*A*DELT))))/(2.*A)
      DELP=DEL-DELT
5     WRITE(6,10) P,DEG,DELP
10    FORMAT(3E20.10)
      IF(((B**2.)-(4.*A*DELT)).LE.0.) GO TO 20
      P=10000.0+P
      GO TO 1
20    STOP
      END
```

Computer Program Number 3

```
      REAL I1,I2,L,L1,L2,K1,K2,K3,K4,K5,K6,DELP
      P=10000.0
      READ(5,7) DEG
 1    E=30000000.
      I1=56.98451
      I2=274.5979
      L=194.
      L1=98.
      L2=96.
      DELP=1.0
 7    FORMAT(F4.1)
      THE=3.14159*DEG/180.
      DELT=L1*THE
      DEL=DELP+DELT
      K1=SQRT(P/(E*I1))
      K2=SQRT(P/(E*I2))
      K3=SQRT((0.33608*P)/(E*I1))
      K4=SQRT((0.66392*P)/(E*I1))
      K5=SQRT((0.33608*P)/(E*I2))
      K6=SQRT((0.66392*P)/(E*I2))
      F=(((K2**2.)*(COS(K5*L2))/(K5**2.))-(((K6**2.)*(SIN(
     1K5*L2)))/((K5**3.)*L)))*(DEL/(SIN(K3*L1)))
      B=F*(COS(K3*L))
      A=F*(SIN(K3*L))
      C=(A*K3*(SIN(K3*L2)))+(B*K3*(COS(K3*L2)))-(((K4**2.)
     1*DEL)/((K3**2.)*L))
      D=((K2**2.)*DEL*(SIN(K5*L2))/K5)+((K6**2.)*DEL*((COS
     1(K5*L2))-1.)/((K5**2.)*L))
      H=C-D-((DELT*((C*D)+1.))/L1)
 10   FORMAT(3E20.10)
 5    IF(H.LE.0.) GO TO 12
      P1=P
      DEG1=DEG
      H1=H
      GO TO 13
 12   WRITE(6,10) P1,DEG1,H1
      WRITE(6,10) P,DEG,H
 13   IF(H.LE.0.) GO TO 20
      P=10000.0+P
      GO TO 1
 20   STOP
      END
```

Computer Program Number 4.

```
      REAL I1,I2,L,L1,L2,K1,K2,K3,K4,K5,K6,K7,K8,K9,DELP
      P=10000.0
      READ(5,7) DEG
1     E=30000000.
      I1=56.98451
      I2=274.5979
      L=194.
      L1=98.
      L2=96.
      DELP=1.0
7     FORMAT(F4.1)
      THE=3.14159*DEG/180.
      DELT=L1*THE
      DEL=DELP+DELT
      K1=SQRT(P/(E*I1))
      K2=SQRT(P/(E*I2))
      K3=SQRT((0.33608*P)/(E*I1))
      K4=SQRT((0.66392*P)/(E*I1))
      K5=SQRT((0.33608*P)/(E*I2))
      K6=SQRT((0.66392*P)/(E*I2))
      K7=SQRT((1.11385*P)/(E*I2))
      K8=SQRT((K3**2.)+((K4**2.)*DEL/L))
      K9=SQRT((K5**2.)+((K6**2.)*DEL/L)+((K7**2.)*L2*DEL/(
     1L**2.)))
      F=((K2**2.)/(K9**2.))+(((K6**2.)*DEL)/((K9**2.)*L))+
     1(((K7**2.)+(L2**2.)*DEL)/((K9**2.)*(L**3.)))
      B=((F*DEL*((COS(K9*L2))-1.))+((K6**2.)*DEL*(L2-((SIN
     1(K9*L2))/K9))/((K9**2.)*L))+((K4**2.)*DEL*(DEL-(DEL*
     1(COS(K8*L2))/(COS(K8*L)))-L2)/((K8**2.)*L))+(((COS(K
     18*L2))*DEL*(1.-((K3**2.)/(K8**2.))))/(COS(K8*L)))+((
     1(K1**2.)*DEL)/(K8**2.)))*(COS(K8*L))/(SIN(K8*L1))
      A=(DEL-(B*(SIN(K8*L)))-((K3**2.)*DEL/(K8**2.))-(((K4
     1**2.)*(DEL**2.))/((K8**2.)*L)))/(COS(K8*L))
      C=(K8*B*(COS(K8*L2)))-(K8*A*(SIN(K8*L2)))-(((K4**2.)
     1*DEL)/((K8**2.)*L))
      D=(F*K9*DEL*(SIN(K9*L2)))+(((K6**2.)*DEL*((COS(K9*L2
     1))-1.))/((K9**2.)*L))
      H=C-D-((DELT*((C*D)+1.))/L1)
10    FORMAT(3E20.10)
5     IF(H.LE.0.) GO TO 12
      P1=P
      DEG1=DEG
      H1=H
      GO TO 13
12    WRITE(6,10) P1,DEG1,H1
      WRITE(6,10) P,DEG,H
```

```
13    IF(H.LE.0.) GO TO 20
      P=10000.0+P
      GO TO 1
20    STOP
      END
```

Computer Program Number 5.

```
      DOUBLE PRECISION P,E,I1,I2,L1,L2,DA,DB,DDELT,DTHE
      DOUBLE PRECISION DK1,DK2,DD,DG,DDEG,DDEL,DDELP
      P=1000.0
      READ(5,7) DDEG
1     E=30000000.
      I1=0.12866
      I2=4.23072
      L1=27.625
      L2=37.375
7     FORMAT(D4.1)
      DTHE=3.14159*DDEG/180.
      DDELT=L1*DTHE
      DK1=DSQRT(P/(E*I1))
      DK2=DSQRT(P/(E*I2))
      DD=DK2*(DSIN(DK2*L2))
      DG=DK1*(DCOS(DK2*L2))*(DCOS(DK1*L1))/(DSIN(DK1*L1))
      DB=L1*(DG-DD)
      DA=DDELT*DD*DG
      DDEL=(DB-(DSQRT((DB**2.)-(4.*DA*DDELT))))/(2.*DA)
      DDELP=DDEL-DDELT
5     WRITE(6,10) P,DDEG,DDELP
10    FORMAT(3D20.10)
      IF(((DB**2.)-(4.*DA*DDELT)).LE.0.) GO TO 20
      P=1000.0+P
      GO TO 1
20    STOP
      END
```

APPENDIX A

General Solutions of the Second Order Linear Differential Equations with Constant Coefficients. (Equations (2-9) and (2-10))

Referring to equations (2-9) and (2-10):

$$\frac{d^2y_1}{dx_1^2} + k_1^2 y_1 = k_1^2 s \quad , \tag{2-9}$$

$$\frac{d^2y_2}{dx_2^2} + k_2^2 y_2 = k_2^2 s \quad . \tag{2-10}$$

To determine the complementary solution of the differential equation in the form shown above, equation (2-9) can be written as:

$$\ddot{y}_1 + k_1^2 y_1 = 0 \quad , \tag{A-1}$$

assuming the complementary solution of the differential equation (A-1) is:

$$y_1 = A_1 e^{r x_1} \quad , \tag{A-2}$$

then; $$\ddot{y}_1 = A_1 r^2 e^{r x_1} \quad . \tag{A-3}$$

Substituting equations (A-2) and (A-3) into equation (A-1) gives the non-trivial solution:

$$r^2 + k_1^2 = 0 \quad , \tag{A-4}$$

for which the roots of equation (A-4) are:

$$r = \pm k_1 i \quad , \qquad (A-5)$$

where i is the imaginary unit, i.e. $i = \sqrt{-1}$.

The complementary solution of the differential equation (A-1) is:

$$y_{1_{CF}} = A \cos k_1 x_1 + B \sin k_1 x_1 \quad , \qquad (A-6)$$

where A and B are arbitrary constants.

Assuming the particular solution of the differential equation (2-9) is:

$$y_{1_{PI}} = C_1 \delta \quad , \qquad (A-7)$$

then
$$\ddot{y}_{1_{PI}} = 0 \quad . \qquad (A-8)$$

Substituting equations (A-7) and (A-8) into equation (2-9) gives the constant value:

$$C_1 = 1 \quad ,$$

and equation (A-7) becomes:

$$y_{1_{PI}} = \delta \quad . \qquad (A-9)$$

By combining equations (A-6) and (A-9), yields the general solution of the differential equation (2-9):

$$y_1 \quad = \quad A \cos k_1 x_1 + B \sin k_1 x_1 + \delta \quad . \tag{A-10}$$

Since the differential equation (2-10) is the same form as equation (2-9), then the general solution of the differential equation (2-10) can be shown to be:

$$y_2 \quad = \quad C \cos k_2 x_2 + D \sin k_2 x_2 + \delta \quad , \tag{A-11}$$

where C and D are arbitrary constants.

APPENDIX B

General Solutions of the Second Order Differential Equations with Constant Coefficients. (Equations (3-21) and (3-22))

Referring to equations (3-21) and (3-22):

$$\frac{d^2y_1}{dx_1^2} + k_3^2 y_1 = -k_4^2 \cdot \frac{\delta}{L} \cdot x_1 + k_1^2 \cdot \delta \quad , \tag{3-21}$$

$$\frac{d^2y_2}{dx_2^2} + k_5^2 y_2 = -k_6^2 \cdot \frac{\delta}{L} \cdot x_2 + k_2^2 \cdot \delta \quad . \tag{3-22}$$

To determine the complementary solution of the differential equation as the form shown above, equation (3-21) can be written as:

$$\ddot{y}_1 + k_3^2 y_1 = 0 \quad , \tag{B-1}$$

and the complementary solution is obtained: (see Appendix A)

$$y_{1_{CF}} = A \cos k_3 x_1 + B \sin k_3 x_1 \quad , \tag{B-2}$$

where A and B are arbitrary constants..

Assuming the particular solution of the differential equation (3-21) is:

$$y_{1_{PI}} = C_1 x_1 + D_1 \delta \quad , \tag{B-3}$$

then; $$\ddot{y}_{1_{PI}} = 0 \quad . \tag{B-4}$$

Substituting equations (B-3) and (B-4) into equation (3-21) gives the constant values:

$$C_1 = -\frac{k_4^2}{k_3^2}\cdot\frac{\delta}{L} \quad , \quad D_1 = \frac{k_1^2}{k_3^2} \quad , \qquad \text{(B-5)}$$

and equation (B-3) becomes:

$$y_{1_{PI}} = -\frac{k_4^2}{k_3^2}\cdot\frac{\delta}{L}\cdot x_1 + \frac{k_1^2}{k_3^2}\cdot\delta \quad . \qquad \text{(B-6)}$$

By combining equations (B-2) and (B-6), yields the general solution of the differential equation (2-9):

$$y_1 = A\cos k_3 x_1 + B\sin k_3 x_1 - \frac{k_4^2}{k_3^2}\cdot\frac{\delta}{L}\cdot x_1 + \frac{k_1^2}{k_3^2}\cdot\delta \quad . \qquad \text{(B-7)}$$

Since the differential equation (3-22) is the same form as equation (3-21), then the general solution of the differential equation (3-22) can be shown to be:

$$y_2 = E\cos k_5 x_2 + F\sin k_5 x_2 - \frac{k_6^2}{k_5^2}\cdot\frac{\delta}{L}\cdot x_2 + \frac{k_2^2}{k_5^2}\cdot\delta \quad , \qquad \text{(B-8)}$$

where E and F are arbitrary constants.

APPENDIX C

General Solutions of the Second Order Differential Equations with Constant Coefficients. (Equations (3-40) and (3-41))

Referring to equation (3-40):

$$\frac{d^2y_1}{dx_1^2} + \left[\left(k_3^2 + k_4^2 \cdot \frac{\delta}{L}\right) y_1\right] = -\left[k_4^2 \cdot \frac{\delta}{L} \cdot x_1\right] + \left[\left(k_1^2 + k_4^2 \cdot \frac{\delta}{L}\right) \cdot \delta\right] \quad . \qquad (3\text{-}40)$$

To determine the complementary solution of the differential equation as shown above, equation (3-40) can be written as:

$$\ddot{y}_1 + \left(k_3^2 + k_4^2 \cdot \frac{\delta}{L}\right) y_1 = 0 \quad , \qquad (C\text{-}1)$$

and the complementary solution is obtained: (see Appendix A)

$$y_{1_{CF}} = A \cos\left(\sqrt{k_3^2 + k_4^2 \cdot \frac{\delta}{L}}\right) x_1 + B \sin\left(\sqrt{k_3^2 + k_4^2 \cdot \frac{\delta}{L}}\right) x_1 \quad , \qquad (C\text{-}2)$$

where A and B are arbitrary constants.

Assuming the particular solution of the differential equation (3-40) is:

$$y_{1_{PI}} = C_1 x_1 + D_1 \delta \quad , \qquad (C\text{-}3)$$

then;

$$\ddot{y}_{1_{PI}} = 0 \quad . \qquad (C\text{-}4)$$

Substituting equations (C-3) and (C-4) into equation (3-40) gives the constant values:

$$C_1 = -\frac{k_4^2}{k_3^2 + k_4^2 \cdot \frac{\delta}{L}} \cdot \frac{\delta}{L} ,$$

$$D_1 = \frac{k_1^2}{k_3^2 + k_4^2 \cdot \frac{\delta}{L}} + \frac{k_4^2}{k_3^2 + k_4^2 \cdot \frac{\delta}{L}} \cdot \frac{\delta}{L} , \tag{C-5}$$

and equation (C-3) becomes:

$$y_{1_{PI}} = -\frac{k_4^2}{k_3^2 + k_4^2 \cdot \frac{\delta}{L}} \cdot \frac{\delta}{L} \cdot x_1 + \left(\frac{k_1^2}{k_3^2 + k_4^2 \cdot \frac{\delta}{L}} + \frac{k_4^2}{k_3^2 + k_4^2 \cdot \frac{\delta}{L}} \cdot \frac{\delta}{L} \right) \cdot \delta . \tag{C-6}$$

By combining equations (C-2) and (C-6), yields the general solution of the differential equation (3-40):

$$y_1 = A \cos k_8 x_1 + B \sin k_8 x_1 - \frac{k_4^2}{k_8^2} \cdot \frac{\delta}{L} \cdot x_1 + \left(\frac{k_1^2}{k_8^2} + \frac{k_4^2}{k_8^2} \cdot \frac{\delta}{L} \right) \cdot \delta , \tag{C-7}$$

where; $$k_8 = \sqrt{k_3^2 + k_4^2 \cdot \frac{\delta}{L}} . \tag{C-8}$$

Referring to equation (3-41):

$$\frac{d^2 y_2}{dx_2^2} + \left[\left(k_5^2 + k_6^2 \cdot \frac{\delta}{L} + k_7^2 \cdot \frac{L_2 \delta}{L^2} \right) y_2 \right]$$

$$= -\left[k_6^2 \cdot \frac{\delta}{L} \cdot x_2 \right] + \left[\left(k_2^2 + k_6^2 \cdot \frac{\delta}{L} + k_7^2 \cdot \frac{L_2^2 \delta}{L^3} \right) \cdot \delta \right] . \tag{3-41}$$

To determine the complementary solution of the differential equation as shown above, equation (3-41) can be written as:

$$\ddot{y}_2 + \left(k_5^2 + k_6^2 \cdot \frac{\delta}{L} + k_7^2 \cdot \frac{L_2 \delta}{L^2}\right) y_2 = 0 \quad , \qquad \text{(C-9)}$$

and the complementary solution is obtained: (see Appendix A)

$$y_{2_{CF}} = E \cos k_9 x_2 + F \sin k_9 x_2 \quad , \qquad \text{(C-10)}$$

where E and F are arbitrary constants, and

$$k_9 = \sqrt{k_5^2 + k_6^2 \cdot \frac{\delta}{L} + k_7^2 \cdot \frac{L_2 \delta}{L^2}} \quad . \qquad \text{(C-11)}$$

Assuming the particular solution of the differential equation (3-41) is:

$$y_{2_{PI}} = G_1 x_2 + H_1 \delta \quad , \qquad \text{(C-12)}$$

then; $$\ddot{y}_{2_{PI}} = 0 \quad . \qquad \text{(C-13)}$$

Substituting equations (C-12) and (C-13) into equation (3-41) gives the constants values:

$$\left.\begin{aligned} G_1 &= -\frac{k_6^2}{k_9^2} \cdot \frac{\delta}{L} \quad , \\ H_1 &= \frac{k_2^2}{k_9^2} + \frac{k_6^2}{k_9^2} \cdot \frac{\delta}{L} + \frac{k_7^2}{k_9^2} \cdot \frac{L_2^2 \delta}{L^3} \quad , \end{aligned}\right] \qquad \text{(C-14)}$$

and equation (C-12) becomes:

$$y_{2_{PI}} = -\frac{k_6^2}{k_9^2} \cdot \frac{\delta}{L} \cdot x_2 + \left(\frac{k_2^2}{k_9^2} + \frac{k_6^2}{k_9^2} \cdot \frac{\delta}{L} + \frac{k_7^2}{k_9^2} \cdot \frac{L_2^2 \delta}{L^3}\right) \cdot \delta \quad . \qquad \text{(C-15)}$$

By combining equations (C-10) and (C-15), yields the general solution of the differential equation (3-41):

$$y_2 = E\cos k_9 x_2 + F\sin k_9 x_2 - \frac{k_6^2}{k_9^2}\cdot\frac{\delta}{L}\cdot x_2$$

$$+\left(\frac{k_2^2}{k_9^2} + \frac{k_6^2}{k_9^2}\cdot\frac{\delta}{L} + \frac{k_7^2}{k_9^2}\cdot\frac{L_2^2\delta}{L^3}\right)\cdot\delta \qquad . \qquad \text{(C-16)}$$

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