ABSTRACT

DYNAMIC ANALYSIS OF THE AXIALLY-LOADED TIMOSHENKO BEAM USING QUADRATIC MATRIX EQUATIONS

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The purpose of this thesis is to obtain a power series expansion of the general stiffness matrix for a beam-column element. Solutions are obtained using the general Timoshenko beam theory including the combined effects of bending and shear stress, together with axial force and transverse and rotary inertia.

The general stiffness matrix of this continuous, elastic, vibrating, structural element possesses components which contain complex hyperbolic and trigonometric functions. The idealization of this continuous system into a discrete element system is performed utilizing a power series expansion technique which produces a family of matrices containing algebraic components. The latter matrix form is extremely efficient for matrix computer operations.

The matrix series expansion produces a two term expansion of elastic stiffness matrix, a two term expansion of the consistent mass matrix, and a two term expansion of the geometric stiffness matrix. In addition, all coupling matrices of the latter three parameters are obtained.

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LIST OF NOTATIONS

SYMBOL	DEFINITION
A	Area of cross-section
a	Nodal displacements function
$\left\{a^{y'}_{rs}\right\}$	The first differential of displacement function in Y-direction
$\left\{a''_{ry}\right\}$	The second differential of displacement function in Y-direction
$\left\{a^{IV}_{ry}\right\}$	The fourth differential of displacement function in Y-direction
B	Body forces
b	Strain displacements function
c	The first derivatives of functions in $\left[a\right]$
E	Modulus of elasticity
E_t	Elastic constants
e	Total strains
G	$= \frac{E}{2(1+\nu)}$, Lame' constant
I	$= Ar^2$, Cross-section moment of inertia
k	Shear factor of constants
$\left[K\right]$	Stiffness matrix
$\left[K^b\right]$	Bending stiffness matrix
$\left[K_G\right]$	Geometric stiffness matrix
$\left[K_i\right]$	Inertia stiffness matrix
L	Length of beam and beam-column element
$\left[M\right]$	Mass matrix
P	Axial force

SYMBOL	DEFINITION
$\{q\}$	Column matrix of amplitudes of the displacement
$[S_{BE}]^{C}$	General stiffness matrix of beam for Bernoulli-Euler theory
$[S_{BE}]^{BC}$	General stiffness matrix of beam-column for Bernoulli-Euler theory
$[S_{T}]^{B}$	General stiffness matrix of beam for Timoshenko theory
$[S_{T}]^{BC}$	General stiffness matrix of beam-column for Timoshenko theory
t	Time variable
u	Total displacements
U	Nodal displacements
V_i	Shear forces at the ends of beam and beam-column element
X, Y	Co-ordinates along the axes of a deflected beam and beam-column
ω	Natural frequency of beam
Ω	Natural frequency of beam-column
α	Coefficient of thermal expansion
β	Shearing rotation
ψ	Bending rotation
θ_i	Angular deflection at the ends of beam and beam-column
ρ	Density

SYMBOL	DEFINITION
q_s	Surface forces
Σ	Summation of the mathematical terms that follow

LIST OF FIGURES

CHAPTER I

INTRODUCTION

1.1 Historical Review

The mathematical idealizing of vibrating continuous elastic systems into discrete element systems, as used in matrix computer analysis, leads to the equations of motion which define a general stiffness matrix with components that are in the form of an infinite matrix series. The classical solutions of the equations of motion of a beam-column form a structural stiffness matrix with coefficients defined in terms of hyperbolic and trigonometric functions. This form of the stiffness matrix is inefficient in the usual computer operations which most efficiently and effectively utilize matrix forms which are algebraic in form.

As a result, a number of authors have develope techniques to circumvent this problem. The stiffness matrix has been separated into a number of parts: the bending stiffness matrix of a beam defining the effects of bending and shear stresses, the geometric stiffness matrix expressing the effect of axial force, and a consistent mass matrix defining the effect of inertial forces. All three of the above matrices contain algebraic (numerical) components which allows for efficient computer operations.

Przemieniecki(5)* derived first and second order matrix forms for the mass and stiffness matrices of a beam, utilizing a single infinite power series expansion in ascending powers of frequency.

Paz(4) has expanded the terms of the exact general stiffness matrix for beam-column element derived from the Bernoulli-Euler equations into a power series expansion. By a process of longhand algebraic division, he obtained the bending stiffness matrix, the first and second order terms of the geometric stiffness matrix and the first and second order terms of the mass matrices including transverse inertia only.

Laohasiripunya(3) utilizing a similar power series expansion and long-hand division extended the latter work to include the terms of geometric stiffness and the mass matrices beyond the second order. Terms of the power series are retained up to and including the order of fourth power in natural frequency, the fourth power in axial force, and similar intermediate order coupling terms between the two.

1.2 Thesis Purpose

The purpose of this thesis is to derive the required bending stiffness, mass, and geometric stiffness matrices of the Timoshenko beam using a "double infinite" power series expansion in simultaneous ascending powers of frequency and

* Number in parenthesis refers to literature cited in the Bibliography.

axial force. The following systematic steps are taken to assure and establish proper results:

1.) The problem of a vibrating beam-element derived from the Bernoulli-Euler theory is considered using the Przemieniecki method. The bending stiffness matrix, the first and second order terms of the mass matrices are obtained. A consistent sign convention is introduced which imposes unique properties to the resulting matrices.

2.) The problem of a vibrating beam-column element derived from the Bernoulli-Euler theory is investigated using a novel double infinite power series expansion. The stiffness, mass, and geometric matrices are obtained through the order of fourth power in natural frequency, the fourth power in axial force, and intermediate orders of coupling between the two.

3.) The problem of the vibrating beam-element derived from the Timoshenko beam theory is investigated using the Przemieniecki method. The first and second order terms of the stiffness and the mass matrices are obtained with the added consideration of the effect of shear stress which is not accounted for 1.) above.

4.) The problem of the vibrating beam-column element derived from the Timoshenko theory is solved using the formulated double-infinite power series expansion. The required stiffness, mass, and geometric matrices for the first and second order terms are derived.

1.3 General Theory

The essential feature of the matrix methods of structural-analysis is that a continuous elastic system may be represented by an equivalent discrete element system having a finite number of degrees of freedom. In the discrete system the displacements are specified at points selected arbitrarily on the actual atructure. These displacements are then used to determine the equivalent elastic properties of the discrete element model representing the continuous system (See Figure 1). For static problems the determination of the equivalent elastic properties presents no special difficulty. The displacements $u_i = u_i(x,y,z)$ $i=x,y,z$ at any point P in the continuous system may be related to a finite number of displacements selected on the structure, in the matrix form:

$$\left\{ u \right\} = \left[a \right] \cdot \left\{ U \right\} \tag{1-1}$$

where

$$\left\{ u \right\} = \left\{ u_x \; u_y \; u_z \right\} \tag{1-2}$$

represents displactments in the directions of x, y, and z axes at point P

$$\left\{ U \right\} = \left\{ U_1 \; U_2 \; \ldots \; U_n \right\} \tag{1-3}$$

represents a column matrix of the N displacements and $\left[a \right]$ specified at a specified number of points is a rectangular matrix whose coefficients are functions of x, y, z. Equation (1-1) can be used to obtain the total strain-displacement relationship

$$\{e\} = [b]\cdot\{U\} \qquad (1\text{-}4)$$

where the coefficients in $[b]$ are determined by proper differentiation of the matrix $[a]$

Using the Principle of Virtual Work and d'Alembert's principle and following the procedures of Przemieniecki, one obtains

$$[M]\{\ddot{U}\} + [K]\{U\} = \{P\} - \int_V \{b\}^T\{E_t\}\alpha t\,dV + \int_S \{a\}^T\{\sigma_s\}\,dS + \int_V \{a\}^T\{B\}\,dV \qquad (1\text{-}5)$$

where $$[M] = \int_V \rho\,[a]^T[a]\,dV \qquad (1\text{-}6)$$

represents the mass matrix of the equivalent discrete system

and $$[K] = \int_V [b]^T[E][b]\,dV \qquad (1\text{-}7)$$

is the stiffness matrix for the displacements $\{U\}$. Equation (1-5) represents matrix equation of motion of the equivalent discrete system. The first term on the right hand side of Equation(1-5) is the column matrix of external forces in the directions of $\{U\}$; the second terms represents equivalent concentrated forces due to some specified temperature distribution; the third and fourth terms represent equivalent concentrated force due to surface forces and body forces respectively. Thus, Equation(1-5) serves not only to determine the discrete system inertia and stiffness properties but also to convert distributed loading into a set of discrete loads.

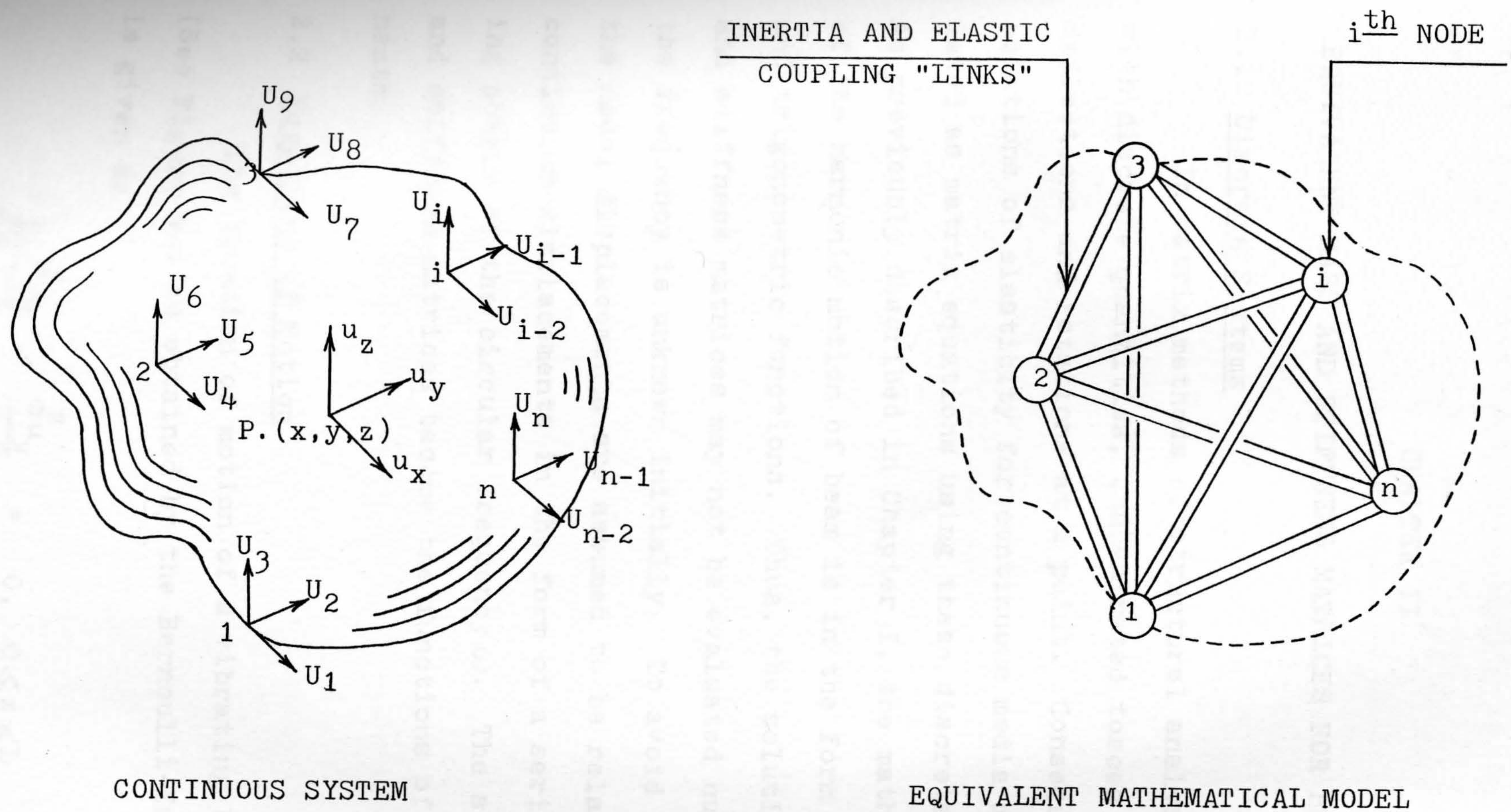

Figure 1. Mathematical Model Used in Matrix Methods of Structural Analysis

CHAPTER II

EQUIVALENT MASS AND STIFFNESS MATRICES FOR BEAM ELEMENTS

2.1 Discrete Systems

In matrix methods of structural analysis one deals with discrete quantities, concentrated forces and moments, deflections and rotations at a point. Consequently, all equations of elasticity for continuous media must be reformulated as matrix equations using these discrete quantities. As previously described in Chapter I, the mathematical form of the harmonic motion of beam is in the form of hyperbolic and trigonometric functions. Thus, the solution for mass and stiffness matrices may not be evaluated numerically since the frequency is unknown initially. To avoid this difficulty the nodal displacements are assumed to be related to the continuous displacements in the form of a series in ascending powers of the circular frequency ω. The solution of mass and stiffness matrices become the functions of nodal displacements.

2.2 Equations of Motion

The equation of motion of a vibrating beam element (See Figure 2a) as obtained by the Bernoulli-Euler theory is given as

$$\frac{EI}{\rho A}\frac{\partial^4 u_y}{\partial x^4} + \frac{\partial^2 u_y}{\partial t^2} = 0, \quad 0 \leqslant x \leqslant L$$

$$t > 0 \qquad (2\text{-}1)$$

with X & Y - the co-ordinates axes of the beam

EI - the flexural stiffness

ρA - the mass per unit length

t - the time variable

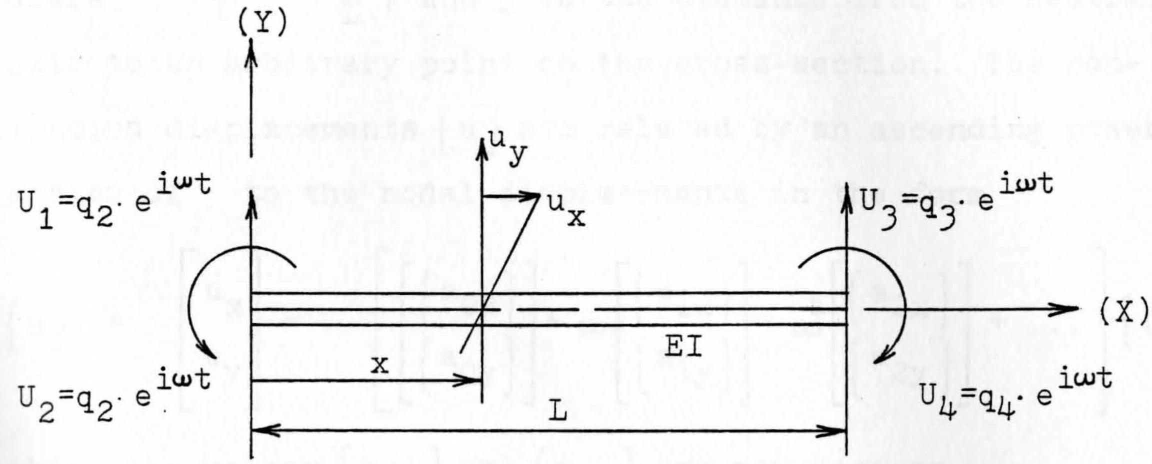

Figure 2a. Sign-Convention for the Beam Element

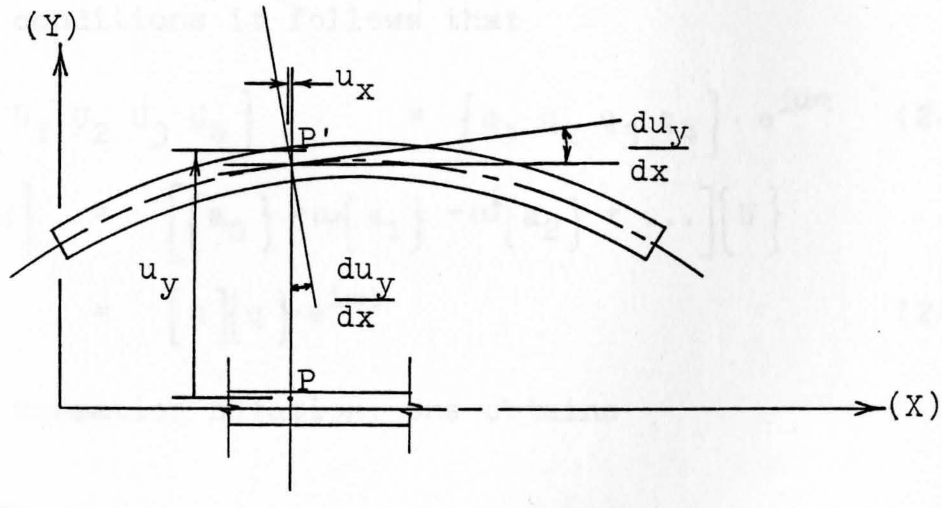

Figure 2b. Deformed Beam Element

Let u_x - the displacement along the X-axes

and u_y - the displacement along the Y-axes

From the Engineering bending theory the relation between u_x and u_y (See Figure 2b) becomes

$$u_x = -\frac{\partial u_y}{\partial x} y = -L\frac{\partial u_y}{\partial x}\eta \qquad (2\text{-}2)$$

where $\eta = \frac{y}{L}$, and y is the distance from the neutral axis to an arbitrary point on the cross-section. The continuous displacements $\{u\}$ are related by an ascending powers series of to the nodal displacements in the form

$$\{u\} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \left[\begin{bmatrix} \{a_{0x}\} \\ \{a_{0y}\} \end{bmatrix} + \omega \begin{bmatrix} \{a_{1x}\} \\ \{a_{1y}\} \end{bmatrix} + \omega^2 \begin{bmatrix} \{a_{2x}\} \\ \{a_{2y}\} \end{bmatrix} + \ldots\right] \cdot \{U\}$$

where the vector $\{a_{ix}\}$ and $\{a_{iy}\}$ are row vectors.
Assuming harmonic variations in the time parameters for free vibration conditions it follows that

$$\{U\} = \{U_1\ U_2\ U_3\ U_4\} = \{q_1\ q_2\ q_3\ q_4\} \cdot e^{i\omega t} \qquad (2\text{-}3)$$

thus $\{u\} = \left[\{a_0\} + \omega\{a_1\} + \omega^2\{a_2\} + \ldots\right]\{U\}$

$$= [a]\{q\} \cdot e^{i\omega t} \qquad (2\text{-}4)$$

Utilizing summation notation, one obtains

$$u_x = \left(\sum_{r=0}^{\infty} \omega^r \{a_{rx}\}\right)\{q\}\ e^{i\omega t} = \{a_x\}\{q\} \cdot e^{i\omega t} \qquad (2\text{-}5a)$$

and

$$u_y = \left(\sum_{r=0}^{\infty} \omega^r \{a_{ry}\}\right)\{q\}\ e^{i\omega t} = \{a_y\}\{q\} \cdot e^{i\omega t} \qquad (2\text{-}5b)$$

Substituting Equation(2-5) into Equation of motion(2-1) gives

$$\frac{EI}{\rho A}\left(\sum_{r=0}^{\infty}\omega^{r}\left\{a_{ry}^{IV}\right\}\right)\left\{q\right\}e^{i\omega t} - \omega^{2}\left(\sum_{r=0}^{\infty}\omega^{r}\left\{a_{ry}\right\}\right)\left\{q\right\}e^{i\omega t} = 0 \qquad (2\text{-}6)$$

Equating to zero coefficients of the same powers of ω in Equation (2-6), the following equations are obtained :

$$\left\{a_{0y}^{IV}\right\} = \left\{0\right\} \qquad (2\text{-}7a)$$

$$\left\{a_{1y}^{IV}\right\} = \left\{0\right\} \qquad (2\text{-}7b)$$

$$\frac{EI}{\rho A}\left\{a_{2y}^{IV}\right\} = \left\{a_{0y}\right\} \qquad (2\text{-}7c)$$

$$\frac{EI}{\rho A}\left\{a_{3y}^{IV}\right\} = \left\{a_{1y}\right\} \ldots \text{ etc} \qquad (2\text{-}7d)$$

2.3 <u>Solution of the Differential Equation</u>

Equations (2-7a) through Equation (2-7d) are solved by direct integration. For Equation (2-7a), the following boundary conditions are applied :

$$\text{at } x = 0 \qquad u_y = U_1 \qquad U_2 = \frac{dU_1}{dx}$$

$$\text{at } x = L \qquad u_y = U_3 \qquad U_4 = -\frac{dU_3}{dx}$$

For the remaining equations containing terms $\left\{a_{1y}\right\}$, $\left\{a_{2y}\right\}$, $\left\{a_{3y}\right\}$ ... the boundary conditions on displacement and slope must all vanish at $x = 0$ and L . Thus it follows that,

$$\left\{a_{0y}\right\} = \left[(1-3\xi^{2}+2\xi^{3}) \quad (\xi-2\xi^{2}+\xi^{3})L \quad (\xi^{2}-\xi^{3})L \quad (3\xi^{2}-2\xi^{3})\right] \qquad (2\text{-}8a)$$

$$\left\{a_{1y}\right\} = \left\{0\right\} \qquad (2\text{-}8b)$$

$$\left\{a_{2y}\right\} = \frac{\rho AL^{4}}{2520EI}\Big[3(22\xi^{2}-52\xi^{3}+35\xi^{4}-7\xi^{6}+2\xi^{7}) \quad (12\xi^{2}-22\xi^{3}+21\xi^{5}-14\xi^{6}+3\xi^{7})L$$

$$(9\xi^{2}-13\xi^{3}+7\xi^{6}-3\xi^{7})L \quad 3(13\xi^{2}-18\xi^{3}+7\xi^{6}-2\xi^{7})\Big] \qquad (2\text{-}8c)$$

$$\left\{a_{3y}\right\} = \left\{0\right\} \tag{2-8d}$$

$$\left\{a_{4y}\right\} = \frac{(\rho A)^2 L^8}{139708800(EI)^2}\Big[3(1784\xi^2-2832\xi^3+3388\xi^6-3432\xi^7+1155\xi^8-77\xi^{10}+14\xi^{11})$$

$$(1136\xi^2-1784\xi^3+1848\xi^6-1452\xi^7+385\xi^9-154\xi^{10}+21\xi^{11})L$$

$$(1097\xi^2-1681\xi^3+1386\xi^6-858\xi^7+77\xi^{10}-21\xi^{11})L$$

$$3(1681\xi^2-2558\xi^3+2002\xi^6-1188\xi^7+77\xi^{10}-14\xi^{11})\Big] \tag{2-8e}$$

$$\left\{a_{5y}\right\} = \left\{0\right\} \ldots \text{ etc} \tag{2-8f}$$

where $\xi = \frac{x}{L}$

The matrix $\left\{a_{0y}\right\}$ represents static transverse deflection distribution due to unit values of displacement U_1, U_2, U_3, U_4 applied independently. The remaining matrices in $\left\{a\right\}$ are determined from Equation (2-2) in the form

$$\left\{a_{0x}\right\} = \eta\Big[6(\xi-\xi^2) \quad (-1+4\xi-3\xi^2)L \quad (-2\xi+3\xi^2)L \quad 6(-\xi+\xi^2)\Big] \tag{2-9a}$$

$$\left\{a_{1x}\right\} = \left\{0\right\} \tag{2-9b}$$

$$\left\{a_{2x}\right\} = \frac{\eta A L^4}{840EI}\Big[2(-22\xi+78\xi^2-70\xi^3+21\xi^5-7\xi^6) \quad (-8\xi+22\xi^2-35\xi^4+28\xi^5-7\xi^6)L$$

$$(-6\xi+13\xi^2-14\xi^5+7\xi^6)L \quad 2(-13\xi+27\xi^2-21\xi^5+7\xi^6)\Big] \tag{2-9c}$$

$$\left\{a_{3x}\right\} = \left\{0\right\} \tag{2-9d}$$

$$\left\{a_{4x}\right\} = \frac{\eta(\rho A)^2 L^8}{139708800(EI)^2}\Big[6(1784\xi-4248\xi^2+10164\xi^5-12012\xi^6+4620\xi^7-385\xi^9+77\xi^{10})$$

$$(2272\xi-5352\xi^2+11088\xi^5-10164\xi^6+3465\xi^8-1540\xi^9+231\xi^{10})L$$

$$(2194\xi-5043\xi^2+8316\xi^5-6006\xi^6+770\xi^9-231\xi^{10})L$$

$$6(1681\xi-3837\xi^2+6006\xi^5-4158\xi^6+385\xi^9-77\xi^{10})\Big] \tag{2-9e}$$

$$\left\{a_{5x}\right\} = \left\{0\right\} \dots \text{ etc} \tag{2-9f}$$

2.4 Total Strains-Displacement Relationship

The strains in the beam element are derived from

$$\left\{e\right\} = \frac{\partial u_x}{\partial x} = -\frac{\partial^2 u_y}{\partial x^2}\, y = -L\frac{\partial^2 u_y}{\partial x^2}$$

$$= \left[\left\{b_0\right\} + \omega\left\{b_1\right\} + \omega^2\left\{b_2\right\} + \dots\right]\left\{U\right\}$$

$$= \left[b\right]\left\{U\right\} \tag{2-10}$$

Hence, using Equations (2-8a) through (2-8f) and Equation (2-10), it follows that

$$\left\{b_0\right\} = -L\eta\left\{a''_{0y}\right\}$$

$$= \frac{\eta}{L}\left[(6-12\xi) \quad (4-6\xi)L \quad (-2+6\xi)L \quad (-6+12\xi)\right] \tag{2-11a}$$

$$\left\{b_1\right\} = -L\eta\left\{a''_{1y}\right\} = \left\{0\right\} \tag{2-11b}$$

$$\left\{b_2\right\} = -L\eta\left\{a''_{2y}\right\}$$

$$= \frac{\eta\rho AL^3}{420EI}\Big[(-22+156\xi-210\xi^2+105\xi^4-42\xi^5) \quad (-4+22\xi-70\xi^3+70\xi^4-21\xi^5)L$$

$$(-3+13\xi-35\xi^4+21\xi^5)L \quad (-13+54\xi-105\xi^4+42\xi^5)\Big] \tag{2-11c}$$

$$\left\{b_3\right\} = -L\eta\left\{a''_{3y}\right\} = \left\{0\right\} \tag{2-11d}$$

$$\left\{b_4\right\} = -L\eta\left\{a''_{4y}\right\}$$

$$= \frac{\eta(\rho A)^2L^7}{69854400(EI)^2}\Big[3(1784 -8496\xi+50820\xi^4-72072\xi^5+32340\xi^6 -3465\xi^8+770\xi^9)$$

$$(1136 -5352\xi+27720\xi^4-30492\xi^5+13860\xi^7-6930\xi^8+1155\xi^9)L$$

$$(1097-5043\xi+20790\xi^4-18018\xi^5+3465\xi^8-1155\xi^9)L$$

$$3(1681-7674\xi+30030\xi^4-24948\xi^5+3465\xi^8-770\xi^9)\Big] \qquad (2\text{-}11e)$$

$$\left\{b_5\right\} = -L\eta\left\{a''_{5y}\right\} = \left\{0\right\} \ldots \text{ etc} \qquad (2\text{-}11f)$$

2.5 <u>Mass Matrix</u>

From Equation (1-1), the matrix $\left[a\right]$ is expanded into an infinite series in ascending powers of ω, thus

$$\left[a\right] = \sum_{r=0}^{\infty} \omega^r \left\{a_r\right\} \qquad (2\text{-}12)$$

Substituting the Equation (2-8a) through Equation (2-9f) into Equation (2-12), the final form of the matrix $\left[a\right]$ is

$$\left[a\right] = \left\{a_0\right\} + \omega^2\left\{a_2\right\} + \omega^4\left\{a_4\right\} + \ldots \qquad (2\text{-}13)$$

so $$\left[a\right]^T\left[a\right] = \left[\left\{a_0\right\}^T\left\{a_0\right\}\right] + \omega^2\left[\left\{a_0\right\}^T\left\{a_2\right\} + \left\{a_2\right\}^T\left\{a_0\right\}\right]$$

$$+ \omega^4\left[\left\{a_0\right\}^T\left\{a_4\right\} + \left\{a_2\right\}^T\left\{a_2\right\} + \left\{a_4\right\}^T\left\{a_0\right\}\right] + \ldots \qquad (2\text{-}14)$$

Substituting the Equation (2-14) into Equation (1-6), the equivalent mass matrices are in the form

$$\left[M\right] = \left[M_0\right] + \omega^2\left[M_2\right] + \omega^4\left[M_4\right] + \ldots \qquad (2\text{-}15)$$

The first two component matrices appearing in the expansion of Equation (2-15) are calculated as

$$\left[M_0\right] = \frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 13L & 54 \\ 22L & 4L^2 & 3L^2 & 13L \\ 13L & 3L^2 & 4L^2 & 22L \\ 54 & 13L & 22L & 156 \end{bmatrix} + \frac{\rho I}{30L}\begin{bmatrix} 36 & 3L & -3L & -36 \\ 3L & 4L^2 & L^2 & -3L \\ -3L & L^2 & 4L^2 & 3L \\ -36 & -3L & 3L & 36 \end{bmatrix}$$

(2-16)

$$\left[M_2\right] = \frac{(\rho AL)^2 L^3}{2520EI}\begin{bmatrix} \frac{708}{385} & \frac{446}{1155}L & \frac{1681}{4620}L & \frac{1279}{770} \\ \frac{446}{1155}L & \frac{284}{3465}L^2 & \frac{1097}{13860}L^2 & \frac{1681}{4620}L \\ \frac{1681}{4620}L & \frac{1097}{13860}L^2 & \frac{284}{3465}L^2 & \frac{446}{1155}L \\ \frac{1279}{770} & \frac{1681}{4620}L & \frac{446}{1155}L & \frac{708}{385} \end{bmatrix}$$

$$+ \frac{(\rho A)(\rho I)L^3}{2520EI}\begin{bmatrix} \frac{4}{5} & 2L & \frac{3}{2}L & -\frac{4}{5} \\ 2L & \frac{4}{5}L^2 & \frac{7}{10}L^2 & \frac{3}{2}L \\ \frac{3}{2}L & \frac{7}{10}L^2 & \frac{4}{5}L^2 & 2L \\ -\frac{4}{5} & \frac{3}{2}L & 2L & \frac{4}{5} \end{bmatrix} \quad (2\text{-}17)$$

2.6 Stiffness Matrix

From Equation (1-4), the total strain-displacement relationship is in the form of matrix $\left[b\right]$. Similarly, it may be expanded into an infinite series in ascending powers of ω.

$$\left[b\right] = \sum_{r=0}^{\infty} \omega^r \left\{b_r\right\} \quad (2\text{-}18)$$

Expanding Equation (2-18), one obtains

$$\left[b\right] = \left\{b_0\right\} + \omega\left\{b_1\right\} + \omega^2\left\{b_2\right\} + \omega^3\left\{b_3\right\} + \omega^4\left\{b_4\right\} + \ldots \quad (2\text{-}19)$$

Substituting the Equations (2-11a) through Equation (2-11f) into Equation (2-19) and forming the multiplication of $\begin{bmatrix} b \end{bmatrix}^T\begin{bmatrix} b \end{bmatrix}$, it follows that

$$\begin{bmatrix} b \end{bmatrix}^T\begin{bmatrix} b \end{bmatrix} = \left[\{b_0\}^T\{b_0\}\right] + \omega^2\left[\{b_0\}^T\{b_2\} + \{b_2\}^T\{b_0\}\right] + \omega^4\left[\{b_0\}^T\{b_4\} + \{b_2\}^T\{b_2\} + \{b_4\}^T\{b_0\}\right] + \ldots \quad (2\text{-}20)$$

Substituting the Equation (2-20) into Equation (1-7), the equivalent stiffness matrices take the form

$$\begin{bmatrix} K \end{bmatrix} = \begin{bmatrix} K_0 \end{bmatrix} + \omega^2\begin{bmatrix} K_2 \end{bmatrix} + \omega^4\begin{bmatrix} K_4 \end{bmatrix} + \ldots \quad (2\text{-}21)$$

The components of the stiffness matrix in Equation (2-21) are calculated for the first three terms as

$$\begin{bmatrix} K_0 \end{bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -6L & -12 \\ 6L & 4L^2 & -2L^2 & -6L \\ -6L & -2L^2 & 4L^2 & 6L \\ -12 & -6L & 6L & 12 \end{bmatrix} \quad (2\text{-}22)$$

$$\begin{bmatrix} K_2 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \quad (2\text{-}23)$$

$$\begin{bmatrix} K_4 \end{bmatrix} = \frac{(\rho AL)^2 L^3}{2520EI}\begin{bmatrix} \frac{354}{385} & \frac{223}{1155}L & \frac{1681}{9240}L & \frac{1279}{1540} \\ \frac{223}{1155}L & \frac{142}{3465}L^2 & \frac{1097}{27720}L^2 & \frac{1681}{9240}L \\ \frac{1681}{9240}L & \frac{1097}{27720}L^2 & \frac{142}{3465}L^2 & \frac{223}{1155}L \\ \frac{1279}{1540} & \frac{1681}{9240}L & \frac{223}{1155}L & \frac{354}{385} \end{bmatrix} \quad (2\text{-}24)$$

The matrices $\left[M_0\right]$ and $\left[K_0\right]$ represent the mass inertia and stiffness of the beam element based on static displacement functions. The matrices $\left[M_2\right]$ and $\left[K_4\right]$ and higher order terms represent dynamic corrections. Recall the final dynamic equilibrium equation (1-5). We shall consider a completely unconstrained (free) structure undergoing free oscillations. Equation (1-5) becomes

$$\left[M\right]\left\{\ddot{U}\right\} + \left[K\right]\left\{U\right\} = 0 \qquad (2\text{-}25)$$

Substituting Equation (2-3) into Equation (2-25), one obtains

$$\left[-\omega^2\left[M\right] + \left[K\right]\right]\left\{q\right\} = \left\{0\right\} \qquad (2\text{-}26)$$

Substituting the frequency-dependent mass and stiffness matrices from Equations (2-15) and (2-21) into Equation (2-26) yields the equation of motion in the form of a matrix series in ascending powers of as

$$\left[\left[K_0\right] - \omega^2\left[M_0\right] - \omega^4\left[\left[M_2\right] - \left[K_4\right]\right] + \cdots\right]\left\{q\right\} = 0 \qquad (2\text{-}27)$$

For reasons of comparison of results in future chapters, the following definition of general stiffness matrix is introduced :

$$\left[S_{BE}\right]^B\left\{q\right\} \equiv \left[\left[K\right] - \omega^2\left[M\right]\right]\left\{q\right\} = \left\{0\right\}$$

where $\left[S_{BE}\right]^B$ is defined as the general stiffness matrix, and is given by Equation (2-27).

CHAPTER III

FORMULATION OF MASS AND STIFFNESS MATRICES FOR BEAM-COLUMN

3.1 Mathematic Analysis Procedures

Recalling the results of Chapter II, the differential equation of motion of a beam element is solved by using a single infinite power series expansion in ascending powers of frequency. The solution for the mass and stiffness matrices are in the form of power series expansion of algebraic terms instead of complex trigonometric and hyperbolic functions which are inefficient in the usual computer operations currently in use. In this chapter, we analyse in the case of beam-column element. The differential equation of motion is augmented due to the presence of the axial load. The new technique is utilized to solve the equation of motion the basis of which is formulated using a double infinite power series expansion in ascending powers of frequency and axial load. Similarly, the mass, stiffness, and geometric matrices are expanded in the form of power series of frequency and axial load. The details are described in the next paragraph.

3.2 Equations of Motion of Beam-Column

The dynamic stiffness matrix of a general beam-column is formulated by using the Bernoulli-Euler theory excluding the effects of rotatory inertia and shear deformation. The differential equation of motion for a uniform beam-column(6) is given as

$$EI\frac{\partial^4 u_y}{\partial x^4} + P\frac{\partial^2 u_y}{\partial x^2} + \rho A\frac{\partial^2 u_y}{\partial t^2} = 0 \qquad (3\text{-}1)$$

with X & Y - the co-ordinates axes of the beam-column as shown in Figure 3

P - the axial force (compression)

EI - the flexural stiffness

ρA - the mass per unit length

t - the time variable

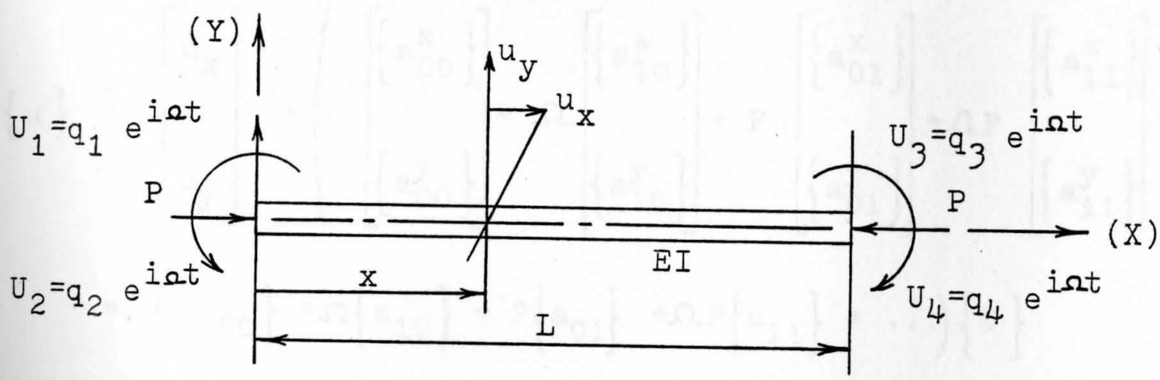

Figure 3a. Sign-Convention for Beam-Column Element

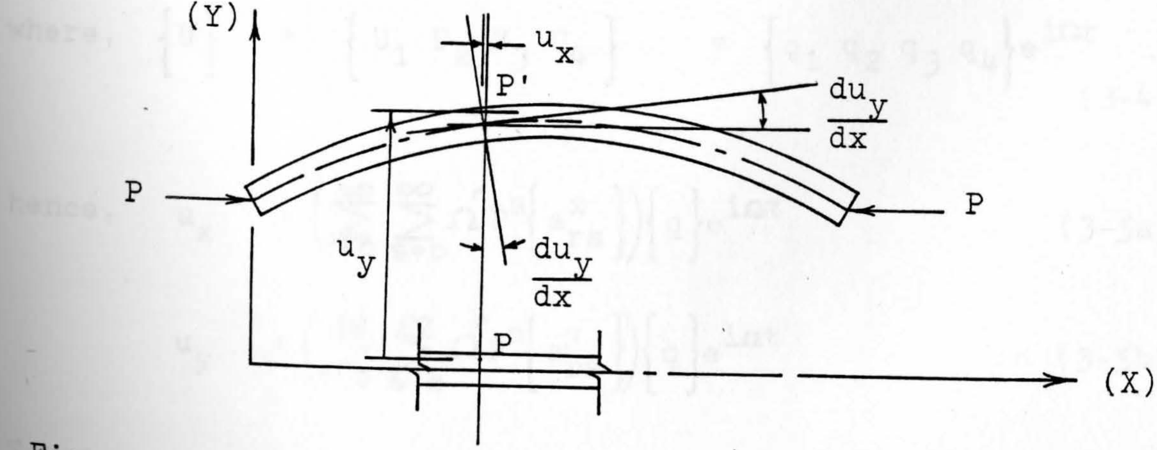

Figure 3b. Deformed Geometry for Beam-Column Element

Letting u_x - displacement along the X-axes

and u_y - displacement along the Y-axes

the usual Engineering bending theory requires that

$$u_x = -\frac{\partial u_y}{\partial x} y = -L\frac{\partial u_y}{\partial x}\eta \qquad (3\text{-}2)$$

where $\eta = \frac{y}{L}$

As described in Paragraph 3.1, it is assumed that the displacements $\{u\}$ are expanded in ascending powers of Ω and P so that

$$\{u\} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \left(\begin{bmatrix} \{a^x_{00}\} \\ \{a^y_{00}\} \end{bmatrix} + \Omega \begin{bmatrix} \{a^x_{10}\} \\ \{a^y_{10}\} \end{bmatrix} + P \begin{bmatrix} \{a^x_{01}\} \\ \{a^y_{01}\} \end{bmatrix} + \Omega P \begin{bmatrix} \{a^x_{11}\} \\ \{a^y_{11}\} \end{bmatrix} + \dots \right) \{U\}$$

$$= \left(\{a_{00}\} + \Omega\{a_{10}\} + P\{a_{01}\} + \Omega P\{a_{11}\} + \dots \right)\{U\}$$

$$= \{a\}\{q\}e^{i\Omega t} \qquad (3\text{-}3)$$

where, $\{U\} = \{U_1\ U_2\ U_3\ U_4\} = \{q_1\ q_2\ q_3\ q_4\}e^{i\Omega t}$ (3-4)

hence, $$u_x = \left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\Omega^r P^s\{a^x_{rs}\}\right)\{q\}e^{i\Omega t} \qquad (3\text{-}5a)$$

$$u_y = \left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\Omega^r P^s\{a^y_{rs}\}\right)\{q\}e^{i\Omega t} \qquad (3\text{-}5b)$$

Subtituting Equation (3-5) into Equation of Motion (3-1), one obtains

$$EI\left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\Omega^{r}P^{s}\left\{a_{rs}^{y}\right\}\right)\left\{q\right\}e^{i\Omega t}+\left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\Omega^{r}P^{(s+1)}\left\{a_{rs}^{y}\right\}\right)\left\{q\right\}e^{i\Omega t}$$

$$-\rho A\left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\Omega^{(r+2)}P^{s}\left\{a_{rs}^{y}\right\}\right)\left\{q\right\}e^{i\Omega t} = 0 \qquad (3\text{-}6)$$

Equating to zero coefficients of the same powers of Ω and P in Equation (3-6), the following equations are obtained :

$$EI\left\{\overset{IV}{a_{00}^{y}}\right\} = \left\{0\right\} \qquad (3\text{-}7a)$$

$$EI\left\{\overset{IV}{a_{01}^{y}}\right\} = -\left\{\overset{II}{a_{00}^{y}}\right\} \qquad (3\text{-}7b)$$

$$EI\left\{\overset{IV}{a_{10}^{y}}\right\} = \left\{0\right\} \qquad (3\text{-}7c)$$

$$EI\left\{\overset{IV}{a_{11}^{y}}\right\} = -\left\{\overset{II}{a_{10}^{y}}\right\} \qquad (3\text{-}7d)$$

$$EI\left\{\overset{IV}{a_{20}^{y}}\right\} = \rho A\left\{a_{00}^{y}\right\} \qquad (3\text{-}7e)$$

$$EI\left\{\overset{IV}{a_{02}^{y}}\right\} = -\left\{\overset{II}{a_{01}^{y}}\right\} \qquad (3\text{-}7f)$$

$$EI\left\{\overset{IV}{a_{21}^{y}}\right\} + \left\{\overset{II}{a_{20}^{y}}\right\} = \rho A\left\{a_{01}^{y}\right\} \qquad (3\text{-}7g)$$

$$EI\left\{\overset{IV}{a_{12}^{y}}\right\} = -\left\{\overset{II}{a_{11}^{y}}\right\} \qquad (3\text{-}7h)$$

$$EI\left\{\overset{IV}{a_{03}^{y}}\right\} = -\left\{\overset{II}{a_{02}^{y}}\right\} \qquad (3\text{-}7i)$$

$$EI\left\{\overset{IV}{a_{30}^{y}}\right\} = \rho A\left\{a_{10}^{y}\right\} \qquad (3\text{-}7j)$$

$$EI\left\{\overset{IV}{a_{22}^{y}}\right\} + \left\{\overset{II}{a_{21}^{y}}\right\} = \rho A\left\{a_{02}^{y}\right\} \qquad (3\text{-}7k)$$

$$EI\left\{\overset{IV}{a_{23}^{y}}\right\} + \left\{\overset{II}{a_{22}^{y}}\right\} = \rho A\left\{a_{03}^{y}\right\} \qquad (3\text{-}7l)$$

$$EI\left\{ \overset{IV}{a^{y}_{40}} \right\} = \rho A\left\{ a^{y}_{20} \right\} \qquad (3\text{-}7m)$$

$$EI\left\{ \overset{IV}{a^{y}_{04}} \right\} = -\left\{ \overset{II}{a^{y}_{03}} \right\} \ \ldots \text{ etc} \qquad (3\text{-}7n)$$

3.3 Solution of the Differential Equation

Solutions of Equations (3-7a) through Equation (3-7n) are obtained by direct integration. For Equation (3-7a) the following boundary conditions are applied :

$$\text{at} \quad x = 0 \qquad u_y = U_1 \qquad U_2 = \frac{dU_1}{dx}$$

$$\text{at} \quad x = L \qquad u_y = U_3 \qquad U_4 = -\frac{dU_3}{dx}$$

For the remaining Equations, the boundary condition that $\left\{a^{y}_{01}\right\}$, $\left\{a^{y}_{10}\right\}$, $\left\{a^{y}_{11}\right\}$ ... must all vanish at $x = 0$ and L is applied; this yields

$$\left\{a^{y}_{00}\right\} = \left[(1-3\xi^2+2\xi^3) \quad (\xi-2\xi^2+\xi^3)L \quad (\xi^2-\xi^3)L \quad (3\xi^2-2\xi^3) \right] \qquad (3\text{-}8a)$$

$$\left\{a^{y}_{01}\right\} = \frac{L^2}{60EI}\Big[3(\xi^2-4\xi^3+5\xi^4-2\xi^5) \quad (4\xi^2-11\xi^3+10\xi^4-3\xi^5)L$$
$$(\xi^2+\xi^3-5\xi^4+3\xi^5)L \quad -3(\xi^2-4\xi^3+5\xi^4-2\xi^5) \Big] \qquad (3\text{-}8b)$$

$$\left\{a^{y}_{10}\right\} = \left\{a^{y}_{11}\right\} = \left\{0\right\} \qquad (3\text{-}8c,d)$$

$$\left\{a^{y}_{20}\right\} = \frac{\rho AL^4}{2520EI}\Big[3(22\xi^2-52\xi^3+35\xi^4-7\xi^6+2\xi^7) \quad (12\xi^2-22\xi^3+21\xi^5-14\xi^6+3\xi^7)L$$
$$(9\xi^2-13\xi^3+7\xi^6-3\xi^7)L \quad 3(13\xi^2-18\xi^3+7\xi^6-2\xi^7) \Big] \qquad (3\text{-}8e)$$

$$\left\{a^y_{21}\right\} = \frac{\rho AL^6}{151200(EI)^2}\Big[(60\xi^2-8\xi^3-330\xi^4+468\xi^5-189\xi^6-36\xi^7+45\xi^8-10\xi^9) \quad (24\xi^2-20\xi^3-60\xi^4+66\xi^5+28\xi^6-63\xi^7+30\xi^8-5\xi^9)L$$

$$(21\xi^2-15\xi^3-45\xi^4+39\xi^5+7\xi^6+3\xi^7-15\xi^8+5\xi^9)L \quad (45\xi^2+8\xi^3-195\xi^4+162\xi^5-21\xi^6+36\xi^7-45\xi^8+10\xi^9)\Big] \qquad (3\text{-}8g)$$

$$\left\{a^y_{12}\right\} = \left\{0\right\} \qquad (3\text{-}8h)$$

$$\left\{a^y_{22}\right\} = \frac{\rho AL^8}{698544000(EI)^3}\Big[3(12856\xi^2-12472\xi^3-7700\xi^4+616\xi^5+17171\xi^6-17226\xi^7+4620\xi^8+1540\xi^9+385\xi^{10}+210\xi^{11})$$

$$(3576\xi^2-2076\xi^3-9240\xi^4+4620\xi^5+10934\xi^6-7359\xi^7-4620\xi^8+6160\xi^9-2310\xi^{10}+315\xi^{11})L$$

$$(3354\xi^2-1774\xi^3-8085\xi^4+3465\xi^5+7931\xi^6-4191\xi^7-1155\xi^8-385\xi^9+1155\xi^{10}-315\xi^{11})L$$

$$3(1774\xi^2+152\xi^3-5775\xi^4-616\xi^5+9779\xi^6-5874\xi^7+1155\xi^8-1540\xi^9+1155\xi^{10}-210\xi^{11})\Big] \qquad (3\text{-}8i)$$

$$\left\{a^y_{30}\right\} = \left\{0\right\} \qquad (3\text{-}8j)$$

$$\left\{a^y_{03}\right\} = \frac{L^6}{1512000(EI)^3}\Big[(6\xi^2-4\xi^3-45\xi^4+18\xi^5+210\xi^6-360\xi^7+225\xi^8-50\xi^9) \quad (28\xi^2-2\xi^3-110\xi^4+9\xi^5+280\xi^6-330\xi^7+150\xi^8-25\xi^9)L$$

$$(22\xi^2+2\xi^3-65\xi^4-9\xi^5+70\xi^6+30\xi^7-75\xi^8+25\xi^9)L \quad -(6\xi^2-4\xi^3-45\xi^4+18\xi^5+210\xi^6-360\xi^7+225\xi^8-50\xi^9)\Big] \qquad (3\text{-}8k)$$

$$\left\{a^y_{31}\right\} = \left\{a^y_{13}\right\} = \left\{a^y_{32}\right\} = \left\{a^y_{33}\right\} = \left\{0\right\} \qquad (3\text{-}8l,m,n,o)$$

$$\left\{ a^{y}_{23} \right\} = \frac{\rho A L^{10}}{544864320000(EI)^4} \left[\begin{array}{l} 3(217504\xi^{2}+49408\xi^{3}-835640\xi^{4}+486408\xi^{5}+202202\xi^{6}-12012\xi^{7}-242385\xi^{8}+187330\xi^{9} \\ \qquad -35035\xi^{10}-16380\xi^{11}-1400\xi^{13}) \\ (86640\xi^{2}-39144\xi^{3}-232440\xi^{4}+80964\xi^{5}+268268\xi^{6}-86658\xi^{7}-175890\xi^{8}+80795\xi^{9}+60060\xi^{10}-58695\xi^{11}+18200\xi^{12} \\ \qquad -2100\xi^{13})L \\ (83985\xi^{2}-35931\xi^{3}-218010\xi^{4}+69186\xi^{5}+232232\xi^{6}-63492\xi^{7}-124410\xi^{8}+44330\xi^{9}+15015\xi^{10}+4095\xi^{11}-9100\xi^{12} \\ \qquad +2100\xi^{13})L \\ 3(35931\xi^{2}+1552\xi^{3}-115310\xi^{4}-5928\xi^{5}+148148\xi^{6}+12012\xi^{7}-132990\xi^{8}+62920\xi^{9}-15015\xi^{10}+16380\xi^{11}-9100\xi^{12} \\ \qquad +1400\xi^{13}) \end{array} \right] \tag{3-8p}$$

$$\left\{ a^{y}_{40} \right\} = \frac{(\rho A L^{4})^{2}}{139708800(EI)^{2}} \left[\begin{array}{l} 3(1784\xi^{2}-2832\xi^{3}+3388\xi^{6}-3432\xi^{7}+1155\xi^{8}-77\xi^{10}+14\xi^{11}) \\ (1136\xi^{2}-1784\xi^{3}+1848\xi^{6}-1452\xi^{7}+385\xi^{9}-154\xi^{10}+21\xi^{11})L \\ (1097\xi^{2}-1681\xi^{3}+1386\xi^{6}-858\xi^{7}+77\xi^{10}-21\xi^{11})L \\ 3(1681\xi^{2}-2558\xi^{3}+2002\xi^{6}-1188\xi^{7}+77\xi^{10}-14\xi^{11}) \end{array} \right] \tag{3-8q}$$

$$\left\{a^{y}_{04}\right\} = \frac{L^{8}}{6985440000(EI)^{4}}\left[\begin{array}{l} 3(111\xi^{2}-74\xi^{3}-770\xi^{4}+308\xi^{5}+2310\xi^{6}-660\xi^{7}-5775\xi^{8}+7700\xi^{9}-3850\xi^{10}+700\xi^{11}) \\ (3054\xi^{2}-111\xi^{3}-10780\xi^{4}+462\xi^{5}+16940\xi^{6}-990\xi^{7}-23100\xi^{8}+21175\xi^{9}-7700\xi^{10}+1050\xi^{11})L \\ (2721\xi^{2}+111\xi^{3}-8470\xi^{4}-462\xi^{5}+10010\xi^{6}+990\xi^{7}-5775\xi^{8}-1925\xi^{9}+3850\xi^{10}-1050\xi^{11})L \\ -3(111\xi^{2}-74\xi^{3}-770\xi^{4}+308\xi^{5}+2310\xi^{6}-660\xi^{7}-5775\xi^{8}+7700\xi^{9}-3850\xi^{10}+700\xi^{11}) \end{array}\right] \ldots \text{etc}$$

(3-8r)

where $\xi = \frac{x}{L}$

The matrix $\left\{ a_{00}^{y} \right\}$ represents static transverse deflection distribution due to unit values of U_1, U_2, U_3, U_4 . The remaining matrices in $\left\{ a \right\}$ are determined from Equation (3-2). This gives

$$\left\{ a_{00}^{x} \right\} = \eta \Big[\; 6(\xi-\xi^2) \quad (-1+4\xi-3\xi^2)L \quad (-2\xi+3\xi^2)L \quad -6(\xi-\xi^2) \; \Big] \tag{3-9a}$$

$$\left\{ a_{01}^{x} \right\} = \frac{\eta L^2}{60EI} \Big[-6(\xi-6\xi^2+10\xi^3-5\xi^4) \quad (-8\xi+33\xi^2-40\xi^3+15\xi^4)L \quad (-2\xi-3\xi^2+20\xi^3-15\xi^4)L \quad 6(\xi-6\xi^2+10\xi^3-5\xi^4) \Big] \tag{3-9b}$$

$$\left\{ a_{10}^{x} \right\} = \left\{ 0 \right\} \tag{3-9c}$$

$$\left\{ a_{11}^{x} \right\} = \left\{ 0 \right\} \tag{3-9d}$$

$$\left\{ a_{20}^{x} \right\} = \frac{\eta \rho A L^4}{2520EI} \Big[-6(22\xi-78\xi^2+70\xi^3-21\xi^5+7\xi^6) \quad -3(8\xi-22\xi^2+35\xi^4-28\xi^5+7\xi^6)L \quad -3(6\xi-13\xi^2+14\xi^5-7\xi^6)L \quad -6(13\xi-27\xi^2+21\xi^5-7\xi^6) \Big] \tag{3-9e}$$

$$\left\{ a_{02}^{x} \right\} = \frac{\eta L^4}{25200(EI)^2} \Big[-6(3\xi-3\xi^2-70\xi^3+210\xi^4-210\xi^5+70\xi^6) \quad (-44\xi+9\xi^2+560\xi^3-1155\xi^4+840\xi^5-210\xi^6)L \quad (-26\xi-9\xi^2+140\xi^3+105\xi^4-420\xi^5+210\xi^6)L \quad 6(3\xi-3\xi^2-70\xi^3+210\xi^4-210\xi^5+70\xi^6) \Big] \tag{3-9f}$$

$$\left\{ a_{12}^{x} \right\} = \left\{ 0 \right\} \tag{3-9g}$$

$$\left\{a^x_{21}\right\} = \frac{\eta\rho AL^6}{50400(EI)^2}\left[\begin{matrix} -2(20\xi-4\xi^2-220\xi^3+390\xi^4-189\xi^5-42\xi^6+60\xi^7-15\xi^8) \\ (-16\xi+20\xi^2+80\xi^3-110\xi^4-56\xi^5+147\xi^6-80\xi^7+15\xi^8)L \\ (-14\xi+15\xi^2+60\xi^3-65\xi^4-14\xi^5-7\xi^6+40\xi^7-15\xi^8)L \\ -2(15\xi+4\xi^2-130\xi^3+135\xi^4-21\xi^5+42\xi^6-60\xi^7+15\xi^8) \end{matrix}\right] \tag{3-9h}$$

$$\left\{a^x_{22}\right\} = \frac{\eta\rho AL^8}{232848000(EI)^3}\left[\begin{matrix} -2(12856\xi-18708\xi^2-15400\xi^3+1540\xi^4+51513\xi^5-60291\xi^6+18480\xi^7+6930\xi^8 \\ +1925\xi^9+1155\xi^{10}) \\ -(2384\xi-2076\xi^2-12320\xi^3+7700\xi^4+21868\xi^5-17171\xi^6-12320\xi^7+18480\xi^8-7700\xi^9+1155\xi^{10})L \\ -(2236\xi-1774\xi^2-10780\xi^3+5775\xi^4+15862\xi^5-9779\xi^6-3080\xi^7-1155\xi^8+3850\xi^9-1155\xi^{10})L \\ -2(1774\xi+228\xi^2-11550\xi^3-1540\xi^4+29337\xi^5-20559\xi^6+4620\xi^7-6930\xi^8+5775\xi^9-1155\xi^{10}) \end{matrix}\right] \tag{3-9i}$$

$$\left\{a^x_{30}\right\} = \left\{0\right\} \tag{3-9j}$$

$$\left\{a^x_{03}\right\} = \frac{\eta L^6}{1512000(EI)^3}\left[\begin{matrix} -6(2\xi-2\xi^2-30\xi^3+15\xi^4+210\xi^5-420\xi^6+300\xi^7-75\xi^8) \\ -(56\xi-6\xi^2-440\xi^3+45\xi^4+1680\xi^5-2310\xi^6+1200\xi^7-225\xi^8)L \\ -(44\xi+6\xi^2-260\xi^3-45\xi^4+420\xi^5+210\xi^6-600\xi^7+225\xi^8)L \\ 6(2\xi-2\xi^2-30\xi^3+15\xi^4+210\xi^5-420\xi^6+300\xi^7-75\xi^8) \end{matrix}\right] \tag{3-9k}$$

$$\left\{a^{x}_{31}\right\} = \left\{0\right\} \qquad (3\text{-}9l)$$

$$\left\{a^{x}_{13}\right\} = \left\{0\right\} \qquad (3\text{-}9m)$$

$$\left\{a^{x}_{32}\right\} = \left\{0\right\} \qquad (3\text{-}9n)$$

$$\left\{a^{x}_{23}\right\} = \frac{\nu \rho A L^{10}}{18162144 0000(EI)^4}\Big[-2(217504\xi+74112\xi^{2}-1671280\xi^{3}+1216020\xi^{4}+606606\xi^{5}-42042\xi^{6}$$
$$-969540\xi^{7}+842985\xi^{8}-175175\xi^{9}-90090\xi^{10}-9100\xi^{12})$$
$$-(57760\xi-39144\xi^{2}-309920\xi^{3}+134940\xi^{4}+536536\xi^{5}-202202\xi^{6}-469040\xi^{7}+242385\xi^{8}+200200\xi^{9}$$
$$-215215\xi^{10}+72800\xi^{11}-9100\xi^{12})L$$
$$-(55990\xi-35931\xi^{2}-290680\xi^{3}+115310\xi^{4}+464464\xi^{5}+148148\xi^{6}-331760\xi^{7}+132990\xi^{8}+50050\xi^{9}$$
$$+15015\xi^{10}-36400\xi^{11}+9100\xi^{12})L$$
$$-2(35931\xi+2328\xi^{2}-230620\xi^{3}-14820\xi^{4}+444444\xi^{5}+42042\xi^{6}-531960\xi^{7}+283140\xi^{8}-75075\xi^{9}$$
$$+90090\xi^{10}-54600\xi^{11}+9100\xi^{12})\Big] \qquad (3\text{-}9o)$$

$$\left\{a^{x}_{33}\right\} = \left\{0\right\} \qquad (3\text{-}9p)$$

$$
\begin{aligned}
\left\{ a^{x}_{40} \right\} = \frac{\eta\,(\rho A L^{4})^{2}}{139708800\,(EI)^{2}} \Big[& -6(1784\xi-4248\xi^{2}+10164\xi^{5}-12012\xi^{6}+4620\xi^{7}-385\xi^{9}+77\xi^{10}) \\
& -(2272\xi-5352\xi^{2}+11088\xi^{5}-10164\xi^{6}+3465\xi^{8}-1540\xi^{9}+231\xi^{10})L \\
& -(2194\xi-5043\xi^{2}+8316\xi^{5}-6006\xi^{6}+770\xi^{9}-231\xi^{10})L \\
& -6(1681\xi-3837\xi^{2}+6006\xi^{5}-4158\xi^{6}+385\xi^{8}-77\xi^{10}) \Big]
\end{aligned}
\qquad (3\text{-}9q)
$$

$$
\begin{aligned}
\left\{ a^{x}_{04} \right\} = \frac{\eta\, L^{8}}{6985440000\,(EI)^{4}} \Big[& -6(111\xi-111\xi^{2}-1540\xi^{3}+770\xi^{4}+6930\xi^{5}-2310\xi^{6}-23100\xi^{7}+34650\xi^{8}-19250\xi^{9} \\
& \qquad +3850\xi^{10}) \\
& -(6108\xi-333\xi^{2}-43120\xi^{3}+2310\xi^{4}+101640\xi^{5}-6930\xi^{6}-184800\xi^{7}+190575\xi^{8}-77000\xi^{9}+11550\xi^{10})L \\
& -(5442\xi+333\xi^{2}-33880\xi^{3}-2310\xi^{4}+60060\xi^{5}+6930\xi^{6}-46200\xi^{7}-17325\xi^{8}+38500\xi^{9}-11550\xi^{10})L \\
& 6(111\xi-111\xi^{2}-1540\xi^{3}+770\xi^{4}+6930\xi^{5}-2310\xi^{6}-23100\xi^{7}+34650\xi^{8}-19250\xi^{9}+3850\xi^{10}) \Big]
\end{aligned}
$$

... etc (3-9r)

3.4 Total Strains-Displacement Relationship

The induced strains in the beam-column element are derived by noting

$$\{e\} = \frac{\partial u_x}{\partial x} = -\frac{\partial^2 u_y}{\partial x^2}\, y = -L\frac{\partial^2 u_y}{\partial x^2}$$

$$= \left(\{b_{00}\} + \Omega\{b_{10}\} + P\{b_{01}\} + \Omega P\{b_{11}\} + \dots \right)\{U\}$$

$$= \{b\}\{U\} \qquad (3\text{-}10)$$

Hence, using Equation (3-8a) through (3-8r) and Equation (3-10), it follows that,

$$\{b_{00}\} = \frac{\eta}{L}\Big[\, 6(1-2\xi) \quad (4-6\xi)L \quad (-2+6\xi)L \quad -6(1-2\xi) \,\Big] \qquad (3\text{-}11a)$$

$$\{b_{01}\} = \frac{\eta L}{30EI}\Big[-3(1-12\xi+30\xi^2-20\xi^3) \quad (-4+33\xi-60\xi^2+30\xi^3)L$$
$$(-1-3\xi+30\xi^2-30\xi^3)L \quad 3(1-12\xi+30\xi^2-20\xi^3) \Big] \qquad (3\text{-}11b)$$

$$\{b_{10}\} = \{b_{11}\} = \{0\} \qquad (3\text{-}11c,d)$$

$$\{b_{20}\} = \frac{\eta\rho AL^3}{420EI}\Big[(-22+156\xi-210\xi^2+105\xi^4-42\xi^5)$$
$$(-4+22\xi-70\xi^3+70\xi^4-21\xi^5)L$$
$$(-3+13\xi-35\xi^4+21\xi^5)L$$
$$(-13+54\xi-105\xi^4+42\xi^5) \Big] \qquad (3\text{-}11e)$$

$$\{b_{12}\} = \{0\} \qquad (3\text{-}11f)$$

$$\{b_{02}\} = \frac{\eta L^3}{12600(EI)^2}\Big[-9(1-2\xi-70\xi^2+280\xi^3-350\xi^4+140\xi^5) \quad (-22+9\xi+840\xi^2-2310\xi^3+2100\xi^4-630\xi^5)L$$

$$(-13-9\xi+210\xi^2+210\xi^3-1050\xi^4+630\xi^5)L \quad 9(1-2\xi-70\xi^2+280\xi^3-350\xi^4+140\xi^5) \Big] \qquad (3\text{-}11g)$$

$$\{b_{21}\} = \frac{\eta\rho AL^5}{25200(EI)^2}\Big[(-20+8\xi+660\xi^2-1560\xi^3+945\xi^4+252\xi^5-420\xi^6+120\xi^7)$$

$$(-8+20\xi+120\xi^2-220\xi^3-140\xi^4+441\xi^5-280\xi^6+60\xi^7)L$$

$$(-7+15\xi+90\xi^2-130\xi^3-35\xi^4-21\xi^5+140\xi^6-60\xi^7)L$$

$$(-15-8\xi+390\xi^2-540\xi^3+105\xi^4-252\xi^5+420\xi^6-120\xi^7) \Big] \qquad (3\text{-}11h)$$

$$\{b_{22}\} = \frac{\eta\rho AL^7}{116424000(EI)^3}\Big[-(12856-37416\xi-46200\xi^2+6160\xi^3+257565\xi^4-361746\xi^5+129360\xi^6+55440\xi^7$$

$$+17325\xi^8+11550\xi^9)$$

$$-(1192-2076\xi-18480\xi^2+15400\xi^3+54670\xi^4-51513\xi^5-43120\xi^6+73920\xi^7-34650\xi^8+5775\xi^9)L$$

$$-(1118-1774\xi-16170\xi^2+11550\xi^3+39655\xi^4-29337\xi^5-10780\xi^6-4620\xi^7+17325\xi^8-5775\xi^9)L$$

$$-(1774+456\xi-34650\xi^2-6160\xi^3+146685\xi^4-123354\xi^5+32340\xi^6-55440\xi^7+51975\xi^8-11550\xi^9) \Big] \qquad (3\text{-}11i)$$

$$\{b_{30}\} = \{0\} \qquad (3\text{-}11j)$$

$$\left\{b_{03}\right\} = \frac{\eta L^5}{378000(EI)^3}\Big[-3(1-2\xi-45\xi^2+30\xi^3+525\xi^4-1260\xi^5+1050\xi^6-300\xi^7)$$

$$-(14-3\xi-330\xi^2+45\xi^3+2100\xi^4-3465\xi^5+2100\xi^6-450\xi^7)L$$

$$-(11+3\xi-195\xi^2-45\xi^3+525\xi^4+315\xi^5-1050\xi^6+450\xi^7)L$$

$$3(1-2\xi-45\xi^2+30\xi^3+525\xi^4-1260\xi^5+1050\xi^6-300\xi^7)\Big] \qquad (3\text{-}11k)$$

$$\left\{b_{23}\right\} = \frac{\eta \rho A L^9}{90810720000(EI)^4}\Big[-(217504+148224\xi-5013840\xi^2+4864080\xi^3+3033030\xi^4$$

$$-252252\xi^5-6786780\xi^6+6743880\xi^7-1576575\xi^8-900900\xi^9-109200\xi^{11})$$

$$-(28880-39144\xi-464880\xi^2+269880\xi^3+1341340\xi^4-606606\xi^5-1641640\xi^6+969540\xi^7+900900\xi^8$$

$$-1076075\xi^9+400400\xi^{10}-54600\xi^{11})L$$

$$-(27995-35931\xi-436020\xi^2+230620\xi^3+1161160\xi^4-444444\xi^5-1161160\xi^6+531960\xi^7+225225\xi^8$$

$$+75075\xi^9-200200\xi^{10}+54600\xi^{11})L$$

$$-(35931+4656\xi-691860\xi^2-59280\xi^3+2222220\xi^4+252252\xi^5-3723720\xi^6+2265120\xi^7-675675\xi^8$$

$$+900900\xi^9-600600\xi^{10}+109200\xi^{11})\Big] \qquad (3\text{-}11l)$$

$$\left\{b_{31}\right\} = \left\{b_{13}\right\} = \left\{b_{32}\right\} = \left\{b_{33}\right\} = \left\{0\right\} \qquad (3\text{-}11m,n,o,p)$$

$$\left\{b_{40}\right\} = \frac{\eta(\rho A)^2 L^7}{69854400(EI)^2}\Big[-3(1784-8496\xi+50820\xi^4-72072\xi^5+32340\xi^6-3465\xi^8+770\xi^9)$$

$$-(1136-5352\xi+27720\xi^4-30492\xi^5+13860\xi^7-6930\xi^8+1155\xi^9)L$$

$$-(1097-5043\xi+20790\xi^4-18018\xi^5+3465\xi^8-1155\xi^9)L$$

$$-3(1681-7674\xi+30030\xi^4-24948\xi^5+3465\xi^8-770\xi^9)\Big] \qquad (3\text{-}11q)$$

$$\left\{b_{04}\right\} = \frac{\eta L^7}{3492720000(EI)^4}\Big[-3(111-222\xi-4620\xi^2+3080\xi^3+34650\xi^4-13860\xi^5-161700\xi^6+277200\xi^7-173250\xi^8+38500\xi^9)$$

$$-(3054-333\xi-64680\xi^2+4620\xi^3+254100\xi^4-20790\xi^5-646800\xi^6+762300\xi^7-346500\xi^8+57750\xi^9)L$$

$$-(2721+333\xi-50820\xi^2-4620\xi^3+150150\xi^4+20790\xi^5-161700\xi^6-69300\xi^7+173250\xi^8-57750\xi^9)L$$

$$3(111-222\xi-4620\xi^2+3080\xi^3+34650\xi^4-13860\xi^5-161700\xi^6+277200\xi^7-173250\xi^8+38500\xi^9)\Big]$$

...etc $\qquad (3\text{-}11r)$

3.5 Mass Matrix

Recalling Equation (3-3), the matrix $\left[a^y\right]$ is expanded into an infinite series in ascending powers of Ω and P, thus

$$\left[a^y\right] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \Omega^r P^s \left\{a^y_{rs}\right\} \qquad (3\text{-}12a)$$

Substituting the Equations (3-8a) through Equation (3-8r) into Equation (3-12a), the final form of the matrix $\left[a^y\right]$ becomes

$$\left[a^y\right] = \left\{a^y_{00}\right\} + P\left\{a^y_{01}\right\} + P\Omega^2\left\{a^y_{21}\right\} + P^2\left\{a^y_{02}\right\} + \Omega^2\left\{a^y_{20}\right\} + \ldots \qquad (3\text{-}12b)$$

From previous work it follows that

$$\left[M\right] = \int_V \rho \left[a^y\right]^T\left[a^y\right] dV$$

so,

$$\begin{aligned}\left[a^y\right]^T\left[a^y\right] = {} & \left[\left\{a^y_{00}\right\}\left\{a^y_{00}\right\}\right] + P\left[\left\{a^y_{00}\right\}\left\{a^y_{01}\right\}+\left\{a^y_{01}\right\}\left\{a^y_{00}\right\}\right] \\ & +P^2\left[\left\{a^y_{00}\right\}\left\{a^y_{02}\right\}+\left\{a^y_{01}\right\}\left\{a^y_{01}\right\}+\left\{a^y_{02}\right\}\left\{a^y_{00}\right\}\right] \\ & +\Omega^2\left[\left\{a^y_{00}\right\}\left\{a^y_{20}\right\}+\left\{a^y_{20}\right\}\left\{a^y_{00}\right\}\right] \\ & +P^3\left[\left\{a^y_{00}\right\}\left\{a^y_{03}\right\}+\left\{a^y_{01}\right\}\left\{a^y_{02}\right\}+\left\{a^y_{02}\right\}\left\{a^y_{01}\right\}+\left\{a^y_{03}\right\}\left\{a^y_{00}\right\}\right] \\ & + \ldots \end{aligned} \qquad (3\text{-}13)$$

Substituting Equation (3-13) into Equation (1-6), a term by term evaluation is made utiliaing direct integration. The mass matrices are obtained in ascending powers of Ω and P in the form

$$\left[M\right] = \left[M_0\right] + P\left[K_1^i\right] + P^2\left[K_2^i\right] + \Omega^2\left[M_2\right] + P^3\left[K_3^i\right] + \ldots \quad (3\text{-}14a)$$

with

$$\left[M_0\right] = \frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 13L & 54 \\ 22L & 4L^2 & 3L^2 & 13L \\ 13L & 3L^2 & 4L^2 & 22L \\ 54 & 13L & 22L & 156 \end{bmatrix} \quad (3\text{-}14b)$$

$$\left[K_1^i\right] = \frac{\rho AL^3}{EI}\begin{bmatrix} \frac{1}{3150} & \frac{L}{1260} & \frac{L}{1680} & \frac{-1}{1350} \\ \frac{L}{1260} & \frac{L^2}{3150} & \frac{L^2}{3600} & \frac{L}{1680} \\ \frac{L}{1680} & \frac{L^2}{3600} & \frac{L^2}{3150} & \frac{L}{1260} \\ \frac{-1}{1350} & \frac{L}{1680} & \frac{L}{1260} & \frac{1}{3150} \end{bmatrix} \quad (3\text{-}14c)$$

$$\left[K_2^i\right] = \frac{\rho AL^5}{1000(EI)^2}\begin{bmatrix} \frac{19}{4851} & \frac{173L}{9702} & \frac{887L}{58212} & \frac{-19}{4851} \\ \frac{173L}{9702} & \frac{149L^2}{14553} & \frac{559L^2}{58212} & \frac{887L}{58212} \\ \frac{887L}{58212} & \frac{559L^2}{58212} & \frac{149L^2}{14533} & \frac{173L}{9702} \\ \frac{-19}{4851} & \frac{887L}{58212} & \frac{174L}{9702} & \frac{19}{4851} \end{bmatrix}, \quad (3\text{-}14d)$$

$$\left[M_2\right] = \frac{(\rho A)^2 L^5}{EI} \begin{bmatrix} \frac{59}{80850} & \frac{223L}{145530} & \frac{1681L}{11642400} & \frac{1279}{1940400} \\ \frac{223L}{1455300} & \frac{71L^2}{2182950} & \frac{1097L^2}{34927200} & \frac{1681L}{11642400} \\ \frac{1681L}{11642400} & \frac{1097L^2}{34927200} & \frac{71L^2}{2182950} & \frac{223L}{1455300} \\ \frac{1279}{1940400} & \frac{1681L}{11642400} & \frac{223L}{1455300} & \frac{59}{80850} \end{bmatrix} \qquad (3\text{-}11e)$$

$$\left[K_3^i\right] = \frac{\rho A L^7}{1000(EI)^3} \begin{bmatrix} \frac{97}{1891890} & \frac{223L}{540540} & \frac{1711L}{4324320} & \frac{-97}{1891890} \\ \frac{233L}{540540} & \frac{361L^2}{1135134} & \frac{509L^2}{1651104} & \frac{1711L}{4324320} \\ \frac{1711L}{4324320} & \frac{509L^2}{1651104} & \frac{361L^2}{1135134} & \frac{233L}{540540} \\ \frac{-97}{1891890} & \frac{1711L}{4324320} & \frac{233L}{540540} & \frac{97}{1891890} \end{bmatrix} \qquad (3\text{-}11f)$$

3.6 Stiffness Matrix

Recalling the Equation (3-10), the strain-displacement relationship is in the form of matrix $\left[b\right]$. Similarly, it may be expanded into an infinite series in ascending powers of Ω and P, as

$$\left[b\right] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \Omega^r P^s \left\{b_{rs}\right\} \qquad (3\text{-}15a)$$

Substituting the Equations (3-11a) through Equation (3-11r) into Equation (3-15a), the final form of the matrix $\left[b \right]$ is

$$\left[b \right] = \left\{ b_{00} \right\} + P\left\{ b_{01} \right\} + P\Omega^2\left\{ b_{21} \right\} + P^2\left\{ b_{02} \right\} + \Omega^2\left\{ b_{20} \right\} + P^2\Omega^2\left\{ b_{22} \right\} + \ldots \qquad (3\text{-}15b)$$

$$\begin{aligned}
\text{so, } \left[b \right]^T\left[b \right] = {} & \left[\left\{ b_{00} \right\}^T\left\{ b_{00} \right\} \right] + P\left[\left\{ b_{00} \right\}^T\left\{ b_{01} \right\} + \left\{ b_{01} \right\}^T\left\{ b_{00} \right\} \right] \\
& + P^2\left[\left\{ b_{00} \right\}^T\left\{ b_{02} \right\} + \left\{ b_{01} \right\}^T\left\{ b_{01} \right\} + \left\{ b_{02} \right\}^T\left\{ b_{00} \right\} \right] \\
& + \Omega^2\left[\left\{ b_{00} \right\}^T\left\{ b_{20} \right\} + \left\{ b_{20} \right\}^T\left\{ b_{00} \right\} \right] \\
& + P\Omega^2\left[\left\{ b_{00} \right\}^T\left\{ b_{21} \right\} + \left\{ b_{01} \right\}^T\left\{ b_{20} \right\} + \left\{ b_{20} \right\}^T\left\{ b_{01} \right\} + \left\{ b_{21} \right\}^T\left\{ b_{00} \right\} \right] \\
& + P^2\Omega^2\left[\left\{ b_{00} \right\}^T\left\{ b_{22} \right\} + \left\{ b_{01} \right\}^T\left\{ b_{21} \right\} + \left\{ b_{02} \right\}^T\left\{ b_{20} \right\} + \left\{ b_{20} \right\}^T\left\{ b_{02} \right\} \right. \\
& \qquad \left. + \left\{ b_{21} \right\}^T\left\{ b_{01} \right\} + \left\{ b_{22} \right\}^T\left\{ b_{00} \right\} \right] \\
& + P^3\left[\left\{ b_{00} \right\}^T\left\{ b_{03} \right\} + \left\{ b_{01} \right\}^T\left\{ b_{02} \right\} + \left\{ b_{02} \right\}^T\left\{ b_{01} \right\} + \left\{ b_{03} \right\}^T\left\{ b_{00} \right\} \right] \\
& + P^3\Omega^2\left[\left\{ b_{00} \right\}^T\left\{ b_{23} \right\} + \left\{ b_{01} \right\}^T\left\{ b_{22} \right\} + \left\{ b_{02} \right\}^T\left\{ b_{21} \right\} + \left\{ b_{20} \right\}^T\left\{ b_{03} \right\} \right. \\
& \qquad \left. + \left\{ b_{21} \right\}^T\left\{ b_{02} \right\} + \left\{ b_{22} \right\}^T\left\{ b_{01} \right\} + \left\{ b_{03} \right\}^T\left\{ b_{20} \right\} + \left\{ b_{23} \right\}^T\left\{ b_{00} \right\} \right] \\
& + P^4\left[\left\{ b_{00} \right\}^T\left\{ b_{04} \right\} + \left\{ b_{01} \right\}^T\left\{ b_{03} \right\} + \left\{ b_{02} \right\}^T\left\{ b_{02} \right\} + \left\{ b_{03} \right\}^T\left\{ b_{01} \right\} \right. \\
& \qquad \left. + \left\{ b_{04} \right\}^T\left\{ b_{00} \right\} \right] \\
& + \Omega^4\left[\left\{ b_{00} \right\}^T\left\{ b_{40} \right\} + \left\{ b_{20} \right\}^T\left\{ b_{20} \right\} + \left\{ b_{40} \right\}^T\left\{ b_{00} \right\} \right] \\
& + \ldots \qquad (3\text{-}16)
\end{aligned}$$

Substituting Equation (3-16) into Equation (1-7), the equivalent stiffness matrices are

$$\left[K\right] = \left[K_0\right] + P\left[K_1^b\right] + P^2\left[K_2^b\right] + \Omega^2\left[K_3^b\right] + P\Omega^2\left[K_4^b\right] + P^2\Omega^2\left[K_5^b\right]$$

$$+ P^3\left[K_6^b\right] + P^3\Omega^2\left[K_7^b\right] + P^4\left[K_8^b\right] + \Omega^4\left[K_9^b\right] + \ldots \qquad (3\text{-}17a)$$

with

$$\left[K_0\right] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -6L & -12 \\ 6L & 4L^2 & -2L^2 & -6L \\ -6L & -2L^2 & 4L^2 & 6L \\ -12 & -6L & 6L & 12 \end{bmatrix} \qquad (3\text{-}17b)$$

$$\left[K_1^b\right] = \left[0\right] \qquad (3\text{-}17c)$$

$$\left[K_2^b\right] = \frac{L}{EI}\begin{bmatrix} \frac{1}{700} & \frac{L}{1400} & \frac{-L}{1400} & \frac{-1}{700} \\ \frac{L}{1400} & \frac{11L^2}{6300} & \frac{13L^2}{12600} & \frac{-L}{1400} \\ \frac{-L}{1400} & \frac{13L^2}{12600} & \frac{11L^2}{6300} & \frac{L}{1400} \\ \frac{-1}{700} & \frac{-L}{1400} & \frac{L}{1400} & \frac{1}{700} \end{bmatrix} \qquad (3\text{-}17d)$$

$$\left[K_3^b\right] = \left[0\right] \qquad (3\text{-}17e)$$

$$\left[K_4^b\right] = \frac{\rho AL^3}{EI}\begin{bmatrix} \frac{1}{3150} & \frac{L}{1260} & \frac{L}{1680} & \frac{-1}{1350} \\ \frac{L}{1260} & \frac{L^2}{3150} & \frac{L^2}{3600} & \frac{L}{1680} \\ \frac{L}{1680} & \frac{L^2}{3600} & \frac{L^2}{3150} & \frac{L}{1260} \\ \frac{-1}{1350} & \frac{L}{1680} & \frac{L}{1260} & \frac{1}{3150} \end{bmatrix} \qquad (3\text{-}17f)$$

$$\left[K_5^b\right] = \frac{\rho A L^5}{1000(EI)^2}\begin{bmatrix} \frac{38}{4851} & \frac{173L}{4851} & \frac{887L}{29106} & \frac{-38}{4851} \\ \frac{173L}{4851} & \frac{298L^2}{14553} & \frac{559L^2}{29106} & \frac{887L}{29106} \\ \frac{887L}{29106} & \frac{559L^2}{29106} & \frac{298L^2}{14553} & \frac{173L}{4851} \\ \frac{-38}{4851} & \frac{887L}{29106} & \frac{173L}{4851} & \frac{38}{4851} \end{bmatrix} \quad (3\text{-}17g)$$

$$\left[K_6^b\right] = \frac{L^3}{189000(EI)^2}\begin{bmatrix} 6 & 3L & -3L & -6 \\ 3L & 14L^2 & 11L^2 & -3L \\ -3L & 11L^2 & 14L^2 & 3L \\ -6 & -3L & 3L & 6 \end{bmatrix} \quad (3\text{-}17h)$$

$$\left[K_7^b\right] = \frac{\rho A L^7}{3024000(EI)^3}\begin{bmatrix} \frac{-48632}{5005} & \frac{21128L}{2145} & \frac{62719L}{4290} & \frac{23152}{5005} \\ \frac{21128L}{2145} & \frac{2888L^2}{1001} & \frac{509L^2}{182} & \frac{5133L}{1430} \\ \frac{62719L}{4290} & \frac{509L^2}{182} & \frac{2888L^2}{1001} & \frac{2796L}{715} \\ \frac{23152}{5005} & \frac{5133L}{1430} & \frac{2796L}{715} & \frac{2328}{5005} \end{bmatrix}$$

(3-17i)

$$\left[K_8^b\right] = \frac{L^5}{388080000(EI)^3}\begin{bmatrix} 222 & 111L & -111L & -222 \\ 111L & 1018L^2 & 907L^2 & -111L \\ -111L & 907L^2 & 1018L^2 & 111L \\ -222 & -111L & 111L & 222 \end{bmatrix}$$

(3-17j)

$$\left[K_9^b\right] = \frac{(\rho A)^2 L^5}{2EI}\begin{bmatrix} \frac{59}{80850} & \frac{223L}{1455300} & \frac{1681L}{11642400} & \frac{1279}{1940400} \\ \frac{223L}{1455300} & \frac{71L^2}{2182950} & \frac{1097L^2}{34927200} & \frac{1681L}{11642400} \\ \frac{1681L}{11642400} & \frac{1097L^2}{34927200} & \frac{71L^2}{2182950} & \frac{223L}{1455300} \\ \frac{1279}{1940400} & \frac{1681L}{11642400} & \frac{223L}{1455300} & \frac{59}{80850} \end{bmatrix} \quad (3\text{-}17k)$$

3.7 Geometric Matrix

The geometric stiffness property of a beam element identifies the increase in stiffness (tension) or decrease in stiffness (compression) of a beam-column element due to the presence of axially directed loads. This stiffness concept(1) is derived by application of virtual displacements upon equating the internal and external virtual work components. The coefficient of the geometric stiffness becomes

$$\left[K_G\right] = \int_0^L \left[c\right]^T \left[c\right] dx \quad (3\text{-}18)$$

where

$$\left[c\right] = \frac{d\left[A\right]}{dx} \quad (3\text{-}19)$$

Evaluation of the matrix $\left[c\right]$, is obtained by taking first derivatives of the functions in Equation (3-8a) through Equation (3-8r) and yields

$$\left\{c_{00}\right\} = \frac{1}{L}\left[\; -6(\xi-\xi^2) \quad (1-4\xi+3\xi^2)L \quad (2\xi-3\xi^2)L \quad 6(\xi-\xi^2) \;\right] \quad (3\text{-}20a)$$

$$\left\{ c_{01} \right\} = \frac{L}{60EI}\Big[\ 6(\xi-6\xi^2+10\xi^3-5\xi^4) \quad (8\xi-33\xi^2+40\xi^3-15\xi^4)L$$

$$(2\xi+3\xi^2-20\xi^3+15\xi^4)L \quad -6(\xi-6\xi^2+10\xi^3-5\xi^4)\ \Big] \qquad (3\text{-}20b)$$

$$\left\{ c_{10} \right\} = \left\{ c_{11} \right\} = \left\{ 0 \right\} \qquad (3\text{-}20c,d)$$

$$\left\{ c_{20} \right\} = \frac{\rho A L^3}{840EI}\Big[\ 2(22\xi-78\xi^2+70\xi^3-21\xi^5+7\xi^6)$$

$$(8\xi-22\xi^2+35\xi^4-28\xi^5+7\xi^6)L$$

$$(6\xi-13\xi^2+14\xi^5-7\xi^6)L \quad 2(13\xi-27\xi^2+21\xi^5-7\xi^6)\ \Big]$$

$$(3\text{-}20e)$$

$$\left\{ c_{02} \right\} = \frac{L^3}{25200(EI)^2}\Big[\ 6(3\xi-3\xi^2-70\xi^3+210\xi^4-210\xi^5+70\xi^6)$$

$$(44\xi-9\xi^2-560\xi^3+1155\xi^4-840\xi^5+210\xi^6)L$$

$$(26\xi+9\xi^2-140\xi^3-105\xi^4+420\xi^5-210\xi^6)L$$

$$-6(3\xi-3\xi^2-70\xi^3+210\xi^4-210\xi^5+70\xi^6)\ \Big] \qquad (3\text{-}20f)$$

$$\left\{ c_{21} \right\} = \frac{\rho A L^5}{50400(EI)^2}\Big[\ 2(20\xi-4\xi^2-220\xi^3+390\xi^4-189\xi^5-42\xi^6+60\xi^7-15\xi^8)$$

$$(16\xi-20\xi^2-80\xi^3+110\xi^4+56\xi^5-147\xi^6+80\xi^7-15\xi^8)L$$

$$(14\xi-15\xi^2-60\xi^3+65\xi^4+14\xi^5+7\xi^6-40\xi^7+15\xi^8)L$$

$$2(15\xi+4\xi^2-130\xi^3+135\xi^4-21\xi^5+42\xi^6-60\xi^7+15\xi^8)\ \Big] \qquad (3\text{-}20g)$$

$$\left\{ c_{12} \right\} = \left\{ 0 \right\} \qquad (3\text{-}20h)$$

$$\{c_{22}\} = \frac{\rho A L^7}{232848000(EI)^3}\Big[\,2(12856\xi - 18708\xi^2 - 15400\xi^3 + 1540\xi^4 + 51513\xi^5 - 60291\xi^6 + 18480\xi^7 + 6930\xi^8 + 1925\xi^9 + 1155\xi^{10})$$

$$(2384\xi - 2076\xi^2 - 12320\xi^3 + 7700\xi^4 + 21868\xi^5 - 17171\xi^6 - 12320\xi^7 + 18480\xi^8 - 7700\xi^9 + 1155\xi^{10})L$$

$$(2236\xi - 1774\xi^2 - 10780\xi^3 + 5775\xi^4 + 15862\xi^5 - 9779\xi^6 - 3080\xi^7 - 1155\xi^8 + 3850\xi^9 - 1155\xi^{10})L$$

$$2(1774\xi + 228\xi^2 - 11550\xi^3 - 1540\xi^4 + 29337\xi^5 - 20559\xi^6 + 4620\xi^7 - 6930\xi^8 + 5775\xi^9 - 1155\xi^{10})\Big] \qquad (3\text{-}20i)$$

$$\{c_{30}\} = \{0\} \qquad (3\text{-}20j)$$

$$\{c_{03}\} = \frac{L^5}{1512000(EI)^3}\Big[\,6(2\xi - 2\xi^2 - 30\xi^3 + 15\xi^4 + 210\xi^5 - 420\xi^6 + 300\xi^7 - 75\xi^8)$$

$$(56\xi - 6\xi^2 - 440\xi^3 + 45\xi^4 + 1680\xi^5 - 2310\xi^6 + 1200\xi^7 - 225\xi^8)L$$

$$(44\xi + 6\xi^2 - 260\xi^3 - 45\xi^4 + 420\xi^5 + 210\xi^6 - 600\xi^7 + 225\xi^8)L$$

$$-6(2\xi - 2\xi^2 - 30\xi^3 + 15\xi^4 + 210\xi^5 - 420\xi^6 + 300\xi^7 - 75\xi^8)\Big] \qquad (3\text{-}20k)$$

$$\left\{c_{31}\right\} = \left\{c_{13}\right\} = \left\{c_{32}\right\} = \left\{0\right\} \qquad (3\text{-}20\text{l,m,n})$$

$$\begin{aligned}\left\{c_{23}\right\} = \frac{\rho A L^9}{181621440000(EI)^4}\Big[\, & 2(217504\xi+74112\xi^2-1671280\xi^3+1216020\xi^4+606606\xi^5-42042\xi^6-969540\xi^7 \\ & +842985\xi^8-175175\xi^9-90090\xi^{10}-9100\xi^{12}) \\ & (57760\xi-39144\xi^2-309920\xi^3+134940\xi^4+536536\xi^5-202202\xi^6-469040\xi^7+242385\xi^8+200200\xi^9 \\ & -215215\xi^{10}+72800\xi^{11}-9100\xi^{12})L \\ & (55990\xi-35931\xi^2-290680\xi^3+115310\xi^4+464464\xi^5-148148\xi^6-331760\xi^7+132990\xi^8+50050\xi^9 \\ & +15015\xi^{10}-36400\xi^{11}+9100\xi^{12})L \\ & 2(35931\xi+2328\xi^2-230620\xi^3-14820\xi^4+444444\xi^5+42042\xi^6-531960\xi^7+283140\xi^8-75075\xi^9 \\ & +90090\xi^{10}-54600\xi^{11}+9100\xi^{12})\Big]\end{aligned} \qquad (3\text{-}20\,\text{o})$$

$$\left\{c_{33}\right\} = \left\{0\right\} \qquad (3\text{-}20\,\text{p})$$

$$\left\{c_{40}\right\} = \frac{(\rho A)^2 L^7}{139708800\,(EI)^2}\begin{bmatrix} 6(1784\xi-4248\xi^2+10164\xi^5-12012\xi^6+4620\xi^7-385\xi^9+77\xi^{10}) \\ (2272\xi-5352\xi^2+11088\xi^5-10164\xi^6+3465\xi^8-1540\xi^9+231\xi^{10})L \\ (2194\xi-5043\xi^2+8316\xi^5-6006\xi^6+770\xi^9-231\xi^{10})L \\ 6(1681\xi-3837\xi^2+6006\xi^5-4158\xi^6+385\xi^9-77\xi^{10}) \end{bmatrix} \qquad (3\text{-}20q)$$

$$\left\{c_{04}\right\} = \frac{L^7}{6985440000\,(EI)^4}\begin{bmatrix} 6(111\xi-111\xi^2-1540\xi^3+770\xi^4+6930\xi^5-2310\xi^6-23100\xi^7+34650\xi^8-19250\xi^9 \\ +3850\xi^{10}) \\ (6108\xi-333\xi^2-43120\xi^3+2310\xi^4+101640\xi^5-6930\xi^6-184800\xi^7+190575\xi^8-77000\xi^9+11550\xi^{10})L \\ (5442\xi+333\xi^2-33880\xi^3-2310\xi^4+60060\xi^5+6930\xi^6-46200\xi^7-17325\xi^8+38500\xi^9-11550\xi^{10})L \\ -6(111\xi-111\xi^2-1540\xi^3+770\xi^4+6930\xi^5-2310\xi^6-23100\xi^7+34650\xi^8-19250\xi^9+3850\xi^{10}) \end{bmatrix}$$

... etc (3-20r)

From Equation (3-19), since the matrix $\left[c \right]$ is the first differentiated value of the matrix $\left[a^y \right]$, we can expand the matrix $\left[c \right]$ in an infinite series in ascending powers of Ω and P, Hence

$$\left[c \right] = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \Omega^r P^s \left\{ c_{rs} \right\} \qquad (3\text{-}21)$$

Substituting the Equations (3-20a) through Equation (3-20r) into Equation (3-21), the matrix $\left[c \right]$ becomes

$$\left[c \right] = \left\{ c_{00} \right\} + P\left\{ c_{01} \right\} + P\Omega\left\{ c_{21} \right\} + P^2\left\{ c_{02} \right\} + \Omega^2\left\{ c_{20} \right\} + P^2\Omega^2\left\{ c_{22} \right\} + \ldots \qquad (3\text{-}22a)$$

$$\begin{aligned}
\text{so, } \left[c \right]^T\left[c \right] = & \left[\left\{ c_{00} \right\}^T \left\{ c_{00} \right\} \right] + P\left[\left\{ c_{00} \right\}^T \left\{ c_{01} \right\} + \left\{ c_{01} \right\}^T \left\{ c_{00} \right\} \right] \\
& + P^2 \left[\left\{ c_{00} \right\}^T \left\{ c_{02} \right\} + \left\{ c_{01} \right\}^T \left\{ c_{01} \right\} + \left\{ c_{02} \right\}^T \left\{ c_{00} \right\} \right] \\
& + \Omega^2 \left[\left\{ c_{00} \right\}^T \left\{ c_{20} \right\} + \left\{ c_{20} \right\}^T \left\{ c_{00} \right\} \right] \\
& + P\Omega^2 \left[\left\{ c_{00} \right\}^T \left\{ c_{21} \right\} + \left\{ c_{01} \right\}^T \left\{ c_{20} \right\} + \left\{ c_{20} \right\}^T \left\{ c_{01} \right\} + \left\{ c_{21} \right\}^T \left\{ c_{00} \right\} \right] \\
& + P^2\Omega^2 \left[\left\{ c_{00} \right\}^T \left\{ c_{22} \right\} + \left\{ c_{01} \right\}^T \left\{ c_{21} \right\} + \left\{ c_{02} \right\}^T \left\{ c_{20} \right\} \right. \\
& \qquad \left. + \left\{ c_{20} \right\}^T \left\{ c_{02} \right\} + \left\{ c_{21} \right\}^T \left\{ c_{01} \right\} + \left\{ c_{22} \right\}^T \left\{ c_{00} \right\} \right] \\
& + P^3 \left[\left\{ c_{00} \right\}^T \left\{ c_{03} \right\} + \left\{ c_{01} \right\}^T \left\{ c_{02} \right\} + \left\{ c_{02} \right\}^T \left\{ c_{01} \right\} + \left\{ c_{03} \right\}^T \left\{ c_{00} \right\} \right] \\
& + \ldots
\end{aligned} \qquad (3\text{-}22b)$$

Substituting Equation (3-22b) into Equation (3-18), the geometric matrices are obtained by direct integration, thus

$$\left[K_G\right] = \left[K_{G1}\right] + P\left[K_{G2}\right] + P^2\left[K_{G3}\right] + \Omega^2\left[K_{G4}\right] + P\Omega^2\left[K_{G5}\right] + P^2\Omega^2\left[K_{G6}\right] + P^3\left[K_{G7}\right] + \ldots \quad (3\text{-}23a)$$

with

$$\left[K_{G1}\right] = \begin{bmatrix} \frac{6}{5L} & \frac{1}{10} & \frac{-1}{10} & \frac{-6}{5L} \\ \frac{1}{10} & \frac{2L}{15} & \frac{L}{30} & \frac{-1}{10} \\ \frac{-1}{10} & \frac{L}{30} & \frac{2L}{15} & \frac{1}{10} \\ \frac{-6}{5L} & \frac{-1}{10} & \frac{1}{10} & \frac{6}{5L} \end{bmatrix} \quad (3\text{-}23b)$$

$$\left[K_{G2}\right] = \frac{2L}{EI} \begin{bmatrix} \frac{1}{700} & \frac{L}{1400} & \frac{-L}{1400} & \frac{-1}{700} \\ \frac{L}{1400} & \frac{11L^2}{6300} & \frac{13L^2}{12600} & \frac{-L}{1400} \\ \frac{-L}{1400} & \frac{13L^2}{12600} & \frac{11L^2}{6300} & \frac{L}{1400} \\ \frac{-1}{700} & \frac{-L}{1400} & \frac{L}{1400} & \frac{1}{700} \end{bmatrix} \quad (3\text{-}23c)$$

$$\left[K_{G3}\right] = \frac{L^3}{126000(EI)^2} \begin{bmatrix} 6 & 3L & -3L & -6 \\ 3L & 14L^2 & 11L^2 & -3L \\ -3L & 11L^2 & 14L^2 & 3L \\ -6 & -3L & 3L & 6 \end{bmatrix} \quad (3\text{-}23d)$$

$$\left[K_{G4}\right] = \frac{\rho A L^3}{EI}\begin{bmatrix} \frac{1}{3150} & \frac{L}{1260} & \frac{L}{1680} & \frac{-1}{1350} \\ \frac{L}{1260} & \frac{L^2}{3150} & \frac{L^2}{3600} & \frac{L}{1680} \\ \frac{L}{1680} & \frac{L^2}{3600} & \frac{L^2}{3150} & \frac{L}{1260} \\ \frac{-1}{1350} & \frac{L}{1680} & \frac{L}{1260} & \frac{1}{3150} \end{bmatrix} \qquad (3\text{-}23e)$$

$$\left[K_{G5}\right] = \frac{\rho A L^5}{1000(EI)^2}\begin{bmatrix} \frac{38}{4851} & \frac{173L}{4851} & \frac{887L}{29106} & \frac{-38}{4851} \\ \frac{173L}{4851} & \frac{298L^2}{14553} & \frac{559L^2}{29106} & \frac{887L}{29106} \\ \frac{887L}{29106} & \frac{559L^2}{29106} & \frac{298L^2}{14553} & \frac{173L}{4851} \\ \frac{-38}{4851} & \frac{887L}{29106} & \frac{173L}{4851} & \frac{38}{4851} \end{bmatrix} \qquad (3\text{-}23f)$$

$$\left[K_{G6}\right] = \frac{\rho A L^7}{3024000(EI)^3}\begin{bmatrix} -\frac{48632}{5005} & \frac{21128L}{2145} & \frac{62719L}{4290} & \frac{23152}{5005} \\ \frac{21128L}{2145} & \frac{2888L^2}{1001} & \frac{509L^2}{182} & \frac{5133L}{1430} \\ \frac{62719L}{4290} & \frac{509L^2}{182} & \frac{2888L^2}{1001} & \frac{2796L}{715} \\ \frac{23152}{5005} & \frac{5133L}{1430} & \frac{2796L}{715} & \frac{2328}{5005} \end{bmatrix} \qquad (3\text{-}23g)$$

$$\left[K_{G7}\right] = \frac{L^5}{291060000(EI)^3}\begin{bmatrix} 222 & 111L & -111L & -222 \\ 111L & 1018L^2 & 907L^2 & -111L \\ -111L & 907L^2 & 1018L^2 & 111L \\ -222 & -111L & 111L & 222 \end{bmatrix} \qquad (3\text{-}23h)$$

3.8 <u>General Stiffness Matrix</u>

For the dynamic analysis of a beam-column, the general stiffness matrix possesses a form of a linear combination of bending stiffness, mass, and geometric stiffness matrices as

$$\left[S_{BE}\right]^{(BC)} = \left[K\right] - \Omega^2\left[M\right] - P\left[K_G\right] \qquad (3\text{-}24)$$

These matrices are used in the finite element method of dynamic and stability problems for frame structures. Substituting the Equation (3-14a), (3-17a), and (3-23a) into Equation (3-24), arranging the final terms of the similar powers of Ω and P, the general stiffness matrix $\left[S\right]$ becomes

$$\begin{aligned}\left[S_{BE}\right]^{(BC)} = {} & \left[K_0\right] + P\left[\left[K_1^b\right]-\left[K_{G1}\right]\right] + P^2\left[\left[K_2^b\right]-\left[K_{G2}\right]\right] + \Omega^2\left[\left[K_3^b\right]-\left[M_0\right]\right] \\ & + P\Omega^2\left[\left[K_4^b\right]-\left[K_1^i\right]-\left[K_{G4}\right]\right] + P^2\Omega^2\left[\left[K_5^b\right]-\left[K_2^i\right]-\left[K_{G5}\right]\right] \\ & + P^3\left[\left[K_6^b\right]-\left[K_{G3}\right]\right] + P^3\Omega^2\left[\left[K_7^b\right]-\left[K_3^i\right]-\left[K_{G6}\right]\right] + P^4\left[\left[K_8^b\right]-\left[K_{G7}\right]\right] \\ & + \Omega^4\left[\left[K_9^b\right]-\left[M_2\right]\right] + \ldots \text{etc}\end{aligned} \qquad (3\text{-}25)$$

Substituting the Equations (3-14b) through Equation (3-14f), Equations (3-17b) through Equation (3-17k) and Equations (3-23b) through Equation (3-23h) into Equation (3-25), the final form of the general stiffness matrix becomes

$$\left[S_{BE}\right]^{(BC)} = \left[K_0\right] - \left[G_0\right]P - \left[M_0\right]\Omega^2 - \left[A_1\right]P\Omega^2 - \left[G_1\right]P^2 - \left[M_1\right]\Omega^4$$

$$- \left[A_2\right]P^2\Omega^2 - \left[G_2\right]P^3 - \left[A_3\right]P^3\Omega^2 - \left[G_3\right]P^4 - \ldots \quad (3\text{-}26a)$$

with

$$\left[K_0\right] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -6L & -12 \\ 6L & 4L^2 & -2L^2 & -6L \\ -6L & -2L^2 & 4L^2 & 6L \\ -12 & -6L & 6L & 12 \end{bmatrix} \quad (3\text{-}26b)$$

$$\left[G_0\right] = \begin{bmatrix} \frac{6}{5L} & \frac{1}{10} & \frac{-1}{10} & \frac{-6}{5L} \\ \frac{1}{10} & \frac{2L}{15} & \frac{L}{30} & \frac{-1}{10} \\ \frac{-1}{10} & \frac{L}{30} & \frac{2L}{15} & \frac{1}{10} \\ \frac{-6}{5L} & \frac{-1}{10} & \frac{1}{10} & \frac{6}{5L} \end{bmatrix} \quad (3\text{-}26c)$$

$$\left[M_0\right] = \frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 13L & 54 \\ 22L & 4L^2 & 3L^2 & 13L \\ 13L & 3L^2 & 4L^2 & 22L \\ 54 & 13L & 22L & 156 \end{bmatrix} \quad (3\text{-}26d)$$

$$\left[A_1\right] = \frac{\rho A L^3}{EI}\begin{bmatrix} \frac{1}{3150} & \frac{L}{1260} & \frac{L}{1680} & \frac{-1}{1350} \\ \frac{L}{1260} & \frac{L^2}{3150} & \frac{L^2}{3600} & \frac{L}{1680} \\ \frac{L}{1680} & \frac{L^2}{3600} & \frac{L^2}{3150} & \frac{L}{1260} \\ \frac{-1}{1350} & \frac{L}{1680} & \frac{L}{1260} & \frac{1}{3150} \end{bmatrix} \qquad (3\text{-}26e)$$

$$\left[G_1\right] = \begin{bmatrix} \frac{1}{700} & \frac{L}{1400} & \frac{-L}{1400} & \frac{-1}{700} \\ \frac{L}{1400} & \frac{11L^2}{6300} & \frac{13L^2}{12600} & \frac{-L}{1400} \\ \frac{-L}{1400} & \frac{13L^2}{12600} & \frac{11L^2}{6300} & \frac{L}{1400} \\ \frac{-1}{700} & \frac{-L}{1400} & \frac{L}{1400} & \frac{1}{700} \end{bmatrix} \qquad (3\text{-}26f)$$

$$\left[M_1\right] = \frac{(\rho A)^2 L^5}{EI}\begin{bmatrix} \frac{59}{80850} & \frac{223L}{1455300} & \frac{1681L}{11642400} & \frac{1279}{1940400} \\ \frac{223L}{1455300} & \frac{71L^2}{2182950} & \frac{1097L^2}{34927200} & \frac{1681L}{11642400} \\ \frac{1681L}{11642400} & \frac{1097L^2}{34927200} & \frac{71L^2}{2182950} & \frac{223L}{1455300} \\ \frac{1279}{1940400} & \frac{1681L}{11642400} & \frac{223L}{1455300} & \frac{59}{80850} \end{bmatrix}$$

(3-26g)

$$[A_2] = \frac{\rho A L^5}{1000(EI)^2}\begin{bmatrix} \frac{19}{4851} & \frac{173L}{9702} & \frac{887L}{58212} & \frac{-19}{4851} \\ \frac{173L}{9702} & \frac{149L^2}{14553} & \frac{559L^2}{58212} & \frac{887L}{58212} \\ \frac{887L}{58212} & \frac{559L^2}{58212} & \frac{149L^2}{14553} & \frac{173L}{9702} \\ \frac{-19}{4851} & \frac{887L}{58212} & \frac{173L}{9702} & \frac{19}{4851} \end{bmatrix} \quad (3\text{-}26h)$$

$$[G_2] = \frac{L^3}{1000(EI)^2}\begin{bmatrix} \frac{1}{63} & \frac{L}{126} & \frac{-L}{126} & \frac{-1}{63} \\ \frac{L}{126} & \frac{L^2}{27} & \frac{11L^2}{378} & \frac{-L}{126} \\ \frac{-L}{126} & \frac{11L^2}{378} & \frac{L^2}{27} & \frac{L}{126} \\ \frac{-1}{63} & \frac{-L}{126} & \frac{L}{126} & \frac{1}{63} \end{bmatrix} \quad (3\text{-}26i)$$

$$[A_3] = \frac{\rho A L^7}{1000(EI)^3}\begin{bmatrix} \frac{97}{1891890} & \frac{223L}{540540} & \frac{1711L}{4324320} & \frac{-97}{1891890} \\ \frac{223L}{540540} & \frac{361L^2}{1135134} & \frac{509L^2}{1651104} & \frac{1711L}{4324320} \\ \frac{1711L}{4324320} & \frac{509L^2}{1651104} & \frac{361L^2}{1135134} & \frac{223L}{540540} \\ \frac{-97}{1891890} & \frac{1711L}{4324320} & \frac{223L}{540540} & \frac{97}{1891890} \end{bmatrix}$$

(3-26j)

$$
\left[G_3\right] = \frac{L^5}{1000(EI)^3}
\begin{bmatrix}
\frac{111}{582120} & \frac{111L}{1164240} & \frac{-111L}{1164240} & \frac{-37}{194040} \\
\frac{111L}{1164240} & \frac{509L^2}{582120} & \frac{907L^2}{1164240} & \frac{-111L}{1164240} \\
\frac{-111L}{1164240} & \frac{907L^2}{1164240} & \frac{509L^2}{582120} & \frac{111L}{1164240} \\
\frac{-37}{194040} & \frac{-111L}{1164240} & \frac{111L}{1164240} & \frac{111}{582120}
\end{bmatrix}
\tag{3-26k}
$$

CHAPTER IV

TIMOSHENKO BEAM THEORY

4.1 General Theory

The general Bernoulli-Euler theory of beams does not include the effect of shear deformation. The Timoshenko theory of beams includes the added effects of both transverse shear and rotatory inertia. Let us accordingly consider shear deformation alone. We assume that line elements such as ab in Figure 4.1 normal to the centerline of the beam in the undeformed state move only in the vertical direction and also remain vertical during deformation.

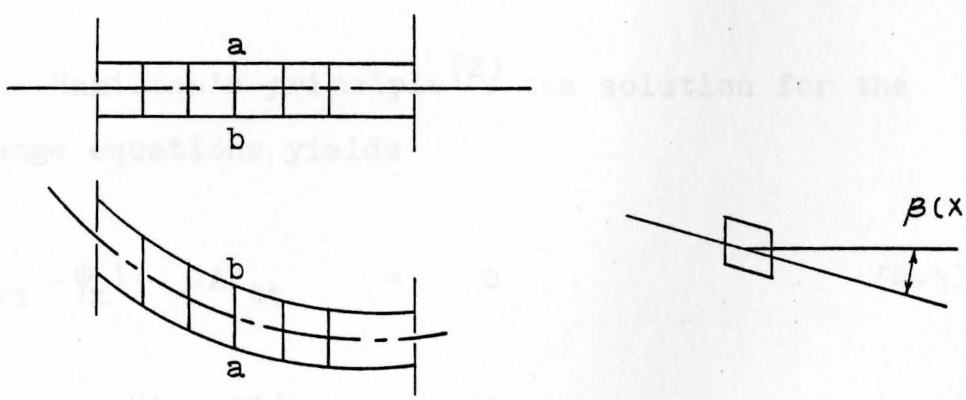

Figure 4.1 Line Element in Shear Beam

Line elements tangent to the centerline meanwhile undergo a shear rotation $\beta(x)$ as has been shown in an exaggerated manner in the diagram. The total slope $\frac{dw}{dx}$ of the centerline resulting respectively from shear deformation and bending deformation is given as the sum of two parts. Hence,

$$\frac{dV}{dx} = \psi(x) + \beta(x) \qquad (4\text{-}1)$$

where $\psi(x)$ = rotation of line elements due to bending only

and $\beta(x)$ = rotation of line elements due to shear only

For no axial force we assume the following displacement field for such beams

$$u_x(x,y,z,t) = -y\psi(x,t) \qquad (4\text{-}2a)$$

$$u_z(x,y,z,t) = 0 \qquad (4\text{-}2b)$$

$$u_y(x,y,z,t) = V(x,t) \qquad (4\text{-}2c)$$

Applying the Hamilton's principle[2], the solution for the Euler-Lagrange equations yields

$$kGA(V_{xx} - \psi_x) - \rho AV_{tt} = 0 \qquad (4\text{-}3)$$

$$EI\psi_{xx} + kGA(V_x - \psi) - \rho I\psi_{tt} = 0 \qquad (4\text{-}4)$$

4.2 Solution of the Differential Equation

Recall the Przemieniecki method from Chapter II, we expand the continuous displacements in ascending power series of frequency, so that

$$\psi = \sum_{r=0}^{\infty} \omega^r \left\{a_{rx}\right\}\left\{q\right\} e^{i\omega t} \tag{4-5}$$

$$V = \sum_{r=0}^{\infty} \omega^r \left\{a_{ry}\right\}\left\{q\right\} e^{i\omega t} \tag{4-6}$$

Since the axial force P is not present a single infinite series is assumed for each variable. Substituting Equations (4-5) and (4-6) into Equations (4-3) and (4-4) respectively, yields

$$-kGA\left[\sum_{r=0}^{\infty}\omega^r\left\{a''_{ry}\right\}\left\{q\right\} e^{i\omega t} - \sum_{r=0}^{\infty}\omega^r\left\{a'_{rx}\right\}\left\{q\right\} e^{i\omega t}\right]$$

$$-\rho A\sum_{r=0}^{\infty}\omega^{(2+r)}\left\{a_{ry}\right\}\left\{q\right\} e^{i\omega t} = 0 \tag{4-7}$$

$$EI\sum_{r=0}^{\infty}\omega^r\left\{a''_{rx}\right\}\left\{q\right\} e^{i\omega t} + kGA\left[\sum_{r=0}^{\infty}\omega^r\left\{a'_{ry}\right\}\left\{q\right\} e^{i\omega t} - \sum_{r=0}^{\infty}\omega^r\left\{a_{rx}\right\}\left\{q\right\} e^{i\omega t}\right]$$

$$+\rho I\sum_{r=0}^{\infty}\omega^{(2+r)}\left\{a_{rx}\right\}\left\{q\right\} e^{i\omega t} = 0 \tag{4-8}$$

Equating to zero coefficients of the similar powers of ω in Equation (4-7) and Equation (4-8), the following equations are obtained :

$$\left\{a''_{0y}\right\} - \left\{a'_{0x}\right\} = 0 \tag{4-9a}$$

and

$$EI\left\{a''_{0x}\right\} + kGA\left[\left\{a'_{0y}\right\} - \left\{a_{0x}\right\}\right] = 0 \tag{4-9b}$$

$$\left\{a''_{1y}\right\} - \left\{a'_{1x}\right\} = 0 \tag{4-9c}$$

and

$$EI\left\{a''_{1x}\right\} + kGA\left[\left\{a'_{1y}\right\} - \left\{a_{1x}\right\}\right] = 0 \tag{4-9d}$$

$$\left.\begin{aligned} \rho A\left\{\ddot{a}_{0y}\right\} + kGA\left[\left\{a''_{2y}\right\}-\left\{a'_{2x}\right\}\right] &= 0 \qquad (4\text{-}9e)\\ \text{and} \quad \rho I\left\{\ddot{a}_{0x}\right\} + EI\left\{a''_{2x}\right\} + kGA\left[\left\{a'_{2y}\right\}-\left\{a_{2x}\right\}\right] &= 0 \qquad (4\text{-}9f)\\ &\vdots \end{aligned}\right\}$$

Equations (4-9a) through Equation (4-9f) are solved by direct integration. For Equation (4-9a) and Equation (4-9b), the following boundary conditions are applied

at $x = 0$ $\quad V(0) = V_1 \quad \psi(0) = \theta_1$

at $x = L$ $\quad V(L) = V_2 \quad \psi(L) = -\theta_2$

For the remaining coupled equations containing terms $\left\{a_{1x}\right\}$, $\left\{a_{1y}\right\}$, $\left\{a_{2x}\right\}$, $\left\{a_{2y}\right\}$ ... displacement and slope must all vanish at $x=0$ and L. Thus,

$$\left\{a_{0x}\right\} = \frac{1}{(1+\phi)}\left[\frac{6}{L}\xi(\xi-1) \quad (3\xi^2-(4+\phi)\xi+(1+\phi)) \quad (-3\xi^2+(2-\phi)\xi) \quad \frac{-6}{L}\xi(\xi-1)\right] \qquad (4\text{-}10a)$$

$$\left\{a_{0y}\right\} = \frac{1}{(1+\phi)}\left[(2\xi^3-3\xi^2-\phi\xi+(1+\phi)) \quad (\xi^3-(2+\frac{\phi}{2})\xi^2+(1+\frac{\phi}{2})\xi)L \quad (-\xi^3+(1-\frac{\phi}{2})\xi^2+\frac{\phi}{2}\xi)L \quad (-2\xi^3+3\xi^2+\phi\xi)\right] \qquad (4\text{-}10b)$$

$$\left\{a_{1x}\right\} = \left\{0\right\} \qquad (4\text{-}10c)$$

$$\left\{a_{1y}\right\} = \left\{0\right\} \qquad (4\text{-}10d)$$

$$\left\{a_{2x}\right\} = \frac{1}{(1+\phi)^2}\left[\ \{F\}\ \{H\}\ \{N\}\ \{O\}\ \right] \qquad (4\text{-}10e)$$

with

$$\{F\} = \left[\left\{\frac{\rho A}{EI}\frac{L^3}{6}\left(\frac{11}{35}+\frac{11\phi}{20}+\frac{\phi^2}{4}\right)+\frac{\rho I}{EI}\frac{L}{10}(1-5\phi)\right\}\xi\right.$$
$$-\left\{\frac{\rho A}{EI}\frac{L^3}{10}\left(\frac{13}{7}+\frac{7\phi}{2}+\frac{5\phi^2}{3}\right)+\frac{\rho I}{EI}\frac{3L}{5}\right\}\xi^2+(1+\phi)\left\{\frac{\rho A}{EI}\frac{L^3}{6}(1+\phi)+\frac{\rho I}{EI}L\right\}\xi^3$$
$$\left.-(1+\phi)\left\{\frac{\rho A}{EI}\frac{\phi L^3}{24}+\frac{\rho I}{EI}\frac{L}{2}\right\}\xi^4-\frac{\rho A}{EI}\frac{L^3}{20}(1+\phi)\xi^5+\frac{\rho A}{EI}\frac{L^3}{60}(1+\phi)\xi^6\right]$$

$$\{H\} = \left[\left\{\frac{\rho A}{EI}\left(\frac{L^4}{30}\ \frac{2}{7}+\frac{\phi}{2}+\frac{\phi^2}{4}\right)+\frac{\rho I}{EI}\left(\frac{L^2}{2}\ \frac{4}{15}+\frac{\phi}{3}+\frac{2\phi^2}{3}\right)\right\}\xi\right.$$
$$-\left\{\frac{\rho A}{EI}\frac{L^4}{12}\left(\frac{11}{35}+\frac{11\phi}{20}+\frac{\phi^2}{4}\right)+\frac{\rho I}{EI}\frac{L^2}{2}\left(\frac{11}{10}+\frac{3\phi}{2}+\phi^2\right)\right\}\xi^2$$
$$+\frac{\rho I}{EI}\frac{L^2}{3}\left(2+\frac{\phi}{2}\right)(1+\phi)\xi^3+(1+\phi)\left\{\frac{\rho A}{EI}\frac{L^4}{24}\left(1+\frac{\phi}{2}\right)-\frac{\rho I}{EI}\frac{L^2}{4}\right\}\xi^4$$
$$\left.-\frac{\rho A}{EI}\frac{L^4}{30}\left(1+\frac{\phi}{4}\right)(1+\phi)\xi^5+\frac{\rho A}{EI}\frac{L^4}{120}(1+\phi)\xi^6\right]$$

$$\{N\} = \left[\left\{\frac{\rho A}{EI}\frac{L^4}{20}\left(\frac{1}{7}+\frac{\phi}{3}+\frac{\phi^2}{6}\right)+\frac{\rho I}{EI}\frac{L^2}{6}\left(\frac{1}{5}+\phi-\phi^2\right)\right\}\xi\right.$$
$$-\left\{\frac{\rho A}{EI}\frac{L^4}{8}\left(\frac{13}{105}+\frac{3\phi}{10}+\frac{\phi^2}{6}\right)-\frac{\rho I}{EI}\frac{L^2}{4}\left(\frac{1}{5}-\phi\right)\right\}\xi^2-\frac{\rho I}{EI}\frac{L^2}{3}\left(1-\frac{\phi}{2}\right)(1+\phi)\xi^3$$
$$+(1+\phi)\left(\frac{\rho A}{EI}\frac{\phi L^4}{48}+\frac{\rho I}{EI}\frac{L^2}{4}\right)\xi^4+\frac{\rho A}{EI}\frac{L^4}{60}(1+\phi)\left(1-\frac{\phi}{2}\right)\xi^5$$
$$\left.-\frac{\rho A}{EI}\frac{L^4}{120}(1+\phi)\xi^6\right]$$

$$\{O\} = \left[\left\{\frac{\rho A}{EI}\frac{L^3}{4}\left(\frac{13}{105}+\frac{3\phi}{10}+\frac{\phi^2}{6}\right)-\frac{\rho I}{EI}\frac{L}{2}\left(\frac{1}{5}-\phi\right)\right\}\xi\right.$$
$$-\left\{\frac{\rho A}{EI}\frac{3L^3}{2}\left(\frac{3}{70}+\frac{\phi}{10}+\frac{\phi^2}{18}\right)-\frac{\rho I}{EI}\frac{3L}{5}\right\}\xi^2-\frac{\rho I}{EI}L(1+\phi)\xi^3$$
$$\left.+(1+\phi)\left\{\frac{\rho A}{EI}\frac{\phi L^3}{24}+\frac{\rho I}{EI}\frac{L}{2}\right\}\xi^4+\frac{\rho A}{EI}\frac{L^3}{20}(1+\phi)\xi^5-\frac{\rho A}{EI}\frac{L^3}{60}(1+\phi)\xi^6\right]$$

$$\left\{a_{2y}\right\} = \frac{1}{(1+\phi)^2}\left[\ \{Q\}\ \{R\}\ \{S\}\ \{T\}\ \right]\ \ldots\ \text{etc} \qquad (4\text{-}10f)$$

$$\left\{ Q \right\} = \Bigg[\left\{ \frac{\rho A}{EI} \frac{L^4}{12} \left(\frac{13\phi}{35} + \frac{7\phi^2}{10} + \frac{\phi^3}{3} \right) + \frac{\rho I}{EI} \frac{\phi L^2}{10} \right\} \xi$$

$$+ \left\{ \frac{\rho A}{EI} \frac{L^4}{4} \left(\frac{11}{105} + \frac{\phi}{60} - \frac{\phi^2}{4} - \frac{\phi^3}{6} \right) + \frac{\rho I}{EI} \frac{L^2}{4} \left(\frac{1}{5} - \phi \right) \right\} \xi^2$$

$$- \left\{ \frac{\rho A}{EI} \frac{L^4}{6} \left(\frac{13}{35} + \frac{7\phi}{10} + \frac{\phi^2}{4} - \frac{\phi^3}{12} \right) + \frac{\rho I}{EI} \frac{L^2}{5} \right\} \xi^3$$

$$+ (1+\phi) \left\{ \frac{\rho A}{EI} \frac{L^4}{24} \left(1 + \frac{3\phi}{2} \right) + \frac{\rho I}{EI} \frac{L^2}{4} \right\} \xi^4 - (1+\phi) \left\{ \frac{\rho A}{EI} \frac{\phi L^4}{60} + \frac{\rho I}{EI} \frac{L^2}{10} \right\} \xi^5$$

$$- \frac{\rho A}{EI} \frac{L^4}{120} (1+\phi) \xi^6 + \frac{\rho A}{EI} \frac{L^4}{420} (1+\phi) \xi^7 \Bigg]$$

$$\left\{ R \right\} = \Bigg[\left\{ \frac{\rho A}{EI} \frac{L^5}{72} \left(\frac{11\phi}{35} + \frac{11\phi^2}{20} + \frac{\phi^3}{4} \right) + \frac{\rho I}{EI} \frac{L^3}{12} \left(\frac{\phi}{10} - \frac{\phi^2}{2} \right) \right\} \xi$$

$$+ \left\{ \frac{\rho A}{EI} \frac{L^5}{60} \left(\frac{2}{7} + \frac{\phi}{2} + \frac{\phi^2}{4} \right) + \frac{\rho I}{EI} \frac{L^3}{4} \left(\frac{4}{15} + \frac{\phi}{3} + \frac{2\phi^2}{3} \right) \right\} \xi^2$$

$$- \left\{ \frac{\rho A}{EI} \frac{L^5}{72} \left(\frac{22}{35} + \frac{21\phi}{10} + 2\phi^2 + \frac{\phi^3}{2} \right) + \frac{\rho I}{EI} \frac{L^3}{6} \left(\frac{11}{10} + \frac{3\phi}{2} + \phi^2 \right) \right\} \xi^3$$

$$+ (1+\phi) \left\{ \frac{\rho A}{EI} \frac{L^5}{72} \left(\phi + \frac{\phi^2}{4} \right) + \frac{\rho I}{EI} \frac{L^3}{6} \left(1 + \frac{\phi}{4} \right) \right\} \xi^4 + (1+\phi) \left\{ \frac{\rho A}{EI} \frac{L^5}{120} - \frac{\rho I}{EI} \frac{L^3}{20} \right\} \xi^5$$

$$- \frac{\rho A}{EI} \frac{L^5}{180} \left(1 + \frac{\phi}{4} \right) (1+\phi) \xi^6 + \frac{\rho A}{EI} \frac{L^5}{840} (1+\phi) \xi^7 \Bigg]$$

$$\left\{ S \right\} = \Bigg[\left\{ \frac{\rho A}{EI} \frac{L^5}{48} \left(\frac{13\phi}{105} + \frac{3\phi^2}{10} + \frac{\phi^3}{6} \right) - \frac{\rho I}{EI} \frac{L^3}{24} \left(\frac{\phi}{5} - \phi^2 \right) \right\} \xi$$

$$+ \left\{ \frac{\rho A}{EI} \frac{L^5}{40} \left(\frac{1}{7} + \frac{\phi}{3} + \frac{\phi^2}{6} \right) + \frac{\rho I}{EI} \frac{L^3}{12} \left(\frac{1}{5} + \phi - \phi^2 \right) \right\} \xi^2$$

$$- \left\{ \frac{\rho A}{EI} \frac{L^5}{24} \left(\frac{13}{105} + \frac{3\phi}{10} + \frac{\phi^2}{3} + \frac{\phi^3}{6} \right) - \frac{\rho I}{EI} \frac{L^3}{12} \left(\frac{1}{5} - \phi \right) \right\} \xi^3$$

$$- (1+\phi) \left\{ \frac{\rho A}{EI} \frac{L^5}{144} \left(\phi - \frac{\phi}{2} \right) + \frac{\rho I}{EI} \frac{L^3}{12} \left(1 - \frac{\phi}{2} \right) \right\} \xi^4 + (1+\phi) \left\{ \frac{\rho A}{EI} \frac{L^5}{120} + \frac{\rho I}{EI} \frac{L^3}{20} \right\} \xi^5$$

$$+ \frac{\rho A}{EI} \frac{L^5}{360} \left(1 - \frac{\phi}{2} \right) (1+\phi) \xi^6 - \frac{\rho A}{EI} \frac{L^5}{840} (1+\phi) \xi^7 \Bigg]$$

$$\left\{ T \right\} = \Bigg[\left\{ \frac{\rho A}{EI} \frac{L^4}{4} \left(\frac{3\phi}{70} + \frac{\phi^2}{10} + \frac{\phi^3}{18} \right) - \frac{\rho I}{EI} \frac{\phi L^2}{10} \right\} \xi$$

$$+ \left\{ \frac{\rho A}{EI} \frac{L^4}{8} \left(\frac{13}{105} + \frac{3\phi}{10} + \frac{\phi^2}{6} \right) - \frac{\rho I}{EI} \frac{L^2}{4} \left(\frac{1}{5} - \phi \right) \right\} \xi^2$$

$$-\left\{\frac{\rho A}{EI}\frac{L^4}{4}\left(\frac{3}{35}+\frac{\phi}{5}+\frac{\phi^2}{6}+\frac{\phi^3}{18}\right)-\frac{\rho I}{EI}\frac{L^2}{5}\right\}\xi^3-(1+\phi)\left\{\frac{\rho A}{EI}\frac{\phi L^4}{48}+\frac{\rho I}{EI}\frac{L^2}{4}\right\}\xi^4$$

$$+(1+\phi)\left\{\frac{\rho A}{EI}\frac{\phi L^4}{60}+\frac{\rho I}{EI}\frac{L^2}{10}\right\}\xi^5+\frac{\rho A}{EI}\frac{L^4}{120}(1+\phi)\xi^6-\frac{\rho A}{EI}\frac{L^4}{420}(1+\phi)\xi^7\Bigg]$$

where $\xi = \frac{x}{L}$, and $\phi = \frac{12EI}{L^2(kGA)}$

4.3 Mass Matrix

From Equation (4-2a), (4-2b), and (4-2c), arranging the displacements in the matrix form and neglecting the displacement in Z-direction, the nodal displacements are in the form

$$\{u\} = \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \begin{bmatrix} -y\phi \\ V \end{bmatrix}\{U\} = [A]\begin{Bmatrix} V_1 \\ \theta_1 \\ \theta_2 \\ V_2 \end{Bmatrix} \qquad (4\text{-}11)$$

From Equation (4-5) and (4-6), the matrix $[A]$ is expanded into infinite series in ascending powers of as

$$[A] = \sum_{r=0}^{\infty} \omega^r [A_r] \qquad (4\text{-}12)$$

Substituting the Equations(4-10a) through Equation(4-10f) into Equation (4-12), the final form of the matrix $[A]$ becomes

$$[A] = [A_0] + \omega^2[A_2] + \omega^4[A_4] + \ldots \qquad (4\text{-}13)$$

$$\text{so}\quad [A]^T[A] = \Big[[A_0]^T[A_0]\Big] + \omega^2\Big[[A_0]^T[A_2] + [A_2]^T[A_0]\Big]$$

$$+\omega^4\Big[[A_0]^T[A_4] + [A_2]^T[A_2] + [A_4]^T[A_0]\Big] + \ldots \qquad (4\text{-}14)$$

with

$$[A_0] = \frac{1}{(1+\phi)}\begin{bmatrix} -\frac{6}{L}y(\xi^2-\xi) & -y(3\xi^2-(4+\phi)\xi+(1+\phi)) & y(3\xi^2-2\xi+\phi\xi) & \frac{6}{L}y(\xi^2-\xi) \\ (2\xi^3-3\xi^2-\phi\xi+(1+\phi)) & (\xi^3-(2+\frac{\phi}{2})\xi^2+(1+\frac{\phi}{2})\xi)L & (-\xi^3+(1-\frac{\phi}{2})\xi^2+\frac{\phi}{2}\xi)L & (-2\xi^3+3\xi^2+\phi\xi) \end{bmatrix} \qquad (4\text{-}15a)$$

$$[A_2] = \frac{1}{(1+\phi)^2}\begin{bmatrix} -y\{F\} & -y\{H\} & -y\{N\} & -y\{O\} \\ \{Q\} & \{R\} & \{S\} & \{T\} \end{bmatrix} \ldots \text{ etc} \qquad (4\text{-}15b)$$

From the previous work it follows that

$$[M] = \int_V \rho [A]^T[A]\, dV \qquad (4\text{-}16)$$

Substituting the Equation (4-14) into Equation (4-16), the equivalent mass matrices are in the form

$$[M]^{(T)} = [M_0]^{(T)} + \omega^2[M_2]^{(T)} + \omega^4[M_4]^{(T)} + \ldots \qquad (4\text{-}17)$$

The first two component matrices appearing in the expansion of Equation (4-17) are calculated as

$$
\left[M_0\right]^{(T)} = \frac{\rho A L}{(1+\phi)^2}
\begin{bmatrix}
\left(\frac{13}{35}+\frac{7\phi}{10}+\frac{\phi^2}{3}\right) & \left(\frac{11}{210}+\frac{11\phi}{120}+\frac{\phi^2}{24}\right)L & \left(\frac{13}{420}+\frac{3\phi}{40}+\frac{\phi^2}{24}\right)L & \left(\frac{9}{70}+\frac{3\phi}{10}+\frac{\phi^2}{6}\right) \\
\left(\frac{11}{210}+\frac{11\phi}{120}+\frac{\phi^2}{24}\right)L & \left(\frac{1}{105}+\frac{\phi}{60}+\frac{\phi^2}{120}\right)L^2 & \left(\frac{1}{140}+\frac{\phi}{60}+\frac{\phi^2}{120}\right)L^2 & \left(\frac{13}{420}+\frac{3\phi}{40}+\frac{\phi^2}{24}\right)L \\
\left(\frac{13}{420}+\frac{3\phi}{40}+\frac{\phi^2}{24}\right)L & \left(\frac{1}{140}+\frac{\phi}{60}+\frac{\phi^2}{120}\right)L^2 & \left(\frac{1}{105}+\frac{\phi}{60}+\frac{\phi^2}{120}\right)L^2 & \left(\frac{11}{210}+\frac{11\phi}{120}+\frac{\phi^2}{24}\right)L \\
\left(\frac{9}{70}+\frac{3\phi}{10}+\frac{\phi^2}{6}\right) & \left(\frac{13}{420}+\frac{3\phi}{40}+\frac{\phi^2}{24}\right)L & \left(\frac{11}{210}+\frac{11\phi}{120}+\frac{\phi^2}{24}\right)L & \left(\frac{13}{35}+\frac{7\phi}{10}+\frac{\phi^2}{3}\right)
\end{bmatrix}
$$

$$
+ \frac{\rho I}{(1+\phi)^2 L}
\begin{bmatrix}
\frac{6}{5} & \left(\frac{1}{10}-\frac{\phi}{2}\right)L & -\left(\frac{1}{10}-\frac{\phi}{2}\right)L & -\frac{6}{5} \\
\left(\frac{1}{10}-\frac{\phi}{2}\right)L & \left(\frac{2}{15}+\frac{\phi}{6}+\frac{\phi^2}{3}\right)L^2 & \left(\frac{1}{30}+\frac{\phi}{6}-\frac{\phi^2}{6}\right)L^2 & -\left(\frac{1}{10}-\frac{\phi}{2}\right)L \\
-\left(\frac{1}{10}-\frac{\phi}{2}\right)L & \left(\frac{1}{30}+\frac{\phi}{6}-\frac{\phi^2}{6}\right)L^2 & \left(\frac{2}{15}+\frac{\phi}{6}+\frac{\phi^2}{3}\right)L^2 & \left(\frac{1}{10}-\frac{\phi}{2}\right)L \\
-\frac{6}{5} & -\left(\frac{1}{10}-\frac{\phi}{2}\right)L & \left(\frac{1}{10}-\frac{\phi}{2}\right)L & \frac{6}{5}
\end{bmatrix}
\quad (4\text{-}18)
$$

$$
\left[M_2\right]^{(T)} = \frac{(\rho AL)^2 L^3}{(1+\phi)^3 EI}
\begin{bmatrix}
\left(\frac{59}{80850}+\frac{47\phi}{7700}+\frac{689\phi^2}{50400}+\frac{121\phi^3}{10080}+\frac{\phi^4}{270}\right) & & & \\
\left(\frac{223}{145530}+\frac{499\phi}{415800}+\frac{113\phi^2}{43200}+\frac{137\phi^3}{60480}+\frac{\phi^4}{1440}\right)L & \left(\frac{71}{2182950}+\frac{101\phi}{415800}+\frac{79\phi^2}{151200}+\frac{823\phi^3}{181440}+\frac{17\phi^4}{120960}\right)L^2 & & \\
\left(\frac{1681}{11642400}+\frac{257\phi}{237600}+\frac{739\phi^2}{302400}+\frac{19\phi^3}{8640}+\frac{\phi^4}{1440}\right)L & \left(\frac{1097}{34927200}-\frac{13\phi}{110880}-\frac{31\phi^2}{120960}+\frac{569\phi^3}{907200}+\frac{17\phi^4}{120960}\right)L^2 & & \text{Symmetry} \\
\left(\frac{1279}{1940400}+\frac{347\phi}{69300}+\frac{571\phi^2}{50400}+\frac{103\phi^3}{10080}+\frac{7\phi^4}{2160}\right) & \left(\frac{1681}{11642400}+\frac{257\phi}{237600}+\frac{739\phi^2}{302400}+\frac{19\phi^3}{8640}+\frac{\phi^4}{1440}\right)L & & \\
\left(\frac{71}{2182950}+\frac{101\phi}{415800}+\frac{79\phi^2}{151200}+\frac{823\phi^3}{1814400}+\frac{17\phi^4}{120960}\right)L^2 & & & \\
\left(\frac{223}{1455300}+\frac{499\phi}{415800}+\frac{113\phi^2}{43200}+\frac{137\phi^3}{60480}+\frac{\phi^4}{1440}\right)L & & \left(\frac{59}{80850}+\frac{47\phi}{7700}+\frac{689\phi^2}{50400}+\frac{121\phi^3}{10080}+\frac{\phi^4}{270}\right) &
\end{bmatrix}
$$

$$
+ \frac{(\rho I)^2 L}{(1+\phi)^3 EI}
\begin{bmatrix}
\left(\frac{1}{350}+\frac{17\phi}{70}\right) & & & \\
\left(\frac{1}{700}+\frac{3\phi}{140}-\frac{\phi^2}{10}\right)L & \left(\frac{11}{3150}+\frac{3\phi}{280}-\frac{\phi^2}{120}+\frac{2\phi^3}{45}\right)L^2 & \text{Symmetry} & \\
-\left(\frac{1}{700}+\frac{3\phi}{140}-\frac{\phi^2}{10}\right)L & \left(\frac{13}{6300}+\frac{\phi}{168}+\frac{\phi^2}{40}-\frac{7\phi^3}{180}\right)L^2 & \left(\frac{11}{3150}+\frac{3\phi}{280}-\frac{\phi^2}{120}+\frac{2\phi^3}{45}\right)L^2 & \\
-\left(\frac{1}{350}+\frac{17\phi}{70}\right) & -\left(\frac{1}{700}+\frac{3\phi}{140}-\frac{\phi^2}{10}\right)L & \left(\frac{1}{700}+\frac{3\phi}{140}-\frac{\phi^2}{10}\right)L & \left(\frac{1}{350}+\frac{17\phi}{70}\right)
\end{bmatrix}
$$

$$
+\frac{(\rho A)(\rho I)L^3}{(1+\phi)^3 EI}
\begin{bmatrix}
\left(\frac{1}{1575}+\frac{61\phi}{6300}+\frac{\phi^2}{140}\right) & & & \\
\left(\frac{1}{630}+\frac{67\phi}{12600}+\frac{\phi^2}{360}\right)L & \left(\frac{1}{1575}+\frac{\phi}{525}+\frac{\phi^2}{720}+\frac{\phi^3}{1680}\right)L^2 & & \text{Symmetry} \\
\left(\frac{1}{840}+\frac{15\phi}{6300}+\frac{\phi^2}{180}+\frac{\phi^3}{360}\right)L & \left(\frac{1}{1800}+\frac{\phi}{600}+\frac{11\phi^2}{5040}+\frac{\phi^3}{1680}\right)L^2 & \left(\frac{1}{1575}+\frac{\phi}{525}+\frac{\phi^2}{720}+\frac{\phi^3}{1680}\right)L^2 & \\
-\left(\frac{1}{1575}+\frac{61\phi}{6300}+\frac{\phi^2}{140}\right) & \left(\frac{1}{840}+\frac{19\phi}{6300}+\frac{\phi^2}{180}+\frac{\phi^3}{360}\right)L & \left(\frac{1}{630}+\frac{67\phi}{12600}+\frac{\phi^2}{360}\right)L & \left(\frac{1}{1575}+\frac{61\phi}{6300}+\frac{\phi^2}{140}\right)
\end{bmatrix}
\quad (4\text{-}19)
$$

4.4 Stiffness Matrix

The strain components as defined by the strain-displacement equation is in the form of matrix $\left[B\right]$ which derived from

$$\left\{e\right\} = \begin{Bmatrix} e_{xx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial x} \\ \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}y \end{bmatrix} = \begin{bmatrix} -y\psi_x \\ (V_x - \psi) \end{bmatrix}\left\{U\right\} = \left[B\right]\begin{Bmatrix} V_1 \\ \theta_1 \\ \theta_2 \\ V_2 \end{Bmatrix} \tag{4-20}$$

Similarly, the matrix $\left[B\right]$ is expanded into an infinite series in ascending powers of

$$\left[B\right] = \sum_{r=0}^{\infty} \omega^r \left[B_r\right] \tag{4-21}$$

Substituting the Equations (4-10a) through Equation (4-10f) into Equation (4-21), the final form of the matrix $\left[B\right]$ becomes

$$\left[B\right] = \left[B_0\right] + \omega^2\left[B_2\right] + \omega^4\left[B_4\right] + \ldots \tag{4-22}$$

so

$$\left[B\right]^T\left[B\right] = \left[\left[B_0\right]^T\left[B_0\right]\right] + \omega^2\left[\left[B_0\right]^T\left[B_2\right] + \left[B_2\right]^T\left[B_0\right]\right] + \omega^4\left[\left[B_0\right]^T\left[B_4\right] + \left[B_2\right]^T\left[B_2\right] + \left[B_4\right]^T\left[B_0\right]\right] + \ldots \tag{4-23}$$

with

$$\left[B_0\right] = \frac{1}{(1+\phi)}\begin{bmatrix} \frac{6}{L^2}\, y(1-2\xi) & \frac{1}{L}y((4+\phi)-6\xi) & -\frac{1}{L}y((2-\phi)-6\xi) & -\frac{6}{L^2}\, y(1-2\xi) \\ -\frac{\phi}{L} & -\frac{\phi}{2} & \frac{\phi}{2} & \frac{\phi}{L} \end{bmatrix}$$

$$\ldots \text{ etc} \qquad (4\text{-}24)$$

From the previous work it follows that

$$\left[K\right] = \int_V \left[B\right]^T \left[C_E\right]\left[B\right] dV \qquad (4\text{-}25a)$$

where

$$\left[C_E\right] = \begin{bmatrix} E & 0 \\ 0 & kG \end{bmatrix} \qquad (4\text{-}25b)$$

Substituting the Equation (4-23) into Equation (4-25a), the equivalent stiffness matrices take the form

$$\left[K\right]^{(T)} = \left[K_0\right]^{(T)} + \omega^2\left[K_2\right]^{(T)} + \omega^4\left[K_4\right]^{(T)} + \ldots \qquad (4\text{-}26)$$

The components of the stiffness matrix in Equation (4-26) are calculated as

$$\left[K_0\right]^{(T)} = \frac{EI}{(1+\phi)^2 L^3}\begin{bmatrix} 12 & 6L & -6L & -12 \\ 6L & (4+\phi)L^2 & -(2-\phi)L^2 & -6L \\ -6L & -(2-\phi)L^2 & (4+\phi)L^2 & 6L \\ -12 & -6L & 6L & 12 \end{bmatrix} \qquad (4\text{-}27)$$

$$\left[K_2\right]^{(T)} = \left[0\right] \tag{4-28}$$

The matrix $\left[K_4\right]^{(T)}$ and matrices of higher powers of (i.e. four), were not calculated. The process in calculating $\left[K_4\right]^{(T)}$ requires to calculation of $\left[B_4\right]$ as a first step which is a complicated integration process taking functions of power ten to function of power thirteen for each shape function in the matrix $\left[B_4\right]$.

4.5 General Stiffness Matrix

Realling the Chapter II, the general stiffness matrix for the beam element is in the form

$$\left[S_T\right]^B = \left[K\right] - \bar{\omega}^2\left[M\right] \tag{4-28a}$$

Substituting Equation (4-17) and Equation (4-26) into Equation (4-28a), the general stiffness becomes

$$\left[S_T\right]^B = \left[K_0\right]^T - \bar{\omega}^2\left[M_0\right]^T - \bar{\omega}^4\left[\left[M_2\right]^T - \left[K_4\right]^T\right] - \ldots \tag{4-28b}$$

Substituting Equation (4-18), (4-19) and Equation (4-27), (4-28) into Equation (4-28b), the final form of the matrix $\left[S_T\right]^B$ is

$$\left[S_T\right]^B = \left[K_0\right]^T - \bar{\omega}^2\left[M_0\right]^T - \bar{\omega}^4\left[M_1\right]^T + \ldots \tag{4-28c}$$

With $\left[M_0\right]^T$ and $\left[K_0\right]^T$ defined by Equation (4-18) and (4-27) respectively. While the matrix $\left[M_1\right]^T$ equals to $\left[\left[M_2\right]^T - \left[K_4\right]^T\right]$

CHAPTER V

TIMOSHENKO BEAM-COLUMN THEORY

5.1 Beam-Column Equations

Recall the results of Chapter IV. The differential equations of motion of the Timoshenko beam element are solved by using a single infinite power series expansion in ascending powers of frequency, with the effects of both transverse shear and rotatory inertia included. In this chapter, we analyse the case of a beam-column element. The differential equations of motion are augmented due to the presence of the axial load. The technique of Chapter III is utilized to solve the equations of motion, the basis of which is formulated using a double infinite power series expansion in ascending powers of frequency and axial load. Similarly, the mass, stiffness, and geometric matrices are solved and expanded in the form of power series of frequency and axial load.

Application of Hamilton's principle[2] in the Timoshenko beam theory which includes the effect of axial load for beam-column, yields the following differential equations of motion of the Timoshenko beam-column as

$$kGA(V_{xx}-\psi_x) - P\,V_{xx} - \rho A\,V_{tt} \quad = \quad 0 \qquad (5\text{-}1)$$

$$EI\psi_{xx} + kGA(V_x - \psi) - \rho I\psi_{tt} = 0 \qquad (5\text{-}2)$$

5.2 Solution of the Differential Equation

Utilizing the technique of Chapter III in solving the latter two coupled differential equations, we assume the continuous displacements in the double infinite series form

$$\psi = \left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\Omega^r P^s\left\{a_{rs}^x\right\}\right)\left\{q\right\} e^{i\Omega t} \qquad (5\text{-}3)$$

$$V = \left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\Omega^r P^s\left\{a_{rs}^y\right\}\right)\left\{q\right\} e^{i\Omega t} \qquad (5\text{-}4)$$

Substituting Equations (5-3) and (5-4) into Equations (5-1) and (5-2) respectively, yields

$$kGA\left[\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\Omega^r P^s\left\{a_{rs}^{y''}\right\}\left\{q\right\}e^{i\Omega t} - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\Omega^r P^s\left\{a_{rs}^{x'}\right\}\left\{q\right\}e^{i\Omega t}\right]$$

$$- P\left[\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\Omega^r P^s\left\{a_{rs}^{y''}\right\}\left\{q\right\}e^{i\Omega t}\right] + \rho A\left[\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\Omega^{(2+r)} P^s\left\{a_{rs}^{y}\right\}\left\{q\right\}e^{i\Omega t}\right] = 0 \qquad (5\text{-}5)$$

$$EI\left[\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\Omega^r P^s\left\{a_{rs}^{x''}\right\}\left\{q\right\}e^{i\Omega t}\right] + kGA\left[\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\Omega^r P^s\left\{a_{rs}^{y'}\right\}\left\{q\right\}e^{i\Omega t}\right.$$

$$\left. - \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\Omega^r P^s\left\{a_{rs}^{x}\right\}\left\{q\right\}e^{i\Omega t}\right] + \rho I\left[\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\Omega^{(2+r)} P^s\left\{a_{rs}^{x}\right\}\left\{q\right\}e^{i\Omega t}\right] = 0 \qquad (5\text{-}6)$$

Equating to zero coefficients of the same powers of Ω and P in Equation (5-5) and (5-6), the following equations are obtained

$$\left\{a_{00}^{y''}\right\} - \left\{a_{00}^{x'}\right\} = 0 \qquad (5\text{-}7a)$$

and $$EI\left\{a_{00}^{x''}\right\} + kGA\left[\left\{a_{00}^{y'}\right\} - \left\{a_{00}^{x}\right\}\right] = 0 \qquad (5\text{-}7b)$$

$$\left\{a_{10}^{y''}\right\} - \left\{a_{10}^{x'}\right\} = 0 \qquad (5\text{-}7c)$$

and $$EI\left\{a_{10}^{x''}\right\} + kGA\left[\left\{a_{10}^{y'}\right\} - \left\{a_{10}^{x}\right\}\right] = 0 \qquad (5\text{-}7d)$$

$$kGA\left[\left\{a_{01}^{y''}\right\} - \left\{a_{01}^{x'}\right\}\right] - \left\{a_{00}^{y''}\right\} = 0 \qquad (5\text{-}7e)$$

and $$EI\left\{a_{01}^{x''}\right\} + kGA\left[\left\{a_{01}^{y'}\right\} - \left\{a_{01}^{x}\right\}\right] = 0 \qquad (5\text{-}7f)$$

$$kGA\left[\left\{a_{02}^{y''}\right\} - \left\{a_{02}^{x'}\right\}\right] - \left\{a_{01}^{y''}\right\} = 0 \qquad (5\text{-}7g)$$

and $$EI\left\{a_{02}^{x''}\right\} + kGA\left[\left\{a_{02}^{y'}\right\} - \left\{a_{02}^{x}\right\}\right] = 0 \qquad (5\text{-}7h)$$

$$kGA\left[\left\{a_{20}^{y''}\right\} - \left\{a_{20}^{x'}\right\}\right] - \left\{a_{00}^{y}\right\} = 0 \qquad (5\text{-}7i)$$

and, $$EI\left\{a_{20}^{x''}\right\} + kGA\left[\left\{a_{20}^{y'}\right\} - \left\{a_{20}^{x}\right\}\right] + \rho I\left\{a_{00}^{x}\right\} = 0 \qquad (5\text{-}7j)$$

$$\vdots$$

Equations (5-7a) through Equation (5-7j) are solved by direct integration. For Equation (5-7a) and Equation (5-7b), the following boundary conditions are applied

at $x = 0$ $\qquad V(0) = V_1 \qquad \psi(0) = \theta_1$

at $x = L$ $\qquad V(L) = V_2 \qquad \dot{\psi}(L) = -\theta_2$

For the remaining pairs of coupled equations containing terms $\left\{a_{10}^{x}\right\}$, $\left\{a_{10}^{y}\right\}$, $\left\{a_{01}^{x}\right\}$, $\left\{a_{01}^{y}\right\}$, ..., displacement and slope must all vanish at $x = 0$ and L thus,

$$\left\{a^x_{00}\right\} = \frac{1}{(1+\phi)}\left[\frac{6}{L}\xi(\xi-1) \quad (3\xi^2-(4+\phi)\xi+(1+\phi)) \quad (-3\xi^2+(2-\phi)\xi) \quad -\frac{6}{L}\xi(\xi-1)\right] \tag{5-8a}$$

$$\left\{a^y_{00}\right\} = \frac{1}{(1+\phi)}\left[(2\xi^3-3\xi^2-\phi\xi+(1+\phi)) \quad (\xi^3-(2+\tfrac{\phi}{2})\xi^2+(1+\tfrac{\phi}{2})\xi)L \quad (-\xi^3+(1-\tfrac{\phi}{2})\xi^2+\tfrac{\phi}{2}\xi)L \quad (-2\xi^3+3\xi^2+\phi\xi)\right] \tag{5-8b}$$

$$\left\{a^x_{10}\right\} = \left\{a^y_{10}\right\} = \left\{0\right\} \tag{5-8c,d}$$

$$\begin{aligned}\left\{a^x_{01}\right\} = \frac{L}{60(1+\phi)^2 EI}\Big[\; & 6\left\{\xi-(6+5\phi)\xi^2+10(1+\phi)\xi^3-5(1+\phi)\xi^4\right\} \quad \left\{(8+10\phi+5\phi^2)\xi-3(11+15\phi+5\phi^2)\xi^2+10(4+5\phi+\phi^2)\xi^3\right. \\ & \left.-15(1+\phi)\xi^4\right\}L \quad \left\{(2+10\phi+5\phi^2)\xi+3(1-5\phi-5\phi^2)\xi^2-10(2+\phi-\phi^2)\xi^3+15(1+\phi)\xi^4\right\}L \\ & -6\left\{\xi-(6+5\phi)\xi^2+10(1+\phi)\xi^3-5(1+\phi)\xi^4\right\}\Big]\end{aligned} \tag{5-8e}$$

$$\begin{aligned}\left\{a^y_{01}\right\} = \frac{L^2}{120(1+\phi)^2 EI}\Big[\; & \left\{2\phi(6+5\phi)\xi+6(1-5\phi-5\phi^2)\xi^2-4(6-5\phi^2)\xi^3+30(1+\phi)\xi^4-12(1+\phi)\xi^5\right\} \\ & \left\{\phi(11+15\phi+5\phi^2)\xi+(8-10\phi-20\phi^2-5\phi^3)\xi^2-2(11+10\phi)\xi^3+5(4+5\phi+\phi^2)\xi^4-6(1+\phi)\xi^5\right\}L \\ & -\left\{\phi(1-5\phi-5\phi^2)\xi-(2+20\phi+10\phi^2-5\phi^3)\xi^2-2(1-10\phi-10\phi^2)\xi^3+5(2+\phi-\phi^2)\xi^4-6(1+\phi)\xi^5\right\}L \\ & -\left\{2\phi(6+5\phi)\xi+6(1-5\phi-5\phi^2)\xi^2-4(6-5\phi^2)\xi^3+30(1+\phi)\xi^4-12(1+\phi)\xi^5\right\}\Big]\end{aligned} \tag{5-8f}$$

$$\left\{a^{x}_{02}\right\} = \frac{L^3}{25200(1+\phi)^3(EI)^2}\Bigg[6\Big[(3+45\phi+35\phi^2)\xi-(3+255\phi+420\phi^2+175\phi^3)\xi^2-70(1-4\phi-10\phi^2-5\phi^3)\xi^3+35(6+6\phi-5\phi^2-5\phi^3)\xi^4$$

$$-210(1+2\phi+\phi^2)\xi^5+70(1+2\phi+\phi^2)\xi^6\Big]$$

$$\Big[(44+415\phi+735\phi^2+560\phi^3+175\phi^4)\xi-3(3+430\phi+945\phi^2+700\phi^3+175\phi^4)\xi^2-70(8-2\phi-30\phi^2-25\phi^3-5\phi^4)\xi^3$$

$$+105(11+21\phi+10\phi^2)\xi^4-210(4+9\phi+6\phi^2+\phi^3)\xi^5+210(1+2\phi+\phi^2)\xi^6\Big]L$$

$$\Big[(26+145\phi+525\phi^2+560\phi^3+175\phi^4)\xi+3(3+80\phi-105\phi^2-350\phi^3-175\phi^4)\xi^2-70(2+22\phi+30\phi^2+5\phi^3-5\phi^4)\xi^3$$

$$-105(1-9\phi-20\phi^2-10\phi^3)\xi^4+210(2+3\phi-\phi^3)\xi^5-210(1+2\phi+\phi^2)\xi^6\Big]L$$

$$-6\Big[(3+45\phi+35\phi^2)\xi-(3+255\phi+420\phi^2+175\phi^3)\xi^2-70(1-4\phi-10\phi^2-5\phi^3)\xi^3+35(6+6\phi-5\phi^2-5\phi^3)\xi^4-210(1+2\phi+\phi^2)\xi^5$$

$$+70(1+2\phi+\phi^2)\xi^6\Big]\Bigg]$$

(5-8g)

$$\left\{a^{x}_{11}\right\} = \left\{a^{y}_{11}\right\} = \left\{0\right\} \qquad (5\text{-}8h,i)$$

$$\left\{a^{y}_{02}\right\} = \frac{L^4}{50400(1+\phi)^3(EI)^2}\Bigg[\Bigg[2(3\phi+255\phi^2+420\phi^3+175\phi^4)\xi+6(3+80\phi-105\phi^2-350\phi^3-175\phi^4)\xi^2-4(3+465\phi+630\phi^2-175\phi^4)\xi^3$$

$$-105(2-18\phi-40\phi^2-20\phi^3)\xi^4+84(6+\phi-15\phi^2-10\phi^3)\xi^5-420(1+2\phi+\phi^2)\xi^6+120(1+2\phi+\phi^2)\xi^7$$

$$\Big[(3\phi+430\phi^2+945\phi^3+700\phi^4+175\phi^5)\xi+(44+695\phi+665\phi^2-490\phi^3-700\phi^4-175\phi^5)\xi^2-2(3+815\phi+1680\phi^2+1050\phi^3+175\phi^4)\xi^3$$

$$-35(8-22\phi-75\phi^2-55\phi^3-10\phi^4)\xi^4+42(11+16\phi-5\phi^3)\xi^5-70(4+9\phi+6\phi^2+\phi^3)\xi^6+60(1+2\phi+\phi^2)\xi^7\Big]L$$

$$-\Big[(3\phi+80\phi^2-105\phi^3-350\phi^4-175\phi^5)\xi-(26+215\phi+1295\phi^2+1610\phi^3+350\phi^4-175\phi^5)\xi^2-2(3+115\phi-420\phi^2-1050\phi^3-525\phi^4)\xi^3$$

$$+35(2+32\phi+45\phi^2+5\phi^3-10\phi^4)\xi^4+42(1-14\phi-30\phi^2-15\phi^3)\xi^5-70(2+3\phi-\phi^3)\xi^6+60(1+2\phi+\phi^2)\xi^7\Big]L$$

$$-\Big[2(3\phi+255\phi^2+420\phi^3+175\phi^4)\xi+6(3+80\phi-105\phi^2-350\phi^3-175\phi^4)\xi^2-4(3+465\phi+630\phi^2-175\phi^4)\xi^3$$

$$-105(2-18\phi-40\phi^2-20\phi^3)\xi^4+84(6+\phi-15\phi^2-10\phi^3)\xi^5-420(1+2\phi+\phi^2)\xi^6+120(1+2\phi+\phi^2)\xi^7\Big]\Bigg] \quad \ldots \text{ etc} \qquad (5\text{-}8j)$$

where $\xi = \frac{x}{L}$

5.3 Mass Matrix

As the previous work (4.3) in Chapter IV, the nodal displacements are in the form

$$\left\{u\right\} = \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \begin{bmatrix} -y\psi \\ V \end{bmatrix} \left\{U\right\} = \left[A\right] \begin{Bmatrix} V_1 \\ \theta_1 \\ \theta_2 \\ V_2 \end{Bmatrix} \qquad (5\text{-}9)$$

From Equations (5-3) and (5-4), the matrix $\left[A\right]$ is expanded into an infinite series in ascending powers of Ω and P. Thus

$$\left[A\right] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \Omega^r P^s \left[A_{rs}\right] \qquad (5\text{-}10)$$

Substituting the Equations (5-8a) through Equation (5-8j) into Equation (5-10), the final form of the matrix $\left[A\right]$ becomes

$$\left[A\right] = \left[A_{00}\right] + P\left[A_{01}\right] + P\Omega^2\left[A_{21}\right] + P^2\left[A_{02}\right] + \Omega^2\left[A_{20}\right]$$

$$+ \ldots \qquad (5\text{-}11)$$

so,

$$\left[A\right]^T\left[A\right] = \left[\left[A_{00}\right]^T\left[A_{00}\right]\right] + P\left[\left[A_{00}\right]^T\left[A_{01}\right] + \left[A_{01}\right]^T\left[A_{00}\right]\right]$$

$$+ \Omega^2\left[\left[A_{00}\right]^T\left[A_{20}\right] + \left[A_{20}\right]^T\left[A_{00}\right]\right] + \ldots \qquad (5\text{-}12)$$

$$\left[A_{00}\right] = \frac{1}{(1+\phi)} \begin{bmatrix} -\frac{6}{L}y(\xi^2-\xi) & y(3\xi^2-(4+\phi)\xi+(1+\phi)) & y(3\xi^2-2\xi+\phi\xi) & \frac{6}{L}y(\xi^2-\xi) \\ (2\xi^3-3\xi^2-\phi\xi+(1+\phi)) & (\xi^3-(2+\frac{\phi}{2})\xi^2+(1+\frac{\phi}{2})\xi)L & (-\xi^3+(1-\frac{\phi}{2})\xi^2+\frac{\phi}{2}\xi)L & (-2\xi^3+3\xi^2+\phi\xi) \end{bmatrix} \quad (5\text{-}13a)$$

$$\left[A_{20}\right] = \frac{1}{(1+\phi)^2} \begin{bmatrix} -y\{F\} & -y\{H\} & -y\{N\} & -y\{O\} \\ \{Q\} & \{R\} & \{S\} & \{T\} \end{bmatrix} \ldots \text{etc} \quad (5\text{-}13b)$$

with defined by Equation (4-10e) and (4-10f)

Substituting the Equation (5-12) into Equation (4-16), the equivalent mass matrices are in the form

$$\left[M\right]^{(T)} = \left[M_0\right]^{(T)} + P\left[K_1^i\right]^{(T)} + \Omega^2\left[M_2\right]^{(T)} + \ldots \quad (5\text{-}14)$$

The components of the mass matrix in Equation (5-14) are calculated as $\left[M_0\right]^{(T)}$ and $\left[M_2\right]^{(T)}$ which defined by Equations (4-18) and (4-19) respectively.

5.4 Stiffness Matrix

As the previous work (4.4) in Chapter IV, the matrix $\left[B\right]$ is derived from

$$\left\{e\right\} = \begin{Bmatrix} e_{xx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} -y\psi_x \\ (V_x - \psi) \end{bmatrix} \left\{U\right\} = \left[B\right] \begin{Bmatrix} V_1 \\ \theta_1 \\ \theta_2 \\ V_2 \end{Bmatrix} \quad (5\text{-}15)$$

Similarly, the matrix $\begin{bmatrix} B \end{bmatrix}$ may be expanded into an infinite series in ascending powers of Ω and P

$$\begin{bmatrix} B \end{bmatrix} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \Omega^r P^s \begin{bmatrix} B_{rs} \end{bmatrix} \qquad (5\text{-}16)$$

Substituting the Equations (5-8a) through Equation (5-8j) into Equation (5-16), the final form of the matrix $\begin{bmatrix} B \end{bmatrix}$ becomes

$$\begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} B_{00} \end{bmatrix} + P\begin{bmatrix} B_{01} \end{bmatrix} + P\Omega^2\begin{bmatrix} B_{21} \end{bmatrix} + P^2\begin{bmatrix} B_{02} \end{bmatrix} + \Omega^2\begin{bmatrix} B_{20} \end{bmatrix} + \ldots \qquad (5\text{-}17)$$

so,
$$\begin{bmatrix} B \end{bmatrix}^T\begin{bmatrix} B \end{bmatrix} = \left[\begin{bmatrix} B_{00} \end{bmatrix}^T\begin{bmatrix} B_{00} \end{bmatrix}\right] + P\left[\begin{bmatrix} B_{00} \end{bmatrix}^T\begin{bmatrix} B_{01} \end{bmatrix} + \begin{bmatrix} B_{01} \end{bmatrix}^T\begin{bmatrix} B_{00} \end{bmatrix}\right]$$
$$+ P^2\left[\begin{bmatrix} B_{00} \end{bmatrix}^T\begin{bmatrix} B_{02} \end{bmatrix} + \begin{bmatrix} B_{01} \end{bmatrix}^T\begin{bmatrix} B_{01} \end{bmatrix} + \begin{bmatrix} B_{02} \end{bmatrix}^T\begin{bmatrix} B_{00} \end{bmatrix}\right]$$
$$+ \Omega^2\left[\begin{bmatrix} B_{00} \end{bmatrix}^T\begin{bmatrix} B_{20} \end{bmatrix} + \begin{bmatrix} B_{20} \end{bmatrix}^T\begin{bmatrix} B_{00} \end{bmatrix}\right] + \ldots \qquad (5\text{-}18)$$

with

$$\begin{bmatrix} B_{00} \end{bmatrix} = \frac{1}{(1+\phi)}\begin{bmatrix} \frac{6}{L^2}\, y(1-2\xi) & \frac{1}{L}y((4+\phi)-6\xi) & \begin{matrix} -\frac{1}{L}y((2-\phi)-6\xi) \\ -\frac{6}{L^2}\, y(1-2\xi) \end{matrix} \\ \\ -\frac{\phi}{L} & -\frac{\phi}{2} & \begin{matrix} \frac{\phi}{2} \\ \frac{\phi}{L} \end{matrix} \end{bmatrix}$$

(5-19a)

$$
\left[B_{01}\right] = \frac{L}{60(1+\phi)^2 EI}
\begin{bmatrix}
-\frac{y}{L}\left[6-12(6+5\phi)\xi+180(1+\phi)\xi^2-120(1+\phi)\xi^3\right] & -y\left[(8+10\phi+5\phi^2)-6(11+15\phi+5\phi)\xi \right. \\
\left. +30(4+5\phi+\phi^2)\xi^2-60(1+\phi)\xi^3\right] & -y\left[(2+10\phi+5\phi^2)+6(1-5\phi-5\phi^2)\xi-30(2+\phi-\phi^2)\xi^2+60(1+\phi)\xi^3\right] \\
\frac{y}{L}\left[6-12(6+5\phi)\xi+180(1+\phi)\xi^2-120(1+\phi)\xi^3\right] & \\
\left[\phi(6+5\phi)-30\phi(1+\phi)\xi+30\phi(1+\phi)\xi^2\right] & \left[\frac{\phi}{2}(11+15\phi+5\phi^2)-5\phi(4+5\phi+\phi^2)\xi+15\phi(1+\phi)\xi^2\right]L \\
-\left[\frac{\phi}{2}(1-5\phi-5\phi^2)-5\phi(2+\phi-\phi^2)\xi+15\phi(1+\phi)\xi^2\right]L & -\left[\phi(6+5\phi)-30\phi(1+\phi)\xi+30\phi(1+\phi)\xi^2\right]
\end{bmatrix}
$$

... etc (5-19b)

Substituting the Equation (5-18) into Equation (4-25a), the equivalent stiffness matrices take the form

$$
\left[K\right]^{(T)} = \left[K_0\right]^{(T)} + P\left[K_1^b\right]^{(T)} + P^2\left[K_2^b\right]^{(T)} + \left[K_3^b\right]^{(T)} + \dots \qquad (5\text{-}20)
$$

with $\left[K_0\right]^{(T)}$ defined by Equation (4-27), and

$$
\left[K_1^b\right]^{(T)} = \left[0\right] \qquad (5\text{-}21a)
$$

$$\left[K_2^b\right]^{(T)} = \frac{L}{25200(1+\phi)^4 EI}\begin{bmatrix} (36+576\phi+120\phi^2-420\phi^3) & & & \\ (18+183\phi-150\phi^2-315\phi^3)L & (44+179\phi+30\phi^2-70\phi^3+35\phi^4)L^2 & & \text{Symmetry} \\ -(18+183\phi-150\phi^2-315\phi^3)L & (26+801\phi+390\phi^2-350\phi^3+35\phi^4)L^2 & (44+179\phi+30\phi^2-70\phi^3+35\phi^4)L^2 & \\ -(36+576\phi+120\phi^2-420\phi^3) & -(18+183\phi-150\phi^2-315\phi^3)L & (18+183\phi-150\phi^2-315\phi^3)L & (36+576\phi+120\phi^2-420\phi^3) \end{bmatrix} \qquad (5\text{-}21b)$$

$$\left[K_3^b\right]^{(T)} = \left[0\right] \qquad (5\text{-}21c)$$

5.5 Geometric Matrix

The geometric stiffness concept is derived by application of virtual displacements upon equating the internal and external virtual work components. The coefficient of the geometric stiffness becomes

$$\left[K_G\right] = \int_0^L \left[C\right]^T \left[C\right] dx \tag{5-22}$$

where

$$\left[C\right] = \begin{bmatrix} \psi \\ (V_x - \psi) \end{bmatrix} \tag{5-23}$$

Similarly, the matrix $\left[C\right]$ is expanded into an infinite series in ascending powers of Ω and P

$$\left[C\right] = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \Omega^r P^s \left[C_{rs}\right] \tag{5-24}$$

Substituting the Equations (5-8a) through Equation (5-8j) into Equation (5-24), the final form of the matrix $\left[C\right]$ becomes

$$\left[C\right] = \left[C_{00}\right] + P\left[C_{01}\right] + P\Omega^2\left[C_{21}\right] + P^2\left[C_{02}\right] + \Omega^2\left[C_{20}\right] + \ldots \tag{5-25}$$

so,

$$\left[C\right]^T\left[C\right] = \left[\left[C_{00}\right]^T\left[C_{00}\right]\right] + P\left[\left[C_{00}\right]^T\left[C_{01}\right] + \left[C_{01}\right]^T\left[C_{00}\right]\right] + \ldots \tag{5-26}$$

with

$$\left[C_{00}\right] = \frac{1}{(1+\phi)} \begin{bmatrix} -\frac{6}{L}(\xi-\xi^2) & ((1+\phi)-(4+\phi)\xi+3\xi^2) & ((2-\phi)\xi-3\xi^2) & \frac{6}{L}(\xi-\xi^2) \\ -\frac{\phi}{L} & -\frac{\phi}{2} & \frac{\phi}{2} & \frac{\phi}{L} \end{bmatrix}$$

$$\left[C_{01}\right] = \frac{L}{120(1+\phi)^2 EI} \begin{bmatrix} 12\left\{\xi-(6+5\phi)\xi^2+10(1+\phi)\xi^3-5(1+\phi)\xi^4\right\} & 2\left\{(8+10\phi+5\phi^2)\xi-3(11+15\phi+5\phi^2)\xi^2+10(4+5\phi+\phi^2)\xi^3-15(1+\phi)\xi^4\right\}L & 2\left\{(2+10\phi+5\phi^2)\xi+3(1-5\phi-5\phi^2)\xi^2-10(2+\phi-\phi^2)\xi^3+30(1+\phi)\xi^4\right\}L & -12\left\{\xi-(6+5\phi)\xi^2+10(1+\phi)\xi^3-5(1+\phi)\xi^4\right\} \\ 2\phi\left\{(6+5\phi)-30(1+\phi)\xi+30(1+\phi)\xi^2\right\} & \left\{\phi(11+15\phi+5\phi^2)-10\phi(4+5\phi+\phi^2)\xi+30\phi(1+\phi)\xi^2\right\}L & -\left\{\phi(1-5\phi-5\phi^2)-10\phi(2+\phi-\phi^2)\xi+30\phi(1+\phi)\xi^2\right\}L & -2\phi\left\{(6+5\phi)-30(1+\phi)\xi+30(1+\phi)\xi^2\right\} \end{bmatrix}$$

... etc (5-27b)

Substituting the Equation (5-26) into Equation (5-22), the geometric stiffness matrices take the form

$$\left[K_G\right]^{(T)} = \left[K_{G1}\right]^{(T)} + P\left[K_{G2}\right]^{(T)} + \ldots \qquad (5\text{-}28)$$

with

$$
\left[K_{G1}\right]^{(T)} = \frac{1}{(1+\phi)^2}\begin{bmatrix}
\left(\frac{6}{5L}+\frac{\phi^2}{L}\right) & \left(\frac{1}{10}-\frac{\phi}{2}+\frac{\phi^2}{2}\right) & -\left(\frac{1}{10}-\frac{\phi}{2}+\frac{\phi^2}{2}\right) & -\left(\frac{6}{5L}+\frac{\phi^2}{L}\right) \\
\left(\frac{1}{10}-\frac{\phi}{2}+\frac{\phi^2}{2}\right) & \left(\frac{2}{15}+\frac{\phi}{6}+\frac{7\phi^2}{12}\right)L & \left(\frac{1}{30}+\frac{\phi}{6}-\frac{5\phi^2}{12}\right)L & -\left(\frac{1}{10}-\frac{\phi}{2}+\frac{\phi^2}{2}\right) \\
-\left(\frac{1}{10}-\frac{\phi}{2}+\frac{\phi^2}{2}\right) & \left(\frac{1}{30}+\frac{\phi}{6}-\frac{5\phi^2}{12}\right)L & \left(\frac{2}{15}+\frac{\phi}{6}+\frac{7\phi^2}{12}\right)L & \left(\frac{1}{10}-\frac{\phi}{2}+\frac{\phi^2}{2}\right) \\
-\left(\frac{6}{5L}+\frac{\phi^2}{L}\right) & -\left(\frac{1}{10}-\frac{\phi}{2}+\frac{\phi^2}{2}\right) & \left(\frac{1}{10}-\frac{\phi}{2}+\frac{\phi^2}{2}\right) & \left(\frac{6}{5L}+\frac{\phi^2}{L}\right)
\end{bmatrix} \qquad (5\text{-}29a)
$$

$$
\left[K_{G2}\right]^{(T)} = \frac{L}{120(1+\phi)^3 EI}\begin{bmatrix}
\frac{4}{35}(3+45\phi-35\phi^2) & & \text{Symmetry} & \\
\frac{L}{35}(6+55\phi-105\phi^2) & \frac{L^2}{105}(44+135\phi-105\phi^2+35\phi^3) & & \\
\frac{-L}{35}(6+55\phi-105\phi^2) & \frac{L^2}{105}(26+775\phi-385\phi^2+35\phi^3) & \frac{L^2}{105}(44+135\phi-105\phi^2+35\phi^3) & \\
\frac{-4}{35}(3+45\phi-35\phi^2) & \frac{-L}{35}(6+55\phi-105\phi^2) & \frac{L}{35}(6+55\phi-105\phi^2) & \frac{4}{35}(3+45\phi-35\phi^2)
\end{bmatrix} \qquad (5\text{-}29b)
$$

5.6 General Stiffness Matrix

For the dynamic analysis of the Timoshenko beam-column, the general stiffness matrix possesses a form of a linear combination of bending stiffness, mass, and geometric stiffness matrices as

$$\left[S\right] = \left[K\right] - \Omega^2\left[M\right] - P\left[K_G\right] \qquad (5\text{-}30)$$

Substituting the Equations (5-14), (5-20), and (5-28) into Equation (5-30), arranging the final terms of the similar powers of Ω and P, the general stiffness matrix $\left[S\right]$ becomes

$$\left[S_T\right]^{(BC)} = \left[K_0\right] + P\left[\left[K_1^b\right] - \left[K_{G1}\right]\right] + \Omega^2\left[\left[K_3^b\right] - \left[M_0\right]\right] + P^2\left[\left[K_2^b\right] - \left[K_{G1}\right]\right] + \ldots \text{ etc} \qquad (5\text{-}31)$$

Substituting the Equations (4-18) through Equation (4-19), Equations (5-21a) through Equation (5-21c) and Equations (5-29a) through Equation (5-29b) into Equation (5-31), the final form of the general stiffness matrix becomes

$$\left[S_T\right]^{(BC)} = \left[K_0\right] - \left[G_0\right]P - \left[M_0\right]\Omega^2 - \left[G_1\right]P^2 + \ldots \qquad (5\text{-}32)$$

with $\left[M_0\right]^{(T)}$ and $\left[K_0\right]^{(T)}$ defined by Equations (4-18) and (4-27) respectively

$$\left[G_0\right]^{(T)} = \frac{1}{(1+\phi)^2}\begin{bmatrix} \left(\frac{6}{5L}+\frac{\phi^2}{L}\right) & \left(\frac{1}{10}-\frac{\phi}{2}+\frac{\phi^2}{2}\right) & -\left(\frac{1}{10}-\frac{\phi}{2}+\frac{\phi^2}{2}\right) & -\left(\frac{6}{5L}+\frac{\phi^2}{L}\right) \\ \left(\frac{1}{10}-\frac{\phi}{2}+\frac{\phi^2}{2}\right) & \left(\frac{2}{15}+\frac{\phi}{6}+\frac{7\phi^2}{12}\right)L & \left(\frac{1}{30}+\frac{\phi}{6}-\frac{5\phi^2}{12}\right)L & -\left(\frac{1}{10}-\frac{\phi}{2}+\frac{\phi^2}{2}\right) \\ -\left(\frac{1}{10}-\frac{\phi}{2}+\frac{\phi^2}{2}\right) & \left(\frac{1}{30}+\frac{\phi}{6}-\frac{5\phi^2}{12}\right)L & \left(\frac{2}{15}+\frac{\phi}{6}+\frac{7\phi^2}{12}\right)L & \left(\frac{1}{10}-\frac{\phi}{2}+\frac{\phi^2}{2}\right) \\ -\left(\frac{6}{5L}+\frac{\phi^2}{L}\right) & -\left(\frac{1}{10}-\frac{\phi}{2}+\frac{\phi^2}{2}\right) & \left(\frac{1}{10}-\frac{\phi}{2}+\frac{\phi^2}{2}\right) & \left(\frac{6}{5L}+\frac{\phi^2}{L}\right) \end{bmatrix} \qquad (5\text{-}33a)$$

$$\left[G_1\right]^{(T)} = \frac{L}{25200(1+\phi)^2 EI}\begin{bmatrix} 12(3+45\phi-35\phi^2) & & & \text{Symmetry} \\ 3(6+55\phi-105\phi^2)L & (44+135\phi-105\phi^2+35\phi^3)L^2 & & \\ -3(6+55\phi-105\phi^2)L & (26+775\phi-385\phi^2+35\phi^3)L^2 & (44+135\phi-105\phi^2+35\phi^3)L^2 & \\ -12(3+45\phi-35\phi^2) & -3(6+55\phi-105\phi^2)L & 3(6+55\phi-105\phi^2)L & 12(3+45\phi-35\phi^2) \end{bmatrix} \qquad (5\text{-}33b)$$

CHAPTER VI

DISCUSSION AND CONCLUSIONS

6.1 Discussion

In Chapter I, the concept of the replacement of a continuum by an equivalent discrete element system is reviewed. Utilizing discrete displacements and discrete loads, equivalent mass, stiffness, and geometric matrices are developed for each finite element of the discretized system.

In Chapter II, the equivalent discrete element system is formulated for a beam element. The nodal displacements are assumed to be related to the continuous displacements in the form of a single infinite power series expansion in ascending powers of frequency. Similarly, the mass, and bending stiffness matrices are solved in the form of a single infinite series in ascending powers of frequency.

In the Chapter III, the equivalent discrete element system is applied to a beam-column element. For both Chapter II and Chapter III, the equation of motion exclude the effects of rotatory inertia and shear deformation. Due to the presence of the axial load, the novel technique of a double infinite power series expansion in ascending powers of frequency and axial load is utilized to solve the equations of motion. Similarly, the mass, bending stiffness, and geometric matrices are solved in the form of power series of frequency and axial load. By combination of these matrices, the general stiffness matrix is formulated in a form of bending stiffness, mass, and geometric stiffness matrices which are expanded into an infinite series in ascending powers of frequency and axial load.

For the Chapter IV, the Timoshenko theory of beams is utilized including the effects of both transverse shear and rotatory inertia. The same technique as in Chapter II is utilized to solve the mass, and bending stiffness matrices which are in the form of a single infinite series in ascending powers of frequency.

For the Chapter V, the presence of the axial load is combined with the Timoshenko beam theory for the beam-column element. A double infinite power series expansion in ascending powers of frequency and axial load is utilized to solve the equation of motion. Similar to Chapter III, the mass, bending stiffness and geometric matrices are solved and the general stiffness matrix is formulated by the combination of these matrices.

6.2 Conclusions

From the latter paragraph, it is clear that we can utilize a single infinite power series expansion in ascending powers of frequency in a general theory of the beam element. For a general theory of beam-column element, due to the presence of the axial load, we must utilize the double infinite power series expansion in ascending powers of frequency and axial load.

For the Bernoulli-Euler beam element, the general stiffness matrix determined in terms of the mass, and bending stiffness matrices take the form

$$\left[S_{BE}\right]^{(B)} = \left[K_0\right] - \omega^2\left[M_0\right] - \omega^4\left[\left[M_2\right]-\left[K_4\right]\right] + \ldots$$

For the Bernoulli-Euler beam-column element, the bending stiffness, mass, and geometric stiffness matrices are obtained in the form

$$\left[S_{BE}\right]^{(BC)} = \left[K_0\right] - \left[G_0\right]P - \left[M_0\right]\Omega^2 - \left[A_1\right]P\Omega^2 - \left[G_1\right]P^2 - \left[M_1\right]\Omega^4$$

$$-\left[A_2\right]P^2\Omega^2 - \left[G_2\right]P^3 - \left[A_3\right]P^3\Omega^2 - \left[G_3\right]P^4 + \ldots$$

The $\left[K_0\right]$, $\left[M_0\right]$, and $\left[M_1\right] = \left[\left[M_2\right]-\left[K_4\right]\right]$ matrices in the latter equation for the beam-column element are identical to those associated with the beam-element. The $\left[G_i\right]$ matrices (i=0, 1, 2, 3) as well as the $\left[A_j\right]$ matrices (j= 1, 2, 3) are due exclusively to the presence of axial force.

For the Timoshenko beam element, the general stiffness matrix determined in terms of the mass, and bending stiffness matrices takes the form

$$\left[S_T\right]^{(B)} = \left[K_0\right]^{(T)} - \omega^2\left[M_0\right]^{(T)} - \omega^4\left[\left[M_2\right]^{(T)}-\left[K_4\right]^{(T)}\right] + \ldots$$

For the Timoshenko beam-column element, the bending stiffness, mass, and geometric stiffness matrices are obtained in the form

$$\left[S_T\right]^{(BC)} = \left[K_0\right]^{(T)} - \left[G_0\right]^{(T)}P - \left[M_0\right]^{(T)}\Omega^2 - \left[A_1\right]^{(T)}P\Omega^2 - \left[G_1\right]^{(T)}P^2$$

$$- \left[M_1\right]^{(T)}\Omega^4 - \left[A_2\right]^{(T)}P^2\Omega^2 - \left[G_2\right]^{(T)}P^3 - \left[A_3\right]^{(T)}P^3\Omega^2$$

$$- \left[G_3\right]^{(T)}P^4 + \ldots$$

Similarly, the $\left[K_0\right]^{(T)}$, $\left[M_0\right]^{(T)}$, and $\left[M_1\right]^{(T)} = \left[\left[M_2\right]^{(T)} - \left[K_4\right]^{(T)}\right]$ matrices in the latter equation for the beam-column element are identical to those associated with the beam element. The $\left[G_i\right]^{(T)}$ matrices (i = 0, 1, 2, 3) as well as the $\left[A_j\right]^{(T)}$ matrices (j = 1, 2, 3) are due exclusively to the presence of axial force.

It is recommended that this numerical technique be utilized in a future topic of a Master's Thesis to extend the analysis to additional geometric configurations. For instance, this method may be applied to the dynamic analysis of arches, rings, plates, and shells.

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