On the Continuity of Functions and of Their Restrictions to Function Graphs

by

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ABSTRACT

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This paper examines a theorem of N. N. Luzin that a function is continuous on a rectangle if and only if it is continuous on the graph of every continuous function from one side of the rectangle to the other, and presents two original generalizations of that theorem.

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INTRODUCTION

This paper is a report of original research by the author regarding the question of which topological properties are necessary for a function to be continuous on a product of topological spaces if and only if it is continuous on the graphs of all continuous functions from one factor of the product into the other. This research generalizes a theorem of N. N. Luzin which proves the above statement for a function from a product of closed intervals into the real line.

Chapter 0 presents background material that is used in later chapters. The first section of this chapter is optional for readers with a working knowledge of general topology. The second section of the chapter, however, is optional only for those readers who are familiar with the properties of sequential spaces. Chapter 1 presents Luzin's Theorem and describes a generalization to be found in the literature. Chapter 2 presents two original generalizations of that theorem, and Chapter 3 presents some counterexamples to show the necessity of some of the hypotheses in these generalizations.

CHAPTER 0

Introductory Material

This chapter presents some background material on general topology, some of which the reader may already know. The results in the second section of the chapter on sequential spaces may well be unfamiliar to him, however, and he is encouraged to read these. Although they do not directly enter into the proof of our main result, they provide some indication of what makes some spaces sequential and others not.

General Topology

Much more information than is included in this section may be found in most textbooks on general topology, for example [2] or [4]. Such information may also be found in part I of [8], although fewer proofs are given there.

A **topological space** is an ordered pair $(X, \mathcal{T})$ where X is any set and $\mathcal{T}$ is a collection of subsets of X, called a **topology**. The members of $\mathcal{T}$ are called **open sets**, and an open set including a point is called a **neighborhood** of that point. The topology $\mathcal{T}$ includes all (arbitrary) unions and all finite intersections of its members. As such, $\mathcal{T}$ must include the empty union $\emptyset$ and the empty intersection X. Usually, we refer to a topological space by naming the set X without mentioning the topology $\mathcal{T}$.

Common examples of topological spaces are the real line $\mathbf{R}$, the closed unit interval I, and the general closed interval $[a, b]$ with endpoints a and b. The topology on the real line is called the **Euclidean** topology. This topology consists of all unions of open intervals (a, b). The topology of a closed interval $[a, b]$ is the **subspace** topology, the set of intersections $U \cap [a, b]$ for all open sets U in the Euclidean topology on the real line. This also applies to the closed unit interval $I = [0, 1]$.

A **basis** for a topology on X is a family $\mathcal{B}$ of open sets such that each open set in X is the union of open sets included in $\mathcal{B}$. Equivalently, for each point in an open set V there is an open set in $\mathcal{B}$ included in V and including that point. This is because a union of sets includes a point if and only if one of the sets in that union includes that point. A local **basis** at a point x is a family $\mathcal{B}$ of open sets each including x such that any open neighborhood of x includes a member of $\mathcal{B}$. Finally, a **subbasis** for a topology on X is a family of open sets, the intersections of finitely many members of which constitute a basis for that topology.

A function f from a topological space X to a topological space Y ($f : X \to Y$) is **continuous** if and only if for any open set V in Y, the inverse image of V with respect to f (denoted by $f^{-1}(V)$) is open in X. If a function $f : X \to Y$ is continuous, one–to–one, and onto, and its inverse is also continuous, f is called a **homeomorphism** and X and Y are called **homeomorphic**. In this case, X and Y are topologically indistinguishable. For example, all closed intervals $[a, b]$ are homeomorphic to one another, so the closed unit interval I is equivalent to any of them for topological purposes. The real line $\mathbf{R}$ is **not** homeomorphic to I.

A continuous function from the closed unit interval I into a space Y is called a **path**. A space is **path–connected** if, for each pair x, y of points in Y, there is a path p such that $p(0) = x$ and $p(1) = y$. A space is **locally path–connected** if it has a basis of path–connected sets, i.e, if every open set is the union of path–connected open sets. A space is **connected** if there is no pair of disjoint nonempty open sets whose union is the entire space. A space is **locally connected** if it has a basis of connected sets. If a space X is the union of two disjoint nonempty open sets U and V, and x is a point in U and y is a point in V, then there is no path p connecting x and y. Otherwise $p^{-1}(U) \ni 0$ and $p^{-1}(V) \ni 1$ are disjoint nonempty open sets whose union is the closed unit interval. This is impossible, so such a path cannot exist. Therefore path–connectedness implies connectedness, and local path–connectedness implies local connectedness.

A point x in a topological space X is called an **isolated** point if the set $\{x\}$ is open. A space is called **dense in itself** if it includes no isolated points. Since an isolated point corresponds to a one–point open set, a space is dense in itself if and only if each open set includes at least two points.

A number of **separation axioms** are used in topology. A space is called T_1 if for any two distinct points x and y there is an open set including x but not y. The open set is said to **separate** the points x and y, hence the designation "separation axioms." A space is called T_2 or **Hausdorff** if for any two distinct points x and y there are two open sets, one including x and the other including y, which do not intersect one another. (Two sets are called **disjoint** if they do not intersect.) A space is called T_3 if for any point x and any closed set K not including x, there are two disjoint open sets, one including x and the other including K. Equivalently, a space is T_3 if for any point x and any neighborhood V of x, there exists a neighborhood U of x whose closure is included in V. A T_3 space that is also T_1 is called **regular**. Finally, a space is called $T_{3\frac{1}{2}}$ if for any point x and any closed set K not including x, there exists a function $f : X \to I$ such that $f(x) = 1$ and $f(K) = 0$. If the space is also T_1, then it is called **completely regular**. There are several other separation axioms, but we will not have occasion to use them here.

LEMMA 0.0. *A completely regular space is regular; a regular space is Hausdorff; and a Hausdorff space is* T_1.

PROOF: Let x, y be distinct points in a space X and let K be a closed subset of X not including x. Suppose that X is $T_{3\frac{1}{2}}$. Then there is a continuous function $f : X \to I$ where $f(x) = 1$ and $f(K) = 0$. Let $U = f^{-1}((1/2, 1])$ and $V = f^{-1}([0, 1/2))$. Since f is continuous, U and V are open. Furthermore, $x \in f^{-1}(1) \subset U$ and $K \subset f^{-1}(0) \subset V$. Therefore X is T_3. If X is completely regular, then it is also T_1, hence regular. Suppose that X is regular. Then, since X is T_1, each point $z \neq y$ in X has an open neighborhood not including y. The union of these neighborhoods is $X \setminus \{y\}$, and is open because it is the union of open sets. Therefore, the point y is closed, and since X is T_3, there exist disjoint open neighborhoods including x and y respectively. Hence X is Hausdorff. Finally, suppose that X is Hausdorff. Then there exist disjoint open neighborhoods U and V of x and y, respectively. Since $y \notin U$, X is T_1. ∎

Warning! A sequence in a space that is not Hausdorff may have more than one limit. For this reason we will take $\lim x_i$ to be a set rather than a point in such spaces.

A space X is called **compact** if for every collection of open sets whose union is X (an **open cover**), there is a finite subcollection whose union is also X (a finite **subcover**). A space is called **countably compact** if every countable open cover has a finite subcover. Obviously, a compact space is countably compact. A space is called **sequentially**

compact if every sequence has a convergent subsequence.

LEMMA 0.1 **[2,p.266].** *A sequentially compact space is countably compact.*

PROOF: Let X be sequentially compact and let $\{U_n\}$ be a countable open cover. Suppose that $\{U_n\}$ has no finite subcover. We may choose $x_n \in X \setminus \bigcup_{k=1}^{n} U_k$ for each n because the union of finitely many U_k cannot cover X. Then the sequence $\{x_n\}$ has a subsequence $\{x'_n\}$ converging to some point x_0. Since $\{U_n\}$ covers X, $x_0 \in U_{n_0}$ for some number n_0. Then there exists a number N such that $x'_n \in U_{n_0}$ for all $n > N$. But $x_n \notin U_{n_0}$ for all $n \geq n_0$, so at most $n_0 - 1$ points of x'_n belong to U_{n_0}, contradicting the previous statement. Therefore $\{U_n\}$ must have a finite subcover. ∎

Sequential Spaces

A space is called **second countable** if it has a countable basis and **first countable** if it has a countable local basis at each point. Obviously, a second countable space is also first countable. A space is called **Fréchet** if the closure of a set includes precisely the limits of all sequences included in the set. A space is called **sequential** if a set is closed precisely when it includes the limits of all sequences included in the set.

LEMMA 0.2**[2,p.78].** *A first countable space is Fréchet and a Fréchet space is sequential.*

PROOF: Let X be first countable. Let A be a subset of X and let x belong to the closure of A. Let $\{U_n\}$ be the countable local basis at x. Since x is in the closure of A, the intersection of A with any open neighborhood of x is nonempty. Therefore, we may choose $x_n \in A \cap \bigcap_{k=1}^{n} U_k$ for each n. Since any open neighborhood of x includes some U_N, it also includes x_n for all $n \geq N$. Thus $\{x_n\}$ converges to x. Since x and A were arbitrary, any point in the closure of any set is a limit of some sequence in the set. Since a limit of a sequence belongs to the closure of any set including that sequence, X is Fréchet. Now let X be Fréchet and let A be a subset of X as before. If A includes the limits of all sequences that it includes, then A is its own closure, and is closed. Therefore X is sequential. ∎

The next two lemmas characterize sequential spaces as those spaces on which a function is continuous if and only if it preserves convergent sequences.

LEMMA 0.3 [2,p.78]. *A function f from a sequential space X to a topological space Y is continuous if and only if $f(\lim x_i) \subset \lim f(x_i)$ for every sequence $\{x_i\}$ in X.*

PROOF: Fix $\{x_i\}$, let $x \in X$, and suppose that f is continuous and $f(x) \notin \lim f(x_i)$. Then there exists an open set V in Y including $f(x)$, but excluding infinitely many of the $f(x_i)$. Since f is continuous, $f^{-1}(V)$ is open in X. But $f^{-1}(V)$ includes x while excluding infinitely many of the x_i. Therefore $x \notin \lim x_i$ and so $f(x) \notin f(\lim x_i)$. We have shown, then, that $f(\lim x_i) \subset \lim f(x_i)$ for the arbitrary sequence $\{x_i\}$. Conversely, suppose that $f(\lim x_i) \subset \lim f(x_i)$ for every sequence $\{x_i\}$ in X. Let K be any closed subset of Y and let $\{x_i\}$ be a sequence in $f^{-1}(K)$. Then $\{f(x_i)\} \subset K$ and so $f(\lim x_i) \subset \lim f(x_i) \subset K$. Therefore $\lim x_i \subset f^{-1}(K)$, and since X is sequential, $f^{-1}(K)$ is closed. Since this holds for any closed subset K of Y, f is continuous. ∎

LEMMA 0.4. *Let X be a topological space. Suppose that, for all topological spaces Y and all functions $f : X \to Y$, f is continuous if and only if $f(\lim x_i) \subset \lim f(x_i)$ for every sequence $\{x_i\}$ in X. Then X is sequential.*

PROOF: Let A be a subset of X. Let Y be Sierpiński space (the set $\{0,1\}$, with open sets $\emptyset$, $\{0\}$, and $\{0,1\}$) and let $f(x)$ be 1 if $x \in A$ and 0 otherwise. Suppose that $\lim x_i \subset A$ for every sequence $\{x_i\}$ in A. Then $f(\lim x_i) \subset f(A) = \{1\}$ and $\lim f(x_i) = \lim 1 = \{1\}$. Since $\{1\} \subset \{1\}$, we have $f(\lim x_i) \subset \lim f(x_i)$, and so f is continuous. But then $A = f^{-1}(1)$ is closed, and so X is sequential. ∎

The following lemma, together with Lemma 0.1, shows that countable compactness and sequential compactness are equivalent in sequential spaces.

LEMMA 0.5 [2,p.266]. *A countably compact sequential space is sequentially compact.*

PROOF: Let X be sequential and countably compact, and let $\{x_n\}$ be a sequence in X. We will show that this sequence has a convergent subsequence. Let $K_n = \{x_k\}_{k=n}^{\infty}$ and let $U_n = X \setminus K_n$ for all n. Then $\{U_n\}$ is a countable cover of X with no finite subcover. Since X is countably compact, there exists a number m such that U_m is not open, and so its complement K_m is not closed. Let x belong to the closure of K_m, but not to K_m itself. Then there exists a sequence $\{y_n\}$ in K_m converging to x. Since $\{y_n\} \subset K_m = \{x_k\}_{k=m}^{\infty} \subset$

$\{x_n\}$, $\{y_n\}$ has a convergent subsequence that is a subsequence of $\{x_n\}$. Since $\{x_n\}$ was arbitrary, X is sequentially compact. ∎

The **topological product** of two spaces X and Y is the Cartesian product $X \times Y$ with the **product topology**, which is generated by the basis consisting of all open rectangles $U \times V$, where U is open in X and V is open in Y. Associated with a product space are the **projections** which map a point onto its first (resp. second) coordinate.

The following lemmas show two ways in which one can force a product of two spaces to be sequential.

LEMMA 0.6 **[2,p.111]**. *The product of two first countable spaces is first countable.*

PROOF: Let X and Y be topological spaces and let $(x,y) \in X \times Y$. Let W be an open neighborhood of (x,y). Since W is the union of open rectangles, at least one such rectangle includes the point (x,y). Therefore, there exist open neighborhoods U and V of x and y, respectively, such that $U \times V \subset W$. Let $\{U_n\}$ be a local basis at x, and let $\{V_n\}$ be a local basis at y. Then there exist numbers i and j such that $U_i \subset U$ and $V_j \subset V$, and so $U_i \times V_j \subset W$. Since W was arbitrary, the collection of all sets of the form $U_i \times V_j$ is a local basis at (x,y). Since this collection is countable and (x,y) was arbitrary, $X \times Y$ is first countable. ∎

LEMMA 0.7 **[2,p.271]**. *The product of two sequential spaces, one of which is sequentially compact, is sequential.*

The proof of this result requires several intermediate lemmas.

A function is a **quotient mapping** if each set in the range of f is closed if and only if its inverse image with respect to f is closed.

The next two lemmas, together with Lemma 0.2, characterize sequential spaces as the images of first countable spaces under quotient mappings.

LEMMA 0.8 **[2,p.134]**. *The image of a sequential space under a quotient mapping is sequential.*

PROOF: Let X be a sequential space and let Y be topological. Let $f : X \to Y$ be an onto quotient mapping. Let A be a subset of Y and let $\{x_n\}$ be any sequence in $f^{-1}(A)$. Suppose that $\lim f(x_n) \subset A$ for every such sequence $\{x_n\}$. Then $f(\lim x_n) \subset \lim f(x_n) \subset A$, so $\lim x_n \subset f^{-1}(A)$ for every sequence $\{x_n\}$, so that $f^{-1}(A)$ is closed. Since f is a quotient mapping, A is closed also. Since A was arbitrary and any closed set includes the limits of any sequences included in it, Y is a sequential space. ∎

LEMMA 0.9 **[2,p.134]**. *Let X be a sequential space, let A be the subspace $\{0, 1, 1/2, \dots\}$ of the real numbers, and let X^A be the space of continuous functions from A to X with the discrete topology. Then the evaluation mapping $f : X^A \times A \to X$ is a quotient mapping.*

PROOF: Since the space X^A is discrete, the basis for the topology of $X^A \times A$ consists of all sets of the form $\{g\} \times U$, where g is in X^A and U is open in A. Let V be open in X. Then $f^{-1}(V) = \bigcup_{g \in X^A} \{g\} \times g^{-1}(V)$. But $g^{-1}(V)$ is open since every function in X^A is continuous. Therefore $f^{-1}(V)$ is open, being a union of open sets, and so f is continuous. If we take $V = X \setminus K$, then $f^{-1}(K) = (X^A \times A) \setminus f^{-1}(V)$ is closed. Let K be a subset of X and suppose that $f^{-1}(K)$ is closed. Since each $g \in X^A$ is closed as a subset, the set $\{g\} \times A$ is closed, and therefore $(\{g\} \times A) \cap f^{-1}(K) = \{g\} \times g^{-1}(K)$ is closed in $\{g\} \times A$. Since $\{g\} \times A$ and A are isomorphic, $g^{-1}(K)$ is closed in A. Thus either $g^{-1}(K)$ is finite, or it includes the point 0. In other words, either K includes only finitely many points of the sequence $\{g(1/n)\}$, or it includes the point $g(0) = \lim g(1/n)$. Now every convergent sequence $\{x_n\} \subset K$ generates a function $g(1/n) = x_n$ in X^A for each of its limit points ($= g(0)$), for the only sets not closed in A are infinite but do not include 0, and since $\{x_n\}$ converges, no infinite subset of $\{x_n\}$ is closed that does not include $\lim x_n$. Therefore every convergent sequence contained in K converges, and so K is closed since X is a sequential space. Thus f is a quotient mapping. ∎

The space $X^A \times A$ is first countable since each point $(g, 1/n)$ is open, and each point $(g, 0)$ has the countable local basis $\{\{(g,0)\} \cup \{(g, 1/m)\}_{m=n}^{\infty}\}_{n=1}^{\infty}$. Therefore every sequential space is the image of a first countable space under a quotient mapping, and by Lemmas 0.2 and 0.8, the image of a first countable space under a quotient mapping is sequential.

A **closed function** maps each closed set to another closed set.

LEMMA 0.10. *Let X be sequentially compact and let Y be sequential. Then the projection $p : X \times Y \to Y$ is closed.*

PROOF: Let F be a closed subset of $X \times Y$. Let $\{y_n\}$ be a convergent sequence in $p(F)$. Then there exists a sequence $\{x_n\}$ in X such that $(x_n, y_n) \in F$ for each n. Since X is sequentially compact, the sequence $\{(x_n, y_n)\}$ has a subsequence $\{(x'_n, y'_n)\}$ such that $\{x'_n\}$ converges. Since the sequence $\{y_n\}$ converges, the subsequence $\{y'_n\}$ converges to $\lim y_n$. Then the compound sequence $\{(x'_n, y'_n)\}$ converges, so $\lim(x'_n, y'_n) \subset F$ and thus $\lim y'_n = \lim y_n \subset p(F)$. Since $\{y_n\}$ was arbitrary and Y is sequential, $p(F)$ is closed, and since F was arbitrary, the projection p is closed. ∎

A space is called **locally compact** (resp. **locally sequentially compact**) if it has a basis, every member of which has a compact (resp. sequentially compact) closure. The space $X^A \times A$ defined above is locally sequentially compact. A basis at each point $(g, 1/n)$ is the set $\{(g, 1/n)\}$, which is closed as well as open, and in which every sequence converges to $(g, 1/n)$. A basis at each point (g,0) consists of all sets of the form $\{g\} \times ([0, 1/n) \cap A) = \{g\} \times ([0, 1/(n+1)] \cap A)$. Every sequence in one of these sets has a convergent subsequence that is either eventually constant or converges to $(g, 0)$. Therefore $X^A \times A$ is locally sequentially compact.

LEMMA 0.11 **[2,p.271]**. *For every locally sequentially compact space X, every sequential space Y, and every quotient mapping $g : Y \to Z$, the mapping $f : X \times Y \to X \times Z$ defined by $f(x, y) = (x, g(y))$ is a quotient mapping.*

PROOF: Let W be a set in $X \times Z$ and let (x_0, z_0) be a point of W. Suppose that $f^{-1}(W)$ is open. Choose a point $y \in g^{-1}(z_0)$, and let U be a neighborhood of x_0 having sequentially compact closure and such that $\overline{U} \times \{y\} \subset f^{-1}(W)$. Since $(x, y) \in f^{-1}(W)$ is equivalent to $f(x, y) = (x, g(y)) \in W$, which is equivalent to $f^{-1}(x, g(y)) = \{x\} \times g^{-1}(g(y)) \subset f^{-1}(W)$, it follows that $\overline{U} \times \{y\} \subset f^{-1}(W)$ whenever $\overline{U} \times g^{-1}(g(y)) \subset f^{-1}(W)$ for each $y \in Y$. Therefore $\overline{U} \times g^{-1}(z_0) \subset f^{-1}(W)$. Let $V = \{z \in Z | \overline{U} \times g^{-1}(z) \subset f^{-1}(W)\}$. Then $g^{-1}(V) = \{y \in Y | \overline{U} \times g^{-1}(g(y)) \subset f^{-1}(W)\} = \{y \in Y | \overline{U} \times \{y\} \subset f^{-1}(W)\}$ by the equivalence noted above. The set $(\overline{U} \times Y) \setminus f^{-1}(W)$ is closed, being the difference of a closed set and an open one. Since $\overline{U}$ is sequentially compact and Y is sequential, the projection $p : \overline{U} \times Y \to Y$ is closed by Lemma 0.10, so $p((\overline{U} \times Y) \setminus f^{-1}(W))$ is closed. But this set is just the complement of $g^{-1}(V)$, which is therefore open. Since g is a quotient mapping, V is open also, and so $U \times V$ is an open neighborhood of (x_0, z_0) included in W. Since (x_0, z_0) was arbitrary, W is open. Conversely, assume W is open. Then W is the union of open rectangles $U_\alpha \times V_\alpha$, where U_α is open in X and V_α is open in Z. Since g is a quotient mapping, $g^{-1}(V_\alpha)$ is open, and so $f^{-1}(W) = \bigcup f^{-1}(U_\alpha \times V_\alpha) = \bigcup U_\alpha \times g^{-1}(V_\alpha)$ is open as well. Thus f is a quotient mapping. ∎

PROOF OF LEMMA 0.7: The space $(X^A \times A) \times (Y^A \times A)$ is first countable by Lemma 0.6. Since $Y^A \times A$ is locally sequentially compact and the evaluation map $g : X^A \times A \to X$ is a quotient mapping, so too is the map $f : (X^A \times A) \times (Y^A \times A) \to X \times (Y^A \times A)$ defined by $f(x, y) = (x, g(y))$ by Lemma 0.11. Then $X \times (Y^A \times A)$ is sequential by Lemma 0.8. Since X is locally sequentially compact and the evaluation map $g : Y^A \times A \to Y$ is a quotient mapping, so too is the map $f : X \times (Y^A \times A) \to X \times Y$ defined as above by Lemma 0.11. Then $X \times Y$ is sequential by Lemma 0.8. ∎

CHAPTER 1

A Theorem of N. N. Luzin

In [5], N. N. Luzin proves the following theorem:

THEOREM 1.1. *The function $F : [a,b] \times [c,d] \to \mathbf{R}$ is continuous if and only if for every continuous function $f : [a,b] \to [c,d]$, the function $F_f : [a,b] \to \mathbf{R}$ defined by $F_f(x) = F(x, f(x))$ is continuous, and for every continuous function $g : [c,d] \to [a,b]$, the function $F^g : [c,d] \to \mathbf{R}$ defined by $F^g(y) = F(g(y), y)$ is continuous.*

Luzin's proof of necessity shows that F_f and F^g are in fact uniformly continuous, but this result is stronger than we need in order to prove this theorem. We give instead a proof that F_f and F^g are continuous when $[a,b]$, $[c,d]$, and $\mathbf{R}$ are replaced by general topological spaces X, Y, and Z, respectively. The general result then applies to the specific case where $X = [a,b]$, $Y = [c,d]$, and $Z = \mathbf{R}$.

NECESSITY: Suppose F is continuous and fix f and g. We will show that F_f and F^g are also continuous. Fix $x \in X$ and let $y = f(x)$ and $z = F_f(x) = F(x,y)$. Let W be an open neighborhood of z. Since F is continuous, $F^{-1}(W)$ is open. Since the open rectangles form a basis for the product topology on $X \times Y$, there exist open neighborhoods U of x and V of y such that $U \times V \subset F^{-1}(W)$. Since f is continuous, $f^{-1}(V)$ is open, and so $G = U \cap f^{-1}(V)$ is an open neighborhood of x such that $F_f(G) \subset W$. Since W and x were arbitrary, F_f is continuous. A similar argument shows that F^g is continuous also. ∎

The proof of sufficiency requires the following lemma:

LEMMA 1.2 **[1,p.51;2,p.78]**. *Let X be a topological space. Let $\{x_n\} \subset X$ converge to x_0. Let $\{x'_n\}$ be a subsequence of $\{x_n\}$. Then $\{x'_n\}$ converges to x_0.*

PROOF: Let U be an open neighborhood of x_0. Then there exists N such that $n \leq N$ for all $x_n \notin U$. Since $\{x'_n\} \subset \{x_n\}$, $\{x'_n\} \setminus U \subset \{x_n\} \setminus U$ and so $\{x'_n\} \setminus U$ is finite. Thus there exists N' such that $n \leq N'$ for all $x'_n \notin U$. Since U was an arbitrary open neighborhood of x_0, $\{x'_n\}$ converges to x_0. ∎

SUFFICIENCY: Suppose that F_f is continuous for every continuous function f and that F is discontinuous at (x_0, y_0). Let

$$m(x_0, y_0) = \lim_{\epsilon \to 0} \Big(\inf_{(x,y) \in B((x_0,y_0),\epsilon)} F(x, y) \Big)$$

and

$$M(x_0, y_0) = \lim_{\epsilon \to 0} \Big(\sup_{(x,y) \in B((x_0,y_0),\epsilon)} F(x, y) \Big),$$

where $B(p, r)$ is the open ball of radius r about the point p, and $m(x_0, y_0)$ and $M(x_0, y_0)$ may be $+\infty$, $-\infty$, or a real number. Let $z_0 = m(x_0, y_0)$ if $m(x_0, y_0) \neq F(x_0, y_0)$ and let $z_0 = M(x_0, y_0)$ otherwise. Then there exists a sequence $\{(x_n, y_n)\}$ of distinct points converging to (x_0, y_0) such that $\lim_{n\to\infty} F(x_n, y_n) = z_0 \neq F(x_0, y_0)$.

We may assume that $x_n \neq x_0$ for all $n > 0$. Otherwise, let $f_n(x) = y_n$ for all $x \in [a, b]$ and $n > 0$. Since f_n is constant, it is continuous and so F_{f_n} is continuous also. Therefore $U_n = F_{f_n}^{-1}(B(F(x_n, y_n), 1/n))$ is open, hence uncountable, so that we may select $x'_n \in U_n \setminus \{x_0\}$ for all $n > 0$. Then we have

$$\lim_{n\to\infty} |F(x'_n, y_n) - F(x_n, y_n)| \leq \lim_{n\to\infty} 1/n = 0,$$

so that $\lim_{n\to\infty} F(x'_n, y_n) = \lim_{n\to\infty} F(x_n, y_n)$. Therefore we may replace x_n by x'_n for all $n > 0$ and then we have what we wanted to assume—that $x_n \neq x_0$ for all $n > 0$.

In order to construct a continuous function through infinitely many points of $\{(x_n, y_n)\}$, we first discard those points for which $x_n < x_0$ if there are infinitely many points for which $x_n > x_0$, and discard those points such that $x_n > x_0$ otherwise. Suppose that we have discarded those points for which $x_n > x_0$. Then we replace $\{(x_n, y_n)\}$ by the subsequence $\{(x'_n, y'_n)\}$, where $(x'_1, y'_1) = (x_1, y_1)$ and for all $n > 1$, $(x'_n, y'_n) = (x_m, y_m)$ where $m = \min_{x_k > x'_{n-1}} k$. By Lemma 1.2, $\lim_{n\to\infty}(x'_n, y'_n) = \lim_{n\to\infty}(x_n, y_n) = z_0$. For ease of notation we will refer to the subsequence $\{(x'_n, y'_n)\}$ as $\{(x_n, y_n)\}$ for the remainder of this proof.

Let

$$f(x) = \begin{cases} y_1 & x \leq x_1, \\ \frac{y_{n+1}(x - x_n) + y_n(x_{n+1} - x)}{x_{n+1} - x_n} & x_n \leq x \leq x_{n+1}, \\ y_0 & x \geq x_0. \end{cases}$$

This function is well-defined since for all $x < x_0$, there exists a point for which $x_n > x$ because $\{(x_n, y_n)\}$ converges to (x_0, y_0), and

$$\frac{y_n(x_n - x_{n-1}) + y_{n-1}(x_n - x_n)}{x_n - x_{n-1}} = y_n = \frac{y_{n+1}(x_n - x_n) + y_n(x_{n+1} - x_n)}{x_{n+1} - x_n}$$

at each point x_n. Therefore f is well-defined at the points x_n, and $f(x_n) = y_n$.

Obviously f is continuous everywhere except possibly at x_0. Now let $\epsilon > 0$ and let $V = (y_0 - \epsilon, y_0 + \epsilon)$. Then there exists a number N such that $y_n \in V$ for all $n > N$. Then for all $x \geq x_{N+1}$, either $x \geq x_0$, in which case $f(x) = y_0 \in V$, or $x_n \leq x \leq x_{n+1}$ for some $n > N$, in which case $f(x) \in [y_n, y_{n+1}] \subset V$. Thus f is continuous at x_0 as well, so F_f is continuous and therefore

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} F_f(x_n) = F_f(x_0) = F(x_0, y_0).$$

But this contradicts our choice of the original sequence $\{(x_n, y_n)\}$ so that $\lim_{n\to\infty} F(x_n, y_n)$ was z_0. Therefore F must be continuous at (x_0, y_0). ∎

Since the proof of necessity given above holds for general topological spaces X, Y, and Z, the challenge lies in generalizing the proof of sufficiency. Several generalizations are possible. In [7], A. Rosenthal shows that F is continuous if it is continuous on the graph of every convex differentiable function from [a,b] to [c,d] and from [c,d] to [a,b], and gives an example of a discontinuous function that is continuous on the graph of every twice differentiable function from [a,b] to [c,d] and from [c,d] to [a,b]. He then proceeds to generalize his result to n-dimensional Euclidean space. In the next chapter we present a different generalization of this theorem. Although we require the function to be continuous on the graphs of all continuous functions, we generalize the setting from the real line and closed intervals to more general topological spaces.

CHAPTER 2

Generalizations of Luzin's Theorem

In this chapter we present two original generalizations of Theorem 1.1. The first generalization places weaker conditions on the spaces involved and a stronger condition on the function to be proven continuous than does the second generalization. Since their proofs are similar, for the second generalization only the modifications necessary to the proof of the first are given. The proof of these generalizations resembles the proof of Theorem 1.1, except that the function f is constructed differently. It may be thought of as a "bumpy plane" where the elevation of the nth "bump" is given by the function f_n, and the direction of the nth "bump" is given by the function p_n. The sets W_n are constructed to keep the "bumps" separated one from another, and the sets G_n are constructed to keep the "bumps" low near the limit of the sequence through which the function f was constructed. Unfortunately, this construction seems to be limited to sequences due to the difficulty of constructing the sets W_n and G_n in the case of a net.

Our first lemma will be used to construct a subsequence with unique x-coordinates so that a single-valued function can be constructed through the subsequence.

LEMMA 2.1. *Let X be a T_1 space and suppose that the sequence of (not necessarily distinct) points $\{x_n\}$ converges to x_0, and $x_n \neq x_0$ for all n. Then $\{x_n\}$ has an infinite subsequence of distinct points $\{x'_n\}$, which converges to x_0 by Lemma 1.2.*

PROOF: Let $\{x'_n\}$ be the sequence of all x_n such that $x_m \neq x_n$ for all $m < n$. Suppose that $\{x'_n\}$ has finite length r. Since X is T_1, there exist open neighborhoods U_n of x_0 such that $x'_n \notin U_n$ for $1 \leq n \leq r$. Let $W = \bigcap_{n=1}^{r} U_n$. Since $\{x'_n\}$ is finite, W is an open neighborhood of x_0. Then there exists a number N such that $x_n \in W$ for all $n > N$. Let $m = \inf\{n | x_n \in W\}$. Since $x_m \in W$ and $x_n \notin W$ for all $n < m$, $x_m \neq x_n$ for all $n < m$. Therefore x_m is a member of the sequence $\{x'_n\}$. But $x'_n \notin W$ for $1 \leq n \leq r$, contradicting our assumption that $\{x'_n\}$ is finite. Now for all m and n such that $m \neq n$, there exist numbers m' and n' such that $x'_m = x_{m'}$, $x'_n = x_{n'}$, and $m' \neq n'$. Without loss of generality, we may assume that $m' < n'$. Then, since $x_{n'}$ is in the sequence $\{x'_n\}$, $x_{m'} \neq x_{n'}$. Therefore $x'_m \neq x'_n$ for all m and n such that $m \neq n$. ∎

The next lemma constructs the sets W_n that separate the "bumps" in the "bumpy plane" described above. The construction is complicated by the requirement that each W_n be open; otherwise letting $W_n = \bigcap_{i=0}^{n-1} V_{in} \cap \bigcap_{j=n+1}^{\infty} U_{nj}$ would suffice, where U_n and V_n are defined below. But since the sets W_n must be open, we must take them to be the intersection of only finitely many open sets.

LEMMA 2.2. *Let X be Hausdorff and suppose that the sequence of distinct points $\{x_n\} \subset X$ converges to x_0, and that $x_n \neq x_0$ for all n. Then there exist pairwise disjoint open neighborhoods W_n of x_n for all $n > 0$.*

PROOF: Since X is Hausdorff, there exist disjoint open neighborhoods U_{ij} and V_{ij} of x_i and x_j, respectively, for all i and j such that $0 \leq i < j$. Since $\{x_n\}$ converges to x_0 and U_{0j} is an open neighborhood of x_0 for all j, there exists a number N_j for each j such that $x_n \in U_{0j}$ for all $n > N_j$. Let

$$W_n = \bigcap_{i=0}^{n-1} V_{in} \cap \bigcap_{j=n+1}^{N_n} U_{nj} \cap \bigcap \{U_{0k} | 1 \leq k < n, x_n \in U_{0k}\}.$$

Then each W_n is open, being the intersection of finitely many open sets. We show that $W_m \cap W_n = \emptyset$ for all m and n such that $m < n$. Suppose that $n \leq N_m$. Then $W_m \subset U_{mn}$ and $W_n \subset V_{mn}$. Since $U_{mn} \cap V_{mn} = \emptyset$, $W_m \cap W_n = \emptyset$. Suppose, to the contrary, that $n > N_m$. Then $x_n \in U_{0m}$, so $W_n \subset U_{0m}$. Since $W_m \subset V_{0m}$ and $U_{0m} \cap V_{0m} = \emptyset$, $W_m \cap W_n = \emptyset$. Thus the sets W_n are indeed pairwise disjoint. ∎

The next lemma constructs the sets G_n, which control the altitude of the "bumps" near the limit point of the sequence through which we will construct the function f.

LEMMA 2.3. *Let Y be first countable and locally path–connected. Then Y has at each point y_0 a countable local basis of path–connected open sets $\{G_n\}$ such that $G_{n+1} \subset G_n$ for all n.*

PROOF: Let $\{G_n\}$ be a countable local basis for Y at y_0. Let $G_n^1 = \bigcap_{k=1}^{n} G_k$ for all n. Since Y is locally path–connected, there exist path–connected open neighborhoods G_n^2 of

y_0 for each n such that $G_n^2 \subset G_n^1$. Let $G_n^3 = \bigcup_{k=n}^{\infty} G_k^2$ for all n. Then $\{G_n^3\}$ is also a countable local basis at y_0 because $G_k^2 \subset G_k^1 \subset G_n$ for all $k \geq n$ and so $G_n^3 \subset G_n$. Fix two points $x, z \in G_n^3$. Then there exist numbers k, m such that $x \in G_k^2$ and $z \in G_m^2$. Since $y_0 \in G_n^2$ for all n, there exist paths $p, q : I \to Y$ such that $p(0) = x$, $p(1) = q(0) = y_0$, and $q(1) = z$. Let

$$r(x) = \begin{cases} p(2x) & x \leq 1/2 \\ q(2x-1) & x \geq 1/2. \end{cases}$$

Then r is continuous, $r(0) = x$, and $r(1) = z$. Finally, for all n, we have

$$G_n^3 = \bigcup_{k=n}^{\infty} G_k^2 \supset \bigcup_{k=n+1}^{\infty} G_k^2 = G_{n+1}^3.$$

Thus $\{G_n^3\}$ satisfies the conclusion of the lemma. ∎

The first generalization of Luzin's Theorem places fewer conditions on the domain X of the functions f than the second generalization will, and no conditions at all on the range Z of F, but only at the expense of supposing that F has continuous x–sections. We note that the condition that $X \times Y$ be sequential may be satisfied either by taking X to be first countable (by Lemmas 0.6 and 0.2) or by taking X to be sequential and either X or Y to be sequentially compact (by Lemma 0.7).

THEOREM 2.4. *Let X be completely regular. Let Y be first countable and locally path–connected. Let Z be a topological space. Suppose that $X \times Y$ is sequential, that the function $F : X \times Y \to Z$ has continuous x–sections, and that for any continuous function $f : X \to Y$, the function $F_f : X \to Z$ defined by $F_f(x) = F(x, f(x))$ is continuous. Then F is also continuous.*

PROOF: Suppose that F is discontinuous. Then there is an open set $W \subset Z$ such that $F^{-1}(W)$ is not open in $X \times Y$. Since $X \times Y$ is sequential, there must be a sequence $\{(x_n, y_n)\}$ in $(X \times Y) \setminus F^{-1}(W)$ converging to a point (x_0, y_0) in $F^{-1}(W)$. But since W is an open neighborhood of $F(x_0, y_0)$ and $F(x_n, y_n) \notin W$ for all n, $\{F(x_n, y_n)\}$ cannot even accumulate to $F(x_0, y_0)$. Therefore $F(x_0, y_0)$ is not a limit point of any subsequence of $\{F(x_n, y_n)\}$.

Let $\{(x_n^1, y_n^1)\}$ be the sequence of all (x_n, y_n) such that $x_n = x_0$. Suppose that $\{(x_n^1, y_n^1)\}$ is infinite. Then since F has continuous x–sections, by Lemma 1.2 we have $\lim F(x_n^1, y_n^1) =$

$\lim F(x_0, y_n^1) \ni F(x_0, y_0)$. This contradicts our earlier conclusion that $F(x_0, y_0)$ is not a limit point of any subsequence of $\{F(x_n, y_n)\}$. Thus $\{(x_n^1, y_n^1)\}$ is finite. Let $\{(x_n^2, y_n^2)\}$ be the sequence of all (x_n, y_n) such that $x_n \neq x_0$. Then $\{(x_n^2, y_n^2)\} = \{(x_n, y_n)\} \setminus \{(x_n^1, y_n^1)\}$ is infinite, so by Lemma 1.2 it converges to (x_0, y_0). Then, by Lemma 2.1, $\{x_n^2\}$ has a subsequence of distinct points $\{x_n^3\}$. For each n choose m such that $x_n^3 = x_m^2$ and let $y_n^3 = y_m^2$. Then $\{(x_n^3, y_n^3)\}$ is a subsequence of $\{(x_n^2, y_n^2)\}$ with distinct x-coordinates.

Let $\{G_n\}$ be a basis for Y at y_0 as in Lemma 2.3. Since $\{y_n^3\}$ converges to y_0 and G_1 is an open neighborhood of y_0, there exists a number N such that $y_n^3 \in G_1$ for all $n > N$. Let $(x_n^4, y_n^4) = (x_{n+N}^3, y_{n+N}^3)$ for all $n \geq 1$. Then $y_n^4 \in G_1$, so that $m_n = \sup\{m | 1 \leq m \leq n, y_n^4 \in G_m\}$ exists for each n. Since G_{m_n} is path-connected for all n, there exist paths $p_n : I \to Y$ such that $p_n(0) = y_0$, $p_n(1) = y_n^4$, and $p_n(I) \subset G_{m_n}$ for all n. Since $\{x_n^4\} \subset \{x_n^3\}$, $\{x_n^4\}$ is a sequence of distinct points and $x_n^4 \neq x_0$ for all n. Therefore, by Lemma 2.2, there exist pairwise disjoint open neighborhoods W_n of x_n^4 for each n. Furthermore, since X is completely regular, there exist continuous functions $f_n : X \to I$ such that $f_n(x_n^4) = 1$ and $f_n(X \setminus W_n) = 0$ for all n.

Let

$$f(x) = \begin{cases} p_n(f_n(x)) & x \in W_n \\ y_0 & x \notin \bigcup_{n=1}^{\infty} W_n. \end{cases}$$

Let V be open in Y. If $y_0 \notin V$, then $f^{-1}(V) = \bigcup_{n=1}^{\infty} f_n^{-1}(p_n^{-1}(V))$ is open. If $y_0 \in V$, then there exists a number M such that $G_M \subset V$. Since $\{y_n^4\}$ converges to y_0 and G_M is an open neighborhood of y_0, there exists a number N such that $y_n^4 \in G_M$ for all $n > N$. Therefore $p_n(I) \subset G_{m_n} \subset G_M \subset V$ for each $n > \max(M, N)$, where m_n was defined above for each n. Therefore

$$f^{-1}(V) = X \setminus f^{-1}(Y \setminus V) = X \setminus \bigcup_{n=1}^{N} f_n^{-1}(p_n^{-1}(Y \setminus V))$$

is open, being the complement of the union of finitely many closed sets. Thus f is continuous, so

$$\lim F(x_n^4, y_n^4) = \lim F(x_n^4, f(x_n^4)) = \lim F_f(x_n^4) \ni F_f(x_0) = F(x_0, y_0).$$

This contradicts our assumption that F is discontinuous, from which we deduced that $F(x_0, y_0)$ is not a limit point of any subsequence of $F(x_n, y_n)$. ∎

The second generalization of Luzin's Theorem dispenses with the condition that F have continuous x-sections by assuming additional hypotheses on the spaces X and Z. Since much of the proof is identical to the proof of Theorem 2.4, we give only the additions to that proof needed to prove this theorem.

THEOREM 2.5. *Let X be first countable, dense in itself, and completely regular. Let Y be first countable and locally path-connected. Let Z be first countable and T_3. Given a function $F : X \times Y \to Z$, suppose that for every continuous function $f : X \to Y$ the function $F_f : X \to Z$ defined by $F_f(x) = F(x, f(x))$ is continuous. Then F is continuous.*

PROOF: The alterations to the proof of Theorem 1 required to prove Theorem 2 follow. First, since X and Y are first countable, $X \times Y$ is first countable by Lemma 0.6, hence sequential by Lemma 0.2. Next, if the sequence $\{(x_n^1, y_n^1)\}$ is infinite, we argue as follows:

Let $\{U_n\}$ and $\{V_n\}$ be countable bases for X at x_0 and for Z at $F(x_0, y_0)$ respectively. For any open neighborhood W of $F(x_0, y_0)$, $Z \setminus W$ is a closed set not including $F(x_0, y_0)$. Since Z is T_3, there exist open neighborhoods U and V of $F(x_0, y_0)$ and $Z \setminus W$. Since $\{V_n\}$ is a countable basis at $F(x_0, y_0)$, there exists a number m such that $V_m \subset U$. Then, since $U \subset (Z \setminus V) \subset W$ and $Z \setminus V$ is closed, the closure of V_m is a subset of $Z \setminus V$, which is in turn a subset of W. Let K_n be the closure of V_n for each n. Then for any open nbd W of $F(x_0, y_0)$, there exists n such that $K_n \subset W$.

Let $f_n : X \to Y$ be defined by $f_n(x) = y_n$ for each n. Then let

$$W_n = \bigcap\{U_i | 1 \leq i \leq n, x_n \in U_i\} \setminus F_{f_n}^{-1}(\bigcup\{K_j | 1 \leq j \leq n, F(x_n, y_n) \notin K_j\}).$$

Note that W_n is open for each n, being the difference of a finite intersection of open sets and the continuous inverse image of a finite union of closed sets. Since X is dense in itself, this means that each W_n includes at least two points, and in particular, $W_n \setminus \{x_0\}$ is nonempty. Now for all n, choose $x_n^2 \in W_n \setminus \{x_0\}$ and let $y_n^2 = y_n$. First we show that the sequence $\{x_n^2\}$ converges to x_0. Let U be an open neighborhood of x_0. Then there exists m such that $U_m \subset U$. Since U_m is an open neighborhood of x_0 and $\{x_n\}$ converges to x_0, there exists N such that $x_n \in U_m$ for all $n > N$. Then by the choice of x_n^2, $x_n^2 \in U_m \subset U$ for all $n > \max(m, N)$. Thus $\{x_n^2\}$ converges to x_0.

Finally, after having shown that a subsequence of $\{F(x_n^2, y_n^2)\}$ converges to $F(x_0, y_0)$, we show that the corresponding subsequence of $\{F(x_n, y_n)\}$ converges to $F(x_0, y_0)$. Let V be an open neighborhood of $F(x_0, y_0)$. Then there exists a number m such that $K_m \subset V$. If $\{F(x_n^2, y_n^2)\}$ converges to $F(x_0, y_0)$, then there exists a number N such that $F(x_n^2, y_n^2) \subset V_m \subset K_m$ for all $n > N$. Then $F(x_n, y_n) \in K_m \subset V$ for all $n > \max(m, N)$, otherwise the choice of x_n^2 leads to a contradiction. Therefore $\{F(x_n, y_n)\}$ converges to $F(x_0, y_0)$. ∎

CHAPTER 3

Counterexamples

In this chapter, we present two counterexamples to show the necessity for some of the hypotheses in Theorem 2.4. The first counterexample shows that the space Y must be at least locally connected for the conclusion of Theorem 2.4 to hold. Local connectedness is weaker than local path–connectedness, so a stronger counterexample may well exist.

EXAMPLE 3.1. *The Perforated Unit Interval*

Let $Y = I \setminus \{1/n | n \in Z^+\}$. Let $X = Z = I$. Then if $f : X \to Y$ is continuous, $f(X)$ is connected (and locally connected) since X is. If $0 \in f(X)$, then $f(X) = 0$. Otherwise, let $y \in f(X) \setminus 0$. Then there exists a number n such that $1/n < y$, and $f^{-1}([0, 1/n))$ and $f^{-1}((1/n, 1])$ are disjoint nonempty open sets whose union is X. Let

$$F(x, y) = \begin{cases} 0 & (x, y) = (0, 0) \\ \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0). \end{cases}$$

Then F is continuous except at $(0, 0)$, and F is separately continuous everywhere. Since the only continuous function through $(0, 0)$ is constant, F satisfies the hypotheses of the theorem, but is not continuous.

The second counterexample (3.5) shows that the space X must be completely regular. It is very complicated, and we must build up to it by a series of three preliminary examples (3.2, 3.3, and 3.4) At each step we prove the regularity of the space and the constancy of real–valued functions on some set. In Example 3.5, this will be the entire space, and we use this fact to construct a function that is discontinuous, but continuous on every continuous function graph.

EXAMPLE 3.2 **[8,pp.68-70]**. *Open Ordinal Space* $[0, \Gamma)$

Let Γ be an ordinal number. Then the open ordinal space $[0, \Gamma)$ is the set of all ordinal numbers less than Γ, with the **order** topology generated by the subbasis consisting of all sets of the form $[0, \alpha] = [0, \alpha+1) = \{x \in [0, \Gamma) | x < \alpha+1\}$ and $(\alpha, \Gamma) = \{x \in [0, \Gamma) | x > \alpha\}$.

(The closed ordinal space $[0,\Gamma]$ is simply $[0,\Gamma+1)$.) For notational convenience let X denote $[0,\Gamma)$ in the next proof.

We show that X is regular. Let K be closed in X, let $U = X \setminus K$, and let $x \in U$. Since U is open, there exist ordinals α and β such that $x \in (\alpha,\beta] \subset U$, since the sets of this form constitute a basis for X. Observe that $X \setminus (\alpha,\beta] = [0,\alpha] \cup (\beta,\Gamma)$ is open, so that $(\alpha,\beta]$ and $X \setminus (\alpha,\beta]$ are disjoint open neighborhoods of x and K, respectively. Therefore X is T_3. Let α and β be ordinal numbers. Suppose, without loss of generality, that $\alpha < \beta$. Then $[0,\alpha]$ and (α,Γ) are disjoint open neighborhoods of α and β respectively. Thus X is Hausdorff and therefore regular.

Now let Γ be the first uncountable ordinal number Ω. We show that every continuous real-valued function f on X is "eventually" constant, i.e., is constant on some interval (α,Ω). We claim that there exists a sequence $\{\alpha_n\}$ such that $|f(\beta) - f(\alpha_n)| < 1/n$ for each n for all $\beta > \alpha_n$. Otherwise, there exists a number m and an increasing sequence $\{\gamma_n\}$ such that $|f(\gamma_{n+1}) - f(\gamma_n)| \geq 1/m$. But the sequence $\{\gamma_n\}$ converges to its least upper bound γ, whereas the sequence $\{f(\gamma_n)\}$ does not converge. This is impossible since f is continuous. Therefore, there exists a sequence $\{\alpha_n\}$ such that $|f(\beta) - f(\alpha_n)| < 1/n$ for each n for all $\beta > \alpha_n$. This sequence has an upper bound α, and for any $\beta > \alpha$, $f(\beta) = \lim f(\alpha_n)$ since $|f(\beta) - f(\alpha_n)| < 1/n$. Thus f is constant on (α,Ω).

Finally, we show that every continuous real-valued function f on $[0,\Omega]$ is constant on $(\alpha,\Omega]$ for some ordinal α. We have already shown that it is constant on (α,Ω). Now every open neighborhood of Ω includes the set $(\alpha,\Omega]$ for some ordinal $\alpha < \Omega$ by the definition of $(\alpha,\Omega]$. Thus Ω is in the closure of (α,Ω) and so $f(\Omega)$ is in the closure of $f((\alpha,\Omega))$, which consists of a single point x. Thus $f(\Omega) = x$ and so f is constant on $(\alpha,\Omega]$ for some ordinal α.

EXAMPLE 3.3 **[8,pp.106-7]**. *The (Deleted) Tihonov Plank*

Let ω be the first infinite ordinal number, and let Ω be the first uncountable ordinal number. The Tihonov plank T is the space $[0,\Omega] \times [0,\omega]$, and the deleted Tihonov plank T_∞ is the subspace $T \setminus \{(\Omega,\omega)\}$.

We show that T is regular. Let K be closed in T, let $U = T \setminus K$, and let $x \in U$. Since U

is open, there exist ordinal numbers α,β,γ, and δ such that $x \in ((\alpha, \beta] \times (\gamma, \delta]) \subset T$. Let $W = (\alpha, \beta] \times (\gamma, \delta]$. Then

$$X \setminus W = (([0, \alpha] \cup (\beta, \Omega]) \times [0, \omega]) \cup ([0, \Omega] \times ([0, \gamma] \cup (\delta, \omega])),$$

which is open. Since W and $X \setminus W$ are disjoint open neighborhoods of x and K respectively, T is T_3. Let (α, β) and (γ, δ) be distinct points of T. If $\alpha \neq \gamma$, there exists a neighborhood U of α in $[0, \Omega]$ excluding γ; otherwise, $\beta \neq \delta$ since the points are distinct, and there exists a neighborhood V of β in $[0, \omega]$ excluding δ. Then the product $U \times [0, \omega]$ or $[0, \Omega] \times V$ is a neighborhood of (α, β) excluding (γ, δ). Thus T is T_1 and therefore regular.

We show that T_∞ is regular also. Let K be closed in T_∞ and let $x \in T_\infty \setminus K$. Then there exists K' closed in T such that $K = K' \cap T_\infty$. Since $x \notin K$ and $x \in T_\infty$, $x \notin K'$. Therefore there exist disjoint open neighborhoods of x and K' in T, and their intersections with T_∞ are disjoint open neighborhoods of x and K. Thus T_∞ is T_3. Let $y \neq x$ be a point of T^∞. Then x has a neighborhood in T excluding y. But the intersection of this neighborhood with T_∞ also excludes y since it is a subset of the T–neighborhood. Therefore T_∞ is T_1, hence regular.

Finally, we show that every continuous real-valued function f on T_∞ can be continuously extended to T. For each ordinal number n in $[0, \omega)$, there exists an ordinal γ_n such that $f(\alpha, n) = x_n$ for all $\alpha \in (\gamma_n, \Omega]$. Furthermore, there exists an ordinal γ_ω such that $f(\alpha, \omega) = x_\omega$ for all $\alpha \in (\gamma_\omega, \Omega)$. Let $\gamma = \sup_{n \in [0,\omega]} \gamma_n$. Then $\gamma < \Omega$ and $f(\gamma, n) = x_n$ for all $n \in [0, \omega]$. Since $\{f(\gamma, n)\}$ converges to $f(\gamma, \omega)$, $\{x_n\}$ converges to x_ω and so the extension of f to T by taking $f(\Omega, \omega) = x_\omega$ is continuous.

EXAMPLE 3.4 **[8,pp.109-11]**. *The Tihonov Corkscrew*

Let T_I, T_{II}, T_{III}, and T_{IV} be four copies of the deleted Tihonov plank T_∞, corresponding to the four quadrants of the Euclidean plane. For each ordinal number $n \in [0, \omega)$, identify the points (Ω, n) in T_I and T_{II} and in T_{III} and T_{IV}. Also identify the points (α, ω) in T_{II} and T_{III} for each ordinal number $\alpha \in [0, \Omega)$ to obtain a space P. Let P_n be a copy of P for each integer n. For each integer n and each ordinal number $\alpha \in [0, \Omega)$, identify the points (α, ω) in T_I of P_n and T_{IV} of P_{n+1} to obtain an infinite corkscrew S. We say that a point of P_n is at level n if it belongs to T_I and its second coordinate is ω. By the above identification, a point of T_{IV} whose second coordinate is ω is at level $n-1$. All other points of P_n have a level between n and $n-1$, say $n-1/2$ for definiteness. We write L(x) to denote the level of the point x. To S we adjoin the ideal points a^+ and a^- to form the Tihonov corkscrew X. These two points act like infinity points at either end of the corkscrew. We define $L(a^+) = +\infty$ and $L(a^-) = -\infty$. Then the basis neighborhoods for a^+ are the sets $\{x|L(x) > n\}$ for all integers n, and basis neighborhoods of a^- are the sets $\{x|L(x) < n\}$ for all integers n.

Since each quadrant of each level P_n of S is homeomorphic to the regular space T, S is regular. To show that X is regular, we need consider only those cases involving a^+ or a^-. If K is a closed set not including a^+, then $X \setminus K$ is open and therefore includes $\{x|L(x) > n\}$ for some integer n. Then $\{x|L(x) > n+1\}$ and $\{x|L(x) < n+1\}$ are disjoint open neighborhoods of a^+ and K, respectively. The argument for a^- is symmetric. Thus X is regular.

Let f be a continuous real-valued function on X. We show that $f(a^+) = f(a^-)$. Since the restriction of f to each quadrant of S may be extended to the missing point (Ω, ω) and f is eventually constant on each $T_I \cap T_{IV}$ and each $T_{II} \cap T_{III}$, by induction f is constant on a set including $\{(\alpha, \omega)|\alpha \in (\beta, \Omega]\}$ for some ordinal number β in each quadrant of S. Therefore there exists a sequence $\{a_i\}_{-\infty}^{\infty}$ on which f is constant such that $\lim_{i\to\infty} a_i = a^+$ and $\lim_{i\to-\infty} a_i = a^-$. Then $f(a^+) = f(a^-)$.

EXAMPLE 3.5 **[3;8,pp.111-3]**. *Hewitt's Condensed Corkscrew*

Let $T = S \cup \{a^+\} \cup \{a^-\}$ be the Tihonov corkscrew and let A be the Cartesian product $T \times [0, \Omega)$. Let X be the subset $S \times [0, \Omega)$ of A. Let A_λ be the subset $T \times \{\lambda\}$ of A. Let $\Gamma : X \times X \to [0, \Omega)$ be a bijection and let π_1 and π_2 be the coordinate projections from $X \times X$ to X. We define the function ψ from $A \setminus X$ onto X by $\psi(a^+, \lambda) = \pi_1(\Gamma^{-1}(\lambda))$ and $\psi(a^-, \lambda) = \pi_2(\Gamma^{-1}(\lambda))$. Then for any two points x and y of X, the sets $\psi^{-1}(x)$ and $\psi^{-1}(y)$ both intersect $A_{\Gamma(x,y)}$.

The topology of A is generated by basis neighborhoods N of each point $x \in X$, which satisfy $\psi^{-1}(N \cap X) \subset N$. Let σ be the product topology on $A = T \times [0, \Omega)$ where

$[0, \Omega)$ is taken to have the discrete topology. A basis neighborhood of $x \in X$ is constructed inductively by taking N_0 to be a neighborhood of $\{x\} \cup \psi^{-1}(x)$ in the σ topology and taking N_{n+1} to be a neighborhood of $N_n \cup \psi^{-1}(N_n \cap X)$ in the σ topology. Then $N = \bigcup_{n=0}^{\infty} N_n$ is a basis neighborhood of x in the topology of A, and as stated above,

$$\psi^{-1}(N \cap X) = \bigcup_{n=0}^{\infty} \psi^{-1}(N_n \cap X) \subset \bigcup_{n=0}^{\infty} N_{n+1} \subset N.$$

Let x and y be distinct points in X. Since each A_λ is T_1 in the σ topology, we may construct a basis neighborhood of x inductively, as above, choosing each N_n to exclude y. Then the basis neighborhood N also excludes y. Therefore X is T_1.

Fix $x \in X$ and let N be a basis neighborhood of x. Then N is the union of relatively open sets $N^\lambda \subset A_\lambda \cap X$. We claim that the closure of N in X is the union, for each $\lambda \in [0, \Omega)$, of the closure of N^λ in A_λ. Suppose not. Then there exists a point y which is not in the closure of any N^λ but every neighborhood of which intersects N. Let M be a neighborhood of y. Since there exist neighborhoods of y in the σ topology that do not intersect any N^λ, the intersection of M and N must include a neighborhood of an ideal point (a^+, λ) or (a^-, λ). Call this point z. There exist numbers m and n such that $z \in M_m \cap N_n$, where M_m and N_n were defined above in the construction of basis neighborhoods. Now $\psi(z) \in M_{m-1} \cap N_{n-1}$. As before, this intersection must include a neighborhood of an ideal point. Repeating this process eventually yields $x \in M$ or $y \in N$. The former cannot hold for all neighborhoods M since X is T_1, and the latter contradicts the choice of y. Therefore the closure of an open neighborhood is the union of the closures of its parts (which also happens to be its closure in the σ topology).

Since each open neighborhood of a point x in the σ topology includes the closure of another open neighborhood of x, each neighborhood N of a point x contains the closure of some other neighborhood N' of x. Therefore X is T_3. Since X is T_1, it is also regular.

We show that any function f defined on X may be extended to a continuous function $\hat{f}$ defined on A. Let $\hat{f}(x) = f(x)$ for all $x \in X$ and let $\hat{f}(a) = f(\psi(a))$ for all $a \in A \setminus X$. Then for any open set U, $\hat{f}^{-1}(U) = f^{-1}(U) \cup (f \circ \psi)^{-1}(U) = f^{-1}(U) \cup \psi^{-1}(f^{-1}(U))$, and $f^{-1}(U)$ is an open subset of X. Therefore $f^{-1}(U) = X \cap N = X \cap \bigcup_{n=0}^{\infty} N_n$, where N is open in A and the sets N_n are the open sets from the σ topology used to construct N. We may choose each set N_n to include only those ideal points that are required by the choice of N_{n-1}, so that $N_n \setminus X = \psi^{-1}(X \cap N_{n-1})$ or, equivalently, $N_n = (X \cap N_n) \cup \psi^{-1}(X \cap N_{n-1})$. Then

$$\begin{aligned}
\hat{f}^{-1}(U) &= \left(X \cap (\bigcup_{n=0}^{\infty} N_n)\right) \cup \left(\psi^{-1}(X \cap (\bigcup_{n=0}^{\infty} N_n))\right) \\
&= \bigcup_{n=0}^{\infty} (X \cap N_n) \cup \bigcup_{n=0}^{\infty} \psi^{-1}(X \cap N_n) \\
&= \bigcup_{n=0}^{\infty} N_n = N
\end{aligned}$$

because $N_n = (X \cap N_n) \cup \psi^{-1}(X \cap N_{n-1}) \subset (X \cap N_n) \cup \psi^{-1}(X \cap N_n) \subset N_{n+1}$ and so $\bigcup_{n=0}^{\infty} N_n \subset \bigcup_{n=0}^{\infty}(X \cap N_n) \cup \bigcup_{n=0}^{\infty} \psi^{-1}(X \cap N_n) \bigcup_{n=0}^{\infty} N_{n+1} = \bigcup_{n=1}^{\infty} N_n \subset \bigcup_{n=0}^{\infty} N_n$. Thus $\hat{f}^{-1}(U)$ is open and so $\hat{f}$ is continuous.

We show that every real-valued function on X is constant. Let x and y be points of X. Then $\psi(a^+, \Gamma(x,y)) = x$ and $\psi(a^-, \Gamma(x,y)) = y$. Since $\hat{f}$ is continuous on A, hence on $A_{\Gamma(x,y)}$, $f(x) = \hat{f}(a^+, \Gamma) = \hat{f}(a^-, \Gamma) = f(y)$. Since x and y were arbitrary, f is constant.

Now let $Y = I$ and let Z be Sierpiński space. Let $\{x_n\}$ be a convergent sequence in X that does not include its limit, for example the sequence $\{(0,n)\}$ in the quadrant T_I of the plank P_0 of the corkscrew A_0. Define $F(x_n, 1/n) = 1$, and $F(x,y) = 0$ for all other points (x,y) of $X \times Y$. Then F is discontinuous since $\lim F(x_n, 1/n) = \lim 1 = 1$ whereas $F(\lim(x_n, 1/n)) = F(\lim(x_n), 0) = 0$. Nevertheless, we will show that F has continuous x–sections and is continuous on the graph of every continuous function from X to Y. Since $\{1\}$ is the only nontrivial closed set in Sierpiński space, a function f into Sierpiński space is continuous if and only if $f^{-1}(1)$ is closed. Fix x, and let F_x be the restriction of F to $X \times \{x\}$. Then $F_{1/n}^{-1}(1) = \{(x_n, 1/n)\}$. Since $X \setminus \{x_n\}$ is the union of open neighborhoods of each of its points not including x_n, and similarly for $Y \setminus \{1/n\}$, it follows that $X \setminus \{x_n\} \times Y \cup X \times Y \setminus \{1/n\} = X \times Y \setminus \{(x_n, 1/n)\}$ is open and so $\{(x_n, 1/n)\}$ is closed. For other values of x, $F_x^{-1}(1) = \emptyset$, which is trivially closed. Thus F has continuous x–sections. Now let f be a continuous (hence constant) function from X to Y. Then $F_f^{-1}(1) = 1/n$ if $f(x) = x_n$, or $F_f^{-1}(1) = \emptyset$ otherwise. Thus F_f is continuous for every continuous f.

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