An Excursion into Differential Equations

By

Patrick Monahan

Submitted in Partial Fulfillment of the Requirements
for the Degree of
Master of Science
in
Mathematics

YOUNGSTOWN STATE UNIVERSITY
May, 2024

An Excursion into Differential Equations

Patrick Monahan

I hereby release this thesis to the public. I understand that this thesis will be made available from the OhioLINK ETD Center and the Maag Library Circulation Desk for public access. I also authorize the University or other individuals to make copies of this thesis as needed for scholarly research.

Signature:

Patrick Monahan, Student Date

Approvals:

Eric Wingler PhD, Thesis Advisor Date

Padraic Taylor PhD, Committee Member Date

Thomas Smotzer, Committee Member Date

Salvatore A. Sanders, PhD, Dean, College of Graduate Studies Date

Abstract

This paper will be exploring the theory of differential equations through a specific example. We will build up the theorems to guarantee a solution. Then we will explore a power series solution. With the power series solution we are able to notice some similarities to a famous type of differential equation. We will then build up the theory behind these equations know as Bessel functions and finally be able to write our solution in terms of them. We will also explore generalizations of our equations and see if we can say anything about how their solutions may look.

Contents

Introduction

We start with a simple question. Can we find a particular solution to the differential equation $y' = y^2 + x^2$ with $y(0) = 0$? This turns into a dive into the many different areas of nonlinear differential equations. The first question you must ask when examining nonlinear differential equations is if they even are guaranteed to have a solution. This will take us into the world of Banach spaces and contraction mappings. These will allow us to show that all differential equations of a certain form have a solution and we will see that ours will satisfy this. After finding a power series solution, we can recognize our solution in terms of Bessel functions. Finally it will be nice to explore and see if we can say anything about the equation $y' = y^n + x^n$ for any $n \in \mathbb{N}$.

Preliminaries

We will begin our exploration of differential equations first by recalling some facts about the real line building up to $\mathbb{R}$ being a complete metric space. This fact we be useful later when exploring existence and uniqueness of solutions for differential equations.

Definition P.1: A *metric space* (X, d) is a set X equipped with a function called a metric $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$,

$$d(x, y) = 0 \iff x = y \tag{1}$$

$$d(x, y) = d(y, x) \tag{2}$$

$$d(x, z) \leq d(x, y) + d(y, z). \tag{3}$$

Some examples of metric spaces include $(\mathbb{R}, d)$ where $d(x, y) = |x - y|$ and $(\mathbb{C}, d)$ where $d(x, y) = \sqrt{(x - y)\overline{(x - y)}}$.

Definition P.2: Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in a metric space (X, d). $\{x_n\}_{n\in\mathbb{N}}$ *converges* to a point $x \in X$ if for each $\varepsilon > 0$, there is a $N \in \mathbb{N}$ such that if $n \geq N$ then $d(x, x_n) < \varepsilon$.

Definition P.3: Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in a metric space (X, d). $\{x_n\}_{n\in\mathbb{N}}$ is *Cauchy* if for each $\varepsilon > 0$, there is a $N \in \mathbb{N}$ such that if $n, m \geq N$ then $d(x_n, x_m) < \varepsilon$.

Notice the key difference here in these definitions being a sequence converges if the points get arbitrarily close to one point in the space whereas a Cauchy sequence is one whose points get arbitrarily close to each other. We now introduce a concept relating the two ideas.

Definition P.4: A metric space (X, d) is *complete* if every Cauchy sequence converges to a point in the space.

The important part of this definition is that the sequence must converge to a point in the space, this is highlighted in the following example.

Example P.5: Consider $\mathbb{Q}$ with the standard metric being the absolute difference of two elements and the sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ where $x_1 = 3, x_2 = 3.1, x_3 = 3.14, x_4 = 3.141, \ldots$.

This is a Cauchy sequence in $\mathbb{Q}$ that converges to π which is know to be irrational. Hence $\mathbb{Q}$ with the standard metric is not complete.

To build up to showing that $\mathbb{R}$ is complete with the standard metric we will require a few definitions and lemmas beforehand.

Definition P.6: A sequence of points $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ is *bounded* if there exists a number $M > 0$ such that $|x_n| \leq M$ for each $n \in \mathbb{N}$

Definition P.7-10: Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}$. If for each $n \in N$,

(7) $x_n \leq x_{n+1}$ then $\{x_n\}_{n\in\mathbb{N}}$ is *increasing.*
(8) $x_n < x_{n+1}$ then $\{x_n\}_{n\in\mathbb{N}}$ is *strictly increasing.*
(9) $x_n \geq x_{n+1}$ then $\{x_n\}_{n\in\mathbb{N}}$ is *decreasing.*
(10) $x_n > x_{n+1}$ then $\{x_n\}_{n\in\mathbb{N}}$ is *strictly decreasing.*

Definition P.11: A sequence of points $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ is *monotonic* if the sequence is increasing or decreasing.

Lemma P.12: If $\{x_n\}_{n\in\mathbb{N}}$ is a monotonic and bounded sequence of points in $\mathbb{R}$ then it converges.

Proof. First suppose that $\{x_n\}_{n\in\mathbb{N}}$ is increasing and bounded. Since our sequence is bounded there exists a supremum $x \in \mathbb{R}$. Let $\varepsilon > 0$, then $x - \varepsilon$ is not an upper bound for $\{x_n\}_{n\in\mathbb{N}}$. So there is a $N \in \mathbb{N}$ such that $x_N > x - \varepsilon$. Now since $\{x_n\}_{n\in\mathbb{N}}$ is increasing for any $n \geq N$, we have $x - \varepsilon < x_n < x + \varepsilon$. Hence $|x_n - x| < \varepsilon$ and $\{x_n\}_{n\in\mathbb{N}}$ converges to x. The case where our sequence is decreasing follows similarly. □

Definition P.13: Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points and $\{n_i\}_{i\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$. Then $\{x_{n_i}\}_{i\in\mathbb{N}}$ is a *subsequence* of $\{x_n\}_{n\in\mathbb{N}}$.

Lemma P.14: Every sequence in $\mathbb{R}$ has a monotonic subsequence.

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. We say x_m is a turn back point of the sequence if $x_n \leq x_m$ for all $n \geq m$. If there are infinitely many of these points then the subsequence of them is monotonic. If there are finitely many let x_{n_1} be the largest of these points. Then there is a $x_{n_2} > x_{n_1}$ where $n_2 > n_1$. Furthermore since x_{n_1} was the last turn back point there is a $x_{n_3} > x_{n_2}$ with $n_3 > n_2$. Constructing this sequence inductively, we arrive at a increasing subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$. □

Proof. By the previous Lemmas we have that every bounded sequence has a monotonic-subsequence. Then that sequence would be bounded and monotonic and thereby converge. □

Theorem P.16: $\mathbb{R}$ with the standard metric is complete.

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}$. First note that Cauchy sequences are bounded so by the Bolzano Weierstrass Theorem $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ that converges to some $x \in \mathbb{R}$. Now to see that $\{x_n\}_{n\in\mathbb{N}}$ converges to x. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that if $n, m \geq N$ we have,

$$|x_n - x_m| < \frac{\varepsilon}{2}.$$

Now chose $K \in N$ such that if $k \geq K$ then,

$$|x_{n_k} - x| < \frac{\varepsilon}{2}.$$

Now suppose $m \geq N$, if we choose n_k such that $n_k \geq N$ and $k \geq K$ we have that,

$$|x_m - x| \leq |x_m - x_{n_k}| + |x_{n_k} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\{x_n\}_{n\in\mathbb{N}}$ converges to x. Thus since $\{x_n\}_{n\in\mathbb{N}}$ was arbitrary we have that $\mathbb{R}$ is complete with the standard metric. □

One last important idea we will discuss are contraction mappings and fixed points.

Definition P.17: Let (X, d) be a metric space and $f : X \to X$. Then f is a *contraction* if there is a $\alpha \in [0, 1)$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in X$.

Notice by the above definition that all contractions are continuous. An important idea about contraction mappings is highlighted in the next theorem.

Theorem P.18: (Contracting Mapping Theorem) If $f : X \to X$ is a contraction on a complete metric space (X, d) then f has a unique fixed point i.e. $\exists! x \in X$ such that $f(x) = x$.

Proof. First for uniqueness suppose that $f : X \to X$ has two fixed points $x_1, x_2 \in X$ and let α be the contraction constant of f. Then $d(f(x_1), f(x_2)) \leq \alpha d(x_1, x_2)$. Since x_1, x_2 are fixed points we have $d(x_1, x_2) \leq \alpha d(x_1, x_2)$. Since $\alpha < 1$ it follows that $d(x_1, x_2) = 0$ or $x_1 = x_2$. Now for existence let $x_0 \in X$ and for each $n \in \mathbb{N}$ let $x_n = f(x_{n-1})$. Consider the sequence $\{x_n\}_{n=0}^{\infty}$. First if $n \in \mathbb{N}$ then,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f(x_n), f(x_{n-1})) \leq \alpha d(x_n, x_{n-1}) \\ &= \alpha d(f(x_{n-1}), f(x_{n-2}) \leq \alpha^2 d(x_{n-2}, x_{n-3}) \\ &. \\ &. \\ &. \\ &\leq \alpha^n (d(x_1, x_0). \end{aligned}$$

Now if $n, m \in \mathbb{N}$ with $n > m$ then,

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_m) \\
&\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + d(x_{n-2}, x_m) \\
&. \\
&. \\
&. \\
&\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + d(x_{n-2}, x_{n-3}) + \cdots + d(x_{n-(n-m-1)}, x_m) \\
&\leq \alpha^{n-1}(d(x_1, x_0) + \alpha^{n-2}(d(x_1, x_0) + \alpha^{n-3}(d(x_1, x_0) + \ldots \alpha^m(d(x_1, x_0) \\
&= (\alpha^{n-1} + \alpha^{n-2} + \alpha^{n-3} + \cdots + \alpha^m)d(x_1, x_0) \\
&= \alpha^m(\alpha^{n-m-1} + \alpha^{n-m-2} + \alpha^{n-m-3} + \ldots \alpha)d(x_1, x_0) \\
&= \alpha^m d(x_1, x_0) \sum_{k=0}^{n-m-1} \alpha^k \\
&< \alpha^m d(x_1, x_0) \sum_{k=0}^{\infty} \alpha^k.
\end{aligned}$$

Notice that $\sum_{k=0}^{\infty} \alpha^k$ is a convergent geometric series since $\alpha < 1$. Hence for any $n, m \in N$ with $n > m$ we have,

$$d(x_n, x_m) < \alpha^m d(x_1, x_0) \sum_{k=0}^{\infty} \alpha^k = \frac{\alpha^m}{1-\alpha} d(x_1, x_0).$$

Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $\frac{\alpha^N}{1-\alpha} d(x_1, x_0) < \varepsilon$. Thus if $n > m \geq N$ we have,

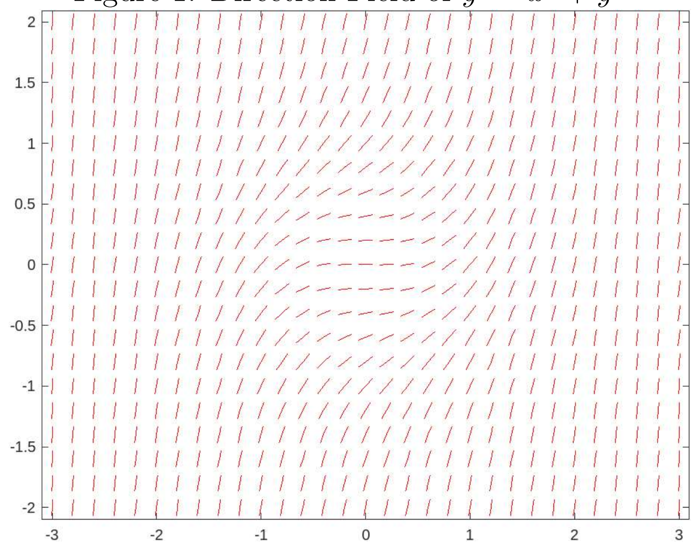
$$d(x_n, x_m) < \varepsilon.$$

Hence $\{x_n\}_{n=0}^{\infty}$ is Cauchy and since (X, d) is complete it converges to some $x \in X$. Finally to see that this x is our desired fixed point observe since contraction maps are continuous we have that $\{f(x_n)\}_{n=0}^{\infty} \to f(x)$, but $\{f(x_n)\}_{n=0}^{\infty} = \{x_n\}_{n=1}^{\infty}$ by how we define our sequence. Thus by uniqueness of limits of sequence we have that $f(x) = x$. □

Picard's Existence and Uniqueness Theorem

To start with our examination of our differential equation, $y' = y^2 + x^2$ with the initial condition $y(0) = 0$, let us look at the direction field of it and see if we can extract any information from it.

Figure 1: Direction Field of $y' = x^2 + y^2$



Notice that as one would expect nearby the origin we have almost zero slope due to the squaring of each term in our differential equation. Furthermore as soon as we cross the line $y = \pm 1$ we begin to shoot of very fast in positive and negative y respectively. So we might be inclined to say that our solution might look like tan or a x^{2n+1} for some $n \in N$.

One the most important ideas in differential equations is when we can guarantee a solution to a given differential equation. In this section we will build up to an important theorem regarding this and then be able to apply this to our differential equation.

Definition 1.1: A *Normed Vector Space* or simply a Normed Space is a vector space V over the complex numbers with a function $||.|| : V \to [0, \infty)$ such that for all $x, y \in V$ and $\alpha \in \mathbb{C}$;

$$||x|| = 0 \iff x = 0_V \tag{4}$$
$$||\alpha * x|| = |\alpha| * ||x|| \tag{5}$$
$$||x + y|| \leq ||x|| + ||y|| \tag{6}$$

Note here $|\alpha| = \sqrt{\alpha\overline{\alpha}}$, so if $\alpha \in \mathbb{R}$ then we get the absolute value of α.

Proposition 1.2: Let $(V, ||.||)$ be a normed space and define

$$d : V \times V \to [0, \infty)$$

by,

$$d(x, y) = ||x - y||$$

then (V, d) is a metric space.

Proof. First note if $x, y \in V$ we already have that $d(x, y) \geq 0$ by how the norm is defined. Now for $x, y \in V$ we have that,

$$d(x, y) = ||x - y|| = 0 \iff x - y = 0 \iff x = y.$$

Now for symmetry if $x, y \in V$ then,

$$d(x,y) = ||x-y|| = ||(-1)(y-x)|| = |(-1)| * ||y-x|| = d(y,x).$$

Lastly for the triangle inequality if $x, y, z \in V$, then

$$d(x,z) = ||x-z|| = ||x-y+y-z|| \leq ||x-y|| + ||y-z|| = d(x,y) + d(y,z).$$

Hence d is metric on V. □

We will say the metric defined above is induced by the norm.

Definition 1.3: A complete normed space with respect to the metric induced by the norm is a *Banach Space*

Proposition 1.4: The space $B[a,b] = \{f : [a,b] \to \mathbb{R} : f \text{ is bounded}\}$ with $||f|| = \sup\{|f(x)| : x \in [a,b]\}$ is a normed space.

Proof. Firstly by how $||.||$ was defined, $||f|| \geq 0$ for all $f \in B[a,b]$. Now observe that,

$$||f|| = 0 \iff f(x) = 0 \ \forall x \in [a,b] \iff f = 0.$$

Now if $\alpha \in \mathbb{R}$ and $f \in B[a,b]$ then,

$$||\alpha f|| = \sup\{|\alpha f(x)| : x \in [a.b]\} = |\alpha| \sup\{|f(x)| : x \in [a.b]\} = |\alpha| * ||f||.$$

Finally if $f, g \in B[a,b]$ and $x \in [a,b]$

$$|(f+g)(x)| \leq |f(x)| + |g(x)| \leq ||f|| + ||g||.$$

And since the above is true for all x we have that $||f+g|| \leq ||f|| + ||g||$. Thus $(B[a,b], ||.||)$ is a normed space. □

Theorem 1.5: $B[a,b]$ is a Banach space.

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $B[a,b]$. Let $\varepsilon > 0$, then there is a $N \in \mathbb{N}$ such that $||f_n - f_m|| < \varepsilon$ for all $n, m \geq N$. Thus we have that,

$$|f_n(x) - f_m(x)| \leq ||f_n - f_m|| < \varepsilon$$

for each $x \in [a,b]$. Thus $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ for each $x \in [a,b]$. Now $\mathbb{R}$ is complete so for each x let $f(x)$ be the limit of $\{f_n(x)\}$. Now Cauchy sequences are bounded so there is $M > 0$ such that $||f_n|| \leq M$ or for each x we have that,

$$|f_n(x)| \leq ||f_n|| \leq M.$$

Thus as we take $n \to \infty$ we see that $|f(x)| \leq M$ for all $x \in [a,b]$. Thereby $f \in B[a,b]$. Finally to see that f is indeed the limit of $\{f_n\}_{n\in\mathbb{N}}$, let $x \in [a,b]$ and $n \geq N$ then,

$$\lim_{k\to\infty} |f_k(x) - f_n(x)| = |f_n(x) - f(x)| < \varepsilon.$$

Since this was true for any $x \in [a,b]$, we have $||f_n - f|| < \varepsilon$. Hence $\{f_n\}_{n\in\mathbb{N}}$ converges to f and $B[a,b]$ is a Banach space. □

The following is an important lemma regarding closed subspaces of Banach Spaces.
Lemma 1.6: A closed subspace of a Banach Space is a Banach Space.

Proof. This follows directly from the fact that closed sets contain all their accumulation points. Let $B^* \subseteq B$ where B is a Banach Space and B^* is closed. If $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in B^* then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in B. So it has a limit point $x \in B$. But as we said $x \in B^*$ since it is closed. Hence B^* is a Banach Space. □

Definition 1.7: Let U be an open subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}$. f satisfies a *Lipschitz condition* in y on U if $\exists M > 0$ such that

$$|f(x, y_2) - f(x, y_1)| \leq M|y_2 - y_1|$$

for all $(x, y_1), (x, y_2) \in U$.

Theorem 1.8: Let U be an open subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ such that the partial derivative in y exists everywhere and is bounded on U. Then f satisfies a Lipschitz condition in y on U.

Proof. Since f_y is bounded on U there is an $M > 0$ such that $|f_y(x, y)| \leq M$ for all $(x, y) \in U$. Let $(x, y_1), (x, y_2) \in U$. Without loss of generality suppose that $y_1 < y_2$. Then by the Mean Value Theorem there is a $c \in (y_1, y_2)$ such that,

$$\frac{f(x, y_1) - f(x, y_2)}{y_2 - y_1} = f_y(x, c).$$

So we have,

$$|f(x, y_1) - f(x, y_2)| = |f_y(x, c)||y_2 - y_1| \leq M|y_2 - y_1|.$$

Hence f satisfies a Lipschitz condition in y on U. □

Let us now take a moment to examine our differential equation. Write $y' = y^2 + x^2$ as $y' = f(x, y)$ where $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = y^2 + x^2$. Furthermore, if we restrict f to some open ball U about the origin with radius $r > 0$. Then $\frac{\partial f}{\partial y} = 2y \leq 2r$. Hence the partial derivative exists and is bounded on U. Thus by Theorem 1.7 $f(x, y) = y^2 + x^2$ satisfies a Lipschitz condition in y on U.

This fact along with the next theorem will show that our differential equation is guaranteed a solution and will give us some insight on what that solution might look like.

Theorem 1.9: (Picard's Existence and Uniqueness Theorem) Let U be an open subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ be a continuous function satisfying a Lipschitz condition in y on U with $(x_0, y_0) \in U$. Then there is an $\varepsilon > 0$ such that $\frac{dy}{dx} = f(x, y)$ has a unique solution $y = p(x)$ on $[x_0 - \varepsilon, x_0 + \varepsilon]$ with $p(x_0) = y_0$.

Proof. Since U is open there is a $\delta > 0$ such that the closed ball, $\overline{B((x_0, y_0), \delta)} \subset U$. Let $K = \max\{|f(x, y)| : (x, y) \in \overline{B((x_0, y_0), \delta)}\}$ and $M > 0$ be the Lipschitz constant for f. Now choose $\varepsilon > 0$ such that $\varepsilon < \frac{1}{M}$ and $N = [x_0 - \varepsilon, x_0 + \varepsilon] \times [y_0 - K\varepsilon, y_0 + K\varepsilon] \subset \overline{B((x_0, y_0), \delta)}$. Define now a map

$$A : C[x_0 - \varepsilon, x_0 + \varepsilon] \to C[x_0 - \varepsilon, x_0 + \varepsilon]$$

by,

$$[A(g)](x) = y_0 + \int_{x_0}^{x} f(t, g(t))\, dt.$$

Unfortunately here A may not be a proper map on $C[x_0 - \varepsilon, x_0 + \varepsilon]$ since there may be functions $g \in C[x_0 - \varepsilon, x_0 + \varepsilon]$ such that $(t, g(t)) \notin U$, i.e. $f(t, g(t))$ is not defined. So we consider a closed subset $\tilde{C} \subseteq C[x_0 - \varepsilon, x_0 + \varepsilon]$ where if $h \in \tilde{C}$ then $h(x_0) = y_0$ and $|h(x) - y_0| \leq K\varepsilon$. Now if $h \in \tilde{C}$ and $x \in [x_0 - \varepsilon, x_0 + \varepsilon]$ then clearly $[A(h)](x_0) = y_0$ and we have,

$$\begin{aligned} |[A(h)](x) - y_0| &= \left| \int_{x_0}^{x} f(t, h(t))\, dt \right| \\ &\leq \int_{x_0}^{x} |f(t, h(t))|\, dt \\ &\leq \int_{x_0}^{x} K\, dt \\ &= (x - x_0)K \\ &\leq |x - x_0|K \\ &\leq K\varepsilon. \end{aligned}$$

Hence $A(h) \in \tilde{C}$ so we have $A : \tilde{C} \to \tilde{C}$. Now let $p_1, p_2 \in \tilde{C}$ and $x \in [x_0 - \varepsilon, x_0 + \varepsilon]$ then,

$$\begin{aligned} |[A(p_1)](x) - [A(p_2)](x)| &= \left| \int_{x_0}^{x} f(t, p_1(t)) - f(t, p_2(t))\, dt \right| \\ &\leq \int_{x_0}^{x} |f(t, p_1(t)) - f(t, p_2(t))|\, dt \\ &\leq \int_{x_0}^{x} M|p_1(t)) - p_2(t))|\, dt \\ &= M \int_{x_0}^{x} |p_1(t)) - p_2(t))|\, dt \\ &\leq |x - x_0| M \max\{|p_1(x^*) - p_2(x^*)| : x^* \in [x_0 - \varepsilon, x_0 + \varepsilon]\} \\ &\leq \varepsilon M ||p_1 - p_2||. \end{aligned}$$

Since this holds for all $x \in [x_0 - \varepsilon, x_0 + \varepsilon]$, we have $||A(p_1) - A(p_2)|| \leq \varepsilon M ||p_1 - p_2||$. Observe that $\varepsilon M < \frac{1}{M} M = 1$, giving us that A is a contraction on $\tilde{C}$. Since as we saw with

$B[a,b]$, $C[a,b]$ is a Banach space and $\tilde{C}$ is a closed subset of $C[a,b]$ and thus a Banach space. Thereby there is a unique $y \in \tilde{C}$ such that $A(y) = y$. In other words y satisfies the equation,

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt.$$

Converting that back into a differential equation we have,

$$\frac{dy}{dx} = f(t, y(t)) \;\; ; \;\; y(x_0) = y_0.$$

□

The theorem above now allows us to guarantee a solution to the equation $y' = y^2 + x^2$ with the initial condition $y(0) = 0$. Now whether or not the solution has a closed form we still have to figure out. However if we take advantage of the proof of Theorem 1.18 we can gain some insight into what the solution will look like. Notice that we showed that if $f(x,y)$ satisfies a Lipschitz condition in y then $y(t) = y(x_0) + \int_{x_0}^{x} f(t, y(t))\, dt$ is a contraction on the space of bounded functions. Thus if we start with an initial guess that goes through our initial condition we can build a sequence that will converge to our solution. This technique is known as Picard Iteration. First, to see this more clearly, let us consider a classic example.

Example 1.10: Consider the differential equation $y' = y$ with the initial condition $y(0) = 1$. We can rewrite this as an integral equation $y = 1 + \int_0^x y(t)\, dt$. Famously this has solution $y = e^x$. If we want to apply the technique of Picard Iteration, let us start with an initial guess $y_0 = 1$. Then,

$$y_1 = 1 + \int_0^x y_0(t)\, dt = 1 + \int_0^x 1\, dt = 1 + x$$

$$y_2 = 1 + \int_0^x y_1(t)\, dt = 1 + \int_0^x 1 + t\, dt = 1 + x + \frac{x^2}{2}$$

$$y_3 = 1 + \int_0^x y_2(t)\, dt = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

.

.

.

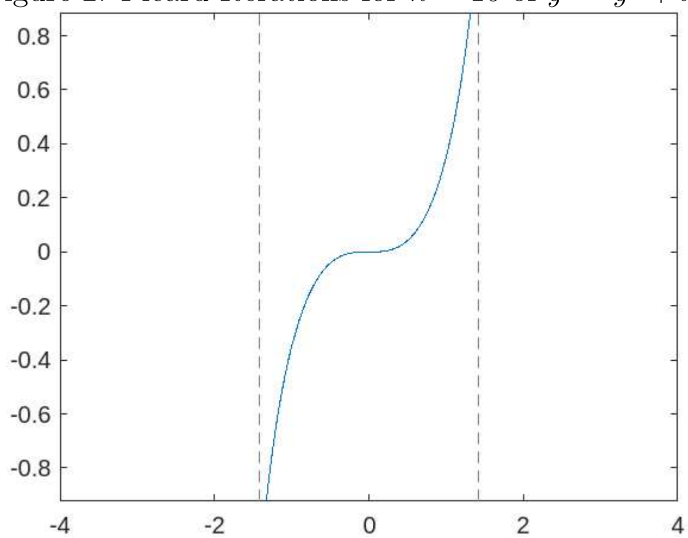
$$y_n = 1 + \int_0^x y_{n-1}(t)\, dt = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots \frac{x^n}{n!}.$$

Notice that the $\lim_{n\to\infty} y_n = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ which is the power series expansion for e^x. Now let us see what happens when we apply this technique to $y' = y^2 + x^2$.

Example 1.11: We will start with an initial guess of $y_0 = x$ since we want $y(0) = 0$. Running a short Matlab code to calculate some of the terms we have,

$$
\begin{aligned}
y_1 &= 0 + \int_0^x y_0(t)^2 + t^2\, dt = \int_0^x t^2 + t^2\, dt = \frac{2x^3}{3} \\
y_2 &= \int_0^x \left(\frac{2t^3}{3}\right)^2 + t^2\, dt = \int_0^x \frac{2^2 t^6}{3^2} + t^2\, dt = \frac{2^2 x^7}{7(3^2)} + \frac{x^3}{3} \\
y_3 &= \int_0^x \left(\frac{2^2 t^7}{7(3^2)} + \frac{t^3}{3}\right)^2 + t^2\, dt = \frac{x^3}{3} + \frac{x^7}{3^2(9)} + \frac{2^3 x^{11}}{3^3(7)(11)} + \frac{2^4 x^{15}}{3^4(5)(7^2)} \\
y_4 &= \int_0^x y_3(t)^2 + t^2\, dt = \frac{x^3}{3} + \frac{x^7}{3^2(7)} + \frac{2x^{11}}{3^3(7)(11)} + \frac{41x^{15}}{3^4(5)(7^2)(11)} \\
&\quad + \frac{2^4(37)x^{19}}{3^6(7^2)(11)(19)} + \frac{2^5 331 x^{23}}{3^7(5)(7^3)(11^2)(23)} + \frac{2^8 x^{27}}{3^{11}(5)(7^3)(11)} + \frac{2^8 x^{31}}{3^{11}(5^2)(7^4)(31)}.
\end{aligned}
$$

Figure 2: Picard Iterations for $n = 10$ of $y' = y^2 + x^2$



As you can see the terms for this sequence blow up a lot faster then in the previous example. So while we may not be able to extract a closed form using this technique we do gain some insight into what a power series solution might look like. Notice that since t^2 is always in our integral we end up with a $\frac{x^3}{3}$ as the first term of our solution. Also we always will go up by powers of 4 and we see we have an asymptote forming around $x = \sqrt{2}$, but slightly larger than directly at $\sqrt{2}$.

Power Series Solution

To employ the technique of power series solutions for differential equations we need to find a way to rewrite our equation into a linear differential equation. Luckily for us $y' = y^2 + x^2$ is a special type of differential equation.

Definition 2.1: A *Riccati Equation* is a first-order differential equation of the form

$$y'(x) = a(x) + b(x)y(x) + c(x)y^2(x)$$

where $a(x) \neq 0$ and $c(x) \neq 0$.

"There is no general way to find a particular solution, which means that one cannot always solve Riccati's equation. Occasionally one can get lucky." [2] Based on the title of this section you would be fair to assume we got lucky. This luck comes from a powerful technique for these equations highlighted in the next lemma.

Lemma 2.2: A Riccati Equation can be converted to a second order linear differential equation.

Proof. Consider the Riccati Equation $y'(x) = a(x) + b(x)y(x) + c(x)y^2(x)$ and let $y = \frac{-u'}{(c)(u)}$. Then $y' = \frac{-u''}{cu} + \frac{(u')^2}{(c)u^2} + \frac{c'u'}{c^2u}$ so we have,

$$\begin{aligned}
&\frac{-u''}{cu} + \frac{(u')^2}{(c)u^2} + \frac{c'u'}{c^2u} = a + b\frac{-u'}{cu} + c\left(\frac{u'}{cu}\right)^2 \\
&\iff \\
&\frac{-u''}{cu} + \frac{c'u'}{c^2u} = a + b\frac{-u'}{cu} \\
&\iff \\
&cu'' - c'u' - bcu' + ac^2u = 0.
\end{aligned}$$

□

Thus for the equation $y' = y^2 + x^2$, since we have $c(x) = 1$, we make the substitution $y = \frac{-u'}{u}$ giving us,

$$u'' + ux^2 = 0.$$

Thus we have converted our first-order nonlinear differential equation in y into a second-order linear differential equation in u. Now before we proceed with our power series solution we need one more definition that will allow us to write our solution in a cleaner way.

Definition 2.3: For $n \in \mathbb{N} \bigcup \{0\}$ the *Factorial Function* $(t)_n$ defined for all $t \in \mathbb{R}$ is given by $(t)_n = \begin{cases} t(t+1)(t+2)\dots(t+n-1) & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases}$

Theorem 2.4 The equation $u'' + ux^2 = 0$ has a power series solution of the form $u(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)(1-\frac{1}{4})_n} \left(\frac{x}{2}\right)^{4n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)(1+\frac{1}{4})_n} \left(\frac{x}{2}\right)^{4n+1}$.

Proof. As with a typical power series solution we will guess that $u(x) = \sum_{n=0}^{\infty} a_n x^n$. This gives us,

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$u'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$$
$$u''(x) = \sum_{n=2}^{\infty} a_n (n)(n-1)x^{n-2}.$$

Substituting these into our differential equation we have,

$$\sum_{n=2}^{\infty} a_n (n)(n-1)x^{n-2} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$
$$\iff$$
$$\sum_{n=2}^{\infty} a_n (n)(n-1)x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

Our goal now is to write the left hand side as one series. This is achieved by re-indexing the second series to $n \to n+2$ and the first series to $n \to n-2$.

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

Now we may pull the first two terms of the first series off to combine the two as one series,

$$3a_2 + 6a_3 x + \sum_{n=2}^{\infty} (a_{n+2}(n+2)(n+1) + a_{n-2})x^n = 0.$$

Since we have the right hand side being equal to zero we know that each term of our series must be zero for all x giving us,

$$
\begin{aligned}
a_0 &= a_0 & a_1 &= a_1 \\
a_2 &= 0 & a_3 &= 0 \\
a_4 &= \frac{-a_0}{4(3)} & a_5 &= \frac{-a_1}{5(4)} \\
a_6 &= 0 & a_7 &= 0 \\
a_8 &= \frac{a_0}{8(7)(4)(3)} & a_9 &= \frac{a_1}{9(8)(5)(4)} \\
&. \\
&. \\
&. \\
a_{4n} &= \frac{(-1)^n a_0}{4n(4n-1)(4n-4)(4n-5)\dots(4)(3)} \\
a_{4n+1} &= \frac{(-1)^n a_1}{(4n+1)(4n)(4n-3)(4n-4)\dots(5)(4)} \\
a_{4n+2} &= 0 \\
a_{4n+3} &= 0.
\end{aligned}
$$

Notice that for the terms not equal to zero we can rewrite them in a clever way as such,

$$
\begin{aligned}
4n(4n-1)(4n-4)(4n-5)\dots(4)(3) &= 4n(4n-4)(4n-8)\dots(4n-1)(4n-5)\dots \\
&= 4^n(n!)(4^n)(n-\frac{1}{4})(n-1-\frac{1}{4})\dots(1-\frac{1}{4}) \\
&= 2^{4n}n!(1-\frac{1}{4})_n.
\end{aligned}
$$

Similarly we get that $(4n+1)(4n)(4n-3)(4n-4)\dots(5)(4) = 2^{4n}n!(1+\frac{1}{4})_n$. So we have that,

$$
\begin{aligned}
a_{4n} &= \frac{(-1)^n a_0}{2^{4n}n!(1-\frac{1}{4})_n} \\
a_{4n+1} &= \frac{(-1)^n a_1}{2^{4n}n!(1+\frac{1}{4})_n}.
\end{aligned}
$$

So we may write our series solution as,

$$
u(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)(1-\frac{1}{4})_n} \left(\frac{x}{2}\right)^{4n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)(1+\frac{1}{4})_n} \left(\frac{x}{2}\right)^{4n+1}.
$$

□

So with the solution above we now have a general solution to $y' = y^2 + x^2$ given by,

$$
y(x) = \frac{a_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}4n}{2(n!)(1-\frac{1}{4})_n} \left(\frac{x}{2}\right)^{4n-1} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(4n+1)}{2(n!)(1+\frac{1}{4})_n} \left(\frac{x}{2}\right)^{4n}}{a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)(1-\frac{1}{4})_n} \left(\frac{x}{2}\right)^{4n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)(1+\frac{1}{4})_n} \left(\frac{x}{2}\right)^{4n+1}}.
$$

You may notice an issue here with this solution. We started with a first order equation so we should only have one unknown. But notice what happens when we plug in our initial condition $y(0) = 0$. The first series on top will go to 0 but the second has the constant term leaving us with $-a_1$. Likewise on the bottom we will be left with a_0. Hence $a_1 = 0$ and so the a_0 terms cancel giving us our particular solution,

$$y(x) = \frac{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}4n}{2(n!)(1-\frac{1}{4})_n} \left(\frac{x}{2}\right)^{4n-1}}{\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)(1-\frac{1}{4})_n} \left(\frac{x}{2}\right)^{4n}}.$$

If we were to expand out the first few terms of each series we have,

$$y(x) = \frac{\frac{x^3}{3} - \frac{x^7}{2^2(3)(7)} + \frac{x^{11}}{2^4(3)(7)(11)} + \dots}{1 - \frac{x^4}{2^2(3)} + \dots}.$$

If we were to cut of the bottom after the first two terms we see we would have a vertical asymptote at $x \approx \sqrt[4]{12}$. Which is indeed slightly larger than $\sqrt{2}$, as $\sqrt[4]{12} = 1.86120971\dots$.

If you have studied differential equations before you may know why we chose to write our solution in that specific way, specifically why we wrote the $(1 - \frac{1}{4})_n$ term in the denominator like that. This idea will be highlighted in the next section.

A Detour into Bessel Functions

As the title suggests we are going to take a slight detour into Bessel Equations and Functions. These functions as we will see will be closely related to our solution to the differential equation we found.

Definition 3.1: For $\nu \geq 0$ a linear second order differential equation of the form,

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0$$

is called a *Bessel Equation of order* ν.

Before we may find solutions to these equations we require a bit of background theory in solving second order equations.

Definition 3.2: For a differential equation of the form $a(x)y''(x)+b(x)y'(x)+c(x)y(x) = 0$, a point x_0 is an *ordinary point* if both $\frac{b(x)}{a(x)}$ and $\frac{c(x)}{a(x)}$ have power series representations about x_0 with a positive radius of convergence i.e. the function is analytic at x_0. Otherwise, we say x_0 is a *singular point*.

Note that in the above definition we are using analytic in regards to real-valued functions only. As this will be all we need in our study here. However, there is an analogous definition for complex-valued functions.

Observe that for the Bessel Equation, $\frac{b(x)}{a(x)} = \frac{x}{x^2} = \frac{1}{x}$ and $\frac{c(x)}{a(x)} = \frac{x^2-\nu^2}{x^2} = 1 - \frac{\nu^2}{x^2}$. So we have a single singular point at 0 since neither function is differentiable at 0.

Definition 3.3: A singular point x_0 of $a(x)y''(x) + b(x)y'(x) + c(x)y(x) = 0$ is a *regular singular point* if both $(x-x_0)\frac{b(x)}{a(x)}$ and $(x-x_0)^2\frac{c(x)}{a(x)}$ are analytic at x_0. Otherwise, x_0 is a *irregular singular point.*

With the singular point $x_0 = 0$ found above for the Bessel equation notice that, $(x-x_0)\frac{b(x)}{a(x)} = 1$ and $(x-x_0)^2 c(x)a(x) = x^2 - \nu^2$. So 0 is a regular singular point.

Definition 3.4: If x_0 is a regular singular point of $a(x)y''(x) + b(x)y'(x) + c(x)y(x) = 0$, then the *indicial equation for x_0* is the equation

$$r(r-1) + pr + q = 0$$

where

$$p = \lim_{x \to x_0} (x-x_0)\frac{b(x)}{a(x)} \quad \text{and} \quad q = \lim_{x \to x_0} (x-x_0)^2\frac{c(x)}{a(x)}.$$

We call the zeros of this equation the *indices of the singularity* x_0.

Circling back to the Bessel Equation and the regular singular point 0, notice $p = \lim_{x\to 0}(x)\frac{1}{x} = 1$ and $q = \lim_{x\to 0} x^2(1 - \frac{\nu^2}{x^2}) = -\nu^2$. So the indicial equation for 0 is $r(r-1) + r - \nu^2 = 0$. So we see our indices are $r = \pm\nu$.

Before we proceed with the Bessel Functions we need will need to consider some ideas and properties of the Gamma Function.

Definition 3.5: The *Gamma Function* is the function $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt$ defined for all complex numbers with real part greater than zero.

We have an important result for this function relating it to the factorial function defined earlier.

Lemma 3.6: For $x \in (0,\infty)$. $\frac{\Gamma(x+n)}{\Gamma(x)} = (x)_n$.

Proof. Proceeding by induction, for $n = 1$

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty e^{-t}t^x\,dt \\
&= \big(-e^{-t}t^x\big|_0^\infty + \int_0^\infty e^{-t}xt^{x-1}\,dt \\
&= x\int_0^\infty e^{-t}t^{x-1}\,dt \\
&= x\Gamma(x).
\end{aligned}$$

So we have $\frac{\Gamma(x+1)}{\Gamma(x)} = (x)_1 = x$. Now if we assume that for some $k \in \mathbb{N}$ that $\frac{\Gamma(x+k)}{\Gamma(x)} = (x)_k$, then

$$\begin{aligned}
\Gamma(x+k+1) &= \int_0^\infty e^{-t} t^{x+k}\, dt \\
&= \big(-e^{-t} t^{x+k} \big|_0^\infty + \int_0^\infty e^{-t}(x+k) t^{x+k-1}\, dt \\
&= (x+k) \int_0^\infty e^{-t} t^{x+k-1}\, dt \\
&= (x+k)\Gamma(x+k) \\
&= (x+k)(x)_k \Gamma(x) \\
&= (x)_{k+1}\Gamma(x).
\end{aligned}$$

Thus for all $n \in \mathbb{N}$ we have $\frac{\Gamma(x+n)}{\Gamma(x)} = (x)_n$. □

Theorem 3.7: For a Bessel equation $x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0$, where $\nu \geq 0$, if $\nu \notin \mathbb{Z}$ then we have two linearly independent solutions of the form;

$$\begin{aligned}
J_\nu(x) &= c_1 \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \\
J_{-\nu}(x) &= c_2 \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}.
\end{aligned}$$

Here $J_\nu(x)$ is called the *Bessel Function of the first kind of order* ν.

Proof. Recall that for the Bessel Function we have one regular singular point at $x_0 = 0$ with the associated indices being $\pm\nu$. We guess a power series solution of the the form $y(x) = \sum_{n=0}^\infty a_n x^{n+r}$ then,

$$\begin{aligned}
y(x) &= \sum_{n=0}^\infty a_n x^{n+r} \\
y'(x) &= \sum_{n=0}^\infty a_n (n+r) x^{n+r-1} \\
y''(x) &= \sum_{n=0}^\infty a_n (n+r)(n+r-1) x^{n+r-2}.
\end{aligned}$$

Substituting these into our equation we have,

$$\sum_{n=0}^\infty a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^\infty a_n (n+r) x^{n+r} + (x^2 - \nu^2) \sum_{n=0}^\infty a_n x^{n+r} = 0.$$

Combining like series we have,

$$\sum_{n=0}^{\infty}[(n+r)(n+r-1)+(n+r)-\nu^2]a_n x^{n+r}+\sum_{n=0}^{\infty}a_n x^{n+r+2}=0.$$

Finally re-indexing allows us to write the powers series as one,

$$[(r)(r-1)+(r)-\nu^2]a_0x^r+[(1+r)(r)+(1+r)-\nu^2]a_1x^{1+r}$$
$$+\sum_{n=2}^{\infty}([(n+r)(n+r-1)+(n+r)-\nu^2]a_n+a_{n-2})x^{n+r}=0.$$

Observe that the piece with the a_0x^r term is the indicial equation which we saw had roots $\pm\nu$. So both ν and $-\nu$ will produce power series solutions to the Bessel equation. Now solving for the other coefficients we have,

$$a_0=a_0 \qquad a_1=0$$
$$a_2=\frac{-a_0}{4+4\nu} \qquad a_3=0$$
$$a_4=\frac{a_0}{(16+8\nu)(4+4\nu)} \qquad a_5=0$$
$$\vdots$$
$$a_{2n}=\frac{(-1)^n a_0}{((2n)^2+4n\nu)((2n-2)^2+2(2n-2)\nu)\dots(4+4\nu)}$$
$$a_{2n+1}=0.$$

Now as we saw with our power series solution to $u''+x^2u=0$ there is a slick way to rewrite the denominator of our terms as follows,

$$((2n)^2+4n\nu)((2n-2)^2+2(2n-2)\nu)\dots(4+4\nu)$$
$$=4^n(n+n\nu)(n^2-2n+1n\nu-\nu)(n^2-4n+4+n\nu-2\nu)\dots(1+\nu)$$
$$=2^{2n}(n)(n+\nu)(n-1)(n+\nu-1)(n-2)(n+\nu-2)\dots(1)(1+\nu)$$
$$=2^{2n}(n!)(1+\nu)_n.$$

Hence we have that,

$$y_1(x)=a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}n!(1+\nu)_n}x^{2n+\nu}$$
$$y_2(x)=b_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}n!(1-\nu)_n}x^{2n-\nu}.$$

Finally if we make the substitution $a_0=\frac{c_1}{2^\nu\Gamma(1+\nu)}$ notice by Lemma 3.7 we have,

$$y_1(x)=\frac{c_1}{2^\nu\Gamma(1+\nu)}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}n!(1+\nu)_n}x^{2n+\nu}=c_1\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu}.$$

A similar substitution of $b_0 = \frac{c_2}{2^{-\nu}\Gamma(1-\nu)}$ gives us,

$$y_2(x) = c_2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}.$$

□

Now there are Bessel functions of other kinds, but for our study we will only need to consider the Bessel Function of the first kind. Below are some graphs of different ν values for the Bessel Function.

Figure 3: Graph of $J_0(x)$

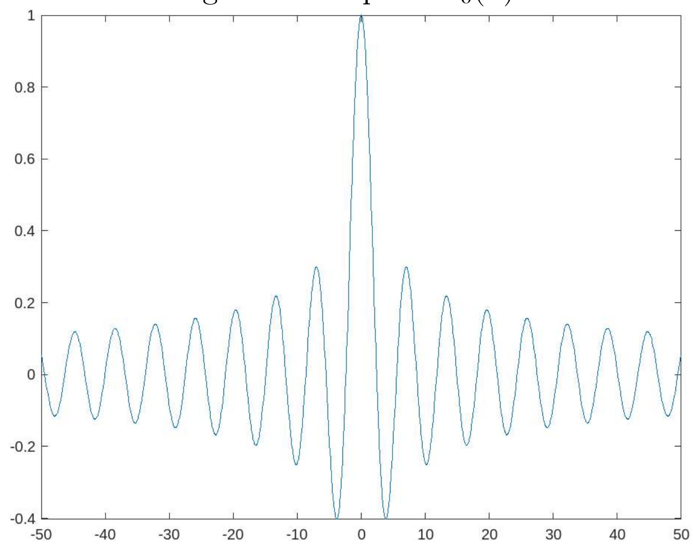
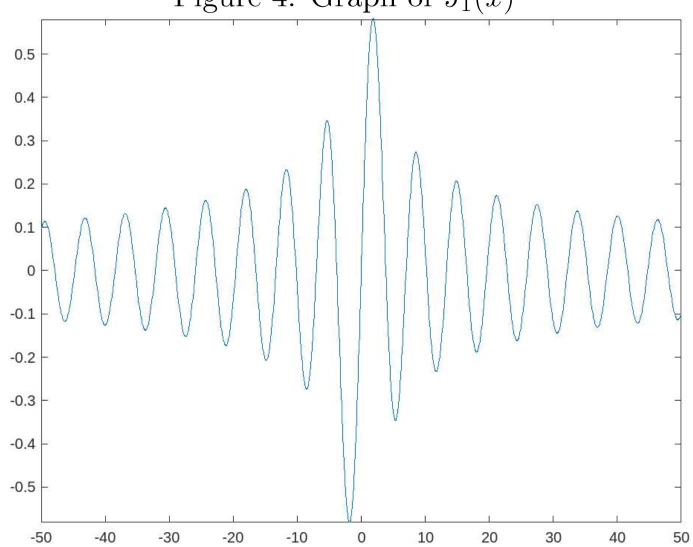


Figure 4: Graph of $J_1(x)$



Back to $y' = y^2 + x^2$

Now let us circle back to our original goal of finding a solution to $y' = y^2 + x^2$. Notice that in the proof of Theorem 3.6 before we introduced the Gamma function, the solutions found for the equations $u'' + x^2u = 0$ are eerily similar. Notice that for the first summation if we let $\nu = \frac{1}{4}$ and $a_0 = \frac{c_0}{\Gamma(1-\nu)}$ then,

$$\begin{aligned}
a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)(1-\frac{1}{4})_n} \left(\frac{x}{2}\right)^{4n} &= c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{4n} \\
&= c_0\sqrt{x/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{4n-\frac{1}{2}} \\
&= c_0\sqrt{x/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)\Gamma(1-\nu+n)} \left(\frac{x^2}{2^2}\right)^{2n-\frac{1}{4}} \\
&= c_0\sqrt{x/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)\Gamma(1-\nu+n)} \left(\frac{x^2}{2^2}\right)^{2n-\nu} \\
&= c_0\sqrt{x/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)\Gamma(1-\nu+n)} \left(\frac{\frac{x^2}{2}}{2}\right)^{2n-\nu} \\
&= c_0\sqrt{x/2} J_{-\nu}\left(\frac{x^2}{2}\right).
\end{aligned}$$

Similarly, $a_1 \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(n!)(1+\frac{1}{4})_n} \left(\frac{x}{2}\right)^{4n+1} = c_1\sqrt{x/2} J_\nu(\frac{x^2}{2})$ where $a_1 = \frac{c_1}{\Gamma(1+\nu)}$.

To further highlight why we are obtaining Bessel functions in our solutions, consider the substitution to $u'' + x^2u = 0$ of $u(x) = (\sqrt{x/2})v(\frac{x^2}{2})$. Notice then that $u''(x) = \frac{1}{\sqrt{2}}\left(\frac{-1}{4}x^{\frac{-3}{2}}v(\frac{x^2}{2}) + 2x^{\frac{1}{2}}v'(\frac{x^2}{2}) + x^{\frac{5}{2}}v''(\frac{x^2}{2})\right)$. Thus substituting that into our differential equation of u we have,

$$\begin{aligned}
0 &= u'' + x^2u \\
&= \frac{1}{\sqrt{2}}\left(\frac{-1}{4}x^{\frac{-3}{2}}v\left(\frac{x^2}{2}\right) + 2x^{\frac{1}{2}}v'\left(\frac{x^2}{2}\right) + x^{\frac{5}{2}}v''\left(\frac{x^2}{2}\right) + x^{\frac{5}{2}}v\left(\frac{x^2}{2}\right)\right).
\end{aligned}$$

Multiplying through by $x^{3/2}$ we get that,

$$\begin{aligned}
0 &= \frac{-1}{4}v\left(\frac{x^2}{2}\right) + 2x^2v'\left(\frac{x^2}{2}\right) + x^4v''\left(\frac{x^2}{2}\right) + x^4v\left(\frac{x^2}{2}\right) \\
&= x^4v''\left(\frac{x^2}{2}\right) + 2x^2v'\left(\frac{x^2}{2}\right) + (x^4 - \frac{1}{4})v\left(\frac{x^2}{2}\right).
\end{aligned}$$

Finally if we introduce the substitution $t = \frac{x^2}{2}$ we arrive at,

$$\begin{aligned}
0 &= 4t^2v''(t) + 4tv'(t) + (4t^2 - \frac{1}{4})v(t) \\
&= 4(t^2v''(t) + tv'(t) + (t^2 - \frac{1}{16})v(t))
\end{aligned}$$

or,

$$0 = t^2v''(t) + tv'(t) + (t^2 - \frac{1}{16})v(t).$$

Notice that the above equation is the Bessel equation of order $\frac{1}{4}$. Since $\nu \notin \mathbb{Z}$ we have solutions of the form $J_{1/4}(t)$ and $J_{-1/4}(t)$. Which going backwards through our substitutions we see that is indeed what we had gotten from our power series solution.

Now if we want to write our solution fully in terms of Bessel functions we first need to see what derivatives of them look like. This is captured in the next lemmas and theorem.

Lemma 4.1: $\frac{d}{dx}(x^\nu J_\nu(x)) = x^\nu J_{\nu-1}(x)$

Proof. Observe that,

$$\begin{aligned}
\frac{d}{dx}(x^\nu J_\nu(x)) &= \frac{d}{dx}x^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \\
&= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x^{2n+2\nu}}{2^{2n+\nu}}\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n(2n+2\nu)}{n!\Gamma(1+\nu+n)} \left(\frac{x^{2n+2\nu-1}}{2^{2n+\nu}}\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n(2n+2\nu)}{n!\Gamma(\nu+n)(\nu+n)} \left(\frac{x^{2n+2\nu-1}}{2^{2n+\nu}}\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 2}{n!\Gamma(\nu+n)} \left(\frac{x^{2n+2\nu-1}}{2^{2n+\nu}}\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n)} \left(\frac{x^{2n+2\nu-1}}{2^{2n+\nu-1}}\right) \\
&= x^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu-1} \\
&= x^\nu J_{\nu-1}(x).
\end{aligned}$$

□

Lemma 4.2: $\frac{d}{dx}(x^{-\nu} J_\nu(x)) = -x^{-\nu} J_{\nu+1}(x)$

Proof. As we saw before notice that,

$$\begin{aligned}
\frac{d}{dx}(x^{-\nu}J_\nu(x)) &= \frac{d}{dx}x^{-\nu}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu}\\
&= \frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\nu+n)}\left(\frac{x^{2n}}{2^{2n+\nu}}\right)\\
&= \sum_{n=1}^{\infty}\frac{(-1)^n 2n}{n!\Gamma(1+\nu+n)}\left(\frac{x^{2n-1}}{2^{2n+\nu}}\right)\\
&= \sum_{n=1}^{\infty}\frac{(-1)^n 2}{(n-1)!\Gamma(1+\nu+n)}\left(\frac{x^{2n-1}}{2^{2n+\nu}}\right)\\
&= \sum_{n=0}^{\infty}\frac{(-1)^{n+1} 2}{(n)!\Gamma(1+\nu+n+1)}\left(\frac{x^{2n+1}}{2^{2n+2+\nu}}\right)\\
&= -x^{-\nu}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n)!\Gamma(1+\nu+n+1)}\left(\frac{x}{2}\right)^{2n+1+\nu}\\
&= -x^{-\nu}J_{\nu+1}(x).
\end{aligned}$$

□

Theorem 4.3: $\frac{d}{dx}J_\nu(x) = \frac{1}{2}(J_{\nu-1}(x) - J_{\nu+1}(x))$.

Proof. By Lemma 4.1,

$$x^\nu(J_\nu(x))' + \nu x^{\nu-1}J_\nu(x) = x^\nu J_{\nu-1}(x)$$
$$\Longleftrightarrow$$
$$(J_\nu(x))' + \nu x^{-1}J_\nu(x) = J_{\nu-1}(x)$$

and by 4.2,

$$x^{-\nu}(J_\nu(x))' - \nu x^{-\nu-1}J_\nu(x) = -x^{-\nu}J_{\nu+1}(x)$$
$$\Longleftrightarrow$$
$$(J_\nu(x))' - \nu x^{-1}J_\nu(x) = -J_{\nu+1}(x).$$

Hence adding these two equations we have,

$$2(J_\nu(x))' = J_\nu(x) - J_{\nu+1}(x).$$

□

With this we can write our solution to $y' = y^2 + x^2$ in terms of Bessel functions,

$$\begin{aligned}
y(x) = &\frac{-c_0(\frac{J_{-1/4}(\frac{x^2}{2})}{2\sqrt{x}} + \frac{x^{3/2}}{2}(J_{-5/4}(\frac{x^2}{2}) - J_{3/4}(\frac{x^2}{2})))}{c_0\sqrt{x}J_{-1/4}(\frac{x^2}{2}) + c_1\sqrt{x}J_{1/4}(\frac{x^2}{2})}\\
&+ \frac{-c_1(\frac{J_{1/4}(\frac{x^2}{2})}{2\sqrt{x}} + \frac{x^{3/2}}{2}(J_{-3/4}(\frac{x^2}{2}) - J_{5/4}(\frac{x^2}{2})))}{c_0\sqrt{x}J_{-1/4}(\frac{x^2}{2}) + c_1\sqrt{x}J_{1/4}(\frac{x^2}{2})}.
\end{aligned}$$

We now look for our particular solution where $y(0) = 0$. Recall that we found $a_1 = 0$, hence $c_1 = 0$ giving us,

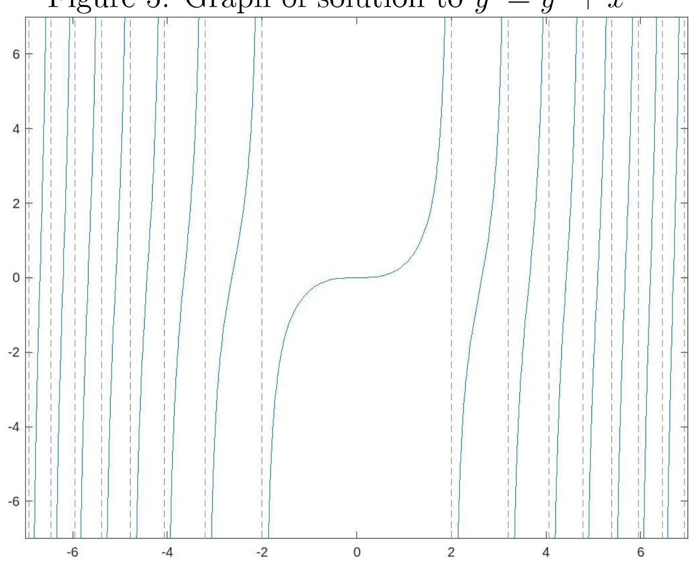
$$\begin{aligned} y(x) &= \frac{-c_0\left(\frac{J_{-1/4}(\frac{x^2}{2})}{2\sqrt{x}} + \frac{x^{3/2}}{2}(J_{-5/4}(\frac{x^2}{2}) - J_{3/4}(\frac{x^2}{2}))\right)}{c_0\sqrt{x}J_{-1/4}(\frac{x^2}{2})} \\ &= \frac{-\left(\frac{J_{-1/4}(\frac{x^2}{2})}{2\sqrt{x}} + \frac{x^{3/2}}{2}(J_{-5/4}(\frac{x^2}{2}) - J_{3/4}(\frac{x^2}{2}))\right)}{\sqrt{x}J_{-1/4}(\frac{x^2}{2})}. \end{aligned}$$

Furthermore if we multiply through on top and bottom by $2\sqrt{x}$ we have,

$$y(x) = \frac{x^2 J_{3/4}(\frac{x^2}{2}) - x^2 J_{-5/4}(\frac{x^2}{2}) - J_{-1/4}(\frac{x^2}{2})}{2x^{J}_{-1/4}(\frac{x^2}{2})}.$$

Using Matlab to graph our solution we have the following graph,

Figure 5: Graph of solution to $y' = y^2 + x^2$



A final note to point out would be our asymptote is now just slightly bigger than 2. Now if we write a short Matlab code employing Newton's Method we find that $x = 2.0031$ is where this vertical aysmptote lies. Here I used a tolerance of 10^{-10} and increasing it did not give us any more accurate of a number. Likewise starting with a guess of $x = -2$ gives us that $x = -2.0031$ is approximately where the left side asymptote lies.

Generalizing our equation

It would be nice now to say something more general about what solutions for differential equations of the form $y' = y^n + x^n$ look like. We just found the solution for $n = 2$. Note that for $n = 1$ we have $y' = y + x$, which if we multiply through by an integrating factor $\mu(x) = e^{-x}$, we have $e^{-x}y' - e^{-x}y = e^{-x}x$. As with a typical integrating factor problem we have on the left hand side the derivative of $e^{-x}y$. Giving us,

$$\begin{aligned} ye^{-x} &= \int xe^{-x}\,dx \\ &= -xe^{-x} + \int e^{-x}\,dx \\ &= -xe^{-x} - e^{-x} + C. \end{aligned}$$

So we have that $y(x) = Ce^x - x - 1$. However, the next case one would like to examine would be $y' = y^3 + x^3$, this one will not be as easy as the $n = 1$ case. Note that by Theorems 1.7/1.8 we do have a solution if we add the initial condition $y(0) = 0$. Furthermore, if we use the method of Picard Iteration as before notice that this equation will blow up a lot faster than for the $n = 2$ case. This is an approximation of our solution after 3 iterations,

$$y(x) = \frac{x^{40}}{44994560} + \frac{3x^{31}}{1341184} + \frac{3x^{22}}{36608} + \frac{x^{13}}{832} + \frac{x^4}{4}.$$

There is at least some information we can extract from how these will begin to look. We always have that $\frac{x^4}{4}$ term for the same reasoning we saw that $\frac{x^3}{3}$ term in the $n = 2$ case. More interestingly we have that we are going up by powers of 9 for each term. This may allow us to carefully chose a substitution we can make to rewrite our equation. Before that as we did with our first example let's look at the direction field and a graph of the Picard iteration.

Figure 6: Picard Iterations for $n = 7$ of $y' = y^3 + x^3$

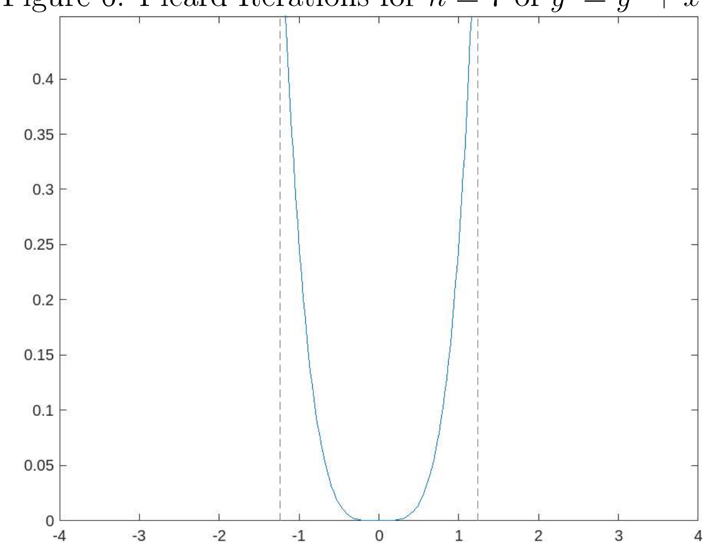
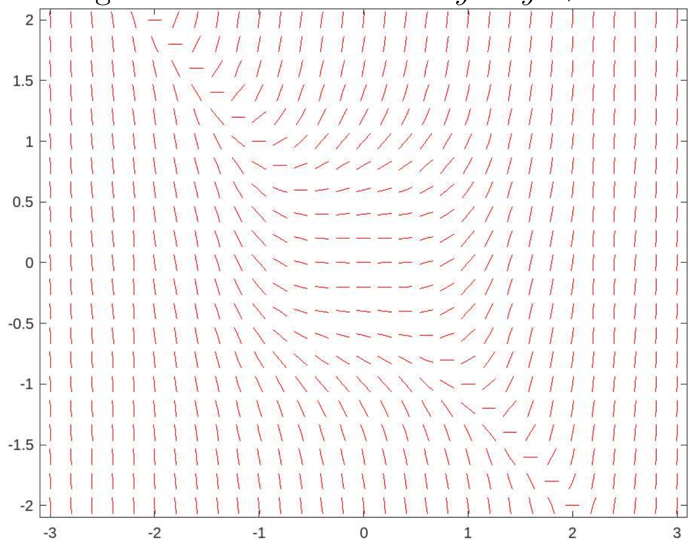


Figure 7: Direction Field of $y' = y^3 + x^3$



As it turns out, this equation is what is known as an Abel equation of the first kind. We can make the substitution $y = \frac{1}{u}$ to transform it into, $\frac{-u'}{u^2} = \frac{1}{u^3} + x^3$ or $u'u + x^3u^3 + 1 = 0$. Note now we can rewrite the first term as $u'u = (\frac{1}{2}u^2)'$. Now another method we can use to rewrite our equation would be to suppose that y_0 is a solution to our differential equation then we make the substitution $y = y_0 + \frac{1}{u}$. This leads to,

$$\begin{aligned} y_0' - \frac{u'}{u^2} &= (y_0 + \frac{1}{u})^3 + x^3 \\ &= y_0^3 + 3\frac{y_0^2}{u} + 3\frac{y_0}{u^2} + \frac{1}{u^3} + x^3. \end{aligned}$$

Note that since y_0 was a solution to the original differential equation this reduces down to,

$$u'u + 3y_0^2u^2 + 3y_0u + 1 = 0.$$

This is now as an Abel equation of the second kind. Unfortunately for us neither of these substitutions lead us to any fruitful way of expressing our solution in terms of known functions.

Conclusion

In conclusion we were able to first show that our differential equation has a solution around the initial condition $y(0) = 0$. We were then able to explore many different techniques in solving differential equations first looking at a power series solution and then rewriting that solution in terms of Bessel functions. Finally we also took a look at generalizing this differential equation to the form $y' = y^n + x^n$ for $n \in \mathbb{N}$. While we were unable to find a closed solution for this equation we did gain some insight into the difficult world of Abel equations and why they are notoriously hard to solve analytically.

References

[1] Thomson, B. S., Bruckner, J. B., & Bruckner, A. M. (2008). Elementary Real Analysis. ClassicalRealAnalysis.com.

[2] Korman, Philip. Lectures on Differential Equations. MAA Press, an Imprint of the AmericanMathematical Society, 2019.

[3] Nagle, R. Kent, et al. Fundamentals of Differential Equations and Boundary Value Problems. Pearson, 2018.