GENERALIZED CONTINUITY, MONOTONICITY, CLOSED GRAPH AND CONTINUITY

by

Roy A. Mimna

Submitted in Partial Fulfillment of the Requirements
for the Degree of
Master of Science
in the
Mathematics
Program

Z. Piotrowski — June 30, 1987
Zbigniew Piotrowski, Adviser — Date

Sally M Hotchkiss — July 1, 1987
Sally M. Hotchkiss, Dean of the Graduate School — Date

YOUNGSTOWN STATE UNIVERSITY

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ABSTRACT

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Roy A. Mimna
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In Chapter I the notion of separate continuity is introduced and explained using various examples, including an example of a real valued separately continuous function which has a dense countably infinite set of points of discontinuity. The latter example is explicitly constructed using a method of densifying points in the real plane.

Chapter II introduces other kinds of generalized continuity and presents theorems on generalized continuity and monotonicity. In particular, the notions of quasi-continuity, symmetric quasi-continuity, and near continuity are introduced. The discussion and analysis deals with real valued functions of two variables which are monotone in one or both of the variables. The general question addressed is what conditions of generalized continuity on such a function will guarantee that the function is continuous. The Lemma on page 8, Theorem 2, Theorem 3, and Corollary I are my results. Theorem 2 states that a function $f\colon \mathbb{R}^2 \to \mathbb{R}$, which is continuous in y for every x, nearly continuous in x for every y, and monotone in x for every y, is continuous. This is a general-

ization of the previously known result presented in Theorem 1. Theorem 3 presents a similar result for a function $f: \mathbb{R}^2 \to \mathbb{R}$ which is jointly nearly continuous.

In Chapter III the closed graph property is introduced, and various theorems are presented concerning this property, generalized continuity and continuity. Theorems 6 and 7 are my results. Theorem 5 states the well-known result that a function $f:X \to Y$, where Y is compact and $G(f)$ is closed in $X \times Y$, is continuous. Theorems 6 and 7 place a different, although related condition on a function $f:X \times Y \to Z$, (namely, that f be bounded), rather than the compactness of the range of f.

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I wish to express my gratitude to Professor Zbigniew Piotrowski of the Department of Mathematical and Computer Sciences, Youngstown State University, whose patience and enthusiasm guided me through my research and writing of this thesis.

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LIST OF SYMBOLS

SYMBOL	DEFINITION
cos	The cosine function
lim	Limit
$\mathbb{R}$	The real numbers
sin	The sine function
W.L.O.G.	Without loss of generality
$\Longrightarrow$	Implies that
$<$	Is less than
$>$	Is greater than
	Is an element of (a set)
	Inclusion
	Intersection

CHAPTER I

SEPARATE CONTINUITY AND JOINT CONTINUITY

Introduction

At least as long ago as 1873 it was known that there are functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ which are continuous on every straight line parallel to the coordinate axes of the domain, but are nevertheless not continuous at a point of their domain.

Definition: Consider a function $f\colon \mathbb{R} \to \mathbb{R}$. The function $f_{x_0}(y)$, given by $f_{x_0}(y) = f(x_0,y)$, is called an x-section of f. Similarly, the function $f_{y_0}(x)$, given by $f_{y_0}(x) = f(x,y_0)$ is called a y-section of f. If all x-sections and all y-sections of $f\colon \mathbb{R}^2 \to \mathbb{R}$ are continuous, we say that f is separately continuous.

Clearly, continuity implies separate continuity, but separate continuity does not imply continuity. Consider, for example, the real valued function

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2 + y^2} & , \text{ if } (x,y) \neq (0,0) \\ 0 & , \text{ if } (x,y) = (0,0) \end{cases}$$

At the point $(0,0)$ f is separately continuous, but not continuous. Clearly, f is separately continuous at $(0,0)$, for $f(0,y) = 0$ and $f(x,0) = 0$. To see that f is not continuous at $(0,0)$, let $x = r\cos\Theta$ and $y = r\sin\Theta$. Then

$$f(x,y) = \frac{2r\cos\ r\sin}{r^2\cos^2\Theta + r^2\sin^2\Theta} = \frac{2r^2\sin\Theta\cos\Theta}{r^2(\sin^2\Theta + \cos^2\Theta)} = 2\sin\Theta\cos\Theta = \sin 2\Theta.$$

Then since f depends only on θ, if the domain of f is a sphere centered at (0,0), no matter how small the radius becomes, f(x,y) takes on all values between -1 and 1. So let $\varepsilon = \frac{1}{2}$ be given. Then there is no δ such that $|f(x,y) - f(0,0)| < \varepsilon$. Notice that f has an oscillation of 2 at the point (0,0). As indicated in the illustration, f "drops off" at the origin from 1 to 0 and from -1 to 0.

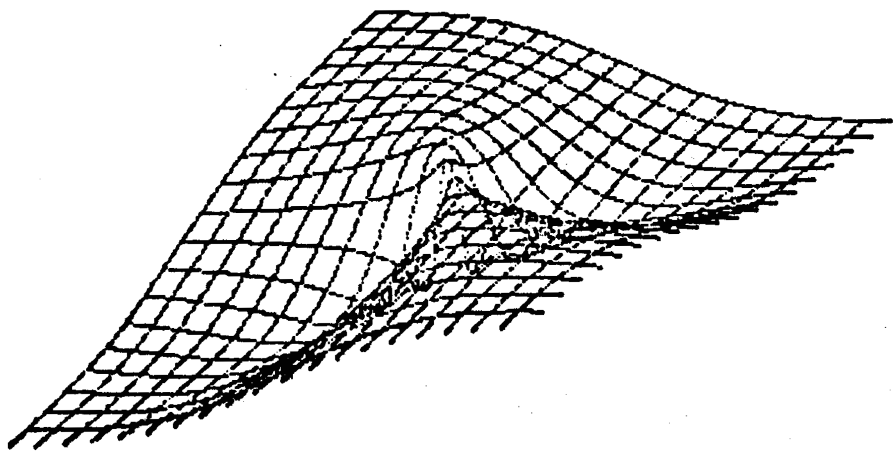

Fig. 1.--Graph of a Separately Continuous Function

We may also use Heine's condition of continuity to show that f is not continuous at (0,0). Recall that Heine's condition of continuity provides that a function $f\colon \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous iff for all $\{a_n\}$, $\lim_{n\to\infty} a_n = (x,y) \Rightarrow \lim_{n\to\infty} f(a_n) = f(x,y)$. So, let $\{a_n\}$ be a sequence, where $a_n = (\frac{1}{n},\frac{1}{n})$. Clearly, $\{a_n\}$ converges to (0,0) as n goes to $+\infty$. However, the corresponding sequence of values of f converges as follows:

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \frac{2(\frac{1}{n})(\frac{1}{n})}{(\frac{1}{n})^2 + (\frac{1}{n})^2} = \lim_{n\to\infty} 1 = 1 \neq f(0,0) = 0.$$

Thus, by Heine's condition of continuity, we see that f is not continuous at $(0,0)$.

An Example of a Separately Continuous Function which is Discontinuous on a Countably Infinite Dense Subset of the Domain

We can now use the function illustrated above to construct a new separately continuous function which has two points of discontinuity. Let

$$f_1(x,y) = \begin{cases} \dfrac{2xy}{x^2 + y^2} & , \text{ if } (x,y) \neq (0,0) \\ 0 & , \text{ if } (x,y) = (0,0) \end{cases}$$

As we have seen, f_1 has a point of discontinuity at the origin. Now, let

$$f_2(x,y) = \begin{cases} \dfrac{2(x-\frac{1}{2})(y-\frac{1}{2})}{(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2} & , \text{ if } (x,y) \neq (\frac{1}{2},\frac{1}{2}) \\ 0 & , \text{ if } (x,y) = (\frac{1}{2},\frac{1}{2}) \end{cases}$$

Observe that f_2 is of the same type as f_1, but with the point $(\frac{1}{2},\frac{1}{2})$ as the "origin". The function $f^* = f_1 + f_2$ is now separately continuous but has two points of discontinuity.

We can continue this process by defining a method of choosing the successive points of origin. In fact, by this method, we can construct a function which is separately continuous, but whose set of points of discontinuity is countably infinite. Consider the closed square $A = [-1,1] \times [-1,1]$ in $\mathbb{R}^2$. Let us define a method of selecting points in A whereby we choose the center of successively smaller squares as shown in the illustration. The first point is the center of A, and the next four points are the centers of the first, second, third, and fourth quadrants in that order. The next sequence of 16 points begins again in the first quadrant, with point number 6 being the center of the first quarter-quadrant, point number 7 being the center of the second quarter-quadrant, and so on.

Observe that the selected points are arranged in a sequence as follows: $(a_1,b_1) = (0,0)$, $(a_2,b_2) = (\frac{1}{2},\frac{1}{2})$, $(a_3,b_3) = (-\frac{1}{2},\frac{1}{2})$, ...

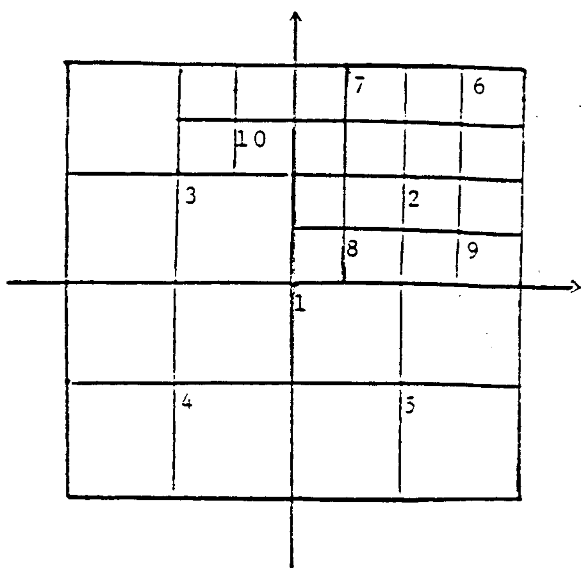

Fig. 2.--Densifying Points in the Rectangle $[-1,1]\times[-1,1]$

Now consider the sequence of all of the points-- arranged in the indicated order; call it $\{(a_n,b_n)\}$, $n = 1$ 2, 3, ... Clearly, the sequence $\{(a_n,b_n)\}$ is countably infinite. Let $f_n(x,y)$ be the function given by

$$f_n(x,y) = \begin{cases} \dfrac{(2(x-a_n)(y-b_n)}{(x-a_n)^2 + (y-b_n)^2}, & \text{if } (x,y) \neq (a_n,b_n) \\ 0, & \text{if } (x,y) = (a_n,b_n) \end{cases}$$

That is, f_n is similar to the function given by

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2 + y^2} & , \text{ if } (x,y) \neq (0,0) \\ 0 & , \text{ if } (x,y) = (0,0) \end{cases}$$

but with the point (a_n, b_n) as the "origin". Now let $F(x,y) = \frac{1}{2}f_1(x,y) + \frac{1}{4}f_2(x,y) + \ldots + \frac{1}{2^n}f_n(x,y) + \ldots$ F is a function series in which each term is less than or equal to the corresponding term of the series $\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} + \ldots$, which is convergent. Thus, the oscillation of F does not exceed two at any point of its domain. Each term f_n of F generates a unique point of discontinuity of F, and the points of discontinuity of F are therefore countably infinite. Yet F is separately continuous everywhere on its domain, for on any straight line parallel to the x-axis or the y-axis, each term of the series is continuous, and therefore F is also continuous on the same line.

We can also show that the points of discontinuity of F form a dense subset of the domain, for let $(x,y) \in [-1,1] \times [-1,1]$ and let $r > 0$ be given. Then $r > \frac{1}{2^k}$ for some $k \in \mathbb{N}$, and if the square $[-1,1] \times [-1,1]$ is divided into quadrants $k + 2$ times, the open sphere $S[(x,y),r]$ will contain at least one point of discontinuity of F.

Thus we have constructed an example of a separately continuous function $f: \mathbb{R}^2 \to \mathbb{R}$ which is discontinuous on a countably infinite dense subset of the domain. It is well known[1] that the set of points of discontinuity of any real-valued function is an F_σ set. Thus, we see that the set of points of discontinuity of f is an F_σ set which is also countably infinite and dense in the domain of F. This result is especially interesting in view of the well-known fact[2] that the set of points of continuity of a real-valued separately continuous function from the product of, say, two separable and complete spaces, is a dense G_δ set. Thus, F has a dense G_δ set of points of continuity and a dense F_σ set of points of discontinuity.

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[1] R. R. Goldberg, Methods of Real Analysis, (N. Y.: John Wiley and Sons, 1976), Second Edition, p. 144.

[2] Z. Piotrowski, "Separate and joint continuity," Real Analysis Exchange, Vol.11, No. 2(1985-1986), pp. 293-322.

CHAPTER II

GENERALIZED CONTINUITY AND MONOTONICITY

Introduction

Although separate continuity does not imply continuity, a separately continuous function $f: \mathbb{R}^2 \to \mathbb{R}$ is continuous if f is monotone in one of the variables. This chapter presents theorems and counterexamples involving monotone functions which exhibit not only separate continuity, but other kinds of generalized continuity as well. In particular, functions $f: \mathbb{R}^2 \to \mathbb{R}$, where f is monotone in one or both variables, are analyzed to determine what additional conditions on the function will result in continuity.

Separate Continuity and Monotonicity

We begin by defining monotonicity:

Definition: Let X and Y be metric spaces. A function $f: X \times Y \to \mathbb{R}$ is nondecreasing [nonincreasing] in x for $y \in Y$ if $x_1 \leq x_2$ implies that $f(x_1,y_0) \leq f(x_2,y_0)$ [$f(x_1,y_0) \geq f(x_2,y_0)$]. We say that $f: X \times Y \to \mathbb{R}$ is monotone in x for $y \in Y$ if f is either nondecreasing or nonincreasing in x for $y \in Y$. The definition of monotonicity in y for $x \in X$ for functions $f: X \times Y \to \mathbb{R}$ is similar to the above.

Continuity clearly implies separate continuity, but as we have seen in Chapter I, the converse is not true.

However, the following theorem presents a well-known result combining the notions of separate continuity and monotonicity:3

Theorem 1: Let f: $\mathbb{R}^2 \to \mathbb{R}$ be separately continuous and suppose that f is monotone in x for $y \in Y$. Then f is continuous.

It is well known[4] that a function f: $\mathbb{R}^2 \to \mathbb{R}$ can be continuous along every analytic curve through a point (x_0, y_0) without being continuous at (x_0, y_0). This stronger kind of generalized continuity clearly implies separate continuity, and thus, when combined with monotonicity with respect to one of the variables, implies continuity.

Near Continuity and Monotonicity

Definition: Let X and Y be metric spaces. A function f: $X \to Y$ is nearly continuous at x_0 if, for every open set V containing $f(x_0)$, $\overline{f^{-1}(V)}$ is a neighborhood of x_0.

In order to proceed further, we need the following:

Lemma: Suppose that a function f: $\mathbb{R} \to \mathbb{R}$ is nearly continuous and monotone. Then f is continuous.

Proof: W.L.O.G., take f to be nondecreasing. Let x_0 be any point in the domain of f. Let V be any open interval

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[3] R. L. Kruse and J. J. Deely, "Joint continuity of monotone functions," Amer. Math Monthly, Vol. 76 (1969), pp. 74-76.

[4] A. Rosenthal, "On the continuity of functions of several variables, "Math. Zeitschr., Vol. 63 (1955), pp. 31-38.

containing $f(x_0)$. By the near continuity of f, $\overline{f^{-1}(V)}$ is a neighborhood of x_0. That is, $f^{-1}(V)$ is dense in some open set, call it G, containing x_0. Choose $r > 0$ such that $S_{opn}(x_0,r) \subset G$. I claim that $f[S(x_0,r)] \subset V$. Suppose, to the contrary, that there exists a point x_1 such that $x_1 \in S(x_0,r)$ and $f(x_1) \notin V$. W.L.O.G., assume that $x_1 > x_0$. Since $f^{-1}(V)$ is dense in $S(x_0,r)$, there exists a point $x^* \in f^{-1}(V)$ such that $x^* > x_1$ and $x^* \in S(x_0,r)$. Since f is nondecreasing, $x_0 < x_1 < x^* \Rightarrow f(x_0) \leq f(x_1) \leq f(x^*)$. But this implies that $f(x^*) \notin V$, which is a contradiction. Thus, for every open set V containing $f(x_0)$, there exists $r > 0$ and there exists an open sphere $S(x_0,r)$ such that $f[S(x_0,r)] \subset V$. Hence f is continuous. □

Applying the above Lemma, we have the following generalization of Theorem 1:

<u>Theorem 2</u>: Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function which is nearly continuous in x for every y and continuous in y for every x. Suppose that f is monotone in x for every y. Then f is continuous.

<u>Proof</u>: Since the y-sections of f are nearly continuous and monotone, by the Lemma, all y-sections are continuous. Since f is monotone in x for every y, by Theorem 1, f is continuous. □

Corollary 1: Let $f: \mathbb{R}^2 \to \mathbb{R}$ be separately nearly continuous (that is, all of the x-sections and all of the y-sections of f are nearly continuous.) Suppose that f is monotone in x for every y and monotone in y for every x. Then f is continuous.

The condition in Theorem 2 (and hence in the corollary) that the function be monotone in x for every y is necessary. To see this, suppose that a function $f: \mathbb{R}^2 \to \mathbb{R}$ is nearly continuous in x for every y, continuous in y for every x, but not monotone in x for every y (even though it is constant - hence monotone in y for every x). We shall construct such a function which will not be continuous. In fact, consider the real plane. Let all of the lines ℓ_x, where ℓ_x is parallel to the y-axis, and where x is rational, be raised to the level one. That is, let

$$f(x,y) = \begin{cases} 1, & \text{if } x \text{ is rational and} \\ 0, & \text{otherwise} \end{cases}$$

Observe that f is monotone in y for every x, but not monotone in x for every y. Clearly, all x-sections of f are continuous. observe further that all y-sections of f are nearly continuous. That is for each y_0 in the domain of f,

$$f_{y_0}(x) = f(x,y_0) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational}. \end{cases}$$

Clearly, for every x in the domain of f, and for every open set V containing $f_{y_0}(x)$, $\overline{f^{-1}(V)}$ is a neighborhood of x. That is every y-section of f is nearly continuous. It is easy to see that f is not continuous, and thus we see the necessity of the condition that f be monotone in x for every y.

It has been shown[5] that separate near continuity does not imply (joint) near continuity, and (joint) near continuity does not imply separate near continuity. In view of the just stated results of T. Neubrunn, it would be interesting to see an analogue of Theorem 2 for (joint) near continuity.

Recall that a function $f: \mathbb{R}^2 \to \mathbb{R}$ is nearly continuous at (p,q) if, for every open set V containing $f(p,q)$, $\overline{f^{-1}(V)}$ is a neighborhood of (p,q).

Theorem 3: Let $f: \mathbb{R}^2 \to \mathbb{R}$ be nearly continuous and suppose that f is increasing [decreasing] in x for every y and is increasing [decreasing] in y for every x. Then f is continuous.

Proof: W.L.O.G., let f be increasing in x for every y and increasing in y for every x. Let (p,q) be any point in the domain of f. Let V be any open interval containing $f(p,q)$. By the near continuity of f, $\overline{f^{-1}(V)}$ is a neighborhood of (p,q). Then, $f^{-1}(V)$ is dense in some open set, call it G containing (p,q). Choose $r > 0$ such that $(p-r,p+r)\times(p-r,q+r) = A \subset G$. I claim that $f(A) \subset V$. Assume, to the contrary, that there exists a point $(x_1,y_1) \in A$ such that $f(x_1,y_1) \notin V$.

[5] T. Neubrunn, "Generalized continuity and separate continuity," Math. Slovaca, Vol. 27 (1977), pp. 307-314.

We now show that this assumption leads to a contradiction. W.L.O.G., let $x_1 > p$ and $y_1 > q$. Since $f^{-1}(V)$ is dense in A, there exists a point (x^*, y^*) in $f^{-1}(V)$ such that $(x^*, y^*) \ni A$ and $x^* > x_1$ and $y^* > y_1$. Since f is increasing in x for every y and increasing in y for every x, $p < x_1 < x^*$ and $q < y_1 < y^* \Rightarrow f(p,q) < f(x_1, y_1) < f(x^*, y^*)$. But this implies that $f(x^*, y^*) \notin V$, a contradiction. Thus, for every open set V containing $f(p,q)$, there exists an open rectangle $(p-r, p+r) \times (q-r, q+r) = A$ such that $f(A) \subset V$. Hence f is continuous.$\square$

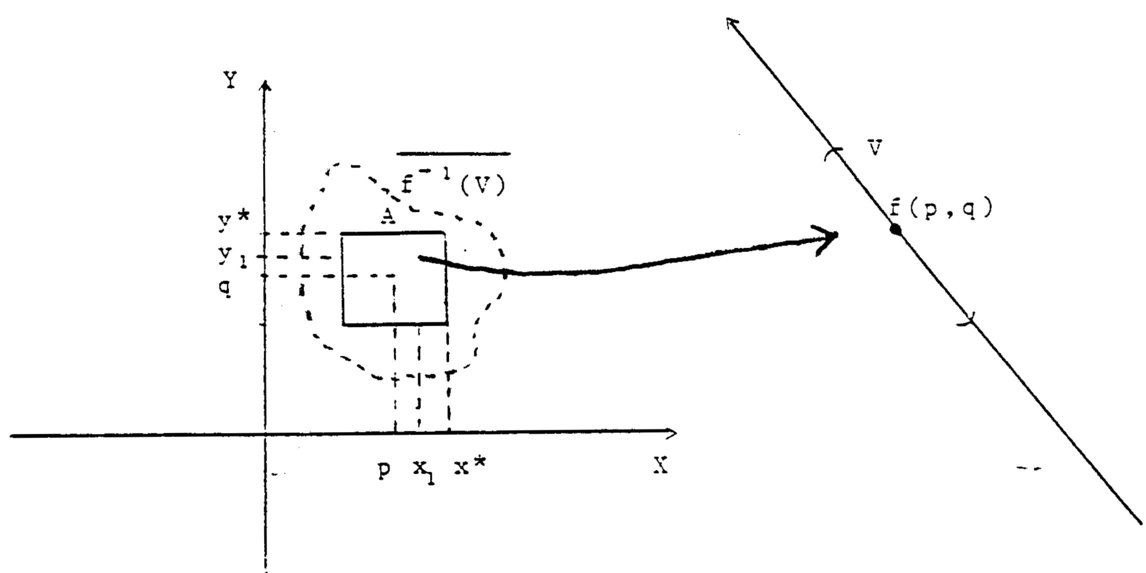

Fig. 3.--An Illustration of the Proof of Theorem 3

Quasi-continuity, Symmetric quasi-continuity, and Monotonicity

S. Kempisty first introduced the notions of quasi-continuity and symmetric quasi-continuity.[6]

Definition: Let X, Y, and Z be topological spaces. A function f: $X \times Y \to Z$ is quasi-continuous at the point (p,q) in its domain if, for every open set V containing f(p,q), and for every open set $U \subset X$ containing p, and for every open set $W \subset Y$ containing q, there exists an open nonempty set G, where $G \subset U \times W$, such that $f(G) \subset V$.

It is well known that separate continuity implies quasi-continuity.[7] An example of a quasi-continuous function is the following: Let $f: [-1,1] \times [-1,1] \to \mathbb{R}$ be defined by:

$$(x,y) = \begin{cases} 0, \text{ if } (0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1) \text{ or } (-1 \leq x \leq 0 \text{ and } -1 \leq y \leq 0) \\ 1, \text{ otherwise} \end{cases}$$

Observe that this function, which is quasi-continuous and is also monotone in x for every y, and monotone in y for every x, would be actually a counterexample to a conjecture that quasi-continuity and monotonicity with respect to both variables, imply continuity.

However, there is another counterexample which will show that a stronger condition of symmetric quasi-continuity

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[6] K. Kempisty, "Sur les fonctions quasicontinues", Fundamenta Mathematicae, Vol. 19 (1932), pp. 184-197.

[7] Z. Piotrowski, "Separate and joint continuity", p. 295.

and monotonicity with respect to both variables, does not imply continuity. First, we define symmetric quasi-continuity.

Definition: A function f: $\mathbb{R}^2 \to \mathbb{R}$ is quasi-continuous with respect to x if for every $(p,q) \in X \times Y$, and for every open set G containing $f(p,q)$, and for every open set $O = U \times V \ni (p,q)$, there exists an open (in X) nonempty set $U' \subset U$, and there exists an open (in Y) set $V' \subset V$, where V contains q, such that $f(U' \times V') \subset G$.

Quasi-continuity with respect to y is similarly defined.

Definition: If a function f: $\mathbb{R}^2 \to \mathbb{R}$ is quasi-continuous with respect to x and quasi-continuous with respect to y, then we say f is symmetrically quasi-continuous.

Again, it is known that separate continuity implies symmetric quasi-continuity and that symmetric quasi-continuity implies quasi-continuity.[8] An example of a symmetrically quasi-continuous function, which turns out to be our counterexample, is the following: Let f: $\mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} 1, & \text{if } y \geq x \\ 0, & \text{otherwise} \end{cases}$$

This function is symmetrically quasi-continuous at every point on the line y = x and is continuous (and thus symmetrically quasi-continuous) at all other points of its

[8] Z. Piotrowski, 'Separate and joint continuity', p. 295.

domain. Observe further that the function is monotone in x for every y and monotone in y for every x. However, the function is not continuous. Thus it is clear that symmetric quasi-continuity, when combined with monotonicity with respect to both variables, does not imply continuity.

CHAPTER III

THE CLOSED GRAPH PROPERTY, GENERALIZED CONTINUITY, AND CONTINUITY

Introduction

A function f: $X \to Y$, where X and Y are arbitrary topological spaces, has a closed graph if the graph of f, denoted by $G(f) = \{(x,f(x)) : x \in X\}$ is a closed subset of the product $X \times Y$. Very little is required in order that a continuous function have a closed graph. In fact, the following is true:

Theorem 4: Let X and Y be topological spaces and Y be Hausdorff. Suppose that f: $X \to Y$ is continuous. Then $G(f)$ is closed in $X \times Y$.

Proof: Let $p = (x_0, y_0)$ be a limit point of $G(f)$. Assume that $p \notin G(f)$. Since Y is Hausdorff, there exists an open set $G \subset Y$ such that G contains y_0 and G does not contain $f(x_0)$; and there exists an open set $V \subset Y$ such that V contains $f(x_0)$ and $G \cap V = \emptyset$. By the continuity of f, there exists an open set $U \subset X$ such that U contains x_0 and $f(U) \subset V$. Since the product of open sets is open in the product of the spaces, $U \times G$ is an open set containing p but no other point of $G(f)$. This is a contradiction and shows that $G(f)$ contains all of its limit points. Hence, $G(f)$ is closed in $X \times Y$. □

The following is a useful characterization of the closed graph property:

<u>Definition</u>: Let $f: X \to Y$, where X and Y are metric spaces. If $\{x_n\}$ converges to x and if $\{f(x_n)\}$ converges to y, then f has a closed graph if $f(x) = y$.

<u>Functions with Closed Graph and Conditions for Continuity</u>

As shown above, continuous functions have closed graphs provided that the range is Hausdorff. We now turn our attention to functions which have the closed graph property. An important problem is to determine, where a function has the closed graph property, what additional conditions on the function are necessary in order that the function be continuous.

First, we observe that the closed graph property does not, of itself, imply continuity. Consider, for example, the function

$$f(x) = \begin{cases} 1/x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Clearly, $G(f)$ is closed in $X \times Y$, but f is discontinuous at the point $x = 0$. Thus, we see that the closed graph property does not imply continuity. However, a well known theorem[9] provides that a function is continuous if it has

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[9] J. Dugundjii, <u>Topology</u>, (Boston: Allyn and Bacon, 1966), p. 228.

the closed graph property and the range is compact:

Theorem 5: Let X and Y be topological spaces and let $f\colon X \to Y$ with Y compact. If $G(f)$ is closed in $X \times Y$, then f is continuous.

We shall now present a theorem which places a condition on the function f rather than on the range of f. We shall require that the function be bounded. This result is proved first for the real numbers and then for more general spaces.

Theorem 6: Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be bounded and suppose that $G(f)$ is closed in $\mathbb{R}^2 \times \mathbb{R}$. Then f is continuous.

Proof: Let (x,y) be any point in the domain of f. Let $\{(x_n,y_n)\}$ be any sequence of points in the domain of f such that $\{(x_n,y_n)\}$ converges to (x,y). I claim that $\{f(x_n,y_n)\}$ converges to $f(x,y)$, and thus by Heine's condition of continuity, f is continuous. Assume the contrary, namely, that $\{f(x_n,y_n)\}$ does not converge to $f(x,y)$. Let $V \subset Z$ be any open interval containing $f(x,y)$. Then since $\{f(x_n,y_n)\}$ does not converge to $f(x,y)$, there exists an infinite set A consisting entirely of points of $\{f(x_n,y_n)\}$ such that $A \cap V = \emptyset$. The set A is bounded because f is bounded. By the Bolzano-Weierstrass Theorem, A has at least one limit point. Let z be a limit point of A. Then A contains a subsequence $\{f(x_{n_i},y_{n_i})\}$ which converges to z. Clearly $z \neq f(x,y)$. Since all subsequences of a convergent sequence of real numbers converge to the same limit as the main sequence, and since $\{(x_{n_i},y_{n_i})\}$ is a subsequence of $\{(x_n,y_n)\}$, then $\{(x_n,y_n)\} \to (x,y) \Rightarrow \{(x_{n_i},y_{n_i})\} \to (x,y)$. But this is a contradiction of the closed graph property of f, because $\{(x_{n_i},y_{n_i})\} \to (x,y)$ and $\{f(x_{n_i},y_{n_i})\} \to z$, but $z \neq f(x,y)$. Hence, the original claim is correct, that $\{f(x_n,y_n)\}$ converges to $f(x,y)$, and by Heine's condition of continuity, f is continuous. □

Observe that Theorem 6 is true for more general spaces. Before demonstrating this, let us recall the following:

<u>Definition</u>: A space X is called a <u>Bolzano-Weierstrass space</u> provided that every infinite subset of X has at least one limit point.

Observe that every compact space is a Bolzano-Weierstrass space, but the converse is not true. Now, we have the following:

<u>Theorem 7</u>: Let X, Y and Z be metric spaces and let Z be Bolzano-Weierstrass. Let f: $X \times Y \to Z$ and suppose that $G(f)$ is closed in $X \times Y \times Z$. Then f is continuous.

<u>Proof</u>: Let (x,y) be any point in the domain of f. Let $\{(x_n,y_n)\}$ be any sequence of points in the domain of f such that $\{(x_n,y_n)\}$ converges to (x,y). I claim that $\{f(x_n,y_n)\}$ converges to $f(x,y)$. Assume the contrary, namely that

$\{f(x_n,y_n)\}$ does not converge to $f(x,y)$. Let $V \subset Z$ be any open interval in Z such that $V \ni f(x,y)$. Then since $\{f(x_n,y_n)\}$ does not converge to $f(x,y)$, there exists an infinite set A consisting entirely of points of $\{f(x_n,y_n)\}$ such that $A \cap V = \emptyset$. Since Z is Bolzano-Weierstrass, the set A has at least one limit point. Let z be a limit point of A. Then A contains a sequence $\{f(x_{n_i},y_{n_i})\}$ which converges to z. Clearly $z \neq f(x,y)$. Since X and Y are metric spaces, and since $\{(x_n,y_n)\}$ converges to (x,y), then the subsequence $\{(x_{n_i},y_{n_i})\}$ also converges to (x,y). But this is a contradiction of the closed graph property of f. Hence, $\{f(x_n,y_n)\}$ converges to $f(x,y)$, and f is continuous. □

Generalized Continuity and the Closed Graph Property

As we implicitly observed above, continuous functions do not necessarily have the closed graph property, but for realvalued functions, continuity does imply closed graph. We shall now show that for a function $f\colon \mathbb{R}^2 \to \mathbb{R}$, separate continuity does not imply closed graph. Consider the function

$$h(x,y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

To show that $G(h)$ is not closed in $X \times Y \times Z$, we shall use the characterization of the closed graph property given on page

17 above. Returning to the function h, observe that the sequence $\{1/n, 1/n\}$ converges to $(0,0)$, and that $\{h(1/n, 1/n)\}$ converges to 1. However, $h(0,0) \neq 1$. Thus, we see that a functian can be separately continuous but not have the closed graph property. Clearly, other kinds of generalized continuity, such as near continuity, do not imply the closed graph property.

Many interesting results have been obtained concerning nearly continuous functions which have the closed graph property. The general problem is to determine what conditions on the domain and range of a function guarantee that if the function is nearly continuous and has a closed graph, then it is continuous. It has been shown, for example, that if the domain and range are both complete metric spaces, then near continuity and closed graph imply continuity.[10] It has also been shown that if f: $X \to Y$ is nearly continuous, Y is locally compact and either regular or Hausdorff, and G(f) is closed, then f is continuous.[11] An open question is the following: Let f: $X \times Y \to Z$ be separately nearly continuous and suppose that Z is locally compact and either regular or Hausdorff. If G(f) is closed in $X \times Y \times Z$, is f continuous?

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[10] A. J. Berner, "Almost continuous functions with closed graphs," Canad. Math. Bull., Vol. 25(4) (1982), pp. 428-434.

[11] E. E. McGehee, Jr. and P. E. Long, "Properties of almost continuous functions," Proc. Amer. Math. Soc., Vol. 24 (1970), pp. 175-180.

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GENERALIZED CONTINUITY, MONOTONICITY,

CLOSED GRAPH AND CONTINUITY

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ABSTRACT

GENERALIZED CONTINUITY, MONOTONICITY,
CLOSED GRAPH AND CONTINUITY

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In Chapter I the notion of separate continuity is introduced and explained using various examples, including an example of a real valued separately continuous function which has a dense countably infinite set of points of discontinuity. The latter example is explicitly constructed using a method of densifying points in the real plane.

Chapter II introduces other kinds of generalized continuity and presents theorems on generalized continuity and monotonicity. In particular, the notions of quasi-continuity, symmetric quasi-continuity, and near continuity are introduced. The discussion and analysis deals with real valued functions of two variables which are monotone in one or both of the variables. The general question addressed is what conditions of generalized continuity on such a function will guarantee that the function is continuous. The Lemma on page 8, Theorem 2, Theorem 3, and Corollary I are my results. Theorem 2 states that a function f: $\mathbb{R}^2 \to \mathbb{R}$, which is continuous in y for every x, nearly continuous in x for every y, and monotone in x for every y, is continuous. This is a general-

ization of the previously known result presented in Theorem 1. Theorem 3 presents a similar result for a function $f: \mathbb{R}^2 \to \mathbb{R}$ which is jointly nearly continuous.

In Chapter III the closed graph property is introduced, and various theorems are presented concerning this property, generalized continuity and continuity. Theorems 6 and 7 are my results. Theorem 5 states the well-known result that a function $f:X \to Y$, where Y is compact and $G(f)$ is closed in $X \times Y$, is continuous. Theorems 6 and 7 place a different, although related condition on a function $f:X \times Y \to Z$, (namely, that f be bounded), rather than the compactness of the range of f.

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