Axiom of Choice: Equivalences and Applications

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Abstract

This paper proves the equivalences of the Axiom of Choice and 7 other well known formulations. It then proves a few notable applications of the Axiom of Choice and discusses its importance in modern mathematics.

Contents

"The Axiom of Choice is obviously true, the Well-Ordering Principle is obviously false; and who can tell about Zorn's Lemma?" - Jerry Bona

1 Introduction

Bertrand Russell is quoted to having likened the Axiom of Choice to choosing from infinitely many pairs of socks and infinitely many pairs of shoes. Choosing a sock from each pair requires choice but choosing a shoe does not. Russell's metaphor is excellent for describing choice to someone unfamiliar with the topic, however I feel it does a disservice to the axiom in causing unfounded assumption about the axiom. The reason that choosing from each pair of socks requires choice is *not* because each pair of socks is indistinguishable. Given any two socks we could find a way to pick one of them. The problem lies in the fact that for a given pair of socks the method of distinguishing a sock is likely unique to that pair. For the shoes we can simplify specify "choose left" and any pair offers up a selection. A similarly universal comparison is not available for the socks and so we must rely on choice.

In modern set theory it is often assumed that we are working in ZFC when needed. While an indication of when and where the Axiom of Choice (AC) is being used is still a good proof etiquette, it is no longer as controversial as it once was [8]. However, the scope of this paper is to demonstrate statements that are equivalent to AC. Were these statements to be made in ZFC they would be theorems; but it isn't as instructive to show that $ZFC \vDash \mathcal{X}$ as it is to show $ZF \vDash AC \Leftrightarrow \mathcal{X}$. For this reason the choice is made to label these equivalents as statements rather than theorems.

Two notes on the structuring of the paper. Many of the definitions used throughout the paper are common enough to avoid redefinition in the course of the proof; as a courtesy to the reader they are formally defined at the end of the paper. Since family $\mathcal{A}$ of sets can be indexed by $\Gamma = \mathcal{A}$; throughout the paper $\mathcal{A}$ or $\{A_\gamma\}_{\gamma \in \Gamma}$ are used interchangeably as appropriate. The only ambiguity in this usage is that if $\mathcal{A}$ is a proper class then Γ would be the indexing class.

2 Axiom of Choice (AC)

Statement 2.1. *Modern Statement of AC*

If $\mathcal{A}$ is a nonempty family of nonempty sets, then there is a choice function f whose domain is $\mathcal{A}$ such that $\forall A \in \mathcal{A}, f(A) \in A$ [2].

Statement 2.2. *Alternative Statement of AC*

For every nonempty family, $\mathcal{A}$, of nonempty pairwise disjoint sets, there exists a selector set $Z \in \bigcup \mathcal{A}$ such that $\forall Y \in \mathcal{A},\ |Y \cap Z| = 1$ [4].

The Axiom of Choice was originally described by Zermelo using this alternative statement.

Proposition 2.3. *(ZF) For $N_0 \in \mathbb{N}$, let $\mathcal{A} = \{A_n\}_{n \leq N_0}$ be a nonempty finite family of nonempty sets. Then there exists a choice function on $\mathcal{A}$.*

Proof. We show this by induction on N_0. Let $N_0 = 1$; then $\mathcal{A} = \{A_1\}$. Let $x \in A_1$ and define $f_1(A_1) = x$. Then f_1 is a choice function on $\mathcal{A}$. Assume that the conclusion of the proposition holds for $N_0 = k$. Let $\mathcal{A} = \{A_n\}_{n \leq k+1}$. Consider $\mathcal{B} = \{A_n\}_{n \leq k}$. There is a choice function $f_k : \mathcal{B} \to \bigcup \mathcal{B}$. Let $y \in A_{k+1}$ and define $f_{k+1} : \mathcal{A} \to \bigcup \mathcal{A}$ as follows:

$$f_{k+1}(A_n) = \begin{cases} f_k(A_n) & \mathcal{A}_n \in \mathcal{B} \\ y & otherwise \end{cases}$$

Then f_{k+1} is a choice function on $\mathcal{A}$. $\square$

Proposition 2.3 shows that AC is unnecessary in the case of a finite family, even if the individual members of the family are themselves infinite.

Example 2.4. Let $\mathbb{E}$ be the even integers and $\mathbb{O}$ be the odd integers. Then $\{1, 2\}$ is a selector set on $\{\mathbb{E}, \mathbb{O}\}$ since $\{1, 2\} \cap \mathbb{E} = \{2\}$ and $\{1, 2\} \cap \mathbb{O} = \{1\}$.

Theorem 2.5. *The two statements of the Axiom of Choice are equivalent.*

Proof.

Claim 2.6. *Statement* 2.1 $\Rightarrow$ *Statement* 2.2.

Let $\mathcal{A}$ be a family of nonempty, pairwise disjoint sets. From the assumption of Statement 2.1 $\exists f : \mathcal{A} \to \bigcup \mathcal{A}$ such that $\forall A \in \mathcal{A}, f(A) \in A$. Define $Z = f^{\to}(\mathcal{A})$.

Assume toward contradiction that $\exists Y \in \mathcal{A},\ |Y \cap Z| = 0$. From the assumptions on f, $f(Y) \in Y$. However, since Z is the image of f over $\mathcal{A}$, $f(Y) \in Z$ and thus $f(Y) \in (Y \cap Z)$. This contradicts the assumption.

Assume toward contradiction $\exists Y \in \mathcal{A},\ |Y \cap Z| \geq 2$. Then $\exists y_1, y_2 \in (Y \cap Z)$ s.t. $y_1 \neq y_2$, so $y_1, y_2 \in Z$. Since Z is an image, $\exists X_1, X_2 \in \mathcal{A}$ s.t. $f(X_1) = y_1$ and $f(X_2) = y_2$. Since f is well-defined, $X_1 \neq X_2$. But $y_1, y_2 \in Y$, so $Y \cap X_1 \neq \varnothing$ and $Y \cap X_2 \neq \varnothing$. This contradicts that $\mathcal{A}$ is pairwise disjoint.

Hence $|Y \cap Z| \neq 0$ and $|Y \cap Z| < 2$, so $|Y \cap Z| = 1$. Thus Statement 2.2 holds.

Claim 2.7. *Statement* 2.2 $\Rightarrow$ *Statement* 2.1.

Let $\mathcal{A} = \{A_\gamma\}_{\gamma\in\Gamma}$ be a nonempty family of nonempty sets. Since Statement 2.2 does not apply to $\mathcal{A}$ we need to produce a pairwise disjoint family that is related to $\mathcal{A}$. To create this family define $B_\gamma = \{(a,\gamma) : a \in \mathcal{A}_\gamma\}$ and $\mathcal{B} = \{B_\gamma\}_{\gamma\in\Gamma}$ [1]. Since $\mathcal{B}$ is pairwise disjoint Statement 2.2 applies to $\mathcal{B}$, namely $\exists Z \subseteq \bigcup \mathcal{B}$ s.t. $\forall \gamma \in \Gamma$, $|Z \cap \mathcal{B}_\gamma| = 1$. Define $f(A_\gamma) = a$ where $(a,\gamma) \in Z$. Since Z only shares one element with any B_γ, f is well defined. Hence f is a choice function as defined in Statement 2.1.

□

3 Infinite Cartesian Product (ICP)

Just and Weese actually use ICP as their version of AC. These are kept separate to reflect that their equivalence is only trivial when using the formal definition of an infinite cartesian product and not when using the intuitive idea of an infinite n-tuple.

Definition 3.1. Cartesian Product

Let $\mathcal{A}$ be a family of sets indexed on Γ. Then the cartesian product of $\mathcal{A}$ over Γ is:

$$\prod_{\gamma\in\Gamma} A_\gamma = \{f : \Gamma \to \bigcup_{\gamma\in\Gamma} A_\gamma \Big| \forall j \in \Gamma, f(j) \in A_j\}$$

Statement 3.2. *Statement of ICP*

Let $\{A_\gamma\}_{\gamma\in\Gamma}$ be a family of nonempty sets. Then $\prod_{\gamma\in\Gamma} A_\gamma \neq \varnothing$ [4].

Theorem 3.3. *ICP* $\Leftrightarrow$ *AC.*

Proof. Let $\{A_\gamma\}_{\gamma\in\Gamma}$ be a family of nonempty sets.

$$\text{Then } f \text{ is a choice function on } \mathcal{A} \iff f \in \prod_{\gamma\in\Gamma} A_\gamma \iff \prod_{\gamma\in\Gamma} A_\gamma \neq \varnothing$$

□

4 Well-Ordering principle (WO)

Statement 4.1. *Statement of WO*

For every set X, there exists a binary relation, $\preceq_w \subseteq X \times X$, such that $\preceq_w$ well-orders X.

Theorem 4.2. *WO $\Rightarrow$AC.*

Attempt of Proof. Let $\{A_\gamma\}_{\gamma\in\Gamma}$ be a nonempty family of nonempty sets. By WO, $\exists \preceq_\gamma$ s.t. $\langle A_\gamma, \preceq_\gamma\rangle$ is well-ordered. Define the function $f : \mathcal{A} \to \bigcup_{\gamma\in\Gamma} A_\gamma$ by $f(A_\gamma) = \min_{\preceq_\gamma}(A_\gamma)$, where $\min_{\preceq_\gamma}(A_\gamma)$ is the $\preceq_\gamma$-smallest element of A_γ. Hence $\forall \gamma \in \Gamma,\ f(A_\gamma) \in A_\gamma$ making f a choice function. □

The above attempt requires that each A_γ be equipped with its own well-order. However, since WO does not require that well-orders be unique, there can be multiple well-orders on each A_γ. This turns the family $W_\gamma = \{\preceq \mid \preceq \text{ is a well-order on } A_\gamma\}$ into a possibly infinitely indexed set of nonempty sets, requiring AC to choose a specific $\preceq_\gamma$. We strengthen the proof by requiring a single well-ordering on $\bigcup_{\gamma\in\Gamma} A_\gamma$. Since the well-orderings on this set comprise only a single set of orders, we do not require AC to choose an order with which to construct our function.

Proof. (from [4]). Let $\{A_\gamma\}_{\gamma\in\Gamma}$ be a nonempty family of nonempty sets. By WO, $\exists \leq$ s.t. $\langle \bigcup\limits_{\gamma\in\Gamma} A_\gamma, \leq\rangle$ is well-ordered. Define the function $f : \mathcal{A} \to \bigcup\limits_{\gamma\in\Gamma} A_\gamma$ by $f(A) = \min_{\leq}(A)$. This element exists and is unique since $\forall\psi \in \Gamma,\ A_\psi \in \bigcup\limits_{\gamma\in\Gamma} A_\gamma$. Hence $\forall\gamma \in \Gamma,\ f(A_\gamma) \in A_\gamma$, making f a choice function. □

5 Disjointification (DIS)

Statement 5.1. *Statement of DIS*

Let $\{A_\gamma\}_{\gamma\in\Gamma}$ be a family of sets. Then there exists $\{B_\gamma\}_{\gamma\in\Gamma}$ such that $\forall i, j \in \Gamma$, $i \neq j \to B_i \cap B_j = \varnothing$. Furthermore, $\forall i \in \Gamma, B_i \subseteq A_i$ and $\bigcup\limits_{\gamma\in\Gamma} B_\gamma = \bigcup\limits_{\gamma\in\Gamma} A_\gamma$ [4].

Theorem 5.2. *WO ⇒ DIS.*

Proof. Let $\{A_\gamma\}_{\gamma\in\Gamma}$ be an family of sets. From the assumption of WO, $\exists \leq$ s.t. $\langle\Gamma, \leq\rangle$ is well-ordered. Define $B_\gamma = A_\gamma - \bigcup\limits_{k<\gamma} A_k$. It follows that $\forall\gamma \in \Gamma, B_\gamma \subseteq A_\gamma$ and then also that $\bigcup\limits_{\gamma\in\Gamma} B_\gamma \subseteq \bigcup\limits_{\gamma\in\Gamma} A_\gamma$. Consider $B_i \neq B_j$; W.L.O.G. assume that $i < j$. Then $B_i \subseteq A_i \subseteq \bigcup\limits_{k<j} A_k$, hence $B_i \cap B_j = \varnothing$. Let $x \in \bigcup\limits_{\gamma\in\Gamma} A_\gamma$ and $\Gamma_x = \{\gamma : x \in A_\gamma\}$. Then Γ_x is a nonempty subset of $\langle\Gamma, \leq\rangle$ so it has a smallest element γ_x. This means that $x \in A_{\gamma_x}$. By definition, $B_{\gamma_x} = A_{\gamma_x} - \bigcup\limits_{k<\gamma_x} A_k$, and since $\forall k < \gamma_x, x \notin A_k$, it must follow that $x \in B_{\gamma_x} \subseteq \bigcup\limits_{\gamma\in\Gamma} B_\gamma$. Thus $\bigcup\limits_{\gamma\in\Gamma} A_\gamma \subseteq \bigcup\limits_{\gamma\in\Gamma} B_\gamma$, and by anti-symmetry of inclusion they are equal. □

Theorem 5.3. *DIS ⇒ AC.*

Proof. Let $\mathcal{A} = \{A_\gamma\}_{\gamma\in\Gamma}$ be a nonempty family of nonempty, pairwise disjoint sets. Define $\mathcal{F} = \{\{(a, 0), (1, A_\gamma)\} : a \in A_\gamma \in \mathcal{A}\}$ [3] and index $\mathcal{F}$ by Ψ. By DIS, there exists a pairwise disjoint family of sets, $\mathcal{B}$, indexed by Ψ such that $\forall\psi \in \Psi, B_\psi \subseteq F_\psi$, and such that $\bigcup\mathcal{B} = \bigcup\mathcal{F}$. Define $Z = \{x \in \bigcup\mathcal{A} : \exists\psi \in \Psi, \{(x, 0), (1, A_\gamma)\} = B_\psi\}$.

Claim 5.4. $\forall \psi \in \Psi, \exists a \in \bigcup \mathcal{A}$ *s.t.* $(a, 0) \in B_\psi$.

Let $\psi \in \Psi$. Then $F_\psi \in \mathcal{F}, \exists Y \in \mathcal{A}$ s.t. $B_\psi \subseteq F_\psi = \{(y \in Y, 0), (1, Y)\}$. $\mathcal{A}$ is pairwise disjoint, so Y is the only element of $\mathcal{A}$ containing y. This means that F_ψ is the only element of $\mathcal{F}$ containing $(y, 0)$. It follows that $(y, 0) \in B_\psi$ since it must be in $\bigcup \mathcal{B}$ and B_ψ is the only subset of F_ψ in $\mathcal{B}$.

Claim 5.5. *Z as defined above is a selector set on $\mathcal{A}$.*

Let $A_\gamma \in \mathcal{A}$, with $A_\gamma \neq \varnothing$. Then $\forall a \in A_\gamma,\ \exists F_\psi = \{(a, 0), (1, A_\gamma)\}$. This implies that $(1, A_\gamma) \in \bigcup \mathcal{F} = \bigcup \mathcal{B}$. Since $\mathcal{B}$ is pairwise disjoint, $\exists! B_s \in \mathcal{B}$ s.t. $(1, A_\gamma) \in B_S$. From Claim 5.4 we know that $(x \in A_\gamma, 0) \in B_s$. So $B_s = \{(x, 0), (1, A_\gamma)\}$, which means that $x \in Z$. Since $x \in A_\gamma,\ x \in A_\gamma \cap Z$. Assume that $x, y \in A_\gamma \cap Z$. Then $\exists B_\psi, B_\delta$ s.t. $B_\psi = \{(x, 0), (1, A_\gamma)\}$ and $B_\delta = \{(y, 0), (1, A_\gamma)\}$. Since $\mathcal{B}$ is pairwise disjoint, $B_\psi \cap B_\delta \neq \varnothing$ implies that $\psi = \delta$. This is true only if $x = y$. We conclude that $|A_\gamma \cap Z| = 1$ and thus Z is a selector set on $\mathcal{A}$.

□

6 Hausdorff's Maximal Principle (HMP)

Statement 6.1. *Statement of HMP*

Let $\langle X, \leq \rangle$ be a partially ordered set and $A \subseteq X$ be a chain. Then $\exists M \subseteq X$ such that the following properties are true:

1. M is a chain
2. $A \subseteq M$
3. $\forall B \subseteq X$, if B is a chain and $M \subseteq B$, then $M = B$.

Conditions (1) and (2) say that M extends A as a chain, and (3) adds that M is maximal w.r.t extending A [4].

Lemma 6.2. *Let X be a set and $\mathcal{C} \subseteq \wp(X)$ be closed under unions of subfamilies that are linearly ordered by inclusion. Let $f : \mathcal{C} \to \mathcal{C}$ be a function s.t. $\forall c \in \mathcal{C}, c \subseteq f(c)$. Then $\exists c \in \mathcal{C}, f(c) = c$ [4].*

Proof. (Heavily from [4]). Let X, $\mathcal{C}$, and f be defined as in Lemma 6.2. Let $\alpha > 0$ be an ordinal s.t. $|\alpha| > |\mathcal{C}|$. This ensures that no function mapping α to $\mathcal{C}$ is injective. Define $F : \alpha \to \wp(X)$ by transfinite recursion as follows:
For $\beta \in \alpha$, assume that $\forall \gamma < \beta,\ f(\gamma)$ is defined.

$$F(\beta) = \begin{cases} f(\bigcup\limits_{\gamma<\beta} F(\gamma)) & \bigcup\limits_{\gamma<\beta} F(\gamma) \in \mathcal{C} \\ X & \text{otherwise} \end{cases}$$

We wish to show that $\forall \beta \in \alpha, F(\beta) \in \mathcal{C}$. We do this in two steps.

Claim 6.3. $\forall \gamma < \beta < \alpha,\ F(\gamma) \subseteq F(\beta)$.

If $F(\beta) = X$, then Claim 6.3 is true since F maps into $\wp(X)$. If $F(\beta) \neq X$, then $\bigcup\limits_{\psi<\beta} F(\psi) \in \mathcal{C}$ from the definition of F. Since $\gamma < \beta$, it follows that $F(\gamma) \subseteq \bigcup\limits_{\psi<\beta} F(\psi)$. And from the assumption on f, that $\forall c \in \mathcal{C}, c \subseteq f(c)$, it follows that

$$F(\gamma) \subseteq \bigcup_{\psi<\beta} F(\psi) \subseteq f(\bigcup_{\psi<\beta} F(\psi)) = F(\beta).$$

Claim 6.4. $\forall \beta \in \alpha, F(\beta) \in \mathcal{C}$.

Claim 6.4 will be shown using transfinite induction on β. Let $\beta < \alpha$. Assume $\forall \gamma < \beta, F(\gamma) \in \mathcal{C}$. Let $\mathcal{F} = \{F(\gamma) : \gamma < \beta\}$. Claim 6.3 shows that $\mathcal{F}$ is linearly ordered, and since each $F(\gamma) \in \mathcal{C}, \mathcal{F} \subseteq \mathcal{C}$. By the assumptions on $\mathcal{C}$ it is closed under unions of linearly ordered subsets, so $\bigcup \mathcal{F} \in \mathcal{C}$. Since f takes $\mathcal{C}$ into $\mathcal{C}$, $F(\beta) = f(\bigcup_{\gamma<\beta} F(\gamma)) = f(\bigcup \mathcal{F}) \in \mathcal{C}$.

To finish proving Lemma 6.2 we restrict the definition of F to $F\big|^{F^{\rightarrow}(\alpha)} : \alpha \to \mathcal{C}$ using Claim 6.4. Since α was chosen so that F could not be one to one, $\exists \gamma, \beta \in \alpha$ s.t. $F(\beta) = F(\gamma)$ and W.L.O.G. $\gamma < \beta$. If $F(\beta) = X$, then $X \in \mathcal{C}$, and $X \subseteq f(X)$ by the restriction on f. Thus $f(X) = X$ and Lemma 6.2 is satisfied. Assume $F(\beta) \neq X$. Let $x \in \mathcal{C}$. Then $F(\beta) = F(\gamma) = x$. From the properties of f we have that $x \subseteq f(x)$. Since $\gamma < \beta$ the following holds:

$$f(x) = f(F(\gamma)) \subseteq \bigcup_{\delta<\beta} f(F(\delta)) = f(\bigcup_{\delta<\beta} F(\delta)) = F(\beta) = x.$$

Thus $f(x) = x$ and Lemma 6.2 is satisfied. $\square$

Theorem 6.5. *ICP $\Rightarrow$ HMP.*

Proof. (Heavily from [4]). Let $\langle X, \preceq \rangle$ be a partially ordered set and $A \subseteq X$ be a chain. Define the set $\mathcal{C}$ s.t. $\mathcal{C} = \{C : A \subseteq C \subseteq X \text{ and } C \text{ is a chain}\}$. Let $\{C_\gamma\}_{\gamma\in\Gamma}$ be a subfamily of $\mathcal{C}$ that is linearly ordered by inclusion. Let $x, y \in \bigcup_{\gamma\in\Gamma} C_\gamma$; then $\exists \delta \in \Gamma, x \in C_\delta$ and $\exists \psi \in \Gamma, y \in C_\psi$. Since $\{C_\gamma\}$ is linearly ordered by inclusion, either $C_\delta \subseteq C_\psi$ or $C_\psi \subseteq C_\delta$. W.L.O.G. assume that it is the former; then both x and y are in C_δ, a chain, and thus x and y are comparable. This makes $\bigcup_{\gamma\in\Gamma} C_\gamma$ a chain containing A, so $\bigcup_{\gamma\in\Gamma} C_\gamma \in \mathcal{C}$. Thus $\mathcal{C}$ is closed under unions of subfamilies that are linearly ordered by inclusion.

For each $C \in \mathcal{C}$ define A_C as follows.

$$A_C = \begin{cases} \{C\} & \text{if } C \text{ is maximal} \\ \{E : E \text{ extends } C \text{ and } E \neq C\} & \text{otherwise} \end{cases}$$

Note that $A_C \subseteq \mathcal{C}$ and that $C \in A_C \Leftrightarrow C$ is maximal. By ICP, $\exists f \in \prod_{C \in \mathcal{C}} A_C$. This f satisfies the conditions of Lemma 6.2, so $\exists M \in \mathcal{C}, f(M) = M$. Thus $M \in A_M$, so M is a maximal chain that extends A. □

Corollary 6.6. *AC ⇒ HMP.*

7 M1 Maximal Principle (M1)

Statement 7.1. *Statement of M1*

If $\langle X, \leq \rangle$ is a non-empty partially ordered set, and $\langle X, \leq \rangle$ has the property that every linearly ordered subset of $\langle X, \leq \rangle$ has an upper bound in $\langle X, \leq \rangle$, then $\langle X, \leq \rangle$ has a maximal element [2].

Kuratowski used a principle related to M1 in 1922 [5]; then in 1935, Zorn used a second related principle [9], was the first to apply this type of principle to algebras, and claimed that this principle was equivalent to AC. What today is referred to as the Kuratowski-Zorn Lemma is actually a slightly different statement that is equivalent to the principles used by both. The naming convention being used is the one used by Rubin and Rubin in [6].

Statement 7.2. *Zorn's Original Theorem*

If $\mathcal{A}$ is a family of sets that is closed under the union of chains, then there is at least one $A^* \in \mathcal{A}$ that is not contained as a proper subset in any other $A \in \mathcal{A}$ [9].

Theorem 7.3. *HMP ⇒ M1.*

Proof. Let $\langle X, \leq \rangle$ be a non-empty partially ordered set s.t. $\langle X, \leq \rangle$ has the property that every linearly ordered subset of $\langle X, \leq \rangle$ has an upper bound in $\langle X, \leq \rangle$. Since $\varnothing$

is vacuously a chain; by HMP there is a maximal chain M s.t. $\varnothing \subseteq M \subseteq X$. By the assumption on $\langle X, \leq \rangle$, M has an upper bound, $x \in X$ s.t. $\forall m \in M, m \leq x$. Assume toward contradiction that x is not a maximal element in X. Then $\exists z \in X,\ z > x$. But $M \subset (M \cup \{z\})$, which contradicts M being maximal. Hence we conclude that x is maximal in $\langle X, \leq \rangle$, satisfying M1. □

Corollary 7.4. *AC* $\Rightarrow$ *M1.*

Theorem 7.5. *M1* $\Rightarrow$ *AC.*

Proof. This proof will rely on Statement 2.2. Let $\{A_\gamma\}_{\gamma \in \Gamma}$ be a nonempty family of pairwise disjoint nonempty sets. Define $X = \{Y : \forall \gamma \in \Gamma,\ |Y \cap A_\gamma| \leq 1\}$ and order X by inclusion. Then $\varnothing \in X$, so X is nonempty. Let $\{Y_\psi\}_{\psi \in \Psi} \subseteq X$ be a chain. We show that $\bigcup\limits_{\psi \in \Psi} Y_\psi \subseteq X$ by contradiction. Assume for some A_γ that $|(\bigcup\limits_{\psi \in \Psi} Y_\psi) \cap A_\gamma| > 1$. Then $\exists x, y \in (\bigcup\limits_{\psi \in \Psi} Y_\psi) \cap A_\gamma$ s.t. $x \neq y$. Since each Y_ψ intersects A_γ exactly once, $\exists \psi_x, \psi_y \in \Psi$ s.t. $x \in Y_{\psi_x}, y \in Y_{\psi_y}$ and $x \notin Y_{\psi_y}, y \notin Y_{\psi_x}$. This implies that $Y_{\psi_x} \neq Y_{\psi_y}$,$Y_{\psi_x} \not\subseteq Y_{\psi_y}$,$Y_{\psi_x} \not\supseteq Y_{\psi_y}$ which contradicts $\{Y_\psi\}_{\psi \in \Psi}$ being a chain. Thus $\bigcup\limits_{\psi \in \Psi} Y_\psi \subseteq X$ is an upper-bound for $\{Y_\psi\}_{\psi \in \Psi}$. Hence $\langle X, \subseteq \rangle$ satisfies the conditions for M1, so $\exists Z \in X$ s.t. Z is maximal.

Claim 7.6. *Z is a selector set on $\mathcal{A}$.*

Let $A_\gamma \in \mathcal{A}$. By the construction of X, $|Z \cap A_\gamma| \leq 1$. Assume toward contradiction that $|Z \cap A_\gamma| = 0$. Since A_γ is nonempty by the assumptions on $\mathcal{A}$, without use of AC we can choose $a \in A_\gamma$. But $(Z \cup a) \in X$ and $Z \subset (Z \cup a)$, contradicting Z being maximal. Hence Z is a selector set on $\mathcal{A}$. □

Theorem 7.7. *M1* $\Rightarrow$ *WO.*

Proof. Let X be a set. Then $\varnothing$ is well-ordered vacuously by any relation, so assume X is nonempty. Define $A = \{\langle Y, \preceq_{Y_\psi}\rangle : Y \subseteq X,\ \langle Y, \preceq_{Y_\psi}\rangle$ is a w.o$\}$. Then $\langle \varnothing, \preceq\rangle \subseteq A$. For any $x \in X$, $\langle \{x\}, \preceq\rangle \in A$ because singletons are trivially well-ordered by any relation. Hence A is nonempty. Define an order on A, $\trianglelefteq$ s.t. $\forall \langle Y, \preceq_{Y_\psi}\rangle, \langle Z, \preceq_{Z_\chi}\rangle \in A$; $\langle Y, \preceq_{Y_\psi}\rangle \trianglelefteq \langle Z, \preceq_{Z_\chi}\rangle$ iff $Y \subseteq Z$ and $\prec_{Y_\psi} \subseteq \prec_{Z_\chi}$ and $\forall y \in Y,\ \forall z \in (Z - Y),\ y \preceq_{Z_\chi} z$. This says that $\langle Y, \preceq_{Y_\psi}\rangle$ is an initial segment of $\langle Z, \preceq_{Z_\chi}\rangle$.

Claim 7.8. $\trianglelefteq$ *is a partial order on* A*.*

Let $\langle Y, \preceq_{Y_\psi}\rangle, \langle Z, \preceq_{Z_\chi}\rangle, \langle W, \preceq_{W_\pi}\rangle \in A$.

(Reflexive) $Y \subseteq Y$, $\preceq_{Y_\psi} \subseteq \preceq_{Y_\psi}$, $\forall y \in Y, \forall n \in (Y - Y) = \varnothing, y \preceq_{Y_\psi} n$. Thus $\langle Y, \preceq_{Y_\psi}\rangle \trianglelefteq \langle Y, \preceq_{Y_\psi}\rangle$.

(Transitive) Assume $\langle Y, \preceq_{Y_\psi}\rangle \trianglelefteq \langle Z, \preceq_{Z_\chi}\rangle \trianglelefteq \langle W, \preceq_{W_\pi}\rangle$. We can also assume that $Y \neq Z \neq W$ otherwise the case is trivial. By transitivity of inclusion $Y \subseteq W$ and $\preceq_{Y_\psi} \subseteq \preceq_{W_\pi}$. Let $y \in Y$ and $w \in (W - Y)$. There are two cases to consider. The first case is that $w \in (W - Z)$; there is a $z \in (Z - Y)$ since $Z \neq Y$, and from the assumption $y \preceq_{Z_\chi} z$ and $z \preceq_{W_\pi} w$. Since $\preceq_{W_\pi}$ extends $\preceq_{Z_\chi}$, $y \preceq_{W_\pi} z$ and since $\preceq_{W_\pi}$ is a well-order it is transitive. So $y \preceq_{W_\pi} w$ and the first case holds. Otherwise, $w \in (Z - Y)$. In this case $\langle Y, \preceq_{Y_\psi}\rangle \trianglelefteq \langle Z, \preceq_{Z_\chi}\rangle$ implies that $y \prec_{Z_\chi} w$. Since $\preceq_{W_\pi}$ extends $\preceq_{Z_\chi}$, $y \preceq_{W_\pi} w$ and the second case holds. Thus the $\trianglelefteq$ is transitive.

(Antisymmetric) Assume $\langle Y, \preceq_{Y_\psi}\rangle \trianglelefteq \langle Z, \preceq_{Z_\chi}\rangle$ and $\langle Z, \preceq_{Z_\chi}\rangle \trianglelefteq \langle Y, \preceq_{Y_\psi}\rangle$. Then $Y = Z$ and $\preceq_{Y_\psi} = \preceq_{Z_\chi}$ by antisymmetry of inclusion. So $\langle Y, \preceq_{Y_\psi}\rangle = \langle Z, \preceq_{Z_\chi}\rangle$.

Claim 7.9. *All chains in* A *have an upper-bound in* A*.*

Let $\{\langle Y_\gamma, \preceq_\gamma\rangle\}_{\gamma\in\Gamma} \subseteq A$ be a chain. Let $\langle M, \preceq\rangle = \langle \bigcup_{\gamma\in\Gamma} Y_\gamma, \bigcup_{\gamma\in\Gamma} \preceq_\gamma\rangle$. We will show that $\langle M, \preceq\rangle$ is well-ordered.

(Reflexive) Let $m \in M$. Then $\exists \gamma \in \Gamma$ s.t. $m \in Y_\gamma$. Since $\langle Y_\gamma, \preceq_\gamma \rangle$ is a w.o. $m \preceq_\gamma m$, thus $m \preceq m$.

(Anti-symmetric) Let $m, n \in M$ s.t. $m \preceq n$ and $n \preceq m$. Then $m \preceq n$ iff $\exists \gamma$ s.t. $m \preceq_\gamma n$. Similarly $\exists \psi$ s.t. $n \preceq_\psi m$. Because $\{\langle Y_\gamma, \preceq_\gamma \rangle\}_{\gamma \in \Gamma}$ is a chain, either $\preceq_\gamma \subseteq \preceq_\psi$ or $\preceq_\psi \subseteq \preceq_\gamma$. W.L.O.G. assume the latter, thus $m \preceq_\gamma n$ and $n \preceq_\gamma m$, since the elements of the chain are all w.o. it follows that $m = n$.

(Transitive) Transitivity will be inherited from $\{\langle Y_\gamma, \preceq_\gamma \rangle\}_{\gamma \in \Gamma}$ in the same manner as anti-symmetry was.

(Total Order) Let $m, n \in M$. Then $\exists \gamma, \psi \in \Gamma,\ m \in Y_\gamma,\ n \in Y_\psi$. W.L.O.G. $Y_\psi \subset Y_\gamma$, so $m, n \in Y_\gamma$ which is a w.o. Thus $m \preceq_\gamma n$ or $n \preceq_\gamma m$. It follows that $n \preceq m$ or $m \preceq n$.

(Well-order) Let $N \subseteq M$ s.t. $N \neq \varnothing$. Then $\exists \gamma \in \Gamma, N \cap Y_\gamma \neq \varnothing$. Since Y_γ is a w.o., let $y \in Y_\gamma$ be the $\preceq$-smallest element of $N \cap Y_\gamma \subseteq Y_\gamma$. Let $x \in N$. If $x \in Y_\gamma$ then $y \preceq x$. Otherwise, $x \notin Y_\gamma$, so $\exists \psi, x \in Y_\psi$ and $\langle Y_\gamma, \preceq_{Y_\gamma} \rangle \trianglelefteq \langle Y_\psi, \preceq_{Y_\psi} \rangle$ since they must be related and $x \in Y_\delta \subset Y_\gamma$ is impossible; so by the definition of $\trianglelefteq$, $y \preceq x$. Thus N has a smallest element, so $\langle M, \preceq \rangle$ is well-ordered. We conclude that $\langle M, \preceq \rangle \in A$.

We now show that $\langle M, \preceq \rangle$ is an upper bound of the chain. Since $\forall \gamma,\ Y_\gamma \subseteq M,\ \preceq_{Y_\gamma} \subseteq \preceq$, the first two conditions of $\trianglelefteq$ are satisfied. Choose $\gamma \in \Gamma$ and $x \in Y_\gamma$. Let $m \in M$ s.t. $m \notin Y_\gamma$. Then $\exists \psi,\ m \in Y_\psi$ and $\langle Y_\gamma, \preceq_{Y_\gamma} \rangle \trianglelefteq \langle Y_\psi, \preceq_{Y_\psi} \rangle$. Thus $x \preceq_{Y_\psi} m$ so $x \preceq m$. Thus each chain in A has an upper bound.

Claim 7.10. *The maximal element of A is $\langle X, \preceq \rangle$ for some ordering $\preceq$.*

By M1, $\exists \langle Z, \preceq \rangle \in A$ s.t. Z is maximal. Assume toward contradiction that $Z \neq X$. Then $\exists x \in X$ s.t. $x \notin Z$. Define $\langle Z \cup \{x\}, \preceq \rangle$ as an extension of $\langle Z, \preceq \rangle$ s.t. $\forall z \in Z, z \preceq x$. Then$\langle Z, \preceq \rangle \triangleleft \langle Z \cup \{x\}, \preceq \rangle \in A$ contradicting $\langle Z, \preceq \rangle$ being maximal in A. Thus $Z = X$, so there is a well-order on X. $\square$

8 Tychonoff's Theorem (TY)

Statement 8.1. *Statement of TY*

The product of every family of compact topological spaces is compact in the product topology.

Theorem 8.2. *TY ⇒ICP [6].*

Proof. Let $\{A_\gamma\}_{\gamma\in\Gamma}$ be an indexed family of nonempty sets. Let $*$ be such that $* \notin \bigcup_{\gamma\in\Gamma} A_\gamma$. Define $X_\gamma = A_\gamma \cup \{*\}$ and $\tau_\gamma = \{\varnothing, \{*\}, X_\gamma\}$. Then $\{(X_\gamma, \tau_\gamma)\}_{\gamma\in\Gamma}$ is a family of compact spaces. Define $\mathcal{Z} = \{Z_\gamma\}_{\gamma\in\Gamma}$ s.t. $Z_\gamma = \{f \in \prod_{\gamma\in\Gamma} X_\gamma : f(\gamma) \in A_\gamma\}$. Then $Z_\gamma \neq \varnothing$ since the function f s.t. $f(\gamma) \in A_\gamma; \forall \psi \neq \gamma,\ f(\psi) = *$ is an element of Z_γ. Furthermore Z_γ is closed as it is the product of closed sets. Also X_γ is closed and $X_\gamma - \{*\} = A_\gamma$ is closed. Let $J \subseteq \Gamma$ be finite and $\{Z_i\}_{i\in J} \subseteq \mathcal{Z}$.

$$\text{Define } f_J \text{ s.t. } \begin{cases} f_J(i) \in A_i & i \in J \\ * & \text{otherwise} \end{cases}$$

Since J is finite, f_J is well-defined without use of AC and $f_J \in \bigcap_{i\in J} Z_i$. Thus $\mathcal{Z}$ is a family of closed sets with the finite intersection property, so by compactness, $\varnothing \neq \bigcap_{\gamma\in\Gamma} Z_\gamma \in \prod_{\gamma\in\Gamma} A_\gamma$. □

Corollary 8.3. *TY ⇒ AC*

Lemma 8.4. *M1 ⇒ Each filter is contained within an ultrafilter [8].*

Proof. Let $\mathcal{F}$ be a filter and $\mathcal{P} = \{\mathcal{A} \text{ a filter} : \mathcal{F} \subseteq \mathcal{A}\}$. Then $\mathcal{P}$ is a poset ordered by inclusion. The union of a chain of filters is a filter, so each chain in $\mathcal{P}$ has an upper bound. By M1, $\mathcal{P}$ has a maximal element. This is an ultrafilter containing $\mathcal{F}$. □

Lemma 8.5. *If each ultrafilter in (X, τ) converges then (X, τ) is compact [8].*

Proof. Let $C = \{U_\gamma\}_{\gamma \in \Gamma}$ s.t. $\forall \gamma \in \Gamma$, $U_\gamma \in \tau$ and no finite subcollection of C covers X. Define $\mathcal{F} = \{A \subseteq X : \bigcap_{i=1}^{n}(X - U_{\gamma_i}) \subseteq A,$ where the U_{γ_i} are any finite collection of $U_\gamma\}$. Then $\mathcal{F}$ is a filter. By Lemma 8.4, $\exists \mathcal{G}$, $\mathcal{F} \subseteq \mathcal{G}$ and $\mathcal{G}$ is an ultrafilter. From the assumption $\mathcal{G}$ converges to some $x \in X$. Assume toward contradiction that $\exists \gamma$, $x \in U_\gamma$. By convergence, $U_\gamma \in \mathcal{G}$, but from construction $(X - U_\gamma) \in \mathcal{G}$. Since $\mathcal{G}$ is a filter $U_\gamma \cap (X - U_\gamma) = \varnothing \in \mathcal{G}$. This however, is a contradiction. Thus no such γ exists, so C does not cover X. We conclude that (X, τ) is compact. □

Lemma 8.6. *$\mathcal{F}$ is an ultrafilter on X if $\forall E \in X; E \in \mathcal{F}$ or $(X - E) \in \mathcal{F}$ [8].*

Proof. Let $\mathcal{F}$ be a filter that fulfills the above condition. Assume $\mathcal{F}$ is not an ultrafilter. Then $\exists \mathcal{G}$, $\mathcal{F} \subset \mathcal{G}$. Thus $\exists A \subset X, A \in \mathcal{G}$ and $\mathcal{A} \notin \mathcal{F}$. By the condition of the lemma, $(X - A) \in \mathcal{F}$ thus $A \in \mathcal{G}$ and $(X - A) \in \mathcal{G}$ which is a contradiction. We conclude that $\mathcal{F}$ was an ultrafilter. □

Definition 8.7. If $\mathcal{F}$ is a filter on X and $f : X \to Y$, then $f(\mathcal{F})$ is a filter on Y s.t. $\{f^{\to}(F) : F \in \mathcal{F}\}$ is a filterbase for $f(\mathcal{F})$.

Lemma 8.8. *If $\mathcal{F}$ is an ultrafilter and π is a surjective function, then $\pi(\mathcal{F})$ is an ultrafilter.*

Proof. Let $E \subset Y$. Then $\pi^{\leftarrow}(E) \in \mathcal{F}$ or $(X - \pi^{\leftarrow}(E)) \in \mathcal{F}$ from Lemma 8.6. We finish considering each case.

Case $[\pi^{\leftarrow}(E) \in \mathcal{F}]$ Since $\pi^{\to} \vdash \pi^{\leftarrow}$, $\pi^{\to}(\pi^{\leftarrow}(E)) \subseteq E$ and by Definition 8.7, $\pi^{\to}(\pi^{\leftarrow}(E)) \in \pi(\mathcal{F})$. Filters are closed under superset, so $E \in \pi(\mathcal{F})$.

Case $[(X - \pi^{\leftarrow}(E)) \in \mathcal{F}]$ Since π is surjective, $\pi^{\to}$ preserve compliments and $\pi^{\to}(\pi^{\leftarrow}(E)) = E$. Thus $(Y - E) \in \pi(\mathcal{F})$.

□

Lemma 8.9. *Let $\{(X_\gamma, \tau_\gamma)\}_{\gamma\in\Gamma}$ be a family of compact topological spaces and $(X, \tau) = \prod_{\gamma\in\Gamma} X_\gamma$ with the product topology. Let $\mathcal{F}$ be an ultrafilter in X. If $\forall\gamma \in \Gamma,\ \pi_\gamma(F) \to \pi(x)$ then $\mathcal{F} \to x$ [8].*

Proof. (Heavily from [8]). Define the sets as above and suppose that $\forall\gamma \in \Gamma,\ \pi_\gamma(F) \to \pi(x)$. Let $U \in \tau$ be a basic nbhd of x. This means that there are finitely many basic nbhds of $\pi_{\gamma_k}(x)$, $U_k \in \tau_{\gamma_k}$ s.t. $U = \bigcap_{k=1}^{n} \pi_{\gamma_k}{}^{\leftarrow}(U_k)$. For each γ_k, $\pi_{\gamma_k}(\mathcal{F}) \to \pi_{\gamma_k}(x)$, so $U_k \in \pi_{\gamma_k}(\mathcal{F})$. Then $\exists F_k \in \mathcal{F},\ \pi_{\gamma_k}{}^{\rightarrow}(F_k) \subseteq U_k$ since this is the filterbase. Thus $\bigcap_{k=1}^{n} F_k \subseteq U$, and since filters are closed under finite intersection, $U \in \mathcal{F}$. Thus $\mathcal{F} \to x$. □

Theorem 8.10. *ICP $\Rightarrow$ TY.*

Proof. Let $\{(X_\gamma, \tau_\gamma)\}_{\gamma\in\Gamma}$ be a family of compact topological spaces and $(X, \tau) = \prod_{\gamma\in\Gamma} X_\gamma$ with the product topology. Let $\mathcal{F}$ be an ultrafilter in X. Then $\forall\gamma \in \Gamma,\ \pi_\gamma(\mathcal{F})$ is an ultrafilter; since X_γ is compact $\pi_\gamma(\mathcal{F})$ converges to some set of points S_γ. From Lemma 8.9, $\mathcal{F} \to \prod_{\gamma\in\Gamma} S_\gamma$. ICP implies that this product is non-empty, so $\mathcal{F}$ converges. By Lemma 8.5, X is compact. □

Corollary 8.11. *AC $\Rightarrow$ TY.*

9 Cardinal Comparability

Statement 9.1 (Statement of Injective Comparability (IC))**.** *For arbitrary sets X and Y, either there exists an injection from X to Y or there exists an injection from Y to X [4].*

Theorem 9.2. *WO $\Rightarrow$ IC.*

Proof. Let X and Y be sets. If X or Y is empty then any function with an empty domain is injective, so we assume both are non-empty. By WO, $\exists \leq_X, \leq_Y$ that well-order X and Y respectively. Let $*$ be such that it is not an element of X or Y.

Define $F : \mathbf{ON} \to X \cup \{*\}$ and $G : \mathbf{ON} \to Y \cup \{*\}$ by transfinite recursion s.t.

$$F(\alpha) = \begin{cases} min_{\leq_X}(X - \bigcup\limits_{\beta<\alpha} F(\beta)) & X - \bigcup\limits_{\beta<\alpha} F(\beta) \neq \varnothing \\ * & \text{otherwise} \end{cases}$$

$$G(\alpha) = \begin{cases} min_{\leq_Y}(Y - \bigcup\limits_{\beta<\alpha} G(\beta)) & Y - \bigcup\limits_{\beta<\alpha} G(\beta) \neq \varnothing \\ * & \text{otherwise} \end{cases}$$

Assume for some α, β that $F(\alpha) = F(\beta) \neq *$. Since $F(\alpha)$ is the min of the elements not removed by β it can't have been removed by β, so $F(\alpha) \notin \bigcup\limits_{\gamma<\beta} F(\gamma)$, so $\alpha \geq \beta$. Similarly $F(\beta) \notin \bigcup\limits_{\gamma<\alpha} F(\gamma)$, so $\beta \geq \alpha$. Thus $\alpha = \beta$.

Suppose $\forall \alpha \in \mathbf{ON},\ F(\alpha) \neq *$. Then F injects $\mathbf{ON}$ into X. Since $\mathbf{ON}$ is a proper class, this is impossible. Let m_X be the smallest ordinal such that $F(m_X) = *$ and $\alpha_X = \bigcup\limits_{\gamma<m_X} \gamma$. Then $F|_{\alpha_X}$ is bijective.

The above holds identically for G; let m_Y be the smallest ordinal such that $G(m_Y) = *$ and $\alpha_Y = \bigcup\limits_{\gamma<m_Y} \gamma$. Then $G|_{\alpha_Y}$ is bijective.

W.L.O.G. $\alpha_X \leq \alpha_Y$, so the identity function $\mathrm{I} : \alpha_X \to \alpha_Y$ is injective. Thus $G|_{\alpha_Y} \circ \mathrm{I} \circ (F|_{\alpha_X})^{-1}$ is an injection from X to Y. $\square$

Alternative Proof. Let X and Y be nonempty sets. Define $\mathcal{F} = \{f \subseteq A \times B \mid A \subseteq X,\ B \subseteq Y,\ f \text{ is an injective function}\}$ and order $\mathcal{F}$ by inclusion, so it is a poset. Let $\{f_\gamma\}_{\gamma\in\Gamma}$ be a chain in $\mathcal{F}$. We wish to show that its upper bound, $\bigcup\limits_{\gamma\in\Gamma} f_\gamma \in \mathcal{F}$. Let

$x \in X$ s.t. $(x, y_1) \in \bigcup_{\gamma \in \Gamma} f_\gamma$ and $(x, y_2) \in \bigcup_{\gamma \in \Gamma} f_\gamma$. Then $\exists \alpha, \beta \in \Gamma$ s.t. $(x, y_1) \in f_\alpha, (x, y_2) \in f_\beta$. Since these are elements of a chain W.L.O.G. $f_\alpha \subseteq f_\beta$ thus $(x, y_1), (x, y_2) \in f_\beta$. This is a well-defined function, so $y_1 = y_2$. Thus the upper bound is a well-defined function and is a member of $\mathcal{F}$. $\mathcal{F}$ meets the criteria for M1, thus it has a maximal element, $F : M \to N$. Assume toward contradiction that $M \neq X$ and $N \neq Y$. Then $\exists x_0 \in X - M$, $\exists y_0 \in Y - N$. Then $F \subset F \cup \{(x_0, y_0)\} \in \mathcal{F}$, contradicting that F was maximal.

Continue by cases:

(Case: $M = X$) Then F is an injection from X into Y.

(Case: $N = Y, M \neq X$) Then $(F\big|_M)^{-1}$ is an injection from Y into X.

□

Statement 9.3 (Statement of Surjective Comparability (SC))**.** *For arbitrary sets X and Y, either there exists an surjection from X to Y or there exists an surjection from Y to X [4].*

Theorem 9.4. *IC $\Rightarrow$ SC.*

Proof. Let X and Y be sets. If either set is empty any function with empty codomain is surjective. So we assume both are non-empty. W.L.O.G. $\exists f$ s.t. $f : X \to Y$ is injective by IC. Let $a \in X$.

Define $g : Y \to X$ s.t.

$$g(y) = \begin{cases} f^{-1}(y) & y \in f^{\to}(X) \\ a & otherwise \end{cases}$$

Then g is a surjection. □

Theorem 9.5. *For every set X there exists an ordinal $\alpha > 0$ such that X cannot be mapped onto α [4].*

Proof in text of [4] on page 160-161.

Theorem 9.6. *$SC \Rightarrow WO$*

Proof. Let X be a set. Let α be an ordinal as described in Theorem 9.5. Then by SC there must exist an $f : \alpha \to X$ s.t. f is surjective. Define $\leq$ on X s.t. $x \leq y \Leftrightarrow min(f^{\leftarrow}(x) \leq min(f^{\leftarrow}(y))$. Let $A \subseteq X$. Then $f^{\leftarrow}(A)$ has a smallest element,γ, since the ordinals are well ordered. Then $f(\gamma)$ is the smallest element of A. Thus $\langle X, \leq \rangle$ is a well-ordered set. □

Statement 9.7 (Law of Trichotomy for Sets)**.** *For any sets X and Y one of the following holds: $X < Y$, $Y < X$, $X \equiv Y$.*

Theorem 9.8. *Law of Trichotomy for Sets is equivalent to AC.*

This theorem is a formalization of the fact that AC ⇒ IC ⇒ SC ⇒ AC.

10 Notable Applications

Theorem 10.1 (M1)**.** *Every vector space has a basis.*

Proof. Let V be a vector space over a field F. Let $P = \{L \subseteq V : L \text{ is linearly independent}\}$. Then $\langle P, \subseteq \rangle$ is a non-empty poset since singletons are linearly independent. Let $\{P_\gamma\}_{\gamma \in \Gamma} \subseteq P$ be a chain and $U = \bigcup_{\gamma \in \Gamma} P_\gamma$. Assume towards contradiction that U is linearly dependent. So $\exists I,\ |I| < \infty,\ \forall i \in I;\ 0 \neq v_i \in U,\ 0 \neq a_i \in F,\ \sum_{i \in I} a_i v_i = 0$. For each i, $\exists \gamma_i \in \Gamma,\ v_i \in P_{\gamma_i}$. But, $\{P_\gamma\}$ is a chain and I is finite, so $\exists \psi \in \Gamma,\ \forall i \in I,\ v_i \in P_\psi$. This would imply that P_ψ is linearly dependent which is a contradiction. Thus $U \in P$ and U is an upper bound for $\{P_\gamma\}$. By M1, P has a maximal element, B. Assume toward contradiction that $span(B) \neq V$. Then $\exists x \in V$ s.t. $x \notin span(B)$. But then $B \cup \{x\}$ is a linearly independent subset of V, contradicting the maximality of B. Thus B is a basis for V. □

Theorem 10.2 (AC)**.** *There exists a Vitali set. (A subset of* $\mathbb{R}$ *that is not Lebesgue measurable) [7].*

Proof. (Heavily from [4]). Define an equivalence relation on the interval $[0,1)$ s.t. $x \equiv y \longleftrightarrow x - y \in \mathbb{Q}$. By AC, $\exists V$ which is a selector set on ${}^{[0,1)}/_{\equiv}$. For $q \in \mathbb{Q}$ define $q + V = \{q + x : x \in V\}$. Let $a, b \in \mathbb{Q}, a \neq b$. Assume toward contradiction that $(a + V) \cap (b + V) \neq \varnothing$. So $\exists v_1, v_2 \in V$ s.t. $a + v_1 = b + v_2$. This implies that $v_1 - v_2$ is rational, and since V was a selector set $v_1 = v_2$; but this contradicts $a \neq b$. Define $W = \bigcup\limits_{q \in [-1,1] \cap \mathbb{Q}} (q + V)$. Since W is a union of disjoint sets and m is translation invariant,

$$m(W) = \sum_{q \in [-1,1] \cap \mathbb{Q}} m(q + V) = \sum_{q \in [-1,1] \cap \mathbb{Q}} m(V).$$

This implies that $m(W) = 0$ or $m(W) = \infty$ depending on $m(V)$. However, $[0,1) \subseteq W \subseteq [-1,2)$, so $1 \leq W \leq 3$. Thus V is a Vitali set.

□

11 Discussion

From the work of Kurt Gödel and Paul Cohen we know that the Axiom of Choice is independent of the other axioms of ZF; neither it nor it's negation can be proven using these axioms. Thus it falls to some other form of intuition to decide whether or not to include AC in our axioms. Though each of the axioms must be accepted by similar intuitive reasons; AC has become much more of a sticking point than the rest.

Informally, an axiom should increase our understanding of the numerical world. The theorems proven by an axiom should give us verifiable results in the limited cases that they can be tested with empirical measure. We can observe addition in action; if our set theory claims to model this then it needs to model it correctly. We also observe that $x \neq y$ and $x = y$ are mutually exclusive statements and so our logical/numerical systems should model this. Taken as a whole the set of axioms should be consistent, that is they do not produce contradictions. The problem occurs in that since AC is dealing strictly with infinite sets that it's applications become very non-intuitive to interpret.

The Banach-Tarski is a perfect example of an application of AC that is seemingly paradoxical when taken as a model for reality. The paradox states that a ball in $\mathbb{R}^3$ can be cut into 5 unmeasurable pieces and reassembled to produce two balls of the same volume as the original. This contradicts our understanding of the laws of conservation of mass, and seems like justification for rejecting AC. My response to this paradox is twofold. The first is explained in [4]. $\mathbb{R}^3$ is uncountably infinite whereas reality seems to be discrete. Therefore to model an idea of mass in $\mathbb{R}^3$ we have developed the idea of measure. The measure of a set is in some sense trying to show its mass. But the pieces used in the paradox are unmeasurable. Since physically we cannot cut a solid ball with mass into massless pieces, the paradox seems to not apply to any physical model of reality. My second response is that this construction is somewhat normal when discussing infinite sets. $\mathbb{Z}$ can be partitioned into $\mathbb{E}$ and $\mathbb{O}$. Both which are the same "size" as the original and each other.

In defense of AC I offer the Law of Trichotomy for Sets. This is the law that allows us to compare the size of sets, and more specifically to compare all cardinal numbers. Again informally, the cardinals were created to put an ordering on the size of the sets in set theory. Rejecting AC in a sense tells us that we can recursively build a chain of ordinals to be well-ordered, and then somehow they eventually stop being well-ordered. This paradox seems to be at least as strong as the Tarski paradox; and by allowing AC we gain all the useful equivalents shown above and more.

Anytime a mathematician proves a theorem the next step is to ask "How can I make my assumptions weaker?" The axiom of choice should be no different. Results proven with AC should be inspected to see if they can be proven without it; however, even if the assumptions of a theorem cannot be weakened (ie it needs AC) that does not negate the value of the original theorem.

12 Miscellaneous Definitions

Definition 12.1. Image

Let $f : \mathcal{X} \to \mathcal{Y}$ be a function and $A \subseteq \mathcal{X}$. The image of f is defined by:

$$f^{\rightarrow}(A) = \{f(x) : x \in A\}$$

Definition 12.2. Pre-image

Let $f : \mathcal{X} \to \mathcal{Y}$ be a function and $B \subseteq \mathcal{Y}$. The pre-image of f is defined by:

$$f^{\leftarrow}(B) = \{x : f(x) \in B\}$$

Definition 12.3. Well-founded relation

Let X be a set and $W \subseteq X \times X$. Then W is a well-founded relation on X iff $\forall A \subseteq X, A \neq \varnothing, \; \exists m \in A$ s.t. $\forall a \in A - \{m\}, (a, m) \notin W$

Definition 12.4. $I_w(x)$

Let X be a set with well-founded relation W. The initial segment of $x \in X$ is defined as $I_w(x) = \{z \in X : (z, x) \in W \text{ and } z \neq x\}$.

Definition 12.5. Partially Ordered Set (poset)

A partially ordered set $\langle X, \leq \rangle$ is a set X equipped with a binary operation $\leq \subseteq X \times X$ such that $\leq$ is reflexive, transitive, and antisymmetric.

Definition 12.6. Linearly Ordered Set

A linear order is a total order.

Definition 12.7. Totally Ordered Set (t.o.)

A totally ordered set $\langle X, \leq \rangle$ is a poset such that $\forall x, y \in X$,$x \leq y$ or $y \leq x$.

Definition 12.8. Chain

Let $\langle X, \leq \rangle$ be a poset and $A \subseteq X$. A is a chain iff $\langle A, \leq \rangle$ is totally ordered.

Definition 12.9. Well-Ordered Set (w.o.)

A well-ordered set $\langle X, \leq \rangle$ is a totally ordered set with the following property; $\forall A \subseteq X$ s.t. $A \neq \varnothing$, A has a smallest element with regard to $\leq$.

Definition 12.10. Adjoint Functions ($\vdash$)

Let L, M be posets. Let $f : L \to \; M$, $g : M \to \; L$ be order preserving functions. Then f is left-adjoint to g, $f \vdash g$, iff $\forall m \in M, \; f(g(m)) \leq m$ and $\forall l \in L, \; l \leq g(f(l))$.

Definition 12.11. Filter

$\mathcal{F}$ is a filter on a set S if $\mathcal{F}$ is a non-empty collection of non-empty subsets of S such that the following hold:

(1)$\forall F_1, F_2 \in \mathcal{F}, F_1 \cap F_2 \in \mathcal{F}$

(2)If $F \in \mathcal{F}$ and $F \subseteq A$ then $A \in \mathcal{F}$

Definition 12.12. Filterbase

$\mathcal{B}$ is a filterbase for filter $\mathcal{F}$ if $\forall F \in \mathcal{F}, \exists B \in \mathcal{B}; B \subseteq F$.

Definition 12.13. Ultrafilter

$\mathcal{F}$ is an ultrafilter if $\mathcal{F}$ is a filter and there are no other filters, $\mathcal{A}$, such that $\mathcal{F} \subset \mathcal{A}$.

Definition 12.14. Filter Convergence

A filter $\mathcal{F}$ on a topological space (X, τ) converges to a point x if $\forall U \in \tau, x \in U; \exists F \in \mathcal{F}, F \subseteq U$.

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