Lie Groups and Lie Algebras

by

Scott M. Eddy

Submitted in Partial Fulfillment of the Requirements

for the Degree of

Masters

in the

Mathematics

Program

YOUNGSTOWN STATE UNIVERSITY

August, 2011

Lie Groups and Lie Algebras

Scott M. Eddy

I hereby release this thesis to the public. I understand that this thesis will be made available from the OhioLINK ETD Center and the Maag Library Circulation Desk for public access. I also authorize the University or other individuals to make copies of this thesis as needed for scholarly research.

Signature:

Scott M. Eddy, Student — Date

Approvals:

Dr. Richard Goldthwait, Thesis Advisor — Date

Dr. Eric Wingler, Committee Member — Date

Dr. Tom Wakefield, Committee Member — Date

Peter J. Kasvinsky, Dean of School of Graduate Studies & Research — Date

ABSTRACT

The subject of Lie groups is one that slips by many a mathematician. Many claim that the topic is not accessible to undergraduate research. The book *Lie Groups* by Harriet Pollatsek came out a few years ago, and it was meant to be a new way to be introduced to the topic. However, the book does not quite get far enough to give a formal definition of a Lie group. The goal of this project is to "bridge the gap." The objective of this thesis is to include all the introductory material required to get to where the definition of a Lie group is no longer something so complicated. We will illustrate the major concepts by examples.

Many matrix groups are Lie groups. Matrix groups are well-known, and they are an ideal place to start learning about what a Lie group can do. We then look at tangent spaces of the matrix groups, or the Lie algebra that is associated with each Lie group. After some motivation behind Lie algebras, we finally get to the feature presentation: a group and a differentiable manifold, put together into one super structure known as a Lie group.

ACKNOWLEDGMENTS

First of all, I would like to thank Dr. Goldthwait for all his help, his insight, and most importantly his patience. For the past year, he listened patiently to everything I had to say, right or wrong. All year long he asked all the right questions to get me thinking, and sometimes just to keep me on track. I knew from the start we would work well together. I'd also like to thank my committee member, Dr. Wingler, who throughout the entire last year took question after question I had, and not only had an answer but was excited to help - and sometimes have something interesting to work on while one of his classes took an exam. Next, I'd like to thank Dr. Wakefield, not only for being on my committee, but for always being supportive, attending my talks early this year, and for just always having such a positive attitude. Dr. Spalsbury deserves my thanks as well. She pushed me right when I needed a push. Not only that, but it was her book that I decided to borrow which got me started on this topic.

I'd also like to thank my many friends in the mathematics department. I don't believe I could ever have done something like this without such a supportive, friendly environment and friends who can listen to you talk for extended periods of time despite not following anything that's being said - that's what they claim anyway, but I haven't seen a proof.

Finally, I'd like to thank my family, who has been extremely supportive of every decision I've ever made throughout my life. I hear all the time how proud my parents are of me. I am truly blessed.

Contents

0 Preliminaries

Definition 0.1. A set G along along with a binary operation (written in multiplicative form) is called a **group** if the following conditions are satisfied:

1. Closure: $ab \in G$ for all $a, b \in G$.

2. Associativity: $(ab)c = a(bc)$ for all $a, b, c \in G$.

3. Identity: there exists an element $e \in G$ such that $ae = ea = a$ for all $a \in G$.

4. Inverse: for all $a \in G$ there exists a $b \in G$ such that $ab = e = ba$.

Definition 0.2. A mapping ϕ of a group G into a group G' ($\phi : G \to G'$) is called a **homomorphism** if it preserves the group operation. Symbolically,

$$\phi(ab) = \phi(a)\phi(b)$$

for all $a, b \in G$.

Definition 0.3. The set of **all n by n matrices with real-valued entries** (which is a vector space over $\mathbb{R}$ under matrix addition and scalar multiplication) is denoted $\mathcal{M}(n, \mathbb{R})$.

Definition 0.4. The **general linear group** is the group of all invertible, n by n matrices with real entries under the group operation matrix multiplication. It is denoted

$$\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathcal{M}(n, \mathbb{R}) : \det A \neq 0\}.$$

Definition 0.5. "The general linear group" also describes the set of all *invertible* linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ and is denoted

$$\mathrm{GL}(\mathbb{R}^n) = \{T : \mathbb{R}^n \to \mathbb{R}^n : T \text{ is linear and invertible}\}.$$

Take special note of the fact that the general linear group of matrices and the general linear group of transformations are essentially "the same." That is, the groups are isomorphic, which we will define later.

Definition 0.6. The **special linear group** is the group of all n by n matrices with determinant 1, denoted

$$\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathcal{M}(n, \mathbb{R}) : \det A = 1\}.$$

Definition 0.7. The **orthogonal group** of n by n matrices (in the Euclidean case) is

$$\mathrm{O}(n, \mathbb{R}) = \{A \in \mathcal{M}(n, \mathbb{R}) : AA^T = I_n\}.$$

Definition 0.8. The **special orthogonal group** of n by n matrices is

$$\mathrm{SO}(n, \mathbb{R}) = \{A \in \mathrm{O}(n, \mathbb{R}) : \det A = 1\}.$$

A few other comments worthy of note:

1. $\leq$ will be used as a subgroup symbol.

2. I_n will be used to denote the n by n identity matrix.

3. 0_n will be used to denote the n by n zero matrix.

1 Tangent Spaces

We will start by taking a look at some tangent spaces of matrix groups with the operation matrix multiplication. Note that we must have invertible matrices since every element in a group must have an inverse. We will start by letting $G \leq \mathrm{GL}(n, \mathbb{R})$. Also let $\gamma : \mathbb{R} \to G$ be a differentiable function with $\gamma(t) = (a_{ij}(t))$. (For each i and j, $a_{ij} : \mathbb{R} \to \mathbb{R}$ is differentiable.) Note also that differentiation of a matrix-valued function is done component-wise, so $\gamma'(t) = (a'_{ij}(t))$.

Definition 1.1. A function f is **smooth** if f is infinitely differentiable.

Definition 1.2. We define the **tangent space at the identity of** G to be the set of matrices having the form $\gamma'(0)$ for a function $\gamma : \mathbb{R} \to G$ such that $\gamma(0) = I_n$.
We use the notation $\mathcal{L}(G) = \{\gamma'(0) | \gamma : \mathbb{R} \to G, \gamma(0) = I_n, \gamma \text{ smooth}\}$.

Proposition 1.3. The product rule holds for derivatives of matrix-valued functions in the cases of $n \times n$ matrices.

The proof follows directly from the use of the product rule of real-valued functions.

Proposition 1.4. The tangent space $\mathcal{L}(G)$ for any matrix group $G \leq \mathrm{GL}(n, \mathbb{R})$ is a vector space.

Proof. Since by its definition, $\mathcal{L}(G) \subseteq \mathcal{M}(n, \mathbb{R})$ and $\mathcal{M}(n, \mathbb{R})$ is itself a vector space, we need only show that $\mathcal{L}(G)$ is nonempty (i), closed under scalar multiplication (ii), and closed under matrix addition (iii).

(i) Let $\gamma(t) = I_n$, so $\gamma(0) = I_n$ as well. Then $\gamma'(t) = 0 \; \forall \, t \in \mathbb{R}$ and $\gamma'(0) = 0_n$. Therefore $0_n \in \mathcal{L}(G)$.

(ii) Let $A \in \mathcal{L}(G)$ and $c \in \mathbb{R}$. We want to show $cA \in \mathcal{L}(G)$. Since $A \in \mathcal{L}(G)$, there exists $\alpha : \mathbb{R} \to G$ smooth and passing through the identity such that $\alpha'(0) = A$. Let $\gamma : \mathbb{R} \to G$ be defined by $\gamma(t) = \alpha(ct)$ for all $t \in \mathbb{R}$. Then clearly γ is smooth and $\gamma(0) = \alpha(c0) = \alpha(0) = I_n$, which implies $\gamma'(0) \in \mathcal{L}(G)$. Finally, $\gamma'(t) = c\alpha'(ct)$ implies $\gamma'(0) = c\alpha'(0) = cA$. Thus $cA \in \mathcal{L}(G)$.

(iii) Let A, $B \in \mathcal{L}(G)$. Then there exists α, $\beta : \mathbb{R} \to G$ smooth and passing through the identity such that $\alpha'(0) = A$ and $\beta'(0) = B$. Let $\gamma : \mathbb{R} \to G$ be defined by

$$\gamma(t) = \alpha(t)\beta(t) \text{ for all } t \in \mathbb{R}.$$

Then because α and β are smooth, γ is smooth (product rule for derivatives), and $\gamma(0) = \alpha(0)\beta(0) = I_n \cdot I_n = I_n$, which implies $\gamma'(0) \in \mathcal{L}(G)$. Finally,

$$\gamma'(t) = \alpha'(t)\beta(t) + \alpha(t)\beta'(t)$$

implies

$$\begin{aligned} \gamma'(0) &= \alpha'(0)\beta(0) + \alpha(0)\beta'(0) \\ &= A \cdot I_n + I_n \cdot B \\ &= A + B. \end{aligned}$$

Thus $A + B \in \mathcal{L}(G)$.

Hence we have $\mathcal{L}(G)$ is a subspace of $\mathcal{M}(n, \mathbb{R})$. $\square$

Now we will determine the tangent spaces of a few particular matrix groups.

Example 1.5. First, suppose $G = \mathrm{GL}(2, \mathbb{R})$. We will determine $\mathcal{L}(G)$.

$$\text{Let } \gamma_1(t) = \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix}, \quad \gamma_2(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \gamma_3(t) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}, \quad \gamma_4(t) = \begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix}.$$

$$\text{Then } \gamma_1'(t) = \begin{bmatrix} e^t & 0 \\ 0 & 0 \end{bmatrix}, \quad \gamma_2'(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \gamma_3'(t) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \gamma_4'(t) = \begin{bmatrix} 0 & 0 \\ 0 & e^t \end{bmatrix},$$

$$\text{and } \gamma_1'(0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \gamma_2'(0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \gamma_3'(0) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \gamma_4'(0) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Notice γ_i is smooth and $\gamma_i(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for each $i = 1, 2, 3, 4$. Also,

$$\{\gamma_1'(0), \gamma_2'(0), \gamma_3'(0), \gamma_4'(0)\} = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$$

is the standard basis for the vector space $\mathcal{M}(2, \mathbb{R})$. Since $\mathcal{L}(G)$ is a vector space, and the basis of $\mathcal{M}(2, \mathbb{R})$ is contained in it, all linear combinations must also be in it. Thus $\mathcal{M}(2, \mathbb{R}) \subseteq \mathcal{L}(G)$. By definition $\mathcal{L}(G) \subseteq \mathcal{M}(2, \mathbb{R})$, which gives subset inclusion in both directions. Therefore the tangent space at the identity of the general linear group of 2 by 2 matrices is *all* 2 by 2 matrices: $\mathcal{L}(G) = \mathcal{M}(2, \mathbb{R})$.

Example 1.6. Now suppose $G = \mathrm{SL}(2, \mathbb{R})$, and we will again determine $\mathcal{L}(G)$. Suppose $\gamma(t) = \begin{bmatrix} \gamma_{11}(t) & \gamma_{12}(t) \\ \gamma_{21}(t) & \gamma_{22}(t) \end{bmatrix}$ gives a smooth curve in G passing through the identity. So, $\gamma_{11}(0) = \gamma_{22}(0) = 1$ and $\gamma_{12}(0) = \gamma_{21}(0) = 0$. Since $\det(\gamma(t)) = 1$,

$$\gamma_{11}(t)\gamma_{22}(t) - \gamma_{12}(t)\gamma_{21}(t) = 1.$$

Differentiating both sides with respect to t, we get

$$\gamma_{11}'(t)\gamma_{22}(t) + \gamma_{11}(t)\gamma_{22}'(t) - \gamma_{12}'(t)\gamma_{21}(t) - \gamma_{12}(t)\gamma_{21}'(t) = 0.$$

And evaluating at $t = 0$,

$$\gamma_{11}'(0)\gamma_{22}(0) + \gamma_{11}(0)\gamma_{22}'(0) - \gamma_{12}'(0)\gamma_{21}(0) - \gamma_{12}(0)\gamma_{21}'(0) = 0$$

$$\gamma_{11}'(0) \cdot 1 + 1 \cdot \gamma_{22}'(0) - \gamma_{12}'(0) \cdot 0 - 0 \cdot \gamma_{21}'(0) = 0$$

$$\gamma_{11}'(0) + \gamma_{22}'(0) = 0.$$

Therefore $\gamma'(0)$ has the form $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ where $a, b, c \in \mathbb{R}$, and

$$\mathcal{L}(G) \subseteq W = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a, b, c \in \mathbb{R} \right\} = \{A \in \mathcal{M}(2, \mathbb{R}) : \mathrm{tr}\, A = 0\}.$$

Next we need to show W is a subspace of $\mathcal{M}(2, \mathbb{R})$. It should be clear that $W \neq \emptyset$. Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and let $\begin{bmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{bmatrix} \in W$, so that we can show closure under scalar multiplication and vector addition in one step. Then

$$\alpha_1 \begin{bmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{bmatrix} + \alpha_2 \begin{bmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 a_1 + \alpha_2 a_2 & \alpha_1 b_1 + \alpha_2 b_2 \\ \alpha_1 c_1 + \alpha_2 c_2 & -(\alpha_1 a_1 + \alpha_2 a_2) \end{bmatrix} \in W.$$

Therefore W is a subspace. Now, we simply need to show the basis of W,

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\},$$

is contained in $\mathcal{L}(G)$ to complete the proof that $\mathcal{L}(G) = W$.

$$\text{Let } \gamma_1(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}, \gamma_2(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \gamma_3(t) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}.$$

$$\text{So } \gamma_1'(0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \gamma_2'(0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \gamma_3'(0) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Therefore $W \subseteq \mathcal{L}(G)$ and $\mathcal{L}(G) = W$.

Example 1.7. For another example, we will determine $\mathcal{L}(G)$ for $G = \mathrm{O}(2, \mathbb{R})$.
First recall that $\mathrm{O}(n, \mathbb{R}) = \{M \in \mathcal{M}(n, \mathbb{R}) : M^T M = I_n\}$. Let $\gamma(t)$ be a smooth curve in G passing through the identity where $\gamma'(0) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\gamma(t)^T \gamma(t) = I_2.$$

Differentiating both sides, we have

$$\gamma'(t)^T \gamma(t) + \gamma(t)^T \gamma'(t) = 0_2.$$

And letting $t = 0$ yields

$$\gamma'(0)^T = -\gamma'(0),$$

or

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}.$$

Hence $\gamma'(0)$ has the form $\begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix}$ where $x \in \mathbb{R}$, and $\mathcal{L}(G) \subseteq W = \left\{ \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} : x \in \mathbb{R} \right\}$.
Using the same method as before, let $\gamma(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$. Then γ is smooth and $\gamma(0) = I_2$. Then $\gamma'(0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathcal{L}(G)$, so $W \subseteq \mathcal{L}(G)$, and $\mathcal{L}(G) = W$.

Definition 1.8. Let $G_1 \leq \mathrm{GL}(n, \mathbb{R})$ and $G_2 \leq \mathrm{GL}(m, \mathbb{R})$, and assume $F : G_1 \to G_2$ is a group homomorphism. Then we can think of F as a function $F : \mathbb{R}^{n^2} \to \mathbb{R}^{m^2}$. Then F is called a **Lie group homomorphism** if F is smooth.

Definition 1.9. A Lie group homomorphism F is called a **Lie group isomorphism** when F is one-to-one and onto.

Example 1.10. Define $F : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^*$, where $\mathbb{R}^*$ is the group of reals without zero under multiplication, by $F(A) = \det(A)$ for all $A \in \mathrm{GL}(n, \mathbb{R})$. Then F is a Lie group

homomorphism and onto, but it is not an isomorphism since it is not one-to-one.
Letting $A, B \in \mathrm{GL}(n, \mathbb{R})$ we have $F(AB) = \det(AB) = \det(A)\det(B) = F(A)F(B)$, which tells us F is a homomorphism.
We will not go into all the detail of why the determinant is smooth, but I will give some justification. Letting S_n be the set of all permutations of $1, 2, \ldots, n$, we can define the determinant of $X \in \mathrm{GL}(n, \mathbb{R})$ as follows:

$$\det(X) = \sum_{\pi \in S_n} (\pm 1) x_{1\pi(1)} x_{2\pi(2)} \cdots x_{n\pi(n)}$$

where the sign is positive when π is an even permutation and odd when π is an odd permutation. The important thing to gather from this "version" of the determinant is that the formula is *linear* with respect to each variable x_{ij}. Since linear functions are smooth, and this is a linear function in each of n^2 variables, the determinant map (F in this case) is smooth and thus a Lie group homomorphism.
To see that F is onto, let $x \in \mathbb{R}^*$. Then choose A to be the matrix $A = (a_{ij})$ where $a_{ij} = 0$ when $i \neq j$, $a_{11} = x$, and $a_{ii} = 1$ for $2 \leq i \leq n$. Then $F(A) = \det(A) = x$, which means F is onto.
To see that F is not one-to-one, consider the case when $n = 2$, and let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Then

$$F(A) = \det(A) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

$$\text{and } F(B) = \det(B) = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1,$$

but $A \neq B$. Therefore F is not one-to-one.

Let $G_1 \leq \mathrm{GL}(n, \mathbb{R})$, $G_2 \leq \mathrm{GL}(m, \mathbb{R})$, and assume $F : G_1 \to G_2$ is a Lie group homomorphism. Now suppose $A_1 \in \mathcal{L}(G_1)$. Then there exists a smooth function $\alpha_1 : \mathbb{R} \to G_1$ such that $\alpha_1(0) = I_n$ and $\alpha_1'(0) = A_1$. Then let $\alpha_2 : \mathbb{R} \to G_2$ be defined by $\alpha_2(t) = F(\alpha_1(t))$.

Definition 1.11. The **differential of F at the identity** is $dF : \mathcal{L}(G_1) \to \mathcal{L}(G_2)$ with $dF(A_1) = \alpha_2'(0)$.

Proposition 1.12. If $F : G_1 \to G_2$ is a Lie group homomorphism with $G_1 \leq \mathrm{GL}(n, \mathbb{R})$ and $G_2 \leq \mathrm{GL}(m, \mathbb{R})$, then the differential $dF : \mathcal{L}(G_1) \to \mathcal{L}(G_2)$ is a linear transformation.

Proof. Let A_1, $B_1 \in \mathcal{L}(G_1)$. Then there exists α_1 and β_1 smooth and passing through the identity such that $A_1 = \alpha_1'(0)$ and $B_1 = \beta_1'(0)$.
Now let

$$\begin{aligned} \gamma_1(t) &= \alpha_1(t)\beta_1(t), \\ \alpha_2(t) &= F(\alpha_1(t)), \\ \beta_2(t) &= F(\beta_1(t)), \\ \text{and } \gamma_2(t) &= F(\gamma_1(t)). \end{aligned}$$

Keep in mind we already know from (1.4) that $\gamma_1(0) = I_n$ and $\gamma_1'(0) = \alpha_1'(0) + \beta_1'(0)$.
Now, we will show $\gamma_2(t) \in G_2$:
First of all,

$$\gamma_2(t) = F(\gamma_1(t)) = F(\alpha_1(t)\beta_1(t)) = F(\alpha_1(t))F(\beta_1(t)) = \alpha_2(t)\beta_2(t).$$

Since G_2 is a group and $\gamma_2(t)$ is the product of two elements of the group, it is contained in G_2 by closure.
Also, γ_2 passes through the identity:

$$\gamma_2(0) = F(\gamma_1(0)) = F(I_n) = I_m.$$

We made use of the well-known fact that a group homomorphism takes the identity in the domain to the identity in the range. (This is also easy to prove.)
Next, we show that $\gamma_2'(0) = \alpha_2'(0) + \beta_2'(0)$:

$$\gamma_2'(t) = [\alpha_2(t)\beta_2(t)]' = \alpha_2'(t)\beta_2(t) + \alpha_2(t)\beta_2'(t).$$

So when we substitute 0 in for t,

$$\begin{aligned}
\gamma_2'(0) &= \alpha_2'(0)\beta_2(0) + \alpha_2(0)\beta_2'(0) \\
&= \alpha_2'(0)F(\beta_1(0)) + F(\alpha_1(0))\beta_2'(0) \\
&= \alpha_2'(0) \cdot F(I_n) + F(I_n) \cdot \beta_2'(0) \\
&= \alpha_2'(0) \cdot I_m + I_m \cdot \beta_2'(0) \\
&= \alpha_2'(0) + \beta_2'(0).
\end{aligned}$$

Finally, we get around to showing dF is linear:

$$\begin{aligned}
dF(A_1) + dF(B_1) &= \alpha_2'(0) + \beta_2'(0) \\
&= \gamma_2'(0) \\
&= dF(\gamma_1'(0)) \\
&= dF(\alpha_1'(0) + \beta_1'(0)) \\
&= dF(A_1 + B_1).
\end{aligned}$$

Now let $c \in \mathbb{R}$, $\gamma_1(t) = \alpha_1(ct)$, and $\gamma_2(t) = F(\gamma_1(t))$.

$$\begin{aligned}
dF(cA_1) &= dF(c\alpha_1'(0)) \\
&= dF(\gamma_1'(0)) \\
&= \gamma_2'(0) \\
&= c\alpha_2'(0) \\
&= c\,dF(\alpha_1'(0)) \\
&= c\,dF(A_1).
\end{aligned}$$

□

Example 1.13. Let $F : \mathrm{GL}(2, \mathbb{R}) \to \mathbb{R}^*$ be given by $F(X) = \det(X)$. We use the notation $X = (x_{ij})$ so that $\det(X) = x_{11}x_{22} - x_{12}x_{21}$. We will show the differential dF at the identity takes X to $\mathrm{tr}(X) \in \mathbb{R}$.
Let $A \in \mathcal{M}(2, \mathbb{R})$, the tangent space of $\mathrm{GL}(2, \mathbb{R})$. Then there exists $\alpha : \mathbb{R} \to \mathrm{GL}(2, \mathbb{R})$

such that α is smooth, $\alpha(0) = I_2$, and $A = \alpha'(0)$. Now let $\beta(t) = F(\alpha(t))$, so $dF(A) = dF(\alpha'(0)) = \beta'(0)$. We have

$$\begin{aligned}\beta(t) &= \alpha_{11}(t)\alpha_{22}(t) - \alpha_{12}(t)\alpha_{21}(t)\\ \beta'(t) &= \alpha'_{11}(t)\alpha_{22}(t) + \alpha_{11}(t)\alpha'_{22}(t) - \alpha'_{12}(t)\alpha_{21}(t) - \alpha_{12}(t)\alpha'_{21}(t)\end{aligned}$$

Hence

$$\begin{aligned}dF(A) = \beta'(0) &= \alpha'_{11}(0)\alpha_{22}(0) + \alpha_{11}(0)\alpha'_{22}(0) - \alpha'_{12}(0)\alpha_{21}(0) - \alpha_{12}(0)\alpha'_{21}(0)\\ &= \alpha'_{11}(0) \cdot 1 + 1 \cdot \alpha'_{22}(0) - \alpha'_{12}(0) \cdot 0 - 0 \cdot \alpha'_{21}(0)\\ &= \alpha'_{11}(0) + \alpha'_{22}(0)\\ &= \mathrm{tr}(A).\end{aligned}$$

2 The Exponential Map

Definition 2.1. Let G be a group. Then a function $\gamma : \mathbb{R} \to G$ is called a **one-parameter subgroup of** G if γ is a continuous group homomorphism.

Proposition 2.2. Every one-parameter subgroup $\gamma : \mathbb{R} \to G$ where $G \leq \mathrm{GL}(n, \mathbb{R})$ satisfies $\gamma(0) = I_n$.

Proof. Let γ be a one-parameter subgroup.
Then since $\gamma(0) = \gamma(0 + 0) = \gamma(0) \cdot \gamma(0)$,

$$\begin{aligned}\gamma(0) &= \gamma(0) \cdot \gamma(0)\\ \gamma(0)^{-1} \cdot \gamma(0) &= \gamma(0)^{-1} \cdot \gamma(0) \cdot \gamma(0)\\ I_n &= I_n \cdot \gamma(0)\\ I_n &= \gamma(0).\end{aligned}$$

□

Lemma 2.3. If $\gamma : \mathbb{R} \to \mathbb{R}$ is continuous, $\gamma(0) = 1$, and $\gamma(t) \neq 0$ for all $t \in \mathbb{R}$, then $\gamma(1) > 0$.

Proof. Let $\gamma(1) = b$ and suppose $b < 0$. Since $b < 0 < 1$ and $\gamma(1) = b < 0 < \gamma(0)$, by the Intermediate Value Theorem from calculus there exists $t \in \mathbb{R}$ such that $\gamma(t) = 0$. This is a contradiction, thus $b > 0$. □

In the following lemma and theorem, we treat $\mathrm{GL}(1, \mathbb{R})$ as if it were $\mathbb{R}^*$, the group of non-zero reals under multiplication. It is easy to see that the two groups are isomorphic to one another.

Lemma 2.4. If $\gamma : \mathbb{R} \to \mathrm{GL}(1, \mathbb{R})$ is a one-parameter subgroup, or continuous group homomorphism, with $\gamma(1) = b > 0$, then $\gamma(t) = b^t$ for all $t \in \mathbb{R}$.

Proof. In order to prove this for all real t, we will show it in six steps:

(i) For all positive integers n, $\gamma(nt) = (\gamma(t))^n$:
We will use induction. If $n = 1$, $\gamma(1t) = \gamma(t) = (\gamma(t))^1$.
Now suppose $\gamma(nt) = (\gamma(t))^n$ for some n.
Then $\gamma((n + 1)t) = \gamma(nt + t) = \gamma(nt)\gamma(t) = (\gamma(t))^n\gamma(t) = (\gamma(t))^{n+1}$.

(ii) For all positive integers n, $\gamma(n) = b^n$:
Just use (i): $\gamma(n) = \gamma(n1) = (\gamma(1))^n = b^n$.

(iii) For all negative integers $-m$, $\gamma(-m) = b^{-m}$:
Since $\gamma(0) = 1$, $1 = \gamma(m - m) = \gamma(m)\gamma(-m) = b^m\gamma(-m)$.
Then $1 = b^m\gamma(-m)$ implies $b^{-m} = \gamma(-m)$.

(iv) For all positive integers k, $\gamma(1/k) = b^{1/k}$:
First, $b = \gamma(1) = \gamma(k(1/k)) = (\gamma(1/k))^k$.
Then raising each side of the equation to the power of $1/k$, we get $b^{1/k} = \gamma(1/k)$.

(v) For all rational numbers n/k, $\gamma(n/k) = b^{n/k}$:
$\gamma(n/k) = (\gamma(1/k))^n = (b^{1/k})^n = b^{n/k}$.

(vi) For all real numbers t, $\gamma(t) = b^t$:
Let $t \in \mathbb{R}$. Then we know that t is the limit of a sequence of rational numbers: Say $t = \lim_{n\to\infty} t_n$ where t_n is rational for all n. Since both γ and exponential functions are continuous, we can conclude

$$\gamma(t) = \gamma(\lim_{n\to\infty} t_n) = \lim_{n\to\infty} \gamma(t_n) = \lim_{n\to\infty} b^{t_n} = b^{\lim_{n\to\infty} t_n} = b^t.$$

□

Theorem 2.5. If $\gamma : \mathbb{R} \to \mathrm{GL}(1, \mathbb{R})$ is a one-parameter subgroup (or continuous group homomorphism), then $\gamma(t) = e^{at}$ for $a = \gamma'(0)$.

Proof. We know $\gamma(0) = 1$, and we will let $\gamma(1) = b$. We have also proven then that $\gamma(t) = b^t$ for all $t \in \mathbb{R}$. So let $a = \ln(b)$. Then

$$\gamma(t) = b^t = (e^{\ln(b)})^t = (e^a)^t = e^{at}.$$

Also, $\gamma'(t) = ae^{at}$ implies $\gamma'(0) = a$. □

Theorem 2.6. Let $G \leq \mathrm{GL}(n, \mathbb{R})$. If $\gamma : \mathbb{R} \to G$ is a one-parameter subgroup of G, then $\gamma(t) = e^{tA}$ for $A = \gamma'(0)$.

The proof of this theorem is an extension of 2.3, 2.4, and 2.5, but involves more in-depth work in matrix-valued functions which we will not get into here.

Next we will learn a very special theorem for matrices, which becomes very useful in discovering tangent spaces of groups of matrices. However, we must first mention a couple of lemmas used in the proof. Proofs of the lemmas are relatively simple and thus left to the reader.

Lemma 2.7. The determinant of an upper triangular matrix is the product of the entries along the main diagonal.

Lemma 2.8. For an upper triangular matrix A with main diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$, e^A has main diagonal entries $e^{a_{11}}, e^{a_{22}}, \ldots, e^{a_{nn}}$.

Making use of the prior two lemmas, we are now able to prove the next theorem.

Theorem 2.9. Let $A \in \mathcal{M}(n, \mathbb{R})$. Then $\det(e^A) = e^{\mathrm{tr}(A)}$.

Proof. For $A \in \mathcal{M}(n, \mathbb{R})$, we know there exists an invertible n by n matrix P such that $A = PUP^{-1}$ where U is (at least) an upper triangular matrix. So,

$$\begin{aligned}
\det(e^A) &= \det(e^{PUP^{-1}}) \\
&= \det(Pe^U P^{-1}) \\
&= \det(P)\det(e^U)\det(P^{-1}) \\
&= \det(P)\det(e^U) 1/\det(P) \\
&= \det(e^U) \\
&= e^{u_{11}} e^{u_{22}} \cdots e^{u_{nn}} \\
&= e^{u_{11}+u_{22}+\cdots+u_{nn}} \\
&= e^{\mathrm{tr}(U)} \\
&= e^{\mathrm{tr}(P^{-1}AP)} \\
&= e^{\mathrm{tr}(P(P^{-1}A))} \\
&= e^{\mathrm{tr}(A)}.
\end{aligned}$$

□

Now we can make special use of the exponential map in determining tangent spaces of our matrix groups. We'll take a look at a more general case of something we've already done. (1.6)

Example 2.10. We will calculate $\mathcal{L}(G)$ for $G = \mathrm{SL}(n, \mathbb{R})$.
We have already seen that when $\gamma : \mathbb{R} \to \mathrm{SL}(n, \mathbb{R})$ is smooth and passes through the identity then $\mathrm{tr}(\gamma'(0)) = 0$ (1.6). Therefore $\mathcal{L}(G) \subseteq W = \{B \in \mathcal{M}(n, \mathbb{R}) : \mathrm{tr}(B) = 0\}$. We only have to show $W \subseteq \mathcal{L}(G)$.
To do this, we will first show W is a vector space. Since by definition $W \subseteq \mathcal{M}(n, \mathbb{R})$ we need only show W is a subspace. Clearly, $0_n \in W$, so W is nonempty.
Now let $A, B \in W$ and $x, y \in \mathbb{R}$. Then

$$\begin{aligned}
\mathrm{tr}(xA + yB) &= xa_{11} + yb_{11} + \cdots + xa_{nn} + yb_{nn} \\
&= xa_{11} + \cdots + xa_{nn} + yb_{11} + \cdots + yb_{nn} \\
&= x(a_{11} + \cdots + a_{nn}) + y(b_{11} + \cdots + b_{nn} \\
&= x\,\mathrm{tr}(A) + y\,\mathrm{tr}(B) \\
&= x \cdot 0 + y \cdot 0 \\
&= 0.
\end{aligned}$$

Therefore W is a subspace of $\mathcal{M}(n, \mathbb{R})$.
Now let $B \in W$. Then if we can show $e^{tB} \in G$ we can say $W \subset \mathcal{L}(G)$. In other words, we have only to show $\det(e^{tB}) = 1$:

$$\det(e^{tB}) = e^{\mathrm{tr}(tB)} = e^0 = 1.$$

Hence $e^{tB} \in G$ for all $B \in W$. Thus $W \subseteq \mathcal{L}(G)$.
Finally, since we have subset inclusion both ways, $\mathcal{L}(G) = W$.

Here's one more example of determining a tangent space by making use of the exponential map. Again, we look at a more general case of an example (1.7) done previously.

Example 2.11. We will calculate $\mathcal{L}(G)$ for $G = \mathrm{O}(n, \mathbb{R})$.
We have already seen that when $\gamma : \mathbb{R} \to \mathrm{O}(n, \mathbb{R})$ is smooth and passes through the identity then by (1.7) $\gamma'(0) + \gamma'(0)^T = 0_n$. Therefore $\mathcal{L}(G) \subseteq W = \{B \in \mathcal{M}(n, \mathbb{R}) : B + B^T = 0_n\}$. We only have to show $W \subseteq \mathcal{L}(G)$.
To do this, we will first show W is a vector space. Since by definition $W \subseteq \mathcal{M}(n, \mathbb{R})$ we need only show W is a subspace. Clearly, $0_n \in W$, so W is nonempty.
Now let $A, B \in W$ and $x, y \in \mathbb{R}$. Then

$$\begin{aligned}(xA + yB) + (xA + yB)^T &= xA + yB + xA^T + yB^T \\ &= x(A + A^T) + y(B + B^T) \\ &= x \cdot 0_n + y \cdot 0_n \\ &= 0_n.\end{aligned}$$

Therefore W is a subspace of $\mathcal{M}(n, \mathbb{R})$.
Now let $B \in W$. Then if we can show $e^{tB} \in G$ we can say $W \subset \mathcal{L}(G)$. In other words, we have only to show $(e^{tB})^T e^{tB} = I_n$:

$$(e^{tB})^T e^{tB} = e^{tB^T} e^{tB} = e^{t(B+B^T)} = e^{0_n} = I_n.$$

Hence $e^{tB} \in G$ for all $B \in W$. Thus $W \subseteq \mathcal{L}(G)$.
Finally, since we have subset inclusion both ways, $\mathcal{L}(G) = W$.

So we have at least a couple of methods with which we can determine the tangent spaces of multiplicative matrix groups. In the next section, we'll take a closer look at the structure such spaces have.

3 Lie Algebras

The multiplicative matrix groups are great examples of Lie groups (The suspense of waiting for that definition must be killing you.) However, there is another structure very closely tied to every Lie group. Each Lie group has an associated Lie algebra. While looking at the tangent spaces of each matrix group in the previous sections, we were in fact looking at the Lie algebra associated with each of the matrix groups as Lie groups. Now we'll take a look at what it means to be a Lie algebra.

Definition 3.1. Let L be a vector space over $\mathbb{R}$. Then L is called a **Lie algebra** if it has a bracket operation satisfying the following properties for all $x, y, z \in L$ and all $c \in \mathbb{R}$:

(1) $[x, y] \in L$ (closure)

(2) $[x, y] = -[y, x]$ (anti-symmetric)

(3) $\left.\begin{array}{r}[x, y + z] = [x, y] + [x, z] \\ [x + y, z] = [x, z] + [y, z] \\ [cx, y] = c[x, y] = [x, cy]\end{array}\right\}$ (bilinear)

(4) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ (Jacobi identity)

We call [,] a **Lie bracket**.

Proposition 3.2. Let L be a Lie algebra. Then for all $v \in L$, $[v,v]=0$ and $[v,0]=0$.

Proof. Let $v \in L$. Then $[v,v]=-[v,v]$ implies $2[v,v]=0$ and $[v,v]=0$.
Also, since $[v,0]=[v,0+0]=[v,0]+[v,0]$,

$$\begin{aligned}[v,0]&=[v,0]+[v,0]\\ [v,0]-[v,0]&=[v,0]+[v,0]-[v,0]\\ 0&=[v,0].\end{aligned}$$

□

Now we'll see an example of a Lie algebra. In this example, we will have to make use of a well-known identity for vectors: for all $x,y,z \in \mathbb{R}^3$, $x\times(y\times z)=(x\cdot z)y-(x\cdot y)z$.

Example 3.3. $\mathbb{R}^3$ is a Lie algebra with the bracket being the cross product.
Since we already know $\mathbb{R}^3$ is a vector space and the cross product of two vectors is a vector, we need only show the cross product is anti-symmetric, bilinear, and satisfies the Jacobi identity.

To see antisymmetric: let $v,w \in \mathbb{R}^3$ with $v=\begin{bmatrix} v_1\\ v_2\\ v_3\end{bmatrix}$ and $w=\begin{bmatrix} w_1\\ w_2\\ w_3\end{bmatrix}$.

Then $v\times w=\begin{bmatrix} v_2w_3-v_3w_2\\ v_3w_1-v_1w_3\\ v_1w_2-v_2w_1\end{bmatrix}=-\begin{bmatrix} w_2v_3-w_3v_2\\ w_3v_1-w_1v_3\\ w_1v_2-w_2v_1\end{bmatrix}=-(w\times v)$.

To see bilinear: let $u,v,w \in \mathbb{R}^3$.

$$\begin{aligned}\text{Then } u\times(v+w)&=\begin{bmatrix} u_2(v_3+w_3)-u_3(v_2+w_2)\\ u_3(v_1+w_1)-u_1(v_3+w_3)\\ u_1(v_2+w_2)-u_2(v_1+w_1)\end{bmatrix}\\ &=\begin{bmatrix} u_2v_3-u_3v_2\\ u_3v_1-u_1v_3\\ u_1v_2-u_2v_1\end{bmatrix}+\begin{bmatrix} u_2w_3-u_3w_2\\ u_3w_1-u_1w_3\\ u_1w_2-u_2w_1\end{bmatrix}\\ &=(u\times v)+(u\times w).\end{aligned}$$

Similarly, $(u+v)\times w=(u\times w)+(v\times w)$ and $cu\times v=c(u\times v)=u\times cv$ for all $c\in\mathbb{R}$.
Finally, to see it satisfies the Jacobi identity:

$$u\times(v\times w)+v\times(w\times u)+w\times(u\times v)$$

$$\begin{aligned}&=(u\cdot w)v-(u\cdot v)w+(v\cdot u)w-(v\cdot w)u+(w\cdot v)u-(w\cdot u)v\\ &=(u\cdot w)v+(v\cdot u)w+(w\cdot v)u-(u\cdot v)w-(v\cdot w)u-(w\cdot u)v\\ &=0.\end{aligned}$$

Therefore the cross product is a Lie bracket and $\mathbb{R}^3$ is a Lie algebra.

Next, we will begin to relate our new knowledge of Lie algebras back to our matrix groups and corresponding tangent spaces.

Definition 3.4. For all matrices $A, B \in \mathcal{M}(n, \mathbb{R})$ define the **matrix bracket** by $[A, B] = AB - BA$.

The question here is obvious. Is the matrix bracket a Lie bracket? And of course the answer is yes.

Proposition 3.5. The matrix bracket is a Lie bracket.

Proof. Closure: let $A, B \in \mathcal{M}(n, \mathbb{R})$. Then $[A, B] = AB - BA \in \mathcal{M}(n, \mathbb{R})$. Anti-Symmetric: let $A, B \in \mathcal{M}(n, \mathbb{R})$. Then

$$[A, B] = AB - BA = -(BA - AB) = -[B, A].$$

Bilinear: let $A, B, C \in \mathcal{M}(n, \mathbb{R})$. Then

$$\begin{aligned}[A, B + C] = A(B + C) - (B + C)A &= AB + AC - BA - CA \\ &= AB - BA + AC - CA \\ &= [A, B] + [A, C].\end{aligned}$$

Similarly, $[A + B, C] = [A, C] + [B, C]$. If $c \in \mathbb{R}$, then

$$\begin{aligned}[cA, B] = (cA)B - B(cA) = A(cB) - (cB)A &= [A, cB] \\ = c(AB - BA) &= c[A, B].\end{aligned}$$

Jacobi: let $A, B, C \in \mathcal{M}(n, \mathbb{R})$. Then

$$\begin{aligned}&[A, [B, C]] + [B, [C, A]] + [C, [A, B]] \\ &= A[B, C] - [B, C]A + B[C, A] - [C, A]B + C[A, B] - [A, B]C \\ &= A(BC - CB) - (BC - CB)A + B(CA - AC) - (CA - AC)B + C(AB - BA) - (AB - BA)C \\ &= ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB + CAB - CBA - ABC + BAC \\ &= 0.\end{aligned}$$

Therefore the matrix bracket is a Lie bracket. □

Notice that this means that when $G = \mathrm{GL}(n, \mathbb{R})$, $\mathcal{L}(G) = \mathcal{M}(n, \mathbb{R})$ is a Lie algebra with the Lie bracket. Notice also that in Definition 3.4 above, when $AB = BA$, $[A, B] = AB - BA = AB - AB = 0$. This motivates the next definition.

Definition 3.6. A Lie algebra L is called **abelian** if $[v, w] = 0$ for all $v, w \in L$.

Proposition 3.7. For any two n by n matrices A and B, $\mathrm{tr}([A, B]) = 0$.

Proof. The proof makes use of the well-known property of matrices that $\mathrm{tr}(XY) = \mathrm{tr}(YX)$ for all $X, Y \in \mathcal{M}(n, \mathbb{R})$:

$$\begin{aligned}\mathrm{tr}([A, B]) &= \mathrm{tr}(AB - BA) \\ &= \mathrm{tr}(AB) - \mathrm{tr}(BA) \\ &= \mathrm{tr}(AB) - \mathrm{tr}(AB) \\ &= 0.\end{aligned}$$

□

Now that we know the matrix bracket is a Lie bracket, we will show some of the tangent spaces of our matrix groups are also Lie algebras. As a matter of fact, all of these tangent spaces are Lie algebras, but we will only show it for a couple of the examples. Since we have already shown the matrix bracket is anti-symmetric, bilinear, and satisfies the Jacobi identity, all that is left in the next couple of examples is to show the closure under the matrix bracket for these particular tangent spaces.

Example 3.8. We have already seen that when $G = \mathrm{SL}(n, \mathbb{R})$, $\mathcal{L}(G) = \{B \in \mathcal{M}(n, \mathbb{R}) : \mathrm{tr}(B) = 0\}$ (1.6). To see that $\mathcal{L}(G)$ is a Lie algebra in this case, all that remains to be shown is closure under the matrix bracket.
Let $A, B \in \mathcal{L}(G)$. Then $\mathrm{tr}([A, B]) = 0$ by (3.7). Therefore $[A, B] \in \mathcal{L}(G)$, and $\mathcal{L}(G)$ is a Lie algebra.

Example 3.9. We have seen that when $G = \mathrm{O}(n, \mathbb{R})$, $\mathcal{L}(G) = \{B \in \mathcal{M}(n, \mathbb{R}) : B^T + B = 0_n\}$ (1.7). Alternatively (and useful in this case), $\mathcal{L}(G) = \{B \in \mathcal{M}(n, \mathbb{R}) : B^T = -B\}$. Now to see that $\mathcal{L}(G)$ is a Lie algebra in this case, once again the only thing left to show is closure under the matrix bracket.
Let $A, B \in \mathcal{L}(G)$. Then

$$\begin{aligned}
([A, B])^T + [A, B] &= (AB - BA)^T + (AB - BA) \\
&= (AB)^T - (BA)^T + AB - BA \\
&= B^T A^T - A^T B^T + AB - BA \\
&= (-B)(-A) - (-A)(-B) + AB - BA \\
&= BA - AB + AB - BA \\
&= 0.
\end{aligned}$$

Therefore $[A, B] \in \mathcal{L}(G)$, and $\mathcal{L}(G)$ is a Lie algebra.

Definition 3.10. Let L_1 and L_2 be Lie algebras. Then a linear transformation $T : L_1 \to L_2$ is called a **Lie algebra homomorphism** if it preserves the bracket.

$$T([v, w]) = [Tv, Tw] \text{ for all } v, w \in L_1.$$

.

Definition 3.11. A Lie algebra homomorphism which is invertible is called a **Lie algebra isomorphism**.

An example of a Lie algebra isomorphism will be found very early on in the next section (4.2).

4 Adjoints

In this section, we explore the origin of the matrix bracket (3.4). It starts with the "big A" Adjoint. From the "big A" Adjoint, we define the "small a" adjoint. Then we will show that the "small a" adjoint agrees with the matrix bracket in the case of Lie algebras of matrices.

Definition 4.1. Fix $n \in \mathbb{N}$ and a matrix $M \in \mathrm{GL}(n, \mathbb{R})$. Define $\mathrm{Ad}(M) : \mathcal{M}(n, \mathbb{R}) \to \mathcal{M}(n, \mathbb{R})$ by

$$\mathrm{Ad}(M)(X) = MXM^{-1}$$

for $X \in \mathcal{M}(n, \mathbb{R})$. The mapping $\mathrm{Ad} : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(\mathbb{R}^{n^2})$ taking the matrix M to the function $\mathrm{Ad}(M)$ is called the **Adjoint**, or **"big A" Adjoint**, map.

Proposition 4.2. $\mathrm{Ad}(M)$ is a Lie algebra isomorphism.

Proof. $\mathrm{Ad}(M)$ is linear: Let $x, y \in \mathbb{R}$ and $A, B \in \mathcal{M}(n, \mathbb{R})$. Then

$$\begin{aligned}
\mathrm{Ad}(M)(xA + yB) &= M(xA + yB)M^{-1} \\
&= M(xA)M^{-1} + M(yB)M^{-1} \\
&= xMAM^{-1} + yMBM^{-1} \\
&= x\,\mathrm{Ad}(M)(A) + y\,\mathrm{Ad}(M)(B).
\end{aligned}$$

$\mathrm{Ad}(M)$ preserves the bracket: Let $A, B \in \mathcal{M}(n, \mathbb{R})$. Then

$$\begin{aligned}
\mathrm{Ad}(M)([A, B]) &= M([A, B])M^{-1} \\
&= M(AB - BA)M^{-1} \\
&= MABM^{-1} - MBAM^{-1} \\
&= MAM^{-1}MBM^{-1} - MBM^{-1}MAM^{-1} \\
&= \mathrm{Ad}(M)(A)\,\mathrm{Ad}(M)(B) - \mathrm{Ad}(M)(B)\,\mathrm{Ad}(M)(A) \\
&= [\mathrm{Ad}(M)(A), \mathrm{Ad}(M)(B)].
\end{aligned}$$

$\mathrm{Ad}(M)$ is invertible: Let $A \in \mathcal{M}(n, \mathbb{R})$. Then

$$\begin{aligned}
\mathrm{Ad}(M^{-1})(\mathrm{Ad}(M)(A)) &= M^{-1}(\mathrm{Ad}(M)(A))M \\
&= M^{-1}MAM^{-1}M \\
&= A.
\end{aligned}$$

Likewise,

$$\begin{aligned}
\mathrm{Ad}(M)(\mathrm{Ad}(M^{-1})(A)) &= M(\mathrm{Ad}(M^{-1})(A))M^{-1} \\
&= MM^{-1}AMM^{-1} \\
&= A.
\end{aligned}$$

Therefore $\mathrm{Ad}(M)$ is linear, bracket preserving, and invertible. Hence $\mathrm{Ad}(M)$ is a Lie algebra isomorphism. □

Proposition 4.3. $\mathrm{Ad} : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(\mathbb{R}^{n^2})$ is a group homomorphism.

Proof. Fix $M_1, M_2 \in \mathrm{GL}(n, \mathbb{R})$ and let $X \in \mathcal{M}(n, \mathbb{R})$. Then

$$\begin{aligned}
\mathrm{Ad}(M_1M_2)(X) &= M_1M_2X(M_1M_2)^{-1} \\
&= M_1M_2XM_2^{-1}M_1^{-1} \\
&= M_1\,\mathrm{Ad}(M_2)(X)M_1^{-1} \\
&= \mathrm{Ad}(M_1)(\mathrm{Ad}(M_2)(X)) \\
&= (\mathrm{Ad}(M_1) \circ \mathrm{Ad}(M_2))(X).
\end{aligned}$$

□

We will assume further that Ad is even a Lie group homomorphism. This leads us to being able to define the "small a" adjoint.

Definition 4.4. The **"small a" adjoint**, is defined as the differential of Ad, so

$$\mathrm{ad} : \mathcal{L}(G_1) \to \mathcal{L}(G_2)$$

where in this case

$$G_1 = \mathrm{GL}(n, \mathbb{R}), G_2 = \mathrm{GL}(\mathbb{R}^{n^2}), \mathcal{L}(G_1) = \mathcal{M}(n, \mathbb{R}), \text{ and } \mathcal{L}(G_2) = \mathcal{M}(n^2, \mathbb{R})$$

where elements of $\mathcal{M}(n^2, \mathbb{R})$ are interpreted as linear transformations from $\mathbb{R}^{n^2}$ to $\mathbb{R}^{n^2}$, or even more to our purposes as linear transformations from $\mathcal{M}(n, \mathbb{R})$ to $\mathcal{M}(n, \mathbb{R})$.

Now we will prove that in the case of matrix groups, the "small a" adjoint map is actually a very familiar map.

Proposition 4.5. For a fixed $A \in \mathcal{M}(n, \mathbb{R})$, $\mathrm{ad}(A)(B) = [A, B]$ for all $B \in \mathcal{M}(n, \mathbb{R})$.

Proof. Using the notation from the definition above (4.4), since $A \in \mathcal{L}(G^1) = \mathcal{M}(n, \mathbb{R})$, we know $\gamma_1(t) = e^{tA}$ is smooth and in $G^1 = \mathrm{GL}(n, \mathbb{R})$ and $\gamma_1'(0) = A$. Let $\gamma_2(t) = \mathrm{Ad}(\gamma_1(t))$. Then by the definition of differential (1.11) $\mathrm{ad}(A) = \gamma_2'(0)$.
Now for any $B \in \mathcal{M}(n, \mathbb{R})$,

$$\begin{aligned}\gamma_2(t)(B) &= \mathrm{Ad}(\gamma_1(t))(B) \\ &= \gamma_1(t)B\gamma_1(t)^{-1}.\end{aligned}$$

Now differentiate both sides of the equation:

$$\begin{aligned}\gamma_2'(t)(B) &= \gamma_1'(t)B\gamma_1(t)^{-1} + \gamma_1(t)B(\gamma_1(t)^{-1})' \\ &= \gamma_1'(t)B\gamma_1(t)^{-1} - \gamma_1(t)B\gamma_1(t)^{-2}\gamma_1'(t).\end{aligned}$$

Finally, we let $t = 0$, so that

$$\begin{aligned}\mathrm{ad}(A)(B) &= \gamma_2'(0)(B) \\ &= \gamma_1'(0)B\gamma_1(0)^{-1} - \gamma_1(0)B\gamma_1(0)^{-2}\gamma_1'(0) \\ &= A \cdot B \cdot I_n - I_n \cdot B \cdot I_n \cdot A \\ &= AB - BA \\ &= [A, B].\end{aligned}$$

□

5 A Lie Group (Finally)

Unfortunately, there are several more definitions we need to get through before we get to the definition of a Lie group. Each definition builds from the last, so with each definition the complexity of such a structure becomes more and more involved. And of course, the Lie group is the most complicated of them all. Although on paper this section may seem short, this is the longest of them all. The formal definition of a Lie group relies on background knowledge of several areas in mathematics. This is why relatively few mathematicians even know exactly what a Lie group is. But as promised, we will press on and find an answer.

Definition 5.1. A **topology** $\mathcal{T}$ on a set X is a collection of subsets of X such that $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$, and $\mathcal{T}$ is closed under finite intersections and arbitrary unions. A set X together with a topology $\mathcal{T}$ on X, is called a **topological space**.

Additionally, sets in $\mathcal{T}$ are referred to as the open sets. I will also refer to an open set which contains a point x as an **open neighborhood of** x.

Definition 5.2. A **Hausdorff space** is a topological space which satisfies one additional axiom: for all $x, y \in X$ with $x \neq y$, there exists $U_x, U_y \in \mathcal{T}$ such that $x \in U_x$, $y \in U_y$, and $U_x \cap U_y = \emptyset$.

Definition 5.3. A **homeomorphism** (not homomorphism) is a one-to-one, onto, and continuous function which also has a continuous inverse.

Definition 5.4. A **manifold** M of dimension n is a Hausdorff space which satisfies the following two properties:

1. For all $x \in M$ there exist an open neighborhood of x, call it U, and map $\phi : U \to \mathbb{R}^n$ such that ϕ is a homeomorphism.

2. M has a countable base of open sets.

Definition 5.5. The pairs U, ϕ and V, ψ where U and V are open neighborhoods, and ϕ and ψ are homeomorphisms from U and V, respectively, to $\mathbb{R}^n$, are called **C^∞-compatible** if when $U \cap V \neq \emptyset$, the functions $\phi \circ \psi^{-1} : \psi(U \cap V) \to \mathbb{R}^n$ and $\psi \circ \phi^{-1} : \phi(U \cap V) \to \mathbb{R}^n$ are C^∞, or infinitely differentiable.

Note that in the case where $U \cap V = \emptyset$, we don't need to worry about the corresponding maps. Also take note that the domain of these functions is always a subspace of $\mathbb{R}^n$, and therefore differentiation is meant in the usual sense.

Definition 5.6. A **differentiable** or C^∞ **manifold** M is a manifold of dimension n along with a family $\mathcal{U} = \{U_\alpha, \phi_\alpha\}$ of neighborhoods paired with an associated homeomorphism such that:

1. $\cup U_\alpha = M$,

2. for any α and β the pairs U_α, ϕ_α and U_β, ϕ_β are C^∞-compatible, and

3. any pair V, ψ which is C^∞-compatible with every $U_\alpha, \phi_\alpha \in \mathcal{U}$ must also be in $\mathcal{U}$.

The third condition for a differentiable manifold makes proving something is one quite challenging. Thankfully, in differential geometry there is a very powerful theorem to help.

Theorem 5.7. Let M be a Hausdorff space which has a countable basis of open sets. If there is a covering of open sets such that for each open set there is a homeomorphism mapping it into $\mathbb{R}^n$, then there is a unique C^∞ structure on M which contains these C^∞-compatible pairs.

Example 5.8. The unit circle in the complex plane, $S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}$, is a differentiable manifold.

The Hausdorff space on S^1 is relatively simple. Take the usual topology on the complex plane and intersect all the open sets with S^1. This gives the topology on S^1. Notice another way of doing this is by taking usual open sets from $\mathbb{R}$ and taking θ over that range. This creates the same topology. We will think of it in these terms as it will become more convenient soon. To make use of Theorem 5.7, we will simply come up with an open cover along with corresponding homeomorphisms for each. For our open sets, let

$$U = \{e^{i\theta} : \theta \in (0, 2\pi)\}$$

and

$$V = \{e^{i\theta} : \theta \in (-\pi, \pi)\}.$$

Notice U is the entire circle except for the point 1, and V is the entire circle except for the point -1. So, U and V cover S^1. Now we need homeomorphisms, $\phi : U \to \mathbb{R}$ and $\psi : V \to \mathbb{R}$, such that U, ϕ and V, ψ are C^∞-compatible. Let

$$\phi(\theta) = \tan(\frac{1}{2}(\theta - \pi)) \text{ and } \psi(\theta) = \tan(\frac{1}{2}\theta).$$

Notice that on the given domains, U and V, ϕ and ψ are one-to-one, onto, and continuous. Their inverses,

$$\phi^{-1}(\theta) = 2\tan^{-1}(\theta) + \pi \text{ and } \psi^{-1}(\theta) = 2\tan^{-1}(\theta),$$

are also continuous. Thus ϕ and ψ are homeomorphisms.

Now all that remains to be shown is that the mappings

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \mathbb{R} \text{ and } \psi \circ \phi^{-1} : \phi(U \cap V) \to \mathbb{R},$$

are C^∞, or infinitely differentiable on their domains. In both cases, the domain is $(-\infty, 0) \cup (0, \infty)$. Now,

$$(\phi \circ \psi^{-1})(\theta) = \tan(\tan^{-1}(\theta) - \frac{\pi}{2}) = -\theta$$

and

$$(\psi \circ \phi^{-1})(\theta) = \tan(\tan^{-1}(\theta) + \frac{\pi}{2}) = -\theta,$$

both of which are well-defined **and differentiable** everywhere but at 0. However, this is not a problem since 0 is also the only real number not in the domain of each function. Therefore S^1 is in fact a differentiable manifold.

Definition 5.9. A **Lie group** is a group G which is also a differentiable manifold such that the group operation and inversion are differentiable. Symbolically,

$$(x, y) \mapsto xy \text{ and } x \mapsto x^{-1}$$

are differentiable. It suffices to assume that $(x, y) \mapsto xy^{-1}$ is differentiable. In fact, it even suffices to assume that only the group operation ($(x, y) \mapsto xy$) is differentiable.

Example 5.10. Let's show that $\mathbb{R}^n$ is a Lie group. First of all, it is clear that it is a group under addition. It should also be clear at this point that $\mathbb{R}^n$ is a differentiable manifold since for any open set in $\mathbb{R}^n$ the identity map will suffice as a homeomorphism from the set into R^n. We need only show that addition in $\mathbb{R}^n$ is differentiable. Let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be the addition map. So for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $f((x, y)) = x + y$. So if $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, then $f((x, y)) = x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$. Each component of the resulting vector is differentiable with respect to any of the x_i's or y_i's, so the entire function is itself differentiable. Thus $\mathbb{R}^n$ is a Lie group.

Now, in a final attempt at connecting our formal definition of a Lie group back to our work with subgroups of $\mathrm{GL}(n, \mathbb{R})$, let's take a look at two last examples.

Example 5.11. Look again at the unit circle, $S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}$. S^1 is a Lie group.
First of all, let's see why S^1 is a group:

- Closure: for all $e^{i\alpha}, e^{i\beta} \in S^1$, we have $e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)} \in S^1$.
- Associativity: for all $e^{i\alpha}, e^{i\beta}, e^{i\gamma} \in S^1$, we have $e^{i\alpha}(e^{i\beta}e^{i\gamma}) = e^{i\alpha}e^{i(\beta+\gamma)} = e^{i(\alpha+(\beta+\gamma))} = e^{i((\alpha+\beta)+\gamma)} = e^{i(\alpha+\beta)}e^{i\gamma} = (e^{i\alpha}e^{i\beta})e^{i\gamma}$.
- Identity: $e^{i\cdot 0} = 1$, so 1 is the identity.
- Inverse: for all $e^{i\alpha} \in S^1$, there is $e^{i(-\alpha)} \in S^1$ and $e^{i\alpha}e^{i(-\alpha)} = e^{i(\alpha-\alpha)} = e^0 = 1$.

We've already seen S^1 is a differentiable manifold (5.8). All that is left is to see why S^1 is a Lie group. We only have left to show that the group operation (multiplication in this case) is differentiable. So, let $x = e^{i\alpha}$ and $y = e^{i\beta}$, then $e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)} \in S^1$. Using a map similar to those in (5.8), we can see that the map $\phi(\theta) = \tan(\frac{1}{2}(\theta - \alpha - \beta))$ is infinitely differentiable in $\mathbb{R}$.

Example 5.12. Now let's finally tie some pieces together by showing that $\mathrm{SO}(2, \mathbb{R})$, the 2 by 2 special orthogonal group (0.8), is a Lie group. We will be taking a slightly different approach this time. Since we already know S^1 is a Lie group from the previous example (5.11), we will simply construct a Lie group isomorphism (1.9) between the two groups, which means the two have the same Lie group structure.
Define $F : \mathrm{SO}(2, \mathbb{R}) \to S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}$ by

$$F\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\right) = e^{i\theta}.$$

Then F is a homomorphism:

$$\begin{aligned}
F\left(\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}\right) &= F\left(\begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}\right) \\
&= e^{i(\alpha+\beta)} \\
&= e^{i\alpha}e^{i\beta} \\
&= F\left(\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\right) F\left(\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}\right).
\end{aligned}$$

Since the exponential function is smooth, it is clear that F is smooth. Thus F is a Lie group homomorphism by (1.8).
Now F is one-to-one:

$$\text{if } F\left(\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\right) = F\left(\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}\right),$$
$$\text{then } e^{i\alpha} = e^{i\beta}$$
$$\text{and } \alpha = \beta + 2k\pi,$$
$$\text{which implies } \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

since the sine and cosine functions have a period of 2π.
Also F is onto: let $e^{i\alpha} \in S^1$. Then

$$F\left(\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\right) = e^{i\alpha}.$$

So F is a Lie group homomorphism, one-to-one, and onto. Hence by (1.9), F is a Lie group ismomorphism. Therefore $\mathrm{SO}(2,\mathbb{R})$ and S^1 are both Lie groups.

6 References

[1] Boothby, William M. (1975). *An Introduction to Differentiable Manifolds and Riemannian Geometry.* San Diego CA: Academic Press.

[2] Gilmore, Robert (1974). *Lie Groups, Lie Algebras, and Some of Their Applications.* Mineola, NY: Dover Publications.

[3] Pollatsek, Harriet (2009). *Lie Groups: A Problem-Oriented Introduction via Matrix Groups.* Washington, DC: The Mathematical Association of America.

[4] Pontrjagin, L. (1946). *Topological Groups.* Princeton: Princeton University Press.

[5] Spivak, Mike (1970). *A Comprehensive Introduction to Differential Geometry: Volume I.* Houston, TX: Publish or Perish.