Carter Subgroups and Carter's Theorem

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ABSTRACT

In 1961 Roger W. Carter proved a theorem about solvable groups similar to Sylow's theorem. He proved that if a group is solvable then it always contains a nilpotent, self-normalizing subgroup called a Carter subgroup, and that all such subgroups are conjugate to each other by an element of the group. The objective of this thesis is to present a proof of Carter's theorem.

Dedication

To my husband, Ishahu Abubakar.

Aknowledgements

My heartfelt gratitude goes to God Almighty for the opportunity, protection, wisdom, and guidance for seeing me through two years of my graduate studies. My most sincere gratitude goes to my thesis advisor Dr. Neil Flowers for his immeasurable suggestions, corrections, patience, enormous encouragement and excellent supervisory role during the entire process of this work. Again, my endless thanks go to my husband, Ishahu Abubakar, for his love, support and encouragement. To my siblings, Abdul Kadir and Sadiq Mohammed, I would like to say God bless you all for helping in one way or the other with my masters' education. In addition, I would like to thank my committee members Dr. Thomas Wakefield and Dr. Eric Wingler for their time and support. I would like to show my profound gratitude to my mum, Hajia Masaratu, my dad, Alhaji Mohammed A. Wahab, my God father, Alhaji Abubakar Musah, my only sister-in-law and her husband, Saeeda Abubakar and Abdalla Mohammed. Finally, my appreciation goes to Mukaddas Alhassan, Yakubu Tahiru, Lordlin D. Owusu, faculty and staff of the Department of Mathematics and Statistics, members of the African Students Union for their support and advice.

CONTENTS

1 Introduction

Let G be a finite group, p be a prime, and $n \in \mathbb{Z}^+ \cup \{0\}$ such that p^n divides $|G|$ but p^{n+1} does not divide $|G|$. In 1872 Ludwig Sylow proved that there is a subgroup P of G such that $|P| = p^n$ and that all such subgroups are conjugate to each other by an element of G. Such a subgroup P is called a Sylow p-subgroup, named after Ludwig Sylow. If G has only one Sylow p-subgroup for each prime p, then G is called a nilpotent group. Now if $H \leq G$ then it is well known that the set

$$N_G(H) = \{g \in G | gHg^{-1} = H\}$$

is a subgroup of G.

Roger W. Carter obtained his PhD in 1960 and his dissertation was entitled "Some Contributions to the Theory of Finite Soluble Groups". He worked as a professor at the University of Warwick in the United Kingdom. He defined Carter subgroups and wrote the standard reference *Simple Groups of Lie Type*. Roger W. Carter in mid 1900s wondered if all groups contained a subgroup H that was nilpotent with the property that H is self-normalizing(ie $= H = N_G(H)$). Well it turns out that not all groups have a nilpotent, self-normalizing subgroup. For example, the alternating group A_5 of order 60 has no such subgroup. A group G is **solvable** if there exists a subnormal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq ... \trianglerighteq G_n = 1$$

such that the factors

$$\frac{G_i}{G_{i+1}}$$

are abelian, for all $0 \leq i \leq n-1$.

In 1961 Roger W. Carter showed a theorem about these subgroups similar to Sylow's theorem. He proved that if a group is solvable then it always contains a nilpotent, self-normalizing subgroup, and that all such subgroups are conjugate to each other by an element of the group [1]. These subgroups have been named Carter subgroups and the theorem, Carter's theorem. The objective of this thesis is to present a proof of Carter's theorem.

2 Preliminaries

Definition A **group** is a non empty set G along with a binary operation $*$ such that the following axioms are satisfied:

1. **Closed** $a * b \in G$ for all a, $b \in G$.

2. **Associativity** $(a * b) * c = a * (b * c)$ for all a, b, $c \in G$.

3. **Identity** There exists $e \in G$ such that for all $a \in G$, $e * a = a * e = a$.

4. **Inverses** For all $a \in G$ there exists $b \in G$ such that $a * b = b * a = e$.

We will write ab instead of $a * b$, 1 instead of e, and a^{-1} instead of b.

Definition A group G is called **abelian** if $ab = ba$ for all a, $b \in G$.

Definition Let G be a group and H be a non empty subset of G. Then H is a subgroup of G if H is a group. We write $H \leq G$.

Theorem 2.1. *(Subgroup test): Let G be a group and H be a non-empty subset of G. Then $H \leq G$ if and only if $ab^{-1} \in H$ for all a, $b \in H$.*

Proof

Suppose $H \leq G$. Let a, $b \in H$. Since $H \leq G$ and $b \in H$, we know $b^{-1} \in H$, and so $ab^{-1} \in H$ by closure. Suppose $ab^{-1} \in H$ for all a, $b \in H$. Let $a \in H$. Then $aa^{-1} \in H$, so $1 \in H$. Now $1a^{-1} \in H$ and so $a^{-1} \in H$ for all $a \in H$. Let a,$b \in H$. Then $b^{-1} \in H$ from above, and so $a(b^{-1})^{-1} \in H$. Thus $ab \in H$ and so H is closed. Since G is associative and $H \subseteq G$, we know H is associative. Therefore H is a group and so $H \leq G$. □

Definition Let G be a group, the **center** of G is

$$Z(G) = \{g \in G | gx = xg \text{ for all } x \in G\}$$

Theorem 2.2. *Let G be a group. Then $Z(G) \leq G$.*

Proof

Now $1x = x$ and $x1 = x$ and so $1x = x1$ for all $x \in G$. Therefore $1 \in Z(G)$ and so $Z(G) \neq \emptyset$. Let a, $b \in Z(G)$ and let $x \in G$ then

$$\begin{aligned} xab^{-1} &= axb^{-1} \text{ since } a, b \in Z(G) \\ &= ab^{-1}bxb^{-1} \\ &= ab^{-1}xbb^{-1} \\ &= ab^{-1}x. \end{aligned}$$

Thus $ab^{-1} \in Z(G)$ and so $Z(G) \leq G$ by the Subgroup test. □

Definition Let G be a group and $a \in G$. Define the **cyclic subgroup generated by** a by

$$\langle a \rangle = \{a^k | k \in Z\}.$$

Theorem 2.3. *Let G be a group and $a \in G$ then $\langle a \rangle \leq G$.*

Proof

Since $1 = a^0 \in \langle a \rangle$ then $\langle a \rangle \neq \emptyset$. Let a^m, $a^n \in \langle a \rangle$. Then $a^m(a^n)^{-1} = a^m a^{-n} =$

$a^{m-n} \in \langle a \rangle$ since $m - n \in \mathbb{Z}$. Therefore $\langle a \rangle \leq G$ by the Subgroup test. $\square$

Definition Let G be a group, $H \leq G$ and $g \in G$. Then the **left coset of H in G containing g** is the set

$$gH = \{gh | h \in H\}.$$

A number of theorems will be listed for (informational purposes) whose proofs are not given here.

Theorem 2.4. *Let G be a group, $H \leq G$, and $a, b \in G$. Then*

1. $|aH| = |H|$.
2. $aH = bH$ *if and only if* $b^{-1}a \in H$.

Theorem 2.5. *(Lagrange): Let G be a group and $H \leq G$. Then $|H|$ divides $|G|$ and*

$$\frac{|G|}{|H|} = \text{ number of left cosets of } H \text{ in } G$$

.

Definition Let G_1 and G_2 be groups and $\phi : G_1 \longrightarrow G_2$. Then ϕ is a **homomorphism** if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G_1$. If, in addition, ϕ is one-to-one and onto, we call ϕ an **isomorphism** and write $G_1 \cong G_2$.

Theorem 2.6. *Let $\phi : G_1 \longrightarrow G_2$ be a homomorphism and $a \in G_1$. Then*

1. $\phi(1) = 1$.

2. $\phi(a^{-1}) = (\phi(a))^{-1}$.

3. $\phi(a^n) = \phi(a)^n$ *for any* $n \in \mathbb{Z}$.

4. If $|a|$ *is finite, then* $|\phi(a)|$ *divides* $|a|$.

5. If $H \leq G_1$*, then* $\phi(H) \leq G_2$.

6. If $K \leq G_2$*, then* $\phi^{-1}(K) \leq G_1$.

Definition Let G_1 and G_2 be groups and $\phi : G_1 \longrightarrow G_2$ be a homomorphism. Define the **kernel** of ϕ by

$$\text{Kern } \phi = \{g \in G_1 | \phi(g) = 1\}.$$

Theorem 2.7. *Let* $\phi : G_1 \longrightarrow G_2$ *be a homomorphism. Then Kern* $\phi \trianglelefteq G_1$.

Definition Let G be a group and $H \leq G$. Then H is a **normal subgroup** of G if $ghg^{-1} \in H$ for all $g \in G$ and for all $h \in H$. We write $H \trianglelefteq G$.

Theorem 2.8. *Let* G *be a group and* $H \trianglelefteq G$*. Define the set* G/H *by*

$$G/H = \{gH | g \in G\}.$$

Then G/H *is a group under the operation* $aHbH = abH$ *for all* aH, $bH \in G/H$. *The group* G/H *is called the quotient group, the factor group, or* G *mod* H.

Theorem 2.9. *(First Isomorphism Theorem): Let G_1 and G_2 be groups and $\phi : G_1 \longrightarrow G_2$ be a homomorphism with Kern $\phi = K$. Then*

$$G_1/K \cong \phi(G_1).$$

Proof

Define a map $\chi : G_1/K \longrightarrow \phi(G_1)$ by $\chi(gK) = \phi(g)$ for all $g \in G$. Let $g_1, g_2 \in G_1$. Suppose $g_1K = g_2K$ then $g_2^{-1}g_1 \in K =$ Kern ϕ and so $\phi(g_2^{-1}g_1) = 1$ or $\phi(g_2^{-1})\phi(g_1) = 1$ since ϕ is a homomorphism. Therefore $\phi(g_2)^{-1}\phi(g_1) = 1$ and so $\phi(g_1) = \phi(g_2)$. Therefore $\chi(g_1K) = \chi(g_2K)$. This implies χ is well defined. Now let $g_1K, g_2K \in G_1/K$. Since ϕ is a homomorphism

$$\chi((g_1K)(g_2K)) = \chi((g_1g_2)K) = \phi(g_1g_2) = \phi(g_1)\phi(g_2) = \chi(g_1K)\chi(g_2K)$$

Implies χ is a homomorphism. Let $g_1K, g_2K \in G_1/K$, suppose $\chi(g_1K) = \chi(g_2K)$. Then $\phi(g_1) = \phi(g_2)$ or $(\phi(g_2))^{-1}\phi(g_1) = 1$ or $\phi(g_2^{-1})\phi(g_1) = 1$ since ϕ is a homomorphism. Hence $\phi(g_2^{-1})\phi(g_1) = \phi(g_2^{-1}g_1) = 1$ since ϕ is a homomorphism. Therefore $g_2^{-1}g_1 \in$ Kern $\phi = K$; hence $g_1K = g_2K$. So χ is one-to-one. Let $y \in \phi(G_1)$. Then there exists $x \in G_1$ such that $y = \phi(x)$. But then $xK \in G/K$ and $\chi(xK) = \phi(x) = y$. Hence χ is onto. Therefore $G_1/K \cong \phi(G_1)$. $\square$

Theorem 2.10. *(Second Isomorphism Theorem): Let G be a group, $H \leq G$, and $N \trianglelefteq G$. Then*

$$\frac{HN}{N} \cong \frac{H}{H \cap N}$$

Proof

Define a map $\phi : H \longrightarrow HN/N$ by $\phi(h) = hN$ for $h \in H$. Let $h_1,\ h_2 \in H$. Then $\phi(h_1h_2) = (h_1h_2)N = h_1Nh_2N = \phi(h_1)\phi(h_2)$. Hence ϕ is a homomorphism. Let $h_1 \in H$. Then

$$\begin{aligned} h_1 &\in \text{Kern } \phi \\ \text{if and only if } \phi(h_1) = h_1N &= 1N \\ \text{if and only if } 1^{-1}h_1 &\in N \\ \text{if and only if } h_1 &\in H \cap N. \end{aligned}$$

Hence $H \cap N =$ Kern ϕ. Let $hnN \in HN/N$ where $h \in H$ and $n \in N$. Then $\phi(h) = hN = hnN$ since $(hn)^{-1}h = n^{-1} \in N$ and so χ is onto. Now by the First Isomorphism Theorem

$$\frac{H}{\text{Kern } \phi} \cong \phi(H)$$

which implies

$$\frac{HN}{N} \cong \frac{H}{H \cap N}.$$

□

Theorem 2.11. *(Third Isomorphism Theorem): Let G be a group, $N \leq H \leq G$, $N \trianglelefteq G$, and $H \trianglelefteq G$. Then*

$$\frac{G/N}{H/N} \cong G/H.$$

Proof

Define $\phi : G/N \longrightarrow G/H$ by $\phi(gN) = gH$ for all $gN \in G/N$. Let $g_1N,\ g_2N \in G/N$ for $g_1, g_2 \in G$. Suppose $g_1N = g_2N$. Then $g_2^{-1}g_1 \in N$. Also $g_2^{-1}g_1 \in H$ since $N \leq H$

and so $g_1H = g_2H$. Therefore $\phi(g_1N) = \phi(g_2N)$ and ϕ is well-defined. Now let g_1N, $g_2N \in G/N$ for some $g_1, g_2 \in G$. Then

$$\phi(g_1Ng_2N) = \phi(g_1g_2N) = g_1g_2H = g_1Hg_2H = \phi(g_1N)\phi(g_2N),$$

and so ϕ is a homomorphism. Let $gH \in G/H$. Then $gN \in G/N$ and so $\phi(gN) = gH$. Therefore ϕ is onto. Let $g_1N \in G/N$. Then

$$\begin{aligned} g_1N &\in Kern\phi \\ \text{if and only if } \phi(g_1N) &= 1H \\ \text{if and only if } g_1H &= 1H \\ \text{if and only if } 1^{-1}g_1 &\in H \\ \text{if and only if } g_1 &\in H \\ \text{if and only if } g_1N &\in H/N. \end{aligned}$$

Thus Kern $\phi = H/N$. Now by the First Isomorphism Theorem

$$\frac{G/N}{\text{Kern } \phi} \cong \phi(G/N);$$

hence

$$\frac{G/N}{H/N} \cong G/H.$$

□

Definition Let G be a group and $S \subseteq G$ be a nonempty subset of G. Then the

subgroup generated by S is

$$\langle S \rangle = \bigcap_{S \subseteq H \leq G} H.$$

Theorem 2.12. *Let G be a group and $S \subseteq G$ be a nonempty subset. Then*

$$\langle S \rangle = \{s_1^{n_1} s_2^{n_2} \cdots s_k^{n_k} | s_i \in S \textit{ and } n_i \in \mathbb{Z} \textit{ for all } 1 \leq i \leq k\}.$$

Proof

Let $T = \{s_1^{n_1} s_2^{n_2} \cdots s_k^{n_k} | s_i \in S$ and $n_i \in Z$ for all $1 \leq i \leq k\}$. We claim that $T \leq G$ Since S is nonempty there exists $s_1 \in S$. Then $1 = s_1^0 \in T$ and so $T \neq \emptyset$. Now let $s_1^{n_1} s_2^{n_2} \cdots s_k^{n_k}$, $r_1^{m_1} r_2^{m_2} \cdots r_l^{m_l} \in T$ where s_i, $r_j \in S$ and n_i, $m_j \in \mathbb{Z}$ for $1 \leq i \leq k$ and $1 \leq j \leq l$. Then

$$\begin{aligned}(s_1^{n_1} s_2^{n_2} \cdots s_k^{n_k})(r_1^{m_1} r_2^{m_2} \cdots r_l^{m_l})^{-1} &= (s_1^{n_1} s_2^{n_2} \cdots s_k^{n_k})(r_l^{-m_l} r_{l-1}^{-m_{l-1}} \cdots r_2^{-m_2} r_1^{-m_1}) \\ &= s_1^{n_1} s_2^{n_2} \cdots s_k^{n_k} r_l^{-m_l} r_{l-1}^{-m_{l-1}} \cdots r_1^{-m_1} \in T.\end{aligned}$$

Thus $T \leq G$ by the subgroup test. Let $s \in S$. Then $s = s^1 \in T$ and so $S \subseteq T \leq G$. Therefore $\langle S \rangle = \bigcap_{S \subseteq H \leq G} H \leq T$. Let $s_1^{n_1} s_2^{n_2} \cdots s_k^{n_k} \in T$ where $k \in \mathbb{Z}^+, s_i \in S$, and $n_i \in \mathbb{Z}$ for all $1 \leq i \leq k$. Suppose that $S \subseteq H \leq G$. Since $s_i \in S \subseteq H$ for all i we know $s_i^{n_i} \in H$ for all i since $H \leq G$. Therefore $s_1^{n_1} s_2^{n_2} \cdots s_k^{n_k} \in H$ since $H \leq G$. Thus $T \leq H$ and so $T \leq \langle S \rangle$ and we have $\langle S \rangle = T$. □

Theorem 2.13. *Let G be a group, $N \trianglelefteq G$, $H \leq G$ and let $\phi : G \longrightarrow G/N$ be defined by $\phi(g) = gN$ for all $g \in G$. Then*

1. *ϕ is a homomorphism;*

2. *Kern $\phi = N$;*

3. *$\phi(H) = HN/N$;*

4. *$\phi^{-1}(HN/N) = HN$;*

5. *if $L \leq G/N$ then $L = K/N$ where $N \leq K \leq G$.*

Proof

For (1), let $g_1, g_2 \in G$. Then $\phi(g_1 g_2) = g_1 g_2 N = g_1 N g_2 N$, so ϕ is a homomorphism. For (2), let $g \in G$. Then

$$\begin{aligned} g &\in \text{Kern } \phi \\ \text{if and only if } \phi(g) &= 1N \\ \text{if and only if } gN &= 1N \\ \text{if and only if } 1^{-1}g &\in N \\ \text{if and only if } g &\in N. \end{aligned}$$

So Kern $\phi = N$. For (3), let $hnN \in HN/N$ for $h \in H$, $n \in N$. Then $hnN = hN$ since $(hn)^{-1}h = n^{-1} \in N$. Therefore $hnN = \phi(h) \in \phi(H)$ and so $HN/N \subseteq \phi(H)$. Let $x \in \phi(H)$. There exists $h \in H$ such that $x = \phi(h)$. Then $x = \phi(h) = hN \in HN/N$. Thus

$$\phi(H) = \frac{HN}{N}.$$

For (4), let $g \in \phi^{-1}(HN/N)$. Then there exists $hnN \in HN/N$ such that $\phi(g) = hnN = hN$. Hence $gN = hN$ and so $h^{-1}g \in N$. But then there exists $n_1 \in N$ such that $h^{-1}g = n_1$ and so $g = hn_1 \in HN$. Hence

$$\phi^{-1}\left(\frac{HN}{N}\right) \subseteq HN.$$

Now let $hn \in HN$. Then $\phi(hn) = hnN \in HN/N$ and so $hn \in \phi^{-1}(HN/N)$. Thus $HN \subseteq \phi^{-1}(HN/N)$, so $\phi^{-1}(HN/N) = HN$. Finally, consider $\phi^{-1}(L) = K$. Since $L \leq G/N$ we know $\phi^{-1}(L) \leq G$. Let $n \in N$, then $\phi(n) = nN = 1N \in L$ since $L \leq G/N$. Hence $n \in \phi^{-1}(L)$ and so $N \leq \phi^{-1}(L)$. We claim that

$$L = \frac{\phi^{-1}(L)}{N}.$$

Let $gN \in L$. Then $\phi(g) = gN \in L$. Hence $g \in \phi^{-1}(L)$ and so $gN \in \phi^{-1}(L)/N$. Therefore $L \leq \phi^{-1}(L)/N$. Let $gN \in \phi^{-1}(L)/N$. Then $g \in \phi^{-1}(L)$ and so $gN = \phi(g) \in L$. Thus $\phi^{-1}(L)/N \leq L$ and so $L = \phi^{-1}(L)/N$. □

Definition Let G be a finite group, p be a prime, and $n \in \mathbb{Z}^+ \cup \{0\}$ such that p^n divides $|G|$ but p^{n+1} does not divide $|G|$. Then

1. A subgroup $P \leq G$ is called a **Sylow p-subgroup** if $|P| = p^n$.

2. $\mathrm{Syl}_p(G) = \{P \leq G | P \text{ is a Sylow } p\text{-subgroup of } G\}$.

Theorem 2.14. *(Sylow's) Let G be a finite group, with $|G| = p^n m$, where p is prime, $n \geq 1$ and p does not divide m. Then*

1. For each i, $1 \leq i \leq n$. There is a subgroup of G of order p^i. Every subgroup *of order p^i is a normal subgroup of some subgroup of order p^{i+1} for all $1 \leq i \leq n-1$;*

2. *Any two Sylow p-subgroups of G are conjugate in G;*

3. *The number n_p of Sylow p-subgroups of G divides $|G|$ and is congruent to 1 mod p.*

Theorem 2.15. *Let G be a group, $H \leq G$, $K \leq G$ and $L \leq G$ such that $K \leq H$. Then,*

$$H \cap KL = K(H \cap L)$$

Proof

Let $x \in K(H \cap L)$. Then there exist $k \in K \leq H$ and also $n \in H \cap L$ such that $x = kn$. Since $n \in H \cap L, n \in H$ and $n \in L$. Therefore $x = kn \in H$ by closure. Also $x = kn \in KL$. Hence $x \in H \cap KL$ and so $K(H \cap L) \subseteq H \cap KL$. Now let $y \in H \cap KL$. Then $y \in H$ and $y \in KL$. Therefore there exist $k \in K$ and $l \in L$ such that $y = kl$. Since $y \in H$ we have $kl \in H$. But since $k \in K \leq H$ and $H \leq G$ we know $k^{-1} \in H$. Thus $l = k^{-1}kl \in H$, and so $l \in H \cap L$. Thus $y = kl \in K(H \cap L)$. Therefore $H \cap KL \subseteq K(H \cap L)$ and so $H \cap KL = K(H \cap L)$. □

3 Solvable Groups

Definition A **subnormal series** of a group G is a sequence of subgroups, each a normal subgroup of the next one. In a standard notation

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = 1.$$

Definition A group G is **solvable** if there exists a subnormal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = 1$$

such that the factors

$$\frac{G_i}{G_{i+1}}$$

are abelian for all $0 \leq i \leq n-1$.

Lemma 3.1. *If G is an abelian group then G is solvable.*

Proof

Consider the subnormal series $G = G_0 \trianglerighteq G_1 = 1$. Then $G_0/G_1 = G/1 \cong G$ is abelian.

□

Examples.

$\mathbb{Z}_n$ and $\mathbb{Z}_m \times \mathbb{Z}_n$ are solvable for all m, $n \in \mathbb{Z}^+$ by Lemma 3.1.

Lemma 3.2. *If G is a p-group then G is solvable.*

Proof

We use induction on $|G|$. If $|G| = p^0 = 1$ then $G = \{1\}$. Hence G is abelian and so G is solvable by Lemma 3.1. Suppose the lemma holds for all p-groups of order less than $|G|$. Since G is a p-group we know $1 \neq Z(G) \trianglelefteq G$. Then $|G/Z(G)| < |G|$ and $G/Z(G)$ is a p-group. Hence $G/Z(G)$ is solvable and so there exists a subnormal series

$$G/Z(G) = G_0/Z(G) \trianglerighteq G_1/Z(G) \trianglerighteq G_2/Z(G) \trianglerighteq \cdots \trianglerighteq G_n/Z(G) = 1$$

such that

$$\frac{G_i/Z(G)}{G_{i+1}/Z(G)}$$

is abelian for all $0 \leq i \leq n-1$. Taking preimages we get

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq Z(G) \trianglerighteq 1,$$

a subnormal series. By the Third Isomorphism Theorem

$$\frac{G_i}{G_{i+1}} \cong \frac{G_i/Z(G)}{G_{i+1}/Z(G)}$$

and so G_i/G_{i+1} is abelian for all $0 \leq i \leq n-1$. Finally, $Z(G)/1 \cong Z(G)$ is abelian and so G is solvable. □

Examples. D_4, Q_8, $\mathbb{Z}_{16} \times D_8$ are all solvable groups.

Theorem 3.3. *Let G be a solvable group and $H \leq G$. Then H is solvable.*

Proof

Since G is solvable, there exists a subnormal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = 1$$

such that G_i/G_{i+1} is abelian for all $0 \leq i \leq n-1$. Now we have the series

$$H = H \cap G \geq H \cap G_1 \geq H \cap G_2 \geq \cdots \geq H \cap G_n = 1.$$

If $g \in H \cap G_{i+1}$ and $x \in H \cap G_i$, then $xgx^{-1} \in H$ since $g, x \in H$ and $H \leq G$. Also since $g \in G_{i+1}$, $x \in G_i$ and $G_{i+1} \trianglelefteq G_i$, we get $xgx^{-1} \in G_{i+1}$. Thus $xgx^{-1} \in H \cap G_{i+1}$; so $H \cap G_{i+1} \trianglelefteq H \cap G_i$ for all $0 \leq i \leq n-1$. Therefore we have a subnormal series

$$H = H \cap G_0 \trianglerighteq H \cap G_1 \trianglerighteq H \cap G_2 \trianglerighteq \cdots \trianglerighteq H \cap G_n = 1.$$

Also

$$\frac{H \cap G_i}{H \cap G_{i+1}} = \frac{H \cap G_i}{H \cap G_i \cap G_{i+1}} \cong \frac{(H \cap G_i)G_{i+1}}{G_{i+1}}$$

by the Second Isomorphism Theorem. Now

$$\frac{(H \cap G_i)G_{i+1}}{G_{i+1}} \leq \frac{G_i}{G_{i+1}}$$

and G_i/G_{i+1} is abelian. Therefore $H \cap G_i / H \cap G_{i+1}$ is abelian and so H is solvable. □

Theorem 3.4. *If G is solvable and $N \trianglelefteq G$ then G/N is solvable.*

Proof

Since G is solvable, there exists a subnormal series $G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = 1$ such that G_i/G_{i+1} is abelian for all $0 \leq i \leq n-1$. Taking the image of this series under the natural map $\phi : G \longrightarrow G/N$ we get

$$\frac{G}{N} = \frac{G_0}{N} \trianglerighteq \frac{G_1 N}{N} \trianglerighteq \cdots \trianglerighteq \frac{G_n N}{N} = N.$$

Now by the Second and Third Isomorphism Theorems,

$$\frac{G_i N/N}{G_{i+1}N/N} \cong \frac{G_i N}{G_{i+1}N} = \frac{G_i G_{i+1} N}{G_{i+1}N} \cong \frac{G_i}{G_i \cap G_{i+1}N} \cong \frac{G_i/G_{i+1}}{(G_i \cap G_{i+1}N)/G_{i+1}}.$$

Since G_i/G_{i+1} is abelian we get

$$\frac{G_i N/N}{G_{i+1}N/N}$$

is abelian for all $0 \leq i \leq n-1$. Therefore G/N is solvable. □

Theorem 3.5. *Let G be a solvable group and $N \trianglelefteq G$. If N is solvable and G/N is solvable then G is solvable.*

Proof

Since N is solvable there exists a subnormal series $N = N_0 \trianglerighteq N_1 \trianglerighteq N_2 \trianglerighteq \cdots \trianglerighteq N_n = 1$ such that N_i/N_{i+1} is abelian for all $0 \leq i \leq n-1$. Also since G/N is solvable then there exists a subnormal series

$$\frac{G}{N} = \frac{G_0}{N} \trianglerighteq \frac{G_1}{N} \trianglerighteq \frac{G_2}{N} \trianglerighteq \cdots \trianglerighteq \frac{G_m}{N} = N$$

such that

$$\frac{G_i/N}{G_{i+1}/N}$$

is abelian for all $0 \le i \le m-1$. Taking preimages we get

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq N = N_0 \trianglerighteq N_1 \trianglerighteq N_2 \trianglerighteq \cdots \trianglerighteq N_n = 1$$

. By the Third Isomorphism Theorem

$$\frac{G_i}{G_{i+1}} \cong \frac{G_i/N}{G_{i+1}/N}$$

and so G_i/G_{i+1} is abelian for all $0 \le i \le m-1$. Therefore G is solvable. □

Definition Let G be a group, $H \le G$, $K \le G$ and a, $b \in G$. Then

1. $[a,b] = aba^{-1}b^{-1}$ is called the **commutator** of a and b.
2. $[H,K] = \langle [h,k] | h \in H, k \in K \rangle$.
3. $G' = \langle [x,y] | x, y \in G \rangle$ is called the **commutator subgroup of** G.

Theorem 3.6. *Let G be a group, $N \trianglelefteq G$, $H \le G$ and a, $b \in G$. Then*

1. *$[a,b] = 1$ if and only if $ab = ba$.*
2. *$G' \trianglelefteq G$.*
3. *G/G' is abelian.*
4. *If $G' \le H$ then $H \trianglelefteq G$.*

Proof

For (1): Now $[a,b] = 1$ if and only if $aba^{-1}b^{-1} = 1$ if and only if $ab = ba$. For (2) :

We know that $G' \leq G$. Now let $g \in G$ and $\prod_{i=1}^{n}[a_i, b_i] \in G'$. Since conjugation is a homomorphism,

$$\begin{aligned} g(\prod_{i=1}^{n}[a_i, b_i])g^{-1} &= \prod_{i=1}^{n} g[a_i, b_i]g^{-1} \\ &= \prod_{i=1}^{n}[ga_ig^{-1}, gb_ig^{-1}] \in G'. \end{aligned}$$

Hence $G' \trianglelefteq G$. For (3): Let aG', $bG' \in G/G'$. Then $(ba)^{-1}ab = a^{-1}b^{-1}ab = [a^{-1}, b^{-1}] \in G'$. Therefore $abG' = baG'$ and so $aG'bG' = bG'aG'$. Hence G/G' is abelian. For (4): Let $h \in H$ and $g \in G$. Then $[h^{-1}, g] \in G' \leq H$ and so $[h^{-1}, g] \in H$. Now since $h \in H$ and $H \leq G$ we get $h(h^{-1}ghg^{-1}) \in H$. Therefore $H \trianglelefteq G$. $\square$

Lemma 3.7. *Let G be a group and $N \trianglelefteq G$ such that G/N is abelian. Then $G' \leq N$.*

Let a, $b \in G$. Then $a^{-1}N$, $b^{-1}N \in G/N$. Since G/N is abelian, $a^{-1}Nb^{-1}N = b^{-1}Na^{-1}N$ and so $a^{-1}b^{-1}N = b^{-1}a^{-1}N$. Hence $(b^{-1}a^{-1})^{-1}a^{-1}b^{-1} \in N$ and so $aba^{-1}b^{-1} \in N$ or $[a, b] \in N$. Now since $N \leq G$ we get $G' \leq N$. $\square$

Definition Let G be a group. Define the **derived series** of G by
$G^{(0)} = G$, $G^{(1)} = (G^{(0)})' = G'$, $G^{(2)} = (G^{(1)})' = G''$, and inductively by $G^{(n)} = (G^{(n-1)})'$.

Lemma 3.8. *Let G be a group. Then*

1. $G^{(i+1)} \leq G^{(i)}$ *for all i.*

2. *$G^{(i)} \trianglelefteq G$ for all i.*

3. *G is solvable if and only if there exists $n \in \mathbb{Z}^+ \cup \{0\}$ such that $G^{(n)} = 1$.*

Proof

By definition of derived series, $G^{(i+1)} = (G^{(i)})' \leq G^{(i)}$ for all $i \in \mathbb{Z}^+$. Statement (2) is true for $i = 1$ since $G^{(1)} = (G^{(0)})' = (G)' = G' \trianglelefteq G$. Suppose the statement is true for i i.e $G^{(i)} \trianglelefteq G$. Let $g \in G$. then

$$\begin{aligned} gG^{(i+1)}g^{-1} &= g(G^{(i)})'g^{-1} \\ &= g[G^{(i)}, G^{(i)}]g^{-1} \\ &= [gG^{(i)}g^{-1}, gG^{(i)}g^{-1}] \\ &= [G^{(i)}, G^{(i)}] \\ &= G^{(i+1)}. \end{aligned}$$

And (2) is proven. Therefore $G^{(i+1)} \trianglelefteq G$. Suppose $G^{(n)} = 1$. Then we have

$$G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq \cdots \trianglerighteq G^{(n)} = 1.$$

Also

$$\frac{G^{(i)}}{G^{(i+1)}} = \frac{G^{(i)}}{(G^{(i)})'}$$

is abelian for $0 \leq i \leq n-1$. Thus G is solvable. Next suppose G is solvable. Then there exists a subnormal series $G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = 1$ such that G_i/G_{i+1} is abelian for all $0 \leq i \leq n-1$. We claim that $G^{(i)} \leq G_i$ for all $0 \leq i \leq n-1$. If $i = 0$ then $G^{(0)} = G \leq G = G_0$ and so $G^{(0)} \leq G_0$. Suppose $G^{(i)} \leq G_i$. Then $G^{(i+1)} = (G^{(i)})' \leq G_i' \leq G_{i+1}$ since G_i/G_{i+1} is abelian. Therefore $G^{(n)} \leq G_n = 1$ and so $G^{(n)} = 1$. $\square$

Definition Let G be a group. Then $\phi : G \longrightarrow G$ is a **automorphism** if ϕ is one-to-one, onto, and a homomorphism.

Definition Let G be a group and $H \leq G$. Then H is a **characteristic subgroup** if $\phi(H) \leq H$ for all automorphisms ϕ of G. We write H char G.

Theorem 3.9. *Let G be a group. Then*

1. *$Z(G)$ char G.*
2. *G' char G.*
3. *If $P \in Syl_p(G)$ such that $P \trianglelefteq G$, then P char G.*

Proof

Let ϕ be a automorphism of G, $x \in Z(G)$, and $g \in G$. Since ϕ is onto, there exists $y \in G$ such that $\phi(y) = g$. Then

$$\phi(x)g = \phi(x)\phi(y) = \phi(xy) = \phi(yx) = \phi(y)\phi(x) = g\phi(x)$$

since $x \in Z(G)$ and ϕ is a homomorphism. Therefore $\phi(x) \in Z(G)$ and so $\phi(Z(G)) \leq Z(G)$. Hence $Z(G)$ char G. Next let ϕ be a automorphism of G and $\prod_{i=1}^{n}[a_i, b_i] \in G'$. Then

$$\phi(\prod_{i=1}^{n}[a_i, b_i]) = \prod_{i=1}^{n}\phi([a_i, b_i]) = \prod_{i=1}^{n}[\phi(a_i), \phi(b_i)] \in G'.$$

Thus $\phi(G') \leq G'$ and so G' char G. Finally, since $P \trianglelefteq G$ we know $N_G(P) = \{g \in G | gPg^{-1} = P\} = G$. Thus by Sylow's Theorem,

$$n_p = \frac{|G|}{|N_G(P)|} = 1.$$

Since ϕ is one-to-one and onto, $|\phi(P)| = |P|$. Hence $\phi(P) \in \text{Syl}_p(G)$. Therefore $\phi(P) = P$ which implies P char G. □

Definition Let G be a group and $N \trianglelefteq G$. Then N is a **minimal normal subgroup** if whenever there exist $M \leq N$ such that $M \trianglelefteq G$ then $M = 1$ or $M = N$.

Example. Note $A_3 \trianglelefteq S_3$ and $|A_3| = 3$. Hence A_3 has no nontrivial subgroups and so A_3 is a minimal normal subgroup of S_3.

Example. Let $H = \{1, (13), (24), (13)(24)\}$. Then $|D_4|/|H| = 8/4 = 2$ and so $H \trianglelefteq D_4$. But H is not a minimal normal subgroup since $1 \neq Z(D_4) \leq H$ and $Z(D_4) \trianglelefteq D_4$.

Theorem 3.10. *Let G be a group and $H \leq K \leq G$. If H char K and K char G. Then H char G.*

Proof

Let ϕ be a automorphism of G. Then since K char G we have $\phi(K) \leq K$. Also since ϕ is one-to-one, $|\phi(K)| = |K|$ and so $\phi(K) = K$. Hence $\phi|_K$ is a automorphism of K. Since H char K we get $\phi|_K(H) \leq H$ or $\phi(H) \leq H$. Hence H char G. □

Theorem 3.11. *Let G be a group, H char K, and $K \trianglelefteq G$. Then $H \trianglelefteq G$.*

Proof

For $g \in G$ define $\phi : K \longrightarrow K$ by $\phi(k) = gkg^{-1}$ for all $k \in K$. Then ϕ is a homomorphism and ϕ is one-to-one. If $k \in K$, and $K \trianglelefteq G$ we have $g^{-1}kg \in K$. Also $\phi(g^{-1}kg) = g(g^{-1}kg)g^{-1} = k$ and so ϕ is onto. Thus ϕ is a automorphism of K. Since H char K we get $\phi(H) \leq H$. But $|gHg^{-}| \leq |H|$. Now since $|gHg^{-1}| = |H|$ we get $gHg^{-1} = H$ and so $H \trianglelefteq G$. □

Definition A group G is called **characteristically simple** if 1 and G are its only characteristic subgroups.

Theorem 3.12. *Let G be a characteristically simple group. Then*

$$G \cong G_1 \times G_2 \times \cdots \times G_n$$

such that G_is are simple isomorphic groups.

Proof

Let $G_1 \trianglelefteq G$ such that $G_1 \neq 1$ and $|G_1|$ is minimal. Also let $H = \prod_{i=1}^{s} G_i$ such that

1. $G_i \trianglelefteq G$ for all $1 \leq i \leq s$;
2. $G_i \cong G_1$ for all $1 \leq i \leq s$;
3. $G_i \bigcap \prod_{j \neq i} G_j = 1$ for all $1 \leq i \leq s$;
4. s is maximal.

Since $G_i \trianglelefteq G$ for all $1 \leq i \leq s$, we get $H = \prod_{i=1}^{s} G_i \trianglelefteq G$. We claim that H char G. If not, there exists an automorphism ϕ of G and $1 \leq i \leq s$ such that $\phi(G_i) \nleq H$.

Then $\phi(G_i)\bigcap H < \phi(G_i)$. Since $G_i \trianglelefteq G$ we get $\phi(G_i) \trianglelefteq G$. But then $H \trianglelefteq G$ implies $\phi(G_i)\bigcap H \trianglelefteq G$. Since ϕ is an automorphism of G we get $G_i \cong \phi(G_i)$, so $|\phi(G_i)\bigcap H| < |\phi(G_i)| = |G_i| = |G_1|$. Therefore by the minimality of G_1 we get $\phi(G_i)\bigcap H = 1$. Now $\phi(G_i) \trianglelefteq G, \phi(G_i) \cong G_i \cong G_1$, and $\phi(G_i)\bigcap \prod_{i=1}^{s} G_i \leq \phi(G_i)\bigcap H = 1$. But then we get $H = \prod_{i=1}^{s} G_i < \phi(G_i)\prod_{i=1}^{s} G_i$ a contradiction to the maximality of s. Therefore H char G. Since G is characteristically simple, $H = 1$ or $H = G$. But $1 \neq G_1 \leq H$ and so $H \neq 1$. Thus $G = H = \prod_{i=1}^{s} G_i$ and G_is are isomorphic groups. Let $1 \leq i \leq s$ and $N \trianglelefteq G_i$. If $1 \leq j \leq s$ and $j \neq i$ then $[G_j, N] \leq [G_j, G_i] \leq G_j \bigcap G_i \leq G_i \bigcap \prod_{j\neq i} G_j = 1$ and so $[G_j, N] = 1$. Hence $G_j \leq N_G(N)$ for all $1 \leq j \leq s$ such that $j \neq i$. Also, since $N \trianglelefteq G_i$ we know $G_i \leq N_G(N)$. Hence $G = \prod_{i=1}^{s} G_i \leq N_G(N)$ and so $N = 1$ or $|N| = |G_1|$ by the minimality of G_1. Thus $N = 1$ or $N = G_i$ and so G_i is simple for all $1 \leq i \leq s$. But then $G = \prod_{i=1}^{s} G_i \cong G_1 \times G_2 \times \cdots \times G_s$ when we consider the map $\theta : G \longrightarrow G_1 \times G_2 \times \cdots \times G_s$ defined by

$$\theta(g_1 g_2 \cdots g_s) = (g_1, g_2, \cdots, g_s)$$

Let $g_1 g_2 \cdots g_s$, $h_1 h_2 \cdots h_s \in G$. Then

$$\begin{aligned}
\theta((g_1 g_2 \cdots g_s)(h_1 h_2 \cdots h_s)) &= \theta(g_1 g_2 \cdots g_s h_1 h_2 \cdots h_s) \\
&= \theta(g_1 h_1 g_2 h_2 \cdots g_s h_s) \\
&= (g_1, g_2, \cdots, g_s)(h_1, h_2, \cdots, h_s) \\
&= \theta(g_1 g_2 \cdots g_s)\theta(h_1 h_2 \cdots h_s).
\end{aligned}$$

Hence θ is homomorphism. Let $g_1 g_2 \cdots g_s$, $h_1 h_2 \cdots h_s \in G$ Now $\theta(g_1 g_2 \cdots g_s) = \theta(h_1 h_2 \cdots h_s)$. This implies that $(g_1, g_2, \cdots, g_s) = (h_1, h_2, \cdots, h_s)$ or $g_i = h_i$ for all $1 \leq i \leq s$. Hence θ is one-to-one. Let $(g_1, g_2, \cdots, g_s) \in G_1 \times G_2 \times \cdots \times G_s$. Since $g_i \in G_i$ for each i we know $(g_1 g_2 \cdots g_s) \in G$ and $\theta(g_1 g_2 \cdots g_s) = (g_1, g_2, \cdots, g_s)$. Therefore θ is onto and so $G \cong G_1 \times G_2 \times \cdots \times G_n$ where the G_is are simple isomorphic groups. □

Theorem 3.13. *Let G be a group and N be a minimal normal subgroup of G. Then*

$$N \cong N_1 \times N_2 \times \cdots \times N_n$$

such that the N_is are simple non-abelian isomorphic groups or $N_i \cong \mathbb{Z}_p$ for all $1 \leq i \leq n$, and for some prime p.

Proof

If M char N then, since $N \trianglelefteq G$, we get $M \trianglelefteq G$. Hence $M = 1$ or $M = N$ by the minimality of N. Therefore N is characteristically simple and so by previous theorem $N \cong N_1 \times N_2 \times \cdots \times N_n$, where the N_is are simple isomorphic groups.

Case 1: N_i is abelian for all $1 \leq i \leq n$. Since N_i is simple we get 1 and N_i as the only subgroups of N_i. By Cauchy's theorem there exist a prime p such that $|N_i| = p^m$. But then by Sylow's theorem $m = 1$ and so $|N_i| = p$; hence $N_i \cong \mathbb{Z}_p$ for all $1 \leq i \leq n$.

Case 2: N_i is non abelian for all $1 \leq i \leq n$. Then $N \cong N_1 \times N_2 \times \cdots \times N_n$ is the direct product of simple non-abelian isomorphic groups. □

Definition Let G be a group. Define the **lower central series** of G by $K_0(G) =$

$G, K_1(G) = [K_0(G), G] = [G, G] = G', K_2(G) = [K_1(G), G] = [[G, G], G]$, and inductively by $K_n(G) = [K_{n-1}(G), G]$.

Theorem 3.14. *Let G be a group. Then*

1. *$K_i(G) \trianglelefteq G$ for all i.*

2. *$K_{i+1}(G) \leq K_i(G)$ for all i.*

Proof

Proceed by using induction on i. If $i = 0$, then $K_0(G) = G \trianglelefteq G$. Assume $K_i(G) \trianglelefteq G$ and let $g \in G$. Then

$$\begin{aligned} gK_{i+1}(G)g^{-1} &= g[K_i(G), G]g^{-1} \\ &= [gK_i(G)g^{-1}, gGg^{-1}] \\ &= [K_i(G), G] \\ &= K_{i+1}(G). \end{aligned}$$

Thus, $K_{i+1}(G) \trianglelefteq G$ and we have (1) by induction. Now $K_{i+1}(G) = [K_i(G), G] \leq K_i(G)$, since $K_i(G) \trianglelefteq G$. Hence we get $K_{i+1}(G) \leq K_i(G)$ for all i. □

4 Nilpotent Groups

Definition A group G is called **nilpotent** if there exists $n \in \mathbb{Z}^+ \cup \{0\}$ such that $K_n(G) = 1$.

Lemma 4.1. *If G is abelian, then $K_1(G) = [K_0(G), G] = [G, G] = 1$. Hence G is nilpotent.*

Example $\mathbb{Z}_{10}$, $\mathbb{Z}_8 \times \mathbb{Z}_{12}$, $\mathbb{R}$, $\mathbb{Q}$ *are nilpotent groups.*

Theorem 4.2. *Let G be a p-group. Then G is nilpotent.*

Proof

We use induction on $|G|$. If $|G| = p$ then G is cyclic. It fellows that G is abelian and by Lemma 4.1 G is nilpotent. Suppose all p-groups of order less than $|G|$ are nilpotent. We claim G is nilpotent. Since G is a p-group, we know $1 \neq Z(G) \trianglelefteq G$. So $G/Z(G)$ is a p-group and $|G/Z(G)| < |G|$. Then by assumption $G/Z(G)$ is nilpotent. So there exists $n \in \mathbb{Z}^+ \cup \{0\}$ such that

$$K_n\left(\frac{G}{Z(G)}\right) = 1.$$

We claim

$$\frac{K_i(G)Z(G)}{Z(G)} \leq K_i\left(\frac{G}{Z(G)}\right) \text{ for all } i$$

Use induction on i. If $i = 0$ then

$$\frac{K_0(G)Z(G)}{Z(G)} = \frac{GZ(G)}{Z(G)} = \frac{G}{Z(G)} \leq K_0\left(\frac{G}{Z(G)}\right) = \frac{G}{Z(G)}.$$

Suppose $K_i(G)Z(G)/Z(G) \leq K_i(G/Z(G))$. Then

$$\begin{aligned} \frac{K_{i+1}(G)Z(G)}{Z(G)} &= \frac{[K_i(G), G]Z(G)}{Z(G)} \\ &\leq \left[\frac{K_i(G)Z(G)}{Z(G)}, \frac{G}{Z(G)}\right] \\ &\leq \left[K_i(\frac{G}{Z(G)}), \frac{G}{Z(G)}\right] \\ &= K_{i+1}\left(\frac{G}{Z(G)}\right). \end{aligned}$$

Thus

$$\frac{K_i(G)Z(G)}{Z(G)} \leq K_i\left(\frac{G}{Z(G)}\right)$$

for all i. Hence

$$\frac{K_n(G)Z(G)}{Z(G)} \leq K_n\left(\frac{G}{Z(G)}\right) = 1Z(G)$$

. And so $K_n(G) \leq Z(G)$. Then $K_{n+1}(G) = [K_n(G), G] \leq [Z(G), G] = 1$. Therefore $K_{n+1}(G) = 1$ and so G is nilpotent. □

Theorem 4.3. *Let G be a nilpotent group and $H \leq G$. Then H is nilpotent.*

Proof

Since G is nilpotent there exists $n \in \mathbb{Z}^+ \cup \{0\}$ such that $K_n(G) = 1$. Claim: $K_i(H) \leq K_i(G)$ for all i. We use induction on i. If $i = 0$ then $K_0(H) = H \leq G = K_0(G)$. Suppose $K_i(H) \leq K_i(G)$. Then $K_{i+1}(H) = [K_i(H), H] \leq [K_i(G), G] = K_{i+1}(G)$, which implies $K_{i+1}(H) \leq K_{i+1}(G)$, and so $K_i(H) \leq K_i(G)$ for all i. Hence $K_n(H) \leq K_n(G) = 1$ and so H is nilpotent. □

Theorem 4.4. *Let G be a nilpotent group and $N \trianglelefteq G$. Then G/N is nilpotent.*

Proof

Since G is nilpotent there exists $n \in \mathbb{Z}^+ \cup \{0\}$ such that $K_n(G) = 1$. As before

$$K_i\left(\frac{G}{N}\right) \leq \frac{K_i(G)N}{N} \text{ for all } i.$$

Thus

$$K_n\left(\frac{G}{N}\right) \leq \frac{K_n(G)N}{N} = \frac{1N}{N} = 1N.$$

Hence G/N is nilpotent. □

Lemma 4.5. *Let G be a nilpotent group and $H < G$. Then $H < N_G(H)$*

Proof

Clearly $H \leq N_G(H)$. Since G is nilpotent there exists $n \in \mathbb{Z}^+$ such that $K_n(G) = 1$. Since $H \neq G$ there exists a maximal i such that $K_i(G)$ is not contained in H. Then

$$[K_i(G), H] \leq [K_i(G), G] = K_{i+1}(G) \leq H$$

by the maximality of i. Let $k \in K_i(G)$ and $h \in H$. Then $[k, h] \in [K_i(G), H] \leq H$ and so $[k, h] \in H$. But $h \in H$ and so $[k, h]h = khk^{-1} \in H$. Thus, $K_i(G) \leq N_G(H)$. Therefore, since $K_i(G)$ is not contained in H, $H < N_G(H)$. □

Definition Let G be a group and $M \leq G$. Then M is a **maximal subgroup** of G if $M \neq G$ and, whenever there exists a subgroup H of G such that $M \leq H \leq G$, then $H = M$ or $H = G$.

Example $\langle(12)\rangle, \langle(13)\rangle, \langle(23)\rangle$, and $\langle(123)\rangle$ are all maximal subgroups of S_3.

Lemma 4.6. *Let G be a nilpotent group and M be a maximal subgroup of G. Then $M \trianglelefteq G$.*

Proof

Now since M is maximal we know $M < G$. Hence, by Lemma 4.5 $M < N_G(M) \leq G$. Thus, $G = N_G(M)$ by the maximality of M. Hence $M \trianglelefteq G$. □

Theorem 4.7. Frattini's argument *Let G be a group, $H \trianglelefteq G$, and $P \in Syl_p(H)$, then $G = N_G(P)H$.*

Proof

Clearly, $N_G(P)H \subseteq G$. Let $g \in G$. Then since $P \leq H$ we get $gPg^{-1} \leq gHg^{-1}$. But since $H \trianglelefteq G$, we have $gHg^{-1} = H$. Thus, $gPg^{-1} \leq H$. Now since $P \in \mathrm{Syl}_p(H)$ and $|gPg^{-1}| = |P|$ we get $gPg^{-1} \in \mathrm{Syl}_p(H)$. Then by Sylow's theorem $gPg^{-1} = hPh^{-1}$ for some $h \in H$. So $h^{-1}gPg^{-1}h = P$, or $hgP(hg)^{-1} = P$. But then $hg \in N_G(P)$ and so $g \in N_G(P)H$. Therefore $G = N_G(P)H$. □

Lemma 4.8. *Let G be a nilpotent group and $P \in Syl_p(G)$. Then $P \trianglelefteq G$.*

Proof

If P is not normal in G then $N_G(P) < G$. Let M be a maximal subgroup of G such that $N_G(P) \leq M$. Since G is nilpotent, by maximality of M, we know $M \trianglelefteq G$. Now $P \leq N_G(P) \leq M$ and $P \in \mathrm{Syl}_p(G)$ implies $P \in \mathrm{Syl}_p(M)$. Therefore by the Frattini Argument $G = N_G(P)M = M$. This is a contradiction to the maximality of M.

Therefore $P \trianglelefteq G$. □

Theorem 4.9. *Let G be a nilpotent group. Then G is solvable.*

Proof

Since G is a nilpotent group, there exists $n \in \mathbb{Z}^+ \cup \{0\}$ such that $K_n(G) = 1$. We know from Theorem 3.15 that $K_i(G) \trianglelefteq G$ for all i and $K_{i+1}(G) \leq K_i(G)$ for all i. Then we have a subnormal series

$$G = K_0(G) \trianglerighteq K_2(G) \trianglerighteq \cdots \trianglerighteq K_n(G) = 1.$$

We claim that $K_i(G)/K_{i+1}(G)$ is abelian for all $1 \leq i \leq n-1$. Let x^{-1}, $y^{-1} \in K_i(G)$. Now $K_i(G)/K_{i+1}(G)$ is abelian if and only if

$$\begin{aligned}
x^{-1}K_{i+1}(G)y^{-1}K_{i+1}(G) &= y^{-1}K_{i+1}(G)x^{-1}K_{i+1}(G) \\
x^{-1}y^{-1}K_{i+1}(G) &= y^{-1}x^{-1}K_{i+1}(G) \\
xyx^{-1}y^{-1} = [x, y] &\in K_{i+1}(G) \\
[K_i(G), K_i(G)] &\leq K_{i+1}(G) \\
K_i(G)' = [K_i(G), K_i(G)] &\leq K_{i+1}(G)
\end{aligned}$$

So by Theorem 3.6 $K_{i+1}(G) \trianglelefteq K_i(G)$ and $K_i(G)/K_{i+1}(G)$ is abelian for all $0 \leq i \leq n-1$. □

Lemma 4.10. *Let G be a nilpotent group such that $G \neq 1$. Then $Z(G) \neq 1$.*

Proof

Since G is nilpotent, there exists a minimal $n \in \mathbb{Z}^+$ such that $K_n(G) = 1$. Then

$$1 = K_n(G) = [K_{n-1}(G), G],$$

and so $K_{n-1}(G) \leq Z(G)$. But $1 \neq K_{n-1}(G)$ by the minimality of n and so $Z(G) \neq 1$.□

Lemma 4.11. *Let G be a nilpotent group and $1 \neq N \trianglelefteq G$. Then $N \cap Z(G) \neq 1$.*

Proof

Since G is nilpotent, there exists $n \in \mathbb{Z}^+$ such that $K_n(G) = 1$. Define $N_0 = N, N_1 = [N_0, G] = [N, G]$, and inductively by $N_k = [N_{k-1}, G]$ for all $k \in \mathbb{Z}^+ \cup \{0\}$. Then we have a normal series

$$N = N_0 \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \cdots$$

Claim $N_i \leq K_i(G)$ for all $i \in \mathbb{Z}^+ \cup \{0\}$. We use induction on i. If $i = 0$, then $N_0 = N \leq G = K_0(G)$. Now suppose $N_i \leq K_i(G)$. Then $N_{i+1} = [N_i, G] \leq [K_i(G), G] = K_{i+1}(G)$. Hence the claim holds by induction. Thus,

$$N_n \leq K_n(G) = 1 \text{ and so } N_n = 1.$$

Let $m \in \mathbb{Z}^+$ be minimal such that $N_m = 1$. Then $1 = N_m = [N_{m-1}, G]$ and so $N_{m-1} \leq Z(G)$. But $N_{m-1} \leq N$ and $N_{m-1} \neq 1$ by the minimality of m. Thus, $1 \neq N_{m-1} \leq N \cap Z(G)$. □

Lemma 4.12. *Let $G = HK$ be a group such that $H \trianglelefteq G, K \trianglelefteq G$ and H and K are nilpotent. Then G is nilpotent.*

Proof

Use induction on $|G|$. If $|G| = 1$ then $K_0(G) = G = 1$ and so G is nilpotent. Assume $|G| > 1$ and that the theorem holds for all groups of order less than $|G|$. We want to show the theorem holds for G. Since H is nilpotent, by Lemma 4.9 $Z(H) \neq 1$. Let $N = [Z(H), K]$. If $N = 1$ then $[Z(H), K] = 1$. Thus

$$1 \neq Z(H) \leq C_G(H) \cap C_G(K) = Z(G).$$

Now $Z(G) \trianglelefteq G$ and so

$$\frac{G}{Z(G)} = \frac{HZ(G)}{Z(G)}\frac{KZ(G)}{Z(G)}$$

is a group. Since $H \trianglelefteq G$ and $K \trianglelefteq G$ we know

$$\frac{HZ(G)}{Z(G)} \trianglelefteq \frac{G}{Z(G)} \text{ and } \frac{KZ(G)}{Z(G)} \trianglelefteq \frac{G}{Z(G)}.$$

Also since H is nilpotent, $\dfrac{HZ(G)}{Z(G)} \cong \dfrac{H}{H \cap Z(G)}$ is nilpotent and similarly $\dfrac{KZ(G)}{Z(G)}$ is nilpotent.

Finally,

$$\left|\frac{G}{Z(G)}\right| = \frac{|G|}{|Z(G)|} < |G|$$

and so $G/Z(G)$ is nilpotent by induction. Therefore there exists $n \in \mathbb{Z}^+$ such that $K_n(G/Z(G)) = 1$. But then

$$\frac{K_n(G)Z(G)}{Z(G)} = K_n\left(\frac{G}{Z(G)}\right) = 1 \text{ and so } K_n(G) \leq Z(G).$$

Hence

$$K_{n+1}(G) = [K_n(G), G] \leq [Z(G), G] = 1 \text{ and so } G \text{ is nilpotent.}$$

If $N \neq 1$, as $K \trianglelefteq G$, we know $N \leq K$. Also since $H \trianglelefteq G, Z(H) \trianglelefteq G$. Thus, $N = [Z(H), K] \trianglelefteq K$. Now since K is nilpotent $N \cap Z(K) \neq 1$ by Lemma 4.10. Hence since $Z(H) \trianglelefteq G$ we get $1 \neq N \cap Z(K) \leq Z(H) \cap Z(K) \leq Z(G)$. Therefore $Z(G) \neq 1$ again and so G is nilpotent using the above argument. □

Definition A group G is called an **elementary abelian p-group** if $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ for some prime p.

Theorem 4.13. *Let G be a solvable group and N be a minimal normal subgroup of G. Then N is an elementary abelian p-group for some prime p.*

Proof

By Theorem **3.9**, $N \cong N_1 \times N_2 \times \cdots \times N_n$ where the N_is are non-abelian simple isomorphic groups or $N_i \cong \mathbb{Z}_p$ for all $1 \leq i \leq n$. If N_i is nonabelian for some $1 \leq i \leq n$ then $1 \neq N_i' \trianglelefteq N_i$ and so $N_i' = N_i^{(1)} = N_i$. Suppose $N_i^{(k)} = N_i$. Then $N_i^{(k+1)} = (N_i^{(k)})' = N_i' = N_i$. Thus, $N_i^{(k)} = N_i$ for all k by induction. But then N_i is not solvable. Now G is solvable and $N_i \leq G$ which implies that N_i is solvable, a contradiction. Hence there exists a prime p such that $N_i \cong \mathbb{Z}_p$ for all i and so

$$N \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$$

is a elementary abelian p-group. □

5 The Hall and Schur-Zassenhaus Theorems

Definition Let G be a group and π be a set of primes. Then

1. $\pi' = \{p | p \text{ is prime and } p \notin \pi\}$.

2. $\pi(G) = \{p | p \text{ is prime and } p || G|\}$.

3. G is called a π**-group** if $\pi(G) \subseteq \pi$.

4. A subgroup $H \leq G$ is called a **Hall** π**-subgroup** if H is a π-group and $\pi(S) \subseteq \pi'$ where $S = \{gH | g \in G\}$.

5. $\text{Hall}_\pi(G) = \{H \leq G | H$ is a Hall $\pi -$ subgroup of $G\}$.

Example 1 $|S_3| = 3{=}\ 3 \cdot 2$ and $\pi(S_3) = \{2, 3\}$. Now $|A_3| = 3$; so A_3 is a 3-group and $\pi(S_3/A_3) \subseteq \{3\}'$. Hence $A_3 \in \text{Hall}_{\{3\}}(S_3)$.

Example 2 $|A_5| = 5!/2 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1/2 = 2^2 \cdot 3 \cdot 5$. Let $H = (A_5)_1$. Then $H \cong A_4$ and $|H| = 4!/2 = 2^2 \cdot 3$. Therefore H is a $\{2, 3\}$-group. Also $\pi(A_5/H) = 5 \in \{2, 3\}'$. Hence $H \in \text{Hall}_{\{2,3\}}(A_5)$.

Example 3 If G is a group, p is a prime, and $\pi = \{p\}$, then $Syl_p(G) = \text{Hall}_\pi(G)$. For some groups G and certain sets of primes π, $\text{Hall}_\pi(G) = \emptyset$.

Example $\text{Hall}_{\{2,5\}}(A_5) = \emptyset$.

Proof

Suppose $H \in \text{Hall}_{\{2,5\}}(A_5)$. Then H is a $\{2, 5\}$-group and $\pi(A_5/H) \subseteq \{2, 5\}'$. Since

$|A_5| = 2^2 \cdot 3 \cdot 5$ we get $|H| = 2^2 \cdot 5$. Let A_5 act on $S = \{gH | g \in A_5\}$ by left multiplication via $\phi : A_5 \longrightarrow \text{Sym}(S)$, where ϕ is defined by $\phi(g)(xH) = gxH$ for all $g \in A_5$ and for all $xH \in S$. Now by Lagrange's Theorem $|S| = |A_5|/|H| = 3$ and so $\text{Sym}(S) \cong S_3$. Now $K = \text{Kern } \phi \trianglelefteq A_5$. Since A_5 is simple either $K = 1$ or $K = A_5$. If $K = A_5$ then

$$A_5 = K = \bigcap_{x \in A_5} xHx^{-1} \leq H$$

and we get $A_5 = H$, a contradiction. If $K = 1$ then, by the First Isomorphism Theorem,

$$A_5 \cong \frac{A_5}{1} = \frac{A_5}{K} \cong \phi(A_5) \leq \text{Sym}(S).$$

But then we get $60 = |A_5| = |\phi(A_5)|$ divides $|\text{Sym}(S)| = 6$, a contradiction. Thus $\text{Hall}_{\{2,5\}}(A_5) = \emptyset$. □

Theorem 5.1. *(Hall's): Let G be a solvable group and π be a set of primes. Then*

1. *$Hall_\pi(G) \neq \emptyset$*
2. *G acts transitively on $Hall_\pi(G)$ by conjugation.*

Definition Let G be a group and $H \leq G$. Then G splits over H if there exists $K < G$ such that $G = HK$ and $H \cap K = 1$. The subgroup K is called the complement of H in G.

Example: S_3 splits over A_3 since $S_3 = A_3\langle(12)\rangle$ and $A_3 \cap \langle(12)\rangle = 1$.

Theorem 5.2. *Let G be a solvable group, $H \in Hall_\pi(G)$, and suppose $N_G(H) \leq K \leq G$. Then $K = N_G(K)$.*

Proof

Clearly $K \leq N_G(K)$. Let $g \in N_G(K)$. Then $H \leq N_G(H) \leq K$; so $H \in$ Hall$_\pi(G)$, so $H \in$ Hall$_\pi(K)$. Now $H \leq K$ implies $gHg^{-1} \leq gKg^{-1} = K$. But $|gHg^{-1}| = |H|$ and so $gHg^{-1} \in$ Hall$_\pi(K)$. Now since G is solvable, K is also solvable. Thus by Hall's theorem there exists $k \in K$ such that $kgHg^{-1}k^{-1} = H$ or $kgH(kg)^{-1} = H$. But then $kg \in N_G(H)$ and so $g \in K$. Therefore $K = N_G(K)$. In this case we say K is self-normalizing. □

Theorem 5.3. *(Schur-Zassenhaus) Let G be a group and $H \in$ Hall$_\pi(G)$ such that $H \trianglelefteq G$. Then G splits over H. In addition if either H or G/H is solvable, then G acts transitively on the complements of H in G by conjugation.*

6 Carter's Theorem

Definition Let G be a group and $H \leq G$. Then H is a **Carter subgroup** of G if

1. H is nilpotent;

2. $N_G(H) = H$.

In this case we write H cart G. When condition (2) holds, we say H is self-normalizing.

Example Any nilpotent group G has a Carter subgroup, namely, G itself is a Carter subgroup since $N_G(G) = G$, and G is nilpotent.

Example $\langle(12)\rangle$ cart S_3 since $\langle(12)\rangle$ is abelian implies $\langle(12)\rangle$ is nilpotent. Also $\langle(12)\rangle \leq N_{S_3}(\langle(12)\rangle) \leq S_3$ and so $2 = |\langle(12)\rangle|$ which divide $|N_{S_3}(\langle(12)\rangle)|$ divides $|S_3| = 6$. Hence $|N_{S_3}(\langle(12)\rangle)| = 2$. But $N_{S_3}(\langle(12)\rangle) \neq S_3$ since $\langle(12)\rangle$ is not a normal subgroup of S_3. And so $\langle(12)\rangle = |N_{S_3}(\langle(12)\rangle)|$.

But not all groups have Carter subgroups.

Example A_5 has no Carter subgroups. $|A_5| = \frac{5!}{2} = 60 = 2^2 \cdot 3 \cdot 5$. A table showing 57 subgroups of A_5 is below.

$Structure$	$Subgroup, H$	$Number$	$Reason$
$\mathbb{Z}_2$	$\{1, (12)(34)\}$	15	$(13)(24) \in N_{A_5}(H) \setminus H$
$\mathbb{Z}_3$	$\{1, (123), (132)\}$	10	$(23)(45) \in N_{A_5}(H) \setminus H$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\{1, (12)(34), (14)(23), (13)(24)\}$	5	$(123) \in N_{A_5}(H) \setminus H$
$\mathbb{Z}_5$	$\{1, (12345), (13524), (14253), (15432)\}$	6	$(15)(24) \in N_{A_5}(H) \setminus H$
S_3	$\{1, (123), (132), (12)(45), (13)(45), (23)(45)\}$	10	Not nilpotent $n_2 = 3$
D_5	$\langle(12345), (15)(24)\rangle$	6	Not nilpotent $n_2 = 5$
A_4	$(A_5)_1$	5	Not nilpotent $n_3 = 4$

Theorem 6.1. *(Carter): Let G be a solvable group. Then*

1. *G has a Carter subgroup;*

2. *If $N \trianglelefteq G$ and H cart G then HN/N cart G/N;*

3. *If H_1 cart G and H_2 cart G then there exists $g \in G$ such that $H_2 = gH_1g^{-1}$.*

Proof

We will use induction on $|G|$. If $|G| = 1$ then $\{1\}$ cart G and (1), (2) and (3) hold. Also if G is nilpotent, then G cart G and (1), (2) and (3) hold. Without loss of generality, assume that $|G| > 1, G$ is not nilpotent, and the result holds for all groups of order less than $|G|$. For (1): Let N be a minimal normal subgroup of G. Since G is solvable, N is an elementary p-group for some prime p. Since G is solvable, by Theorem 3.4 we know G/N is solvable. Also

$$|G/N| = \frac{|G|}{|N|} < |G|$$

and so by induction there exists K/N cart G/N. Now let $S/N \in \mathrm{Syl}_p(K/N)$. Since K/N cart G/N, we know K/N is nilpotent. Thus by Lemma 4.7, $S/N \trianglelefteq K/N$. But then $S \trianglelefteq K$. Now

$$\frac{|K|}{|S|} = \frac{|K|/|N|}{|S|/|N|} = \frac{|K/N|}{|S/N|}$$

and so p does not divide $|K|/|S|$ since $S/N \in \mathrm{Syl}_p(K/N)$. Also,

$$|S| = \frac{|S|}{|N|}|N| = |S/N||N|$$

is a power of p since $S/N \in \mathrm{Syl}_p(K/N)$ and N is an elementary p-group. Hence $S \in \mathrm{Syl}_p(K)$ and so K splits over S by the Schur-Zassenhaus Theorem. But then there exists $R \leq K$ such that $K = RS$ and $R \cap S = 1$. Now by the Second Isomorphism Theorem

$$R \cong \frac{R}{1} = \frac{R}{R \cap S} \cong \frac{RS}{S} = \frac{K}{S}.$$

From the above, p does not divide $|K/S|$ and so p does not divide $|R|$. Also

$$\frac{|K|}{|R|} = \frac{|RS|}{|R|} = \frac{|S|}{|R \cap S|} = |S|$$

is a power of p. Thus $R \in \mathrm{Hall}_{p'}(K)$. Let $H = N_K(R)$ and $g \in N_G(H)$. Now $N_K(R) \leq HN \leq K$, $R \in \mathrm{Hall}_{p'}(K)$, and K is solvable. Thus by Theorem 5.2 $HN = N_K(HN)$. But then

$$\frac{HN}{N} = \frac{N_K(HN)N}{N} = N_{K/N}\left(\frac{HN}{N}\right).$$

Now $HN/N \leq K/N$ and K/N is nilpotent. Hence we get $K/N = HN/N$ and so $K = HN$. Since $N \trianglelefteq G$ and $g \in N_G(H)$ we have $g \in N_G(HN) = N_G(K)$. Hence $gN \in N_{G/N}(K/N)$. But $K/N = N_{G/N}(K/N)$ since K/N cart G/N. Therefore $gN \in K/N$ and so $g \in K$. But then $g \in N_K(H)$. Also $N_K(R) \leq H \leq K$, $R \in \mathrm{Hall}_{p'}(K)$, and K is solvable. Thus by Theorem 5.2, $H = N_K(H)$. Therefore $g \in H$ and so

$H = N_G(H)$. Now

$$H = H \cap K = H \cap RS = R(H \cap S).$$

Since $S \trianglelefteq K$ and $H \leq K$ we know $S \cap H \trianglelefteq H$. Also since $R \leq H \leq N_G(R)$ we know $R \trianglelefteq H$. Since S is a p-group we know $S \cap H$ is a p-group. Therefore $S \cap H$ is nilpotent. Also by the Second and Third Isomorphism Theorems,

$$R \cong \frac{R}{1} = \frac{R}{R \cap S} \cong \frac{RS}{S} = \frac{K}{S} \cong \frac{K/N}{S/N}.$$

But since

$$\frac{K}{N} \text{ cart } \frac{G}{N}$$

K/N is nilpotent. Thus R is nilpotent by Theorem 4.4. Therefore $H = R(H \cap S)$ is nilpotent by Lemma 4.11 and so H cart G.

For (2) : Let H cart G and $N \trianglelefteq G$. Then

$$\frac{HN}{N} \leq \frac{G}{N}.$$

Also since H is nilpotent,

$$\frac{HN}{N} \cong \frac{H}{H \cap N}$$

is nilpotent. Clearly

$$\frac{HN}{N} \leq N_{G/N}\left(\frac{HN}{N}\right)$$

Let $gN \in N_{G/N}(HN/N)$. Then $g^{-1}N \in N_{G/N}(HN/N)$ and also

$$\begin{aligned}\frac{HN}{N} = g^{-1}N(\frac{HN}{N})gN &= \frac{g^{-1}(HN)g}{N} \\ &= \frac{g^{-1}HgN}{N}.\end{aligned}$$

By taking preimages we get $g^{-1}HgN = HN$. If $G = HN$ then $G/N = HN/N$. Hence

$$\frac{HN}{N} = \frac{G}{N} = N_{G/N}\left(\frac{G}{N}\right) = N_{G/N}\left(\frac{HN}{N}\right).$$

Therefore we may assume $HN < G$. Now $g^{-1}Hg \cong H$ and so $g^{-1}Hg$ is nilpotent. Also,

$$N_G(g^{-1}Hg) = g^{-1}N_G(H)g = g^{-1}Hg \text{ since } H \text{ cart } G.$$

Thus, $g^{-1}Hg$ cart HN and H cart HN. Therefore by induction there exists $n \in N$ such that $ng^{-1}Hgn^{-1} = H$. But then $ng^{-1} \in N_G(H) = H$ since H cart G. So $gn^{-1} \in H$ since $H \leq G$. Then $gN = gn^{-1}N \in HN/N$ and so

$$\frac{HN}{N} = N_{G/N}\left(\frac{HN}{N}\right) \text{ and so } \frac{HN}{N} \text{ cart } \frac{G}{N}.$$

For (3): Let H_1 cart G and H_2 cart G. Let N be a minimal normal subgroup of G. Since G is solvable, by Theorem 3.6, N is an elementary p-group. By (2),

$$\frac{H_1N}{N} \text{ cart } \frac{G}{N} \text{ and } \frac{H_2N}{N} \text{ cart } \frac{G}{N}.$$

Since $|G/N| < |G|$, by induction there exists $gN \in G/N$ such that

$$\frac{H_2N}{N} = gN\left(\frac{H_1N}{N}\right)g^{-1}N \quad = \quad \frac{gH_1g^{-1}N}{N}.$$

Therefore $gH_1g^{-1}N = H_2N$. If $H_2N < G$ then gH_1g^{-1} cart H_2N and H_2 cart H_2N. Hence by induction there exists $g_1 \in H_2N$ such that $g_1gH_1g^{-1}g_1^{-1} = H_2$. We may assume $G = gH_1g^{-1}N = H_2N$. Since gH_1g^{-1} and H_2 are nilpotent, there exist $gR_1g^{-1} \in \mathrm{Hall}_{p'}(gH_1g^{-1})$ and $R_2 \in \mathrm{Hall}_{p'}(H_2)$. Now

$$\frac{|G|}{|R_2|} = \frac{|G|}{|H_2|}\cdot\frac{|H_2|}{|R_2|} = \frac{|H_2N|}{|H_2|}\cdot\frac{|H_2|}{|R_2|} = \frac{|N|}{|N \cap H_2|}\cdot\frac{|H_2|}{|R_2|}$$

is a power of p. Thus $R_2 \in \mathrm{Hall}_{p'}(G)$ and similarly $gR_1g^{-1} \in \mathrm{Hall}_{p'}(G)$. Since G is solvable, by Hall's Theorem, there exists $g_2 \in G$ such that $g_2gR_1g^{-1}g_2^{-1} = R_2$. Now gR_1g^{-1} and H_2 are nilpotent implies $gR_1g^{-1} \trianglelefteq gH_1g^{-1}$ and $R_2 \trianglelefteq H_2$. Thus $g_2gR_1g^{-1}g_2^{-1} \trianglelefteq g_2gH_1g^{-1}g_2^{-1}$ and so

$$g_2gH_1g^{-1}g_2^{-1} \leq N_G(g_2gR_1g^{-1}g_2^{-1}) = N_G(R_2) \geq H_2.$$

Let $K = N_G(R_2)$. Now $R_2 \trianglelefteq K$ and so K/R_2 is a group. Since $g_2gH_1g^{-1}g_2^{-1}$ cart K, by part (2)

$$\frac{g_2gH_1g^{-1}g_2^{-1}R_2}{R_2} \text{ cart } \frac{K}{R_2} \text{ and } \frac{H_2}{R_2} \text{ cart } \frac{K}{R_2}.$$

If $R_2 = 1$ then H_2 is a p-group. Since N is a p-group, we get $G = H_2N$ is a p-group. Thus, G is nilpotent and so $G = H_1 = H_2$. We may assume $R_2 \neq 1$ and $|K/R_2| < |G|$.

So by induction there exists $kR_2 \in K/R_2$ such that

$$\frac{H_2}{R_2} = kR_2 \left(\frac{g_2 g H_1 g^{-1} g_2^{-1} R_2}{R_2} \right) k^{-1} R_2 = \frac{k g_2 g H_1 g^{-1} g_2^{-1} k^{-1} R_2}{R_2}.$$

Thus

$$k g_2 g H_1 g^{-1} g_2^{-1} k^{-1} R_2 = H_2.$$

Now $kR_2 \in K/R_2$ implies $k \in K = N_G(R_2)$. But

$$R_2 = g_2 g R_1 g^{-1} g_2^{-1} \leq g_2 g H_1 g^{-1} g_2^{-1}$$

and so

$$R_2 = k R_1 k^{-1} \leq k g_2 g H_1 g^{-1} g_2^{-1} k^{-1}.$$

Therefore

$$k g_2 g H_1 g^{-1} g_2^{-1} k^{-1} R_2 = H_2 = k g_2 g H_1 g^{-1} g_2^{-1} k^{-1} = H_2$$

and so we have (3). □

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