# The Generalized Riemann Integral in $\mathbb{R}^2$

by

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## ABSTRACT

In their undergradate studies, new calculus students learn how to integrate a function using the Riemann integral. The Riemann integral is used in Mathematics and Engineering and is a fairly simple concept to understand. However, the Riemann integral has some drawbacks and limitations. These limitations include improper integrals, application of the Fundamental theorem of Calculus, and others that will be discussed. By using the generalized Riemann integral, we can eliminate these limitations by only slightly changing our definition of the Riemann integral. However, what has been written on it isn't particularly easy to understand. Because of this, the theory is only taught at the higher undergraduate or graduate level. This paper will take a look at the generalized Riemann integral. Being a more general version of the standard Riemann integral, it is not subject to these restrictions. The focus of this paper will be in one and two dimensions, but the theory from there can be easily generalized into higher dimensions. In this paper we will look at the generalized Riemann integral and will show all the ways all in which it is more powerful than the standard Riemann integral taught in calculus courses. While showing that the generalized Riemann is superior, we will also break the theory down for both the one and two-dimensional cases in a more understandable way.

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# Nomenclature

$[c,d] \subset [a,b]$: $[c,d]$ is a subinterval of $[a,b]$

$\delta$: gauge

$\int_a^b$: Integral over interval $[a,b]$

$\int_I$: Integral over set $I$

$\mathbb{N}$: Set of natural numbers

$\mathbb{Q}$: Set of rational numbers

$\mathbb{R}$: Set of real numbers

$\mu(I)$: Measure of an interval $I$

$A \subset B$: Set $A$ is a subset of $B$

$A(f,P)$: Riemann sum of $f$ associated with $P$

$l(I)$: Length of an interval $I$

# Contents

# 1 Introduction

Starting with the definition of a Riemann Sum is an appropriate way to start since it is used in both the standard Riemann integral as well as the generalized Riemann integral. Suppose a function $f$ is to be integrated over a closed interval $[a,b]$. Form a partition of $[a,b]$ into nonoverlapping intervals $[x_k, x_{k+1}]$ by choosing $x_k$ such that

$$a = x_o < x_1 < x_2 < ..... < x_n = b.$$

Then create a tagged partition from this division by choosing a number $y_k \in [x_{k-1}, x_k]$ for each k. This number $y_k$ is called a tag. In calculus, they are often called sample points. Then $\sum_{k=1}^{n} f(y_k)(x_k - x_{k-1})$ is a Riemann sum for $f$ on $[a,b]$.

Though both methods use Riemann sums to approximate area, the generalized Riemann integral puts more emphasis on tags than the standard Riemann approach. While standard Riemann integration is the method taught to new undergraduate students entering calculus, we will see that there are restrictions and drawbacks that limit it.

The first of these drawbacks is that there are a great many functions that are not integrable using the standard method of Riemann integration. One of the classic examples is Dirichlet's function, also called by some as the "salt and pepper function".

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

The rational and irrational numbers are both dense in a set $[a,b]$, so this function is discontinuous everywhere on this interval. On every interval the supremum of $f$ is 1 and the infimum of $f$ is 1, so this function cannot be Riemann integrable.

Another drawback is that not all derivatives are Riemann integrable. Looking at another classic example, consider the function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

On the interval $[-1,1]$, $f(x)$ has the derivative

$$f'(x) = \begin{cases} -\frac{2}{x}\cos\left(\frac{1}{x^2}\right) + 2x\left(\sin\left(\frac{1}{x^2}\right)\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

However, $f'(x)$ is unbounded on the interval $[-1,1]$, so $f'(x)$ is not Riemann integrable.

Integrals that are considered "improper" in the standard Riemann sense are not an issue for the generalized Riemann. For example the function $\log x$ is not integrable in the Riemann case since the logarithm function is not bounded on the interval $(0,1]$. Also we can not apply the Fundamental Theorem of Calculus to this function directly. In this case we use a special property to define our integral,

$$\int_a^b f = \lim_{c \to a} \int_c^b f.$$

We refer to this as a improper integral. However in the theory of the generalized Riemann integral there is no such thing as an improper integral. This fact alone shows us that the class of functions which are generalized Riemann integrable is far greater. These ideas will be discussed later in the paper.

Another type of integral, commonly learned at the graduate level, is called the Lebesgue integral, introduced by Henri Lebesgue in 1902. This integral is more powerful than the standard Riemann integral. The Lebesgue technique integrates over the function's range instead of its domain. This does allow for a larger class of functions to be Lebesgue integrable, but even the Lebesgue integral has its drawbacks. The one major drawback is that the Lebesgue integral, much like the standard Riemann, does not always guarantee that a function's derivative is integrable.

These drawbacks present another issue that needs to be addressed. The Fundamental Theorem of Calculus requires in its statement that a function $f$ is continuous. However as discussed above, not every derivative is Riemann integrable. As will be shown later, the restrictions to continuous functions is unnecessary if the generalized Riemann integral is used.

The generalized Riemann integral, also known as the Henstock-Kurzweil integral or "The Integral", is a solution to a lot of these issues. First introduced in the 1960's, Ralph Henstock and Jaroslav Kurzweil developed this integral, with Kurzweil introducing the definition and Henstock fully developing the integral. The Henstock or "HK" integral is defined on an even larger class of functions than Riemann or Lebesgue. Figure 1 below is a Venn diagram of the different types of integration. As you can see, the generalized Riemann includes the class of Riemann integrable functions and the class of Lebesgue integrable functions. If a function is Riemann or Lebesgue integrable, then it is also HK integrable . With the Henstock integral, every derivative is integrable, and there is a more general version of the Fundamental Theorem of Calculus that will be discussed later. The generalized Riemann integral can integrate the salt and pepper function and a number of others. Once some of the basics of the generalized Riemann integral are discussed, we will come back to some examples of this integral's power.

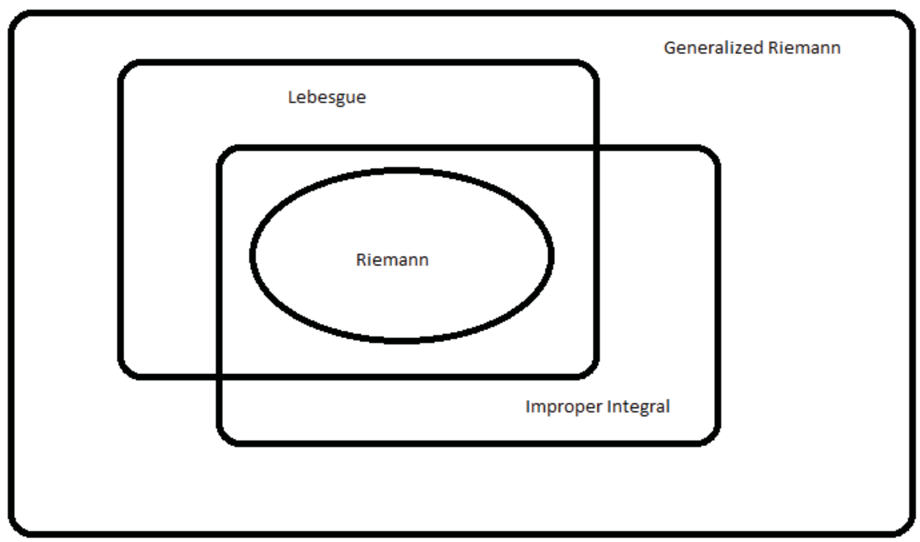


Figure 1: Relation between different types of integration

The theorems, exercises and definitions show in this paper where taken from different sources included in the reference section.

### 1.1 Basic Definitions

Next are some definitions and terms that will be referred to throughout the paper. The first definition we will present is a gauge. You can think about a gauge geometrically as the radius of a neighborhood.

**Definition 1.1.** *If $I = [a, b] \subset \mathbb{R}$, then a function $\delta : I \mapsto \mathbb{R}$ is a gauge on $I$ if $\delta(t) > 0$ for all $t \in I$. The interval $[t - \delta(t), t + \delta(t)]$ is known as the interval controlled by the gauge.*

We will see later in this paper that even though not all gauges that can be constructed are useful, defining any gauge on an interval will give us a $\delta$-fine partition of said interval.

**Definition 1.2.** *If $\{[u_1, v_1], ..., [u_n, v_n]\}$ is a finite collection of pairwise non-overlapping subintervals of $[a, b]$ such that $[a, b] = \bigcup_{k=1}^{n} [u_k, v_k]$, we say that $\{[u_1, v_1], ..., [u_n, v_n]\}$ is a partition of $[a, b]$.*

**Definition 1.3.** *A point-interval pair $(t, [u, v])$ consists of a point $t \in \mathbb{R}$ and an interval $[u, v]$ in $\mathbb{R}$. Here $t$ is known as the tag of $[u, v]$.*

**Definition 1.4.** *A tagged partition of $[a, b]$ is a finite collection of point-interval pairs $\{(t_1, [u_1, v_1]), ..., (t_n, [u_n, v_n])\}$, where $\{[u_1, v_1], ..., [u_n, v_n]\}$ is a partition of $[a, b]$ and $t_k \in [u_k, v_k]$ for $k = 1, 2, ..., n$.*

**Definition 1.5.** *Let $\delta$ be a gauge on $[a, b]$. A tagged partition $\{(t_1, [u_1, v_1]), ..., (t_n, [u_n, v_n])\}$ of $[a, b]$ is said to be $\delta$-fine if $[u_k, v_k] \subset (t_k - \delta(t_k), t_k + \delta(t_k))$ for $k = 1, 2, ..., n$.*

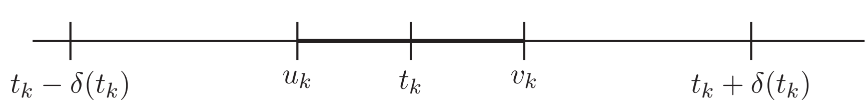


Figure 2: A geometric representation of $\delta$-fineness

Above is a geometric representation of a $\delta$-fine tagged partition. We should note that the $[u_k, v_k]$ is a subset of the interval controlled by the gauge $\delta(t_k)$.

Next we present a proof that for every gauge $\delta$ on a closed, bounded interval, there exists a $\delta$-fine tagged partition. The following theorem is sometimes referred to as the "Fineness Theorem". This proof will also aid in proving the two dimensional case later in the paper.

**Theorem 1.6.** *Cousin's Theorem: Let $I \subset \mathbb{R}$ be a closed bounded interval. For every gauge $\delta$ there exists a $\delta$-fine tagged partition of $P$ on $I$.*

Proof: Let I = [a,b] be a bounded interval. Define the set E by

$$E = \{x \in I : a < x \leq b \text{ and there exists a } \delta\text{-fine tagged partition of } [a, x] \}.$$

$E \neq \emptyset$ since for any $x$ in $(a, a + \delta(a))$, $\{(a, [a, x])\}$ is a $\delta$-fine partition of $[a, x]$

Let $y = \sup(E)$. By definition of the supremum, there exists $x \in E$ such that $y - \delta(y) < x \leq y$. Since $x \in E$, there exists a $\delta$-fine partition $P_1$ of $[a, x]$. If $y = b$, then $P_1 \cup \{(y, [x, y])\}$ is a $\delta$-fine partition of $[a, b]$. If $y < b$, choose $z$ in $(y, y + \delta(y)) \cap [a, b]$. Then $P_1 \cup \{(y, [x, z])\}$ is a $\delta$-fine partition of $[a, z]$, which is not possible since $z \in E$ and $z > y =$.

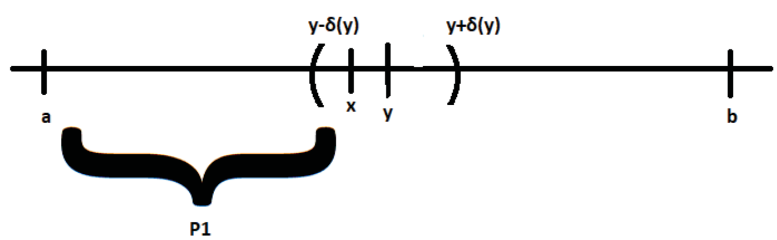


Figure 3: Defining a set $E$

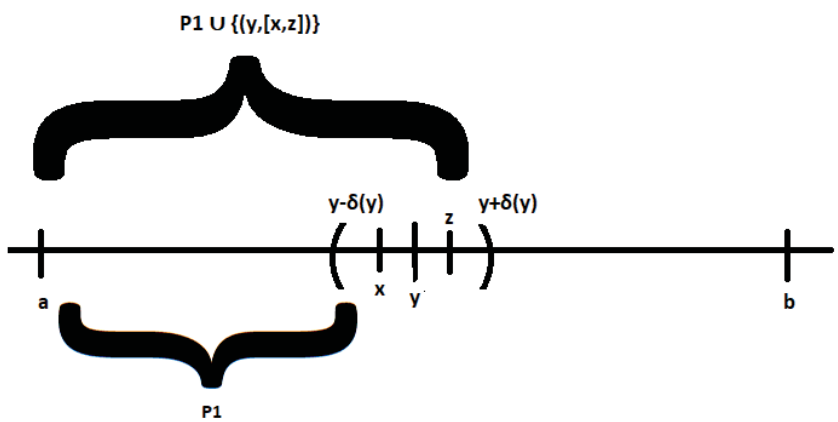


Figure 4: Cousin's lemma for $y < b$

The figures above give a visual idea of whats happening in this proof. Figure 3 show what is taking place after we define our set $E$ with $\delta$-fine partition $P_1$ of $[a, x]$. Figure 4 shows the $y < b$ situation where our contradiction takes place.

We will show later that this same method can be used to prove the two dimensional case.

It is important to address a question that many might have. Why is a gauge that is not a constant better than the traditional constant gauge. This question has a somewhat complicated answer.

In an attempt to answer this question, first let us look at how the approach to the standard Riemann integral defines fineness. For the standard Riemann integral, the measure of fineness defined for a partition is the maximum length of its subintervals. So all subinterval lengths are less than or equal to some number. In addition to this, the tag for each subinterval is chosen after the partition has been created. Using the generalized Riemann approach one has more variation in how they can choose the length of subintervals. This is possible if the function is changing fast, either increasing or decreasing, over a subinterval of small length. The length of a subinterval would not need to be small if the function over those subintervals was constant or close to it. Knowing this, let us explore the question.

One of the great advantage of using a non-constant gauge is that you can take intervals of a set of points that is finite or countable whose union has small length. Because the total length of these intervals are arbitrarily small, their lengths end up not contributing to the Riemann sum in any significant way. An example of this would be the 'salt and pepper" function discussed above. As discussed the function is not integrable by the standard Riemann method of integration, however we can integrate this function on the entire interval $[0, 1]$ using the generalized Riemann theory. This requires us to pick an appropriate gauge $\delta$ that can make the Riemann sums less than epsilon, for any $\delta$-fine partition. This is done by choosing $\delta$ to be arbitrarily small at the rationals. For all the irrational point it does not matter how we define $\delta$, since the function value at the irrational points is zero, thus those terms do not contribute to the Riemann sum. The proof of this function is discussed in full in a later section.

Another advantage to using a non constant gauge is that by choosing a certain gauge we can force a partition to have a specific tag. By doing this we can control certain "bad" points that would normally make the function non-integrable. An example of this would be the function $f(x) = \dfrac{1}{\sqrt{x}}$ for $x \in (0, 1]$ and $f(0) = 0$. This function falls apart at zero so it is not integrable through the standard Riemann method. However if we choose a gauge

$\delta$ correctly, we can force the function value at the first tag to be zero. This will make any terms of the Riemann sum with this tag equal to zero. On the remaining part of the intervals the function $f$ will be bounded and continuous and thus the function is now integrable. The full proof of this problem is given later in the paper.

A final note would be that a non-constant gauge will help us define a improved, more general statement for the Fundamental Theorem of Calculus without some of the restrictions that come with the standard Riemann approach. This is discussed in the Fundamental Theorem section, section 1.4, of the paper.

### 1.2 Definition of the generalized Riemann integral

It is important for us to take some time to go over the definitions of both the standard Riemann integral and the generalized Riemann integral. Much like the standard Riemann integral, the generalized Riemann is defined as a limit of Riemann sums with only one slight difference. The difference in the definitions may appear to be small, but it has an enormous consequence.

Note that this definition is not the standard formulation of the Riemann integral. This definition has been modified so that the differences in the two definitions will be easier to notice. The definition is an equivalent definition to the Riemann integral you would see in an analysis textbook.

**Definition 1.7.** *Riemann sum: Let* $P = \{(t_1, [u_1, v_1]), ..., (t_n, [u_n, v_n])\}$ *be a* $\delta$*-fine tagged partition of* $[a, b]$*. If* $f : [a, b] \to \mathbb{R}$ *we define the Riemann sum* $A(f, P)$ *by*

$$A(f, P) = \sum_{k=1}^{n} f(t_k)(v_k - u_k)$$

**Definition 1.8.** *Standard Riemann integral: Let* $f : [a, b] \to \mathbb{R}$ *be some function. Let* $L$ *be some number. We say that* $L$ *is the Riemann integral of* $f$*, written* $L = \int_a^b f(t)dt$*, if for each number* $\epsilon > 0$ *there exists a number* $\delta > 0$ *such that for any* $\delta$*-fine partition* $P$ *of* $[a, b]$*,*

$$|A(f, P) - L| < \epsilon.$$

Now a small change to the definition of the Standard Riemann integral will produce the definition of generalized Riemann integral.

**Definition 1.9.** *Generalized Riemann integral: Let $f : [a, b] \to \mathbb{R}$ be some function. Let $L$ be some number. We say that $L$ is the generalized Riemann integral or gauge integral of f, written $L = \int_a^b f(t)dt$, if for each number $\epsilon > 0$ there exists a* ***function*** *$\delta(x) > 0$ such that for any $\delta$-fine partition $P$ of $[a, b]$,*

$$|A(f, P) - L| < \epsilon.$$

The only difference between the two definitions is how $\delta$ is defined. For the standard Riemann integral $\delta$ is defined as some constant. The generalized definition allows $\delta$ to be a positive valued function. This function $\delta$ is called the gauge. This is but one way to define the generalized Riemann integral, but for the purposes of this paper, this is the definition that will be used.

Before we give some examples, we will show that the value of the generalized Riemann integral (if it exists) of a function is unique.

**Theorem 1.10.** *Uniqueness: There is at most one number $L$ that satisfies the definition of the generalized Riemann integral of a function $f : [a, b] \to \mathbb{R}$.*

Proof: Let $\epsilon > 0$ and suppose some numbers $L_1$ and $L_2$ satisfy the definition of the generalized Riemann integral.

We wish to show that $L_1 = L_2$.

$L_1$ is a number that satisfies the definition of the generalized Riemann integral, so we know that there exists a gauge $\delta_1$ on $[a, b]$ such that

$$|A(f, P_1) - L_1| < \frac{\epsilon}{2}$$

for each $\delta_1$-fine tagged partition $P_1$ of $[a, b]$. Also $L_2$ is a number that satisfies the definition of the generalized Riemann integral, we know that there exist a gauge $\delta_2$ on $[a, b]$ such that

$$|A(f, P_2) - L_2| < \frac{\epsilon}{2}$$

for each $\delta_2$-fine tagged partition $P_2$ of $[a, b]$. Now let a gauge $\delta$ on $[a, b]$ be defined as

$$\delta(x) = \min\{\delta_1(x), \delta_2(x)\}.$$

Let $P$ be a $\delta$-fine tagged partition of $[a, b]$. Then $P$ is $\delta_1$-fine and $\delta_2$-fine, so

$$
\begin{aligned}
|L_1 - L_2| &= |L_1 - A(f,P) + A(f,P) - L_2| \\
&\leq |L_1 - A(f,P)| + |A(f,P) - L_2| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary it follows that $L_1 = L_2$.

Throughout the rest of the paper

$$L = \int_a^b f(x)dx = \int_a^b f.$$

## 1.3 Some examples of the generalized Riemann integral

Now that we have introduced the definition of the generalized Riemann integral, it is time to see some examples of its utility. Consider the Dirichlet function mentioned above.

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

As discussed above, this function is nowhere continuous and not Riemann integrable. This function however can be integrated using the generalized Riemann method.

We will show that $\int_0^1 f = 0$

As discussed above, the idea here is that we will define a gauge $\delta$ such that $\delta$ is sufficiently small on the rational numbers of the interval $[0,1]$. The value of gauge does not matter at the irrationals.

Proof: Let $\epsilon > 0$ and let $\delta(x)$ be a positive-valued function for all $x \in [0,1]$ defined by,

$$\delta(x) = \begin{cases} \frac{\epsilon}{2^{i+1}}, & \text{if } t = r_i \in \mathbb{Q} \\ 1, & \text{if } t \in \mathbb{R} - \mathbb{Q} \end{cases}.$$

Let $P$ be a $\delta$-fine partition of $[0,1]$ where $(t_k, [u_k, v_k])$ is a point-interval pair and $t_k$ is a tag. Let the sequence $\{r_i : i \in \mathbb{N}\}$ be the set of rationals on $[0,1]$.

If a tag $t_k \in [u_k, v_k]$ is rational then $f(t_k) = 1$ and if $t_k$ is irrational then $f(t_k) = 0$. If a rational number $r_i$ is the tag for $[u_k, v_k]$, then $[u_k, v_k] \subset [r_i - \delta(r_i), r_i + \delta(r_i)]$, implying $l([u_k, v_k]) \leq \frac{\epsilon}{2^i}$. Note that $r_i$ could be in two intervals $I$ and $J$ if the intervals are adjacent. In this case $I \cup J \subset [r_i - \delta(r_i), r_i + \delta(r_i)]$ and $l(I \cup J) \leq \frac{\epsilon}{2^i}$. Therefore,

$$|A(f,P)-0| = \sum_{k=1}^{n} f(t_k)(v_k-u_k) \le \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon$$

.

Since $\epsilon > 0$ is arbitrary, the function is integrable and the claim holds.

Going back to an example discussed earlier, we can also force tags to be at certain points that would normally give us difficulty. The next proof is a simple example of this process.

**Example:** We will show that the function $f$, defined by $f(x) = \frac{1}{\sqrt{x}}$ for $x \neq 0$ and $f(0) = 0$, is integrable on the interval $[0,1]$.

In particular we will show that $\int_0^1 f = 2$. Let $\epsilon > 0$. We may assume that $\epsilon < 3/4$. Define the gauge $\delta$ on $[0,1]$ as

$$\delta(x) = \begin{cases} \epsilon x^2, & \text{if } x \in (0,1] \\ \epsilon^2, & \text{if } x = 0. \end{cases}$$

Let $P$ be a $\delta$-fine tagged partition of $[0,1]$. For each $x > 0$, $x - \epsilon x^2 > 0$, so one of our tags must be zero. Let $(x,[c,d]) \in P$ such that $x > 0$. Then $c > x - \epsilon x^2 \geq x - \epsilon x > x/4$, so $\sqrt{x}(\sqrt{x}+\sqrt{c})^2 > \sqrt{x}(\sqrt{x}+\sqrt{x/4})^2 = \frac{9x\sqrt{x}}{4} > x\sqrt{x} \geq x^2$.

Now it follows that

$$\left|\frac{2\sqrt{x}-2\sqrt{c}}{x-c} - \frac{1}{\sqrt{x}}\right| = \left|\frac{2(\sqrt{x}-\sqrt{c})}{x-c} - \frac{1}{\sqrt{x}}\right|$$

$$\left|\frac{2}{\sqrt{x}+\sqrt{c}} - \frac{1}{\sqrt{x}}\right|$$

$$= \left|\frac{2\sqrt{x}-\sqrt{x}-\sqrt{c}}{\sqrt{x}(\sqrt{x}+\sqrt{c})}\right|$$

$$= \frac{\sqrt{x}-\sqrt{c}}{\sqrt{x}(\sqrt{x}+\sqrt{c})}$$

$$= \frac{x-c}{\sqrt{x}(\sqrt{x}+\sqrt{c})^2}$$

$$< \frac{\epsilon x^2}{x^2}$$

$$= \epsilon.$$

A similar argument can be made to show that

$$\left|\frac{2\sqrt{d}-2\sqrt{x}}{d-x}-\frac{1}{\sqrt{x}}\right|<\epsilon.$$

Now we combine these inequalities together and use the triangle inequality,

$$\begin{aligned}
&\left|\frac{1}{\sqrt{x}}(d-c)-(2\sqrt{d}-2\sqrt{c})\right| \\
&\leq\left|\frac{1}{\sqrt{x}}(d-x)-(2\sqrt{d}-2\sqrt{x})\right|+\left|\frac{1}{\sqrt{x}}(x-c)-(2\sqrt{x}-2\sqrt{c})\right| \\
&=\left|\frac{1}{\sqrt{x}}-\frac{(2\sqrt{d}-2\sqrt{x})}{d-x}\right|(d-x)+\left|\frac{1}{\sqrt{x}}-\frac{(2\sqrt{x}-2\sqrt{c})}{x-c}\right|(x-c) \\
&<\epsilon(d-x)+\epsilon(x-c) \\
&=\epsilon(d-c).
\end{aligned}$$

Now let $E=\{(x_i,[c_i,d_i]):1\leq i\leq n\}$ and $P=(x_o,[c_o,d_o])\cup E$, where $x_o=0$. Then

$$\begin{aligned}
|A(f,P)-2| &= \left|\sum_{i=1}^{n} f(x_i)(d_i-c_i)-\sum_{i=0}^{n}(2\sqrt{d_i}-2\sqrt{c_i})\right| \\
&\leq 2\sqrt{d_o}+\sum_{i=0}^{n}\left|\frac{1}{\sqrt{x_i}}(d_i-c_i)-2(\sqrt{d_i}-\sqrt{c_i})\right| \\
&< 2\epsilon+\sum_{i=0}^{n}\epsilon(d_i-c_i) \\
&< 3\epsilon.
\end{aligned}$$

Since $\epsilon>0$ is arbitrary, the statement holds.

## 1.4 Fundamental Theorem for the generalized Riemann integral

We can also formulate a Fundamental Theorem of Calculus for the generalized Riemann integral. One might assume that it would be more difficult to evaluate a more general integral than it would to evaluate a standard Riemann integral. However this is not true, and you will see that many of the evaluation rules are the same for both. It is also the case that while not all derivatives of functions are Riemann or even Lebesgue integrable, they are integrable using the generalized theory. Thinking back to the standard Riemann integral, a function $f$ must be continuous. For the Lebesgue integral, $f$ must be absolutely continuous. These restrictions force us to make these a requirement when applying the Fundamental Theorem of Calculus to either the Riemann or Lebesgue integrals. There are no such requirements on the generalized Riemann integral. This distinguishes the generalized Riemann from the others. As a consequence the generalized Riemann allows a larger body of functions not integrable by Riemann or Lebesgue to be integrable. The reader will see that the Fundamental Theorem of Calculus under the generalized Riemann integral is far superior to anything the standard Riemann integral has to offer.

It is important to mention two lemmas that will need to be used. The Straddle lemma and Henstock lemma play key parts in establishing a fundamental theorem. The first lemma, the Straddle Lemma, does literally what the name suggests. Here two points $s$ and $t$ "straddle" a point of differentiability $x$. As the interval of $[s,t]$ becomes smaller as it straddles $x$ the slope of the secant line will become arbitrarily close to that of the tangent. The other lemma needed is the Henstock lemma, or Saks-Henstock lemma. This lemma contributes not only to the proof of the Fundamental Theorem, but also to more of the advanced theorems concerning the generalized Riemann integral. The statement and proof of the Henstock lemma will appear in a subsequent section.

**Theorem 1.11.** *Let $f:[a,b]\to\mathbb{R}$. If $f=0$ almost everywhere on $[a,b]$ then $f$ is integrable on $[a,b]$ and $\int_a^b f=0$.*

Proof: Let $N\subset[a,b]$ on which $f(x)\neq 0$. Then $\mu(N)=0$. So for $\epsilon>0$ there exists intervals $I_i$ such that $N\subset\cup I_i$ and $\sum l(I_i)<\epsilon$. Define the gauge $\delta$ on $[a,b]$ by

$$\delta(x)=\begin{cases}1, \text{if } x\notin N\\ \dfrac{l(I_i)}{f(x)}, \text{if } x\in I_i\cap N\end{cases}$$

Then for any $\delta$-fine partition $P=\{(x_i,J_i):i=1,2,...,n\}$

$$\begin{aligned}|A(f,P)|=\left|\sum_{i=1}^{n} f(x_i)l(J_i)\right| &= \left|\textstyle\sum_{x_i\in N} f(x_i)l(J_i)\right|\\ &\leq \sum_{x_i\in N}\left|f(x_i)\frac{l(I_i)}{|f(x_i)|}\right|\\ &=\sum l(I_i)\\ &<\epsilon.\end{aligned}$$

**Theorem 1.12.** *Straddle Lemma: Let f: $[a,b] \to \mathbb{R}$ be differentiable at $x \in [a,b]$. For any $\epsilon > 0$, there exists a $\delta > 0$ such that if $[s,t] \in [a,b]$, and $x - \delta \leq s \leq x \leq t \leq t + \delta$, then*

$$|f(t) - f(s) - f'(x)(t-s)| \leq \epsilon(t-s).$$

Proof: Let $\epsilon > 0$. By the definition of the derivative $f'(x)$ there exists $\delta > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y} - f'(x)\right| < \epsilon$$

for all $y \in [a,b]$ such that $0 < |x - y| < \delta$

Rearranging the inequalilty above we get

$$|f(x) - f(y) - f'(x)(x-y)| \leq \epsilon|x-y|$$

By the hypothesis $s \leq x \leq t$, which implies $t - x \geq 0$ and $x - s \geq 0$. So

$$\begin{aligned}
|f(t) - f(s) - f'(x)(t-s)| &= |f(t) - f(s) - f'(x)(t-s) - f'(x) + f'(x)| \\
&\leq |f(t) - f'(x) - f'(x)(t-x)| + |f(s) - f'(x) - f'(x)(s-x)| \\
&\leq \epsilon|t-x| + \epsilon|x-s| \\
&= \epsilon(t-x) + \epsilon(x-s) \\
&= \epsilon(t-s).
\end{aligned}$$

This proves the inequality above.

**Theorem 1.13.** *Fundamental Theorem of Calculus: Let $f : [a,b] \to \mathbb{R}$ and let $F$ be a continuous function in $[a,b]$. If $F$ is differentiable on $(a,b)$ such that $F'(t) = f(t)$ for all $t \in (a,b)$, then $f$ is integrable on $[a,b]$ and*

$$\int_a^b f = F(b) - F(a).$$

Proof: Let $\epsilon > 0$ and let $F$ be differentiable on $[a,b]$. For $t \in [a,b]$, let $\delta(t) > 0$ be the value given by the Straddle Lemma for the given $\frac{\epsilon}{(b-a)}$. Let $P = \{(t_k, [u_k, v_k]) : k = 1, 2, ..., n\}$ be a $\delta$-fine partition of $[a,b]$. These points $u_k$ and $v_k$ straddle the tag $t_k$, so by the Straddle Lemma we lemma

$$\begin{aligned}
&|F(v_k) - F(u_k) - F'(t_k)(v_k - u_k)| \\
&\leq \frac{\epsilon(v_k - u_k)}{2(b-a)}.
\end{aligned}$$

Summing all the terms together over our tagged partition, we get,

$$\begin{aligned}
&|F(b) - F(a) - A(f, P)| \\
&= \left|\sum_{k=1}^{n}(F(v_k) - F(u_k)) - \sum_{k=1}^{n} F'(t_k)(v_k - u_k)\right| \\
&= \left|\sum_{k=1}^{n} F(v_k) - F(u_k) - F'(t_k)(v_k - u_k)\right| \\
&\leq \sum_{k=1}^{n} \left|F(v_k) - F(u_k) - F'(t_k)(v_k - u_k)\right| \\
&\leq \sum_{k=1}^{n} \frac{\epsilon(v_k - u_k)}{2(b-a)} \\
&= \frac{\epsilon(b-a)}{2(b-a)} \\
&\leq \frac{\epsilon}{2} \\
&< \epsilon.
\end{aligned}$$

As stated earlier, the generalized Riemann integral is more powerful than its standard Riemann and Lebesgue counterparts. This theorem shows why.

## 1.5 Basic Properties of the generalized Riemann Integral

So far we have reviewed basic terminology and definitions of the generalized Riemann integral and some examples of it in action. The next step is to talk about some basic properties of the integral. Using the definition of the generalized Riemann integral, we will show some of its important properties. Fortunately, a lot of these properties are similar to properties of the standard Riemann integral. We will use this section to highlight some of these useful properties.

**Theorem 1.14.** *If $f$ is integrable on an interval $[a, b]$, then $f$ is integrable on every subinterval of $[a, b]$.*

Proof: Let $\epsilon > 0$, and let the interval $[c, d] \subset [a, b]$. Choose a gauge $\delta$ on [a,b] such that for any $\delta$-fine tagged partition $P$ of $[a, b]$,

$$|A(f, P_1) - \int_a^b f| < \frac{\epsilon}{2},$$

Then choose $\delta$-fine tagged partitions $P_{ac}$ of the interval $[a, c]$ and $P_{db}$ of $[d, b]$. Now let $P_1'$ and $P_2'$ be $\delta$-fine tagged partitions of $[c, d]$. Then

$$P_1 = P_{ac} \cup P_1' \cup P_{db} \text{ and } P_2 = P_{ac} \cup P_2' \cup P_{db}$$

are $\delta$-fine tagged partitons of $[a,b]$. Since $P_1$ and $P_2$ are tagged partitions that are $\delta$-fine,

$$|A(f,P_1') - A(f,P_2')| = |A(f,P_1) - A(f,P_2)| < \epsilon.$$

So by the Cauchy criterion (to be proven later), $f$ is integrable on $[c,d]$. Therefore the $f$ is integrable on all subintervals of $[a,b]$.

**Theorem 1.15.** *Let $f$, $g$ be integrable on $[a,b]$, then $f+g$ is integrable on $[a,b]$ and*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Proof: Let $\epsilon > 0$. $f$ and $g$ are integrable functions, so there exist gauges $\delta_1(x)$ and $\delta_2(x)$ such that

$$\left| A(f,P_1) - \int_a^b f \right| < \frac{\epsilon}{2}$$

and

$$\left| A(g,P_2) - \int_a^b g \right| < \frac{\epsilon}{2}$$

where $P_1$ is a $\delta_1$-fine tagged partition of $[a,b]$ and $P_2$ is a $\delta_2$-fine tagged partition of $[a,b]$.

Define the gauge

$$\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$$

on $[a,b]$, and let $P$ be a $\delta$-fine tagged partition of $[a,b]$. $P$ is both $\delta_1$-fine and $\delta_2$-fine, so $A(f+g,P) = A(f,P) + A(g,P)$ and

$$\begin{aligned} &\left| A(f+g,P) - \left( \int_a^b f + \int_a^b g \right) \right| \\ \leq &\left| A(f,P) - \int_a^b f \right| + \left| A(g,P) - \textstyle\int_a^b g \right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence, $f+g$ is integrable and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

**Theorem 1.16.** *Let $f$ be integrable on $[a,b]$ and $c \in \mathbb{R}$, then $cf$ is integrable on $[a,b]$ and*

$$\int_a^b cf = c\left(\int_a^b f\right)$$

Proof: Let $\epsilon > 0$. For $c = 0$ the result is clear. For $c \neq 0$ we know that $f$ is integrable on $[a, b]$, so there exists a gauge $\delta$ on $[a, b]$ such that

$$\left|A(f,P) - \int_a^b f\right| < \frac{\epsilon}{|c|}$$

for each $\delta$-fine tagged partition $P$of $[a, b]$. For such a partition of $[a, b]$ we have $A(cf, P) = cA(f, P)$; so

$$\begin{aligned}&\left|A(cf,P) - c\int_a^b f\right| \\ &= |c|\left|A(f,P) - \int_a^b f\right| \\ &< |c|\frac{\epsilon}{|c|} \\ &= \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that $cf$ is integrable and

$$\int_a^b cf = c\left(\int_a^b f\right).$$

**Theorem 1.17.** *Let $f$, $g$ be integrable on $[a, b]$ and let $f(x) \leq g(x)$. Then*

$$\int_a^b f \leq \int_a^b g$$

Proof: Let $\epsilon > 0$. Since $f$ and $g$ are integrable functions, we know there exist gauges $\delta_1(x)$ and $\delta_2(x)$ such that

$$\left|A(f,P_1) - \int_a^b f\right| < \frac{\epsilon}{2}$$

and

$$\left|A(g,P_2) - \int_a^b g\right| < \frac{\epsilon}{2}$$

where $P_1$ and $P_2$ are $\delta_1$-fine and $\delta_2$-fine tagged partitions of $[a, b]$ respectively.

Define the gauge

$$\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$$

on $[a, b]$. Let $P$ be a $\delta$-fine tagged partition of $[a, b]$ which is also $\delta_1$-fine and $\delta_2$-fine. Since $f \leq g$, we have the inequality

$$A(f, P) \leq A(g, P).$$

We know that

$$\left| A(f, P_1) - \int_a^b f \right| < \frac{\epsilon}{2}$$

which implies

$$-\frac{\epsilon}{2} < A(f, P) - \int_a^b f < \frac{\epsilon}{2}$$
$$\implies -\frac{\epsilon}{2} + \int_a^b f < A(f, P) < \int_a^b f + \frac{\epsilon}{2}.$$

A similar inequality holds for $A(g, P)$. So

$$-\frac{\epsilon}{2} + \int_a^b f < A(f, P) \leq A(g, P) < \int_a^b g + \frac{\epsilon}{2}$$
$$\implies -\frac{\epsilon}{2} + \int_a^b f \leq \int_a^b g + \frac{\epsilon}{2}$$
$$\implies \int_a^b f \leq \int_a^b g + \epsilon$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\int_a^b f \leq \int_a^b g.$$

This Cauchy criterion is a common tool used when trying to show existence of an integral even when there is no given value of an integral. The Cauchy criterion will be used in the PROOFS of some theorems later on.

**Theorem 1.18.** *Cauchy Criterion: A function* $f : [a, b] \to \mathbb{R}$ *is integrable on* $[a, b]$ *if and only if for each* $\epsilon > 0$ *there exists a gauge* $\delta > 0$ *on* $[a, b]$ *such that*

$$|A(f, P) - A(f, Q)| < \epsilon$$

*for all* $\delta$*-fine tagged partitions* $P$ *and* $Q$ *of* $[a, b]$.

Proof: ($\implies$) Let $\epsilon > 0$. Since $f$ is integrable, there exists a gauge $\delta$ on $[a, b]$ such that

$$\left| A(f, P_o) - \int_a^b f \right| < \frac{\epsilon}{2}$$

for any $\delta$-fine tagged partition $P_o$. Let $P$ and $Q$ be $\delta$-fine tagged partitions of $[a, b]$. Then

$$\begin{aligned}
&|A(f,P) - A(f,Q)| \\
&\leq \left|A(f,P) - \int_a^b f\right| + \left|\int_a^b f - A(f,Q)\right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

($\Longleftarrow$) For each $n \in \mathbb{N}$, let $\delta_n$ be a gauge on $[a,b]$ such that

$$|A(f,P_n) - A(f,Q_n)| < \frac{1}{n}$$

for $\delta_n$-fine tagged partitions $P_n$ and $Q_n$. Let $\delta(x) = \min(\delta_1(x), ..., \delta_n(x))$. Choose a number $N \in \mathbb{Z}^+$ such that $0 < \frac{1}{N} < \epsilon$. Consider the sequence $\{A(f,P_n)\}_{n=1}^{\infty}$ in $\mathbb{R}$, where each $P_n$ is a $\delta'$-fine tagged partition of $[a,b]$. If $n_1, n_2 \in \mathbb{Z}^+$ such that $\min\{n_1,n_2\} \geq N$, then $P_{n_1}$ and $P_{n_2}$ are $\delta'_{\min\{n_1,n_2\}}$-fine tagged partitions of $[a,b]$ and

$$|A(f,P_{n_1}) - A(f,P_{n_2})| \leq \frac{1}{N} < \epsilon$$

Hence $\{A(f,P_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and since $\mathbb{R}$ is complete this sequence converges to a real number M.

Now let $P$ be a $\delta_N$-fine tagged partition of $[a,b]$. The sequence $\{\delta'_n\}_{n=1}^{\infty}$ of gauges is non-increasing we can see that $P_n$ is $\delta'_N$-fine for integers $n \geq N$.

So, we have

$$|A(f,P) - M| = \lim_{n\to\infty} |A(f,P) - A(f,P_n)| \leq \frac{1}{N} < \epsilon$$

Since $\epsilon > 0$ is arbitrary, $f$ is integrable and $M = \displaystyle\int_a^b f$.

**Theorem 1.19.** *Let $f$ be integrable on each interval $[a,c]$ and $[c,b]$. Then $f$ is integrable on $[a,b]$ and*

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof: Let $\epsilon > 0$. By hypothesis $f$ is integrable on $[a,c]$ and $[c,b]$, so there exists a gauge $\delta_1(x)$ on $[a,c]$ such that

$$\left|A(f,P_1) - \int_a^c f\right| < \frac{\epsilon}{2}$$

for each $\delta_1$-fine tagged partition $P_1$ of $[a,c]$. Similarly there exists a gauge $\delta_2(x)$ on $[c,b]$ such that

$$\left|A(f,P_2) - \int_c^b f\right| < \frac{\epsilon}{2}$$

for each $\delta_2$-fine tagged partition $P_2$ of $[c, b]$.

Define the gauge $\delta$ on $[a, b]$ by

$$\delta(x) = \begin{cases} \min\{\delta_1(x), c - x\}, & \text{when } a < x < c \\ \min\{\delta_1(c), \delta_2(c)\}, & \text{when } x = c \\ \min\{\delta_2(x), x - c\}, & \text{when } c < x < b \end{cases}.$$

Suppose that each of our tags only occur once. Note that this definition of $\delta$ forces $c$ to be a tag in any tagged $\delta$-fine partition of $[a, b]$ and $P_1$ and $P_2$ be $\delta$-fine tagged partitions of $[a, c]$ and $[c, b]$, respectively. We can express these as $P_1 = P_a \cup \{(c, [u, c])\}$ and $P_2 = P_b \cup \{(c, [c, v])\}$ since both $P_1$ and $P_2$ contain a tagged interval whose tag is $c$. Let $P = P_1 \cup P_2$. Then $A(f, P_1) + A(f, P_2) = A(f, P)$.

So,

$$\begin{aligned} &\left|A(f,P) - \int_a^c f - \int_c^b f\right| \\ \leq &\left|A(f,P_1) - \int_a^c f\right| + \left|A(f,P_2) - \int_c^b f\right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Hence, the function $f$ is integrable on $[a, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

It is important to comment on a part of the proof above. In that proof we split our Riemann sums into two separate Riemann sums, one Riemann sum over the interval $[a, c]$ and one Riemann sum over the interval $[c, b]$. This can be done because we have chosen our gauge such that $c$ is forced to be the tag in the partitions. This is a useful method when working with the generalized Riemann integral.

## 1.6 Saks-Henstock Lemma

The next lemma is the Henstock Lemma, also known as the Saks-Henstock Lemma. This lemma is vital for ideas later on. Almost all of the major results for the generalized Riemann integral will use this lemma in their proofs.

Before stating and proving the Saks-Henstock lemma, we first must take care of some preliminaries. We will start by stating a few definitions.

**Definition 1.20.** *A finite collection of point interval pairs* $\{(t_1, [u_1, v_1], ...(t_p, [u_p, v_p])\}]$ *is called a tagged subpartition of* $[a, b]$ *if* $t_i \in [u_i, v_i]$ *for* $i = 1, 2, ..., p$ *and* $\{[u_1, v_1], ..., [u_p, v_p]\}$ *is a collection of finite non-overlapping subintervals of* $[a, b]$*.*

**Definition 1.21.** *Let* $\{(t_1, [u_1, v_1), ...(t_p, [u_p, v_p])\}$ *be a tagged subpartition of* $[a, b]$ *and let a gauge* $\delta$ *be defined on* $\{t_1, ..., t_p\}$*. The tagged subpartition* $\{(t_1, [u_1, v_1], ...(t_p, [u_p, v_p])\}$ *is* $\delta$*-fine if* $[u_i, v_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))]$ *for* $i = 1, 2, ..., p$*.*

**Theorem 1.22.** *Let* $f$ *be integrable on* $[a, b]$ *and let* $\epsilon > 0$*. If* $\delta$ *is a gauge on* $[a, b]$ *such that*

$$\left| \sum_{(x,[c,d])\in P} f(x)(d-c) - \int_a^b f \right| < \epsilon$$

*for each* $\delta$*-fine tagged partition* $P$ *of* $[a, b]$*, then*

$$\left| \sum_{(t,[u,v])\in Q} (f(t)(v-u) - \int_u^v f) \right| \leq \epsilon$$

*for each* $\delta$*-fine subpartition* $Q$ *of* $[a, b]$*.*

Proof: Let $Q$ be a $\delta$-fine tagged subpartition of $[a, b]$. Let $[c_1, d_1], ..., [c_n, d_n]$ be non-overlapping intervals such that

$$[a, b] - \bigcup_{(t,[u,v])\in Q} (u, v) = \bigcup_{i=1}^{n} [c_i, d_i]$$

Since f is integrable on $[a, b]$, f is integrable on each of its subintervals $[c_i, d_i]$. So for $\gamma > 0$ there exist a gauge $\delta_i$ on $[c_i, d_i]$ such that

$$\left| A(f, Q_i) - \int_{c_i}^{d_i} f \right| < \frac{\gamma}{n}$$

for each $\delta$-fine tagged partition $Q_i$ of $[c_i, d_i]$. Define $\delta_o$ by

$$\delta_o = \min\{\delta, \delta_i\}.$$

Let $P_i$ be a $\delta_o$-fine tagged partition of $[c_i, d_i]$. So now there is a $\delta$-fine tagged partition $P$ of $[a, b]$, defined by $P = Q \cup \left( \bigcup_{i=1}^{n} P_i \right)$ such that

$$A(f,P) = A(f,Q) + \sum_{i=1}^{n} A(f,P_i).$$

Also

$$\int_a^b f = \sum_{(t,[u,v])\in Q} \int_u^v f + \sum_{i=1}^{n} \int_{c_i}^{d_i} f$$

As a result of this, we get

$$\begin{aligned}
&\left| \sum_{(t,[u,v])\in Q} (f(t)(v-u) - \int_u^v f) \right| \\
&= \left| \left( A(f,P) - \sum_{i=1}^{n} A(f,P_i) \right) - \left( \int_a^b f - \sum_{i=1}^{n} \int_{c_i}^{d_i} f \right) \right| \\
&\leq \left| A(f,P) - \int_a^b f \right| + \sum_{i=1}^{n} \left| A(f,P_i) - \int_{c_i}^{d_i} f \right| \\
&< \epsilon + \sum_{i=1}^{n} \frac{\gamma}{n} \\
&= \epsilon + \gamma.
\end{aligned}$$

Since $\gamma$ can be chosen to be sufficiently small, the lemma follows.

In our definition of the generalized Riemann integral, a function on an interval requires that for $\epsilon > 0$ there exists a gauge $\delta$ on that interval such that if any partition $P$ is $\delta$-fine then $|A(f,P) - L| < \epsilon$. The Saks-Henstock lemma takes this a step further and states that we can approximate with equal accuracy the difference of any subset of terms from a Riemann sum and the sum of the integrals of $f$ over the subintervals of $I$.

**Theorem 1.23.** *Saks-Henstock: If $f$ is integrable on $[a,b]$, then for each $\epsilon > 0$ there exists a gauge $\delta$ on $[a,b]$ such that*

$$\sum_{(t,[u,v])\in P} \left| f(t)(v-u) - \int_u^v f \right| < \epsilon$$

*for every $\delta$-fine tagged subpartition $P$ of $[a,b]$.*

Proof: Since $f$ is integrable on the interval $[a,b]$, there exists a gauge $\delta$ on $[a,b]$ such that

$$\sum_{(x,[c,d])\in Q} \left| f(x)(d-c) - \int_c^d f \right| < \frac{\epsilon}{2}$$

for each $\delta$-fine tagged subpartition $Q$ of $[a,b]$. Let $P$ be a $\delta$-fine tagged subpartition of $[a,b]$ and define

$$P_o = \{(t,[u,v]) \in P : f(t)(v-u) - \int_u^v f \geq 0\}.$$

Let $P' = P - P_o$. It is clear that $P_o \cup P'$ is a $\delta$-fine tagged subpartition of $[a,b]$. Then

$$\begin{aligned} &\sum_{(t,[u,v])\in P} \left| f(t)(v-u) - \int_u^v f \right| \\ &= \sum_{(t,[u,v])\in P_o} \left( f(t)(v-u) - \int_u^v f \right) - \sum_{(t,[u,v])\in P'} \left( f(t)(v-u) - \int_u^v f \right) \\ &= \left| \sum_{(t,[u,v])\in P_o} f(t)(v-u) - \int_u^v f \right| + \left| \sum_{(t,[u,v])\in P'} f(t)(v-u) - \int_u^v f \right| \\ &< \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the theorem holds.

# 2 Generalized Riemann integral in two dimensions

## 2.1 Definitions

Now that we have covered some of the basic theory of the generalized Riemann integral in one dimension, the next step is to make the jump into higher dimensions. There are far more applications for integration in two or three dimensions than for the one dimensional case. This section investigates the theory of integration over two dimensions. The move to two dimensions will be the difficult step, but after we establish the theory for two dimensions, it would not be difficult to generalize it to higher dimensions. Most of the basic properties will be similar to those for the one dimensional case, however some of the proofs will require different approaches.

To start off the next section we will review some of the definitions that we defined in the first section, where the notation has changed due to the fact that we are working in multiple dimensions.

A typical element in $\mathbb{R}^2$ looks like $(x_1, x_2)$. This notation can be extended to $\mathbb{R}^n$, where $n$ is any integer greater than one. To do the multiple integration we will express $\mathbb{R}^2$ as a Cartesian product of two subspaces $\mathbb{R} \times \mathbb{R}$. We can also extend this into this into n-dimensions by letting $\mathbb{R}^n = \mathbb{R}^t \times \mathbb{R}^u$, where $t$ and $u$ are positive integers with $n = t + u$. For example, if $n = 3$ one way to express $\mathbb{R}^3$ would be as $\mathbb{R} \times \mathbb{R}^2$. Each number $z \in \mathbb{R}^n$ corresponds to a point $(x, y) \in \mathbb{R}^t \times \mathbb{R}^u$. So each point $(z_1, z_2, z_3) \in \mathbb{R}^3$ corresponds to a point $(z_1, (z_2, z_3)) \in \mathbb{R} \times \mathbb{R}^2$. It is very important to specify which of the factors of $\mathbb{R}^n$ go into $\mathbb{R}^t$ and $\mathbb{R}^u$ and their order when there a more than two.

An interval $I \subset \mathbb{R}^2$ is a two-dimensional interval $I = [a, b] \times [c, d]$. Since we are focusing on the two-dimensional case, our region of interest will be a rectangle with boundaries parallel to the coordinate axes. After the factors and their order have been set, then each of the intervals $I \subset \mathbb{R}^n$ will be expressed as $G \times H$ where $G \subset \mathbb{R}^t$ and $H \subset \mathbb{R}^u$.

The two two-dimensional iterated integrals we are investigating are

$$\int_H \int_G f(x, y) dx dy$$

and

$$\int_G \int_H f(x, y) dy dx.$$

In proving any theorems using double integrals, it is superfluous to prove them with both of the iterated integrals, so we will use $\int_H \int_G f(x, y) dx dy$. The values $x$ and $y$ may be vectors, so we should define the roles of $dx$ and $dy$. The the only purpose of $dx$ and $dy$ is to show which argument of our function is being used in an integration. For example, using the two fold integral agreed upon above, $\int_G f(x, y) dx$ is integrating $x$ in the function $f(x, y)$ for a fixed $y$ over $G$. That result will be a function $y \mapsto \int_G f(x, y) dx$.

**Definition 2.1.** *A function $\delta : I \mapsto \mathbb{R}^+$ is known as a gauge on $I$ where $I \subset \mathbb{R}^n$.*

**Definition 2.2.** *A point-interval pair $(t, J)$ consists of a point $t \in \mathbb{R}^n$ and a closed interval $J \in \mathbb{R}^n$ with $t \in J$.*

**Definition 2.3.** *A tagged partition of $I$ is a finite collection $\{(t_1, J_1), ..., (t_n, J_n)\}$ of point-interval pairs, where $I = J_1 \cup J_2 \cup ... \cup J_n$.*

**Definition 2.4.** *Riemann sum: Let $f$ be defined on $I$ and let $\{(t_1, J_1), ..., (t_n, J_n)\}$ be a tagged partition of $I$. Define the Riemann sum by*

$$A(f,P) = \sum_{i=1}^{n} f(t_i)\mu(J_i)$$

*where* $\mu([a,b] \times [c,d]) = l([a,b])l([c,d]) = (b-a)(d-c)$ *is the measure of the intervals corresponding to the length function.*

The next definitions are of the standard Riemann integral and the generalized Riemann integral. Since we have already defined them in the one dimensional case and there are not many differences for higher dimensions, the definitions will be abbreviated.

**Definition 2.5.** *Standard Riemann Integral: A function* $f : I \to \mathbb{R}$ *is said to be Riemann integrable on* $I \subset \mathbb{R}^n$ *if there exists an* $L \in \mathbb{R}$ *with the following property: for every* $\epsilon > 0$ *there exists a constant* $\delta > 0$ *on* $I$ *such that*

$$|A(f,P) - L| < \epsilon$$

*for each* $\delta$*-fine tagged partition* $P$ *of* $I$*.*

Now notice again that the only change from the standard definition to the generalized Riemann definition is how the gauge $\delta$ is defined.

**Definition 2.6.** *Generalized Riemann Integral: A function* $f : I \mapsto \mathbb{R}$ *is said to be generalized Riemann integrable on* $I \subset \mathbb{R}^n$ *if there exists an* $L \in \mathbb{R}$ *with the following properties: for* $\epsilon > 0$ *there exists a function* $\delta(x) > 0$ *on* $I$ *such that*

$$|A(f,P) - L| < \epsilon$$

*for each* $\delta$*-fine tagged partition* $P$ *of* $I$*.*

As in the one dimensional case, if a function is Riemann integrable on the interval $I$, then that function is also Henstock integrable on $I$.

Like the one-dimensional case above, we will prove uniqueness of the value of the generalized Riemann integral. As before, we will define this unique number $L$ as

$$L = \int_I f$$

**Theorem 2.7.** *Uniqueness: There is at most one number* $L$ *that satisfies the definition of the generalized Riemann integral of* $f$ *on* $I$*.*

Proof: Let $\epsilon > 0$ and let some numbers $L_1$ and $L_2$ satisfy the definition of the generalized Riemann integral $\int_I f$.

Since $L_1$ is a number satisfying the definition, we know that there exists a gauge $\delta_1$ on $I$ such that

$$|A(f, P_1) - L_1| < \frac{\epsilon}{2}$$

for each $\delta_1$-fine tagged partition $P_1$ of $[a, b]$. Also since $L_2$ is a number satisfying the definition, we know that there exists a gauge $\delta_2$ on $I$ such that

$$|A(f, P_2) - L_2| < \frac{\epsilon}{2}$$

for each $\delta_2$-fine tagged partition $P_2$ of $I$. Now let a gauge $\delta$ on $[a, b]$ be defined as

$$\delta(x, y) = \min\{\delta_1(x, y), \delta_2(x, y)\}.$$

Let $P$ be a $\delta$-fine tagged partition of $I$.Then $P$ is $\delta_1$-fine and $\delta_2$-fine, so

$$\begin{aligned} |L_1 - L_2| &= |L_1 - A(f, P) + A(f, P) - L_2| \\ &\leq |L_1 - A(f, P)| + |A(f, P) - L_2| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that $L_1 = L_2$.

Next we will discuss Cousin's Lemma in two dimensions. The existence of a $\delta$-fine tagged partition for any gauge on an interval is crucial, so it is best we establish this result early. This is a key cornerstone in the theory of the generalized Riemann integral and is one of the proofs that needs to be approached differently from the one-dimensional version. The move from one to two dimensions presents several difficulties, but after showing that Cousin's Lemma holds in two dimensions, we can easily generalize it into higher dimensions. For our purposes, we will focus on the two dimensional case in preparation for proving Fubini's Theorem.

**Theorem 2.8.** *Cousin's Lemma for $\mathbb{R}^2$: Let $I \subset \mathbb{R}^2$ and let $\delta$ be a gauge on $I$. Then there exists a $\delta$-fine partition $P$ of $I$.*

Proof: Let $I = [a,b] \times [c,d]$. We know that there exists a $\delta$-fine partition of $[a,b] \times \{c\}$. Define a set

$$T = \{u \in [c,d] : [a,b] \times [c,u] \text{ has a } \delta\text{-fine partition}\}.$$

Note: $T$ is a non-empty set since there exist an $h > 0$ such that $[a,b] \times [c, c+h]$ has a $\delta$-fine partition.

Now let $u = \sup T$.

We want to show that $u = d$. Assume $u < d$. On $[a,b] \times \{u\}$ there exists a $\delta$-fine partition. The rectangles associated with this partition can be chosen such that they are centered vertically around the tags $\{x_1, x_2, ..., x_n\}$ on the line $[a,b] \times \{u\}$. We will denote the rectangles as $(J_i \times K_i)$ each with tag $x_i$. Let $K = \cap K_i$. Then $J_i \times K \subset J_i \times K_i$ for each $i$, and $\{(x_i, J_i \times K) : i = 1, 2, ..., n\}$ is a $\delta$-fine partition of the strip $[a,b] \times K$, which is centered around the line segment $[a,b] \times u$. Let's call this partition $Q$. Note that $K = [u-h, u+h]$ for some $h > 0$.

We know that $[a,b] \times [c, u-h]$ has a $\delta$-fine partition $P$. Hence $[a,b] \times [c, u+h]$ has a $\delta$-fine partition $P \cup Q$. This produces a contradiction in our assumption since $u+h > u$. Therefore $u$ must equal $d$. We use this same construction technique with $u = d$ to demonstrate that there exists a $\delta$-fine partition of $[a,b] \times [c,d]$.

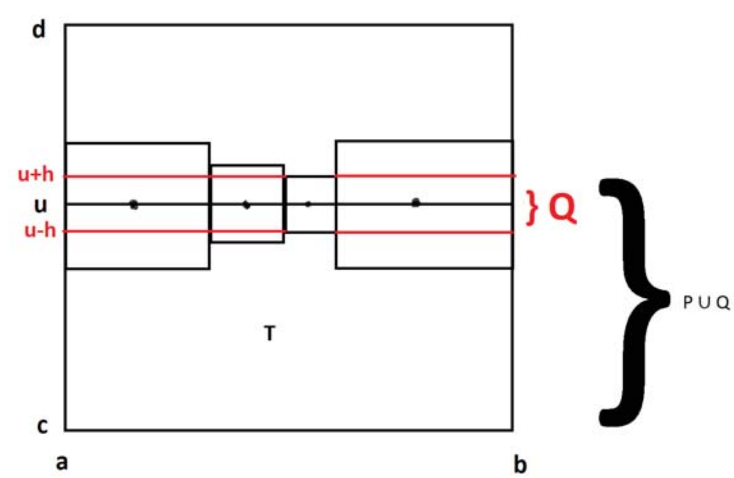


Figure 5: Cousin's lemma in $\mathbb{R}^2$

**Remarks:** The same idea of picking a supremum and then going past that point is used in both the one and two-dimensional proofs. We should however take a moment to examine the implications of the construction used in the proof. One of the features that makes this a special construction is how tags are picked. In the proof, all tags lie on horizontal lines. The construction also allows us to place some tags along a vertical line such as the y-axis if needed. The construction in the proof can be compared to laying bricks while building a wall. That is to say there exists a $\delta$-fine tagged partition of $[a, b] \times [c, d]$ that "tiles" horizontal strips but not necessarily vertical strips. This is a special case of a compounded partition.

## 2.2 Basic Properties in $\mathbb{R}^2$

In this next section, we will prove some of the basic properties touched on earlier using the definition of the generalized Riemann integral that has been defined for two dimensions.

**Theorem 2.9.** *Let $f$, $g$ be integrable on $I \subset \mathbb{R}^2$, then $f + g$ is integrable on $I$ and*

$$\int_I (f + g) = \int_I f + \int_I g.$$

Proof: Let $\epsilon > 0$ and suppose functions $f$ and $g$ are integrable. Then there exist gauges $\delta_1(x, y)$ and $\delta_2(x, y)$ such that

$$\left| A(f, P_1) - \int_I f \right| < \frac{\epsilon}{2}$$

and

$$\left| A(g, P_2) - \int_I g \right| < \frac{\epsilon}{2}$$

where $P_1$ is a $\delta_1$-fine tagged partition of $I$ and $P_2$ is a $\delta_2$-fine tagged partition of $I$.

Define the gauge $\delta$ on $I$ by

$$\delta(x, y) = \min\{\delta_1(x, y), \delta_2(x, y)\},$$

and let $P$ be a $\delta$-fine tagged partition of $I$. $P$ is $\delta_1$-fine and $\delta_2$-fine, so $A(f + g, P) = A(f, P) + A(g, P)$ and

$$\begin{aligned} &\left| A(f + g, P) - \left( \int_I f + \int_I g \right) \right| \\ \leq &\left| A(f, P) - \int_I f \right| + \left| A(g, P) - \textstyle\int_I g \right| \end{aligned}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$= \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $f + g$ is integrable and

$$\int_I (f+g) = \int_I f + \int_I g.$$

**Theorem 2.10.** *Let $f$ be integrable on $I \subset \mathbb{R}^2$ and $c \in \mathbb{R}$, then $cf$ is integrable on $I$ and*

$$\int_I cf = c\left(\int_I f\right).$$

Proof: Let $\epsilon > 0$. For $c = 0$ the result is clear. For $c \neq 0$ we know that $f$ is integrable on $I$, so there exists a gauge $\delta$ on $I$ such that

$$\left| A(f, P_1) - \int_I f \right| < \frac{\epsilon}{|c|}$$

for each $\delta$-fine tagged partition $P_1$ of $I$. Thus, if $P$ is a $\delta$-fine tagged partition of $I$, then

$$\left| A(cf, P) - c\int_I f \right|$$
$$= |c| \left| A(f, P) - \int_I f \right|$$
$$< |c| \frac{\epsilon}{|c|}$$
$$= \epsilon.$$

We can choose $\epsilon > 0$ arbitrarily, so $cf$ is integrable and

$$\int_I cf = c\left(\int_I f\right).$$

**Theorem 2.11.** *Let $f$, $g$ be integrable on $I \subset \mathbb{R}^2$ and suppose $f(x) \leq g(x)$. Then*

$$\int_I f \leq \int_I g.$$

Proof: Let $\epsilon > 0$. Since $f$ and $g$ are integrable functions, we know there exist gauges $\delta_1(x, y)$ and $\delta_2(x, y)$ such that

$$\left| A(f, P_1) - \int_I f \right| < \frac{\epsilon}{2}$$

and

$$\left| A(g, P_2) - \int_I g \right| < \frac{\epsilon}{2}$$

where $P_1$ and $P_2$ are respectively $\delta_1$-fine and $\delta_2$-fine tagged partitions of $I$.

Define the gauge $\delta$ on $I$ by

$$\delta(x,y) = \min\{\delta_1(x,y), \delta_2(x,y)\}.$$

Let $P$ be a $\delta$-fine tagged partition of $I$, which is also $\delta_1$-fine and $\delta_2$-fine. This implies the inequality

$$A(f,P) \leq A(g,P).$$

We know that

$$\left| A(f,P) - \int_I f \right| < \frac{\epsilon}{2},$$

which implies

$$\begin{aligned} -\frac{\epsilon}{2} < A(f,P) - \int_I f < \frac{\epsilon}{2} \\ \implies -\frac{\epsilon}{2} + \int_I f < A(f,P) < \int_I f + \frac{\epsilon}{2} \end{aligned}$$

Similarly for $A(g,P)$. So

$$\begin{aligned} -\frac{\epsilon}{2} + \int_I f < A(f,P) \leq A(g,P) < \int_I g + \frac{\epsilon}{2} \\ \implies -\frac{\epsilon}{2} + \int_I f \leq \int_I g + \frac{\epsilon}{2} \\ \implies \int_I f \leq \int_I g + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\int_I f \leq \int_I g.$$

**Theorem 2.12.** *Cauchy Criterion: A function $f : I \to \mathbb{R}$ is integrable on $I \subset \mathbb{R}^2$ if and only if for each $\epsilon > 0$ there exists a gauge $\delta(x) > 0$ on $I$ such that*

$$|A(f,P) - A(f,Q)| < \epsilon$$

*for all $\delta$-fine tagged partitions $P$ and $Q$ of $I$.*

Proof: ($\Longrightarrow$) Let $\epsilon > 0$. Since $f$ is integrable, there exists a gauge $\delta$ on $I$ such that

$$\left| A(f,P_o) - \int_I f \right| < \frac{\epsilon}{2}$$

for any $\delta$-fine tagged partition $P_o$. Let $P$ and $Q$ be $\delta$-fine tagged partitions of $I$. Then

$$\begin{aligned}
&|A(f,P) - A(f,Q)| \\
&\leq \left|A(f,P) - \int_I f\right| + \left|\int_I f - A(f,Q)\right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

($\Longleftarrow$) For each $n \in \mathbb{N}$, let $\delta_n$ be a gauge on $I$ such that

$$|A(f,P_n) - A(f,Q_n)| < \frac{1}{n}$$

for $\delta_n$-fine tagged partitions $P_n$ and $Q_n$. Let $\delta(x) = \min(\delta_1(x), ..., \delta_n(x))$. Choose a number $N \in \mathbb{Z}^+$ such that $0 < \frac{1}{N} < \epsilon$.Consider the sequence $\{A(f,P_n)\}_{n=1}^{\infty}$ in $\mathbb{R}$, where each $P_n$ is a $\delta'$-fine tagged partition of $I$. If $n_1, n_2 \in \mathbb{Z}^+$ such that $\min\{n_1, n_2\} \geq N$, then $P_{n_1}$ and $P_{n_2}$ are $\delta'_{\min\{n_1,n_2\}}$-fine tagged partitions of $[a,b]$ and

$$|A(f,P_{n_1}) - A(f,P_{n_2})| \leq \frac{1}{N} < \epsilon$$

Hence $\{A(f,P_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and since $\mathbb{R}$ is complete this sequence converges to a real number M.

Now let $P$ be a $\delta_N$-fine tagged partition of $I$. The sequence $\{\delta_n\}_{n=1}^{\infty}$ of gauges is non-increasing we can see that $P_n$ is $\delta_N$-fine for integers $n \geq N$.

So, we have

$$|A(f,P) - M| = \lim_{n\to\infty} |A(f,P) - A(f,P_n)| \leq \frac{1}{N} < \epsilon.$$

$\epsilon > 0$ is arbitrary, so $f$ is integrable and $M = \int_I f$.

## 3 Fubini's Theorem

In the first sections of this paper we focused on integration theory for one dimension. The Fundamental Theorem of Calculus above states the integrability of $f$ (under certain conditions) as well as giving a way to evaluate $\int_a^b f$. However in two-dimensions and higher, we tackle existence and evaluation of an integral as separate steps. Because of this we commonly evaluate integrals using iterated integrals. Suppose $I = [a,b] \times [c,d] \subset \mathbb{R}^2$ and $f$ is a function that is well behaved. An integral $\int_I f$ can be represented as an iterated integral in two ways,

$$\int_I f = \int_c^d \left[\int_a^b f(x,y)dy\right] dx = \int_a^b \left[\int_c^d f(x,y)dx\right] dy.$$

This tool for evaluating higher dimensional integrals takes care of the evaluation aspect of the problem. Breaking an integral down into iterated integrals allows us to just apply the Fundamental Theorem of Calculus repeatedly as many times as needed. The existence of $\int_I f$ is a sufficient condition to show $\int_I f$ is equal to the iterated integrals. The result that gives us this is Fubini's Theorem. Fubini's Theorem is a central theorem for integration in dimensions higher than one. It should be stated that existence of iterated integrals is not enough on its own to guarantee the integrability of $f$. Equality of iterated integrals is stated in Fubini's Theorem, so any function whose iterated integrals are not equal must be a function that is not integrable. The following example from Mcleod shows what this looks like.

Calculate both iterated integrals of $f(x,y) = (x^2-y^2)(x^2+y^2)^{-2}$ over the region $[0,1]\times[0,1]$.

$$\frac{\partial}{\partial x}\left[\frac{-x}{x^2+y^2}\right] = \frac{x^2-y^2}{(x^2+y^2)^2}$$

except at the point $(0,0)$. When $0 < y \leq 1$, $\frac{-x}{x^2+y^2}$ is a anti-derivative on $[0,1]$ of $x \mapsto f(x,y)$. So

$$\int_0^1 f(x,y)dx = \frac{-1}{1+y^2}$$

for $0 < y \leq 1$. $-\arctan(y)$ is the primitive for $\frac{-1}{1+y^2}$; therefore

$$\int_0^1 \left[\int_0^1 f dx\right] dy = \int_0^1 \frac{-1}{1+y^2} dy = \frac{-\pi}{4}.$$

Since $f(x,y) = -f(y,x)$, we conclude that

$$\int_0^1 \left[\int_0^1 f dy\right] dx = \frac{\pi}{4}$$

Since these two iterated integrals are not equal, we can conclude that $\int_{[0,1]\times[0,1]} f$ does not exist. [1]

## 3.1 Proving Fubini's Theorem

Before proving Fubini's theorem it is necessary to discuss a lemma that was used in the proof of Fubini's Theorem, which is found in Mcleod's text[1]. The lemma he introduces lays the ground work for the construction of compound partitions. He constructs his compound partitions in the text and then turns the constuction into a precise lemma. This lemma is then used in the first part of Fubini's Theorem to help show that the integral of $f(x, y)$ is integrable over $G$ by the Cauchy criterion. The lemma stated below is from Mcleod's text.

**Lemma:** Let $\delta$ be a gauge on $I = G \times H$, with $G \subset \mathbb{R}$ and $H \subset \mathbb{R}$. For each $y$ in $H$ let $P_o$ be a partition of $G$ which is $\delta$-fine. Let the gauge $\delta'$ be defined as $\delta'(y) = \bigcap_{(x,J)\in P_o} \delta(x, y)$ for all $y$ in $H$. Let $E$ be a $\delta'$-fine division of $H$. Then the compound partition $P$ formed from $E$ and the partitions $P_o$ for which $y$ is a tag in $E$ is $\delta$-fine. [1]

Next we want to go forward with the proof of Fubini's Theorem. The theorem uses only one piece of information, that the integral of $f$ over $I$ exists. This will be sufficient to prove the two statements. We will show two things in the proof. The first is that the integral of $f(x, y)$ over $G$ exists almost everywhere except on a subset of $H$ that has measure zero, also referred to as M-Null. This will be accomplished by using the Cauchy criterion from earlier in the paper since we have no value for the integral of $f(x, y)$ over $G$. After this we will show that rhe iterated integral exists and that it equals the integral of $f$ over $I$.

**Theorem 3.1.**

*Fubini's Theorem: Let $I = G \times H \subset \mathbb{R}^2$. Suppose $\int_I f$ exists. Then $f(\cdot, y)$ is integrable over $G$ almost everywhere on $H$. Moreover $\int_G f(x, \cdot)dx$ is integrable over $H$ and*

$$\int_I f = \int_H \int_G f(x, y)dxdy.$$

Proof: We will use the Cauchy criterion to show that $\int_G f(x, y)dx$ exists. Let $N$ be the set of all $y \in H$ such that the integral of $f$ over $G$ does not exist. We want to show that N is $\mu$-Null; i.e. $\mu(N) = 0$. By the Cauchy criterion, for each $y \in N$ there exists $z(y) > 0$ such that for every gauge $\delta_1$ on $G$ there exist $\delta_1$-fine partitions $P_o$ and $Q_o$ of $G$ such that

$$z(y) \leq |A(f, P_o) - A(f, Q_o)|$$

Letting $z(y) = 0$ when $y \in H - N$, defines a function $z : H \mapsto \mathbb{R}$. Using the fact that $\int_I f(z)$ exists, for $\epsilon > 0$ there exists a gauge $\delta$ on $I$ such that

$$|A(f,P) - A(f,Q)| < \epsilon$$

whenever $P$ and $Q$ are $\delta$-fine. Select a $\delta_o$ in place of $\delta_1$. Now fix the partitions $P_o$ and $Q_o$ for all $y \in H$ so that $z(y) \leq |A(f,P_o) - A(f,Q_o)|$ is true.

Now we will use Mcleod's lemma above to create a compound partition. The family of partitions of $P_o$ determines the gauge $\delta_1'$ on $H$ from $\delta$. Similarly the gauge $\delta_2'$ can be obtained using the partitions of $Q_o$ and the gauge $\delta$.

Let

$$\delta' = \min\{\delta_1', \delta_2'\}.$$

Now any $\delta'$-fine partition $E$ of $H$ gives us $\delta$-fine partitions $P$ and $Q$ from the partitions of $P_o$ and $Q_o$. Let $E$ be a $\delta'$-fine partition. Then

$$0 \leq A(z,E) \leq |A(f,P) - A(f,Q)| < \epsilon$$

since the partitions $P$ and $Q$ are $\delta$-fine from above. Therefore the integral of $z(y)$ over $H$ is zero and $N$ is M-Null.

Therefore f(x,y) is integrable over $G$.

Now we must show that $f(x,\cdot)$ is integrable over $H$. Let $\epsilon > 0$. The function $f$ is integrable on $I$, so there exists a gauge $\delta$ on $I$ such that

$$\left| A(f,P) - \int_I f \right| < \frac{\epsilon}{2}.$$

The function $x \mapsto f(x,y)$ is defined on $G$, for each $y \in H$. So for each $y$ there exists a $\delta(\cdot,y)$-fine tagged partition $P(y)$ of $G$ such that

$$\left| A(f,P(y)) - \int_G f(x,y) \right| < \frac{\epsilon}{2\mu(H)}.$$

Define a gauge $\delta_o$ of $H$ such that

$$\delta_o(y) = min\{\delta(x,y) : (x,J) \in P(y)\}.$$

Let $P_1$ be any $\delta_o$-fine tagged partition of $H$. Then

$$P' = \{((x,y),(J\times K)) : (x,J) \in P(y) \text{ and } (y,K) \in P_1\}$$

is a $\delta$-fine partition of $I$. Then

$$\begin{aligned}
&\left|\sum_{(y,K)\in P_1}\left(\int_G f(x,y)dx\right)\mu(K) - \int_I f\right| \\
&\leq \left|\sum_{(y,K)\in P_1}\left(\int_G f(x,y)dx\right)\mu(K) - A(f,P')\right| + \left|A(f,P') - \textstyle\int_I f\right| \\
&< \left|\sum_{(y,K)\in P_1}\left(\int_G f(x,y)dx\right)\mu(K) - A(f,P')\right| + \frac{\epsilon}{2} \\
&= \left|\sum_{(y,K)\in P_1}\left(\int_G f(x,y)dx\right)\mu(K) - \sum_{(y,K)\in P_1}\sum_{(x,J)\in P_o} f(x,y)\mu(J)\mu(K)\right| + \frac{\epsilon}{2} \\
&\leq \sum_{(y,K)\in P_1}\left|\left(\int_G f(x,y)dx\right)\mu(K) - \sum_{(x,J)\in P_o} f(x,y)\mu(J)\mu(K)\right| + \frac{\epsilon}{2} \\
&= \sum_{(y,K)\in P_1}\left|\int_G f(x,y)dx - \sum_{(x,J)\in P_o} f(x,y)\mu(J)\right|\mu(K) + \frac{\epsilon}{2} \\
&< \frac{\epsilon}{2\mu(H)}\sum_{(y,K)\in P_1}\mu(K) + \frac{\epsilon}{2} \\
&= \frac{\epsilon}{2\mu(H)}\mu(H) + \frac{\epsilon}{2} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

$\epsilon > 0$ is arbitrary, so

$$\int_H\left(\int_G f(x,y)dx\right)dy = \int_I f.$$

We can go through a similar process to obtain

$$\int_G\left(\int_H f(x,y)dy\right)dx = \int_I f.$$

# 4 Conclusion

This paper is just a small look into the power of the generalized Riemann integral. But with just this tiny look the reader comes away with a greater understanding about how the generalized Riemann integral is superior to its counterparts. We have seen that the generalized Riemann integral can be used to integrate all functions which have anti-derivatives, all Riemann/Lebesgue integrable functions as well as all the functions whose integrals are considered improper. We also explored higher dimensions to better understand the usefulness of this integral. While I feel that the generalized Riemann integral would be a useful tool for undergraduates to learn, it seems unlikely it will be taught at that level, since it relies on some understanding of the concepts of Lebesgue theory and measure theory.

To conclude, the generalized Riemann integral is an effective tool to use in integration that is overlooked but if developed correctly could be a great addition to any undergraduate curriculum.

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