A NONLINEAR THEORY FOR THIN ELASTIC SHELLS

INCLUDING THE EFFECTS OF TRANSVERSE SHEAR STRESS,

TRANSVERSE NORMAL STRESS AND TRANSVERSE AND ROTARY INERTIA

by

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ABSTRACT

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The purpose of this thesis is to derive a nonlinear theory of thin elastic shells including the effects of transverse normal stress, transverse shear stress, and transverse and rotary inertia.

Using a variation theorem due to Reissner, the equations of motion, the stress-strain relationships, and the associated natural boundary conditions are simultaneously determined. The resulting equations may be applied to a certain group of shell problems where the applied dynamic loads produce deformations which are of such an order that only an appropriate nonlinear theory can account for them.

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LIST OF SYMBOLS

SYMBOL	DEFINITION
x, y, z	Surface & normal coordinates
X, Y, Z	Rectangular coordinates
α, β, γ	Lame's coefficient
r_1, r_2	Radii of curvature in direction of x,y respectively
$r_{\hat{t}}$	Radius of curvature for the edge of the shell
$\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}$	Normal strain components in direction of x,y,z respectively
$\gamma_{xz}, \gamma_{yz}, \gamma_{xy}$	Shearing strain components
$\tau_{xx}, \tau_{yy}, \tau_{zz}$	Normal stress components
$\tau_{xy}, \tau_{xz}, \tau_{yz}$	Shearing stress components
$\tau_{nn}, \tau_{n\hat{t}}, \tau_{nz}$	Normal & shear stress on the edge of the shell
U, V, W	Displacement components in direction of x,y,z respectively for any arbitrary point in the shell
$\bar{U}, \bar{V}, \bar{W}$	Displacement components at the middle surface of the shell
$\bar{W}', \bar{W}''$	Displacement components which contribute to the transverse normal displacement
$\bar{U}_{nn}, \bar{U}_{n\hat{t}}$	Displacement at the edge of the shell in the normal and tangential direction respectively
ϕ, ψ	Change of slope of the normal to the middle surface
ρ	Mass density per unit mass
ν	Poisson's ratio
ω	Natural frequency of vibration in radian per sec.
z	Normal coordinate in z-direction of any arbitrary point in the shell
h	Thickness of the shell
D	Flexural rigidity of the shell

E	Modulus of elasticity
F_x, F_y, F_z	Body force per unit volumn in direction of x,y,z respectively
G	Shear modulus of elasticity
N_{xx}, N_{yy}, N_{xy}, N_{yx}	Normal stress resultants in unit of force per unit length
M_{xx}, M_{yy}, M_{xy}, M_{yx}	Bending stress couples in unit of moment per unit length
Q_{xz}, Q_{yz}	Shear stress resultants in z-direction
N_{nn}, $N_{n\hat{t}}$, M_{nn}, $M_{n\hat{t}}$, Q_{nz}	Normal stress resultants, bending stress couples and shear stress resultants at the edge of the shell
N^*_{nn}, $N^*_{n\hat{t}}$, Q^*_{nz}	Prescribed stress resultants at the edge of the shell
M^*_{nn}, $M^*_{n\hat{t}}$	Prescribed stress couples at the edge of the shell
P_x, P_y, P_z	External shear and normal stress components in the direction of x,y,z respectively
R_x, R_y	External bending stress couples
P^+_1, P^-_1	Stress component τ_{xz} at the upper and lower surfaces of the shell respectively
P^+_2, P^-_2	Stress component τ_{yz} at the upper and lower surfaces of the shell respectively
q^+, q^-	Stress component τ_{zz} at the upper and lower surfaces of the shell respectively
S, T	Parameters used in stress distribution
ϕ_{nn}, $\phi_{n\hat{t}}$	Change of slope of the normal to the middle surface at the edge of the shell

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Nonlinear theories for thin elastic shells as derived by using the theory of finite displacements differ greatly depending on the restrictive assumptions placed on the resulting deformations.

A linear theory for thin elastic shells including the effects of transverse normal stress, transverse shear stress and rotary inertia is considered by Naghdi.[3] A group of existing theories is summarized by Sanders,[5] where he derives a set of nonlinear theories which include as a special case the Donnell-Mushtari-Vlosov theory. A nonlinear shear deformation theory for thin elastic shells is presented by Archer.[1] This paper derives a nonlinear theory of the Donnell type which includes shear deformations, transverse and rotary inertia effects, but does not include the effect of transverse normal stress.

A direct application of the resulting equations play an important role in wave propagation problems, where the effects of transverse normal stress and transverse shear stress are of primary importance.

The objective of this thesis is to derive a nonlinear theory of thin elastic shells of the Donnell type based on Reissner's variational theorem of finite elastic displacement.[4]

CHAPTER II

METHOD OF ANALYSIS

In analysis of shell, not only basic assumptions of the analysis in beam and plate were used, but also some more restrictly assumptions to get the useful results were included as follows:

1. The thickness of the shell assumes uniform and is small when compared with the least radius of curvature. Terms of the order $(h/r)^2$ are retained in comparision to unity.
2. Points on the lines which are normal to the middle surface before deformation do not remain normal to the middle surface after deformation (i.e., shear deformations are accounted for).
3. Linear elastic stress-strain relationships are assumed to hold, and the component of stress normal to the middle surface is considered to be of the same order as the other components of stress.

2.1 The Coordinate System and Notation

The notation used throughout the paper is similar to that given by Langhaar.[2] Where the middle surface of shell is defined as the equations of $X = X(x,y)$, $Y = Y(x,y)$ and $Z = Z(x,y)$ where the parameters x,y are called middle surface coordinates and X,Y,Z are rectangular cartesian coordinates. The normal distance from the middle surface is denoted by $\pm$ z, the normal coordinate.

The unit normal vector at a point of the middle surface is defined as $\bar{n}_j$ and tangent vectors to the curves of constant x and y curves by $\bar{r}_x$ and $\bar{r}_y$ respectively.

For the special of orthogonal middle lines, the coordinate curves align with the curves of principle curvature.

The distance ds between points is given by the equation:

$$ds^2 = \alpha^2 dx^2 + \beta^2 dy^2 + \gamma^2 dz^2 \quad , \qquad (1)$$

where

$$\left.\begin{array}{lll} \alpha = A(1+ \frac{z}{r_1}) \,, & \beta = B(1+ \frac{z}{r_2}) \,, & \gamma = 1 \,, \\ A^2 = \bar{r}_x \cdot \bar{r}_x \,, & B^2 = \bar{r}_y \cdot \bar{r}_y \quad . & \end{array}\right] \qquad (2)$$

and $\frac{1}{r_1}$, $\frac{1}{r_2}$ are the principle curvatures of the middle surface.

2.2 Stress Resultants and Stress Couples

Stress resultants and stress couples applied to a differential shell element are shown in Figs. 2.1b & 2.1c. These stress resultants and stress couples are defined as total forces and moments acting per unit length of the middle surface.

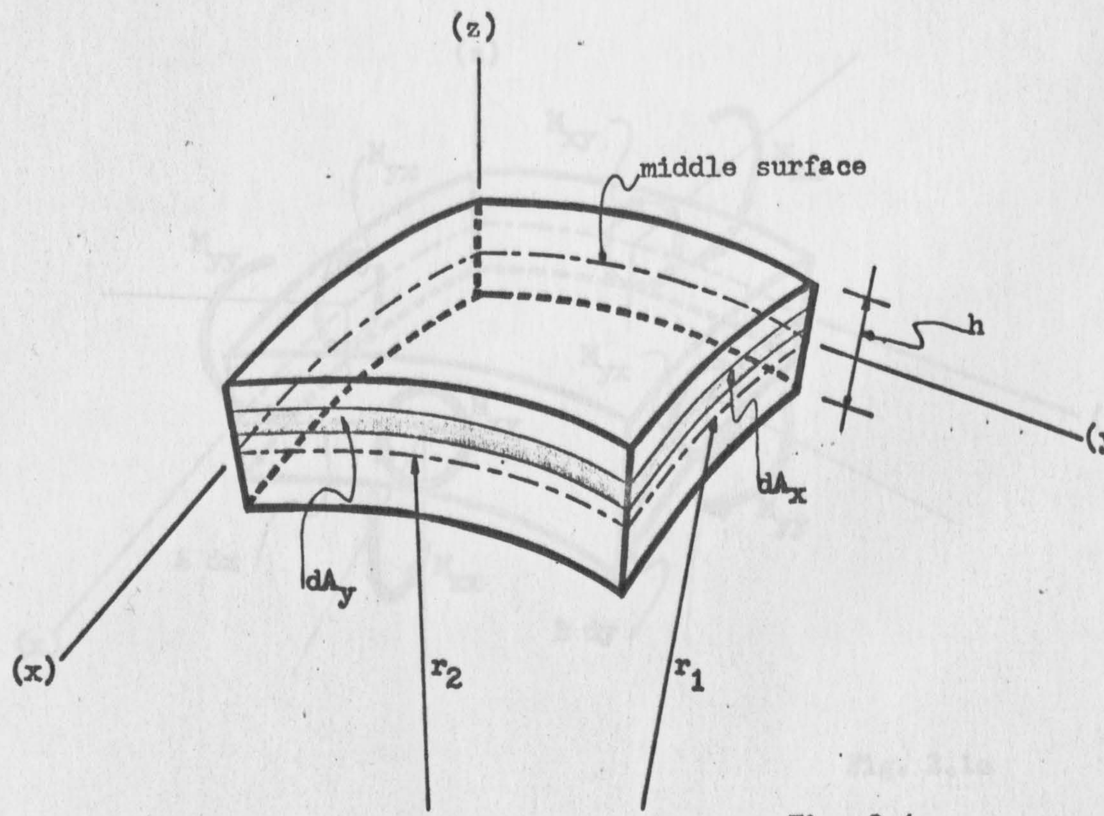

Fig. 2.1a

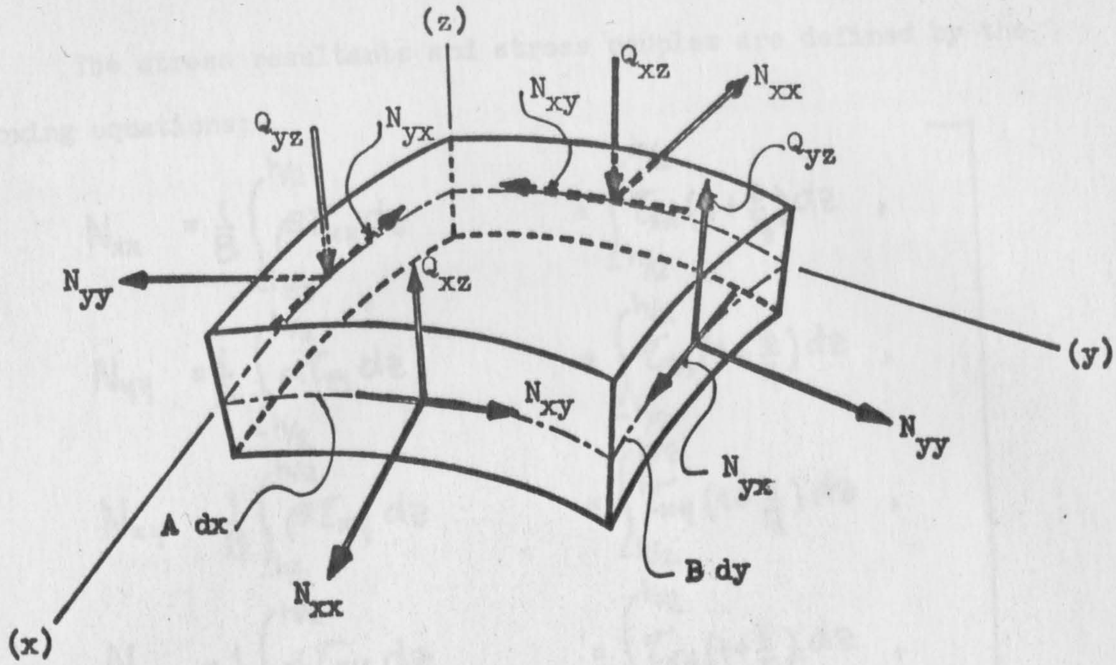

Fig. 2.1b

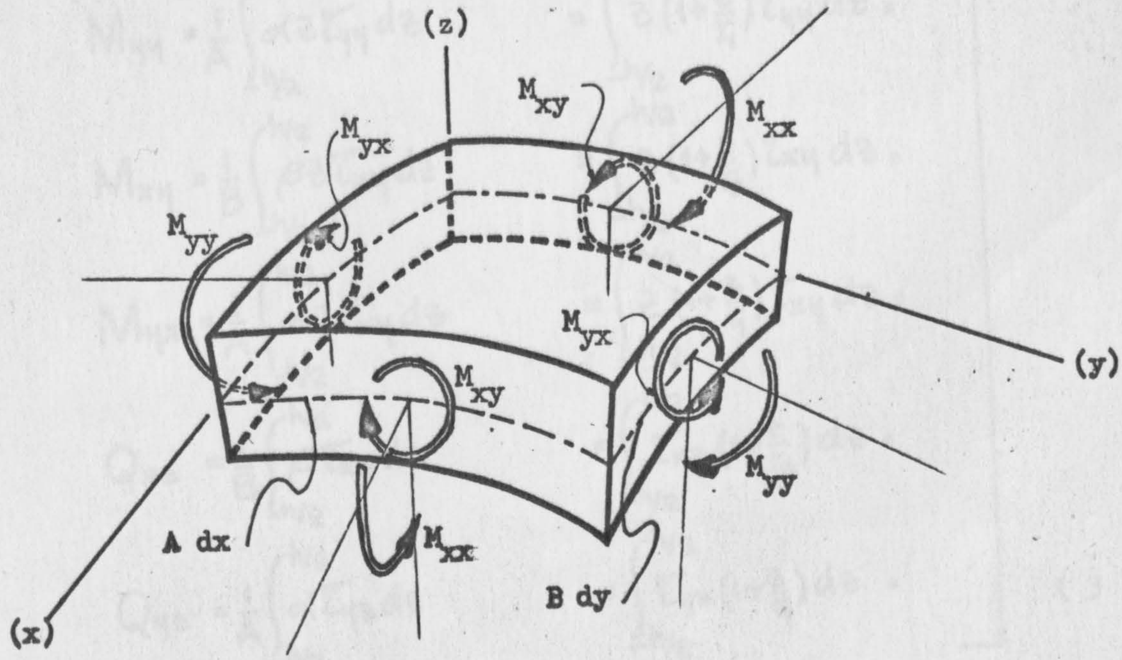

Fig. 2.1c

The stress resultants and stress couples are defined by the following equations;

$$
\left.
\begin{aligned}
N_{xx} &= \frac{1}{B}\int_{-h/2}^{h/2} \beta \tau_{xx}\, dz &&= \int_{-h/2}^{h/2} \tau_{xx}\left(1+\frac{z}{r_2}\right) dz \,, \\
N_{yy} &= \frac{1}{A}\int_{-h/2}^{h/2} \alpha \tau_{yy}\, dz &&= \int_{-h/2}^{h/2} \tau_{yy}\left(1+\frac{z}{r_1}\right) dz \,, \\
N_{xy} &= \frac{1}{B}\int_{-h/2}^{h/2} \beta \tau_{xy}\, dz &&= \int_{-h/2}^{h/2} \tau_{xy}\left(1+\frac{z}{r_2}\right) dz \,, \\
N_{yx} &= \frac{1}{A}\int_{-h/2}^{h/2} \alpha \tau_{xy}\, dz &&= \int_{-h/2}^{h/2} \tau_{xy}\left(1+\frac{z}{r_1}\right) dz \,, \\
M_{xx} &= \frac{1}{B}\int_{-h/2}^{h/2} \beta z \tau_{xx}\, dz &&= \int_{-h/2}^{h/2} z\left(1+\frac{z}{r_2}\right) \tau_{xx}\, dz \,, \\
M_{yy} &= \frac{1}{A}\int_{-h/2}^{h/2} \alpha z \tau_{yy}\, dz &&= \int_{-h/2}^{h/2} z\left(1+\frac{z}{r_1}\right) \tau_{yy}\, dz \,, \\
M_{xy} &= \frac{1}{B}\int_{-h/2}^{h/2} \beta z \tau_{xy}\, dz &&= \int_{-h/2}^{h/2} z\left(1+\frac{z}{r_2}\right) \tau_{xy}\, dz \,, \\
M_{yx} &= \frac{1}{A}\int_{-h/2}^{h/2} \alpha z \tau_{xy}\, dz &&= \int_{-h/2}^{h/2} z\left(1+\frac{z}{r_1}\right) \tau_{xy}\, dz \,, \\
Q_{xz} &= \frac{1}{B}\int_{-h/2}^{h/2} \beta \tau_{xz}\, dz &&= \int_{-h/2}^{h/2} \tau_{xz}\left(1+\frac{z}{r_2}\right) dz \,, \\
Q_{yz} &= \frac{1}{A}\int_{-h/2}^{h/2} \alpha \tau_{yz}\, dz &&= \int_{-h/2}^{h/2} \tau_{yz}\left(1+\frac{z}{r_1}\right) dz \,.
\end{aligned}
\right] \qquad (3)
$$

Equation (3) yields the relationship of

$$
N_{xy} + \frac{M_{xy}}{r_1} = N_{yx} + \frac{M_{yx}}{r_2} \,. \qquad (4)
$$

2.3 Strain-Displacement Relations

The equations of the general three dimensional nonlinear strain-displacement are given as:

$$
\left.
\begin{aligned}
\epsilon_{xx} &= \frac{1}{\alpha}\left[U_x + \frac{\alpha_y}{\beta}V + \frac{\alpha_z}{\gamma}W + \frac{1}{2\alpha}\left(U_x + \frac{\alpha_y}{\beta}V + \frac{\alpha_z}{\gamma}W\right)^2 + \frac{1}{2\alpha}\left(V_x - \frac{\alpha_y}{\beta}U\right)^2 + \frac{1}{2\alpha}\left(W_x - \frac{\alpha_z}{\gamma}U\right)^2\right], \\
\epsilon_{yy} &= \frac{1}{\beta}\left[V_y + \frac{\beta_z}{\gamma}W + \frac{\beta_x}{\alpha}U + \frac{1}{2\beta}\left(V_y + \frac{\beta_z}{\gamma}W + \frac{\beta_x}{\alpha}U\right)^2 + \frac{1}{2\beta}\left(W_x - \frac{\beta_z}{\gamma}V\right)^2 + \frac{1}{2\beta}\left(U_y - \frac{\beta_x}{\alpha}V\right)^2\right], \\
\epsilon_{zz} &= \frac{1}{\gamma}\left[W_z + \frac{\gamma_x}{\alpha}U + \frac{\gamma_y}{\beta}V + \frac{1}{2\gamma}\left(W_z + \frac{\gamma_x}{\alpha}U + \frac{\gamma_y}{\beta}V\right)^2 + \frac{1}{2\gamma}\left(U_z - \frac{\gamma_x}{\alpha}W\right)^2 + \frac{1}{2\gamma}\left(V_z - \frac{\gamma_y}{\beta}W\right)^2\right], \\
\gamma_{xy} &= \frac{U_y}{\beta} + \frac{V_x}{\alpha} - \frac{\beta_x V}{\alpha\beta} - \frac{\alpha_y U}{\alpha\beta} + \frac{1}{\alpha\beta}\left(U_x + \frac{\alpha_y}{\beta}V + \frac{\alpha_z}{\gamma}U\right)\left(U_y - \frac{\beta_x}{\gamma}V\right) \\
&\quad + \frac{1}{\alpha\beta}\left(V_y + \frac{\beta_x}{\alpha}U + \frac{\beta_z}{\gamma}W\right)\left(V_x - \frac{\alpha_y}{\beta}U\right) + \frac{1}{\alpha\beta}\left(W_x - \frac{\alpha_z}{\gamma}U\right)\left(W_y - \frac{\beta_z}{\gamma}V\right), \\
\gamma_{yz} &= \frac{V_z}{\gamma} + \frac{W_y}{\beta} - \frac{\gamma_y W}{\beta\gamma} - \frac{\beta_z V}{\beta\gamma} + \frac{1}{\beta\gamma}\left(V_y + \frac{\beta_z}{\gamma}W + \frac{\beta_x}{\alpha}U\right)\left(V_z - \frac{\gamma_y}{\beta}W\right) \\
&\quad + \frac{1}{\beta\gamma}\left(W_z + \frac{\gamma_y}{\beta}V + \frac{\gamma_x}{\alpha}U\right)\left(W_y - \frac{\beta_z}{\gamma}V\right) + \frac{1}{\beta\gamma}\left(U_y - \frac{\beta_x}{\alpha}V\right)\left(U_z - \frac{\gamma_x}{\alpha}W\right), \\
\gamma_{xz} &= \frac{W_x}{\alpha} + \frac{U_z}{\gamma} - \frac{\alpha_z U}{\gamma\alpha} + \frac{1}{\gamma\alpha}\left(W_z + \frac{\gamma_x}{\alpha}U + \frac{\gamma_y}{\beta}V\right)\left(W_x - \frac{\alpha_z}{\gamma}U\right) + \frac{\gamma_x W}{\gamma\alpha} \\
&\quad + \frac{1}{\gamma\alpha}\left(U_x + \frac{\alpha_z}{\gamma}W + \frac{\alpha_y}{\beta}V\right)\left(U_z - \frac{\gamma_x}{\alpha}W\right) + \frac{1}{\gamma\alpha}\left(V_z - \frac{\gamma_y}{\beta}W\right)\left(V_x - \frac{\alpha_y}{\beta}U\right).
\end{aligned}
\right]
\tag{5}
$$

Retaining all linear terms together with the second order of rotation terms W_x^2, W_y^2, & W_xW_y.

Equation (5) reduces to the following:

$$
\left.
\begin{aligned}
\epsilon_{xx} &= \frac{1}{\alpha}\left[U_x + \frac{\alpha_y}{\beta} V + \alpha_z W + \frac{1}{2\alpha} W_x^2 \right] , \\
\epsilon_{yy} &= \frac{1}{\beta}\left[\frac{\beta_x}{\alpha} U + V_y + \beta_z W + \frac{1}{2\beta} W_y^2 \right] , \\
\epsilon_{zz} &= W_z , \\
\gamma_{xy} &= \frac{U_y}{\beta} + \frac{V_x}{\alpha} - \frac{\beta_x}{\alpha\beta} V - \frac{\alpha_y}{\alpha\beta} U + \frac{W_x W_y}{\alpha\beta} , \\
\gamma_{yz} &= V_z + \frac{W_y}{\beta} - \frac{\beta_z}{\beta} V , \\
\gamma_{xz} &= U_z + \frac{W_x}{\alpha} - \frac{\alpha_z}{\alpha} U .
\end{aligned}
\right] \qquad (6)
$$

The differential equation of Codazzi for orthogonal shell coordinates are written as:

$$
\left.
\begin{aligned}
\frac{\partial}{\partial y}\left(\frac{A}{r_1} \right) &= \frac{1}{r_2} A_y , \\
\frac{\partial}{\partial x}\left(\frac{B}{r_2} \right) &= \frac{1}{r_1} B_x .
\end{aligned}
\right] \qquad (7)
$$

The following equalities are obtained using equation (7)

$$
\left.
\begin{aligned}
\frac{\alpha_y}{\beta} &= \frac{A_y}{B} , \\
\frac{\beta_x}{\alpha} &= \frac{B_x}{A} , \\
\alpha_z &= \frac{A}{r_1} , \\
\beta_z &= \frac{B}{r_2} .
\end{aligned}
\right] \qquad (8)
$$

Substituting equation (8) into equation (6) the following reduced form is obtained:

$$\left.\begin{aligned}
\epsilon_{xx} &= \frac{1}{\alpha}\left[U_x + \frac{A_y}{B}V + \frac{A}{r_1}W + \frac{1}{2\alpha}W_x^2\right], \\
\epsilon_{yy} &= \frac{1}{\beta}\left[V_y + \frac{B_x}{A}U + \frac{B}{r_2}W + \frac{1}{2\beta}W_y^2\right], \\
\epsilon_{zz} &= W_z \ , \\
\gamma_{xy} &= \frac{U_y}{\beta} + \frac{V_x}{\alpha} - \frac{B_x}{A\beta}V - \frac{A_y}{B\alpha}U + \frac{W_x W_y}{\alpha\beta} \ , \\
\gamma_{yz} &= V_z + \frac{W_y}{\beta} - \frac{\beta_z}{\beta}V \ , \\
\gamma_{xz} &= U_z + \frac{W_x}{\alpha} - \frac{\alpha_z}{\alpha}U \ .
\end{aligned}\right] \qquad (9)$$

To obtain the appropriate stress-strain relation, the following approximate equations are assumed:

$$\left.\begin{aligned}
U &= \bar{U}(x,y) + z\phi(x,y) \ , \\
V &= \bar{V}(x,y) + z\psi(x,y) \ , \\
W &= \bar{W}(x,y) + z\bar{W}'(x,y) + \frac{z^2}{2}\bar{W}''(x,y) \ .
\end{aligned}\right] \qquad (10)$$

where $\bar{U}$ and $\bar{V}$ are the components of displacement at the middle surface, $\phi(x,y)$ and $\psi(x,y)$ are the change of slope of the normal to the middle surface along the x and y coordinates lines respectively, and $\bar{W}'(x,y)$ and $\bar{W}''(x,y)$ are the contributions to the transverse normal strain.

Substituting equation (10) into equation (9), the following are obtained:

$$\epsilon_{xx} = \frac{1}{\alpha}\left[\bar{u}_x + z\phi_x + \frac{A_y}{B}(\bar{v}+z\psi) + \frac{A}{r_1}(\bar{w}+z\bar{w}'+\frac{z^2}{2}\bar{w}'') + \frac{1}{2\alpha}(\bar{w}_x+z\bar{w}_x'+\frac{z^2}{2}\bar{w}_x'')^2\right],$$

$$\epsilon_{yy} = \frac{1}{\beta}\left[\bar{v}_y + z\psi_y + \frac{B_x}{A}(\bar{u}+z\phi) + \frac{B}{r_2}(\bar{w}+z\bar{w}'+\frac{z^2}{2}\bar{w}'') + \frac{1}{2\beta}(\bar{w}_y+z\bar{w}_y'+\frac{z^2}{2}\bar{w}_y'')^2\right],$$

$$\epsilon_{zz} = \bar{w}' + z\bar{w}'',$$

$$\gamma_{xy} = \frac{1}{\alpha\beta}\Big[\alpha(\bar{u}_y+z\phi_y) + \beta(\bar{v}_x+z\psi_x) - \frac{B_x}{A}\alpha(\bar{v}+z\psi) - \frac{A_y}{B}\beta(\bar{u}+z\phi)$$
$$+(\bar{w}_x+z\bar{w}_x'+\frac{z^2}{2}\bar{w}_x'')(\bar{w}_y+z\bar{w}_y'+\frac{z^2}{2}\bar{w}_y'')\Big],$$

$$\gamma_{yz} = \frac{1}{\beta}\left[\beta\psi + (\bar{w}_y+z\bar{w}_y'+\frac{z^2}{2}\bar{w}_y'') - \beta_z(\bar{v}+z\psi)\right],$$

$$\gamma_{xz} = \frac{1}{\alpha}\left[\alpha\phi + (\bar{w}_x+z\bar{w}_x'+\frac{z^2}{2}\bar{w}_x'') - \alpha_z(\bar{u}+z\phi)\right].$$

(11)

If the terms $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are replaced respectively by the terms $\frac{1}{A}$ and $\frac{1}{B}$ in the first and second equations given in equation (11), the following equations for the components of strain are rewritten as:

$$(1+\frac{z}{r_1})\epsilon_{xx} = \mathring{\epsilon}_{xx} + zK_x + z^2C_x + \frac{z^3}{2A^2}\bar{w}_x'\bar{w}_x'' + \frac{z^4}{8A^2}(\bar{w}_x'')^2,$$

$$(1+\frac{z}{r_2})\epsilon_{yy} = \mathring{\epsilon}_{yy} + zK_y + z^2C_y + \frac{z^3}{2B^2}\bar{w}_y'\bar{w}_y'' + \frac{z^4}{8B^2}(\bar{w}_y'')^2,$$

$$\epsilon_{zz} = \bar{w}' + z\bar{w}'',$$

$$(1+\frac{z}{r_1})(1+\frac{z}{r_2})\gamma_{xy} = (1+\frac{z}{r_2})(\overset{\circ}{\gamma}_{xx}+z\delta_{xx}) + (1+\frac{z}{r_1})(\overset{\circ}{\gamma}_{yy}+z\delta_{yy}) + \frac{1}{AB}\bar{W}_x\bar{W}_y + zD_{xy} + z^2E_{xy} + z^3F_{xy} + \frac{z^4}{4AB}(\bar{W}_x''\bar{W}_y'')$$

$$(1+\frac{z}{r_2})\gamma_{yz} = \overset{\circ}{\gamma}_{yz} + \frac{z}{B}(\bar{W}_y' + \frac{z}{2}\bar{W}_y'') ,$$

$$(1+\frac{z}{r_1})\gamma_{xz} = \overset{\circ}{\gamma}_{xz} + \frac{z}{A}(\bar{W}_x' + \frac{z}{2}\bar{W}_x'') . \qquad (12)$$

where

$$\overset{\circ}{\epsilon}_{xx} = \frac{1}{A}(\bar{U}_x + \frac{A_y}{B}\bar{V}) + \frac{\bar{W}}{r_1} + \frac{1}{2A^2}(\bar{W}_x')^2 , \qquad \overset{\circ}{\gamma}_{xx} = \frac{1}{A}(\bar{V}_x - \frac{A_y}{B}\bar{U}) ,$$

$$\overset{\circ}{\epsilon}_{yy} = \frac{1}{B}(\bar{V}_y + \frac{B_x}{A}\bar{U}) + \frac{\bar{W}}{r_2} + \frac{1}{2B^2}(\bar{W}_y')^2 , \qquad \overset{\circ}{\gamma}_{yy} = \frac{1}{B}(\bar{U}_y - \frac{B_x}{A}\bar{V}) ,$$

$$\delta_{xx} = \frac{1}{A}(\psi_x - \frac{A_y}{B}\phi) , \qquad \delta_{yy} = \frac{1}{B}(\phi_y - \frac{B_x}{A}\psi) ,$$

$$\overset{\circ}{\gamma}_{xz} = \frac{\bar{W}_x}{A} - \frac{\bar{U}}{r_1} + \phi , \qquad \gamma_{yz} = \frac{\bar{W}_y}{B} - \frac{\bar{V}}{r_2} + \psi ,$$

$$K_x = \frac{1}{A}(\phi_x + \frac{A_y}{B}\psi) + \frac{\bar{W}'}{r_1} + \frac{1}{A^2}\bar{W}_x\bar{W}_x' ,$$

$$K_y = \frac{1}{B}(\psi_y + \frac{B_x}{A}\phi) + \frac{\bar{W}'}{r_2} + \frac{1}{B^2}\bar{W}_y\bar{W}_y' ,$$

$$C_x = \frac{1}{2}\left[\frac{\bar{W}''}{r_1} + \frac{1}{A^2}(\bar{W}_x\bar{W}_x'' + \bar{W}_x'^2)\right] ,$$

$$C_y = \frac{1}{2}\left[\frac{\bar{W}''}{r_2} + \frac{1}{B^2}(\bar{W}_y\bar{W}_y'' + \bar{W}_y'^2)\right] ,$$

$$D_{xy} = \frac{1}{AB}(\bar{W}_x\bar{W}_y' + \bar{W}_x'\bar{W}_y) ,$$

$$E_{xy} = \frac{1}{AB}\left(\frac{\bar{W}_x\bar{W}_y''}{2} + \bar{W}_x'\bar{W}_y' + \frac{\bar{W}_x''\bar{W}_y}{2}\right) ,$$

$$F_{xy} = \frac{1}{2AB}(\bar{W}_x'\bar{W}_y'' + \bar{W}_x''\bar{W}_y') . \qquad (13)$$

2.4 The Components of Stress

Noting equations (3), the components of stress are assumed to take the form,

$$\left.\begin{aligned}
(1+\frac{z}{r_2})\tau_{xx} &= \frac{N_{xx}}{h} + \frac{12}{h^3} z M_{xx} \quad , \\
(1+\frac{z}{r_1})\tau_{yy} &= \frac{N_{yy}}{h} + \frac{12}{h^3} z M_{yy} \quad , \\
(1+\frac{z}{r_2})\tau_{xy} &= \frac{N_{xy}}{h} + \frac{12}{h^3} z M_{xy} \quad , \\
(1+\frac{z}{r_1})\tau_{yx} &= \frac{N_{yx}}{h} + \frac{12}{h^3} z M_{yx} \quad .
\end{aligned}\right] \qquad (14)$$

the components of shearing stress of τ_{xz}, τ_{yz} and τ_{zz} are determined by direct solution of the first three equilibrium equation of stress which are:

$$\left.\begin{aligned}
&\frac{\partial}{\partial x}(\beta\gamma\tau_{xx}) + \frac{\partial}{\partial y}(\alpha\gamma\tau_{yx}) + \frac{\partial}{\partial z}(\alpha\beta\tau_{zx}) + \gamma\alpha_y\tau_{xy} + \beta\alpha_z\tau_{xz} \\
&\qquad - \gamma\beta_x\tau_{yy} - \beta\gamma_x\tau_{zz} + \rho\alpha\beta\gamma F_x = 0 \quad , \\
&\frac{\partial}{\partial x}(\beta\gamma\tau_{xy}) + \frac{\partial}{\partial y}(\alpha\gamma\tau_{yy}) + \frac{\partial}{\partial z}(\alpha\beta\tau_{zy}) + \gamma\beta_x\tau_{yx} + \alpha\beta_z\tau_{yz} \\
&\qquad - \gamma\alpha_y\tau_{xx} - \alpha\gamma_y\tau_{zz} + \rho\alpha\beta\gamma F_y = 0 \quad , \\
&\frac{\partial}{\partial x}(\beta\gamma\tau_{xz}) + \frac{\partial}{\partial y}(\alpha\gamma\tau_{yz}) + \frac{\partial}{\partial z}(\alpha\beta\tau_{zz}) + \beta\gamma_x\tau_{zx} + \alpha\gamma_y\tau_{zy} \\
&\qquad - \beta\alpha_z\tau_{xx} - \alpha\beta_z\tau_{yy} + \rho\alpha\beta\gamma F_z = 0 \quad .
\end{aligned}\right]$$

(15)

For thin shell theory in orthogonal coordinates $\gamma = 1$; the previous equations reduce to,

$$\begin{aligned}
&\frac{\partial}{\partial x}(\beta\tau_{xx}) + \frac{\partial}{\partial y}(\alpha\tau_{yx}) + \frac{\partial}{\partial z}(\alpha\beta\tau_{zx}) + \alpha_y\tau_{xy} + \beta\alpha_z\tau_{xz} - \beta_x\tau_{yy} = 0 \,, \\
&\frac{\partial}{\partial x}(\beta\tau_{xy}) + \frac{\partial}{\partial y}(\alpha\tau_{yy}) + \frac{\partial}{\partial z}(\alpha\beta\tau_{zy}) + \beta_x\tau_{yx} + \alpha\beta_z\tau_{yz} - \alpha_y\tau_{xx} = 0 \,, \\
&\frac{\partial}{\partial x}(\beta\tau_{xz}) + \frac{\partial}{\partial y}(\alpha\tau_{yz}) + \frac{\partial}{\partial z}(\alpha\beta\tau_{zz}) - \beta\alpha_z\tau_{xx} - \alpha\beta_z\tau_{yy} = 0 \,.
\end{aligned} \tag{16}$$

Substituting equations (3) into equations (16), and integrating over the thickness of the shell yields respectively,

$$\begin{aligned}
&\frac{\partial}{\partial x}(BN_{xx}) + \frac{\partial}{\partial y}(AN_{yx}) + A_yN_{xy} - B_xN_{yy} + \frac{AB}{r_1}Q_{xz} + ABP_x = 0 \,, \\
&\frac{\partial}{\partial x}(BN_{xy}) + \frac{\partial}{\partial y}(AN_{yy}) + B_xN_{yx} - A_yN_{xx} + \frac{AB}{r_2}Q_{yz} + ABP_y = 0 \,, \\
&\frac{\partial}{\partial x}(BQ_{xz}) + \frac{\partial}{\partial y}(AQ_{yz}) - \frac{AB}{r_1}N_{xx} - \frac{AB}{r_2}N_{yy} + ABP_z = 0 \,.
\end{aligned} \tag{17}$$

Multiplying the first two equations (16) by z and preforming the same operations as in the previous set of equations yields respectively,

$$\begin{aligned}
&\frac{\partial}{\partial x}(BM_{xx}) + \frac{\partial}{\partial y}(AM_{yx}) + A_yM_{xy} - B_xM_{yy} - ABQ_{xz} + ABR_y = 0 \,, \\
&\frac{\partial}{\partial x}(BM_{xy}) + \frac{\partial}{\partial y}(AM_{yy}) + B_xM_{yx} - A_yM_{xx} - ABQ_{yz} - ABR_x = 0 \,.
\end{aligned} \tag{18}$$

where,

$$AB\,P_x = \alpha\beta\tau_{xz}\Big|_{-h/2}^{h/2}, \qquad AB\,P_y = \alpha\beta\tau_{yz}\Big|_{-h/2}^{h/2},$$

$$AB\,P_z = \alpha\beta\tau_{zz}\Big|_{-h/2}^{h/2},$$

$$AB\,R_x = -\alpha\beta z\tau_{yz}\Big|_{-h/2}^{h/2}, \qquad AB\,R_y = \alpha\beta z\tau_{xz}\Big|_{-h/2}^{h/2}. \tag{19}$$

Substituting equations (14) into the first equation (16) and noting equations (3), yields

$$\frac{\partial}{\partial z}(\alpha\beta\tau_{xz}) + \beta\alpha_z\tau_{xz} = \frac{1}{h}\left[-\frac{\partial}{\partial x}(BN_{xx}) - \frac{\partial}{\partial y}(AN_{yx}) - A_yN_{xy} + B_xN_{yy}\right] + \frac{12z}{h^3}\left[-\frac{\partial}{\partial x}(BM_{xx}) - \frac{\partial}{\partial y}(AM_{yx}) - A_yM_{xy} + B_xM_{yy}\right]. \tag{20}$$

Rearranging the left hand side of equation (20) and substituting equations (17) & (18) into the right hand side gives,

$$\frac{1}{\alpha}\frac{\partial}{\partial z}(\alpha^2\beta\tau_{xz}) = \frac{AB}{h}Q_{xz}\left(\frac{1}{r_1} - \frac{12}{h^2}z\right) + \frac{1}{h}\left|\alpha\beta\tau_{xz}\right|_{-h/2}^{h/2} + \frac{12}{h^3}z\left|\alpha\beta z\tau_{xz}\right|_{-h/2}^{h/2}. \tag{21}$$

The integration of equation (21) is carried out over the function z. Applying the boundary condition of @ $z = +\frac{h}{2}$, $\tau_{xz} = P_1^+$, and neglecting terms containing the quantity $\frac{h}{r}$ and all higher order terms, the transverse shearing stress τ_{xz} becomes,

$$\left(1+\frac{z}{r_2}\right)\tau_{xz} = \frac{3}{2}\frac{Q_{xz}}{h}\left[1-\left(\frac{z}{h/2}\right)^2\right] - \frac{1}{4}\left\{H_1^+P_1^+\left[1-2\left(\frac{z}{h/2}\right)-3\left(\frac{z}{h/2}\right)^2\right] + H_1^-P_1^-\left[1+2\left(\frac{z}{h/2}\right)-3\left(\frac{z}{h/2}\right)^2\right]\right\}. \tag{22}$$

where P_1^+ and P_1^- are the values of τ_{xz} at the upper and lower surfaces of the shell respectively,

and where

$$
\left.
\begin{aligned}
H^+ &= \left(1+\frac{h}{2r_1}\right)\left(1+\frac{h}{2r_2}\right) \; , \\
H^- &= \left(1-\frac{h}{2r_1}\right)\left(1-\frac{h}{2r_2}\right) , \\
H_1^+ &= \left(1+\frac{h}{2r_2}\right) \; , \\
H_1^- &= \left(1-\frac{h}{2r_2}\right) \; , \\
H_2^+ &= \left(1+\frac{h}{2r_1}\right) \; , \\
H_2^- &= \left(1-\frac{h}{2r_1}\right) \; .
\end{aligned}
\right] \qquad (23)
$$

In a similar manner, using the second equation of equilibrium of stress, equation (16), the expression of τ_{yz} is written,

$$
\left(1+\frac{z}{r_1}\right)\tau_{yz} = \frac{3}{2}\frac{Q_{yz}}{h}\left[1-\left(\frac{z}{h/2}\right)^2\right] - \frac{1}{4}\left\{H_2^+P_2^+\left[1-2\left(\frac{z}{h/2}\right)-3\left(\frac{z}{h/2}\right)^2\right]+H_2^-P_2^-\left[1+2\left(\frac{z}{h/2}\right)-3\left(\frac{z}{h/2}\right)^2\right]\right\} \; . \qquad (24)
$$

Using the third equation of equilibrium, equation (16), together with equations (22) and (24), and noting the boundary condition of @ $z = \pm\frac{h}{2}$, $\tau_{zz} = q^{\pm}$, the transverse normal stress, τ_{zz}, becomes

$$
\begin{aligned}
\left(1+\frac{z}{r_1}\right)\left(1+\frac{z}{r_2}\right)\tau_{zz} = C\Bigg\{&\left[\frac{3}{2}\frac{S}{h}+\frac{T}{4h}\left(\frac{z}{h/2}\right)\right]\left[1-\left(\frac{z}{h/2}\right)^2\right] + \frac{1}{2}H^+q^+\left[1+\frac{3}{2}\left(\frac{z}{h/2}\right)-\frac{1}{2}\left(\frac{z}{h/2}\right)^3\right] \\
&+\frac{1}{2}H^-q^-\left[1-\frac{3}{2}\left(\frac{z}{h/2}\right)+\frac{1}{2}\left(\frac{z}{h/2}\right)^3\right]\Bigg\} \; ,
\end{aligned}
$$

(25)

where q^+ and q^- are the values of τ_{zz} at the top and bottom surfaces of the shell.

The parameters S and T are to be determined in the variational problem in the following section.

The coefficient c on the right hand side of equation (25) is introduced in order to distinguish the terms introduced by the transverse normal stress. In the final result, the value of c is set as unity.

The completed set of the approximation equations of stress distribution is given by equations (14), (22), (24) and (25).

2.5 Reissner's Variational Theorem

Reissner's variational theorem of three dimensional elasticity is written in the form

$$\delta I = \int_{t_1}^{t_2}\Bigg\{\iiint_V\Big\{\Big[\tau_{xx}\epsilon_{xx}+\tau_{yy}\epsilon_{yy}+\tau_{zz}\epsilon_{zz}+\tau_{xy}\gamma_{xy}+\tau_{xz}\gamma_{xz}+\tau_{yz}\gamma_{yz}\Big]$$
$$-\frac{1}{2E}\Big[\tau_{xx}^2+\tau_{yy}^2+\tau_{zz}^2-2\nu(\tau_{xx}\tau_{yy}+\tau_{xx}\tau_{zz}+\tau_{yy}\tau_{zz})$$
$$+2(1+\nu)(\tau_{xy}^2+\tau_{xz}^2+\tau_{yz}^2)\Big]$$
$$-\frac{\rho}{2}\Big[U_t^2+V_t^2+W_t^2\Big]\Big\}\left(1+\frac{z}{r_1}\right)\left(1+\frac{z}{r_2}\right)AB\,dx\,dy\,dz$$
$$-\iint_{S_1}\Big[(\overset{+}{P_1}\overset{+}{U}+\overset{+}{P_2}\overset{+}{V}+\overset{+}{q}\overset{+}{W})\left(1+\frac{h}{2r_1}\right)\left(1+\frac{h}{2r_2}\right)+(\overset{-}{P_1}\overset{-}{U}+\overset{-}{P_2}\overset{-}{V}+\overset{-}{q}\overset{-}{W})\left(1-\frac{h}{2r_1}\right)\left(1-\frac{h}{2r_2}\right)\Big]AB\,dx\,dy$$
$$-\oint\Big[\int_{-h/2}^{h/2}(\tau_{nn}U_{nn}+\tau_{n\hat{t}}U_{n\hat{t}}+\tau_{nz}W)\left(1+\frac{z}{r_{\hat{t}}}\right)dz\Big]A_{\hat{t}}\,ds_{\hat{t}}\Bigg\}dt = 0\,, \tag{26}$$

where E = Modulus of elasticity,

ν = Poisson's ratio.

The first term in the integrand represents twice the strain energy, the second term - the complementary energy, the third term - the kinetic energy, the fourth term - the workdone by the external forces on the upper and lower surfaces of the shell, the last term - the workdone by the edge forces.

Substituting the relation of stress and strain from equations (12),(13),(14),(22),(24) and (25) into equation (26), we obtain.

The variational equation then becomes;

$$
\begin{aligned}
\delta\int_{t_1}^{t_2}\Big\{\iiint_V\Big\{\Big[&\Big(\frac{N_{xx}}{h}+\frac{12}{h^3}zM_{xx}\Big)\Big(\overset{\circ}{\epsilon}_{xx}+zK_x+z^2C_x+\frac{z^3}{2A^2}\bar{W}'_x\bar{W}''_x+\frac{z^4}{8A^2}\bar{W}''^2_x\Big)\\
&+\Big(\frac{N_{yy}}{h}+\frac{12}{h^3}zM_{yy}\Big)\Big(\overset{\circ}{\epsilon}_{yy}+zK_y+z^2C_y+\frac{z^3}{2B^2}\bar{W}'_y\bar{W}''_y+\frac{z^4}{8B^2}\bar{W}''^2_y\Big)\\
&+c\Big\{\Big[\frac{3}{2}\frac{S}{h}+\frac{T}{4h}\Big(\frac{z}{h/2}\Big)\Big]\Big[1-\Big(\frac{z}{h/2}\Big)^2\Big]+\frac{1}{2}\overset{+}{q}\overset{+}{H}\Big[1+\frac{3}{2}\Big(\frac{z}{h/2}\Big)-\frac{1}{2}\Big(\frac{z}{h/2}\Big)^3\Big]\\
&\qquad+\frac{1}{2}\overset{-}{q}\overset{-}{H}\Big[1-\frac{3}{2}\Big(\frac{z}{h/2}\Big)+\frac{1}{2}\Big(\frac{z}{h/2}\Big)^3\Big]\Big\}(\bar{W}'+z\bar{W}'')\\
&+\Big(1+\frac{z}{r_2}\Big)^{-1}\Big\{\Big[\frac{N_{xy}}{h}+\frac{12}{h^3}zM_{xy}\Big]\Big[\Big(1+\frac{z}{r_2}\Big)(\overset{\circ}{\gamma}_{xx}+z\delta_{xx})+\Big(1+\frac{z}{r_1}\Big)(\overset{\circ}{\gamma}_{yy}+z\delta_{yy})+\frac{1}{AB}\bar{W}'_x\bar{W}'_y\\
&\qquad+zD_{xy}+z^2E_{xy}+z^3F_{xy}+\frac{z^4}{4AB}\bar{W}''_x\bar{W}''_y\Big]\Big\}\\
&+\Big\{\frac{3}{2}\frac{Q_{xz}}{h}\Big[1-\Big(\frac{z}{h/2}\Big)^2\Big]-\frac{1}{4}\overset{+}{P}_1\overset{+}{H}_1\Big[1-2\Big(\frac{z}{h/2}\Big)-3\Big(\frac{z}{h/2}\Big)^2\Big]-\frac{1}{4}\overset{-}{P}_1\overset{-}{H}_1\Big[1+2\Big(\frac{z}{h/2}\Big)-3\Big(\frac{z}{h/2}\Big)^2\Big]\Big\}\\
&\qquad\Big[\overset{\circ}{\gamma}_{xz}+\frac{z}{A}\Big(\bar{W}'_x+\frac{z}{2}\bar{W}''_x\Big)\Big]\\
&+\Big\{\frac{3}{2}\frac{Q_{yz}}{h}\Big[1-\Big(\frac{z}{h/2}\Big)^2\Big]-\frac{1}{4}\overset{+}{P}_2\overset{+}{H}_2\Big[1-2\Big(\frac{z}{h/2}\Big)-3\Big(\frac{z}{h/2}\Big)^2\Big]-\frac{1}{4}\overset{-}{P}_2\overset{-}{H}_2\Big[1+2\Big(\frac{z}{h/2}\Big)-3\Big(\frac{z}{h/2}\Big)^2\Big]\Big\}\\
&\qquad\Big[\overset{\circ}{\gamma}_{yz}+\frac{z}{B}\Big(\bar{W}'_y+\frac{z}{2}\bar{W}''_y\Big)\Big]
\end{aligned}
$$

$$-\frac{1}{2E}\Bigg\{(1+\frac{z}{r_1})(1+\frac{z}{r_2})^{-1}\Big[\frac{N_{xx}}{h}+\frac{12}{h^3}zM_{xx}\Big]^2+(1+\frac{z}{r_1})^{-1}(1+\frac{z}{r_2})\Big[\frac{N_{yy}}{h}+\frac{12}{h^3}zM_{yy}\Big]^2$$

$$+(1+\frac{z}{r_1})^{-1}(1+\frac{z}{r_2})^{-1}c^2\Big\{\Big[\frac{3}{2}\frac{S}{h}+\frac{T}{4h}(\frac{z}{h/2})\Big]\Big[1-(\frac{z}{h/2})^2\Big]+\frac{1}{2}q^+H^+\Big[1+\frac{3}{2}(\frac{z}{h/2})-\frac{1}{2}(\frac{z}{h/2})^3\Big]$$

$$+\frac{1}{2}q^-H^-\Big[1-\frac{3}{2}(\frac{z}{h/2})+\frac{1}{2}(\frac{z}{h/2})^3\Big]\Big\}^2$$

$$-2\nu\Big\{(\frac{N_{xx}}{h}+\frac{12}{h^3}zM_{xx})(\frac{N_{yy}}{h}+\frac{12}{h^3}zM_{yy})$$

$$+c(1+\frac{z}{r_2})^{-1}\Big[\frac{N_{xx}}{h}+\frac{12}{h^3}zM_{xx}\Big]\Big\{\Big[\frac{3}{2}\frac{S}{h}+\frac{T}{4h}(\frac{z}{h/2})\Big]\Big[1-(\frac{z}{h/2})^2\Big]$$

$$+\frac{1}{2}q^+H^+\Big[1+\frac{3}{2}(\frac{z}{h/2})-\frac{1}{2}(\frac{z}{h/2})^3\Big]$$

$$+\frac{1}{2}q^-H^-\Big[1-\frac{3}{2}(\frac{z}{h/2})+\frac{1}{2}(\frac{z}{h/2})^3\Big]\Big\}$$

$$+c(1+\frac{z}{r_1})^{-1}\Big[\frac{N_{yy}}{h}+\frac{12}{h^3}zM_{yy}\Big]\Big\{\Big[\frac{3}{2}\frac{S}{h}+\frac{T}{4h}(\frac{z}{h/2})\Big]\Big[1-(\frac{z}{h/2})^2\Big]$$

$$+\frac{1}{2}q^+H^+\Big[1+\frac{3}{2}(\frac{z}{h/2})-\frac{1}{2}(\frac{z}{h/2})^3\Big]$$

$$+\frac{1}{2}q^-H^-\Big[1-\frac{3}{2}(\frac{z}{h/2})+\frac{1}{2}(\frac{z}{h/2})^3\Big]\Big\}\Big\}$$

$$+2(1+\nu)\Big\{(1+\frac{z}{r_1})(1+\frac{z}{r_2})^{-1}\Big[\frac{N_{xy}}{h}+\frac{12}{h^3}zM_{xy}\Big]^2$$

$$+(1+\frac{z}{r_1})(1+\frac{z}{r_2})^{-1}\Big\{\frac{3}{2}\frac{Q_{xz}}{h}\Big[1-(\frac{z}{h/2})^2\Big]-\frac{1}{4}P_1^+H_1^+\Big[1-2(\frac{z}{h/2})-3(\frac{z}{h/2})^2\Big]$$

$$-\frac{1}{4}P_1^-H_1^-\Big[1+2(\frac{z}{h/2})-3(\frac{z}{h/2})^2\Big]\Big\}^2$$

$$+(1+\frac{z}{r_1})^{-1}(1+\frac{z}{r_2})\Big\{\frac{3}{2}\frac{Q_{yz}}{h}\Big[1-(\frac{z}{h/2})^2\Big]-\frac{1}{4}P_2^+H_2^+\Big[1-2(\frac{z}{h/2})-3(\frac{z}{h/2})^2\Big]$$

$$-\frac{1}{4}P_2^-H_2^-\Big[1+2(\frac{z}{h/2})-3(\frac{z}{h/2})^2\Big]\Big\}^2\Big\}$$

$$\Bigg\}AB\,dx\,dy\,dz$$

$$-\frac{\rho}{2}\iiint_V\Big\{[\bar{u}_t+z\phi_t]^2+[\bar{v}_t+z\psi_t]^2+[\bar{w}_t+z\bar{w}'_t+\frac{z^2}{2}\bar{w}''_t]^2\Big\}(1+\frac{z}{r_1})(1+\frac{z}{r_2})AB\,dx\,dy\,dz$$

$$-\iint_{S_1}\Big\{\big[\overset{+}{P_1}(\bar{U}+z\Phi)^{+}+\overset{+}{P_2}(\bar{V}+z\Psi)^{+}+\overset{+}{q}(\bar{W}+z\bar{W}'+\frac{z^2}{2}\bar{W}'')\big](1+\frac{h}{2r_1})(1+\frac{h}{2r_2})$$

$$+\big[\overset{-}{P_1}(\bar{U}+z\Phi)^{-}+\overset{-}{P_2}(\bar{V}+z\Psi)^{-}+\overset{-}{q}(\bar{W}+z\bar{W}'+\frac{z^2}{2}\bar{W}'')\big](1-\frac{h}{2r_1})(1-\frac{h}{2r_2})\Big\}AB\,dx\,dy$$

$$-\oint\Big\{\int_{-h/2}^{h/2}\big[\tau_{nn}(\bar{U}_{nn}+z\Phi_{nn})+\tau_{n\hat{t}}(\bar{U}_{n\hat{t}}+z\Phi_{n\hat{t}})+\tau_{nz}(\bar{W}+z\bar{W}'+\frac{z^2}{2}\bar{W}'')\big](1+\frac{z}{r_{\hat{t}}})\,dz$$

$$\Big\}A_{\hat{t}}\,ds_{\hat{t}}\Big\}dt = 0 \quad .$$

(27)

Before carrying out the variation in equation (27), the following approximation is introduced. In the expansion of the function $(1+\frac{z}{r})^n$ only the terms up to $\frac{z^2}{r^2}$ are retained, thus

$$\int_{-h/2}^{h/2}(1+\frac{z}{r_2})(1+\frac{z}{r_1})^{-1}dz \cong h\Big[1+\frac{h^2}{12r_1}(\frac{1}{r_1}-\frac{1}{r_2})\Big] \; ,$$

$$\int_{-h/2}^{h/2}(1+\frac{z}{r_2})(1+\frac{z}{r_1})^{-1}z^2dz \cong \frac{h^3}{12}\Big[1+\frac{3}{20}\frac{h^2}{r_1}(\frac{1}{r_1}-\frac{1}{r_2})\Big] \; . \qquad (28)$$

Using equation (14), the expression for N_{yx} and M_{yx} is written as

$$\begin{Bmatrix} N_{yx} \\ M_{yx} \end{Bmatrix} = \int_{-h/2}^{h/2}\Big[\frac{N_{xy}}{h}+\frac{12}{h^3}zM_{xy}\Big](1+\frac{z}{r_2})^{-1}\begin{Bmatrix} 1 \\ z \end{Bmatrix}(1+\frac{z}{r_1})\,dz \; .$$

(29)

The following integral relating the inplane forces and bending moments, can be shown to hold,

$$\iiint(\frac{N_{xy}}{h}+\frac{12}{h^3}z\,M_{xy})(\overset{\circ}{\gamma}_{yy}+z\,\delta_{yy})(1+\frac{z}{r_1})(1+\frac{z}{r_2})^{-1}AB\,dx\,dy\,dz$$

$$=\iint(N_{yx}\overset{\circ}{\gamma}_{yy}+M_{yx}\delta_{yy})AB\,dx\,dy \; .$$

(30)

Carrying out the integration in equation (27) with respect to z in the limit of $\pm \frac{h}{2}$ yields

$$\delta \int_{t_1}^{t_2} \Big\{ \iint_S \Big\{ \Big[\frac{N_{xx}}{h} \Big(h\overset{\circ}{\epsilon}_{xx} + \frac{h^3}{12} C_x + \frac{h^5}{640A^2} \bar{W}''^2_x \Big) + \frac{12}{h^3} M_{xx} \Big(\frac{h^3}{12} K_x + \frac{h^5}{160A^2} \bar{W}'_x \bar{W}''_x \Big) \Big]$$

$$+ \Big[\frac{N_{yy}}{h} \Big(h\overset{\circ}{\epsilon}_{yy} + \frac{h^3}{12} C_y + \frac{h^5}{640B^2} \bar{W}''^2_y \Big) + \frac{12}{h^3} M_{yy} \Big(\frac{h^3}{12} K_y + \frac{h^5}{160B^2} \bar{W}'_y \bar{W}''_y \Big) \Big]$$

$$+ c \Big[\Big(S + \frac{h}{2} \overset{+}{q}\overset{+}{H} + \frac{h}{2} \overset{-}{q}\overset{-}{H} \Big) \bar{W}' + \Big(\frac{T}{4h} \frac{h^2}{15} + \frac{h^2}{10} \overset{+}{q}\overset{+}{H} - \frac{h^2}{10} \overset{-}{q}\overset{-}{H} \Big) \bar{W}'' \Big]$$

$$+ \Big[\frac{N_{xy}}{h} \Big\{ h\overset{\circ}{\gamma}_{xx} + \frac{\bar{W}_x \bar{W}_y}{AB} \Big(h + \frac{h^3}{12r_2^2} \Big) + D_{xy} \Big(-\frac{h^3}{12r_2} \Big) + E_{xy} \Big(\frac{h^3}{12} + \frac{h^5}{80r_2^2} \Big) + F_{xy} \Big(-\frac{h^5}{80r_2} \Big)$$

$$+ \frac{\bar{W}''_x \bar{W}''_y}{4AB} \Big(\frac{h^5}{80} + \frac{h^7}{448r_2^2} \Big) \Big\} + \frac{12}{h^3} M_{xy} \Big\{ \frac{h^3}{12} \delta_{xx} + \frac{\bar{W}_x \bar{W}_y}{AB} \Big(-\frac{h^3}{12r_2} \Big) + D_{xy} \Big(\frac{h^3}{12} + \frac{h^5}{80r_2^2} \Big)$$

$$+ E_{xy} \Big(-\frac{h^5}{80r_2} \Big) + F_{xy} \Big(\frac{h^5}{80} + \frac{h^7}{448r_2^2} \Big) + \frac{\bar{W}'_x \bar{W}''_y}{4AB} \Big(-\frac{h^7}{448r_2} \Big) \Big\} + \Big\{ N_{yx} \overset{\circ}{\gamma}_{yy} + M_{yx} \delta_{yy} \Big\} \Big]$$

$$+ \Big[Q_{xz} \overset{\circ}{\gamma}_{xz} + \Big(\frac{h^2}{12} \overset{+}{P}_1 \overset{+}{H}_1 - \frac{h^2}{12} \overset{-}{P}_1 \overset{-}{H}_1 \Big) \frac{\bar{W}'_x}{A} + \Big(\frac{3}{2} \frac{Q_{xz}}{h} \Big(\frac{h^3}{30} \Big) + \frac{h^3}{60} \overset{+}{P}_1 \overset{+}{H}_1 + \frac{h^3}{60} \overset{-}{P}_1 \overset{-}{H}_1 \Big) \frac{\bar{W}''_x}{2A} \Big]$$

$$+ \Big[Q_{yz} \overset{\circ}{\gamma}_{yz} + \Big(\frac{h^2}{12} \overset{+}{P}_2 \overset{+}{H}_2 - \frac{h^2}{12} \overset{-}{P}_2 \overset{-}{H}_2 \Big) \frac{\bar{W}'_y}{B} + \Big(\frac{3}{2} \frac{Q_{yz}}{h} \Big(\frac{h^3}{30} \Big) + \frac{h^3}{60} \overset{+}{P}_2 \overset{+}{H}_2 + \frac{h^3}{60} \overset{-}{P}_2 \overset{-}{H}_2 \Big) \frac{\bar{W}''_y}{2B} \Big]$$

$$- \frac{1}{2E} \Big\{ \Big[\Big(\frac{N_{xx}}{h} \Big)^2 \Big[h + \frac{h^3}{12r_2} \Big(\frac{1}{r_2} - \frac{1}{r_1} \Big) \Big] + \frac{24}{h^3} \Big(\frac{N_{xx}}{h} \Big) M_{xx} \Big[\frac{h^3}{12} \Big(\frac{1}{r_1} - \frac{1}{r_2} \Big) \Big] + \Big(\frac{12}{h^3} M_{xx} \Big)^2 \Big[\frac{h^3}{12} + \frac{h^5}{80r_2} \Big(\frac{1}{r_2} - \frac{1}{r_1} \Big) \Big] \Big]$$

$$+ \Big[\Big(\frac{N_{yy}}{h} \Big)^2 \Big[h + \frac{h^3}{12r_1} \Big(\frac{1}{r_1} - \frac{1}{r_2} \Big) \Big] + \frac{24}{h^3} \Big(\frac{N_{yy}}{h} \Big) M_{yy} \Big[\frac{h^3}{12} \Big(\frac{1}{r_2} - \frac{1}{r_1} \Big) \Big] + \Big(\frac{12}{h^3} M_{yy} \Big)^2 \Big[\frac{h^3}{12} + \frac{h^5}{80r_1} \Big(\frac{1}{r_1} - \frac{1}{r_2} \Big) \Big] \Big]$$

$$+ c^2 \Big[\Big(\frac{3}{2} \frac{S}{h} \Big)^2 \Big[\frac{8h}{15} + \frac{2h^3}{105} \Big(\frac{1}{r_1^2} + \frac{1}{r_1 r_2} + \frac{1}{r_2^2} \Big) \Big] + \overset{+}{q}\overset{+}{H} \Big(\frac{3}{2} \frac{S}{h} \Big) \Big[\frac{2}{3} h - \frac{3}{35} h^2 \Big(\frac{1}{r_1} + \frac{1}{r_2} \Big) + \frac{h^3}{30} \Big(\frac{1}{r_1^2} + \frac{1}{r_1 r_2} + \frac{1}{r_2^2} \Big) \Big]$$

$$+ \overset{-}{q}\overset{-}{H} \Big(\frac{3}{2} \frac{S}{h} \Big) \Big[\frac{2}{3} h + \frac{3}{35} h^2 \Big(\frac{1}{r_1} + \frac{1}{r_2} \Big) + \frac{h^3}{30} \Big(\frac{1}{r_1^2} + \frac{1}{r_1 r_2} + \frac{1}{r_2^2} \Big) \Big]$$

$$+\left(\frac{T}{4h}\right)^2\left[\frac{8h}{105}+\frac{2h^3}{315}\left(\frac{1}{r_1^2}+\frac{1}{r_1r_2}+\frac{1}{r_2^2}\right)\right]+\overset{+}{q}\overset{+}{H}\left(\frac{T}{4h}\right)\left[\frac{6h}{35}-\frac{h^2}{15}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)+\frac{11}{630}h^3\left(\frac{1}{r_1^2}+\frac{1}{r_1r_2}+\frac{1}{r_2^2}\right)\right]$$

$$-\overset{-}{q}\overset{-}{H}\left(\frac{T}{4h}\right)\left[\frac{6}{35}h+\frac{h^2}{15}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)+\frac{11}{630}\left(\frac{1}{r_1^2}+\frac{1}{r_1r_2}+\frac{1}{r_2^2}\right)\right]$$

$$+\left(\frac{3}{2}\frac{S}{h}\right)\left(\frac{T}{4h}\right)\left[-\frac{4}{105}h^2\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right]\Bigg]$$

$$-2\nu\Bigg\{\left[\left(\frac{N_{xx}}{h}\right)\left(\frac{N_{yy}}{h}\right)h+\left(\frac{12}{h^3}\right)^2\frac{h^3}{12}(M_{xx})(M_{yy})\right]$$

$$+c\Bigg[\left(\frac{N_{xx}}{h}\right)\left\{\frac{3}{2}\frac{S}{h}\left(\frac{2h}{3}+\frac{h^3}{20r_2^2}\right)-\frac{T}{4h}\left(\frac{h^2}{15r_2}\right)+\frac{1}{2}\overset{+}{q}\overset{+}{H}\left(h-\frac{h^2}{5r_2}+\frac{h^3}{12r_2^2}\right)+\frac{1}{2}\overset{-}{q}\overset{-}{H}\left(h+\frac{h^2}{5r_2}+\frac{h^3}{12r_2^2}\right)\right\}$$

$$+\left(\frac{12}{h^3}M_{xx}\right)\left\{\frac{3}{2}\frac{S}{h}\left(-\frac{h^3}{20r_2}\right)+\frac{T}{4h}\left(\frac{h^2}{15}+\frac{h^4}{140r_2^2}\right)+\frac{1}{2}\overset{+}{q}\overset{+}{H}\left(\frac{h^2}{5}-\frac{h^3}{12r_2}+\frac{h^4}{35r_2^2}\right)-\frac{1}{2}\overset{-}{q}\overset{-}{H}\left(\frac{h^2}{5}+\frac{h^3}{12r_2}+\frac{h^4}{35r_2^2}\right)\right\}\Bigg]$$

$$+c\Bigg[\left(\frac{N_{yy}}{h}\right)\left\{\frac{3}{2}\frac{S}{h}\left(\frac{2h}{3}+\frac{h^3}{30r_1^2}\right)-\frac{T}{4h}\left(\frac{h^2}{15r_1}\right)+\frac{1}{2}\overset{+}{q}\overset{+}{H}\left(h-\frac{h^2}{5r_1}+\frac{h^3}{12r_1^2}\right)+\frac{1}{2}\overset{-}{q}\overset{-}{H}\left(h+\frac{h^2}{5r_1}+\frac{h^3}{12r_1^2}\right)\right\}$$

$$+\left(\frac{12}{h^3}M_{yy}\right)\left\{\frac{3}{2}\frac{S}{h}\left(-\frac{h^3}{20r_1}\right)+\frac{T}{4h}\left(\frac{h^2}{15}+\frac{h^4}{140r_1^2}\right)+\frac{1}{2}\overset{+}{q}\overset{+}{H}\left(\frac{h^2}{5}-\frac{h^3}{12r_1}+\frac{h^4}{35r_1^2}\right)-\frac{1}{2}\overset{-}{q}\overset{-}{H}\left(\frac{h^2}{5}+\frac{h^3}{12r_1}+\frac{h^4}{35r_1^2}\right)\right\}\Bigg]\Bigg\}$$

$$+2(1+\nu)\Bigg\{\left[\left(\frac{N_{xy}}{h}\right)^2\left[h+\frac{h^3}{12r_2}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\right]+\frac{24}{h^3}\left(\frac{N_{xy}}{h}\right)M_{xy}\left[\frac{h^3}{12}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]+\left(\frac{12}{h^3}M_{xx}\right)^2\left[\frac{h^3}{12}+\frac{h^5}{80r_2}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\right]\right]$$

$$+\Bigg[\left(\frac{3}{2}\frac{Q_{xz}}{h}\right)^2\left[\frac{8h}{15}+\frac{4h^3}{210r_2}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\right]-\left(\frac{3}{2}\frac{Q_{xz}}{h}\right)\left(\frac{1}{4}\overset{+}{P_1}\overset{+}{H_1}\right)\left[\frac{4h}{15}-\frac{2h^2}{15}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)-\frac{h^3}{105r_2}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\right]$$

$$-\left(\frac{3}{2}\frac{Q_{xz}}{h}\right)\left(\frac{1}{4}\overset{-}{P_1}\overset{-}{H_1}\right)\left[\frac{4h}{15}+\frac{2h^2}{15}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)-\frac{h^3}{105r_2}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\right]\Bigg]$$

$$+\Bigg[\left(\frac{3}{2}\frac{Q_{yz}}{h}\right)^2\left[\frac{8h}{15}+\frac{4h^3}{210r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]-\left(\frac{3}{2}\frac{Q_{yz}}{h}\right)\left(\frac{1}{4}\overset{+}{P_2}\overset{+}{H_2}\right)\left[\frac{4h}{15}-\frac{2h^2}{15}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)-\frac{h^3}{105r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]$$

$$-\left(\frac{3}{2}\frac{Q_{yz}}{h}\right)\left(\frac{1}{4}\overset{-}{P_2}\overset{-}{H_2}\right)\left[\frac{4h}{15}+\frac{2h^2}{15}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)-\frac{h^3}{105r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]\Bigg]\Bigg\}$$

$$\Bigg\}\Bigg\}AB\,dx\,dy$$

$$-\frac{\rho}{2}\iint_S\Big\{(\bar{u}_t^2+\bar{v}_t^2+\bar{w}_t^2)\Big[h(1+\frac{h^2}{12r_1r_2})\Big]+2\Big[\bar{u}_t\phi_t+\bar{v}_t\psi_t+\bar{w}_t\bar{w}'_t\Big]\frac{h^3}{12}(\frac{1}{r_1}+\frac{1}{r_2})$$

$$+(\phi_t^2+\psi_t^2+\bar{w}_t'^2+\bar{w}_t\bar{w}''_t)\Big[\frac{h^3}{12}(1+\frac{3h^2}{20r_1r_2})\Big]+(\bar{w}'_t\bar{w}''_t)\frac{h^5}{80}(\frac{1}{r_1}+\frac{1}{r_2})+\frac{\bar{w}_t''^2}{4}\Big[\frac{h^5}{80}(1+\frac{5h^2}{28r_1r_2})\Big]$$

$$\Big\}AB\,dx\,dy$$

$$-\iint_{S_1}\Big\{\Big[\overset{+}{P_1}(\bar{u}+z\phi)^{+}+\overset{+}{P_2}(\bar{v}+z\psi)^{+}+\overset{+}{q}(\bar{w}+z\bar{w}'+\frac{z^2}{2}\bar{w}'')^{+}\Big](1+\frac{h}{2r_1})(1+\frac{h}{2r_2})$$

$$+\Big[\overset{-}{P_1}(\bar{u}+z\phi)^{-}+\overset{-}{P_2}(\bar{v}+z\psi)^{-}+\overset{-}{q}(\bar{w}+z\bar{w}'+\frac{z^2}{2}\bar{w}'')^{-}\Big](1-\frac{h}{2r_1})(1-\frac{h}{2r_2})\Big\}AB\,dx\,dy$$

$$-\oint\Big\{N_{nn}\bar{u}_{nn}+M_{nn}\phi_{nn}+N_{n\hat{t}}\bar{u}_{n\hat{t}}+M_{n\hat{t}}\phi_{n\hat{t}}+Q_{nz}\bar{w}+\frac{h^2}{12}P_n\bar{w}'+\frac{h^2}{40}Q_{nz}\bar{w}''$$

$$+\frac{h^2}{60}m_n\bar{w}''\Big\}A_{\hat{t}}\,ds_{\hat{t}}\Big\}dt=0\,, \qquad (31)$$

where,

$$P_{\hat{n}} = (\overset{+}{H}_{\hat{n}}\overset{+}{P}_{\hat{n}} - \overset{-}{H}_{\hat{n}}\overset{-}{P}_{\hat{n}}),$$

$$m_{\hat{n}} = \frac{h}{2}(\overset{+}{H}_{\hat{n}}\overset{+}{P}_{\hat{n}} + \overset{-}{H}_{\hat{n}}\overset{-}{P}_{\hat{n}}), \quad (\hat{n} = n,1,2). \qquad (32)$$

Integrating by parts and carrying out the variational procedure yields

$$\int_{t_1}^{t_2}\Big\{\iint_S\Big\{-\Big[\frac{\partial}{\partial x}(BN_{xx})+\frac{\partial}{\partial y}(AN_{yx})+A_yN_{xy}-B_xN_{yy}+AB(\frac{Q_{xz}}{r_1}+P_1)$$

$$-\rho hAB\Big\{(1+\frac{h^2}{12r_1r_2})\bar{u}_{tt}+\frac{h^2}{12}(\frac{1}{r_1}+\frac{1}{r_2})\phi_{tt}\Big\}\Big]\delta\bar{u}$$

$$-\Big[\frac{\partial}{\partial x}(BN_{xy})+\frac{\partial}{\partial y}(AN_{yy})+B_xN_{yx}-A_yN_{xx}+AB(\frac{Q_{yz}}{r_2}+P_2)$$

$$-\rho hAB\Big\{(1+\frac{h^2}{12r_1r_2})\bar{v}_{tt}+\frac{h^2}{12}(\frac{1}{r_1}+\frac{1}{r_2})\psi_{tt}\Big\}\Big]\delta\bar{v}$$

$$-\Big[\frac{\partial}{\partial x}(BM_{xx})+\frac{\partial}{\partial y}(AM_{yx})+A_yM_{xy}-B_xM_{yy}-AB(Q_{xz}-m_1)$$

$$-\rho\frac{h^3}{12}AB\Big\{(\frac{1}{r_1}+\frac{1}{r_2})\bar{u}_{tt}+(1+\frac{3h^2}{20r_1r_2})\phi_{tt}\Big\}\Big]\delta\phi$$

$$-\Big[\frac{\partial}{\partial x}(BM_{xy}) + \frac{\partial}{\partial y}(AM_{yy}) + B_x M_{yx} - A_y M_{xx} - AB(Q_{yz} - m_2)$$

$$-\rho\frac{h^3}{12}AB\Big\{\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)\bar V_{tt} + \Big(1+\frac{3h^2}{20r_1r_2}\Big)\psi_{tt}\Big\}\Big]\delta\psi$$

$$-\Big[\frac{\partial}{\partial x}(BQ_{xz}) + \frac{\partial}{\partial y}(AQ_{yz}) - \frac{AB}{r_1}N_{xx} - \frac{AB}{r_2}N_{yy} + AB(\overset{+}{q}\overset{+}{H} - \overset{-}{q}\overset{-}{H})$$

$$+\frac{\partial}{\partial x}\Big\{BN_{xx}\Big(\frac{\bar W_x}{A} + \frac{h^2}{24}\frac{\bar W''_x}{A}\Big) + BM_{xx}\Big(\frac{\bar W'_x}{A}\Big) + N_{xy}\Big[\Big(1+\frac{h^2}{12r_2^2}\Big)\bar W_y - \frac{h^2}{12r_2}\bar W'_y + \frac{h^2}{24}\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar W''_y\Big]$$

$$- M_{xy}\Big[\frac{\bar W_y}{r_2} - \Big(1+\frac{3h^2}{20r_2^2}\Big)\bar W'_y + \frac{3h^2}{40r_2}\bar W''_y\Big]\Big\}$$

$$+\frac{\partial}{\partial y}\Big\{AN_{yy}\Big(\frac{\bar W_y}{B} + \frac{h^2}{24}\frac{\bar W''_y}{B}\Big) + AM_{yy}\Big(\frac{\bar W'_y}{B}\Big) + N_{xy}\Big[\Big(1+\frac{h^2}{12r_2^2}\Big)\bar W_x - \frac{h^2}{12r_2}\bar W'_x + \frac{h^2}{24}\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar W''_x\Big]$$

$$- M_{xy}\Big[\frac{\bar W_x}{r_2} - \Big(1+\frac{3h^2}{20r_2^2}\Big)\bar W'_x + \frac{3h^2}{40r_2}\bar W''_x\Big]\Big\}$$

$$-\rho h AB\Big\{\Big(1+\frac{h^2}{12r_1r_2}\Big)\bar W_{tt} + \frac{h^2}{12}\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)\bar W'_{tt} + \frac{h^2}{24}\Big(1+\frac{3h^2}{20r_1r_2}\Big)\bar W''_{tt}\Big\}\Big]\delta\bar W$$

$$+\Big\{\Big[\frac{M_{xx}}{r_1} + \frac{M_{yy}}{r_2} + c\Big[S + \frac{h}{2}(\overset{+}{q}\overset{+}{H} + \overset{-}{q}\overset{-}{H})\Big] - \frac{h}{2}(\overset{+}{q}\overset{+}{H} + \overset{-}{q}\overset{-}{H}) - \frac{h^2}{12AB}\Big[\frac{\partial}{\partial x}(BP_1) + \frac{\partial}{\partial y}(AP_2)\Big]\Big]AB$$

$$-\frac{\partial}{\partial x}\Big\{BN_{xx}\Big(\frac{h^2}{12}\frac{\bar W'_x}{A}\Big) + BM_{xx}\Big[\frac{\bar W_x}{A} + \frac{3h^2}{40}\frac{\bar W''_x}{A}\Big] - N_{xy}\Big[\frac{h^2}{12r_2}\bar W_y - \frac{h^2}{12}\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar W'_y + \frac{h^4}{160r_2}\bar W''_y\Big]$$

$$+ M_{xy}\Big[\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar W_y - \frac{3h^2}{20r_2}\bar W'_y + \frac{3h^2}{40}\Big(1+\frac{5h^2}{28r_2^2}\Big)\bar W''_y\Big]\Big\}$$

$$-\frac{\partial}{\partial y}\Big\{AN_{yy}\Big(\frac{h^2}{12}\frac{\bar W'_y}{B}\Big) + AM_{yy}\Big[\frac{\bar W_y}{B} + \frac{3h^2}{40}\frac{\bar W''_y}{B}\Big] - N_{xy}\Big[\frac{h^2}{12r_2}\bar W_x - \frac{h^2}{12}\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar W'_x + \frac{h^4}{160r_2}\bar W''_x\Big]$$

$$+ M_{xy}\Big[\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar W_x - \frac{3h^2}{20r_2}\bar W'_x + \frac{3h^2}{40}\Big(1+\frac{5h^2}{28r_2^2}\Big)\bar W''_x\Big]\Big\}$$

$$+\rho\frac{h^3}{12}AB\Big\{\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)\bar W_{tt} + \Big(1+\frac{3h^2}{20r_1r_2}\Big)\bar W'_{tt} + \frac{3h^2}{40}\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)\bar W''_{tt}\Big\}\Big\}\delta\bar W'$$

$$
+\Bigg\{\Bigg[\frac{h^2}{24}\Big(\frac{N_{xx}}{r_1}+\frac{N_{yy}}{r_2}\Big)+c\Big[\frac{h}{60}T+\frac{h^2}{10}\big(\overset{+}{q}\overset{+}{H}-\overset{-}{q}\overset{-}{H}\big)\Big]-\frac{h^2}{8}\big(\overset{+}{q}\overset{+}{H}-\overset{-}{q}\overset{-}{H}\big)-\frac{h^2}{60AB}\Big[\frac{\partial}{\partial x}(Bm_1)+\frac{\partial}{\partial y}(Am_2)\Big]\Bigg]AB
$$

$$
-\frac{\partial}{\partial x}\Big\{BN_{xx}\Big[\frac{h^2}{24}\frac{\bar{W}_x}{A}+\frac{h^4}{320}\frac{\bar{W}''_x}{A}\Big]+\frac{3h^2}{40}BM_{xx}\Big(\frac{\bar{W}'_x}{A}\Big)+N_{xy}\Big[\frac{h^2}{24}\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar{W}_y-\frac{h^4}{160r_2}\bar{W}'_y+\frac{h^4}{80}\Big(1+\frac{5h^2}{28r_2^2}\Big)\frac{\bar{W}''_y}{4}\Big]
$$

$$
-M_{xy}\Big[\frac{3h^2}{40r_2}\bar{W}_y-\frac{3h^2}{40}\Big(1+\frac{5h^2}{28r_2^2}\Big)\bar{W}'_y+\frac{3h^4}{112r_2}\frac{\bar{W}''_y}{4}\Big]+\frac{h^2}{40}BQ_{xz}\Big\}
$$

$$
-\frac{\partial}{\partial y}\Big\{AN_{yy}\Big[\frac{h^2}{24}\frac{\bar{W}_y}{B}+\frac{h^4}{320}\frac{\bar{W}''_y}{B}\Big]+\frac{3h^2}{40}AM_{yy}\Big(\frac{\bar{W}'_y}{B}\Big)+N_{xy}\Big[\frac{h^2}{24}\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar{W}_x-\frac{h^4}{160r_2}\bar{W}'_x+\frac{h^4}{80}\Big(1+\frac{5h^2}{28r_2^2}\Big)\frac{\bar{W}''_x}{4}\Big]
$$

$$
-M_{xy}\Big[\frac{3h^2}{40r_2}\bar{W}_x-\frac{3h^2}{40}\Big(1+\frac{5h^2}{28r_2^2}\Big)\bar{W}'_x+\frac{3h^4}{112r_2}\frac{\bar{W}''_x}{4}\Big]+\frac{h^2}{40}AQ_{yz}\Big\}
$$

$$
+\rho\frac{h^3}{24}AB\Big\{\Big(1+\frac{3h^2}{20r_1r_2}\Big)\bar{W}_{tt}+\frac{3h^2}{20}\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)\bar{W}'_{tt}+\frac{3h^2}{40}\Big(1+\frac{5h^2}{28r_1r_2}\Big)\bar{W}''_{tt}\Big\}\Bigg\}\delta\bar{W}''\Bigg\}dx\,dy\Bigg\}dt
$$

$$
+\int_{t_1}^{t_2}\Bigg\{\iint_S\Bigg\{\Bigg[\mathring{\epsilon}_{xx}+\frac{h^2}{24}\Big[\frac{\bar{W}''}{r_1}+\frac{1}{A^2}\big(\bar{W}_x\bar{W}''_x+\bar{W}'^2_x\big)\Big]+\frac{h^4}{640}\frac{\bar{W}''^2_x}{A^2}-\frac{1}{Eh}\Big\{\Big[1-\frac{h^2}{12r_2}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big]N_{xx}-\nu N_{yy}
$$

$$
+\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)M_{xx}-\nu c\Big[\Big(1+\frac{h^2}{20r_2^2}\Big)S-\frac{h}{60r_2}T+\frac{h}{2}\overset{+}{q}\overset{+}{H}\Big(1-\frac{h}{5r_2}+\frac{h^2}{12r_2^2}\Big)+\frac{h}{2}\overset{-}{q}\overset{-}{H}\Big(1+\frac{h}{5r_2}+\frac{h^2}{12r_2^2}\Big)\Big]\Big\}\Bigg]\delta N_{xx}
$$

$$
+\Bigg[\mathring{\epsilon}_{yy}+\frac{h^2}{24}\Big[\frac{\bar{W}''}{r_2}+\frac{1}{B^2}\big(\bar{W}_y\bar{W}''_y+\bar{W}'^2_y\big)\Big]+\frac{h^4}{640}\frac{\bar{W}''^2_y}{B^2}-\frac{1}{Eh}\Big\{\Big[1+\frac{h^2}{12r_1}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big]N_{yy}-\nu N_{xx}
$$

$$
-\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)M_{yy}-\nu c\Big[\Big(1+\frac{h^2}{20r_1^2}\Big)S-\frac{h}{60r_1}T+\frac{h}{2}\overset{+}{q}\overset{+}{H}\Big(1-\frac{h}{5r_1}+\frac{h^2}{12r_1^2}\Big)+\frac{h}{2}\overset{-}{q}\overset{-}{H}\Big(1+\frac{h}{5r_1}+\frac{h^2}{12r_1^2}\Big)\Big]\Big\}\Bigg]\delta N_{yy}
$$

$$
+\Bigg[\mathring{\gamma}_{xx}+\Big[1-\frac{h^2}{12r_2}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big]\mathring{\gamma}_{yy}+\frac{h^2}{12}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\delta_{yy}-\frac{2(1+\nu)}{Eh}\Big\{\Big[1-\frac{h^2}{12r_2}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big]N_{xy}
$$

$$
+\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)M_{xy}\Big\}+\Big(1+\frac{h^2}{12r_2^2}\Big)\frac{\bar{W}_x\bar{W}_y}{AB}-\frac{h^2}{12r_2}\Big[\frac{1}{AB}\big(\bar{W}_x\bar{W}'_y+\bar{W}'_x\bar{W}_y\big)\Big]
$$

$$
+\frac{h^2}{12}\Big(1+\frac{3h^2}{20r_2^2}\Big)\Big[\frac{1}{AB}\Big(\frac{\bar{W}_x\bar{W}''_y}{2}+\bar{W}'_x\bar{W}'_y+\frac{\bar{W}''_x\bar{W}_y}{2}\Big)\Big]-\frac{h^4}{80r_2}\Big[\frac{1}{2AB}\big(\bar{W}'_x\bar{W}''_y+\bar{W}''_x\bar{W}'_y\big)\Big]
$$

$$
+\frac{h^4}{80}\Big(1+\frac{5h^2}{28r_2^2}\Big)\frac{\bar{W}''_x\bar{W}''_y}{4AB}\Bigg]\delta N_{xy}
$$

$$
\begin{aligned}
&+\Bigg[\Big[\frac{1}{A}\Big(\Phi_x+\frac{A_y}{B}\psi\Big)+\frac{\bar{W}'}{r_1}+\frac{\bar{W}_x\bar{W}'_x}{A^2}\Big]+\frac{3h^2}{40A^2}\bar{W}'_x\bar{W}''_x-\frac{1}{Eh}\Big\{\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)N_{xx}-\nu\frac{12}{h^2}M_{yy}\\
&\quad+\frac{12}{h^2}\Big[1-\frac{3h^2}{20r_2}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big]M_{xx}-\nu c\Big[\frac{1}{5h}\Big(1+\frac{3h^2}{28r_2^2}\Big)T-\frac{3}{5r_2}S+\frac{6}{5}\overset{+}{q}\overset{+}{H}\Big(1-\frac{5h}{12r_2}+\frac{h^2}{7r_2^2}\Big)\\
&\qquad-\frac{6}{5}\bar{q}\bar{H}\Big(1+\frac{5h}{12r_2}+\frac{h^2}{7r_2^2}\Big)\Big]\Big\}\Bigg]\delta M_{xx}
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg[\Big[\frac{1}{B}\Big(\psi_y+\frac{B_x}{A}\Phi\Big)+\frac{\bar{W}'}{r_2}+\frac{\bar{W}_y\bar{W}'_y}{B^2}\Big]+\frac{3h^2}{40B^2}\bar{W}'_y\bar{W}''_y-\frac{1}{Eh}\Big\{\frac{12}{h^2}\Big[1+\frac{3h^2}{20r_1}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big]M_{yy}\\
&\quad-\nu\frac{12}{h^2}M_{xx}-\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)N_{yy}-\nu c\Big[\frac{1}{5h}\Big(1+\frac{3h^2}{28r_1^2}\Big)T-\frac{3}{5r_1}S+\frac{6}{5}\overset{+}{q}\overset{+}{H}\Big(1-\frac{5h}{12r_1}+\frac{h^2}{7r_1^2}\Big)\\
&\qquad-\frac{6}{5}\bar{q}\bar{H}\Big(1+\frac{5h}{12r_1}+\frac{h^2}{7r_1^2}\Big)\Big]\Big\}\Bigg]\delta M_{yy}
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg[\delta_{xx}+\Big[1-\frac{3h^2}{20r_2}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big]\delta_{yy}+\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\overset{\circ}{\gamma}_{yy}-\frac{2(1+\nu)}{Eh}\Big\{\frac{12}{h^2}\Big[1-\frac{3h^2}{20r_2}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big]M_{xy}\\
&\quad+\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)N_{xy}\Big\}-\frac{1}{r_2}\frac{\bar{W}_x\bar{W}_y}{AB}+\Big(1+\frac{3h^2}{20r_2^2}\Big)\Big[\frac{1}{AB}\big(\bar{W}_x\bar{W}'_y+\bar{W}'_x\bar{W}_y\big)\Big]\\
&\quad-\frac{3h^2}{20r_2}\Big[\frac{1}{AB}\Big(\frac{\bar{W}_x\bar{W}''_y}{2}+\bar{W}'_x\bar{W}'_y+\frac{\bar{W}''_x\bar{W}_y}{2}\Big)\Big]+\frac{3h^2}{20}\Big(1+\frac{5h^2}{28r_2^2}\Big)\Big[\frac{1}{2AB}\big(\bar{W}'_x\bar{W}''_y+\bar{W}''_x\bar{W}'_y\big)\Big]\\
&\qquad-\frac{3h^4}{112r_2}\frac{\bar{W}''_x\bar{W}''_y}{4AB}\Bigg]\delta M_{xy}
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg[\overset{\circ}{\gamma}_{xz}+\frac{h^2}{40A}\bar{W}''_x-\frac{(1+\nu)}{Eh}\Big\{\frac{12}{5}\Big[1-\frac{3h^2}{84r_2}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big]Q_{xz}-\frac{2}{5}\Big[1+\frac{h^2}{28r_2}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big]m_1\\
&\qquad+\frac{h^2}{10}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)P_1\Big\}\Bigg]\delta Q_{xz}
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg[\overset{\circ}{\gamma}_{yz}+\frac{h^2}{40B}\bar{W}''_y-\frac{(1+\nu)}{Eh}\Big\{\frac{12}{5}\Big[1+\frac{3h^2}{84r_1}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big]Q_{yz}-\frac{2}{5}\Big[1-\frac{h^2}{28r_1}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big]m_2\\
&\qquad-\frac{h^2}{10}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)P_2\Big\}\Bigg]\delta Q_{yz}
\end{aligned}
$$

$$+\Bigg[c\bar{W}' - \frac{1}{Eh}\Big\{\frac{9}{4}c^2\Big[\frac{8}{15}+\frac{2h^2}{105}\Big(\frac{1}{r_1^2}+\frac{1}{r_1r_2}+\frac{1}{r_2^2}\Big)\Big]S - c^2\frac{h}{70}\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)T$$

$$+c^2\frac{h}{2}\overset{+}{q}\overset{+}{H}\Big[1-\frac{9h}{70}\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)+\frac{h^2}{20}\Big(\frac{1}{r_1^2}+\frac{1}{r_1r_2}+\frac{1}{r_2^2}\Big)\Big]$$

$$+c^2\frac{h}{2}\bar{q}\bar{H}\Big[1+\frac{9h}{70}\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)+\frac{h^2}{20}\Big(\frac{1}{r_1^2}+\frac{1}{r_1r_2}+\frac{1}{r_2^2}\Big)\Big]$$

$$+\vartheta c\Big[\frac{3}{5}\Big(\frac{M_{xx}}{r_2}+\frac{M_{yy}}{r_1}\Big)-\Big(1+\frac{h^2}{20r_2^2}\Big)N_{xx}-\Big(1+\frac{h^2}{20r_1^2}\Big)N_{yy}\Big]\Big\}\Bigg]\delta S$$

$$+\Bigg[c\frac{h}{60}\bar{W}'' - \frac{1}{Eh}\Big\{\frac{c^2}{16}\Big[\frac{8}{105}+\frac{2h^2}{315}\Big(\frac{1}{r_1^2}+\frac{1}{r_1r_2}+\frac{1}{r_2^2}\Big)\Big]T - c^2\frac{h}{70}\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)S$$

$$+c^2\frac{3h}{140}\overset{+}{q}\overset{+}{H}\Big[1-\frac{7h}{18}\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)+\frac{77h^2}{756}\Big(\frac{1}{r_1^2}+\frac{1}{r_1r_2}+\frac{1}{r_2^2}\Big)\Big]$$

$$-c^2\frac{3h}{140}\bar{q}\bar{H}\Big[1+\frac{7h}{18}\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)+\frac{77h^2}{756}\Big(\frac{1}{r_1^2}+\frac{1}{r_1r_2}+\frac{1}{r_2^2}\Big)\Big]$$

$$+\vartheta c\Big[\frac{h}{60}\Big(\frac{N_{xx}}{r_2}+\frac{N_{yy}}{r_1}\Big)-\frac{1}{5h}\Big(1+\frac{3h^2}{28r_2^2}\Big)M_{xx}-\frac{1}{5h}\Big(1+\frac{3h^2}{28r_1^2}\Big)M_{yy}\Big]\Big\}\Bigg]\delta T$$

$$\Big\}AB\,dx\,dy\Big\}\,dt$$

$$-\oint\int_{t_1}^{t_2}\Big\{(N^*_{nn}-N_{nn})\delta\bar{u}_{nn}+(N^*_{n\hat{t}}-N_{n\hat{t}})\delta\bar{u}_{n\hat{t}}+(M^*_{nn}-M_{nn})\delta\phi_{nn}$$

$$+(M^*_{n\hat{t}}-M_{n\hat{t}})\delta\phi_{n\hat{t}}+(Q^*_{nz}-Q_{nz})\delta\bar{W}+\frac{h^2}{40}(Q^*_{nz}-Q_{nz})\delta\bar{W}''$$

$$\Big\}A_{\hat{t}}\,ds_{\hat{t}}\Big\}\,dt=0. \qquad (33)$$

Boundary conditions take the form,

$$\int_{t_1}^{t_2}\Big\{\int\Big|BN_{xx}\delta\bar{u}+BN_{xy}\delta\bar{v}+BM_{xx}\delta\phi+BM_{xy}\delta\psi\Big|_{x_1}^{x_2}dy$$

$$+\int\Big|AN_{yx}\delta\bar{u}+AN_{yy}\delta\bar{v}+AM_{yx}\delta\phi+AM_{yy}\delta\psi\Big|_{y_1}^{y_2}dx$$

$$+\int\Big|\Big\{BN_{xx}\Big(\frac{\bar{W}_x}{A}+\frac{h^2}{24}\frac{\bar{W}''_x}{A}\Big)+BM_{xx}\Big(\frac{\bar{W}'_x}{A}\Big)+N_{xy}\Big[\Big(1+\frac{h^2}{12r_2^2}\Big)\bar{W}_y-\frac{h^2}{12r_2}\bar{W}'_y+\frac{h^2}{24}\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar{W}''_y\Big]$$

$$+M_{xy}\Big[\Big(-\frac{1}{r_2}\Big)\bar{W}_y+\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar{W}'_y-\frac{3h^2}{40r_2}\bar{W}''_y\Big]+BQ_{xz}\Big\}\delta\bar{W}\Big|_{x_1}^{x_2}dy$$

$$+\int\Big|\Big\{AN_{yy}\Big(\frac{\bar{W}_y}{B}+\frac{h^2}{24}\frac{\bar{W}''_y}{B}\Big)+AM_{yy}\Big(\frac{\bar{W}'_y}{B}\Big)+N_{xy}\Big[\Big(1+\frac{h^2}{12r_2^2}\Big)\bar{W}_x-\frac{h^2}{12r_2}\bar{W}'_x+\frac{h^2}{24}\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar{W}''_x\Big]$$

$$+M_{xy}\Big[\Big(-\frac{1}{r_2}\Big)\bar{W}_x+\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar{W}'_x-\frac{3h^2}{40r_2}\bar{W}''_x\Big]+AQ_{yz}\Big\}\delta\bar{W}\Big|_{y_1}^{y_2}dx$$

$$+\int\Big|\Big\{BN_{xx}\Big(\frac{h^2}{12}\frac{\bar{W}'_x}{A}\Big)+BM_{xx}\Big(\frac{\bar{W}_x}{A}+\frac{3h^2}{40}\frac{\bar{W}''_x}{A}\Big)-N_{xy}\Big[\frac{h^2}{12r_2}\bar{W}_y-\frac{h^2}{12}\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar{W}'_y+\frac{h^4}{160r_2}\bar{W}''_y\Big]$$

$$+M_{xy}\Big[\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar{W}_y-\frac{3h^2}{20r_2}\bar{W}'_y+\frac{3h^2}{40}\Big(1+\frac{5h^2}{28r_2^2}\Big)\bar{W}''_y\Big]+\frac{h^2}{12}BP_1\Big\}\delta\bar{W}'\Big|_{x_1}^{x_2}dy$$

$$+\int\Big|\Big\{AN_{yy}\Big(\frac{h^2}{12}\frac{\bar{W}'_y}{B}\Big)+AM_{yy}\Big(\frac{\bar{W}_y}{B}+\frac{3h^2}{40}\frac{\bar{W}''_y}{B}\Big)-N_{xy}\Big[\frac{h^2}{12r_2}\bar{W}_x-\frac{h^2}{12}\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar{W}'_x+\frac{h^4}{160r_2}\bar{W}''_x\Big]$$

$$+M_{xy}\Big[\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar{W}_x-\frac{3h^2}{20r_2}\bar{W}'_x+\frac{3h^2}{40}\Big(1+\frac{5h^2}{28r_2^2}\Big)\bar{W}''_x\Big]+\frac{h^2}{12}AP_2\Big\}\delta\bar{W}'\Big|_{y_1}^{y_2}dx$$

$$+\int\Big|\Big\{BN_{xx}\Big[\frac{h^2}{24}\frac{\bar{W}_x}{A}+\frac{h^4}{320}\frac{\bar{W}''_x}{A}\Big]+\frac{3h^2}{40}BM_{xx}\Big(\frac{\bar{W}'_x}{A}\Big)+N_{xy}\Big[\frac{h^2}{24}\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar{W}_y-\frac{h^4}{160r_2}\bar{W}'_y+\frac{h^4}{80}\Big(1+\frac{5h^2}{28r_2^2}\Big)\frac{\bar{W}''_y}{4}\Big]$$

$$-M_{xy}\Big[\frac{3h^2}{40r_2}\bar{W}_y-\frac{3h^2}{40}\Big(1+\frac{5h^2}{28r_2^2}\Big)\bar{W}'_y+\frac{3h^4}{112r_2}\frac{\bar{W}''_y}{4}\Big]+\frac{1}{2}\Big(\frac{h^2}{20}BQ_{xz}+\frac{h^2}{30}Bm_1\Big)\Big\}\delta\bar{W}''\Big|_{x_1}^{x_2}dy$$

$$+\int\Big|\Big\{AN_{yy}\Big[\frac{h^2}{24}\frac{\bar{W}_y}{B}+\frac{h^4}{320}\frac{\bar{W}''_y}{B}\Big]+\frac{3h^2}{40}AM_{yy}\Big(\frac{\bar{W}'_y}{B}\Big)+N_{xy}\Big[\frac{h^2}{24}\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar{W}_x-\frac{h^4}{160r_2}\bar{W}'_x+\frac{h^4}{80}\Big(1+\frac{5h^2}{28r_2^2}\Big)\frac{\bar{W}''_x}{4}\Big]$$

$$-M_{xy}\Big[\frac{3h^2}{40r_2}\bar{W}_x-\frac{3h^2}{40}\Big(1+\frac{5h^2}{28r_2^2}\Big)\bar{W}'_x+\frac{3h^4}{112r_2}\frac{\bar{W}''_x}{4}\Big]+\frac{1}{2}\Big(\frac{h^2}{20}AQ_{yz}+\frac{h^2}{30}Am_2\Big)\Big\}\delta\bar{W}''\Big|_{y_1}^{y_2}dx\Big\}dt$$

$$-\iiint_S\Big|\rho\Big\{\Big[h\Big(1+\frac{h^2}{12r_1r_2}\Big)\bar{u}_t+\frac{h^3}{12}\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)\phi_t\Big]\delta\bar{u}+\Big[h\Big(1+\frac{h^2}{12r_1r_2}\Big)\bar{v}_t+\frac{h^3}{12}\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)\psi_t\Big]\delta\bar{v}$$

$$+\Big[h\Big(1+\frac{h^2}{12r_1r_2}\Big)\bar{W}_t+\frac{h^3}{12}\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)\bar{W}'_t+\frac{h^3}{24}\Big(1+\frac{3h^2}{20r_1r_2}\Big)\bar{W}''_t\Big]\delta\bar{W}$$

$$+\Big[\frac{h^3}{12}\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)\bar{u}_t+\frac{h^3}{12}\Big(1+\frac{3h^2}{20r_1r_2}\Big)\phi_t\Big]\delta\phi+\Big[\frac{h^3}{12}\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)\bar{v}_t+\frac{h^3}{12}\Big(1+\frac{3h^2}{20r_1r_2}\Big)\psi_t\Big]\delta\psi$$

$$+\left[\frac{h^3}{12}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\bar{W}_t+\frac{h^3}{12}\left(1+\frac{3h^2}{20r_1r_2}\right)\bar{W}_t'+\frac{h^5}{160}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\bar{W}_t''\right]\delta\bar{W}'$$

$$+\left[\frac{h^3}{24}\left(1+\frac{3h^2}{20r_1r_2}\right)\bar{W}_t+\frac{h^5}{160}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\bar{W}_t'+\frac{h^5}{320}\left(1+\frac{5h^2}{28r_1r_2}\right)\bar{W}_t''\right]\delta\bar{W}''\Big\}\Bigg|_{t_1}^{t_2} AB\,dx\,dy=0.$$

(34)

Also the independent vanishing of each term of the line integral in equation (33) furnishes the required boundary conditions along each edge of the shell; these are either the stress or displacement prescribed for the boundary conditions.

When the stress is prescribed, boundary conditions are

$$\left.\begin{aligned}
\bar{U}_{nn} &= U_{nn}(x_{nn}, x_{n\hat{t}}, 0)\,, \\
\bar{U}_{n\hat{t}} &= U_{n\hat{t}}(x_{nn}, x_{n\hat{t}}, 0)\,, \\
\Phi_{nn} &= \Phi_{nn}(x_{nn}, x_{n\hat{t}}, 0)\,, \\
\Phi_{n\hat{t}} &= \Phi_{n\hat{t}}(x_{nn}, x_{n\hat{t}}, 0)\,, \\
\bar{W} &= \bar{W}(x_{nn}, x_{n\hat{t}}, 0)\,, \\
\bar{W}'' &= \bar{W}''(x_{nn}, x_{n\hat{t}}, 0)\,.
\end{aligned}\right] \qquad (35)$$

When the displacement is prescribed, boundary conditions are

$$\left.\begin{aligned}
N^*_{nn} &= N_{nn}\,, \\
N^*_{n\hat{t}} &= N_{n\hat{t}}\,, \\
M^*_{nn} &= M_{nn}\,, \\
M^*_{n\hat{t}} &= M_{n\hat{t}}\,, \\
Q^*_{nz} &= Q_{nz}\,,
\end{aligned}\right] \qquad (36)$$

where each condition in equations (35) and (36) are respectively along the x_{nn} = constant and $x_{n\hat{t}}$ = constant.

2.6 Equations of Equilibrium and Stress-Strain Relations

Since $\bar{U}$, $\bar{V}$, $\bar{W}$, Φ, and Ψ are independent functions, the coefficients of the functions $\delta\bar{U}$, $\delta\bar{V}$, $\delta\bar{W}$, $\delta\Phi$, & $\delta\Psi$ are set equal to zero. These conditions yield the set of five equilibrium equations in the form,

$$\frac{\partial}{\partial x}(BN_{xx})+\frac{\partial}{\partial y}(AN_{yx})+A_yN_{xy}-B_xN_{yy}+AB\left(\frac{Q_{xz}}{r_1}+P_1\right)=\rho hAB\left[\left(1+\frac{h^2}{12r_1r_2}\right)\bar{U}_{tt}+\frac{h^2}{12}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\Phi_{tt}\right],$$

$$\frac{\partial}{\partial x}(BN_{xy})+\frac{\partial}{\partial y}(AN_{yy})+B_xN_{yx}-A_yN_{xx}+AB\left(\frac{Q_{yz}}{r_2}+P_2\right)=\rho hAB\left[\left(1+\frac{h^2}{12r_1r_2}\right)\bar{V}_{tt}+\frac{h^2}{12}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\Psi_{tt}\right],$$

$$\frac{\partial}{\partial x}(BM_{xx})+\frac{\partial}{\partial y}(AM_{yx})+A_yM_{xy}-B_xM_{yy}-AB(Q_{xz}-m_1)=\rho\frac{h^3}{12}AB\left[\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\bar{U}_{tt}+\left(1+\frac{3h^2}{20r_1r_2}\right)\Phi_{tt}\right],$$

$$\frac{\partial}{\partial x}(BM_{xy})+\frac{\partial}{\partial y}(AM_{yy})+B_xM_{yx}-A_yM_{xx}-AB(Q_{yz}-m_2)=\rho\frac{h^3}{12}AB\left[\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\bar{V}_{tt}+\left(1+\frac{3h^2}{20r_1r_2}\right)\Psi_{tt}\right],$$

$$\begin{aligned}
&\frac{\partial}{\partial x}(BQ_{xz})+\frac{\partial}{\partial y}(AQ_{yz})-\frac{AB}{r_1}N_{xx}-\frac{AB}{r_2}N_{yy}+AB(\overset{+}{q}\overset{+}{H}-\overset{-}{q}\overset{-}{H})\\
&+\frac{\partial}{\partial x}\Big\{BN_{xx}\Big(\frac{\bar{W}_x}{A}+\frac{h^2}{24}\frac{\bar{W}''_x}{A}\Big)+BM_{xx}\Big(\frac{\bar{W}'_x}{A}\Big)+N_{xy}\Big[\Big(1+\frac{h^2}{12r_2^2}\Big)\bar{W}_y-\frac{h^2}{12r_2}\bar{W}'_y+\frac{h^2}{24}\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar{W}''_y\Big]\\
&\qquad -M_{xy}\Big[\frac{\bar{W}_y}{r_2}-\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar{W}'_y+\frac{3h^2}{40r_2}\bar{W}''_y\Big]\Big\}\\
&+\frac{\partial}{\partial y}\Big\{AN_{yy}\Big(\frac{\bar{W}_y}{B}+\frac{h^2}{24}\frac{\bar{W}''_y}{B}\Big)+AM_{yy}\Big(\frac{\bar{W}'_y}{B}\Big)+N_{xy}\Big[\Big(1+\frac{h^2}{12r_2^2}\Big)\bar{W}_x-\frac{h^2}{12r_2}\bar{W}'_x+\frac{h^2}{24}\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar{W}''_x\Big]\\
&\qquad -M_{xy}\Big[\frac{\bar{W}_x}{r_2}-\Big(1+\frac{3h^2}{20r_2^2}\Big)\bar{W}'_x+\frac{3h^2}{40r_2}\bar{W}''_x\Big]\Big\}\\
&=\rho hAB\Big[\Big(1+\frac{h^2}{12r_1r_2}\Big)\bar{W}_{tt}+\frac{h^2}{12}\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)\bar{W}'_{tt}+\frac{h^2}{24}\Big(1+\frac{3h^2}{20r_1r_2}\Big)\bar{W}''_{tt}\Big].
\end{aligned}$$

(37)

In addition, the coefficients of the functions δN_{xx}, δN_{yy}, δN_{xy}, δM_{xx}, δM_{yy}, δM_{xy}, δQ_{xz} and δQ_{yz} are set equal to zero. These conditions yield the following eight stress-strain relationships,

$$\begin{aligned}
\overset{\circ}{\epsilon}_{xx}+\frac{h^2}{24}\Big[\frac{\bar{W}''}{r_1}+\frac{1}{A^2}(\bar{W}_x\bar{W}''_x+\bar{W}'^2_x)\Big]+\frac{h^4}{640}\frac{\bar{W}''^2_x}{A^2}=\frac{1}{Eh}\Big\{&\Big[1-\frac{h^2}{12r_2}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big]N_{xx}-\nu N_{yy}+\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)M_{xx}\\
&-\nu c\Big[\Big(1+\frac{h^2}{20r_2^2}\Big)S-\frac{h}{60r_2}T+\frac{h}{2}\overset{+}{q}\overset{+}{H}\Big(1-\frac{h}{5r_2}+\frac{h^2}{12r_2^2}\Big)\\
&+\frac{h}{2}\overset{-}{q}\overset{-}{H}\Big(1+\frac{h}{5r_2}+\frac{h^2}{12r_2^2}\Big)\Big]\Big\},
\end{aligned}$$

$$\overset{\circ}{\epsilon}_{yy} + \frac{h^2}{24}\left[\frac{\overline{W}''}{r_2} + \frac{1}{B^2}\left(\overline{W}_y\overline{W}_y'' + \overline{W}_y'^2\right)\right] + \frac{h^4}{640}\frac{\overline{W}_y''^2}{B^2} = \frac{1}{Eh}\Big\{\left[1+\frac{h^2}{12r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]N_{yy} - \nu N_{xx} - \left(\frac{1}{r_1}-\frac{1}{r_2}\right)M_{yy}$$

$$-\nu c\Big[\left(1+\frac{h^2}{20r_1^2}\right)S - \frac{h}{60r_1}T + \frac{h}{2}\overset{+}{q}\overset{+}{H}\left(1-\frac{h}{5r_1}+\frac{h^2}{12r_1^2}\right)$$

$$+\frac{h}{2}\overset{-}{q}\overset{-}{H}\left(1+\frac{h}{5r_1}+\frac{h^2}{12r_1^2}\right)\Big]\Big\},$$

$$\overset{\circ}{\gamma}_{xx} + \left[1-\frac{h^2}{12r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]\overset{\circ}{\gamma}_{yy} + \frac{h^2}{12}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\delta_{yy} + \left(1+\frac{h^2}{12r_2^2}\right)\frac{\overline{W}_x\overline{W}_y}{AB} - \frac{h^2}{12r_2}\left[\frac{1}{AB}\left(\overline{W}_x\overline{W}_y' + \overline{W}_x'\overline{W}_y\right)\right]$$

$$+\frac{h^2}{12}\left(1+\frac{3h^2}{20r_2^2}\right)\left[\frac{1}{AB}\left(\frac{\overline{W}_x\overline{W}_y''}{2} + \overline{W}_x'\overline{W}_y' + \frac{\overline{W}_x''\overline{W}_y}{2}\right)\right] - \frac{h^4}{80r_2}\left[\frac{1}{2AB}\left(\overline{W}_x'\overline{W}_y'' + \overline{W}_x''\overline{W}_y'\right)\right] + \frac{h^4}{80}\left(1+\frac{5h^2}{28r_2^2}\right)\frac{\overline{W}_x''\overline{W}_y''}{4AB}$$

$$= \frac{2(1+\nu)}{Eh}\left\{\left[1-\frac{h^2}{12r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]N_{xy} + \left(\frac{1}{r_1}-\frac{1}{r_2}\right)M_{xy}\right\},$$

$$\left[\frac{1}{A}\left(\phi_x + \frac{A_y}{B}\psi\right) + \frac{\overline{W}'}{r_1} + \frac{\overline{W}_x\overline{W}_x'}{A^2}\right] + \frac{3h^2}{40A^2}\overline{W}_x'\overline{W}_x'' = \frac{1}{Eh}\Big\{\left(\frac{1}{r_1}-\frac{1}{r_2}\right)N_{xx} + \frac{12}{h^2}\left[1-\frac{3h^2}{20r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]M_{xx}$$

$$-\nu\frac{12}{h^2}M_{yy} - \nu c\Big[\frac{1}{5h}\left(1+\frac{3h^2}{28r_2^2}\right)T - \frac{3}{5r_2}S$$

$$+\frac{6}{5}\overset{+}{q}\overset{+}{H}\left(1-\frac{5h}{12r_2}+\frac{h^2}{7r_2^2}\right) - \frac{6}{5}\overset{-}{q}\overset{-}{H}\left(1+\frac{5h}{12r_2}+\frac{h^2}{7r_2^2}\right)\Big]\Big\},$$

$$\left[\frac{1}{B}\left(\psi_y + \frac{B_x}{A}\phi\right) + \frac{\overline{W}'}{r_2} + \frac{\overline{W}_y\overline{W}_y'}{B^2}\right] + \frac{3h^2}{40B^2}\overline{W}_y'\overline{W}_y'' = \frac{1}{Eh}\Big\{-\left(\frac{1}{r_1}-\frac{1}{r_2}\right)N_{yy} + \frac{12}{h^2}\left[1+\frac{3h^2}{20r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]M_{yy}$$

$$-\nu\frac{12}{h^2}M_{xx} - \nu c\Big[\frac{1}{5h}\left(1+\frac{3h^2}{28r_1^2}\right)T - \frac{3}{5r_1}S$$

$$+\frac{6}{5}\overset{+}{q}\overset{+}{H}\left(1-\frac{5h}{12r_1}+\frac{h^2}{7r_1^2}\right) - \frac{6}{5}\overset{-}{q}\overset{-}{H}\left(1+\frac{5h}{12r_1}+\frac{h^2}{7r_1^2}\right)\Big]\Big\},$$

$$\delta_{xx} + \left[1-\frac{3h^2}{20r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]\delta_{yy} + \left(\frac{1}{r_1}-\frac{1}{r_2}\right)\overset{\circ}{\gamma}_{yy} - \frac{1}{r_2}\frac{\overline{W}_x\overline{W}_y}{AB} + \left(1+\frac{3h^2}{20r_2^2}\right)\left[\frac{1}{AB}\left(\overline{W}_x\overline{W}_y' + \overline{W}_x'\overline{W}_y\right)\right]$$

$$-\frac{3h^2}{20r_2}\left[\frac{1}{AB}\left(\frac{\overline{W}_x\overline{W}_y''}{2} + \overline{W}_x'\overline{W}_y' + \frac{\overline{W}_x''\overline{W}_y}{2}\right)\right] + \frac{3h^2}{20}\left(1+\frac{5h^2}{28r_2^2}\right)\left[\frac{1}{2AB}\left(\overline{W}_x'\overline{W}_y'' + \overline{W}_x''\overline{W}_y'\right)\right] - \frac{3h^4}{112r_2}\frac{\overline{W}_x''\overline{W}_y''}{4AB}$$

$$= \frac{2(1+\nu)}{Eh}\left\{\frac{12}{h^2}\left[1-\frac{3h^2}{20r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]M_{xy} + \left(\frac{1}{r_1}-\frac{1}{r_2}\right)N_{xy}\right\},$$

$$\mathring{\gamma}_{xz} + \frac{h^2}{40}\frac{\bar{W}''_x}{A} = \frac{2(1+\nu)}{Eh}\left\{\frac{6}{5}\left[1-\frac{h^2}{28r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]Q_{xz} - \frac{1}{5}\left[1+\frac{h^2}{28r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]\mathcal{M}_1 + \frac{h^2}{20}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)P_1\right\},$$

$$\mathring{\gamma}_{yz} + \frac{h^2}{40}\frac{\bar{W}''_y}{B} = \frac{2(1+\nu)}{Eh}\left\{\frac{6}{5}\left[1+\frac{h^2}{28r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]Q_{yz} - \frac{1}{5}\left[1-\frac{h^2}{28r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]\mathcal{M}_2 - \frac{h^2}{20}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)P_2\right\}.$$

(38)

Finally, the coefficients of the functions $\delta\bar{W}'$, $\delta\bar{W}''$, δS and δT are set equal to zero. These equations, which yield the functions $\bar{W}'$, $\bar{W}''$, S and T, are written

$$\bar{W}' = -\frac{\nu}{Eh}\left[(N_{xx}+N_{yy}) + \frac{h^2}{20}\left(\frac{N_{xx}}{r_2^2}+\frac{N_{yy}}{r_1^2}\right) - \frac{3}{5}\left(\frac{M_{xx}}{r_2}+\frac{M_{yy}}{r_1}\right)\right]$$
$$+\frac{c}{Eh}\left\{S\left[\frac{6}{5}+\frac{3h^2}{70}\left(\frac{1}{r_1^2}+\frac{1}{r_1r_2}+\frac{1}{r_2^2}\right)\right] - \frac{h^2}{70}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\frac{T}{h} + \frac{h}{2}(\overset{+}{q}\overset{+}{H}+\overset{-}{q}\overset{-}{H})\left[1+\frac{h^2}{5}\left(\frac{1}{r_1^2}+\frac{1}{r_1r_2}+\frac{1}{r_2^2}\right)\right]\right.$$
$$\left. - \frac{9h^2}{140}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)(\overset{+}{q}\overset{+}{H}-\overset{-}{q}\overset{-}{H})\right\},$$

$$\bar{W}'' = -\frac{12\nu}{Eh^3}\left[(M_{xx}+M_{yy}) + \frac{3h^2}{28}\left(\frac{M_{xx}}{r_2^2}+\frac{M_{yy}}{r_1^2}\right)\right] + \frac{\nu}{Eh}\left(\frac{N_{xx}}{r_2}+\frac{N_{yy}}{r_1}\right)$$
$$+\frac{c}{Eh}\left\{\frac{T}{h}\left[\frac{2}{7}+\frac{h^2}{42}\left(\frac{1}{r_1^2}+\frac{1}{r_1r_2}+\frac{1}{r_2^2}\right)\right] - \frac{6}{7}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)S + \frac{9}{7}(\overset{+}{q}\overset{+}{H}-\overset{-}{q}\overset{-}{H})\left[1+\frac{11h^2}{108}\left(\frac{1}{r_1^2}+\frac{1}{r_1r_2}+\frac{1}{r_2^2}\right)\right]\right.$$
$$\left. - \frac{h}{2}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)(\overset{+}{q}\overset{+}{H}+\overset{-}{q}\overset{-}{H})\right\},$$

$$S = -\frac{1}{c}\left(\frac{M_{xx}}{r_1}+\frac{M_{yy}}{r_2}\right) + \frac{h^2}{c12AB}\left[\frac{\partial}{\partial x}(BP_1)+\frac{\partial}{\partial y}(AP_2)\right] + \frac{h}{2c}(1-c)(\overset{+}{q}\overset{+}{H}+\overset{-}{q}\overset{-}{H})$$
$$+\frac{1}{cAB}\frac{\partial}{\partial x}\left\{BN_{xx}\left(\frac{h^2}{12}\frac{\bar{W}'_x}{A}\right) + BM_{xx}\left[\frac{\bar{W}_x}{A}+\frac{3h^2}{40}\frac{\bar{W}''_x}{A}\right] - N_{xy}\left[\frac{h^2}{12r_2}\bar{W}_y - \frac{h^2}{12}\left(1+\frac{3h^2}{20r_2^2}\right)\bar{W}'_y + \frac{h^4}{160r_2}\bar{W}''_y\right]\right.$$
$$\left. + M_{xy}\left[\left(1+\frac{3h^2}{20r_2^2}\right)\bar{W}_y - \frac{3h^2}{20r_2}\bar{W}'_y + \frac{3h^2}{40}\left(1+\frac{5h^2}{28r_2^2}\right)\bar{W}''_y\right]\right\}$$
$$+\frac{1}{cAB}\frac{\partial}{\partial y}\left\{AN_{yy}\left(\frac{h^2}{12}\frac{\bar{W}'_y}{B}\right) + AM_{yy}\left[\frac{\bar{W}_y}{B}+\frac{3h^2}{40}\frac{\bar{W}''_y}{B}\right] - N_{xy}\left[\frac{h^2}{12r_2}\bar{W}_x - \frac{h^2}{12}\left(1+\frac{3h^2}{20r_2^2}\right)\bar{W}'_x + \frac{h^4}{160r_2}\bar{W}''_x\right]\right.$$
$$\left. + M_{xy}\left[\left(1+\frac{3h^2}{20r_2^2}\right)\bar{W}_x - \frac{3h^2}{20r_2}\bar{W}'_x + \frac{3h^2}{40}\left(1+\frac{5h^2}{28r_2^2}\right)\bar{W}''_x\right]\right\}$$
$$-\rho\frac{h^3}{12c}\left\{\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\bar{W}_{tt} + \left(1+\frac{3h^2}{20r_1r_2}\right)\bar{W}'_{tt} + \frac{3h^2}{40}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\bar{W}''_{tt}\right\},$$

$$
\begin{aligned}
\frac{T}{h} = & -\frac{5}{2c}\left(\frac{N_{xx}}{r_1} + \frac{N_{yy}}{r_2}\right) + \frac{1}{cAB}\left[\frac{\partial}{\partial x}(B\mathcal{M}_1) + \frac{\partial}{\partial y}(A\mathcal{M}_2)\right] + \frac{3}{2c}(5-4c)(\overset{+}{q}\overset{+}{H} - \overset{-}{q}\overset{-}{H}) \\
& + \frac{60}{ch^2}\Bigg\{\frac{1}{AB}\frac{\partial}{\partial x}\Big\{BN_{xx}\left[\frac{h^2}{24}\frac{\bar{W}_x}{A} + \frac{h^4}{320}\frac{\bar{W}''_x}{A}\right] + \frac{3h^2}{40}BM_{xx}\left(\frac{\bar{W}'_x}{A}\right) + \frac{h^2}{40}BQ_{xz} \\
& \qquad + N_{xy}\left[\frac{h^2}{24}\left(1+\frac{3h^2}{20r_2^2}\right)\bar{W}_y - \frac{h^4}{160r_2}\bar{W}'_y + \frac{h^4}{80}\left(1+\frac{5h^2}{28r_2^2}\right)\frac{\bar{W}''_y}{4}\right] \\
& \qquad - M_{xy}\left[\frac{3h^2}{40r_2}\bar{W}_y - \frac{3h^2}{40}\left(1+\frac{5h^2}{28r_2^2}\right)\bar{W}'_y + \frac{3h^4}{112r_2}\frac{\bar{W}''_y}{4}\right]\Big\} \\
& + \frac{1}{AB}\frac{\partial}{\partial y}\Big\{AN_{yy}\left[\frac{h^2}{24}\frac{\bar{W}_y}{B} + \frac{h^4}{320}\frac{\bar{W}''_y}{B}\right] + \frac{3h^2}{40}AM_{yy}\left(\frac{\bar{W}'_y}{B}\right) + \frac{h^2}{40}AQ_{yz} \\
& \qquad + N_{xy}\left[\frac{h^2}{24}\left(1+\frac{3h^2}{20r_2^2}\right)\bar{W}_x - \frac{h^4}{160r_2}\bar{W}'_x + \frac{h^4}{80}\left(1+\frac{5h^2}{28r_2^2}\right)\frac{\bar{W}''_x}{4}\right] \\
& \qquad - M_{xy}\left[\frac{3h^2}{40r_2}\bar{W}_x - \frac{3h^2}{40}\left(1+\frac{5h^2}{28r_2^2}\right)\bar{W}'_x + \frac{3h^4}{112r_2}\frac{\bar{W}''_x}{4}\right]\Big\} \\
& - \rho\frac{h^3}{24}\left\{\left(1+\frac{3h^2}{20r_1r_2}\right)\bar{W}_{tt} + \frac{3h^2}{20}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\bar{W}'_{tt} + \frac{3h^2}{40}\left(1+\frac{5h^2}{28r_1r_2}\right)\bar{W}''_{tt}\right\}\Bigg\} .
\end{aligned}
\tag{39}
$$

Retaining terms of the order $\frac{h}{r_1}$ and $\frac{h}{r_2}$ only (see appendix A) the stress-strain relationships are written as,

$$
\begin{aligned}
N_{xx} = & \frac{Eh}{(1-\nu^2)}\Bigg\{(\mathring{\epsilon}_{xx} + \nu\mathring{\epsilon}_{yy}) + \frac{h^2}{24}\Big[\bar{W}''\left(\frac{1}{r_1}+\frac{\nu}{r_2}\right) + \frac{1}{A^2}\left(\bar{W}_x\bar{W}''_x - 2\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\bar{W}_x\bar{W}'_x + \bar{W}'^2_x\right) \\
& + \frac{\nu}{B^2}\left(\bar{W}_y\bar{W}''_y + \bar{W}'^2_y\right)\Big] + \frac{h^4}{640}\left[\frac{1}{A^2}\left(\bar{W}''^2_x - 4\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\bar{W}'_x\bar{W}''_x\right) + \frac{\nu}{B^2}\bar{W}''^2_y\right] \\
& - \frac{h^2}{12}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\left[\frac{1}{A}\left(\Phi_x + \frac{A_y}{B}\psi\right)\right]\Bigg\} + \frac{\nu c}{(1-\nu^2)}\Big\{(1+\nu)S - \frac{h}{60r_1}(1+\nu)T \\
& + \frac{h}{2}(1+\nu)(\overset{+}{q}\overset{+}{H} + \overset{-}{q}\overset{-}{H}) - \frac{h^2}{10r_1}(1+\nu)(\overset{+}{q}\overset{+}{H} - \overset{-}{q}\overset{-}{H})\Big\} ,
\end{aligned}
$$

$$
\begin{aligned}
N_{yy} = & \frac{Eh}{(1-\nu^2)}\Bigg\{(\mathring{\epsilon}_{yy} + \nu\mathring{\epsilon}_{xx}) + \frac{h^2}{24}\Big[\bar{W}''\left(\frac{\nu}{r_1}+\frac{1}{r_2}\right) + \frac{\nu}{A^2}\left(\bar{W}_x\bar{W}''_x + \bar{W}'^2_x\right) + \frac{1}{B^2}\Big(\bar{W}_y\bar{W}''_y + 2\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\bar{W}_y\bar{W}'_y \\
& + \bar{W}'^2_y\Big)\Big] + \frac{h^4}{640}\left[\frac{\nu}{A^2}\bar{W}''^2_x + \frac{1}{B^2}\left(\bar{W}''^2_y + 4\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\bar{W}'_y\bar{W}''_y\right)\right] + \frac{h^2}{12}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\left[\frac{1}{B}\left(\psi_y + \frac{B_x}{A}\Phi\right)\right]\Bigg\} \\
& + \frac{\nu c}{(1-\nu^2)}\Big\{(1+\nu)S - \frac{h}{60r_2}(1+\nu)T + \frac{h}{2}(1+\nu)(\overset{+}{q}\overset{+}{H} + \overset{-}{q}\overset{-}{H}) - \frac{h^2}{10r_2}(1+\nu)(\overset{+}{q}\overset{+}{H} - \overset{-}{q}\overset{-}{H})\Big\} ,
\end{aligned}
$$

$$N_{xy} = Gh\Big\{\mathring{\gamma}_{xx}+\mathring{\gamma}_{yy}-\frac{h^2}{12}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\delta_{xx}+\frac{\bar{W}_x\bar{W}_y}{AB}-\frac{h^2}{12r_1}\Big[\frac{1}{AB}(\bar{W}_x\bar{W}'_y+\bar{W}'_x\bar{W}_y)\Big]$$
$$+\frac{h^2}{12}\Big[\frac{1}{AB}\Big(\frac{\bar{W}_x\bar{W}''_y}{2}+\bar{W}'_x\bar{W}'_y+\frac{\bar{W}''_x\bar{W}_y}{2}\Big)\Big]-\frac{h^4}{160r_1}\Big[\frac{1}{AB}(\bar{W}'_x\bar{W}''_y+\bar{W}''_x\bar{W}'_y)\Big]+\frac{h^4}{320}\frac{\bar{W}''_x\bar{W}''_y}{AB}\Big\},$$

$$M_{xx} = D\Big\{\Big[\frac{1}{A}\Big(\phi_x+\frac{A_y}{B}\psi\Big)+\frac{\bar{W}'}{r_1}+\frac{\bar{W}_x\bar{W}'_x}{A^2}\Big]+\nu\Big[\frac{1}{B}\Big(\psi_y+\frac{B_x}{A}\phi\Big)+\frac{\bar{W}'}{r_2}+\frac{\bar{W}_y\bar{W}'_y}{B^2}\Big]-\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\mathring{\epsilon}_{xx}$$
$$-\frac{h^2}{24}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big[\frac{1}{A^2}(\bar{W}_x\bar{W}''_x+\bar{W}'^2_x)\Big]-\frac{h^4}{640}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\frac{\bar{W}''^2_x}{A^2}+\frac{3h^2}{40}\Big[\frac{\bar{W}'_x\bar{W}''_x}{A^2}+\nu\frac{\bar{W}'_y\bar{W}''_y}{B^2}\Big]\Big\}$$
$$+\frac{\nu c}{(1-\nu^2)}\frac{h^2}{12}\Big\{(1+\nu)\frac{T}{5h}-\Big[\frac{(3\nu+5)}{5r_1}-\frac{2}{5r_2}\Big]S+\frac{6}{5}(1+\nu)(\overset{+}{q}\overset{+}{H}-\overset{-}{q}\overset{-}{H})-\frac{h}{2}\Big[\frac{2}{r_2}-\frac{(1-\nu)}{r_1}\Big](\overset{+}{q}\overset{+}{H}+\overset{-}{q}\overset{-}{H})\Big\},$$

$$M_{yy} = D\Big\{\nu\Big[\frac{1}{A}\Big(\phi_x+\frac{A_y}{B}\psi\Big)+\frac{\bar{W}'}{r_1}+\frac{\bar{W}_x\bar{W}'_x}{A^2}\Big]+\Big[\frac{1}{B}\Big(\psi_y+\frac{B_x}{A}\phi\Big)+\frac{\bar{W}'}{r_2}+\frac{\bar{W}_y\bar{W}'_y}{B^2}\Big]+\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\mathring{\epsilon}_{yy}$$
$$+\frac{h^2}{24}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big[\frac{1}{B^2}(\bar{W}_y\bar{W}''_y+\bar{W}'^2_y)\Big]+\frac{h^4}{640}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\frac{\bar{W}''^2_y}{B^2}+\frac{3h^2}{40}\Big[\nu\frac{\bar{W}'_x\bar{W}''_x}{A^2}+\frac{\bar{W}'_y\bar{W}''_y}{B^2}\Big]\Big\}$$
$$+\frac{\nu c}{(1-\nu^2)}\frac{h^2}{12}\Big\{(1+\nu)\frac{T}{5h}-\Big[\frac{(3\nu+5)}{5r_2}-\frac{2}{5r_1}\Big]S+\frac{6}{5}(1+\nu)(\overset{+}{q}\overset{+}{H}-\overset{-}{q}\overset{-}{H})-\frac{h}{2}\Big[\frac{2}{r_1}-\frac{(1-\nu)}{r_2}\Big](\overset{+}{q}\overset{+}{H}+\overset{-}{q}\overset{-}{H})\Big\},$$

$$M_{xy} = \frac{(1-\nu)}{2}D\Big\{(\delta_{xx}+\delta_{yy})-\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\mathring{\gamma}_{xx}-\frac{1}{r_1}\frac{\bar{W}_x\bar{W}_y}{AB}+\Big[\frac{1}{AB}(\bar{W}_x\bar{W}'_y+\bar{W}'_x\bar{W}_y)\Big]$$
$$-\frac{h^2}{12}\Big(\frac{1}{r_1}+\frac{4}{5r_2}\Big)\Big[\frac{1}{AB}\Big(\frac{\bar{W}_x\bar{W}''_y}{2}+\bar{W}'_x\bar{W}'_y+\frac{\bar{W}''_x\bar{W}_y}{2}\Big)\Big]+\frac{3h^2}{40}\Big[\frac{1}{AB}(\bar{W}'_x\bar{W}''_y+\bar{W}''_x\bar{W}'_y)\Big]$$
$$-\frac{h^4}{320}\Big(\frac{1}{r_1}+\frac{8}{7r_2}\Big)\frac{\bar{W}''_x\bar{W}''_y}{AB}\Big\},$$

$$Q_{xz} = \frac{5}{6}Gh\Big(\mathring{\gamma}_{xz}+\frac{h^2}{40}\frac{\bar{W}''_x}{A}\Big)+\frac{1}{6}\Big[m_1-\frac{h^2}{4}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)P_1\Big],$$

$$Q_{yz} = \frac{5}{6}Gh\Big(\mathring{\gamma}_{yz}+\frac{h^2}{40}\frac{\bar{W}''_y}{B}\Big)+\frac{1}{6}\Big[m_2+\frac{h^2}{4}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)P_2\Big], \qquad (40)$$

where, G = Shear modulus of elasticity $\left(=\dfrac{E}{2(1+\nu)}\right)$,

D = Flexural rigidity of the shell $\left(=\dfrac{Eh^3}{12(1-\nu^2)}\right)$.

Where the quantities of N_{yx} and M_{yx} can be computed by the identity given in equation (4).

If the effect of transverse normal stress is neglected, that is, the coefficient $c = 0$ and the terms $\overline{W}'$ and $\overline{W}''$, are neglected, then the stress-strain relationships are the same as those given by R. Archer.[1]

If the nonlinear terms are neglected, the resulting equations are as derived by P.M. Naghdi.[3]

2.7 Example of Application in a Beam Problem

In this section, the results from the analysis are applied to a one dimensional problem, a beam simply supported at both ends is subjected to the following axial-load conditions,

1. Free and Forced Vibration of a Beam
2. Free and Forced Vibration of a Beam-Column
3. Parametric Excitation of a Beam- Column

Thus, only terms N_{xx}, M_{xx}, and Q_{xz} are retained. Also, $\tau_{zz} = 0$ which implies the parameters S, T, $\overline{W}'$ and $\overline{W}''$ equal zero. All functions of y are eliminated together with the terms $\overline{V}$ and ψ. The external shearing forces (P^+, P^-) on upper and lower surfaces are neglected together with the external normal force at the lower surface (q^-). The radii of curvature approach infinity for this problem which implies that $\frac{1}{r_1} = \frac{1}{r_2} = 0$.

In rectangular cartesian coordinates, A = B = 1. If Poisson's ratio (ν) also is set equal to zero, the elastic constants $\frac{Eh}{(1-\nu^2)}$ and $\frac{Eh^3}{12(1-\nu^2)}$ correspond to following conditions,

extensional rigidity $\quad \frac{Eh}{(1-\nu^2)} \rightarrow EA \quad (A = bh)$,

flexural rigidity $\quad \frac{Eh^3}{12(1-\nu^2)} \rightarrow EI \quad (I = \frac{bh^3}{12})$.

Also, for convenience N_{xx}, M_{xx} and Q_{xz} are written in shorthand form as N, M and Q respectively and the sign of the term N is changed so that the stability condition may be investigated.

Therefore, the five equations of equilibrium (37) reduces to

$$\begin{aligned} -\frac{\partial N}{\partial x} &= 0 \, , \\ \frac{\partial M}{\partial x} - Q &= \rho I \frac{\partial^2 \phi}{\partial t^2} \, , \\ \frac{\partial Q}{\partial x} - \frac{\partial}{\partial x}\left(N \frac{\partial W}{\partial x}\right) + q &= \rho A \frac{\partial^2 W}{\partial t^2} \, . \end{aligned} \qquad (41)$$

Also, the eight stress-strain relationships reduce to the following two equations

$$\begin{aligned} M &= EI \frac{\partial \phi}{\partial x} \, , \\ Q &= \frac{5}{6} GA \left[\frac{\partial W}{\partial x} + \phi\right] . \end{aligned} \qquad (42)$$

Thus, there are five equations for five functions N, M, Q, W and ϕ. The first equation of equations (41) restricts the function N to be a constant function, and independent with function x. For convenience, the function N is replaced by the function P to correspond to current practice in the literature.

CASE I. Free and Forced Vibration of a Beam

For this case the conditions $P = q = 0$ hold. Equations (41) and (42) become,

$$\begin{aligned} \frac{\partial M}{\partial x} - Q &= \rho I \frac{\partial^2 \phi}{\partial t^2} \, , \\ \frac{\partial Q}{\partial x} &= \rho A \frac{\partial^2 W}{\partial t^2} \, , \\ M &= EI \frac{\partial \phi}{\partial x} \, , \\ Q &= \frac{5}{6} GA \left[\frac{\partial W}{\partial x} + \phi\right] . \end{aligned} \qquad (43)$$

The variables $W(x,t)$ and $\phi(x,t)$ are assumed harmonic in time,

$$\left.\begin{aligned} \text{or} \quad W_m(x,t) &= W_m(x)e^{i\omega_m t}, \text{ and} \\ \phi_m(x,t) &= \phi_m(x)e^{i\omega_m t} \text{ for the } m^{\text{ th}} \text{ mode of vibration.} \end{aligned}\right] \qquad (44)$$

Substituting $W_m(x,t)$ and $\phi_m(x,t)$ into the first two equations (43) and noting the last two equations of equations (43), the following equations are obtained,

$$\left.\begin{aligned} \frac{\partial M_m(x,t)}{\partial x} - Q_m(x,t) + \rho I \omega_m^2 \phi_m(x) &= 0 \ , \\ \frac{\partial Q_m(x,t)}{\partial x} + \rho A \omega_m^2 W_m(x) &= 0 \ . \end{aligned}\right] \qquad (45)$$

The orthogonality condition for the functions $W(x)$ and $\phi(x)$ is obtained by operation on equation (45), as

$$(\omega_m^2 - \omega_n^2)\int_0^L \left[\rho I \phi_m(x)\phi_n(x) + \rho A W_m(x) W_n(x)\right] dx$$
$$= \Big|\left[M_n(x,t)\phi_m(x) + Q_n(x,t)W_m(x)\right] - \left[M_m(x,t)\phi_n(x) + Q_m(x,t)W_n(x)\right]\Big|_{x=0}^{x=L} \qquad (46)$$

The right hand side of equation (46) contains both natural boundary conditions and forced boundary conditions which become zero for simply supported, fixed and free boundary conditions.

Thus, we have,

$$\int_0^L \left[\rho I \phi_m(x)\phi_n(x) + \rho A W_m(x) W_n(x)\right] dx = 0 \ , \qquad (47)$$

provided $\omega_m^2 \neq \omega_n^2$, for $m(1,2,3,\ldots) \neq n(1,2,3,\ldots)$.

Combining the four equations (43), we obtain the fourth order partial differential equation of the function $W(x,t)$ as,

$$EI\frac{\partial^4 W}{\partial x^4} - \rho I\left(\frac{6E}{5G}+1\right)\frac{\partial^4 W}{\partial x^2 \partial t^2} + \rho A\frac{\partial^2 W}{\partial t^2} + \frac{6}{5}\rho^2\frac{I}{G}\frac{\partial^4 W}{\partial t^4} = 0 \ , \qquad (48)$$

where bending stress, transverse shear stress, transverse and rotary inertia terms are included.

A similar fourth order equation is obtained for the shear parameter where $W(x,t)$ is replaced by $\phi(x,t)$.

Using the method of separation of variables, and noting equation (44) and assuming the time function as harmonic, we obtain,

$$\frac{\partial^4 W(x)}{\partial x^4} + \mu^2 \frac{\partial^2 W(x)}{\partial x^2} - \eta^4 W(x) = 0 , \qquad (49)$$

where

$$\left.\begin{aligned} \mu^2 &= \frac{\rho\omega^2}{E}\left(\frac{6}{5}\frac{E}{G}+1\right) , \\ \eta^4 &= \frac{\rho\omega^2}{E}\left(\frac{A}{I} - \frac{6}{5}\frac{\rho\omega^2}{G}\right) . \end{aligned}\right] \qquad (50)$$

the four roots of the differential equation are obtained as follows,

$$\left.\begin{aligned} R_{1,2} &= \pm\left[\left(\frac{\mu^4}{4}+\eta^4\right)^{1/2} - \frac{\mu^2}{2}\right]^{1/2} = \pm z , \\ R_{3,4} &= \pm i\left[\left(\frac{\mu^4}{4}+\eta^4\right)^{1/2} + \frac{\mu^2}{2}\right]^{1/2} = \pm i z . \end{aligned}\right] \qquad (51)$$

Then, it follows that

$$W(x) = A_1 \cosh zx + A_2 \sinh zx + A_3 \cos zx + A_4 \sin zx . \qquad (52)$$

A similar solution for function $\phi(x)$ is obtained in the same manner.

The boundary conditions for a simply supported beam with length L are (see equation (46))

$$\left.\begin{aligned} W &= 0 \Big|_{x=0}^{x=L} , \\ M &= EI \frac{d\phi}{dx} = 0 \Big|_{x=0}^{x=L} . \end{aligned}\right] \qquad (53)$$

Noting equations (43), we obtain,

$$\frac{d\phi}{dx} = -\left[\frac{6}{5}\rho\frac{\omega^2}{G}W + \frac{d^2W}{dx^2}\right] \quad . \tag{54}$$

Thus, the boundary condition for moment written as a function of W only, is

$$M = -EI\left[\frac{6}{5}\rho\frac{\omega^2}{G}W + \frac{d^2W}{dx^2}\right] = 0\,\Bigg|_{x=0}^{x=L} \quad . \tag{55}$$

Using the preceding two boundary conditions, the following set of eigen functions are obtained,

$$W_n(x) = A_{4n}\sin\xi x \quad , \tag{56}$$

where, $\xi = \frac{n\pi}{L}$ (n = 1,2,3,...) .

Substituting ξ into the roots $R_{3,4}$ in equation (51) yields the natural frequency of free vibration defined as,

$$\omega_n^2 = \frac{1}{2}\left(\frac{n\pi}{L}\right)^2\left\{\left[\frac{E}{\rho} + \frac{5G}{6\rho} + \frac{5}{6}\frac{GA}{\rho I}\left(\frac{L}{n\pi}\right)^2\right] \pm \left\{\left[\frac{E}{\rho} + \frac{5G}{6\rho} + \frac{5}{6}\frac{GA}{\rho I}\left(\frac{L}{n\pi}\right)^2\right]^2 - 4\left(\frac{5}{6}\frac{GE}{\rho^2}\right)\right\}^{1/2}\right\} . \tag{57}$$

A similar result is obtained for the function $\phi(x)$. The boundary conditions are taken with the aid of equation (43) as follows,

$$\left.\begin{aligned} W &= -\left[\frac{EI}{\rho A\omega^2}\frac{d^3\phi}{dx^3} + \frac{I}{A}\frac{d\phi}{dx}\right] = 0\,\Bigg|_{x=0}^{x=L} \; , \\ M &= EI\frac{d\phi}{dx} = 0\,\Bigg|_{x=0}^{x=L} \; . \end{aligned}\right] \tag{58}$$

The solution of the forced vibration is obtained from the following equations,

$$\left.\begin{aligned} \frac{\partial M}{\partial x} - Q &= \rho I\frac{\partial^2\phi}{\partial t^2} \; , \\ \frac{\partial Q}{\partial x} + q &= \rho A\frac{\partial^2 W}{\partial t^2} \; , \\ M &= EI\frac{\partial\phi}{\partial x} \; , \\ Q &= \frac{5}{6}GA\left[\frac{\partial W}{\partial x} + \phi\right] . \end{aligned}\right] \tag{59}$$

Combining equation (59) yields a fourth order partial differential equation in the form

$$EI\frac{\partial^4 W}{\partial x^4} - \rho I\left(\frac{6E}{5G}+1\right)\frac{\partial^4 W}{\partial x^2 \partial t^2} + \rho A\frac{\partial^2 W}{\partial t^2} + \frac{6}{5}\rho^2\frac{I}{G}\frac{\partial^4 W}{\partial t^4}$$
$$= q(x,t) + \frac{6EI}{5GA}\frac{\partial^2 q(x,t)}{\partial x^2} - \frac{6\rho I}{5GA}\frac{\partial^2 q(x,t)}{\partial t^2}. \tag{60}$$

A similar equation is obtained for the function (x)

$$EI\frac{\partial^4 \Phi}{\partial x^4} - \rho I\left(\frac{6E}{5G}+1\right)\frac{\partial^4 \Phi}{\partial x^2 \partial t^2} + \rho A\frac{\partial^2 \Phi}{\partial t^2} + \frac{6}{5}\rho^2\frac{I}{G}\frac{\partial^4 \Phi}{\partial t^4} = -\frac{\partial q}{\partial x}(x,t). \tag{61}$$

Since the free vibration problem yields a complete set of orthogonal functions for both the functions W(X) and $\Phi(x)$, a normal-mode type solution for the forced vibration problem is assumed to take the form,

$$\left.\begin{aligned} \Phi(x,t) &= \sum_m a_m(t)\,\Phi_m(x) \;, \\ W(x,t) &= \sum_m a_m(t)\,W_m(x) \;, \end{aligned}\right] \tag{62}$$

and also, by equations (43)

$$\left.\begin{aligned} M(x,t) &= \sum_m a_m(t)\,M_m(x) \;, \\ Q(x,t) &= \sum_m a_m(t)\,Q_m(x) \;. \end{aligned}\right] \tag{63}$$

Substituting these conditions into equation (59) integrating over the length of the beam, and making use of equation (45) gives the following result,

$$\sum_m\left(\ddot{a}_m(t) + \omega_m^2 a_m(t)\right)\int_0^L\left[\rho I\,\Phi_m(x)\Phi_n(x) + \rho A\,W_m(x)W_n(x)\right]dx$$
$$= \int_0^L q(x,t)\,W_n(x)\,dx \quad . \tag{64}$$

Applying the orthogonality conditions of equation (47) reduced the form of equation (64) to the form,

$$\ddot{a}_m(t) + \omega_m^2 a_m(t) = \frac{\int_0^L q(x,t) W_m(x)\, dx}{\int_0^L \left[\rho I \phi_m^2(x) + \rho A W_m^2(x)\right] dx} . \tag{65}$$

Using the method of variation parameters the solution of equation (65) is assumed as,

$$a_m(t) = A_m(t) \cos \omega_m t + B_m(t) \sin \omega_m t \quad , \tag{66}$$

where $A_m(t)$ and $B_m(t)$ are arbitrary functions of time.

This results in the solution for $a_m(t)$ as follows,

$$a_m(t) = A_m(0) \cos \omega_m t + B_m(0) \sin \omega_m t + \int_{\tau=0}^{\tau=t} \frac{f_m(x,\tau)}{\omega_m} \sin \omega_m (t-\tau)\, d\tau , \tag{67}$$

where,

$$f_m(x,\tau) = \frac{\int_0^L q(x,\tau) W_m(x)\, dx}{\int_0^L \left[\rho I \phi_m^2(x) + \rho A W_m^2(x)\right] dx} , \tag{68}$$

and the last integral called Duhamel's Integral.

The parameters $A_m(0)$ and $B_m(0)$ are obtained by applying the necessary initial conditions on displacement and velocity.

Combining equations (67) with equations (62), we write,

$$W(x,t) = \sum_m \left\{ \left\{ \left[A_m(0) \cos \omega_m t + B_m(0) \sin \omega_m t \right] + \frac{1}{\omega_m} \int_{\tau=0}^{\tau=t} f_m(x,\tau) \sin \omega_m (t-\tau)\, d\tau \right\} W_m(x) \right\} ,$$

$$\phi(x,t) = \sum_m \left\{ \left\{ \left[A_m(0) \cos \omega_m t + B_m(0) \sin \omega_m t \right] + \frac{1}{\omega_m} \int_{\tau=0}^{\tau=t} g_m(x,t) \sin \omega_m (t-\tau)\, d\tau \right\} \phi_m(x) \right\} , \tag{69}$$

where $f_m(x,\tau)$ and $g_m(x,\tau)$ are the right hand side of equation (60)&(61).

Using Liebnitz's rule which is defined as

$$\frac{d}{dt}\int_{\tau=C_1(t)}^{\tau=C_2(t)} f(\tau,t)\,d\tau = \int_{\tau=C_1(t)}^{\tau=C_2(t)} \frac{\partial}{\partial t} f(\tau,t)\,d\tau + f[C_2(t)]\frac{dC_2(t)}{dt} - f[C_1(t)]\frac{dC_1(t)}{dt} \qquad (70)$$

and noting the initial conditions on displacement and velocity as,

$$\left.\begin{aligned} @\ t = 0 \quad W(x,t) &= W(x,0)\,, \\ \mathring{W}(x,t) &= \mathring{W}(x,0)\,, \\ \phi(x,t) &= \phi(x,0)\,, \\ \mathring{\phi}(x,t) &= \mathring{\phi}(x,0)\,, \end{aligned}\right] \qquad (71)$$

and together with the orthogonality conditions given in equations (47), it follows that,

$$\left.\begin{aligned} A_m(0) &= \frac{\int_0^L \left[\rho I \phi(x,0)\phi_m(x) + \rho A W(x,0) W_m(x)\right] dx}{\int_0^L \left[\rho I \phi_m^2(x) + \rho A W_m^2(x)\right] dx}\,, \\ B_m(0) &= \frac{\int_0^L \left[\rho I \mathring{\phi}(x,0)\phi_m(x) + \rho A \mathring{W}(x,0) W_m(x)\right] dx}{\omega_m \int_0^L \left[\rho I \phi_m^2(x) + \rho A W_m^2(x)\right] dx}\,. \end{aligned}\right] \qquad (72)$$

The general solution is written in final form as,

$$\begin{aligned} W(x,t) = \sum_m \Bigg\{ \Bigg\{ & \frac{\int_0^L \left[\rho I \phi(x,0)\phi_m(x) + \rho A W(x,0) W_m(x)\right] dx}{\int_0^L \left[\rho I \phi_m^2(x) + \rho A W_m^2(x)\right] dx} \cos\omega_m t \\ & + \frac{\int_0^L \left[\rho I \mathring{\phi}(x,0)\phi_m(x) + \rho A \mathring{W}(x,0) W_m(x)\right] dx}{\omega_m \int_0^L \left[\rho I \phi_m^2(x) + \rho A W_m^2(x)\right] dx} \sin\omega_m t \\ & + \frac{1}{\omega_m} \int_{\tau=0}^{\tau=t} \frac{\int_0^L q(x,\tau) W_m(x)\,dx}{\int_0^L \left[\rho I \phi_m^2(x) + \rho A W_m^2(x)\right] dx} \sin\omega_m (t-\tau)\,d\tau \Bigg\} W_m(x) \Bigg\}, \end{aligned} \qquad (73)$$

and,

$$\dot{W}(x,t) = \sum_m \left\{ \left\{ -\frac{\omega_m \int_0^L [\rho I \phi(x,0)\phi_m(x) + \rho A W(x,0) W_m(x)]\,dx}{\int_0^L [\rho I \phi_m^2(x) + \rho A W_m^2(x)]\,dx} \sin\omega_m t \right.\right.$$

$$+ \frac{\int_0^L [\rho I \dot{\phi}(x,0)\phi_m(x) + \rho A \dot{W}(x,0) W_m(x)]\,dx}{\int_0^L [\rho I \phi_m^2(x) + \rho A W_m^2(x)]\,dx} \cos\omega_m t$$

$$\left.\left. + \int_{\tau=0}^{\tau=t} \frac{\int_0^L q(x,\tau) W_m(x)\,dx}{\int_0^L [\rho I \phi_m^2(x) + \rho A W_m^2(x)]\,dx} \cos\omega_m(t-\tau)\,d\tau \right\} W_m(x) \right\} . \qquad (74)$$

Similar solutions for $\phi(x,t)$ and $\dot{\phi}(x,t)$ are obtained, if the proper load functions are substituted.

CASE II. Free and Forced Vibration of a Beam-Column

Refering to the equilibrium equations (41), we write

$$\left.\begin{aligned} \frac{\partial M}{\partial x} - Q - \rho I \frac{\partial^2 \phi}{\partial t^2} &= 0 , \\ P_0 \frac{\partial^2 W}{\partial x^2} - \frac{\partial Q}{\partial x} + \rho A \frac{\partial^2 W}{\partial t^2} &= q(x,t) . \end{aligned}\right] \qquad (75)$$

Proceeding in the same manner as in CASE I, (see equation (47)) the orthogonality condition for the beam-column are determined as,

$$(\omega_m^2 - \omega_n^2) \int_0^L [\rho I \phi_m(x)\phi_n(x) + \rho A W_m(x) W_n(x)]\,dx \qquad (76)$$

$$= \left| \left[M_n(x,t)\phi_m(x) + \left(Q_n(x,t) - P_0 \frac{dW_n}{dx}(x)\right) W_m(x) \right] - \left[M_m(x,t)\phi_n(x) + \left(Q_m(x,t) - P_0 \frac{dW_m}{dx}(x)\right) W_n(x) \right] \right|_0^L$$

For the special cases of simple supports, free and fixed boundary conditions, the orthogonality condition reduces to

$$\int_0^L \left[\rho I \phi_m(x)\phi_n(x) + \rho A W_m(x) W_n(x) \right] dx = 0 \,, \tag{77}$$

where $\omega_m^2 \neq \omega_n^2$

The fourth order differential equation for the functions W(x,t) and ϕ(x,t) are respectively,

$$EI\left(1-\frac{6P_o}{5GA}\right)\frac{\partial^4 W}{\partial x^4} + P_o\frac{\partial^2 W}{\partial x^2} - \rho I\left(1-\frac{6P_o}{5GA}+\frac{6E}{5G}\right)\frac{\partial^4 W}{\partial x^2 \partial t^2} + \rho A\frac{\partial^2 W}{\partial t^2} + \frac{6}{5}\frac{\rho^2 I}{G}\frac{\partial^4 W}{\partial t^4}$$
$$= q(x,t) - \frac{6EI}{5GA}\frac{\partial^2 q(x,t)}{\partial x^2} + \frac{6\rho I}{5GA}\frac{\partial^2 q(x,t)}{\partial t^2} \tag{78}$$

and

$$EI\left(1-\frac{6P_o}{5GA}\right)\frac{\partial^4 \phi}{\partial x^4} + P_o\frac{\partial^2 \phi}{\partial x^2} - \rho I\left(1-\frac{6P_o}{5GA}+\frac{6E}{5G}\right)\frac{\partial^4 \phi}{\partial x^2 \partial t^2} + \rho A\frac{\partial^2 \phi}{\partial t^2} + \frac{6}{5}\frac{\rho^2 I}{G}\frac{\partial^4 \phi}{\partial t^4}$$
$$= \frac{\partial q(x,t)}{\partial x} \tag{79}$$

Assuming the function W(x,t) is harmonic function of time, the free vibration form of equations (75) are

$$\left.\begin{aligned} \frac{dM(x)}{dx} - Q(x) &= -\rho I \Omega^2 \phi(x) \,, \\ -P_o\frac{d^2W(x)}{dx^2} + \frac{dQ(x)}{dx} &= -\rho A \Omega^2 W(x) \,. \end{aligned}\right] \tag{80}$$

In a similar manner, the free vibration form of equation (78) takes the form,

$$\frac{\partial^4 W}{\partial x^4} + k^2\frac{\partial^2 W}{\partial x^2} - j^4 W = 0 \tag{81}$$

where

$$k^2 = \rho\frac{\Omega^2}{E}\left[1+\frac{\left(\frac{6E}{5G}+\frac{P_o}{\rho I\Omega^2}\right)}{\left(1-\frac{6P_o}{5GA}\right)}\right] \tag{82}$$

$$j^4 = \rho\frac{\Omega^2}{E}\frac{\left(\frac{A}{I}-\frac{6\rho\Omega^2}{5G}\right)}{\left(1-\frac{6P_o}{5GA}\right)} \tag{83}$$

For the special case of simply-supported boundaries, that is $M(0) = M(L) = 0$ and $W(0) = W(L) = 0$; the equation for natural frequency is determined as

$$\sin jL = 0, \qquad j = 1,2,3,\ldots$$

thus, $jL = n\pi$

and the natural frequency is obtained in the form,

$$\Omega_n^2 = \frac{1}{2}\left(\frac{n\pi}{L}\right)^2 \left\{\left[\frac{E}{\rho} + \frac{5G}{6\rho} + \frac{5GA}{6\rho I}\left(\frac{L}{n\pi}\right)^2 - \frac{P_0}{A}\right] \pm \left\{\left[\frac{E}{\rho} + \frac{5G}{6\rho} + \frac{5GA}{6\rho I}\left(\frac{L}{n\pi}\right)^2 - \frac{P_0}{A}\right]^2 - 4\left[\frac{5GE}{6\rho^2} - \frac{P_0 E}{\rho^2 A} - \frac{5P_0 G}{6\rho^2 I}\left(\frac{L}{n\pi}\right)^2\right]\right\}^{1/2}\right\} . \tag{84}$$

The solution of the forced vibration problem is obtained in a similar manner as in equation (73)&(74), in the form

$$W(x,t) = \sum_m \left\{\left\{ \frac{\int_0^L [\rho I \phi(x,0)\phi_m(x) + \rho A W(x,0) W_m(x)]\,dx}{\int_0^L [\rho I \phi_m^2(x) + \rho A W_m^2(x)]\,dx} \cos \omega_m t + \frac{\int_0^L [\rho I \dot{\phi}(x,0)\phi_m(x) + \rho A \dot{W}(x,0) W_m(x)]\,dx}{\Omega_m \int_0^L [\rho I \phi_m^2(x) + \rho A W_m^2(x)]\,dx} \sin \omega_m t + \frac{1}{\Omega_m}\int_{\tau=0}^{\tau=t} \frac{\int_0^L q(x,\tau) W_m(x)\,dx}{\int_0^L [\rho I \phi_m^2(x) + \rho A W_m^2(x)]\,dx} \sin \omega_m (t-\tau)\,d\tau \right\} W_m(x)\right\},$$

and,

(85)

$$\dot{W}(x,t) = \sum_m \left\{\left\{ -\frac{\Omega_m \int_0^L [\rho I \phi(x,0)\phi_m(x) + \rho A W(x,0) W_m(x)]\,dx}{\int_0^L [\rho I \phi_m^2(x) + \rho A W_m^2(x)]\,dx} \sin \omega_m t \right.\right.$$

$$+\frac{\int_0^L\left[\rho I\mathring{\phi}(x,0)\phi_m(x)+\rho A\mathring{W}(x,0)W_m(x)\right]dx}{\int_0^L\left[\rho I\phi_m^2(x)+\rho AW_m^2(x)\right]dx}\cos\omega_m t$$

$$+\left\{\int_{\tau=0}^{\tau=L}\frac{\int_0^L q(x,\tau)W_m(x)dx}{\int_0^L\left[\rho I\phi_m^2(x)+\rho AW_m^2(x)\right]dx}\cos\omega_m(t-\tau)d\tau\right\}W_m(x)\Big\} .$$

(86)

CASE III. Parametric Excitation of the Beam-Column

The axial load on the beam is assumed to be a harmonic function of time, as $P(t) = P_0 + P_t \cos \theta t$ (see Figure 2.2).

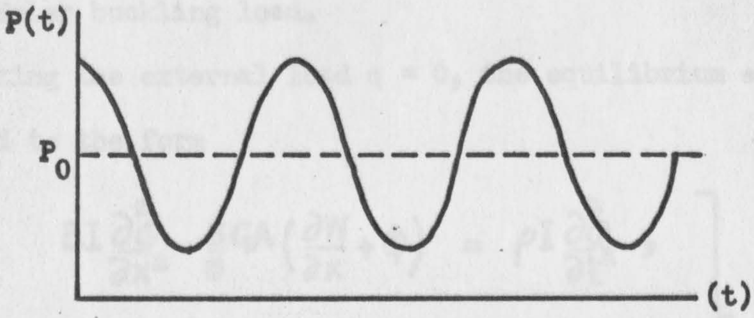

Figure 2.2 Time Variation of the Parametric Force

As a first approximation to the solution, we neglect the fourth order derivative with respect to t in equation (78).

The equation becomes

$$EI\left(1-\frac{6P(t)}{5GA}\right)\frac{\partial^4 W}{\partial x^4}+\left[P(t)+\rho I\left(1-\frac{6P(t)}{5GA}+\frac{6E}{5G}\right)\Omega^2\right]\frac{\partial^2 W}{\partial x^2}-\rho A\Omega^2 W=0 \quad . \quad (87)$$

The natural frequency of free vibration from CASE I and CASE II respectively reduce to the form

$$\bar{\omega}_n^2 = \frac{\left(\frac{n\pi}{L}\right)^4 EI}{\left[\rho A+\rho I\left(\frac{n\pi}{L}\right)^2\left(1+\frac{6E}{5G}\right)\right]} \quad (88)$$

and

$$\bar{\Omega}_n^2 = \bar{\omega}_n^2 \left[\frac{1}{1 - \frac{6}{5GA} \frac{P_0}{\left(1 + \frac{AE}{P_{nCR}}\right)}} \right] \left(1 - \frac{P_0}{P_{nCR}}\right) , \qquad (89)$$

where

$$P_{nCR} = \frac{P_{nE}}{\left(1 + \frac{6 P_{nE}}{5GA}\right)} , \qquad (90)$$

and

$$P_{nE} = \left(\frac{n\pi}{L}\right)^2 EI , \qquad (91)$$

P_{nCR} is defined the critical buckling load and

P_{nE} is the Euler buckling load.

Setting the external load q = 0, the equilibrium equations (75) are simplified to the form

$$\left.\begin{aligned} EI\frac{\partial^2 \phi}{\partial x^2} - \frac{5}{6}GA\left(\frac{\partial W}{\partial x} + \phi\right) &= \rho I \frac{\partial^2 \phi}{\partial t^2} , \\ P(t)\frac{\partial^2 W}{\partial x^2} + \frac{5}{6}GA\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial \phi}{\partial x}\right) &= \rho A \frac{\partial^2 W}{\partial t^2} . \end{aligned}\right] \qquad (92)$$

Taking one derivative with respect to x for the first equation in equation (92) and combining the result with the second equation yields

$$EI\frac{\partial^3 \phi}{\partial x^3} + P(t)\frac{\partial^2 W}{\partial x^2} = \rho I \frac{\partial^3 \phi}{\partial x \partial t^2} + \rho A \frac{\partial^2 W}{\partial t^2} . \qquad (93)$$

Since the eigen functions for the both quantities of $\phi(x)$ and W(x)are sine functions for the special case of a simply supported beam at both ends, the function $\phi(x,t)$ is assumed in the form,

$$\phi(x,t) = f_n(t) \sin \frac{n\pi x}{L} \quad , \qquad (94)$$

where $f_n(t)$ is a pure function of time.

Refering to equation (58) the relationship between the functions $\phi(x,t)$ and $W(x,t)$ is given as,

$$W(x,t) = -\left[\frac{EI}{\rho A \bar{\omega}^2}\frac{\partial^3 \phi}{\partial x^3} + \frac{I}{A}\frac{\partial \phi}{\partial x}\right] . \qquad (95)$$

Substituting equation (94) into equation (95) yields,

$$W(x,t) = \left(\frac{n\pi}{L}\right)\left[\frac{EI}{\rho A \bar{\omega}_n^2}\left(\frac{n\pi}{L}\right)^2 - \frac{I}{A}\right] f_n(t) \cos \frac{n\pi x}{L} \qquad (96)$$

Substituting equations (94) and (96) into equation (93) yields

$$\ddot{f}_n(t) + \left\{\bar{\omega}_n^2 + \left[\frac{1}{\rho A}\left(\frac{n\pi}{L}\right)^2 - \frac{\bar{\omega}_n^2}{AE}\right](P_o + P_t \cos \theta t)\right\} f_n(t) = 0 \; . \qquad (97)$$

Noting the value of $\bar{\omega}_n^2$ from equation (88), equation (97) reduces to

$$\ddot{f}_n(t) + \bar{\omega}_n^2\left(1 - \frac{P_o}{P_{nCR}}\right)\left[1 - \frac{P_t}{(P_{nCR} - P_o)} \cos \theta t\right] f_n(t) = 0 \; . \qquad (98)$$

Substituting equation (89) into above equation, and rearranging terms yields the form of Mathieu's equation as,

$$\ddot{f}_n(t) + \hat{\bar{\Omega}}_n^2 \left[1 - 2\bar{\mu} \cos \theta t\right] f_n(t) = 0 \; , \qquad (99)$$

where

$$\hat{\bar{\Omega}}_n^2 = \bar{\Omega}_n^2 \left[1 - \frac{6 P_o}{5 G A} \frac{1}{\left(1 + \frac{AE}{P_{nCR}}\right)}\right] \; , \qquad (100)$$

and

$$2\bar{\mu} = \frac{P_t}{P_{nCR} - P_o} \; . \qquad (101)$$

Exact solutions of these types of equations are not possible. However, the stability characteristics of the equations are known. The stable and unstable regions are shown by use of Floguet's theory of differential equations to be separated by solutions of periodic functions of period T and 2T of the parameter θt.

To determine regions bounded by periodic functions of period T we assume Fourier's series solution in the form

$$f(t) = b_0 + \sum_{k=1}^{\infty} (b_k \cos k\theta t + a_k \sin k\theta t) \qquad (102)$$

Substituting into equation (99) by equation (102) and equating the coefficients of similar trigonometric functions gives,

$$\begin{bmatrix} [\hat{\hat{\Omega}}^2-\theta^2] & -\bar{\mu}\hat{\hat{\Omega}}^2 & 0 & 0 & \cdots \\ -\bar{\mu}\hat{\hat{\Omega}}^2 & [\hat{\hat{\Omega}}^2-(2\theta)^2] & -\bar{\mu}\hat{\hat{\Omega}}^2 & 0 & \cdots \\ 0 & -\bar{\mu}\hat{\hat{\Omega}}^2 & [\hat{\hat{\Omega}}^2-(3\theta)^2] & -\bar{\mu}\hat{\hat{\Omega}}^2 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0_m \end{bmatrix}$$

(103)

and

$$\begin{bmatrix} \hat{\hat{\Omega}}^2 & -\bar{\mu}\hat{\hat{\Omega}}^2 & 0 & 0 & \cdot & \cdot \\ -2\bar{\mu}\hat{\hat{\Omega}}^2 & [\hat{\hat{\Omega}}^2-\theta^2] & -\bar{\mu}\hat{\hat{\Omega}}^2 & 0 & \cdot & \cdot \\ 0 & -\bar{\mu}\hat{\hat{\Omega}}^2 & [\hat{\hat{\Omega}}^2-(2\theta)^2] & -\bar{\mu}\hat{\hat{\Omega}}^2 & \cdot & \cdot \\ 0 & 0 & -\bar{\mu}\hat{\hat{\Omega}}^2 & [\hat{\hat{\Omega}}^2-(3\theta)^2] & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0_m \end{bmatrix}$$

(104)

To determine regions bounded by periodic functions of period 2T the Fourier's series take the form

$$f(t) = \sum_{k=1,3,5,7,\ldots}^{\infty} \left(a_k \sin \frac{k\theta t}{2} + b_k \cos \frac{k\theta t}{2} \right) \quad . \qquad (105)$$

Substituting equation (105) into equation (99), gives

$$\begin{bmatrix} \left\{\left[1-\left(\frac{\theta}{2\hat{\Omega}}\right)^2\right]+\bar{\mu}\right\} & -\bar{\mu} & 0 & 0 & \cdot & \cdot \\ -\bar{\mu} & \left[1-\left(\frac{3\theta}{2\hat{\Omega}}\right)^2\right] & -\bar{\mu} & 0 & \cdot & \cdot \\ 0 & -\bar{\mu} & \left[1-\left(\frac{5\theta}{2\hat{\Omega}}\right)^2\right] & -\bar{\mu} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \cdot \\ \cdot \\ a_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0_m \end{bmatrix}$$

(106)

and

$$\begin{bmatrix} \left\{\left[1-\left(\frac{\theta}{2\hat{\Omega}}\right)^2\right]-\bar{\mu}\right\} & -\bar{\mu} & 0 & 0 & \cdots \\ -\bar{\mu} & \left[1-\left(\frac{3\theta}{2\hat{\Omega}}\right)^2\right]-\bar{\mu} & & 0 & \cdots \\ 0 & -\bar{\mu} & \left[1-\left(\frac{5\theta}{2\hat{\Omega}}\right)^2\right]-\bar{\mu} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \cdot \\ \cdot \\ b_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0_m \end{bmatrix} \tag{107}$$

As a first approximation, the coefficient of a_1 in equations (103) and (106) yield the solutions as $\frac{\theta}{2\hat{\Omega}} = 0.5$ and $\frac{\theta}{2\hat{\Omega}} = 1.0$ respectively. If additional terms are included, the curves of stable and unstable are shown in Figure 2.3 as follows.

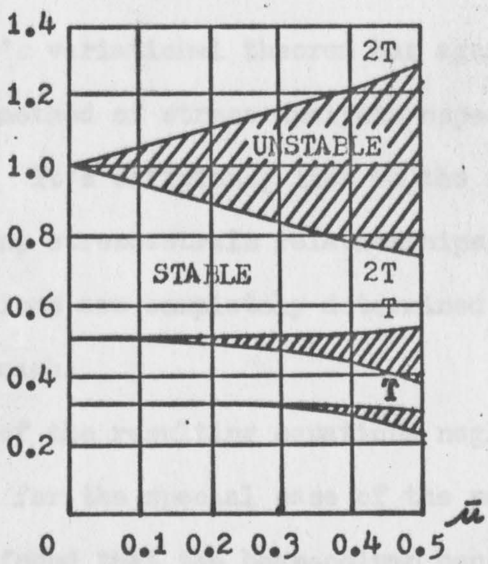

Figure 2.3 Location of Stability and Instability Zone

The shaded area shown represent the first two instability zones.

CHAPTER III

CONCLUSIONS

The equations of motion, the stress-strain relationships and the natural and forced boundary conditions are determined for the special case of a nonlinear shell theory including the effects of transverse normal stress, transverse shear stress and transverse and rotary inertia.

The addition of the transverse normal stress into the stress analysis problem produces a set of highly coupled differential equations which do not easily extend themselves to the usual uncoupling procedures. The uncoupling of the equations is not preformed in this thesis. An extension of this thesis is the determination of the proper procedure for this condition.

The Reissner's variational theorem has again proven itself as an extremely powerful method of stress analysis especially when applied to nonlinear problems. It's efficiency lies in the fact that the resulting equations of motion, stress-strain relationships and natural and forced boundary conditions are completely determined without use of a free body diagram approach.

Application of the resulting equations neglecting the transverse normal stress, is made for the special case of the parametric stability of a beam-column. It is found that the beam-column becomes unstable at a much lower frequency when the effects of shear and rotary and transverse inertia are included.

APPENDIX A

The eight stress-strain relationships given in equation (38) are recasted in the following matrix form.

$$\begin{bmatrix} A & B & 0 & -C & 0 & 0 & 0 & 0 \\ B & D & 0 & 0 & C & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 & F & 0 & 0 \\ -C & 0 & 0 & G & H & 0 & 0 & 0 \\ 0 & C & 0 & H & J & 0 & 0 & 0 \\ 0 & 0 & F & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & M \end{bmatrix} \begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \\ Q_{xz} \\ Q_{yz} \end{bmatrix} = \begin{bmatrix} \overline{A} \\ \overline{B} \\ \overline{C} \\ \overline{D} \\ \overline{E} \\ \overline{F} \\ \overline{G} \\ \overline{H} \end{bmatrix} \tag{A 1}$$

where,

$$\left.\begin{aligned} A &= \frac{1}{Eh}\left[1-\frac{h^2}{12r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right], & J &= \frac{12}{Eh^3}\left[1+\frac{3h^2}{20r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right], \\ B &= -\frac{\nu}{Eh}, & K &= 2(1+\nu)\frac{12}{Eh^3}\left[1-\frac{3h^2}{20r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right], \\ C &= -\frac{1}{Eh}\left(\frac{1}{r_1}-\frac{1}{r_2}\right), & L &= \frac{2(1+\nu)}{Eh}\,\frac{6}{5}\left[1-\frac{h^2}{28r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right], \\ D &= \frac{1}{Eh}\left[1+\frac{h^2}{12r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right], & M &= \frac{2(1+\nu)}{Eh}\,\frac{6}{5}\left[1+\frac{h^2}{28r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right], \\ E &= \frac{2(1+\nu)}{Eh}\left[1-\frac{h^2}{12r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right], \\ F &= \frac{2(1+\nu)}{Eh}\left(\frac{1}{r_1}-\frac{1}{r_2}\right), \\ G &= \frac{12}{Eh^3}\left[1-\frac{3h^2}{20r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right], \\ H &= -\nu\frac{12}{h^3}, \end{aligned}\right] \tag{A 2}$$

and where,

$$\bar{A} = \overset{\circ}{\epsilon}_{xx} + \frac{h^2}{24}\left[\frac{\bar{W}''}{r_1} + \frac{1}{A^2}\left(\bar{W}_x\bar{W}_x'' + \bar{W}_x'^2\right)\right] + \frac{h^4}{640}\frac{\bar{W}_x''^2}{A^2} + \frac{\vartheta C}{Eh}\left[\left(1+\frac{h^2}{20r_2^2}\right)S - \frac{h}{60r_2}T + \frac{h}{2}\overset{+}{q}\overset{+}{H}\left(1-\frac{h}{5r_2}+\frac{h^2}{12r_2^2}\right) + \frac{h}{2}\overset{-}{q}\overset{-}{H}\left(1+\frac{h}{5r_2}+\frac{h^2}{12r_2^2}\right)\right] ,$$

$$\bar{B} = \overset{\circ}{\epsilon}_{yy} + \frac{h^2}{24}\left[\frac{\bar{W}''}{r_2} + \frac{1}{B^2}\left(\bar{W}_y\bar{W}_y'' + \bar{W}_y'^2\right)\right] + \frac{h^4}{640}\frac{\bar{W}_y''^2}{B^2} + \frac{\vartheta C}{Eh}\left[\left(1+\frac{h^2}{20r_1^2}\right)S - \frac{h}{60r_1}T + \frac{h}{2}\overset{+}{q}\overset{+}{H}\left(1-\frac{h}{5r_1}+\frac{h^2}{12r_1^2}\right) + \frac{h}{2}\overset{-}{q}\overset{-}{H}\left(1+\frac{h}{5r_1}+\frac{h^2}{12r_1^2}\right)\right] ,$$

$$\bar{C} = \overset{\circ}{\gamma}_{xx} + \left[1-\frac{h^2}{12r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]\overset{\circ}{\gamma}_{yy} + \frac{h^2}{12}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\delta_{yy} + \left(1+\frac{h^2}{12r_2^2}\right)\frac{\bar{W}_x\bar{W}_y}{AB} - \frac{h^2}{12r_2}\left[\frac{1}{AB}\left(\bar{W}_x\bar{W}_y' + \bar{W}_x'\bar{W}_y\right)\right] + \frac{h^2}{12}\left(1+\frac{3h^2}{20r_2^2}\right)\left[\frac{1}{AB}\left(\frac{\bar{W}_x\bar{W}_y''}{2} + \bar{W}_x'\bar{W}_y' + \frac{\bar{W}_x''\bar{W}_y}{2}\right)\right] - \frac{h^4}{160r_2}\left[\frac{1}{AB}\left(\bar{W}_x'\bar{W}_y'' + \bar{W}_x''\bar{W}_y'\right)\right] + \frac{h^4}{320}\left(1+\frac{5h^2}{28r_2^2}\right)\frac{\bar{W}_x''\bar{W}_y''}{AB} ,$$

$$\bar{D} = \left[\frac{1}{A}\left(\Phi_x + \frac{A_y}{B}\Psi\right) + \frac{\bar{W}'}{r_1} + \frac{\bar{W}_x\bar{W}_x'}{A^2}\right] + \frac{3h^2}{40}\frac{\bar{W}_x'\bar{W}_x''}{A^2} + \frac{\vartheta C}{Eh}\left[\frac{1}{5h}\left(1+\frac{3h^2}{28r_2^2}\right)T - \frac{3}{5r_2}S + \frac{6}{5}\overset{+}{q}\overset{+}{H}\left(1-\frac{5h}{12r_2}+\frac{h^2}{7r_2^2}\right) - \frac{6}{5}\overset{-}{q}\overset{-}{H}\left(1+\frac{5h}{12r_2}+\frac{h^2}{7r_2^2}\right)\right] ,$$

$$\bar{E} = \left[\frac{1}{B}\left(\Psi_y + \frac{B_x}{A}\Phi\right) + \frac{\bar{W}'}{r_2} + \frac{\bar{W}_y\bar{W}_y'}{B^2}\right] + \frac{3h^2}{40}\frac{\bar{W}_y'\bar{W}_y''}{B^2} + \frac{\vartheta C}{Eh}\left[\frac{1}{5h}\left(1+\frac{3h^2}{28r_1^2}\right)T - \frac{3}{5r_1}S + \frac{6}{5}\overset{+}{q}\overset{+}{H}\left(1-\frac{5h}{12r_1}+\frac{h^2}{7r_1^2}\right) - \frac{6}{5}\overset{-}{q}\overset{-}{H}\left(1+\frac{5h}{12r_1}+\frac{h^2}{7r_1^2}\right)\right] ,$$

$$\bar{F} = \delta_{xx} + \left[1-\frac{3h^2}{20r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]\delta_{yy} + \left(\frac{1}{r_1}-\frac{1}{r_2}\right)\overset{\circ}{\gamma}_{yy} - \frac{1}{r_2}\frac{\bar{W}_x\bar{W}_y}{AB} + \left(1+\frac{3h^2}{20r_2^2}\right)\left[\frac{1}{AB}\left(\bar{W}_x\bar{W}_y' + \bar{W}_x'\bar{W}_y\right)\right] - \frac{3h^2}{20r_2}\left[\frac{1}{AB}\left(\frac{\bar{W}_x\bar{W}_y''}{2} + \bar{W}_x'\bar{W}_y' + \frac{\bar{W}_x''\bar{W}_y}{2}\right)\right] + \frac{3h^2}{40}\left(1+\frac{5h^2}{28r_2^2}\right)\left[\frac{1}{AB}\left(\bar{W}_x'\bar{W}_y'' + \bar{W}_x''\bar{W}_y'\right)\right] - \frac{3h^4}{448r_2}\frac{\bar{W}_x''\bar{W}_y''}{AB} ,$$

$$\bar{G} = \overset{\circ}{\gamma}_{xz} + \frac{h^2}{40}\frac{\bar{W}_x''}{A} + \frac{(1+\vartheta)}{Eh}\left[\frac{2}{5}\left[1+\frac{h^2}{28r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]m_1 - \frac{h^2}{10}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)P_1\right] ,$$

$$\bar{H} = \mathring{\gamma}_{yz} + \frac{h^2}{40}\frac{\bar{W}_y''}{B} + \frac{(1+\nu)}{Eh}\left[\frac{2}{5}\left[1-\frac{h^2}{28r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]\mathcal{M}_2 + \frac{h^2}{10}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)P_2\right] .$$

(A 3)

The solutions are given as,

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \\ Q_{xz} \\ Q_{yz} \end{Bmatrix} = \frac{1}{|D|}\begin{bmatrix} 11 & 12 & 0 & -14 & 15 & 0 & 0 & 0 \\ -12 & 22 & 0 & 24 & -25 & 0 & 0 & 0 \\ 0 & 0 & 33 & 0 & 0 & -36 & 0 & 0 \\ -14 & 24 & 0 & 44 & -45 & 0 & 0 & 0 \\ 15 & -25 & 0 & -45 & 55 & 0 & 0 & 0 \\ 0 & 0 & -36 & 0 & 0 & 66 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 77 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 88 \end{bmatrix}\begin{Bmatrix} \bar{A} \\ \bar{B} \\ \bar{C} \\ \bar{D} \\ \bar{E} \\ \bar{F} \\ \bar{G} \\ \bar{H} \end{Bmatrix}$$

(A 4)

where,

$$\begin{aligned}
|D| &= ML(KE - F^2)\left[(GJ - H^2)(AD - B^2) - C^2(AG-2BH+DJ-C^2)\right], \\
11 &= ML(KE - F^2)\left[D(GJ - H^2) - C^2G\right], \\
12 &= ML(KE - F^2)\left[B(GJ - H^2) - C^2H\right], \\
14 &= -ML(KE - F^2)\left[C(DJ - C^2) - BHC\right], \\
15 &= -ML(KE - F^2)\left[C(DH - BG)\right], \\
22 &= ML(KE - F^2)\left[A(JG - H^2) - C^2J\right], \\
24 &= -ML(KE - F^2)\left[C(BJ - AH)\right], \\
25 &= -ML(KE - F^2)\left[C(C^2 - AG) + BHC\right], \\
33 &= MLK\left[(GJ - H^2)(AD - B^2) - C^2(AG-2BH+DJ-C^2)\right], \\
36 &= MLF\left[(GJ - H^2)(AD - B^2) - C^2(AG-2BH+DJ-C^2)\right], \\
44 &= ML(KE - F^2)\left[A(DJ - C^2) - B^2J\right], \\
45 &= ML(KE - F^2)\left[DAH - B(BH + C^2)\right],
\end{aligned}$$

$$
\left.\begin{aligned}
55 &= ML(\ KE - F^2\ )\left[ADG - B^2G - \ C^2D\right], \\
66 &= MLE\left[(GJ - H^2)(AD - B^2) - C^2(AG-2BH+DJ-C^2)\right], \\
77 &= M(\ KE - F^2\ )\left[(GJ - H^2)(AD - B^2) - C^2(AG-2BH+DJ-C^2)\right], \\
88 &= L(\ KE - F^2\ )\left[(GJ - H^2)(AD - B^2) - C^2(AG-2BH+DJ-C^2)\right].
\end{aligned}\right] \qquad (\ A\ 5\ )
$$

Note that each term in the matrix is divided by the term $|D|$ which will resulting terms containing ratios of the form $\frac{h}{r_1}$, $\frac{h}{r_2}$, and up to the order of $\frac{h^8}{r^8}$.

Division of each term of equation (A 5) by the determinant yields,

$$
\left.\begin{aligned}
\frac{11}{|D|} &= \frac{\left[D(\ GJ - H^2\ ) - C^2G\right]}{|D|^*}, \\
\frac{12}{|D|} &= \frac{\left[B(\ GJ - H^2\ ) - CH^2\right]}{|D|^*}, \\
\frac{14}{|D|} &= -\frac{\left[C(\ DJ - C^2\ ) - BHC\right]}{|D|^*}, \\
\frac{15}{|D|} &= -\frac{\left[C(\ DH - BG\ )\right]}{|D|^*}, \\
\frac{22}{|D|} &= \frac{\left[A(\ GJ - H^2\ ) - C^2J\right]}{|D|^*}, \\
\frac{24}{|D|} &= -\frac{\left[C(\ BJ - AH\ )\right]}{|D|^*}, \\
\frac{25}{|D|} &= -\frac{\left[C(\ C^2 - AG\ ) + BHC\right]}{|D|^*}, \\
\frac{33}{|D|} &= \frac{K}{(\ KE - F^2\ )}, \\
\frac{36}{|D|} &= \frac{F}{(\ KE - F^2\ )}, \\
\frac{44}{|D|} &= \frac{\left[A(\ DJ - C^2\ ) - B^2J\right]}{|D|^*}, \\
\frac{45}{|D|} &= \frac{\left[ADH - B(\ BH + C^2\ )\ \right]}{|D|^*},
\end{aligned}\right]
$$

$$\frac{55}{|D|} = \frac{[ADG - B^2G - C^2D]}{|D|^*} ,$$

$$\frac{66}{|D|} = \frac{E}{(KE - F^2)} ,$$

$$\frac{77}{|D|} = \frac{1}{L} ,$$

$$\frac{88}{|D|} = \frac{1}{M} , \qquad (A\ 6)$$

where, $$|D|^* = \left[(GJ - H^2)(AD - B^2) - C^2(AG-2BH+DJ-C^2)\right] . \qquad (A\ 7)$$

Substituting the value of A,B,C,D,E,F,G,H,J,K,L, and M from equations (A 2) into equations (A 6) gives the following set of equations with the restrictions of the orders of the terms $\frac{h}{r}$.

I. Retaining terms up to $\frac{h^8}{r^8}$

$$a_{11} = Eh\frac{\left\{(1-\nu^2)\left[1+\frac{h^2}{12r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]+\frac{h^2}{15}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\left[1-\frac{27h^2}{80r_1r_2}\right]+\frac{h^4}{80}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^3\left[\frac{1}{r_1}+\frac{1}{r_2}\left(1-\frac{3h^2}{20r_1r_2}\right)\right]\right\}}{a'} ,$$

$$a_{12} = -\nu Eh\frac{\left\{(1-\nu^2)+\frac{7h^2}{30}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\left[1-\frac{27h^2}{280r_1r_2}\right]\right\}}{a'} ,$$

$$a_{14} = \frac{Eh^3}{12}\frac{\left\{(1-\nu^2)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)-\frac{h^2}{12}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\left[\left(1-\frac{3h^2}{20r_1^2}\right)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)-\frac{14}{5r_1}\right]\right\}}{a'} ,$$

$$a_{15} = -\nu\frac{Eh^3}{12}\frac{\left\{\frac{h^2}{12}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\left(\frac{1}{r_1}+\frac{9}{5r_2}\right)\right\}}{a'} ,$$

$$a_{22} = Eh\frac{\left\{(1-\nu^2)\left[1-\frac{h^2}{12r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]+\frac{h^2}{15}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\left[1-\frac{27h^2}{80r_1r_2}\right]-\frac{h^4}{80}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^3\left[\frac{1}{r_1}+\frac{1}{r_2}-\frac{3h^2}{20r_1r_2^2}\right]\right\}}{a'} ,$$

$$a_{24} = -\nu\frac{Eh^3}{12}\frac{\left\{\frac{h^2}{12}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\left[\frac{9}{5r_1}+\frac{1}{r_2}\right]\right\}}{a'} ,$$

$$a_{25} = -\frac{Eh^3}{12}\frac{\left\{(1-\nu^2)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)-\frac{h^2}{12}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\left[\frac{1}{r_1}+\frac{9}{5r_2}-\frac{3h^2}{20r_2^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]\right\}}{a'} ,$$

$$a_{33} = \frac{Eh}{2(1+\nu)} \frac{\left\{1 - \frac{3h^2}{20r_2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\right\}}{a''} ,$$

$$a_{36} = \frac{Eh^3}{24(1+\nu)} \frac{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}{a''} ,$$

$$a_{44} = \frac{Eh^3}{12} \frac{\left\{(1-\nu)^2\left[1 + \frac{3h^2}{20r_1}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\right] + \frac{h^4}{80r_1}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)^3\left[1 - \frac{h^2}{12r_1r_2}\right] - \frac{h^4}{114r_2^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)^2\right\}}{a'} ,$$

$$a_{45} = -\nu \frac{Eh^3}{12} \frac{\left\{(1-\nu^2) - \frac{h^4}{114r_1r_2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)^2\right\}}{a'} ,$$

$$a_{55} = \frac{Eh^3}{12} \frac{\left\{(1-\nu)^2\left[1 - \frac{3h^2}{20r_2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\right] - \frac{h^4}{114r_1r_2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)^2\left[1 - \frac{3h^2}{20r_2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\right] - \frac{h^4}{144}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)^3\left[\frac{1}{r_1} + \frac{9}{5r_2}\right]\right\}}{a'} ,$$

$$a_{66} = \frac{Eh^3}{24(1+\nu)} \frac{\left[1 - \frac{h^2}{12r_2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\right]}{a''} ,$$

$$a_{77} = \frac{Eh}{2(1+\nu)} \frac{5}{6} \frac{1}{\left[1 - \frac{h^2}{28r_2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\right]} ,$$

$$a_{88} = \frac{Eh}{2(1+\nu)} \frac{5}{6} \frac{1}{\left[1 + \frac{h^2}{28r_1}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\right]} , \qquad \text{(A 8)}$$

where,

$$a' = \left\{(1-\nu^2)\left[(1-\nu^2) + \frac{h^2}{15}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)^2\left(1 - \frac{53h^2}{120r_1r_2}\right)\right] + \frac{h^4}{114}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)^4\left[1 - \frac{3h^2}{20}\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right) - \frac{21h^2}{50r_1r_2}\left(1 - \frac{3h^2}{56r_1r_2}\right)\right]\right\} ,$$

$$a'' = \left\{1 - \frac{7h^2}{30r_2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) - \frac{h^2}{12}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)^2\left(1 - \frac{3h^2}{20r_2^2}\right)\right\} ,$$

(A 9)

II. Retaining terms up to $\frac{h^6}{r^6}$

$$b_{11} = Eh\frac{\left\{(1-\nu^2)\left[1+\frac{h^2}{12r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]+\frac{h^2}{15}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\left[1-\frac{27h^2}{80r_1r_2}\right]+\frac{h^4}{80}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^3\left[\frac{1}{r_1}+\frac{1}{r_2}\left(1-\frac{3h^2}{20r_1r_2}\right)\right]\right\}}{b'},$$

$$b_{12} = -\nu Eh\frac{\left\{(1-\nu^2)+\frac{7h^2}{30}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\left[1-\frac{27h^2}{280r_1r_2}\right]\right\}}{b'},$$

$$b_{14} = \frac{Eh^3}{12}\frac{\left\{(1-\nu^2)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)-\frac{h^2}{12}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\left[\left(1-\frac{3h^2}{20r_2^2}\right)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)-\frac{14}{5r_1}\right]\right\}}{b'},$$

$$b_{15} = -\nu\frac{Eh^3}{12}\frac{\left\{\frac{h^2}{12}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\left(\frac{1}{r_1}+\frac{9}{5r_2}\right)\right\}}{b'},$$

$$b_{22} = Eh\frac{\left\{(1-\nu^2)\left[1-\frac{h^2}{12r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]+\frac{h^2}{15}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\left[1-\frac{27h^2}{80r_1r_2}\right]-\frac{h^4}{80}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^3\left[\frac{1}{r_1}+\frac{1}{r_2}-\frac{3h^2}{20r_1r_2^2}\right]\right\}}{b'},$$

$$b_{24} = -\nu\frac{Eh^3}{12}\frac{\left\{\frac{h^2}{12}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\left[\frac{9}{5r_1}+\frac{1}{r_2}\right]\right\}}{b'},$$

$$b_{25} = -\frac{Eh^3}{12}\frac{\left\{(1-\nu^2)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)-\frac{h^2}{12}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\left[\frac{1}{r_1}+\frac{9}{5r_2}-\frac{3h^2}{20r_2^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]\right\}}{b'},$$

$$b_{33} = a_{33},$$

$$b_{36} = a_{36},$$

$$b_{44} = \frac{Eh^3}{12}\frac{\left\{(1-\nu^2)\left[1+\frac{3h^2}{20r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]+\frac{h^4}{80r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^3\left[1-\frac{h^2}{12r_1r_2}\right]-\frac{h^4}{144r_2^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right\}}{b'},$$

$$b_{45} = -\nu\frac{Eh^3}{12}\frac{\left\{(1-\nu^2)-\frac{h^4}{144r_1r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right\}}{b'},$$

$$b_{55} = \frac{Eh^3}{12}\frac{\left\{(1-\nu^2)\left[1-\frac{3h^2}{20r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]-\frac{h^4}{144r_1r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\left[1-\frac{3h^2}{20r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]-\frac{h^4}{144}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^3\left[\frac{1}{r_1}+\frac{9}{5r_2}\right]\right\}}{b'},$$

$$b_{66} = a_{66} \quad ,$$

$$b_{77} = a_{77} \quad ,$$

$$b_{88} = a_{88} \quad , \qquad (A\ 10)$$

where, $$b' = \Big\{(1-\nu)^2\Big[(1-\nu)^2 + \frac{h^2}{15}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)^2\Big(1-\frac{53h^2}{120r_1r_2}\Big)\Big] + \frac{h^4}{144}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)^4\Big[1-\frac{3h^2}{20}\Big(\frac{1}{r_1^2}+\frac{1}{r_2^2}\Big) - \frac{21h^2}{50r_1r_2}\Big]\Big\} \ . \qquad (A\ 11)$$

III. Retaining terms up to $\frac{h^4}{r^4}$

$$C_{11} = Eh\,\frac{\Big\{(1-\nu)^2\Big[1+\frac{h^2}{12r_1}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big] + \frac{h^2}{15}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)^2\Big(1-\frac{27h^2}{80r_1r_2}\Big) + \frac{h^4}{80}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)^3\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)\Big\}}{C'} \ ,$$

$$C_{12} = -\nu Eh\,\frac{\Big\{(1-\nu)^2 + \frac{7h^2}{30}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)^2\Big[1-\frac{27h^2}{280r_1r_2}\Big]\Big\}}{C'} \ ,$$

$$C_{14} = \frac{Eh^3}{12}\,\frac{\Big\{(1-\nu)^2\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big) + \frac{h^2}{12}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)^2\Big(\frac{9}{5r_1}+\frac{1}{r_2}\Big)\Big\}}{C'} \ ,$$

$$C_{15} = -\nu\frac{Eh^3}{12}\,\frac{\Big\{\frac{h^2}{12}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big(\frac{1}{r_1}+\frac{9}{5r_2}\Big)\Big\}}{C'} \ ,$$

$$C_{22} = Eh\,\frac{\Big\{(1-\nu)^2\Big[1-\frac{h^2}{12r_2}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big] + \frac{h^2}{15}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)^2\Big(1-\frac{27h^2}{80r_1r_2}\Big) - \frac{h^4}{80}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)^3\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)\Big\}}{C'} \ ,$$

$$C_{24} = -\nu\frac{Eh^3}{12}\,\frac{\Big\{\frac{h^2}{12}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)^2\Big[\frac{9}{5r_1}+\frac{1}{r_2}\Big]\Big\}}{C'} \ ,$$

$$C_{25} = \frac{Eh^3}{12}\,\frac{\Big\{(1-\nu)^2\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big) - \frac{h^2}{12}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)^2\Big[\frac{1}{r_1}+\frac{9}{5r_2}\Big]\Big\}}{C'} \ ,$$

$$C_{33} = a_{33} \quad ,$$

$$C_{36} = a_{36} \quad ,$$

$$C_{44} = \frac{Eh^3}{12} \frac{\left\{(1-\nu)^2\left[1+\frac{3h^2}{20r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]+\frac{h^4}{80r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^3-\frac{h^4}{144r_2^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right\}}{C'} ,$$

$$C_{45} = -\nu\frac{Eh^3}{12} \frac{\left\{(1-\nu)^2-\frac{h^4}{144r_1r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right\}}{C'} ,$$

$$C_{55} = \frac{Eh^3}{12} \frac{\left\{(1-\nu)^2\left[1-\frac{3h^2}{20r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]-\frac{h^4}{144r_1r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2-\frac{h^4}{144}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^3\left[\frac{1}{r_1}+\frac{9}{5r_2}\right]\right\}}{C'} ,$$

$$C_{66} = a_{66} \quad ,$$

$$C_{77} = a_{77} \quad ,$$

$$C_{88} = a_{88} \quad , \tag{A 12}$$

where,
$$C' = \left\{(1-\nu)^2\left[(1-\nu)^2+\frac{h^2}{15}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\left(1-\frac{53h^2}{120r_1r_2}\right)\right]+\frac{h^4}{144}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^4\right\}$$

(A 13)

IV. Retaining terms up to $\frac{h^2}{r^2}$

$$d_{11} = \frac{Eh}{(1-\nu^2)} \frac{\left\{(1-\nu)^2\left[1+\frac{h^2}{12r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]+\frac{h^2}{15}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right\}}{d'} ,$$

$$d_{12} = -\nu\frac{Eh}{(1-\nu^2)} \frac{\left\{(1-\nu)^2+\frac{7h^2}{30}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right\}}{d'} ,$$

$$d_{14} = \frac{Eh^3}{12} \frac{\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}{d'} ,$$

$$d_{15} = -\frac{\nu E h^3}{12(1-\nu^2)} \frac{\left\{\frac{h^2}{12}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\left(\frac{1}{r_1}+\frac{9}{5r_2}\right)\right\}}{d'} ,$$

$$d_{22} = \frac{Eh}{(1-\nu^2)} \frac{\left\{(1-\nu)^2\left[1-\frac{h^2}{12r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right] + \frac{h^2}{15}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right\}}{d'} ,$$

$$d_{24} = 0 ,$$

$$d_{25} = -\frac{Eh^3}{12} \frac{\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}{d'} ,$$

$$d_{33} = \frac{Eh}{2(1+\nu)} \frac{\left\{1-\frac{3h^2}{20r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right\}}{d''} ,$$

$$d_{36} = \frac{Eh^3}{24(1+\nu)} \frac{\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}{d''} ,$$

$$d_{44} = \frac{Eh^3}{12} \frac{\left\{1+\frac{3h^2}{20r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right\}}{d'} ,$$

$$d_{45} = -\nu \frac{Eh^3}{12} \frac{1}{d'} ,$$

$$d_{55} = \frac{Eh^3}{12} \frac{\left\{1-\frac{3h^2}{20r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right\}}{d'} ,$$

$$d_{66} = \frac{Eh^3}{24(1+\nu)} \frac{\left\{1-\frac{h^2}{12r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right\}}{d''} ,$$

$$d_{77} = a_{77} ,$$

$$d_{88} = a_{88} , \qquad \text{(A 14)}$$

where,
$$d' = \left\{(1-\nu^2) + \frac{h^2}{15}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right\} ,$$

$$d'' = \left\{1 - \frac{7h^2}{30r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) - \frac{h^2}{12}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right\} . \qquad (A\ 15)$$

V. Retaining terms up to $\frac{h}{r}$

$$e_{11} = \frac{Eh}{(1-\nu^2)} ,$$

$$e_{12} = -\frac{\nu Eh}{(1-\nu^2)} ,$$

$$e_{14} = \frac{Eh^3}{12(1-\nu^2)}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) ,$$

$$e_{15} = 0 ,$$

$$e_{22} = \frac{Eh}{(1-\nu^2)} ,$$

$$e_{24} = 0 .$$

$$e_{25} = -\frac{Eh^3}{12(1-\nu^2)}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) ,$$

$$e_{33} = \frac{Eh}{2(1+\nu)} ,$$

$$e_{36} = \frac{Eh^3}{24(1+\nu)}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) ,$$

$$e_{44} = \frac{Eh^3}{12(1-\nu^2)} ,$$

$$e_{45} = -\nu \frac{Eh^3}{12(1-\nu^2)} \,,$$

$$e_{55} = \frac{Eh^3}{12(1-\nu^2)} \,,$$

$$e_{66} = \frac{Eh^3}{24(1+\nu)} \,,$$

$$e_{77} = \frac{Eh}{2(1+\nu)} \frac{5}{6} \,,$$

$$e_{88} = \frac{Eh}{2(1+\nu)} \frac{5}{6} \,, \qquad \text{(A 16)}$$

Where the terms of a, b, c, d and e are particular terms in the matrix as the order of the quantity ($\frac{h}{r}$) changes.

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