A NONLINEAR THEORY FOR THIN ELASTIC PLATES

by

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ABSTRACT

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The purpose of this thesis is to derive a nonlinear theory of thin elastic rectangular plates including the effects of transverse normal stress, transverse shear stress, and transverse and rotary inertia. This nonlinear theory also includes the effects of the square of the rotation terms in the strain-displacement equations and the product of stress times rotation terms in the equilibrium equations.

Using a variation theorem due to Reissner, the equations of motion, the stress-strain relationships, and the associated natural and forced boundary conditions are simultaneously determined. The resulting equations may be applied to a certain group of rectangular plate problems where the applied dynamic loads produce deformations which are of such an order that only an appropriate nonlinear theory can account for them.

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LIST OF SYMBOLS

SYMBOL	DEFINITION
x, y, z	Surface & normal coordinates
X, Y, Z	Rectangular coordinates
α, β, γ	Lame's coefficient
r_1, r_2	Radii of curvature in direction of x,y respectively
$\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}$	Normal strain components in direction of x,y,z respectively
$\gamma_{xz}, \gamma_{yz}, \gamma_{xy}$	Shearing strain components
$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$	Normal stress components
$\tau_{xy}, \tau_{xz}, \tau_{yz}$	Shearing stress components
U, V, W	Displacement components in direction of x,y,z respectively for any arbitrary point
$\overline{U}, \overline{V}, \overline{W}$	Displacement components at the middle surface
$\overline{W}', \overline{W}''$	Displacement components which contribute to the transverse normal displacement
ω_x, ω_y	Change of slope of the normal to the middle surface
ρ	Mass density per unit mass
ν	Poisson's ratio
z	Normal coordinate in z-direction of any arbitrary point
h	Thickness of the plate or shell
D	Flexural rigidity of the plate
E	Modulus of elasticity
F_x, F_y, F_z	Body force per unit volume in direction of x,y,z respectively
G	Shear modulus of elasticity
$N_{xx}, N_{yy}, N_{xy}, N_{yx}$	Normal stress resultants in unit of force per unit length

M_{xx}, M_{yy}, M_{xy}, M_{yz}	Bending stress couples in unit of moment per unit length
Q_{xz}, Q_{yz}	Shear stress resultants in z-direction
P_1^+, P_1^-	Stress component τ_{xz} at the upper and lower surfaces of the plate respectively
P_2^+, P_2^-	Stress component τ_{yz} at the upper and lower surfaces of the plate respectively
P_3^+, P_3^-	Stress component σ_{zz} at the upper and lower surfaces of the plate respectively
C	Parameter used in variational procedure
K	Parameter used in stress-strain relationships

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A.1c Directions and components of stress couple on the element of the shell 34

CHAPTER I

INTRODUCTION

Nonlinear theories for thin elastic plates as derived by using the theory of finite displacements differ greatly depending on the restrictive assumptions placed on the resulting deformations.

The classical nonlinear theory for thin elastic plates is presented by Von-Kármán[5], where the nonlinear rotation terms are retained in the strain-displacement relationships only.

The classical nonlinear theory has been expanded to include the shell theory by Naghdi[3], where the effects of transverse normal stress and transverse shear stress are accounted for.

A nonlinear shear deformation theory for thin elastic shells and thin elastic plates is presented by Archer[2]. This paper derives a nonlinear theory of the Donnell type which includes shear deformations, transverse and rotary inertia effects, but does not include the effect of transverse normal stress.

A direct application of the resulting equations play an important role in wave propagation problems, where the effects of transverse normal stress and transverse shear stress are of primary importance.

The objective of this thesis is to derive a nonlinear theory of thin elastic plates based on Von-Kármán's[5] approach. Additional nonlinear terms are retained, i.e., the product of stress times rotation terms are retained in the equations stress equilibrium. These terms are neglected in the classical nonlinear theory.

Reissner's[4] Variational Theorem is used to develop equations of equilibrium and stress-strain relationships together with natural and forced boundary conditions.

CHAPTER II

METHOD OF ANALYSIS

The basic assumptions used in the analysis of thin elastic plate are as follows:

1. The thickness of the plate is assumed uniform and small.
2. Lines which are normal to the middle surface before deformation do not remain normal to the middle surface after deformation (i.e., shear deformations are accounted for).
3. Linear elastic stress-strain relationships are assumed to hold; and the component of stress normal to the middle surface is considered to be of the same order as the other components of stress.

The following steps are taken to achieve the end results:

1. Assume stresses σ_{xx}, σ_{yy} & $\tau_{xy} = \tau_{yx}$
2. Solve for τ_{xz}, τ_{yz} & σ_{zz} which satisfies nonlinear equilibrium equations and establish assumed stress field.
3. Formulate the reduced form of nonlinear strain-displacement equations.
4. Combine the assumed stress field and the strain-displacement equations in the Reissner's Variational Theorem.
5. Deduce equations of motion, stress-strain relationships together with natural and forced boundary conditions.

2.1 THE COORDINATE SYSTEM AND NOTATION

The notation used through the paper is similar to that given by Langhaar(1). A middle surface of plate is defined as the x-y plane. The normal distance from the middle surface is defined by z, the normal coordinate.

2.2 STRESS RESULTANTS AND STRESS COUPLES

Stress resultants and stress couples applied to a differential plate element are shown in Figs. 2.1b and 2.1c. These stress resultants and stress couples are defined as total forces acting per unit length of the middle surface.

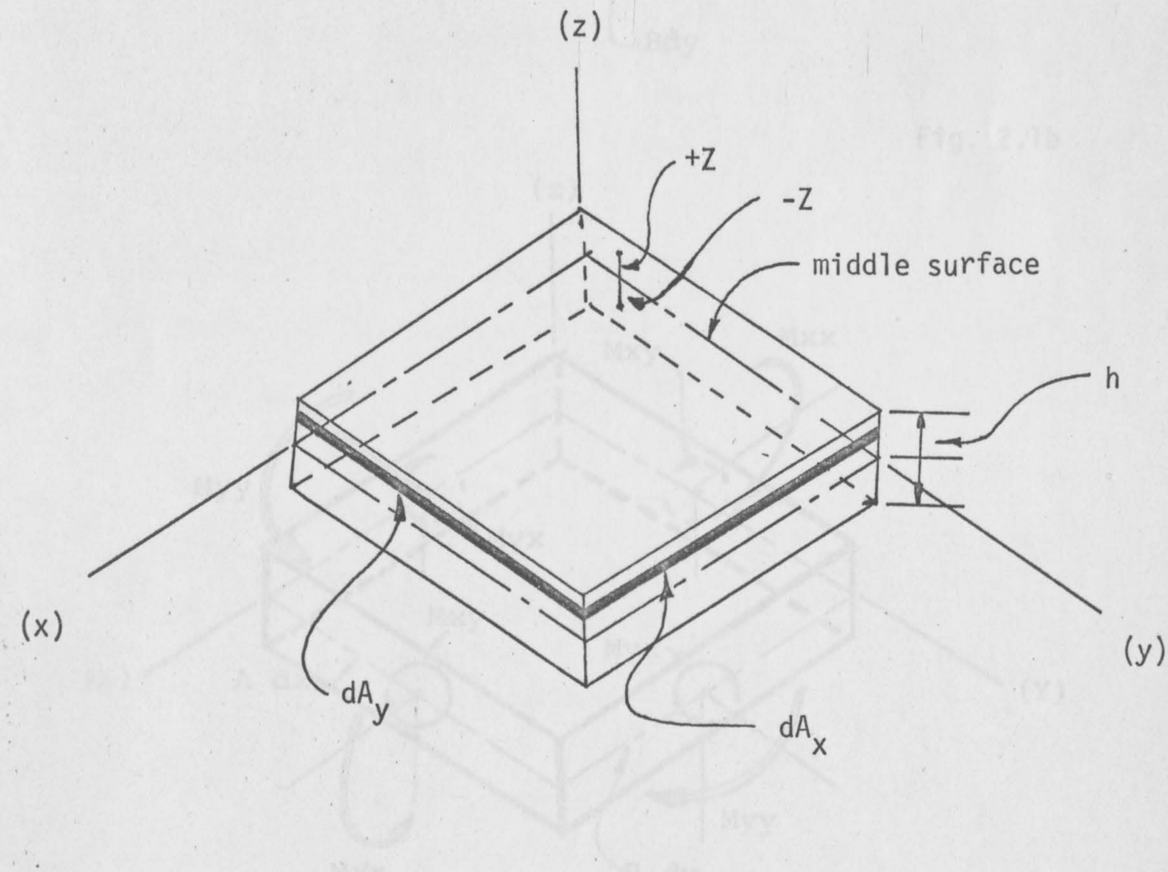

Fig. 2.1a

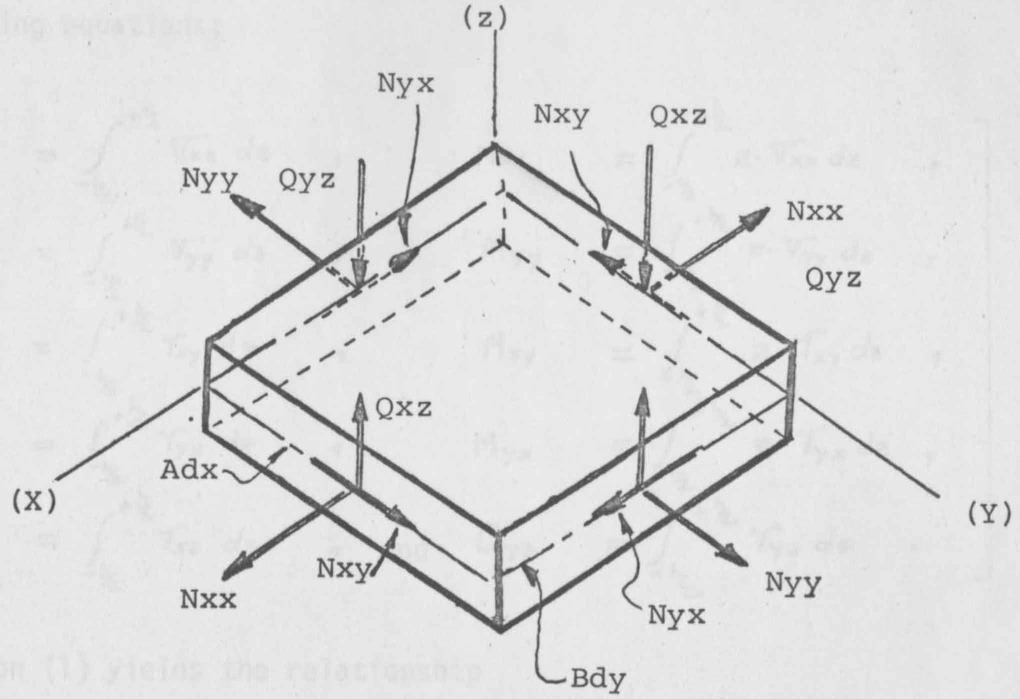

Fig. 2.1b

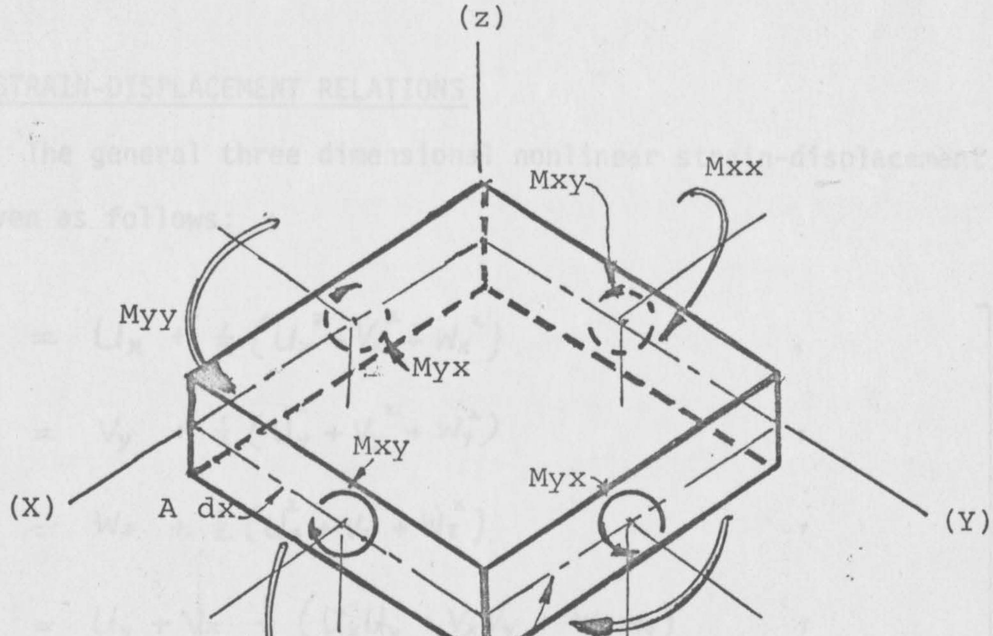

Fig. 2.1c

The stress resultants and stress couples are defined by the following equations;

$$
\left.
\begin{aligned}
N_{xx} &= \int_{-h/2}^{+h/2} \sigma_{xx}\, dz \;, & M_{xx} &= \int_{-h/2}^{+h/2} z\cdot\sigma_{xx}\, dz \;, \\
N_{yy} &= \int_{-h/2}^{+h/2} \sigma_{yy}\, dz \;, & M_{yy} &= \int_{-h/2}^{+h/2} z\cdot\sigma_{yy}\, dz \;, \\
N_{xy} &= \int_{-h/2}^{+h/2} \tau_{xy}\, dz \;, & M_{xy} &= \int_{-h/2}^{+h/2} z\cdot\tau_{xy}\, dz \;, \\
N_{yx} &= \int_{-h/2}^{+h/2} \tau_{yx}\, dz \;, & M_{yx} &= \int_{-h/2}^{+h/2} z\cdot\tau_{yx}\, dz \;, \\
Q_{xz} &= \int_{-h/2}^{+h/2} \tau_{xz}\, dz \;, \text{ and } & Q_{yz} &= \int_{-h/2}^{+h/2} \tau_{yz}\, dz \;.
\end{aligned}
\right] \quad (1)
$$

Equation (1) yields the relationship

$$N_{xy} = N_{yx} \quad \text{and} \quad M_{xy} = M_{yx} \;. \qquad (2)$$

2.3 STRAIN-DISPLACEMENT RELATIONS

The general three dimensional nonlinear strain-displacement are given as follows:

$$
\left.
\begin{aligned}
\epsilon_{xx} &= U_x + \tfrac{1}{2}\left(U_x^2 + V_x^2 + W_x^2\right) \;, \\
\epsilon_{yy} &= V_y + \tfrac{1}{2}\left(U_y^2 + V_y^2 + W_y^2\right) \;, \\
\epsilon_{zz} &= W_z + \tfrac{1}{2}\left(U_z^2 + V_z^2 + W_z^2\right) \;, \\
\gamma_{xy} &= U_y + V_x + \left(U_x U_y + V_x V_y + W_x W_y\right) \;, \\
\gamma_{yz} &= V_z + W_y + \left(U_y U_z + V_y V_z + W_y W_z\right) \;, \\
\text{and} & \\
\gamma_{xz} &= W_x + U_z + \left(U_x U_z + V_x V_z + W_x W_z\right) \;.
\end{aligned}
\right] \quad (3)
$$

Retaining all linear terms together with the second order rotation terms W_x^2, W_y^2, & W_xW_y.

Equation (3) reduces to the following:

$$\left.\begin{aligned}
\epsilon_{xx} &= U_x + \tfrac{1}{2} W_x^2 , \\
\epsilon_{yy} &= V_y + \tfrac{1}{2} W_y^2 , \\
\epsilon_{zz} &= W_z , \\
\gamma_{xy} &= U_y + V_x + W_x W_y , \\
\gamma_{yz} &= V_z + W_y , \\
\text{and} & \\
\gamma_{xz} &= U_z + W_x .
\end{aligned}\right] \quad (4)$$

To obtain the appropriate stress-strain relation, the following approximate equations are assumed for the displacement field,

$$\left.\begin{aligned}
U &= \overline{U}(x,y) + z\omega_x(x,y), \\
V &= \overline{V}(x,y) + z\omega_y(x,y), \\
W &= \overline{W}(x,y) + z\overline{W}'(x,y) + \frac{z^2}{2}\overline{W}''(x,y),
\end{aligned}\right] \quad (5)$$

where $\overline{U}$, $\overline{V}$ & $\overline{W}$ are the components of displacement at the middle surface, $\omega_x(x,y)$ and $\omega_y(x,y)$ are the change of slope of the normal to the middle surface along the x and y coordinates lines respectively, and $\overline{W}'(x,y)$ and $\overline{W}''(x,y)$ are the contributions to the transverse normal strain.

Substituting equation (5) into equation (4), the following equations are obtained:

$$
\begin{aligned}
\epsilon_{xx} &= \bar{U}_x + z\cdot\omega_{x,x} + \tfrac{1}{2}\left(\bar{W}_x + z\bar{W}_x' + \tfrac{z^2}{2}\bar{W}_x''\right)^2 , \\
\epsilon_{yy} &= \bar{V}_y + z\cdot\omega_{y,y} + \tfrac{1}{2}\left(\bar{W}_y + z\bar{W}_y' + \tfrac{z^2}{2}\bar{W}_y''\right)^2 , \\
\epsilon_{zz} &= \bar{W}' + z\cdot\bar{W}'' , \\
\gamma_{xy} &= (\bar{U}_y + z\cdot\omega_{x,y}) + (\bar{V}_x + z\cdot\omega_{y,x}) \\
&\quad + \left(\bar{W}_x + z\bar{W}_x' + \tfrac{z^2}{2}\bar{W}_x''\right)\left(\bar{W}_y + z\bar{W}_y' + \tfrac{z^2}{2}\bar{W}_y''\right) , \\
\gamma_{yz} &= \omega_y + \left(\bar{W}_y + z\bar{W}_y' + \tfrac{z^2}{2}\bar{W}_y''\right) ,
\end{aligned}
$$

and

$$
\gamma_{xz} = \omega_x + \left(\bar{W}_x + z\bar{W}_x' + \tfrac{z^2}{2}\bar{W}_x''\right) . \qquad (6)
$$

Equation (6) for the components of strain are rewritten in the following form:

$$
\begin{aligned}
\epsilon_{xx} &= \overset{\circ}{\epsilon}_{xx} + z\cdot K_x + z^2 C_x + \tfrac{z^3}{2}\bar{W}_x'\bar{W}_x'' + \tfrac{z^4}{8}(\bar{W}_x'')^2 , \\
\epsilon_{yy} &= \overset{\circ}{\epsilon}_{yy} + z\cdot K_y + z^2 C_y + \tfrac{z^3}{2}\bar{W}_y'\bar{W}_y'' + \tfrac{z^4}{8}(\bar{W}_y'')^2 , \\
\epsilon_{zz} &= \bar{W}' + z\bar{W}'' , \\
\gamma_{xy} &= (\overset{\circ}{\gamma}_{xx} + z\,\delta_{xx}) + (\overset{\circ}{\gamma}_{yy} + z\,\delta_{yy}) + \bar{W}_x\bar{W}_y \\
&\quad + z D_{xy} + z^2 E_{xy} + z^3 F_{xy} + \tfrac{z^4}{4}(\bar{W}_x''\bar{W}_y'') , \\
\gamma_{yz} &= \overset{\circ}{\gamma}_{yz} + z\left(\bar{W}_y' + \tfrac{z}{2}\bar{W}_y''\right) , \quad \text{and} \\
\gamma_{xz} &= \overset{\circ}{\gamma}_{xz} + z\left(\bar{W}_x' + \tfrac{z}{2}\bar{W}_x''\right) .
\end{aligned}
\qquad (7)
$$

where,

$$
\left.
\begin{aligned}
\overset{\circ}{\epsilon}_{xx} &= \bar{U}_x + \tfrac{1}{2}(\bar{W}_x)^2 , \\
\overset{\circ}{\epsilon}_{yy} &= \bar{V}_y + \tfrac{1}{2}(\bar{W}_y)^2 , \\
\overset{\circ}{\gamma}_{xx} &= \bar{V}_x , \\
\overset{\circ}{\gamma}_{yy} &= \bar{U}_y , \\
\delta_{xx} &= \omega_{y,x} , \\
\delta_{yy} &= \omega_{x,y} , \\
\overset{\circ}{\gamma}_{xz} &= \bar{W}_x + \omega_x , \\
\overset{\circ}{\gamma}_{yz} &= \bar{W}_y + \omega_y , \\
K_x &= \omega_{x,x} + \bar{W}_x \bar{W}_x' , \\
K_y &= \omega_{y,y} + \bar{W}_y \bar{W}_y' , \\
C_x &= \tfrac{1}{2}(\bar{W}_x \bar{W}_x'' + \bar{W}_x'^2) , \\
C_y &= \tfrac{1}{2}(\bar{W}_y \bar{W}_y'' + \bar{W}_y'^2) , \\
D_{xy} &= (\bar{W}_x \bar{W}_y' + \bar{W}_x' \bar{W}_y) , \\
E_{xy} &= \bar{W}_x' \bar{W}_y' + \tfrac{1}{2}(\bar{W}_x \bar{W}_y'' + \bar{W}_y \bar{W}_x'') , \text{ and} \\
F_{xy} &= \tfrac{1}{2}(\bar{W}_x' \bar{W}_y'' + \bar{W}_x'' \bar{W}_y') .
\end{aligned}
\right\rbrack \quad (8)
$$

2.4 THE COMPONENTS OF STRESS

Noting equations (1), the components of stress are assumed to take the form

$$\left.\begin{aligned}\sigma_{xx} &= \frac{N_{xx}}{h} + \frac{12}{h^3} z M_{xx} \;, \\ \sigma_{yy} &= \frac{N_{yy}}{h} + \frac{12}{h^3} z M_{yy} \;, \\ \text{and} \qquad \tau_{xy} &= \frac{N_{xy}}{h} + \frac{12}{h^3} z M_{xy} \;. \end{aligned}\right] \qquad (9)$$

The components of shearing stress of τ_{xz}, τ_{yz} and σ_{zz} are determined by direct solution of the first three equilibrium equations of stress which are:

$$\frac{\partial}{\partial x}(\sigma_{xx} - \tau_{xz}\omega_y - \tau_{xy}\omega_z) + \frac{\partial}{\partial y}(\tau_{yx} + \tau_{yz}\omega_y - \sigma_{yy}\omega_z) + \frac{\partial}{\partial z}(\tau_{zx} + \sigma_{zz}\omega_y - \tau_{zy}\omega_z) + \rho F_x = 0 \;,$$

$$\frac{\partial}{\partial x}(\tau_{xy} + \sigma_{xx}\omega_z - \tau_{xz}\omega_x) + \frac{\partial}{\partial y}(\sigma_{yy} + \tau_{yx}\omega_z - \tau_{yz}\omega_x) + \frac{\partial}{\partial z}(\tau_{zy} + \tau_{zx}\omega_z - \sigma_{zz}\omega_x) + \rho F_y = 0 \;,$$

and

$$\frac{\partial}{\partial x}(\tau_{xz} + \tau_{xy}\omega_x - \sigma_{xx}\omega_y) + \frac{\partial}{\partial y}(\tau_{yz} + \sigma_{yy}\omega_x - \tau_{yx}\omega_y) + \frac{\partial}{\partial z}(\sigma_{zz} + \tau_{zy}\omega_x - \tau_{zx}\omega_y) + \rho F_z = 0 \;. \qquad (10)$$

It should be noted that each equilibrium equation contains six additional terms as compared to the linear theory. These terms contain the product of the stress times the rotation terms.

For thin plate theory in orthogonal coordinates $\omega_z = 0$. Neglecting the body forces, $F_x = F_y = F_z = 0$, and noting $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$ and $\tau_{yz} = \tau_{zy}$ the previous equation reduces to the following form:

$$\frac{\partial}{\partial x}(\sigma_{xx} - \omega_y \tau_{xz}) + \frac{\partial}{\partial y}(\tau_{xy} + \omega_y \tau_{yz}) + \frac{\partial}{\partial z}(\tau_{xz} + \omega_y \sigma_{zz}) = 0 ,$$

$$\frac{\partial}{\partial x}(\tau_{xy} - \omega_x \tau_{xz}) + \frac{\partial}{\partial y}(\sigma_{yy} - \omega_x \tau_{yz}) + \frac{\partial}{\partial z}(\tau_{yz} - \omega_x \sigma_{zz}) = 0 ,$$

$$\frac{\partial}{\partial x}(\tau_{xz} + \omega_x \tau_{xy} - \omega_y \sigma_{xx}) + \frac{\partial}{\partial y}(\tau_{yz} + \omega_x \sigma_{yy} - \omega_y \tau_{xy}) + \frac{\partial}{\partial z}(\sigma_{zz} + \omega_x \tau_{yz} - \omega_y \tau_{xz}) = 0 . \tag{11}$$

Substituting equations (1) into equations (11), and integrating over the thickness of the plate yields respectively,

$$\frac{\partial}{\partial x}(N_{xx} + \omega_y Q_{xz}) + \frac{\partial}{\partial y}(N_{xy} + \omega_y Q_{yz}) + \omega_y P_3 + P_1 = 0 ,$$

$$\frac{\partial}{\partial x}(N_{xy} - \omega_x Q_{xz}) + \frac{\partial}{\partial y}(N_{yy} - \omega_x Q_{yz}) - \omega_x P_3 + P_2 = 0 ,$$

$$\frac{\partial}{\partial x}(\omega_x N_{xy} - \omega_y N_{xx} + Q_{xz}) + \frac{\partial}{\partial y}(\omega_x N_{yy} - \omega_y N_{xy} + Q_{yz}) + P_3 + \omega_x P_2 - \omega_y P_1 = 0 . \tag{12}$$

Multiplying the equations (11) by z and performing the integration over the thickness of the plate yields respectively,

$$\frac{\partial}{\partial x}(M_{xx}) + \frac{\partial}{\partial y}(M_{xy}) - Q_{xz} - \omega_y \overline{Q_{zz}} + h\omega_y P_3 - hP_1 = 0 ,$$

$$\frac{\partial}{\partial x}(M_{xy}) + \frac{\partial}{\partial y}(M_{yy}) - Q_{yz} + \omega_x \overline{Q_{zz}} - h\omega_x P_3 + hP_2 = 0 ,$$

$$\frac{\partial}{\partial x}(\omega_x M_{xy} - \omega_y M_{xx}) + \frac{\partial}{\partial y}(\omega_x M_{yy} - \omega_y M_{xy}) + \omega_y Q_{xz} - \omega_x Q_{yz} - \overline{Q_{zz}} + hP_3 + h\omega_x P_2 - h\omega_y P_1 = 0 , \tag{13}$$

where,

$$P_1 = \tau_{xz} \Big|_{-\frac{h}{2}}^{+\frac{h}{2}} ,$$

$$P_2 = \tau_{yz} \Big|_{-\frac{h}{2}}^{+\frac{h}{2}} ,$$

$$P_3 = \sigma_{zz} \Big|_{-\frac{h}{2}}^{+\frac{h}{2}} ,$$

and

$$\overline{Q_{zz}} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{zz} \, dz . \tag{14}$$

Substituting equation (9) into equation (11), one obtains,

$$\left.\begin{aligned}
&\frac{\partial}{\partial x}(\omega_y \tau_{xz}) + \frac{\partial}{\partial y}(\omega_y \tau_{yz}) + \frac{\partial}{\partial z}(\tau_{xz} + \omega_y \sigma_{zz}) + \chi_1 = 0 \ , \\
&\frac{\partial}{\partial x}(\omega_x \tau_{xz}) + \frac{\partial}{\partial y}(\omega_x \tau_{yz}) - \frac{\partial}{\partial z}(\tau_{yz} - \omega_x \sigma_{zz}) - \chi_2 = 0 \ , \quad \text{and} \\
&\frac{\partial}{\partial x}(\tau_{xz}) + \frac{\partial}{\partial y}(\tau_{yz}) + \frac{\partial}{\partial z}(\sigma_{zz} + \omega_x \tau_{yz} - \omega_y \tau_{xz}) + \chi_3 = 0 \ ,
\end{aligned}\right] \quad (15)$$

where,

$$\left.\begin{aligned}
\chi_1 &= \chi_{11} + z\chi_{12} \ , \\
\chi_2 &= \chi_{21} + z\chi_{22} \ , \\
\text{and} \quad \chi_3 &= \chi_{31} + z\chi_{32} \ .
\end{aligned}\right] \quad (16)$$

From equations (11) through (16), one obtains,

$$\left.\begin{aligned}
\chi_{11} &= -\frac{1}{h}\left[\frac{\partial}{\partial x}(\omega_y Q_{xz}) + \frac{\partial}{\partial y}(\omega_y Q_{yz}) + \omega_y P_3 + P_1\right] \ , \\
\chi_{21} &= \frac{1}{h}\left[\frac{\partial}{\partial x}(\omega_x Q_{xz}) + \frac{\partial}{\partial y}(\omega_x Q_{yz}) + \omega_x P_3 - P_2\right] \ , \\
\chi_{31} &= -\frac{1}{h}\left[\frac{\partial}{\partial x}(Q_{xz}) + \frac{\partial}{\partial y}(Q_{yz}) + P_3 + \omega_x P_2 - \omega_y P_1\right] \ , \\
\chi_{12} &= \frac{12}{h^3}\left[\omega_y \overline{Q_{zz}} + Q_{xz} - h\omega_y P_3 - hP_1\right] \ , \\
\chi_{22} &= -\frac{12}{h^3}\left[\omega_x \overline{Q_{zz}} - Q_{yz} - h\omega_x P_3 + hP_2\right] \ , \quad \text{and} \\
\chi_{32} &= -\frac{12}{h^3}\left[\omega_y Q_{xz} - \omega_x Q_{yz} - \overline{Q_{zz}} + h(P_3 + \omega_x P_2 - \omega_y P_1)\right] \ .
\end{aligned}\right] \quad (17)$$

Eliminating σ_{zz} from equations (15), there results,

$$\left.\begin{aligned}
&\omega_{y,x}\tau_{xz} + \frac{\partial}{\partial z}(\tau_{xz})(1+\omega_y^2) + \omega_{y,y}\tau_{yz} - \omega_x\omega_y\frac{\partial}{\partial z}(\tau_{yz}) \\
&\qquad\qquad + (\chi_1 - \omega_y\chi_3) = 0 \ , \quad \text{and} \\
&\omega_{x,x}\tau_{xz} + \frac{\partial}{\partial z}(\tau_{xz})(\omega_x\omega_y) + \omega_{x,y}\tau_{yz} - (1+\omega_x^2)\frac{\partial}{\partial z}(\tau_{yz}) \\
&\qquad\qquad - (\chi_2 + \omega_x\chi_3) = 0 \ ,
\end{aligned}\right] \quad (18)$$

Uncoupling equations (18) into two independent equations in τ_{xz} and τ_{yz} and neglecting all terms of the square and higher order of ω , the following equations result:

$$\frac{\partial^2}{\partial z^2}(\tau_{yz}) + \frac{\partial}{\partial z}(\tau_{yz})(\omega_{y,x} - \omega_{x,y}) + \frac{\partial}{\partial z}(\chi_2 + \omega_x \chi_3) + \omega_{y,x}\chi_2 - \omega_{x,x}\chi_1 = 0 ,$$

and

$$\frac{\partial^2}{\partial z^2}(\tau_{xz}) + \frac{\partial}{\partial z}(\tau_{xz})(\omega_{y,x} - \omega_{x,y}) + \frac{\partial}{\partial z}(\chi_1 - \omega_y \chi_3) - \omega_{x,y}\chi_1 - \omega_{y,y}\chi_2 = 0 . \tag{19}$$

The solution of the first equation (19) for τ_{yz}, as detailed in Appendix (B), and the second equation (19) for τ_{xz} , yields respectively ,

$$\begin{aligned}\tau_{yz} = {} & \frac{3Q_{yz}}{2h}\left(1-\frac{4z^2}{h^2}\right) - \frac{1}{4}\left[P_2^+\left(1-\frac{4z}{h}-\frac{12z^2}{h^2}\right) + P_2^-\left(1+\frac{4z}{h}-\frac{12z^2}{h^2}\right)\right] \\ & + \frac{1}{4}\left(1-\frac{4z^2}{h^2}\right)\left\{\omega_{x,x}\left[P_1^+\left(z+\frac{h}{2}\right) + P_1^-\left(z-\frac{h}{2}\right) - \frac{2z}{h}Q_{xz}\right]\right. \\ & \left. - \omega_{y,x}\left[P_2^+\left(z+\frac{h}{2}\right) + P_2^-\left(z-\frac{h}{2}\right) - \frac{2z}{h}Q_{yz}\right]\right\} , \text{ and} \\ \tau_{xz} = {} & \frac{3Q_{xz}}{2h}\left(1-\frac{4z^2}{h^2}\right) - \frac{1}{4}\left[P_1^+\left(1-\frac{4z}{h}-\frac{12z^2}{h^2}\right) + P_1^-\left(1+\frac{4z}{h}-\frac{12z^2}{h^2}\right)\right] \\ & + \frac{1}{4}\left(1-\frac{4z^2}{h^2}\right)\left\{\omega_{y,y}\left[P_2^+\left(z+\frac{h}{2}\right) + P_2^-\left(z-\frac{h}{2}\right) - \frac{2z}{h}Q_{yz}\right]\right. \\ & \left. - \omega_{x,y}\left[P_1^+\left(z+\frac{h}{2}\right) + P_1^-\left(z-\frac{h}{2}\right) - \frac{2z}{h}Q_{xz}\right]\right\} .\end{aligned} \tag{20}$$

Substituting τ_{xz} and τ_{yz} from equations (20) into the third equilibrium equation (15) and solving for σ_{zz} as detailed in Appendix (c), there results

$$\begin{aligned}\sigma_{zz} = {} & \left(1-\frac{4z^2}{h^2}\right)\left\{\frac{1}{4}[R][L_1] - \frac{1}{4}[S][L_2] + [L_6]\right\} - \frac{1}{4}\omega_{x,y}[S'_{h/2} - S']_x \\ & + \frac{1}{4}\omega_{x,x}[S'_{h/2} - S']_y + \frac{1}{4}\omega_{y,y}[R'_{h/2} - R']_x - \frac{1}{4}\omega_{y,x}[R'_{h/2} - R']_y \\ & - \frac{1}{4}[T'_{h/2} - T']_x - \frac{1}{4}[J'_{h/2} - J']_y + P_3^+ + \frac{\omega_y}{4}[T_{h/2} - T] \\ & - \frac{\omega_x}{4}[J_{h/2} - J] + \frac{3}{2h}\left(\frac{h}{3} - z + \frac{4z^3}{3h^2}\right)L_3 + \left(\frac{z}{2h} - \frac{2z^3}{h^3}\right)L_4 \\ & + \left(\frac{3}{2h} - \frac{6z^2}{h^3}\right)L_5 ,\end{aligned} \tag{21}$$

where,

$$[R] = P_2^+\left(z+\frac{h}{2}\right) + P_2^-\left(z-\frac{h}{2}\right) - \frac{2z}{h}Q_{yz} ,$$

$$[S] = P_1^+\left(z+\frac{h}{2}\right) + P_1^-\left(z-\frac{h}{2}\right) - \frac{2z}{h}Q_{xz} ,$$

$$[R'_{h/2} - R'] = P_2^+\left(\frac{11}{48}h^2 - \frac{hz}{2} - \frac{z^2}{2} + \frac{2z^3}{3h} + \frac{z^4}{h^2}\right) - P_2^-\left(\frac{5}{48}h^2 - \frac{hz}{2} + \frac{z^2}{2} + \frac{2z^3}{3h} - \frac{z^4}{h^2}\right) - Q_{yz}\left(\frac{h}{8} + \frac{z^2}{h} - \frac{2z^4}{h^3}\right) ,$$

$$[S'_{h/2} - S'] = P_1^+\left(\frac{11}{48}h^2 - \frac{hz}{2} - \frac{z^2}{2} + \frac{2z^3}{3h} + \frac{z^4}{h^2}\right) - P_1^-\left(\frac{5}{48}h^2 - \frac{hz}{2} + \frac{z^2}{2} + \frac{2z^3}{3h} - \frac{z^4}{h^2}\right) - Q_{xz}\left(\frac{h}{8} + \frac{z^2}{h} - \frac{2z^4}{h^3}\right) ,$$

$$[T'_{h/2} - T'] = P_1^+\left(-\frac{h}{2} - z + \frac{2z^2}{h} + \frac{4z^3}{h^2}\right) + P_1^-\left(\frac{h}{2} - z - \frac{2z^2}{h} + \frac{4z^3}{h^2}\right) ,$$

$$[J'_{h/2} - J'] = P_2^+\left(-\frac{h}{2} - z + \frac{2z^2}{h} + \frac{4z^3}{h^2}\right) + P_2^-\left(\frac{h}{2} - z - \frac{2z^2}{h} + \frac{4z^3}{h^2}\right) ,$$

$$[T_{h/2} - T] = P_1^+\left(-5 + \frac{4z}{h} + \frac{12z^2}{h^2}\right) - P_1^-\left(1 + \frac{4z}{h} - \frac{12z^2}{h^2}\right) ,$$

$$[J_{h/2} - J] = P_2^+\left(-5 + \frac{4z}{h} + \frac{12z^2}{h^2}\right) - P_2^-\left(1 + \frac{4z}{h} - \frac{12z^2}{h^2}\right) ,$$

$$[L_1] = \omega_y \omega_{y,y} + \omega_x \omega_{y,x} ,$$

$$[L_2] = \omega_y \omega_{x,y} + \omega_x \omega_{x,x} ,$$

$$[L_3] = -P_3 - \omega_x P_2 + \omega_y P_1 ,$$

$$[L_4] = (N_{xy}\omega_x - N_{xx}\omega_y)_{,x} + (N_{yy}\omega_x - N_{xy}\omega_y)_{,y} ,$$

$$[L_5] = (M_{xy}\omega_x - M_{xx}\omega_y)_{,x} + (M_{yy}\omega_x - M_{xy}\omega_y)_{,y} ,$$

and

$$[L_6] = \frac{3}{2h}(\omega_y Q_{xz} - \omega_x Q_{yz}) . \qquad (22)$$

2.5 REISSNER'S VARIATIONAL THEOREM

Reissner's variational theorem of three dimensional elasticity is written in the form

$$\delta I = \int_{t_1}^{t_2} \Big\{ \iiint_V \Big\{ [\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{zz}\epsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}]$$

$$- \frac{1}{2E}\Big[\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 - 2\nu(\sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz})$$

$$+ 2(1+\nu)(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2) \Big]$$

$$- \frac{\rho}{2}\left[U_t^2 + V_t^2 + W_t^2 \right] \Big\} dx\,dy\,dz$$

$$- \iint_{S_1} \left[(P_1^+ U^+ + P_2^+ V^+ + P_3^+ W^+) - (P_1^- U^- + P_2^- V^- + P_3^- W^-) \right] dx\,dy$$

$$\Big\} dt = 0 , \tag{23}$$

where, E = Modulus of elasticity,

ν = Poisson's ratio.

The first term in the integrand represents twice the strain energy, the second term - the complementary energy, the third term - the kinetic energy, and the last term - the workdone by the external forces on the upper and lower surfaces of the plate.

The assumed stress distributions given by equations (9), (20), and (21), together with the reduced form of the strain-displacement relationships given by equations (7) and (8) are substituted into equation (23).

The resultant equation is integrated over the thickness of the plate, and the variation is carried out, yielding

$$\int_{t_1}^{t_2} \Big\{ \int \Big| N_{xx}\,\delta\bar{u} + N_{xy}\,\delta\bar{v} + C_{\bar{w}_x}\,\delta\bar{w} + C_{\bar{w}'_x}\,\delta\bar{w}' + C_{\bar{w}''_x}\,\delta\bar{w}''$$

$$- C_{N_{xxx}}\,\delta N_{xx} + C_{N_{xyx}}\,\delta N_{xy} - C_{M_{xxx}}\,\delta M_{xx} + C_{M_{xyx}}\,\delta M_{xy}$$

$$+ C_{Q_{xzx}}\,\delta Q_{xz} - C_{Q_{yzx}}\,\delta Q_{yz} + C_{\omega_{xx}}\,\delta\omega_x + C_{\omega_{yx}}\,\delta\omega_y \Big|_{x_1}^{x_2} dy$$

$$+ \Big| N_{xy}\,\delta\bar{u} + N_{yy}\,\delta\bar{v} + C\bar{w}_y\,\delta\bar{w} + C\bar{w}_y'\,\delta\bar{w}' + C\bar{w}_y''\,\delta\bar{w}''$$

$$- C_{N_{xyy}}\,\delta N_{xy} + C_{N_{yyy}}\,\delta N_{yy} - C_{M_{xyy}}\,\delta M_{xy} + C_{M_{yyy}}\,\delta M_{yy}$$

$$- C_{Q_{xzy}}\,\delta Q_{xz} + C_{Q_{yzy}}\,\delta Q_{yz} + C_{\omega_{xy}}\,\delta\omega_x + C_{\omega_{yy}}\,\delta\omega_y \Big|_{y_1}^{y_2} dx \Big\}\, dt$$

$$- \rho h \iint_S \Big| \bar{u}_t\,\delta\bar{u} + \bar{v}_t\,\delta\bar{v} + \left(\bar{w}_t + \frac{h^2}{24}\bar{w}_t''\right)\delta\bar{w} + \frac{h^2}{12}\bar{w}_t'\,\delta\bar{w}'$$

$$+ \frac{h^2}{24}\left(\bar{w}_t + \frac{3h^2}{40}\bar{w}_t''\right)\delta\bar{w}'' + \frac{h^2}{12}\omega_{x,t}\,\delta\omega_x + \frac{h^2}{12}\omega_{y,t}\,\delta\omega_y \Big|_{t_1}^{t_2} dx\,dy$$

$$+ \int_{t_1}^{t_2} \Big\{ \iint_S \Big\{ -\frac{\partial}{\partial x}\Big[N_{xx}\,\delta\bar{u} + N_{xy}\,\delta\bar{v} + C\bar{w}_x\,\delta\bar{w} + C\bar{w}_x'\,\delta\bar{w}' + C\bar{w}_x''\,\delta\bar{w}''$$

$$- C_{N_{xxx}}\,\delta N_{xx} + C_{N_{xyx}}\,\delta N_{xy} - C_{M_{xxx}}\,\delta M_{xx} + C_{M_{xyx}}\,\delta M_{xy}$$

$$+ C_{Q_{xzx}}\,\delta Q_{xz} - C_{Q_{yzx}}\,\delta Q_{yz} + C_{\omega_{xx}}\,\delta\omega_x + C_{\omega_{yx}}\,\delta\omega_y \Big]$$

$$- \frac{\partial}{\partial y}\Big[N_{xy}\,\delta\bar{u} + N_{yy}\,\delta\bar{v} + C\bar{w}_y\,\delta\bar{w} + C\bar{w}_y'\,\delta\bar{w}' + C\bar{w}_y''\,\delta\bar{w}''$$

$$- C_{N_{xyy}}\,\delta N_{xy} + C_{N_{yyy}}\,\delta N_{yy} - C_{M_{xyy}}\,\delta M_{xy} + C_{M_{yyy}}\,\delta M_{yy}$$

$$- C_{Q_{xzy}}\,\delta Q_{xz} + C_{Q_{yzy}}\,\delta Q_{yz} + C_{\omega_{xy}}\,\delta\omega_x + C_{\omega_{yy}}\,\delta\omega_y \Big]$$

$$+ \rho h \Big[\bar{u}_{tt}\,\delta\bar{u} + \bar{v}_{tt}\,\delta\bar{v} + \left(\bar{w}_{tt} + \frac{h^2}{24}\bar{w}_{tt}''\right)\delta\bar{w}$$

$$+ \frac{h^2}{12}\bar{w}_{tt}'\,\delta\bar{w}' + \frac{h^2}{24}\left(\bar{w}_{tt} + \frac{3h^2}{40}\bar{w}_{tt}''\right)\delta\bar{w}''$$

$$+ \frac{h^2}{12}\omega_{x,tt}\,\delta\omega_x + \frac{h^2}{12}\omega_{y,tt}\,\delta\omega_y \Big]$$

$$- \left[P_3 \delta \bar{w} + \frac{h}{2} P_3 \delta \bar{w}' + \frac{h^2}{8} P_3 \delta \bar{w}'' \right]$$

$$+ \left[C_{\bar{w}'} \delta \bar{w}' + C_{\bar{w}''} \delta \bar{w}'' + C_{N_{xx}} \delta N_{xx} + C_{N_{xy}} \delta N_{xy} \right.$$

$$+ C_{N_{yy}} \delta N_{yy} + C_{M_{xx}} \delta M_{xx} + C_{M_{xy}} \delta M_{xy} + C_{M_{yy}} \delta M_{yy}$$

$$\left. + C_{Q_{xz}} \delta Q_{xz} + C_{Q_{yz}} \delta Q_{yz} + C_{\omega_x} \delta \omega_x + C_{\omega_y} \delta \omega_y \right] \Big\}$$

$$dx\,dy \Big\} dt = 0 \qquad (24)$$

Where the functional constants C are defined in Appendix E.

2.6 EQUATIONS OF EQUILIBRIUM AND STRESS-STRAIN RELATIONSHIPS TOGETHER WITH NATURAL AND FORCED BOUNDARY CONDITIONS

Since $\bar{U}$, $\bar{V}$, $\bar{W}$, ω_x and ω_y are independent functions, the coefficients of the functions $\delta\bar{U}$, $\delta\bar{V}$, $\delta\bar{W}$, $\delta\omega_x$, and $\delta\omega_y$ are set equal to zero. These conditions yield the following set of five equilibrium equations:

$$\frac{\partial}{\partial x} N_{xx} + \frac{\partial}{\partial y} N_{xy} = \rho h \bar{U}_{tt} \quad ,$$

$$\frac{\partial}{\partial x} N_{xy} + \frac{\partial}{\partial y} N_{yy} = \rho h \bar{V}_{tt} \quad ,$$

$$\frac{\partial}{\partial x} C_{\bar{w}x} + \frac{\partial}{\partial y} C_{\bar{w}y} = \rho h \left(\bar{W}_{tt} + \frac{h^2}{24} \bar{W}''_{tt} \right) + P_3 \quad ,$$

$$\frac{\partial}{\partial x} C_{\omega xx} + \frac{\partial}{\partial y} C_{\omega xy} - C_{\omega x} = \rho \frac{h^3}{12} \omega_{x,tt} \quad ,$$

and

$$\frac{\partial}{\partial x} C_{\omega yx} + \frac{\partial}{\partial y} C_{\omega yy} - C_{\omega y} = \rho \frac{h^3}{12} \omega_{y,tt} \quad . \qquad (25)$$

In addition, the coefficients of the functions δN_{xx}, δN_{yy}, δN_{xy}, δM_{xx}, δM_{yy}, δM_{xy}, δQ_{xz}, and δQ_{yz} are set equal to zero. These conditions yield the following eight stress-strain equations:

$$\frac{\partial}{\partial x} C_{N_{xxx}} + C_{N_{xx}} = 0 \; ,$$

$$\frac{\partial}{\partial y} C_{N_{yyy}} - C_{N_{yy}} = 0 \; ,$$

$$\frac{\partial}{\partial x} C_{M_{xxx}} + C_{M_{xx}} = 0 \; ,$$

$$\frac{\partial}{\partial y} C_{M_{yyy}} - C_{M_{yy}} = 0 \; ,$$

$$\frac{\partial}{\partial x} C_{N_{xyx}} - \frac{\partial}{\partial y} C_{N_{xyy}} - C_{N_{xy}} = 0 \; ,$$

$$\frac{\partial}{\partial x} C_{M_{xyx}} - \frac{\partial}{\partial y} C_{M_{xyy}} - C_{M_{xy}} = 0 \; ,$$

$$\frac{\partial}{\partial x} C_{Q_{xzx}} - \frac{\partial}{\partial y} C_{Q_{xzy}} - C_{Q_{xz}} = 0 \; ,$$

and

$$\frac{\partial}{\partial x} C_{Q_{yzx}} - \frac{\partial}{\partial y} C_{Q_{y\,y}} + C_{Q_{yz}} = 0 \; . \qquad (26)$$

The functional constants C are substituted from Appendix E, into equations (26). Solving for N_{xx}, N_{yy}, N_{xy}, M_{xx}, M_{yy}, M_{xy}, Q_{xz} and Q_{yz} from the above equations, yields

$$N_{xx} = \nu\left[1 - \frac{\nu D}{G}\cdot\frac{6}{h^3}\right]\Big\{ \frac{5Gh}{6}\left(\omega_y K_7 - \omega_x K_8\right) - \omega_{y,x} K_4$$

$$+ \frac{11}{48}\cdot\frac{Gh^3}{6}\left[\omega_{x,y} K_{7,x} - \omega_{y,y} K_{8,x} - \omega_{x,x} K_{7,y} + \omega_{y,x} K_{8,y}\right]$$

$$+ \frac{Gh^3}{12}\left[\left(\omega_{x,x} - \omega_{y,y}\right) K_6 + \left(\omega_x K_{6,x} - \omega_y K_{6,y}\right)\right]$$

$$+ D\left(K_4 + K_5\right)\left(\omega_{x,y} - \nu\omega_{y,x}\right)\Big\} + EhK_1 - \frac{12}{h^2}\nu D\left(K_2 + \nu K_1\right)$$

$$+ \frac{12}{Eh^3}\cdot\frac{\omega_x}{5}\cdot\nu^2 D K_{4,y} + \frac{\omega_y}{5}\nu D\left(K_{4,x} + K_{5,x}\right)\left[4\nu\left(\frac{6}{Gh^3} - 1\right) + 1\right]$$

$$+ 4\nu\omega_y K_{4,x}\left[3\frac{\nu D}{Eh^3}\left(1 + \frac{4}{5}\nu\right) - \frac{1}{5}\right] - \nu D\omega_x\left(K_{4,y} + K_{5,y}\right)\left[\frac{24}{5h^3}\cdot\frac{\nu}{G} - 1\right] ,$$

$$N_{yy} = \frac{6}{h^3}\frac{\nu D^2}{G}\left(K_4 + K_5\right)\left(\nu\omega_{y,x} - \omega_{x,y}\right) + \frac{6}{h^3}\frac{\nu D}{G}\omega_{y,x} K_4$$

$$+ \frac{24}{5h^3}\frac{\nu D}{G}\omega_y\left(K_{4,x} + K_{5,x}\right) + \nu D\omega_y K_{4,x}\frac{12}{Eh^3}\left(1 + \frac{4}{5}\nu\right)$$

$$- \frac{24}{5h^3}\frac{\nu D}{G}\omega_x\left(K_{4,y} + K_{5,y}\right) + \nu D K_{4,y}\frac{\omega_x}{5}\cdot\frac{12}{Eh^3} - \frac{12}{h^2}D\left(K_2 + \nu K_1\right)$$

$$- \frac{5}{h^2}\nu D\left(\omega_y K_7 - \omega_x K_8\right) - \frac{\nu D}{2}\left[\left(\omega_{x,x} - \omega_{y,y}\right)K_6 + \left(\omega_x K_{6,x} - \omega_y K_{6,y}\right)\right]$$

$$- \frac{11}{48}\nu D\left[\omega_{x,y} K_{7,x} - \omega_{y,y} K_{8,x} - \omega_{x,x} K_{7,y} + \omega_{y,x} K_{8,y}\right] ,$$

$$N_{xy} = GhK_3 + \frac{\nu G}{5E}\left(\omega_y K_{4,y} - \omega_x K_{4,x}\right)$$

$$+ \frac{\nu G}{5E} D\left(1 + \nu\right)\left(\omega_y K_{4,y} - \omega_x K_{4,x} + \omega_y K_{5,y} - \omega_x K_{5,x}\right) ,$$

$$M_{xx} = \nu D\left(K_4 + K_5\right) + K_4 EI - \frac{1}{5}\nu D\left(\nu K_1 + K_2\right)\left(\omega_{x,y} - \nu\omega_{y,x}\right)\left[1 + \nu D + \frac{12}{Eh^3}\nu\right]$$

$$- \frac{1}{5}\nu K_1 \omega_{y,x}\left[\frac{Eh^3}{12}\left(1 + \nu D\right) + \nu D\right] - \frac{6}{5}\nu^2 D\omega_x K_{1,y} - \frac{\omega_x}{5}\nu D K_{2,y}$$

$$+ \nu G K_3 (\omega_{x,x} - \omega_{y,y}) \left[\frac{h^3}{60} (1 + \nu D) + \frac{\nu D}{5E} \right]$$

$$+ \frac{1}{5} \nu G (\omega_x K_{3,x} - \omega_y K_{3,y}) \left[\frac{h^3}{12} (1 + \nu D) + \frac{\nu D}{E} \right]$$

$$- \nu D \omega_y (\nu K_{1,x} + K_{2,x}) \left[\left(\frac{4}{5}\nu + 1 \right) (1 + \nu D) - \frac{1}{5} \nu^2 \frac{E h^3}{12} \right]$$

$$+ \nu D \omega_x (\nu K_{1,y} + K_{2,y}) \left[\nu \cdot \frac{12}{E h^3} \left(\nu + \frac{4}{5} \right) - \frac{1}{5} (1 + \nu D) \right]$$

$$+ \nu \omega_y K_{1,x} \left[\frac{4}{5} \frac{E h^3}{12} (1 + \nu D) - \frac{1}{5} \nu D \right] ,$$

$$M_{yy} = D (K_4 + K_5) - \frac{1}{5} \nu D (\nu K_1 + K_2)(\omega_{x,y} - \nu \omega_{y,x}) \left(D + \frac{12}{E h^3} \right)$$

$$+ \frac{1}{5} \nu D \left(\frac{E h^3}{12} + 1 \right) \left\{ \frac{G K_3}{3} (\omega_{x,x} - \omega_{y,y}) + \frac{G}{E} (\omega_x K_{3,x} - \omega_y K_{3,y}) - \omega_{y,x} K_1 \right\}$$

$$- \nu D \omega_y (\nu K_{1,x} + K_{2,x}) \left[D \left(\frac{4}{5} \nu + 1 \right) - \frac{\nu}{5} \cdot \frac{12}{E h^3} \right]$$

$$+ \nu D \omega_x (\nu K_{1,y} + K_{2,y}) \left[\frac{12}{E h^3} \left(\nu + \frac{4}{5} \right) - \frac{D}{5} \right] + \nu D \omega_y K_{1,x} \left[\frac{4}{5} \frac{E h^3}{12} - \frac{1}{5} \right] ,$$

$$M_{xy} = \nu G \frac{h^3}{12} (\omega_y K_{1,y} - \omega_x K_{1,x}) + G \frac{h^3}{12} K_6$$

$$+ \frac{\nu}{2} D \left[\omega_x (K_{2,x} + \nu K_{1,x}) - \omega_y (K_{2,y} + \nu K_{1,y}) \right] ,$$

$$Q_{xz} = \frac{5h}{6} G K_7 - \frac{5 \nu E D}{h^2} \omega_y (K_2 + \nu K_1) + \frac{5}{6} \nu G E h \omega_y K_1$$

$$+ \frac{11}{48} \nu D \left[\omega_{x,y} (K_{2,x} + \nu K_{1,x}) - \omega_{x,x} (K_{2,y} + \nu K_{1,y}) \right]$$

$$+ \frac{11 h^3}{48} \frac{\nu G}{6} (\omega_{x,x} K_{1,y} - \omega_{x,y} K_{1,x}) ,$$

and

$$Q_{yz} = \frac{5h}{6} G K_8 + \frac{5\nu E D}{h^2}(K_2 + \nu K_1)\omega_x - \frac{5}{6}\nu G E h\, \omega_x K_1 + \frac{11}{48}\nu D\left[\omega_{y,x}(K_{2,y} + \nu K_{1,y}) - \omega_{y,y}(K_{2,x} + \nu K_{1,x})\right] + \frac{11h^3}{48}\frac{\nu G}{6}(\omega_{y,y} K_{1,x} - \omega_{y,x} K_{1,y}) \quad , \tag{27}$$

where,

$$G = \frac{E}{2(1+\nu)} = \text{Shear modulus of elasticity} \quad ,$$

$$D = \frac{Eh^3}{12(1-\nu^2)} = \text{Flexural rigidity of the plate} \quad ,$$

$$K_1 = \frac{h^2}{60}\omega_y \bar{W}''_{,x} + \frac{1}{2E}\frac{3h}{70}\omega_y L_{3,x} + \left(\overset{\circ}{\epsilon}_{xx} + \frac{h^2}{12}C_x + \frac{h^4}{640}\bar{W}''^{2}_x\right) + \frac{\nu}{E}\left(P_3^+ + \frac{L_3}{2}\right) \quad ,$$

$$K_2 = -\frac{h^2}{60}\omega_x \bar{W}''_{,y} - \frac{1}{2E}\frac{3h}{70}\omega_x L_{3,y} + \left(\overset{\circ}{\epsilon}_{yy} + \frac{h^2}{12}C_y + \frac{h^4}{640}\bar{W}''^{2}_y\right) + \frac{\nu}{E}\left(P_3^+ + \frac{L_3}{2}\right) \quad ,$$

$$K_3 = \frac{h^2}{60}(\omega_y \bar{W}''_{,y} - \omega_x \bar{W}''_{,x}) + \frac{1}{2E}\frac{3h}{70}(\omega_y L_{3,y} - \omega_x L_{3,x}) + \left(\overset{\circ}{\gamma}_{xx} + \overset{\circ}{\gamma}_{yy} + \bar{W}_x \bar{W}_y + \frac{h^2}{12}E_{xy} + \frac{h^4}{320}\bar{W}''_x \bar{W}''_y\right) \quad ,$$

$$K_4 = \omega_y \bar{W}'_{,x} - \frac{1}{E}P^+_{3,x}\omega_y - \frac{1}{2E}\omega_y L_{3,x} - \frac{\nu}{E}\frac{6}{5h}L_3 + \left(K_x + \frac{3h^2}{40}\bar{W}'_x \bar{W}''_x\right) \quad ,$$

$$K_5 = -\omega_x \bar{W}'_{,y} + \frac{1}{E}P^+_{3,y}\omega_x + \frac{1}{2E}\omega_x L_{3,y} - \frac{\nu}{E}\frac{6}{5h}L_3 + \left(K_y + \frac{3h^2}{40}\bar{W}'_y \bar{W}''_y\right) \quad ,$$

$$K_6 = -\omega_x \bar{W}'_x + \omega_y \bar{W}'_y + \frac{1}{E}\omega_x P^+_{3,x} - \frac{1}{E}\omega_y P^+_{3,y} + \frac{1}{2E}\omega_x L_{3,x} - \frac{1}{2E}\omega_y L_{3,y} + \left(\delta_{xx} + \delta_{yy} + D_{xy} + \frac{3h^2}{20}F_{xy}\right) \quad ,$$

$$K_7 = \frac{11h^2}{240}(\omega_{x,x}\bar{W}'_{,y} - \omega_{x,y}\bar{W}'_{,x}) + \frac{11h^2}{480}\cdot\frac{1}{2E}(\omega_{x,y}P^+_{3,x} - \omega_{x,x}P^+_{3,y})$$

$$+\frac{11h^2}{240}\cdot\frac{1}{2E}\left(\omega_{y,x}L_{3,x}-\omega_{x,x}L_{3,y}\right)+\omega_y\bar{W}'+\frac{h^2}{60}\bar{W}''[L_2]+\overset{o}{\gamma}_{xz}$$

$$+\frac{h^2}{60}\omega_{x,y}\bar{W}_x'+\frac{h^2}{40}\bar{W}_x''-\frac{h^2}{60}\omega_{x,x}\bar{W}_y'-\frac{1}{E}P_3^+\omega_y+\frac{1}{2E}\frac{3h}{70}[L_2]L_3$$

$$-\frac{1}{2E}\omega_yL_3 \quad ,$$

and

$$K_\gamma=\frac{11h^2}{240}\left(\omega_{y,y}\bar{W}'_{,x}-\omega_{y,x}\bar{W}'_{,y}\right)+\frac{1}{2E}\frac{11h^2}{480}\left(\omega_{y,x}P^+_{3,y}-\omega_{y,y}P^+_{3,x}\right)$$

$$+\frac{1}{2E}\frac{11h^2}{240}\left(\omega_{y,x}L_{3,y}-\omega_{y,y}L_{3,x}\right)-\omega_x\bar{W}'-\frac{h^2}{60}\bar{W}''[L_2]$$

$$+\overset{o}{\gamma}_{yz}-\frac{h^2}{60}\omega_{y,y}\bar{W}_x'+\frac{h^2}{40}\bar{W}_y''+\frac{h^2}{60}\omega_{y,x}\bar{W}_y'+\frac{1}{E}P_3^+\omega_x$$

$$-\frac{1}{2E}\frac{3h}{70}L_3[L_1]+\frac{1}{2E}\omega_xL_3 \quad . \qquad (28)$$

In addition, the coefficients of the functions $\delta\bar{W}'$ and $\delta\bar{W}''$ are set equal to zero, giving

$$\frac{\partial}{\partial x}C\bar{W}_x'+\frac{\partial}{\partial y}C\bar{W}_y'-C\bar{W}'=\frac{\rho h^3}{12}\bar{W}'_{tt}-\frac{h}{2}P_3 \quad ,$$

and

$$\frac{\partial}{\partial x}C\bar{W}_x''+\frac{\partial}{\partial y}C\bar{W}_y''-C\bar{W}''=\frac{\rho h^3}{24}\left(\bar{W}_{tt}+\frac{3h^2}{40}\bar{W}''_{tt}\right)-\frac{h^2}{8}P_3 \quad . \qquad (29)$$

The natural and forced boundary conditions along lines parallel to the Y-axis take the following form:

Either $\quad N_{xx}=0 \quad$ or $\quad \delta\bar{u}=0 \quad ,$

$\qquad N_{xy}=0 \quad$ or $\quad \delta\bar{v}=0 \quad ,$

$$\begin{aligned}
C\bar{w}_x &= 0 &\quad \text{or} \quad& \delta\bar{w} = 0\ , \\
C\bar{w}'_x &= 0 &\quad \text{or} \quad& \delta\bar{w}' = 0\ , \\
C\bar{w}''_x &= 0 &\quad \text{or} \quad& \delta\bar{w}'' = 0\ , \\
C\omega_{xx} &= 0 &\quad \text{or} \quad& \delta\omega_x = 0\ , \\
\text{and} \quad C\omega_{yx} &= 0 &\quad \text{or} \quad& \delta\omega_y = 0\ .
\end{aligned} \tag{30}$$

Similarly, the boundary conditions along lines parallel to the X-axis are written as,

$$\begin{aligned}
\text{Either} \quad N_{xy} &= 0 &\quad \text{or} \quad& \delta\bar{u} = 0\ , \\
N_{yy} &= 0 &\quad \text{or} \quad& \delta\bar{v} = 0\ , \\
C\bar{w}_y &= 0 &\quad \text{or} \quad& \delta\bar{w} = 0\ , \\
C\bar{w}'_y &= 0 &\quad \text{or} \quad& \delta\bar{w}' = 0\ , \\
C\bar{w}''_y &= 0 &\quad \text{or} \quad& \delta\bar{w}'' = 0\ , \\
C\omega_{xy} &= 0 &\quad \text{or} \quad& \delta\omega_x = 0\ , \\
\text{and} \quad C\omega_{yy} &= 0 &\quad \text{or} \quad& \delta\omega_y = 0\ .
\end{aligned} \tag{31}$$

2.7 EXAMPLE OF APPLICATION TO A SPECIAL CASE OF BEAM THEORY

In this section, the results from the analysis are applied to a general case of the beam-column theory and reduced to a classical equation of beam theory in the following steps:

1. General Beam-Column theory
2. Beam-Column theory neglecting the effect of transverse normal strain.
3. Classical Beam theory

The terms containing N_{xx}, M_{xx}, and Q_{xz}, $\overline{U}$, $\overline{W}$, $\overline{W}'$, $\overline{W}''$, ω_x and P_3 are retained. Hence, N_{yy}, N_{xy}, M_{yy}, M_{xy}, Q_{yz} are set equal to zero. Also, the terms containing $\overline{V}$ and ω_y are neglected. The flexural rigidity of the plate $D = \frac{Eh^3}{12(1-\nu^2)}$ reduces to EI. For convenience N_{xx}, M_{xx} and Q_{xz} are written in shorthand form as N, M and Q respectively.

Therefore, the eight stress-strain relationships reduce to the following three equations:

$$
\begin{aligned}
N = {} & \left[1 - \frac{6}{h^3}\frac{\nu EI}{G}\right]\Big\{ \nu EI\,(K_4 + K_5)\,\omega_{x,y} - \nu\omega_x \frac{5h}{6} G K_8 \\
& + \frac{h^3}{12}\nu G\left(\omega_{x,x}K_6 + \omega_x K_{6,x}\right) + \frac{11}{48}\frac{Gh^3}{6}\nu\left(\omega_{x,y}K_{7,x} - \omega_{x,x}K_{7,y}\right)\Big\} \\
& - \nu EI\,\omega_x\,(K_{4,y} + K_{5,y})\left[\frac{24}{5h^3}\frac{\nu}{G} - 1\right] + \frac{12}{Eh^3}\frac{\omega_x}{5}\nu^2 EI\,K_{4,y} \\
& - \frac{12}{h^2}\nu EI\,(K_2 + \nu K_1) + EhK_1 ,
\end{aligned}
$$

$$
\begin{aligned}
M = {} & \nu EI\,(K_4 + K_5) + K_4 EI - \frac{1}{5}\nu EI\,\omega_{x,y}\,(\nu K_1 + K_2)\left[1 + \nu EI + \frac{12}{Eh^3}\nu\right] \\
& + \frac{1}{5}\nu G\left[\frac{h^3}{12}(1 + \nu EI) + \nu I\right]\left(\omega_{x,x}K_3 + \omega_x K_{3,x}\right) \\
& + \nu^2 EI\,\omega_x K_{1,y}\left[\frac{12}{Eh^3}\nu\left(\nu + \frac{4}{5}\right) - \frac{1}{5}(7 + \nu EI)\right]
\end{aligned}
$$

$$+ \nu E I \omega_x K_{2,y} \left[\nu \frac{12}{Eh^3} \left(\nu + \frac{4}{5} \right) - \frac{1}{5} \left(2 + \nu E I \right) \right] ,$$

and

$$Q = \frac{5h}{6} K_7 G + \frac{11}{48} h^3 \frac{\nu G}{6} \cdot (\omega_{x,x} K_{1,y} - \omega_{x,y} K_{1,x})$$

$$- \frac{11}{48} \nu E I \, \omega_{x,x} (K_{2,y} + \nu K_{1,y}) + \frac{11}{48} \nu E I \, \omega_{x,y} (K_{2,x} + \nu K_{1,x}) \quad . \qquad (32)$$

Also, the five equations of equilibrium (25) reduce to three equations given as

$$\frac{\partial}{\partial x} N = \rho h \bar{u}_{tt} ,$$

$$\frac{\partial}{\partial x} \left[(\bar{w}_x + \frac{h^2}{24} \bar{w}_x'') N + \bar{w}_x' M + Q \right] = \rho h \left(\bar{w}_{tt} + \frac{h^2}{24} \bar{w}_{tt}'' \right) + P_3 ,$$

and

$$\frac{\partial}{\partial x} C_{\omega xx} + \frac{\partial}{\partial y} C_{\omega xy} - C_{\omega x} = \rho \frac{h^3}{12} \omega_{x,tt} , \qquad (33)$$

Where, $C_{\omega xx}$, $C_{\omega xy}$ and $C_{\omega x}$ reduce to

$$C_{\omega xx} = \frac{1}{2E} \frac{179 h^3}{16 \times 24 \times 105} Q_{,y} \left[\omega_{x,y} Q_{,x} - \omega_{x,x} Q_{,y} \right]$$

$$- \frac{11 h^2}{240} Q_{,y} \left[\bar{w}' + \frac{\nu}{Eh} N + \frac{1}{2E} P_3 \right] + M - \frac{h^2}{60} \bar{w}_y' Q + \frac{1}{2E} \frac{11 h^2}{480} P_3^+ Q_{,y}$$

$$+ \omega_x Q \left\{ \frac{h^2}{60} \left(\bar{w}'' + \frac{\nu}{E} \frac{12}{h^3} M \right) - \frac{1}{2E} \left[\frac{h}{105} \omega_x \omega_{x,x} Q + \frac{3h}{70} P_3 \right] \right\}$$

$$- \frac{(1+\nu)}{E} \cdot \frac{h}{105} \omega_{x,x} Q^2 ,$$

$$C_{\omega xy} = \frac{1}{2E} \frac{179 h^3}{16 \times 24 \times 105} Q_{,x} \left[\omega_{x,x} Q_{,y} - \omega_{x,y} Q_{,x} \right]$$

$$+ \frac{11 h^2}{240} Q_{,x} \left[\bar{w}' + \frac{\nu}{Eh} N + \frac{1}{2E} P_3 \right] - \frac{h^2}{60} \bar{w}_x' Q - \frac{1}{2E} \cdot \frac{11 h^2}{480} P_3^+ Q_{,x}$$

$$+ \omega_y Q \left\{ \frac{h^2}{60} \left(\bar{w}'' + \frac{\nu}{E} \frac{12}{h^3} M \right) - \frac{1}{2E} \left[\frac{h}{105} \omega_x \omega_{x,x} Q + \frac{3h}{70} P_3 \right] \right\}$$

$$- \frac{(1+\nu)}{E} \cdot \frac{h}{105} \omega_{x,y} Q^2 ,$$

$$C_{\omega_x} = Q + \omega_{x,x} Q \left\{ \frac{h^2}{60}\left(\bar{W}'' + \frac{\nu}{E}\frac{12}{h^3}M\right) - \frac{1}{2E}\left[\frac{h}{105}\omega_x \omega_{x,x} Q + \frac{3h}{70}P_3\right]\right\} \quad . \qquad (34)$$

Also, equations (29) reduce to

$$\frac{\partial}{\partial x}\left[\frac{h^2}{12}\bar{W}_x' N + \left(\bar{W}_x + \frac{3h^2}{40}\bar{W}_x''\right)M\right] - \frac{7h^2}{240}\left[\omega_{x,y} Q_{,x} - \omega_{x,x} Q_{,y}\right]$$

$$= \frac{\rho h^3}{12}\bar{W}_{tt}' - hP_3^- \quad ,$$

and

$$\frac{\partial}{\partial x}\left[\frac{h^2}{12}\left(\bar{W}_x + \frac{3h^2}{40}\bar{W}_x''\right)N + \frac{3h^2}{20}\bar{W}_x' M + \frac{h^2}{20}Q\right] - \frac{h^2}{30}\omega_x \omega_{x,x} Q$$

$$= \frac{\rho h^3}{12}\left(\bar{W}_{tt} + \frac{3h^2}{40}\bar{W}_{tt}'\right) - \frac{h^2}{20}P_3 \quad , \qquad (35)$$

Futher, if the effects of transverse normal strain (i.e., $\bar{W}' = \bar{W}'' = 0$) are neglected, the stress-strain relationships remain the same in form, as those given by equations (32), however, the definition of the functional constants K simplify to

$$K_1 = \bar{U}_x + \frac{1}{2}(\bar{W}_x)^2 + \frac{\nu}{2E}\left(P_3^+ + P_3^-\right) \quad ,$$

$$K_2 = \frac{1}{2E}\frac{3h}{70}\omega_x P_{3,y} + \frac{1}{2}(\bar{W}_y)^2 + \frac{\nu}{2E}\left(P_3^+ + P_3^-\right) \quad ,$$

$$K_3 = \bar{U}_y + \bar{W}_x \bar{W}_y + \frac{1}{2E}\frac{3h}{70}\omega_x P_{3,x} \quad ,$$

$$K_4 = \omega_{x,x} + \frac{6}{5h}\frac{\nu}{E}P_3 \quad ,$$

$$K_5 = \frac{6}{5h}\frac{\nu}{E}P_3 + \frac{1}{2E}\omega_x\left(P_{3,y}^+ + P_{3,y}^-\right) \quad ,$$

$$K_6 = \omega_{x,y} + \frac{1}{2E}\omega_x\left(P_{3,x}^+ + P_{3,x}^-\right) \quad ,$$

$$K_7 = \frac{1}{2E}\frac{11h^2}{480}\left(\omega_{x,y}P_{3,x}^+ - \omega_{x,x}P_{3,y}^+\right) + \frac{1}{2E}\frac{11h^2}{240}\left(\omega_{x,x}P_{3,y} - \omega_{x,y}P_{3,x}\right)$$

$$+ \overset{\circ}{\gamma}_{xz} - \frac{1}{2E}\frac{3h}{70}\omega_x \omega_{x,x} P_3 \quad ,$$

and

$$K_8 = \bar{W}_y + \frac{1}{2E}\,\omega_x\,(P_3^{+} + P_3^{-}) \quad . \tag{36}$$

The equilibrium equations (33) reduce to

$$\begin{aligned}
\frac{\partial}{\partial x} N &= \rho h \bar{U}_{tt} \quad , \\
\frac{\partial}{\partial x}(\bar{W}_x N + Q) &= \rho h \bar{W}_{tt} + (P_3^{+} + P_3^{-}) \quad , \\
\frac{\partial}{\partial x} C\omega_{xx} + \frac{\partial}{\partial y} C\omega_{xy} - C\omega_x &= \frac{\rho h^3}{12}\,\omega_{x,tt} \quad ,
\end{aligned} \tag{37}$$

where,

$$\begin{aligned}
C\omega_{xx} = {} & \frac{1}{2E}\cdot\frac{179h^3}{16\times24\times105}\,Q_{,y}\left[\omega_{x,y} Q_{,x} - \omega_{x,x} Q_{,y}\right] - \frac{11h^2}{240E}\,Q_{,y}\left[\frac{\nu}{h} N + \frac{1}{2} P_3\right] \\
& + M + \frac{1}{2E}\,\frac{11h^2}{480}\,P_3^{+} Q_{,y} + \omega_x Q\left[\frac{\nu}{5Eh} M - \frac{1}{2E}\left(\frac{h}{105}\,\omega_x \omega_{x,x} Q + \frac{3h}{70} P_3\right)\right] \\
& - \frac{(1+\nu)}{E}\cdot\frac{h}{105}\,\omega_{x,x} Q^2 \quad , \\
C\omega_{xy} = {} & \frac{1}{2E}\,\frac{179h^3}{16\times24\times105}\,Q_{,x}\left[\omega_{x,x} Q_{,y} - \omega_{x,y} Q_{,x}\right] + \frac{11h^2}{240E}\,Q_{,x}\left[\frac{\nu}{h} N + \frac{1}{2} P_3\right] \\
& - \frac{1}{2E}\,\frac{11h^2}{480}\,P_3^{+} Q_{,x} + \omega_y Q\left[\frac{\nu}{5Eh} M - \frac{1}{2E}\left(\frac{h}{105}\,\omega_x \omega_{x,x} Q + \frac{3h}{70} P_3\right)\right] \\
& - \frac{(1+\nu)}{E}\cdot\frac{h}{105}\,\omega_{x,y} Q^2 \quad ,
\end{aligned}$$

and

$$C\omega_x = Q + \omega_{x,x} Q\left[\frac{\nu}{5Eh} M - \frac{1}{2E}\left(\frac{h}{105}\,\omega_x \omega_{x,x} Q + \frac{3h}{70} P_3\right)\right] \quad . \tag{38}$$

The equations still remain highly coupled and mathematically complex. To deduce classical beam theory, the effect of rotation, ω_x, is assumed small and neglected in the stress equilibrium equations along with the axial force N. Also, the effects of Poisson's ratio, ν , is neglected.

The three equations of equilibrium takes the form

$$\left.\begin{aligned} \bar{U}_{tt} &= 0 \quad , \\ \frac{\partial}{\partial x} M &= Q \quad , \\ \text{and} \quad \frac{\partial}{\partial x} Q &= \rho h \bar{W}_{tt} + (P_3^+ + P_3^-) . \end{aligned}\right] \quad (39)$$

The stress-strain relationships reduce to

$$\left.\begin{aligned} Q &= \frac{5}{6} GA \left[\bar{W}_x + \omega_x \right] , \\ \text{and} \quad M &= EI \, \omega_{x,x} \; . \end{aligned}\right] \quad (40)$$

CHAPTER III

SUMMARY

The linear theory for thin elastic plates limits the maximum deflection of the plate to approximately one half the thickness of the plate.

A classical nonlinear theory is derived by Von-Kármán[5]. The maximum allowable deflection of the plate as per this theory ranges from twice the thickness to twenty times the thickness of the plate. This classical theory takes into account shear deformations in the strain-displacement equations, but does not include the effect of the nonlinear rotation terms in the equilibrium equations.

In this thesis an attempt is made to derive a nonlinear theory for plates which includes the effects of the square of the rotation terms in the strain-displacement equations and the product of stress times rotation terms in the equilibrium equations.

As observed in Appendix A the author attempted to derive a nonlinear theory for thin elastic shell. However, the application of the assumed stress distribution of Reissener's Theorem in orthogonal curvilinear coordinate form results in extreme complexity in the necessary mathematical manipulations. As a result, the problem is reduced from the shell theory approach to the theory of rectangular plates.

At this stage it is very difficult to say about the maximum allowable deflection of the plate. The use of the nonlinear theory derived in this thesis, is expected to range well beyond the upper limit

of the classical Von-Kármán theory. The upper limit of the theory derived in this can be determined only by numerical method-type solutions to the coupled partial differential equations. No attempt is made to determine the characteristics of the limit in this thesis.

The resulting nonlinear equations reduce to a special case of beam theory. If the effect of the transverse normal strain, i.e. ϵ_{zz}, is neglected, the equations still remain highly coupled and mathematically complex. If in addition, the rotation terms are assumed small the theory reduces to the classical beam theory.

CHAPTER IV

CONCLUSIONS

The equations of motion, the stress-strain relationships and the natural and forced boundary conditions are determined for the special case of a nonlinear plate theory including the effects of transverse normal stress, transverse shear stress and transverse and rotary inertia.

The addition of the transverse normal stress into the stress analysis problem produces a set of highly coupled differential equations which do not easily extend themselves to the usual uncoupling procedures. The uncoupling of the equations is not preformed in this thesis. An extension of this thesis is the determination of the proper procedure for this condition.

The Reissner's variational theroem has again proven itself as an extremely powerful method of stress analysis especially when applied to nonlinear problems. It's efficiency lies in the fact that the resulting equations of motion, stress-strain relationships and natural and forced boundary conditions are completely determined without use of a free body diagram approach.

APPENDIX A

Strain-Displacement relations and the components of stress in the curvilinear coordinate system

A.1 The coordinate system and notation

The middle surface of the shell is defined by the equations of X=X(x,y), Y=Y(x,y) and Z=Z(x,y) where the parameters x,y are called middle surface coordinates and X,Y,Z are rectangular cartesian coordinates. The normal distance from the middle surface is denoted by $\pm z$, the normal coordinate.

The unit normal vector at a point of the middle surface is defined as $\bar{n}_j$ and tangent vectors to the curves of constant x and y curves by $\bar{r}_x$ and $\bar{r}_y$ respectively.

For the special case of orthogonal middle lines, the coordinate curves align with the curves of the principle curvature.

The distance ds between points is given by the equation:

$$ds^2 = \alpha^2 dx^2 + \beta^2 dy^2 + \gamma^2 dz^2 , \qquad \text{(A1)}$$

where,

$$\left.\begin{aligned} &\alpha = A(1 + \frac{z}{r_1}) , \quad \beta = B(1 + \frac{z}{r_2}) , \quad \gamma = 1, \\ &A^2 = \bar{r}_x . \bar{r}_x , \qquad B^2 = \bar{r}_y . \bar{r}_y \ . \end{aligned}\right] \qquad \text{(A2)}$$

and $\frac{1}{r_1}$, $\frac{1}{r_2}$ are the principle curvatures of the middle surface.

A.2 Stress resultants and stress couples

Stress resultants and stress couples applied to a differential shell element are shown in Figs. A.1b and A.1c. These stress resultants and stress couples are defined as total forces and moments acting per unit length of the middle surface.

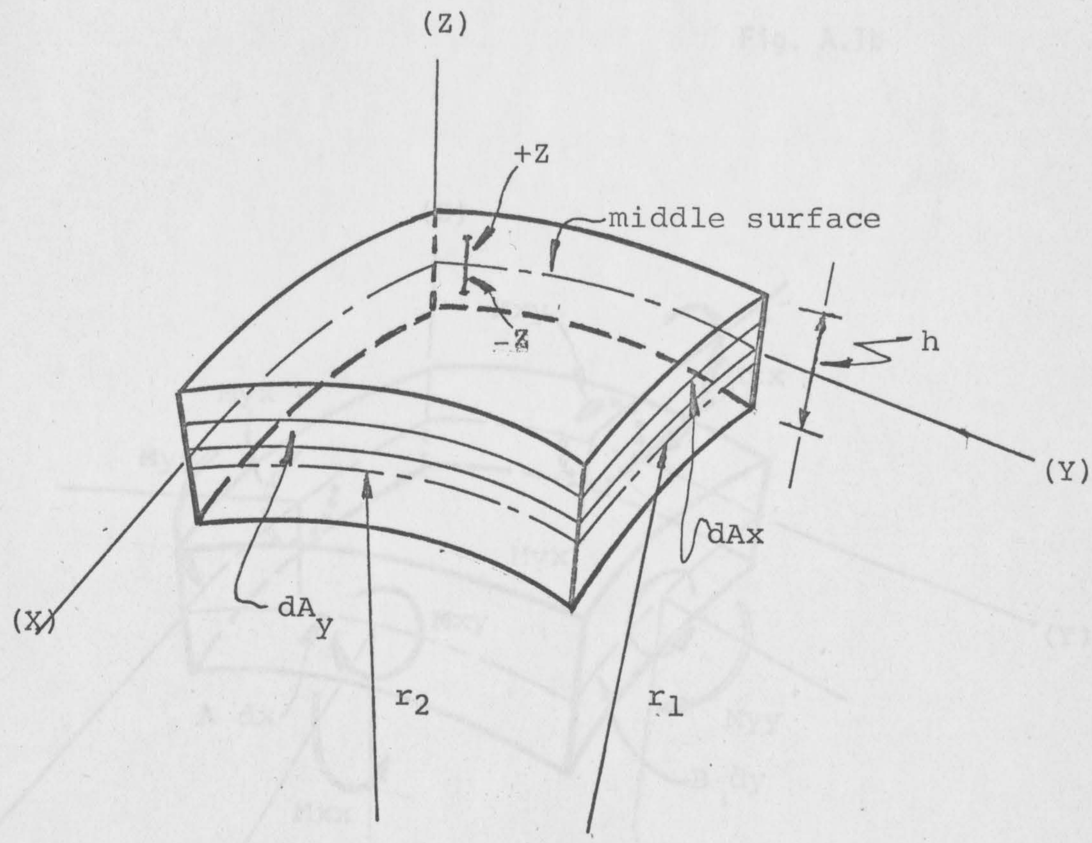

Fig. A.1a

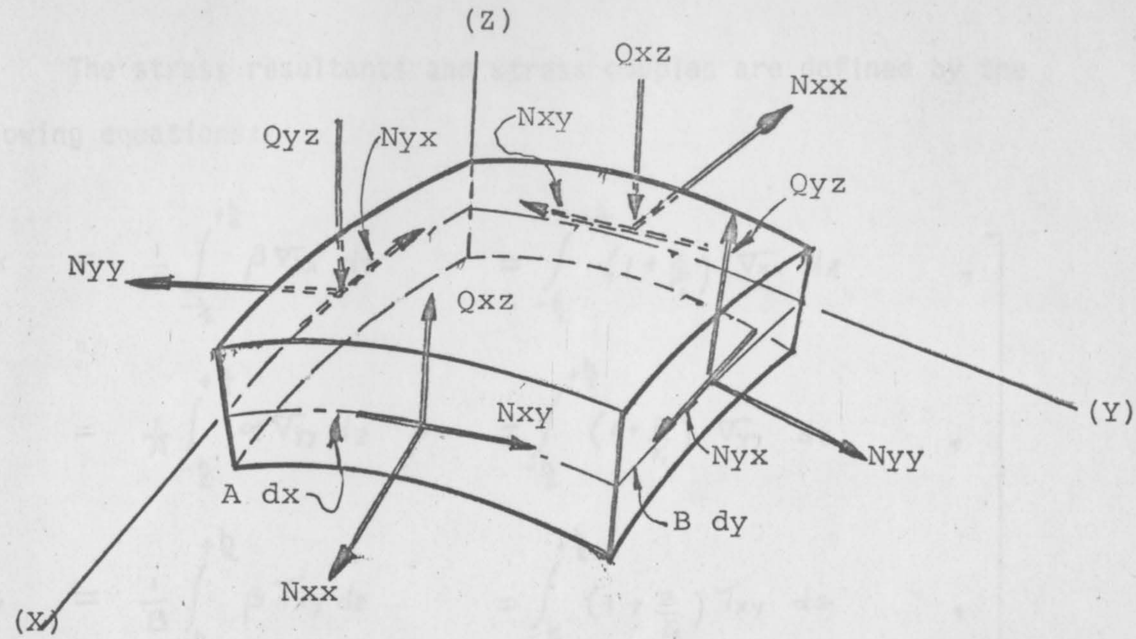

Fig. A.1b

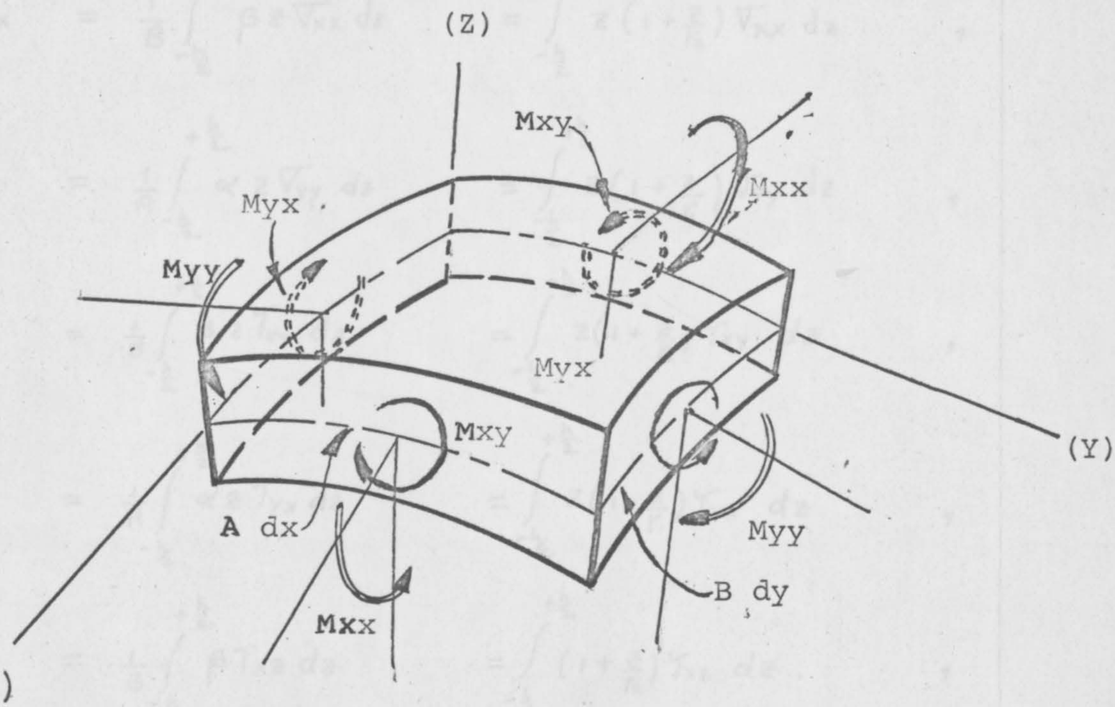

Fig. A.1c

The stress resultants and stress couples are defined by the following equations:

$$
\left.
\begin{aligned}
N_{xx} &= \frac{1}{B}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \beta\,\sigma_{xx}\,dz &&= \int_{-\frac{h}{2}}^{+\frac{h}{2}} \left(1+\frac{z}{r_2}\right)\sigma_{xx}\,dz \;, \\
N_{yy} &= \frac{1}{A}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \alpha\,\sigma_{yy}\,dz &&= \int_{-\frac{h}{2}}^{+\frac{h}{2}} \left(1+\frac{z}{r_1}\right)\sigma_{yy}\,dz \;, \\
N_{xy} &= \frac{1}{B}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \beta\,\tau_{xy}\,dz &&= \int_{-\frac{h}{2}}^{+\frac{h}{2}} \left(1+\frac{z}{r_2}\right)\tau_{xy}\,dz \;, \\
N_{yx} &= \frac{1}{A}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \alpha\,\tau_{yx}\,dz &&= \int_{-\frac{h}{2}}^{+\frac{h}{2}} \left(1+\frac{z}{r_1}\right)\tau_{yx}\,dz \;, \\
M_{xx} &= \frac{1}{B}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \beta\,z\,\sigma_{xx}\,dz &&= \int_{-\frac{h}{2}}^{+\frac{h}{2}} z\left(1+\frac{z}{r_2}\right)\sigma_{xx}\,dz \;, \\
M_{yy} &= \frac{1}{A}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \alpha\,z\,\sigma_{yy}\,dz &&= \int_{-\frac{h}{2}}^{+\frac{h}{2}} z\left(1+\frac{z}{r_1}\right)\sigma_{yy}\,dz \;, \\
M_{xy} &= \frac{1}{B}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \beta\,z\,\tau_{xy}\,dz &&= \int_{-\frac{h}{2}}^{+\frac{h}{2}} z\left(1+\frac{z}{r_2}\right)\tau_{xy}\,dz \;, \\
M_{yx} &= \frac{1}{A}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \alpha\,z\,\tau_{yx}\,dz &&= \int_{-\frac{h}{2}}^{+\frac{h}{2}} z\left(1+\frac{z}{r_1}\right)\tau_{yx}\,dz \;, \\
Q_{xz} &= \frac{1}{B}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \beta\,\tau_{xz}\,dz &&= \int_{-\frac{h}{2}}^{+\frac{h}{2}} \left(1+\frac{z}{r_2}\right)\tau_{xz}\,dz \;, \\
Q_{yz} &= \frac{1}{A}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \alpha\,\tau_{yz}\,dz &&= \int_{-\frac{h}{2}}^{+\frac{h}{2}} \left(1+\frac{z}{r_1}\right)\tau_{yz}\,dz \;.
\end{aligned}
\right] \quad \text{(A3)}
$$

Equations (A3) yields the relationship

$$N_{xy} + \frac{M_{xy}}{r_1} = N_{yx} + \frac{M_{yx}}{r_2} \quad . \tag{A4}$$

A.3 Strain-Displacement relations

The equations of the general three dimensional nonlinear strain-displacement are given as:

$$\epsilon_{xx} = \frac{1}{\alpha}\left[U_x + \frac{\alpha_y}{\beta}V + \frac{\alpha_z}{\gamma}W + \frac{1}{2\alpha}\left(U_x + \frac{\alpha_y}{\beta}V + \frac{\alpha_z}{\gamma}W\right)^2 + \frac{1}{2\alpha}\left(V_x - \frac{\alpha_y}{\beta}U\right)^2 + \frac{1}{2\alpha}\left(W_x - \frac{\alpha_z}{\gamma}U\right)^2\right],$$

$$\epsilon_{yy} = \frac{1}{\beta}\left[V_y + \frac{\beta_z}{\gamma}W + \frac{\beta_x}{\alpha}U + \frac{1}{2\beta}\left(V_y + \frac{\beta_z}{\gamma}W + \frac{\beta_x}{\alpha}U\right)^2 + \frac{1}{2\beta}\left(W_y - \frac{\beta_z}{\gamma}V\right)^2 + \frac{1}{2\beta}\left(U_y - \frac{\beta_x}{\alpha}V\right)^2\right],$$

$$\epsilon_{zz} = \frac{1}{\gamma}\left[W_z + \frac{\gamma_x}{\alpha}U + \frac{\gamma_y}{\beta}V + \frac{1}{2\gamma}\left(W_z + \frac{\gamma_x}{\alpha}U + \frac{\gamma_y}{\beta}V\right)^2 + \frac{1}{2\gamma}\left(U_z - \frac{\gamma_x}{\alpha}W\right)^2 + \frac{1}{2\gamma}\left(V_z - \frac{\gamma_y}{\beta}W\right)^2\right],$$

$$\gamma_{xy} = \frac{U_y}{\beta} + \frac{V_x}{\alpha} - \frac{\beta_x V}{\alpha\beta} - \frac{\alpha_y U}{\alpha\beta} + \frac{1}{\alpha\beta}\left(U_x + \frac{\alpha_y}{\beta}V + \frac{\alpha_z}{\gamma}W\right)\left(U_y - \frac{\beta_x}{\gamma}V\right)$$

$$+ \frac{1}{\alpha\beta}\left(V_y + \frac{\beta_x}{\alpha}U + \frac{\beta_z}{\gamma}W\right)\left(V_x - \frac{\alpha_y}{\beta}U\right) + \frac{1}{\alpha\beta}\left(W_x - \frac{\alpha_z}{\gamma}U\right)\left(W_y - \frac{\beta_z}{\gamma}V\right),$$

$$\gamma_{yz} = \frac{V_z}{\gamma} + \frac{W_y}{\beta} - \frac{\gamma_y W}{\beta\gamma} - \frac{\beta_z V}{\beta\gamma} + \frac{1}{\beta\gamma}\left(V_y + \frac{\beta_z}{\gamma}W + \frac{\beta_x}{\alpha}U\right)\left(V_z - \frac{\gamma_y}{\beta}W\right)$$

$$+ \frac{1}{\beta\gamma}\left(W_z + \frac{\gamma_y}{\beta}V + \frac{\gamma_x}{\alpha}U\right)\left(W_y - \frac{\beta_z}{\gamma}V\right) + \frac{1}{\beta\gamma}\left(U_y - \frac{\beta_x}{\alpha}V\right)\left(U_z - \frac{\gamma_x}{\alpha}W\right),$$

and

$$\gamma_{xz} = \frac{W_x}{\alpha} + \frac{U_z}{\gamma} - \frac{\alpha_z U}{\alpha\gamma} + \frac{\gamma_x W}{\alpha\gamma} + \frac{1}{\alpha\gamma}\left(W_z + \frac{\gamma_x}{\alpha}U + \frac{\gamma_y}{\beta}V\right)\left(W_x - \frac{\alpha_z}{\gamma}U\right)$$

$$+ \frac{1}{\alpha\gamma}\left(U_x + \frac{\alpha_z}{\gamma}W + \frac{\alpha_y}{\beta}V\right)\left(U_z - \frac{\gamma_x}{\alpha}W\right) + \frac{1}{\alpha\gamma}\left(V_z - \frac{\gamma_y}{\beta}W\right)\left(V_x - \frac{\alpha_y}{\beta}U\right) . \tag{A5}$$

Retaining all linear terms together with the second order of rotation terms W_x^2, W_y^2 & W_xW_y ,

equation (A5) reduces to the following:

$$\epsilon_{xx} = \frac{1}{\alpha}\left[U_x + \frac{\alpha_y}{\beta}V + \alpha_z W + \frac{1}{2\alpha}W_x^2\right],$$

$$\epsilon_{yy} = \frac{1}{\beta}\left[\frac{\beta_x}{\alpha}U + V_y + \beta_z W + \frac{1}{2\beta}W_y^2\right],$$

$$\epsilon_{zz} = W_z ,$$

$$\gamma_{xy} = \frac{U_y}{\beta} + \frac{V_x}{\alpha} - \frac{\beta_x V}{\alpha\beta} - \frac{\alpha_y U}{\alpha\beta} + \frac{W_x W_y}{\alpha\beta},$$

$$\gamma_{yz} = V_z + \frac{W_y}{\beta} - \frac{\beta_z}{\beta}V ,$$

and

$$\gamma_{xz} = U_z + \frac{W_x}{\alpha} - \frac{\alpha_z}{\alpha}U . \qquad \text{(A6)}$$

The differential equation of Codazzi for orthogonal shell coordinates are written as

$$\frac{\partial}{\partial y}\left(\frac{A}{r_1}\right) = \frac{1}{r_2}A_y ,$$

and

$$\frac{\partial}{\partial x}\left(\frac{B}{r_2}\right) = \frac{1}{r_1}B_x . \qquad \text{(A7)}$$

The following equations are obtained using equation (7):

$$\frac{\alpha_y}{\beta} = \frac{A_y}{B} ,$$

$$\frac{\beta_x}{\alpha} = \frac{B_x}{A} ,$$

$$\alpha_z = \frac{A}{r_1} ,$$

and

$$\beta_z = \frac{B}{r_2} . \qquad \text{(A8)}$$

Substituting equations (A8) into equations (A6) the following reduced form is obtained:

$$\epsilon_{xx} = \frac{1}{\alpha}\left[U_x + \frac{A_y}{B}V + \frac{A}{r_1}W + \frac{1}{2\alpha}W_x^2\right],$$

$$\epsilon_{yy} = \frac{1}{\beta}\left[V_y + \frac{B_x}{A}U + \frac{B}{r_2}W + \frac{1}{2\beta}W_y^2\right],$$

$$\epsilon_{zz} = W_z ,$$

$$\gamma_{xy} = \frac{U_y}{\beta} + \frac{V_x}{\alpha} - \frac{B_x V}{A\beta} - \frac{A_y U}{B\alpha} + \frac{W_x W_y}{\alpha\beta},$$

$$\gamma_{yz} = V_z + \frac{W_y}{\beta} - \frac{\beta_z}{\beta}V ,$$

and

$$\gamma_{xz} = U_z + \frac{W_x}{\alpha} - \frac{\alpha_z}{\alpha}U . \qquad \text{(A9)}$$

To obtain the appropriate stress-strain relation, the following approximate equations are assumed:

$$U = \bar{U}(x,y) + z\,\omega_x(x,y),$$

$$V = \bar{V}(x,y) + z\,\omega_y(x,y),$$

and

$$W = \bar{W}(x,y) + z\,\bar{W}'(x,y) + \frac{z^2}{2}\bar{W}''(x,y), \qquad \text{(A10)}$$

where, $\bar{U}$, $\bar{V}$ and $\bar{W}$ are the components of displacement at the middle surface, $\omega_x(x,y)$ and $\omega_y(x,y)$ are the change of slope of the normal to the middle surface along the x and y coordinates lines respectively, and $\bar{W}'(x,y)$ and $\bar{W}''(x,y)$ are the contirbutions to the transverse normal strain.

Substituting equations (A10) into equations (A9), the following equations are obtained:

$$\epsilon_{xx} = \frac{1}{\alpha}\left[\bar{U}_x + z\cdot\omega_{x,x} + \frac{A_y}{B}(\bar{V}+z\omega_y) + \frac{A}{r_1}(\bar{W}+z\bar{W}'+\frac{z^2}{2}\bar{W}'') + \frac{1}{2\alpha}(\bar{W}_x+z\bar{W}_x'+\frac{z^2}{2}\bar{W}_x'')^2\right],$$

$$\epsilon_{yy} = \frac{1}{\beta}\left[\bar{V}_y + z\omega_{y,y} + \frac{B_x}{A}(\bar{u}+z\omega_x) + \frac{\beta}{r_2}(\bar{W}+z\bar{W}'+\frac{z^2}{2}\bar{W}'') + \frac{1}{2\beta}(\bar{W}_y+z\bar{W}_y'+\frac{z^2}{2}\bar{W}_y'')^2\right],$$

$$\epsilon_{zz} = \bar{W}' + z\bar{W}'' ,$$

$$\gamma_{xy} = \frac{1}{\alpha\beta}\Big[\alpha(\bar{u}_y+z\omega_{x,y}) + \beta(\bar{V}_x+z\omega_{y,x}) - \frac{B_x}{A}\alpha(\bar{V}+z\omega_y) - \frac{A_y}{B}\beta(\bar{u}+z\omega_x)$$

$$+ (\bar{W}_x+z\bar{W}_x'+\frac{z^2}{2}\bar{W}_x'')(\bar{W}_y+z\bar{W}_y'+\frac{z^2}{2}\bar{W}_y'')\Big] ,$$

$$\gamma_{yz} = \frac{1}{\beta}\left[\beta\omega_y + (\bar{W}_y+z\bar{W}_y'+\frac{z^2}{2}\bar{W}_y'') - \beta_z(\bar{V}+z\cdot\omega_y)\right] ,$$

and

$$\gamma_{xz} = \frac{1}{\alpha}\left[\alpha\omega_x + (\bar{W}_x+z\bar{W}_x'+\frac{z^2}{2}\bar{W}_x'') - \alpha_z(\bar{u}+z\omega_x)\right] . \tag{A11}$$

If the terms $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are replaced respectively by the terms $\frac{1}{A}$ and $\frac{1}{B}$ in the first and second equations given in equations (A11), the following equations for the components of strain are rewritten as:

$$(1+\frac{z}{r_1})\epsilon_{xx} = \overset{\circ}{\epsilon}_{xx} + zK_x + z^2C_x + \frac{z^3}{2A^2}\bar{W}_x'\bar{W}_x'' + \frac{z^4}{8A^2}(\bar{W}_x'')^2 ,$$

$$(1+\frac{z}{r_2})\epsilon_{yy} = \overset{\circ}{\epsilon}_{yy} + zK_y + z^2C_y + \frac{z^3}{2B^2}\bar{W}_y'\bar{W}_y'' + \frac{z^4}{8B^2}(\bar{W}_y'')^2 ,$$

$$\epsilon_{zz} = \bar{W}' + z\bar{W}'' ,$$

$$(1+\frac{z}{r_1})(1+\frac{z}{r_2})\gamma_{xy} = (1+\frac{z}{r_2})(\overset{\circ}{\gamma}_{xx}+z\cdot\delta_{xx}) + (1+\frac{z}{r_1})(\overset{\circ}{\gamma}_{yy}+z\delta_{yy})$$

$$+ \frac{1}{AB}\bar{W}_x\bar{W}_y + zD_{xy} + z^2E_{xy} + z^3F_{xy} + \frac{z^4}{4AB}(\bar{W}_x''\bar{W}_y'') ,$$

$$(1+\frac{z}{r_2})\gamma_{yz} = \overset{\circ}{\gamma}_{yz} + \frac{z}{B}(\bar{W}_y' + \frac{z}{2}\bar{W}_y'') ,$$

and

$$\left(1+\frac{z}{r_1}\right)\gamma_{xz} = \overset{\circ}{\gamma}_{xz} + \frac{z}{A}\left(\bar{W}_x' + \frac{z}{2}\bar{W}_x''\right) , \qquad \text{(A12)}$$

where,

$$\overset{\circ}{\epsilon}_{xx} = \frac{1}{A}\left(\bar{U}_x + \frac{A_y}{B}\bar{V}\right) + \frac{\bar{W}}{r_1} + \frac{1}{2A^2}(\bar{W}_x)^2 ,$$

$$\overset{\circ}{\epsilon}_{yy} = \frac{1}{B}\left(\bar{V}_y + \frac{B_x}{A}\bar{U}\right) + \frac{\bar{W}}{r_2} + \frac{1}{2B^2}(\bar{W}_y)^2 ,$$

$$\overset{\circ}{\gamma}_{xx} = \frac{1}{A}\left(\bar{V}_x - \frac{A_y}{B}\bar{U}\right) ,$$

$$\overset{\circ}{\gamma}_{yy} = \frac{1}{B}\left(\bar{U}_y - \frac{B_x}{A}\bar{V}\right) ,$$

$$\delta_{xx} = \frac{1}{A}\left(\omega_{y,x} - \frac{A_y}{B}\omega_x\right) ,$$

$$\delta_{yy} = \frac{1}{B}\left(\omega_{x,y} - \frac{B_x}{A}\omega_y\right) ,$$

$$\overset{\circ}{\gamma}_{xz} = \frac{\bar{W}_x}{A} - \frac{\bar{U}}{r_1} + \omega_x ,$$

$$\overset{\circ}{\gamma}_{yz} = \frac{\bar{W}_y}{B} - \frac{\bar{V}}{r_2} + \omega_y ,$$

$$K_x = \frac{1}{A}\left(\omega_{x,x} + \frac{A_y}{B}\omega_y\right) + \frac{\bar{W}'}{r_1} + \frac{1}{A^2}\bar{W}_x\bar{W}_x' ,$$

$$K_y = \frac{1}{B}\left(\omega_{y,y} + \frac{B_x}{A}\omega_x\right) + \frac{\bar{W}'}{r_2} + \frac{1}{B^2}\bar{W}_y\bar{W}_y' ,$$

$$C_x = \frac{1}{2}\left[\frac{\bar{W}''}{r_1} + \frac{1}{A^2}\left(\bar{W}_x\bar{W}_x'' + \bar{W}_x'^2\right)\right] ,$$

$$C_y = \frac{1}{2}\left[\frac{\bar{W}''}{r_2} + \frac{1}{B^2}\left(\bar{W}_y\bar{W}_y'' + \bar{W}_y'^2\right)\right] ,$$

$$D_{xy} = \frac{1}{AB}\left(\bar{W}_x\bar{W}_y' + \bar{W}_x'\bar{W}_y\right) ,$$

$$E_{xy} = \frac{1}{AB}\left(\frac{\bar{W}_x\bar{W}_y''}{2} + \bar{W}_x'\bar{W}_y' + \frac{\bar{W}_x''\bar{W}_y}{2}\right) ,$$

and

$$F_{xy} = \frac{1}{2AB}\left(\bar{W}_x'\bar{W}_y'' + \bar{W}_x''\bar{W}_y'\right) . \qquad \text{(A13)}$$

A.4 The Components of Stress

Noting equations (A3), the components of stress are assumed to take the form:

$$(1+\frac{z}{r_2})\sigma_{xx} = \frac{N_{xx}}{h} + \frac{12}{h^3} z M_{xx} ,$$

$$(1+\frac{z}{r_1})\sigma_{yy} = \frac{N_{yy}}{h} + \frac{12}{h^3} z M_{yy} ,$$

$$(1+\frac{z}{r_2})\tau_{xy} = \frac{N_{xy}}{h} + \frac{12}{h^3} z M_{xy} ,$$

and

$$(1+\frac{z}{r_1})\tau_{yx} = \frac{N_{yx}}{h} + \frac{12}{h^3} z M_{yx} . \qquad \text{(A14)}$$

The components of shearing stress of τ_{xz}, τ_{yz} and σ_{zz} are determined by direct solution of the first three equilibrium equations of stress which are:

$$\frac{\partial}{\partial x}[\beta\gamma(\sigma_{xx} + \omega_y\tau_{xz} - \omega_z\tau_{xy})] + \frac{\partial}{\partial y}[\alpha\gamma(\tau_{yx} + \omega_y\tau_{yz} - \omega_z\sigma_{yy})]$$

$$+ \frac{\partial}{\partial z}[\alpha\beta(\tau_{zx} + \omega_y\sigma_{zz} - \omega_z\tau_{zy})] + \gamma\alpha_z(\sigma_{xx}\omega_z + \tau_{xy} - \omega_x\tau_{xz})$$

$$+ \beta\alpha_z(\tau_{xz} - \omega_y\sigma_{xx} + \omega_x\tau_{xy}) + \gamma\beta_x(\omega_x\tau_{yz} - \omega_z\tau_{yx} - \sigma_{yy})$$

$$+ \beta\gamma_x(\omega_y\tau_{yx} - \omega_x\tau_{zy} - \sigma_{zz}) + \alpha\beta\gamma\rho F_x = 0 ,$$

$$\frac{\partial}{\partial x}[\beta\gamma(\tau_{xy} + \omega_z\sigma_{xx} - \omega_x\tau_{xz})] + \frac{\partial}{\partial y}[\alpha\gamma(\sigma_{yy} + \omega_z\tau_{yx} - \omega_x\tau_{yz})]$$

$$+ \frac{\partial}{\partial z}[\alpha\beta(\tau_{zy} + \omega_z\tau_{zx} - \omega_x\sigma_{zz})] + \alpha\beta_z(\sigma_{yy}\omega_x + \tau_{yz} - \omega_y\tau_{yx})$$

$$+ \gamma\beta_x(\tau_{yx} - \omega_z\sigma_{yy} + \omega_y\tau_{yz}) + \alpha\gamma_y(\omega_y\tau_{zx} - \omega_x\tau_{zy} - \sigma_{zz})$$

$$+ \gamma\alpha_y(\omega_z\tau_{xy} - \omega_y\tau_{xz} - \sigma_{xx}) + \alpha\beta\gamma\rho F_y = 0 \quad ,$$

and

$$\frac{\partial}{\partial x}[\beta\gamma(\tau_{xz} + \omega_x\tau_{xy} - \omega_y\sigma_{xx})] + \frac{\partial}{\partial y}[\alpha\gamma(\tau_{yz} + \omega_x\sigma_{yy} - \omega_y\tau_{yx})]$$

$$+ \frac{\partial}{\partial z}[\alpha\beta(\sigma_{zz} + \omega_x\tau_{zy} - \omega_y\tau_{zx})] + \beta\gamma_x(\tau_{zx} + \omega_y\sigma_{zz} - \omega_z\tau_{zy})$$

$$+ \alpha\gamma_y(\tau_{zy} - \omega_x\sigma_{zz} + \omega_z\tau_{zx}) + \beta\alpha_z(\omega_z\tau_{xy} - \omega_y\tau_{xz} - \sigma_{xx})$$

$$+ \alpha\beta_z(\omega_x\tau_{yz} - \omega_z\tau_{yx} - \sigma_{yy}) + \alpha\beta\gamma\rho F_z = 0 \, . \qquad \text{(A15)}$$

It should be noted that each equilibrium equation contains fourteen additional terms as compared to the linear theory. These terms contain the product of the stress times the rotation terms.

For thin shell theory in orthogonal coordinates $\omega_z = 0$ and $\gamma = 1$, hence $\gamma_x = \gamma_y = \gamma_z = 0$. Neglecting the body forces, $F_x = F_y = F_z = 0$, and noting $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$ and $\tau_{yz} = \tau_{zy}$, the previous equations reduce to the following form:

$$\frac{\partial}{\partial x}[\beta(\sigma_{xx} + \omega_y\tau_{xz})] + \frac{\partial}{\partial y}[\alpha(\tau_{xy} + \omega_y\tau_{yz})] + \frac{\partial}{\partial z}[\alpha\beta(\tau_{xz} + \omega_y\sigma_{zz})]$$

$$+ \alpha_y(\tau_{xy} - \omega_x\tau_{xz}) + \beta\alpha_z(\tau_{xz} - \omega_y\sigma_{xx} + \omega_x\tau_{xy})$$

$$+ \beta_x(\omega_x\tau_{yz} - \sigma_{yy}) = 0 \quad ,$$

$$\frac{\partial}{\partial x}\left[\beta(\tau_{xy} - \omega_x \tau_{xz})\right] + \frac{\partial}{\partial y}\left[\alpha(\sigma_{yy} - \omega_x \tau_{yz})\right] + \frac{\partial}{\partial z}\left[\alpha\beta(\tau_{yz} - \omega_x \sigma_{zz})\right]$$

$$+ \alpha\beta_z(\tau_{yz} + \omega_x \sigma_{yy} - \omega_y \tau_{xy}) + \beta_x(\tau_{xy} + \omega_y \tau_{yz})$$

$$- \alpha_y(\sigma_{xx} + \omega_y \tau_{xz}) = 0 ,$$

and

$$\frac{\partial}{\partial x}\left[\beta(\tau_{xz} + \omega_x \tau_{xy} - \omega_y \sigma_{xx})\right] + \frac{\partial}{\partial y}\left[\alpha(\tau_{yz} + \omega_x \sigma_{yy} - \omega_y \tau_{xy})\right]$$

$$+ \frac{\partial}{\partial z}\left[\alpha\beta(\sigma_{zz} + \omega_x \tau_{yz} - \omega_y \tau_{xz})\right] - \beta\alpha_z(\sigma_{xx} + \omega_y \tau_{xz})$$

$$+ \alpha\beta_z(\omega_x \tau_{yz} - \sigma_{yy}) = 0 . \qquad \text{(A16)}$$

Substituting equations (A3) into equations (A16), and integrating over the thickness of the plate yields respectively,

$$\frac{\partial}{\partial x}\left[B(N_{xx} + \omega_y Q_{xz})\right] + \frac{\partial}{\partial y}\left[A(N_{xy} + \omega_y Q_{yz})\right] + A_y(N_{xy} - \omega_x Q_{xz})$$

$$- B_x(N_{yy} - \omega_x Q_{yz}) + \frac{AB}{r_1}(Q_{xz} + \omega_x N_{xy} - \omega_y N_{xx}) + AB\omega_y \bar{P}_3 - AB\bar{P}_1 = 0 ,$$

$$\frac{\partial}{\partial x}\left[B(N_{xy} - \omega_x Q_{xz})\right] + \frac{\partial}{\partial y}\left[A(N_{yy} - \omega_x Q_{yz})\right] + B_x(N_{xy} + \omega_y Q_{yz})$$

$$- A_y(N_{xx} + \omega_y Q_{xz}) + \frac{AB}{r_2}(Q_{yz} + \omega_x N_{yy} - \omega_y N_{xy}) - AB\omega_x \bar{P}_3 + AB\bar{P}_2 = 0 ,$$

and

$$\frac{\partial}{\partial x}\left[B(Q_{xz} + \omega_x N_{xy} - \omega_y N_{xx})\right] + \frac{\partial}{\partial y}\left[A(Q_{yz} + \omega_x N_{yy} - \omega_y N_{xy})\right]$$

$$+ \frac{AB}{r_2}(\omega_x Q_{yz} - N_{yy}) - \frac{AB}{r_1}(N_{xx} + \omega_y Q_{xz})$$

$$+ AB(\bar{P}_3 + \omega_x \bar{P}_2 - \omega_y \bar{P}_1) = 0 , \qquad \text{(A17)}$$

where,

$$AB\overline{P_1} = \alpha\beta\tau_{xz}\Big|_{-\frac{h}{2}}^{+\frac{h}{2}} ,$$

$$AB\overline{P_2} = \alpha\beta\tau_{yz}\Big|_{-\frac{h}{2}}^{+\frac{h}{2}} ,$$

and

$$AB\overline{P_3} = \alpha\beta\sigma_{zz}\Big|_{-\frac{h}{2}}^{+\frac{h}{2}} . \tag{A18}$$

Multiplying the equations (A16) by Z and performing the integration over the thickness of the plate yields respectively,

$$\frac{\partial}{\partial x}(BM_{xx}) + \frac{\partial}{\partial y}(AM_{xy}) + \frac{AB}{r_1}(\omega_x M_{xy} - \omega_y M_{xx}) + A_y M_{xy} - B_x M_{yy} - AB\left[Q_{xz} + \omega_y \overline{\overline{Q_{zz}}} - h(\overline{P_1} + \omega_y \overline{P_3})\right] = 0 ,$$

$$\frac{\partial}{\partial x}(BM_{xy}) + \frac{\partial}{\partial y}(AM_{yy}) + \frac{AB}{r_2}(\omega_x M_{yy} - \omega_y M_{xy}) + B_x M_{xy} - A_y M_{xx} - AB\left[Q_{yz} - \omega_x \overline{\overline{Q_{zz}}} - h(\overline{P_2} - \omega_x \overline{P_3})\right] = 0 ,$$

and

$$\frac{\partial}{\partial x}\left[B(\omega_x M_{xy} - \omega_y M_{xx})\right] + \frac{\partial}{\partial y}\left[A(\omega_x M_{yy} - \omega_y M_{xy})\right] - AB\left[\left(\frac{M_{xx}}{r_1} + \frac{M_{yy}}{r_2}\right) + \left(\overline{\overline{Q_{zz}}} + \omega_x Q_{yz} - \omega_y Q_{xz}\right) + h\left(\omega_y \overline{P_1} - \omega_x \overline{P_2} - \overline{P_3}\right)\right] = 0 , \tag{A19}$$

where ,

$$AB\overline{\overline{Q_{zz}}} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \alpha\beta\,\overline{\sigma_{zz}}\,dz \quad . \tag{A20}$$

Substituting equations (A14) into equations (A16), one obtains,

$$\frac{\partial}{\partial x}(\beta\omega_y\tau_{xz}) + \frac{\partial}{\partial y}(\alpha\omega_y\tau_{yz}) + \frac{\partial}{\partial z}\left[\alpha\beta(\tau_{xz} + \omega_y\overline{\sigma_{zz}})\right]$$
$$- \alpha_y\omega_x\tau_{xz} + \beta\alpha_z\tau_{xz} + \beta_x\omega_x\tau_{yz} + \chi_1 = 0 \quad ,$$

$$\frac{\partial}{\partial x}(\beta\omega_x\tau_{xz}) + \frac{\partial}{\partial y}(\alpha\omega_x\tau_{yz}) + \frac{\partial}{\partial z}\left[\alpha\beta(\tau_{yz} - \omega_x\overline{\sigma_{zz}})\right]$$
$$+ \beta_x\omega_y\tau_{yz} + \alpha\beta_z\tau_{yz} - \alpha_y\omega_y\tau_{xz} + \chi_2 = 0 \quad ,$$

and

$$\frac{\partial}{\partial x}(\beta\tau_{xz}) + \frac{\partial}{\partial y}(\alpha\tau_{yz}) + \frac{\partial}{\partial z}\left[\alpha\beta(\overline{\sigma_{zz}} + \omega_x\tau_{yz} - \omega_y\tau_{xz})\right]$$
$$- \beta\alpha_z\omega_y\tau_{xz} + \alpha\beta_z\omega_x\tau_{yz} + \chi_3 = 0 \quad , \tag{A21}$$

where,

$$\chi_1 = \chi_{11} + z\cdot\chi_{12} \quad ,$$
$$\chi_2 = \chi_{21} + z\cdot\chi_{22} \quad ,$$

and

$$\chi_3 = \chi_{31} + z\cdot\chi_{32} \quad . \tag{A22}$$

From equations (A16) through (A22), one obtains

$$\chi_{11} = -\frac{1}{h}\Big[\frac{\partial}{\partial x}(B Q_{xz}\omega_y) + \frac{\partial}{\partial y}(A Q_{yz}\omega_y) - A_y Q_{xz}\omega_x + B_x Q_{yz}\omega_x + \frac{AB}{r_1} Q_{xz} + AB(\bar{P}_1 + \omega_y \bar{P}_3) \Big] ,$$

$$\chi_{21} = \frac{1}{h}\Big[\frac{\partial}{\partial x}(B Q_{xz}\omega_x) + \frac{\partial}{\partial y}(A Q_{yz}\omega_x) + A_y Q_{xz}\omega_y - B_x Q_{yz}\omega_y - \frac{AB}{r_2} Q_{yz} - AB(\bar{P}_2 - \omega_x \bar{P}_3) \Big] ,$$

$$\chi_{31} = -\frac{1}{h}\Big[\frac{\partial}{\partial x}(B Q_{xz}) + \frac{\partial}{\partial y}(A Q_{yz}) - \frac{AB}{r_1} Q_{xz}\omega_y + \frac{AB}{r_2} Q_{yz}\omega_x + AB(\bar{P}_3 + \omega_x \bar{P}_2 - \omega_y \bar{P}_1) \Big] ,$$

$$\chi_{12} = \frac{12}{h^3} AB\Big[Q_{xz} + \omega_y \overline{\overline{Q_{zz}}} - h\bar{P}_1 - h\omega_y \bar{P}_3 \Big] ,$$

$$\chi_{22} = \frac{12}{h^3} AB\Big[Q_{yz} - \omega_x \overline{\overline{Q_{zz}}} - h\bar{P}_2 + h\omega_x \bar{P}_3 \Big] ,$$

and

$$\chi_{32} = -\frac{12}{h^3} AB\Big[Q_{xz}\omega_y - Q_{yz}\omega_x - \overline{\overline{Q_{zz}}} - h\omega_y \bar{P}_1 + h\omega_x \bar{P}_2 + h\bar{P}_3 \Big] . \qquad \text{(A23)}$$

Eliminating σ_{zz} from equations (A21), there results,

$$\tau_{xz}\big[\beta\,\omega_{y,x} - \alpha_y \omega_x\big] + \frac{1}{\alpha}\frac{\partial}{\partial z}(\alpha^2\beta\,\tau_{xz})(1+\omega_y^2) + \tau_{yz}\big[\alpha\,\omega_{y,y} + \beta_x \omega_x\big] - \frac{1}{\beta}\frac{\partial}{\partial z}(\alpha\beta^2\tau_{yz})(\omega_x\omega_y) - (\chi_1 - \omega_y \chi_3) = 0 ,$$

and

$$\Upsilon_{xz}\left[\beta\,\omega_{x,x} + \alpha_y\,\omega_y\right] + \frac{1}{\alpha}\frac{\partial}{\partial z}(\alpha^2\beta\,\Upsilon_{xz})(\omega_x\,\omega_y)$$
$$+ \Upsilon_{yz}\left[\alpha\,\omega_{x,y} - \beta_x\,\omega_y\right] - \frac{1}{\beta}\frac{\partial}{\partial z}(\alpha\beta^2\Upsilon_{yz})(1+\omega_x^2)$$
$$-(\chi_2 + \omega_x\chi_3) = 0 \quad . \qquad \text{(A24)}$$

Uncoupling equations (A24) into an independent equation in Υ_{yz} and neglecting all the terms of the square and higher order of ω, the following equation results:

$$\alpha\frac{\partial}{\partial z}\left[\beta\frac{\partial}{\partial z}(\alpha\Upsilon_{yz})\right] + \frac{1}{\beta}\frac{\partial}{\partial z}(\alpha\beta^2\Upsilon_{yz})\left[(\omega_{y,x} - \frac{A_y}{B}\omega_x) + 2\,\alpha_z\right]$$
$$+ \frac{1}{\alpha}\frac{\partial}{\partial z}(\alpha^2\Upsilon_{yz})\left(\frac{B_x}{A}\omega_y - \omega_{x,y}\right) + 2\,\alpha\,\beta_z\frac{\partial}{\partial z}(\alpha\,\Upsilon_{yz})$$
$$+\left[(\omega_{y,x} - \frac{A_y}{B}\omega_x) + 2\alpha_z\right]\chi_2 + 2\,\alpha_z\,\omega_x\chi_3$$
$$+\alpha\frac{\partial}{\partial z}(\chi_2 + \omega_x\chi_3) - \chi_1(\omega_{x,x} + \frac{A_y}{B}\omega_y) = 0 \quad . \qquad \text{(A25)}$$

Now, droping all the terms with the product of Υ and ω and similar or higher order and simplifying, one obtains

$$\frac{\partial}{\partial z}\left[\frac{\alpha^2}{\beta}\frac{\partial}{\partial z}(\alpha\beta^2\Upsilon_{yz})\right] + \frac{\partial}{\partial z}\left[\alpha^2(\chi_2 + \omega_x\chi_3)\right]$$
$$+\alpha\,\chi_2(\omega_{y,x} - \frac{A_y}{B}\omega_x) - \alpha\,\chi_1(\omega_{x,x} + \frac{A_y}{B}\omega_y) = 0 \quad . \qquad \text{(A26)}$$

Integrating w.r.t. z, there results,

$$\frac{\partial}{\partial z}(\alpha\beta^2\tau_{yz}) + \beta(\chi_2+\omega_x\chi_3) + \frac{\beta}{\alpha^2} f_1(x,y)$$
$$+ A\frac{\beta}{\alpha^2}(\omega_{y,x} - \frac{A_y}{B}\omega_x)\left[\chi_{21}\cdot z + (\frac{\chi_{21}}{r_1}+\chi_{22})\cdot\frac{z^2}{2} + \frac{\chi_{22}}{r_1}\cdot\frac{z^3}{3}\right]$$
$$- A\frac{\beta}{\alpha^2}(\omega_{x,x} + \frac{A_y}{B}\omega_y)\left[\chi_{11}\cdot z + (\frac{\chi_{11}}{r_1}+\chi_{12})\cdot\frac{z^2}{2} + \frac{\chi_{12}}{r_1}\cdot\frac{z^3}{3}\right] = 0 . \qquad (A27)$$

The integration of equation (A27) is carried out over the function z. Applying the boundary conditions

$$@ \quad z = +\frac{h}{2}, \quad \tau_{yz} = \overline{P_2}^+ ,$$
$$\alpha = AH_2^+ ,$$
$$\beta = BH_1^+ ,$$

And

$$@ \quad z = -\frac{h}{2}, \quad \tau_{yz} = \overline{P_2}^- ,$$
$$\alpha = AH_2^- ,$$
$$\beta = BH_1^- . \qquad (A28)$$

and neglecting the terms containing the quantity $\frac{h}{r}$ and all higher powers in comparision to one, the transverse shearing stress τ_{yz} becomes,

$$\frac{\alpha}{A}\tau_{yz} = \frac{3Q_{yz}}{2h}\left(1-\frac{4z^2}{h^2}\right) - \frac{1}{4}\left[H_2^+\bar{P}_2^+\left(1-\frac{4z}{h}-\frac{12z^2}{h^2}\right) + H_2^-\bar{P}_2^-\left(1+\frac{4z}{h}-\frac{12z^2}{h^2}\right)\right]$$
$$+\frac{1}{4A}\left(1-\frac{4z^2}{h^2}\right)\left\{\left(\omega_{x,x}+\frac{A_y}{B}\omega_y\right)\left[H_1^+\bar{P}_1^+\left(z+\frac{h}{2}\right)+H_1^-\bar{P}_1^-\left(z-\frac{h}{2}\right)-\frac{2z}{h}Q_{xz}\right]\right.$$
$$\left.-\left(\omega_{y,x}-\frac{A_y}{B}\omega_x\right)\left[H_2^+\bar{P}_2^+\left(z+\frac{h}{2}\right)+H_2^-\bar{P}_2^-\left(z-\frac{h}{2}\right)-\frac{2z}{h}Q_{yz}\right]\right\}. \quad \text{(A29)}$$

Similarly, the solution of an uncoupled equation from equations (A24), in τ_{xz} yields

$$\frac{\beta}{B}\tau_{xz} = \frac{3Q_{xz}}{2h}\left(1-\frac{4z^2}{h^2}\right) - \frac{1}{4}\left[H_1^+\bar{P}_1^+\left(1-\frac{4z}{h}-\frac{12z^2}{h^2}\right) + H_1^-\bar{P}_1^-\left(1+\frac{4z}{h}-\frac{12z^2}{h^2}\right)\right]$$
$$+\frac{1}{4B}\left(1-\frac{4z^2}{h^2}\right)\left\{\left(\omega_{y,y}+\frac{B_x}{A}\omega_x\right)\left[H_2^+\bar{P}_2^+\left(z+\frac{h}{2}\right)+H_2^-\bar{P}_2^-\left(z-\frac{h}{2}\right)-\frac{2z}{h}Q_{yz}\right]\right.$$
$$\left.-\left(\omega_{x,y}-\frac{B_x}{A}\omega_y\right)\left[H_1^+\bar{P}_1^+\left(z+\frac{h}{2}\right)+H_1^-\bar{P}_1^-\left(z-\frac{h}{2}\right)-\frac{2z}{h}Q_{xz}\right]\right\}. \quad \text{(A30)}$$

The solution for σ_{zz} is obtained by substituting τ_{xz} from equation (A30) and τ_{yz} from equation (A29) into the third equilibrium equation (A21). The result is integrated twice w.r.t. z and the following boundary conditions are used

$$\sigma_{zz} = \bar{P}_3^+ \quad @ \quad z = +\frac{h}{2},$$

and

$$\sigma_{zz} = \bar{P}_3^- \quad @ \quad z = -\frac{h}{2}. \quad \text{(A31)}$$

The final form of the stress distribution σ_{zz} is written as

$$\alpha\beta \bar{V}_{zz} = \left(1-\frac{4z^2}{h^2}\right)\Big\{\frac{1}{4}[\bar{R}][\bar{L}_1] - \frac{1}{4}[\bar{S}][\bar{L}_2] + \frac{3AB}{2h}[\bar{L}_6]$$

$$+ \bar{S}'\bar{L}_7 + \bar{R}'\bar{L}_8 + \frac{AB\omega_y}{4r_1}\bar{T}' - \frac{AB\omega_x}{4r_2}\bar{J}' - \frac{1}{4}(B\bar{T}')_x$$

$$-\frac{1}{4}(A\bar{J}')_y + ABH^+P_3^+ - \frac{1}{4}\bar{S}'_{,x}M_2 + \frac{1}{4}\bar{S}'_{,y}M_3 + \frac{1}{4}\bar{R}'_{,x}M_1$$

$$-\frac{1}{4}\bar{R}'_{,y}M_4 + \frac{\alpha B\omega_y}{4}T - \beta\frac{A\omega_x}{4}J + \frac{3AB}{2h}\left(\frac{h}{3} - z + \frac{4z^3}{3h^2}\right)\bar{L}_3$$

$$+ \left(\frac{z}{2h} - \frac{2z^3}{h^3}\right)\bar{L}_4 + \left(\frac{3}{2h} - \frac{6z^2}{h^3}\right)\bar{L}_5 \quad , \qquad \text{(A32)}$$

where,

$$\bar{R} = H_2^+\bar{P}_2^+\left(z+\frac{h}{2}\right) + H_2^-\bar{P}_2^-\left(z-\frac{h}{2}\right) - \frac{2z}{h}Q_{yz} \quad ,$$

$$\bar{S} = H_1^+\bar{P}_1^+\left(z+\frac{h}{2}\right) + H_1^-\bar{P}_1^-\left(z-\frac{h}{2}\right) - \frac{2z}{h}Q_{xz} \quad ,$$

$$\bar{T} = H_1^+\bar{P}_1^+\left(1-\frac{4z}{h}-\frac{12z^2}{h^2}\right) + H_1^-\bar{P}_1^-\left(1+\frac{4z}{h}-\frac{12z^2}{h^2}\right) ,$$

and

$$\bar{J} = H_2^+\bar{P}_2^+\left(1-\frac{4z}{h}-\frac{12z^2}{h^2}\right) + H_2^-\bar{P}_2^-\left(1+\frac{4z}{h}-\frac{12z^2}{h^2}\right) \quad ,$$

$$M_1 = \omega_{y,y} + \frac{B_x}{A}\omega_x$$

$$M_2 = \omega_{x,y} - \frac{B_x}{A}\omega_y \quad ,$$

$$M_3 = \omega_{x,x} + \frac{A_y}{B}\omega_y \quad ,$$

$$M_4 = \omega_{y,x} - \frac{A_y}{B}\omega_x \quad ,$$

$$\bar{R}' = H_2^+\bar{P}_2^+\left(\frac{11}{48}h^2 - \frac{hz}{2} - \frac{z^2}{2} + \frac{2z^3}{3h} + \frac{z^4}{h^2}\right) - H_2^-\bar{P}_2^-\left(\frac{5}{48}h^2 - \frac{hz}{2} + \frac{z^2}{2} + \frac{2z^3}{3h} - \frac{z^4}{h^2}\right)$$

$$- Q_{yz}\left(\frac{h}{8} + \frac{z^2}{h} - \frac{2z^4}{h^3}\right) ,$$

$$\bar{S}' = H_1^+\bar{P}_1^+\left(\frac{11}{48}h^2 - \frac{hz}{2} - \frac{z^2}{2} + \frac{2z^3}{3h} + \frac{z^4}{h^2}\right) - H_1^-\bar{P}_1^-\left(\frac{5}{48}h^2 - \frac{hz}{2} + \frac{z^2}{2} + \frac{2z^3}{3h} - \frac{z^4}{h^2}\right)$$

$$- Q_{xz}\left(\frac{h}{8} + \frac{z^2}{h} - \frac{2z^4}{h^3}\right) ,$$

$$\bar{T}' = H_1^+ \bar{P}_1^+ \left(-\frac{h}{2} - z + \frac{2z^2}{h} + \frac{4z^3}{h^2}\right) + H_1^- \bar{P}_1^- \left(\frac{h}{2} - z - \frac{2z^2}{h} + \frac{4z^3}{h^2}\right),$$

$$\bar{J}' = H_2^+ \bar{P}_2^+ \left(-\frac{h}{2} - z + \frac{2z^2}{h} + \frac{4z^3}{h^2}\right) + H_2^- \bar{P}_2^- \left(\frac{h}{2} - z - \frac{2z^2}{h} + \frac{4z^3}{h^2}\right),$$

$$\bar{L}_1 = \alpha \omega_y M_1 + \beta \omega_x M_4 ,$$

$$\bar{L}_2 = \alpha \omega_y M_2 + \beta \omega_x M_3 ,$$

$$\bar{L}_3 = \omega_y \bar{P}_1 - \omega_x \bar{P}_2 - \bar{P}_3 ,$$

$$\bar{L}_4 = [B(N_{xy}\omega_x - N_{xx}\omega_y)]_{,x} + [A(N_{yy}\omega_x - N_{xy}\omega_y)]_{,y} - AB\left(\frac{N_{xx}}{r_1} + \frac{N_{yy}}{r_2}\right),$$

$$\bar{L}_5 = [B(M_{xy}\omega_x - M_{xx}\omega_y)]_{,x} + [A(M_{yy}\omega_x - M_{xy}\omega_y)]_{,y} - AB\left(\frac{M_{xx}}{r_1} + \frac{M_{yy}}{r_2}\right),$$

$$\bar{L}_6 = \frac{3}{2h}(\alpha \omega_y Q_{xz} - \beta \omega_x Q_{yz}),$$

$$\bar{L}_7 = \frac{1}{4}\left[\left(\frac{A_y}{B}\omega_y\right)_{,y} + \left(\frac{B_x}{A}\omega_y\right)_{,x} + \frac{A\omega_y}{r_1} M_2 - \frac{B\omega_x}{r_2} M_3\right],$$

and

$$\bar{L}_8 = \frac{1}{4}\left[\left(\frac{B_x}{A}\omega_x\right)_{,x} + \left(\frac{A_y}{B}\omega_x\right)_{,y} - \frac{A\omega_y}{r_1} M_1 - \frac{B\omega_x}{r_2} M_4\right]. \tag{A33}$$

The application of the assumed stress distribution of Reissner's Theorem in orthogonal curvilinear coordinate form resulted in extreme complexity in the necessary mathematical manipulations.

As a result, upon consultation, it was decided to reduce the problem from the shell theory approach to the theory of rectangular plates.

If the stress distributions are given by (A14), (A29), (A30) and (A32) are simplified for the rectangular cartesian coordinates the reduced form is given by equations (9), (20) and (21) as shown in the main body of the report.

APPENDIX B

Details For Solving Υ_{yz}

From equation (19) droping all the terms with the product of Υ and ω and similar or higher order and simplifying, one obtains

$$\frac{\partial^2}{\partial z^2}(\Upsilon_{yz}) + \frac{\partial}{\partial z}(\chi_2 + \omega_x \chi_3) + \omega_{y,x}\chi_2 - \omega_{x,x}\chi_1 = 0. \tag{B1}$$

Integrating w.r.t. z, yields

$$\frac{\partial}{\partial z}(\Upsilon_{yz}) + (\chi_2 + \omega_x \chi_3) + \omega_{y,x} \cdot z \cdot (\chi_{21} + \tfrac{z}{2}\chi_{22}) - \omega_{x,x} \cdot z \cdot (\chi_{11} + \tfrac{z}{2}\chi_{12}) + f_1(x,y) = 0. \tag{B2}$$

Integrating again w.r.t. z, gives

$$\Upsilon_{yz} + z \cdot f_1(x,y) + f_2(x,y) - \tfrac{z^2}{2}\omega_{x,x}(\chi_{11} + \tfrac{z}{3}\chi_{12}) + z \cdot \chi_{21}(1 + \tfrac{z}{2}\omega_{y,x}) + \tfrac{z^2}{2}\chi_{22}(1 + \tfrac{z}{3}\omega_{y,x}) + z \cdot \omega_x(\chi_{31} + \tfrac{z}{2}\chi_{32}) = 0. \tag{B3}$$

Solving for functional constants $f_1(x,y)$ and $f_2(x,y)$ by using the boundary conditions,

$$@\ z = +\tfrac{h}{2},\ \Upsilon_{yz} = P_2^+ \quad \text{and} \quad @\ z = -\tfrac{h}{2},\ \Upsilon_{yz} = P_2^- \tag{B4}$$

And substituting into equation (B3) and also making use of equation (16) one obtains,

$$\begin{aligned}\Upsilon_{yz} = {} & \frac{3Q_{yz}}{2h}\left(1 - \frac{4z^2}{h^2}\right) - \frac{1}{4}\left[P_2^+\left(1 - \frac{4z}{h} - \frac{12z^2}{h^2}\right) + P_2^-\left(1 + \frac{4z}{h} - \frac{12z^2}{h^2}\right)\right] \\ & + \frac{1}{4}\left(1 - \frac{4z^2}{h^2}\right)\Big\{\omega_{x,x}\left[P_1^+\left(z + \frac{h}{2}\right) + P_1^-\left(z - \frac{h}{2}\right) - \frac{2z}{h}Q_{xz}\right] \\ & - \omega_{y,x}\left[P_2^+\left(z + \frac{h}{2}\right) + P_2^-\left(z - \frac{h}{2}\right) - \frac{2z}{h}Q_{yz}\right]\Big\}.\end{aligned} \tag{B5}$$

APPENDIX C

Details For Solving σ_{zz}

Substituting the results of τ_{xz} and τ_{yz} in the third equilibrium equation (15), we get,

$$
\begin{aligned}
\frac{\partial}{\partial z}(\sigma_{zz}) = & -\frac{3}{2h}\left(1-\frac{4z^2}{h^2}\right)\left[Q_{xz,x}+Q_{yz,y}\right] - \frac{1}{4}\left(1-\frac{4z^2}{h^2}\right)\left\{\omega_{y,y}[R+R_{,x}] - \omega_{x,y}[S+S_{,x}]\right\} \\
& + \frac{1}{4}[T_{,x}] - \frac{1}{4}\left(1-\frac{4z^2}{h^2}\right)\left\{\omega_{x,x}[S+S_{,y}] - \omega_{y,x}[R+R_{,y}]\right\} + \frac{1}{4}[J_{,y}] \\
& + \frac{\partial}{\partial z}\left\{\omega_y\left\{\left(1-\frac{4z^2}{h^2}\right)\left[\frac{3Q_{xz}}{2h} + \frac{1}{4}\omega_{y,y}[R] - \frac{1}{4}\omega_{x,y}[S]\right] - \frac{1}{4}[T]\right\}\right\} \\
& - \frac{\partial}{\partial z}\left\{\omega_x\left\{\left(1-\frac{4z^2}{h^2}\right)\left[\frac{3Q_{yz}}{2h} + \frac{1}{4}\omega_{x,x}[S] - \frac{1}{4}\omega_{y,x}[R]\right] - \frac{1}{4}[J]\right\}\right\} \\
& - \frac{1}{h}L_4 - \frac{12}{h^3}\cdot z\cdot L_5 \,,
\end{aligned}
\qquad \text{(C1)}
$$

where,

$$
\begin{aligned}
[R] &= P_2^{+}\left(z+\frac{h}{2}\right) + P_2^{-}\left(z-\frac{h}{2}\right) - \frac{2z}{h}Q_{yz} \,, \\
[S] &= P_1^{+}\left(z+\frac{h}{2}\right) + P_1^{-}\left(z-\frac{h}{2}\right) - \frac{2z}{h}Q_{xz} \,, \\
[T] &= P_1^{+}\left(1-\frac{4z}{h}-\frac{12z^2}{h^2}\right) + P_1^{-}\left(1+\frac{4z}{h}-\frac{12z^2}{h^2}\right) \,, \\
[J] &= P_2^{+}\left(1-\frac{4z}{h}-\frac{12z^2}{h^2}\right) + P_2^{-}\left(1+\frac{4z}{h}-\frac{12z^2}{h^2}\right) \,, \\
L_4 &= (N_{xy}\omega_x - N_{xx}\omega_y)_{,x} + (N_{yy}\omega_x - N_{xy}\omega_y)_{,y} \,,
\end{aligned}
$$

and

$$
L_5 = (M_{xy}\omega_x - M_{xx}\omega_y)_{,x} + (M_{yy}\omega_x - M_{xy}\omega_y)_{,y} \,. \qquad \text{(C2)}
$$

Using equation (12) and integrating w.r.t. z,

$$
\begin{aligned}
\nabla_{zz} = & -\frac{1}{4}\left\{\omega_{y,y}[R'+R'_{,x}] - \omega_{x,y}[S'+S'_{,x}] - [T'_{,x}]\right\} \\
& -\frac{1}{4}\left\{\omega_{x,x}[S'+S'_{,y}] - \omega_{y,x}[R'+R'_{,y}] - [J'_{,y}]\right\} \\
& +\omega_y\left\{\left(1-\frac{4z^2}{h^2}\right)\left\{\frac{3Q_{xz}}{2h} + \frac{1}{4}\omega_{y,y}[R] - \frac{1}{4}\omega_{x,y}[S]\right\} - \frac{1}{4}[T]\right\} \\
& -\omega_x\left\{\left(1-\frac{4z^2}{h^2}\right)\left\{\frac{3Q_{yz}}{2h} + \frac{1}{4}\omega_{x,x}[S] - \frac{1}{4}\omega_{y,x}[R]\right\} - \frac{1}{4}[J]\right\} \\
& -\frac{3}{2h}\left(z-\frac{4z^3}{3h^2}\right)L_3 + \frac{z}{2h}\left(1-\frac{8z^3}{3h^2}\right)L_4 - \frac{6z^2}{h^3}L_5 + f(x,y) \ ,
\end{aligned} \qquad (C3)
$$

where,

$$
\begin{aligned}
[R'] &= z\cdot P_2^+\left(\frac{z}{2}+\frac{h}{2}-\frac{2z^2}{3h}-\frac{z^3}{h^2}\right) + z\cdot P_2^-\left(\frac{z}{2}-\frac{h}{2}+\frac{2z^2}{3h}-\frac{z^3}{h^2}\right) - \frac{z^2}{h}\left(1-\frac{2z^2}{h^2}\right)Q_{yz} \\
[S'] &= z\cdot P_1^+\left(\frac{z}{2}+\frac{h}{2}-\frac{2z^2}{3h}-\frac{z^3}{h^2}\right) + z\cdot P_1^-\left(\frac{z}{2}-\frac{h}{2}+\frac{2z^2}{3h}-\frac{z^3}{h^2}\right) - \frac{z^2}{h}\left(1-\frac{2z^2}{h^2}\right)Q_{xz} \\
[T'] &= z\cdot P_1^+\left(1-\frac{2z}{h}-\frac{4z^2}{h^2}\right) + z\cdot P_1^-\left(1+\frac{2z}{h}-\frac{4z^2}{h^2}\right) \\
[J'] &= z\cdot P_2^+\left(1-\frac{2z}{h}-\frac{4z^2}{h^2}\right) + z\cdot P_2^-\left(1+\frac{2z}{h}-\frac{4z^2}{h^2}\right)
\end{aligned}
$$

and $L_3 = P_1\omega_y - P_2\omega_x - P_3$, (C4)

and the functional constants $f(x,y)$ is evaluated using the boundary condition,

$$\nabla_{zz} = P_3^+ \qquad @ \quad Z = +\frac{h}{2} \ .$$

Solving for $f(x,y)$ and substituting the result into equation (C3), ∇_{zz} reduces to the form

$$\sigma_{zz} = \left(1-\frac{4z^2}{h^2}\right)\left\{\frac{1}{4}[R][L_1] - \frac{1}{4}[S][L_2] + [L_6]\right\}$$

$$-\frac{1}{4}\omega_{x,y}[S'_{\frac{h}{2}}-S']_x + \frac{1}{4}\omega_{x,x}[S'_{\frac{h}{2}}-S']_y + \frac{1}{4}\omega_{y,y}[R'_{\frac{h}{2}}-R']_x - \frac{1}{4}\omega_{y,x}[R'_{\frac{h}{2}}-R']_y$$

$$-\frac{1}{4}[T'_{\frac{h}{2}}-T']_x - \frac{1}{4}[J'_{\frac{h}{2}}-J']_y + \frac{\omega_y}{4}[T_{\frac{h}{2}}-T] - \frac{\omega_x}{4}[J_{\frac{h}{2}}-J] + P_3^{+}$$

$$+\frac{3}{2h}\left(\frac{h}{3}-z+\frac{4z^3}{3h^2}\right)L_3 + \left(\frac{z}{2h}-\frac{2z^3}{h^3}\right)L_4 + \left(\frac{3}{2h}-\frac{6z^2}{h^3}\right) \quad , \qquad \text{(C6)}$$

where,

$$[L_1] = \omega_y\omega_{y,y} + \omega_x\omega_{y,x} \quad ,$$

$$[L_2] = \omega_y\omega_{x,y} + \omega_x\omega_{x,x} \quad ,$$

and

$$[L_6] = \frac{3}{2h}(\omega_y Q_{xz} - \omega_x Q_{yz}) \quad . \qquad \text{(C7)}$$

APPENDIX D

Integration of the Variational Equation

The assumed stress distributions given by equations (9), (20) and (21), together with the reduced form of the strain-displacement relationships given by equations (7) and (8), are substituted into equation (23). At this stage a simplifying assumption is made $P_1 = P_2 = 0$ and then carrying out the integration in Reissner's variational equation with respect to Z in the limit of $\pm \frac{h}{z}$ yields,

$$\int_{t_1}^{t_2} \Bigg\{ \iint_S \Bigg\{ \Big[N_{xx}\Big(\overset{\circ}{\epsilon}_{xx} + \frac{h^2}{12} C_x + \frac{h^4}{640} \bar{w}_x''^2\Big) + M_{xx}\Big(K_x + \frac{3h^2}{40} \bar{w}_x' \bar{w}_x''\Big)\Big]$$

$$+ \Big[N_{yy}\Big(\overset{\circ}{\epsilon}_{yy} + \frac{h^2}{12} C_y + \frac{h^4}{640} \bar{w}_y''^2\Big) + M_{yy}\Big(K_y + \frac{3h^2}{40} \bar{w}_y' \bar{w}_y''\Big)\Big]$$

$$+ \Big[\bar{w}' + \frac{\nu}{Eh}(N_{xx} + N_{yy})\Big]\Big[\frac{2h}{3} L_6 + h\beta_3^+ + \frac{h}{2} L_3 + L_5$$

$$+ \frac{11h^2}{240}\big(\omega_{x,y} Q_{xz,x} - \omega_{y,y} Q_{yz,x} - \omega_{x,x} Q_{xz,y} + \omega_{y,x} Q_{yz,y}\big)\Big]$$

$$+ \frac{h^2}{60}\Big[\bar{w}'' + \frac{12}{h^3}\frac{\nu}{E}(M_{xx} + M_{yy})\Big]\Big[L_2 Q_{xz} - L_1 Q_{yz} - 6L_3 + L_4\Big]$$

$$+ N_{xy}\Big[\big(\overset{\circ}{\gamma}_{xx} + \overset{\circ}{\gamma}_{yy} + \bar{w}_x \bar{w}_y\big) + \frac{h^2}{12} E_{xy} + \frac{h^4}{320} \bar{w}_x'' \bar{w}_y''\Big]$$

$$+ M_{xy}\Big[\delta_{xx} + \delta_{yy} + D_{xy} + \frac{3h^2}{20} F_{xy}\Big]$$

$$+ \overset{\circ}{\gamma}_{xz} Q_{xz} + \frac{h^2}{60} \bar{w}_x' \big(\omega_{x,y} Q_{xz} - \omega_{y,y} Q_{yz}\big) + \frac{h^2}{40} \bar{w}_x'' Q_{xz}$$

$$+ \overset{\circ}{\gamma}_{yz} Q_{yz} + \frac{h^2}{60} \bar{w}_y' \big(\omega_{y,x} Q_{yz} - \omega_{x,x} Q_{xz}\big) + \frac{h^2}{40} \bar{w}_y'' Q_{yz}$$

$$- \frac{1}{2E}\Bigg\{ \frac{1}{h}\big(N_{xx}^2 + N_{yy}^2\big) + \frac{12}{h^3}\big(M_{xx}^2 + M_{yy}^2\big)$$

$$+\frac{h}{210}\left[L_1^2 Q_{yz}^2 + L_2^2 Q_{xz}^2 - 2L_1L_2 Q_{xz}Q_{yz}\right] + \frac{8h}{15}L_6^2$$

$$+\frac{179h^3}{16\times48\times105}\left[(\omega_{x,y})^2(Q_{xz,x})^2 + (\omega_{x,x})^2(Q_{xz,y})^2 + (\omega_{y,y})^2(Q_{yz,x})^2 + (\omega_{y,x})^2(Q_{yz,y})^2\right]$$

$$+\frac{13h}{35}L_3^2 + \frac{h}{210}L_4^2 + \frac{6}{5h}L_5^2 + \frac{23h^2}{420}\left[\omega_{x,y}Q_{xz,x} - \omega_{x,x}Q_{xz,y}\right.$$

$$\left. - \omega_{y,y}Q_{yz,x} + \omega_{y,x}Q_{yz,y}\right]L_6 + \frac{4h}{3}p_3^+L_6 + \frac{8}{5}L_5L_6$$

$$+\frac{h^2}{105}L_3\left[L_1Q_{yz}\left(\frac{9}{2h}\right) - L_2Q_{xz}\left(\frac{9}{2h}\right) + L_6\left(\frac{70}{h}\right)\right] + \frac{h}{105}L_4\left[L_2Q_{xz} - L_1Q_{yz}\right]$$

$$+\frac{11h^2}{480}p_3^+\left[\omega_{x,y}Q_{xz,x} - \omega_{x,x}Q_{xz,y} - \omega_{y,y}Q_{yz,x} + \omega_{y,x}Q_{yz,y}\right]$$

$$+p_3^+L_3h + 2p_3^+L_5 - \frac{179h^3}{8\times16\times105}\omega_{x,y}Q_{xz,x}\left[\omega_{x,x}Q_{xz,y} + \omega_{y,y}Q_{yz,x}\right.$$

$$\left. - \omega_{y,x}Q_{yz,y}\right] + \frac{77h^2}{16\times105}\omega_{x,y}L_3Q_{xz,x} + \frac{23h}{280}\omega_{x,y}L_5Q_{xz,x}$$

$$+\frac{179h^3}{8\times16\times105}\omega_{x,x}Q_{xz,y}\left[\omega_{y,y}Q_{yz,x} - \omega_{y,x}Q_{yz,y}\right] - \frac{77h^2}{16\times105}\omega_{x,x}L_3Q_{xz,y}$$

$$-\frac{23h}{280}\omega_{x,x}L_5Q_{xz,y} - \frac{179h^3}{8\times16\times105}\omega_{y,y}\omega_{y,x}Q_{yz,x}Q_{yz,y}$$

$$-\frac{77h^2}{16\times105}\omega_{y,y}L_3Q_{yz,x} - \frac{23h}{280}\omega_{y,y}L_5Q_{yz,x} + \frac{77h^2}{16\times105}\omega_{y,x}L_3Q_{yz,y}$$

$$+\frac{23h}{280}\omega_{y,x}L_5Q_{yz,y} - \frac{3h}{70}L_3L_4 + L_3L_5$$

$$-2\nu\left[\frac{1}{h}N_{xx}N_{yy} + \frac{12}{h^3}M_{xx}M_{yy}\right]\Bigg\}$$

$$-\frac{(1+\nu)}{E}\Bigg\{\left[\frac{1}{h}N_{xy}^2 + \frac{12}{h^3}M_{xy}^2\right]$$

$$+\frac{6}{5h}Q_{xz}^2 + \frac{h}{210}\left(\omega_{y,y}Q_{yz} - \omega_{x,y}Q_{xz}\right)^2$$

$$+\frac{6}{5h}Q_{yz}^2 + \frac{h}{210}\left(\omega_{x,x}Q_{xz} - \omega_{y,x}Q_{yz}\right)^2\Bigg\}$$

$$-\frac{\rho}{2}\Bigg\{h\left(\bar{u}_t^2 + \bar{v}_t^2 + \bar{w}_t^2\right) + \frac{h^3}{12}\left[(\omega_{x,t})^2 + (\omega_{y,t})^2 + \bar{w}_t'^2 + \bar{w}_t\bar{w}_t''\right] + \frac{h^5}{320}\bar{w}_t''^2\Bigg\}$$

$$-\left[p_3^+\left(\bar{w} + z\bar{w}' + \frac{z^2}{2}\bar{w}''\right)^+ - p_3^-\left(\bar{w} + z\bar{w}' + \frac{z^2}{2}\bar{w}''\right)^-\right]\Bigg\}dx\,dy\Bigg\}dt = 0. \qquad \text{(D1)}$$

APPENDIX E

Definitions of functional constants C used in equations (24)

$$C_{\bar{W}_x} = (\bar{W}_x + \tfrac{h^2}{24}\bar{W}_x'')N_{xx} + \bar{W}_x' M_{xx} + (\bar{W}_y + \tfrac{h^2}{24}\bar{W}_y'')N_{xy} + \bar{W}_y' M_{xy} + Q_{xz} ,$$

$$C_{\bar{W}_y} = (\bar{W}_y + \tfrac{h^2}{24}\bar{W}_y'')N_{yy} + \bar{W}_y' M_{yy} + (\bar{W}_x + \tfrac{h^2}{24}\bar{W}_x'')N_{xy} + \bar{W}_x' M_{xy} + Q_{yz} ,$$

$$C_{\bar{W}_x'} = \tfrac{h^2}{12}\bar{W}_x' N_{xx} + (\bar{W}_x + \tfrac{3h^2}{40}\bar{W}_x'')M_{xx} + \tfrac{h^2}{12}\bar{W}_y' N_{xy} + (\bar{W}_y + \tfrac{3h^2}{40}\bar{W}_y'')M_{xy} + \tfrac{h^2}{60}(\omega_{x,y} Q_{xz} - \omega_{y,y} Q_{yz}) ,$$

$$C_{\bar{W}_y'} = \tfrac{h^2}{12}\bar{W}_y' N_{yy} + (\bar{W}_y + \tfrac{3h^2}{40}\bar{W}_y'')M_{yy} + \tfrac{h^2}{12}\bar{W}_x' N_{xy} + (\bar{W}_x + \tfrac{3h^2}{40}\bar{W}_x'')M_{xy} + \tfrac{h^2}{60}(\omega_{y,x} Q_{yz} - \omega_{x,x} Q_{xz}) ,$$

$$C_{\bar{W}'} = \tfrac{2h}{3}[L_6] + \tfrac{11h^2}{240}(\omega_{x,y} Q_{xz,x} - \omega_{y,y} Q_{yz,x} - \omega_{x,x} Q_{xz,y} + \omega_{y,x} Q_{yz,y}) + h\beta_3^+ + \tfrac{h}{2}L_3 + L_5 ,$$

$$C_{\bar{W}_x''} = \tfrac{h^2}{24}\left[(\bar{W}_x + \tfrac{3h^2}{40}\bar{W}_x'')N_{xx} + (\bar{W}_y + \tfrac{3h^2}{40}\bar{W}_y'')N_{xy}\right] + \tfrac{3h^2}{40}(\bar{W}_x' M_{xx} + \bar{W}_y' M_{xy}) + \tfrac{h^2}{40} Q_{xz} ,$$

$$C_{\bar{W}_y''} = \tfrac{h^2}{24}\left[(\bar{W}_y + \tfrac{3h^2}{40}\bar{W}_y'')N_{yy} + (\bar{W}_x + \tfrac{3h^2}{40}\bar{W}_x'')N_{xy}\right] + \tfrac{3h^2}{40}(\bar{W}_y' M_{yy} + \bar{W}_x' M_{xy}) + \tfrac{h^2}{40} Q_{yz} ,$$

$$C_{\bar{w}''} = \frac{h^2}{60}[L_2]Q_{xz} - \frac{h^2}{60}[L_1]Q_{yz} - \frac{h^2}{10}L_3 + \frac{h^2}{60}L_4 ,$$

$$C_{N_{xxx}} = \frac{h^2}{60}\omega_y\left[\bar{w}''+\frac{\nu}{E}\cdot\frac{12}{h^3}(M_{xx}+M_{yy})\right] - \frac{1}{2E}\left\{\frac{h}{105}\omega_y\left[L_4 + L_2Q_{xz} - L_1Q_{yz}\right] - \frac{3h}{70}\omega_y L_3\right\} ,$$

$$C_{N_{xx}} = \left(\overset{\circ}{\epsilon}_{xx} + \frac{h^2}{12}C_x + \frac{h^4}{640}\bar{w}_x''^2\right) - \frac{h^2}{60}\omega_{y,x}\left[\bar{w}''+\frac{\nu}{E}\cdot\frac{12}{h^3}(M_{xx}+M_{yy})\right]$$

$$-\frac{1}{Eh}(N_{xx}-\nu N_{yy}) + \frac{\omega_{y,x}}{2E}\left\{\frac{h}{105}\left[L_4+L_2Q_{xz}-L_1Q_{yz}\right] + \frac{3h}{70}L_3\right\} + \frac{\nu}{Eh}\left\{\frac{2h}{3}L_6 + hP_3^+ + \frac{h}{2}L_3 + L_5\right.$$

$$\left. + \frac{11h^2}{240}\left[\omega_{x,y}Q_{xz,x} - \omega_{y,y}Q_{yz,x} - \omega_{x,x}Q_{xz,y} + \omega_{y,x}Q_{yz,y}\right]\right\} ,$$

$$C_{N_{xyx}} = \frac{h^2}{60}\omega_x\left[\bar{w}''+\frac{\nu}{E}\cdot\frac{12}{h^3}(M_{xx}+M_{yy})\right] - \frac{\omega_x}{2E}\left\{\frac{h}{105}\left[L_4 + L_2Q_{xz} - L_1Q_{yz}\right] - \frac{3h}{70}L_3\right\} ,$$

$$C_{N_{xyy}} = \frac{h^2}{60}\omega_y\left[\bar{w}''+\frac{\nu}{E}\cdot\frac{12}{h^3}(M_{xx}+M_{yy})\right] - \frac{\omega_y}{2E}\left\{\frac{h}{105}\left[L_4 + L_2Q_{xz} - L_1Q_{yz}\right] - \frac{3h}{70}L_3\right\} ,$$

$$C_{N_{xy}} = \left(\overset{\circ}{\gamma}_{xx} + \overset{\circ}{\gamma}_{yy} + \bar{w}_x\bar{w}_y + \frac{h^2}{12}E_{xy} + \frac{h^4}{320}\bar{w}_x''\bar{w}_y''\right)$$

$$-\frac{2(1+\nu)}{Eh}N_{xy} + (\omega_{x,x}-\omega_{y,y})\left\{\frac{h^2}{60}\left[\bar{w}''+\frac{\nu}{E}\frac{12}{h^3}(M_{xx}+M_{yy})\right] - \frac{1}{2E}\left\{\frac{h}{105}\left[L_4 + L_2Q_{xz} - L_1Q_{yz}\right] - \frac{3h}{70}L_3\right\}\right\} ,$$

$$C_{N_{yyy}} = \frac{h^2}{60}\omega_x\left[\bar{w}''+\frac{\nu}{E}\cdot\frac{12}{h^3}(M_{xx}+M_{yy})\right] - \frac{\omega_x}{2E}\left\{\frac{h}{105}\left[L_4 + L_2Q_{xz} - L_1Q_{yz}\right] - \frac{3h}{70}L_3\right\} ,$$

$$C_{N_{yy}} = \left(\overset{\circ}{\epsilon}_{yy} + \frac{h^2}{12}C_y + \frac{h^4}{640}\bar{w}_y''^2\right) + \frac{h^2}{60}\omega_{x,y}\left[\bar{w}''+\frac{\nu}{E}\cdot\frac{12}{h^3}(M_{xx}+M_{yy})\right]$$

$$-\frac{1}{Eh}(N_{yy}-\nu N_{xx}) - \frac{\omega_{x,y}}{2E}\left\{\frac{h}{105}\left[L_4+L_2Q_{xz}-L_1Q_{yz}\right]\right.$$

$$- \frac{3h}{70} L_3 \Big\} \quad + \frac{\nu}{Eh} \Big\{ \frac{2h}{3} L_6 + h P_3^+ + \frac{h}{2} L_3 + L_5$$

$$+ \frac{11h^2}{240} \Big[\omega_{x,y} Q_{xz,x} - \omega_{y,y} Q_{yz,x} - \omega_{x,x} Q_{xz,y} + \omega_{y,x} Q_{yz,y} \Big] \Big\} ,$$

$$C_{M_{xxx}} = \omega_y \Big[\bar{w}' + \frac{\nu}{Eh} (N_{xx} + N_{yy}) \Big] - \frac{\omega_y}{2E} \Big\{ \frac{12}{5h} L_5 + L_3 + 2P_3^+ + \frac{8}{5} L_6$$

$$+ \frac{69h}{8 \times 105} \Big[\omega_{x,y} Q_{xz,x} - \omega_{x,x} Q_{xz,y} - \omega_{y,y} Q_{yz,x} + \omega_{y,x} Q_{yz,y} \Big] \Big\} ,$$

$$C_{M_{xx}} = \Big(K_x + \frac{3h^2}{40} \bar{W}_x' \bar{W}_x'' \Big) - \frac{12}{Eh^3} (M_{xx} - \nu M_{yy}) - \omega_{y,x} \Big[\bar{w}' + \frac{\nu}{Eh} (N_{xx} + N_{yy}) \Big]$$

$$+ \frac{\nu}{5Eh} \Big[L_4 + L_2 Q_{xz} - L_1 Q_{yz} - 6 L_3 \Big] + \frac{\omega_{y,x}}{2E} \Big\{ \frac{12}{5h} L_5 + \frac{8}{5} L_6 + 2P_3^+$$

$$+ L_3 + \frac{69h}{8 \times 105} \Big[\omega_{x,y} Q_{xz,x} - \omega_{x,x} Q_{xz,y} - \omega_{y,y} Q_{yz,x} + \omega_{y,x} Q_{yz,y} \Big] \Big\} ,$$

$$C_{M_{xyx}} = \omega_x \Big[\bar{w}' + \frac{\nu}{Eh} (N_{xx} + N_{yy}) \Big] - \frac{\omega_x}{2E} \Big\{ \frac{12}{5h} L_5 + L_3 + 2P_3^+ + \frac{8}{5} L_6$$

$$+ \frac{69h}{8 \times 105} \Big[\omega_{x,y} Q_{xz,x} - \omega_{x,x} Q_{xz,y} - \omega_{y,y} Q_{yz,x} + \omega_{y,x} Q_{yz,y} \Big] \Big\} ,$$

$$C_{M_{xyy}} = \omega_y \Big[\bar{w}' + \frac{\nu}{Eh} (N_{xx} + N_{yy}) \Big] - \frac{\omega_y}{2E} \Big\{ \frac{12}{5h} L_5 + L_3 + 2P_3^+ + \frac{8}{5} L_6$$

$$+ \frac{69h}{8 \times 105} \Big[\omega_{x,y} Q_{xz,x} - \omega_{x,x} Q_{xz,y} - \omega_{y,y} Q_{yz,x} + \omega_{y,x} Q_{yz,y} \Big] \Big\} ,$$

$$C_{M_{xy}} = \Big(\delta_{xx} + \delta_{yy} + D_{xy} + \frac{3h^2}{20} F_{xy} \Big) - \frac{24(1+\nu)}{Eh^3} M_{xy} + (\omega_{x,x} - \omega_{y,y}) \Big\{$$

$$\Big[\bar{w}' + \frac{\nu}{Eh} (N_{xx} + N_{yy}) \Big] - \frac{1}{2E} \Big\{ \frac{12}{5h} L_5 + L_3 + 2P_3^+ + \frac{8}{5} L_6$$

$$+ \frac{69h}{8 \times 105} \Big[\omega_{x,y} Q_{xz,x} - \omega_{x,x} Q_{xz,y} - \omega_{y,y} Q_{yz,x} + \omega_{y,x} Q_{yz,y} \Big] \Big\} \Big\} ,$$

$$C_{M_{yyy}} = \omega_x \Big[\bar{w}' + \frac{\nu}{Eh} (N_{xx} + N_{yy}) \Big] - \frac{\omega_x}{2E} \Big\{ \frac{12}{5h} L_5 + 2P_3^+ + L_3 + \frac{8}{5} L_6$$

$$+ \frac{69h}{8 \times 105} \Big[\omega_{x,y} Q_{xz,x} - \omega_{x,x} Q_{xz,y} - \omega_{y,y} Q_{yz,x} + \omega_{y,x} Q_{yz,y} \Big] \Big\} ,$$

$$C_{M_{yy}} = \Big(K_y + \frac{3h^2}{40} \bar{W}_y' \bar{W}_y'' \Big) - \frac{12}{Eh^3} (M_{yy} - \nu M_{xx}) + \omega_{x,y} \Big[\bar{w}' + \frac{\nu}{Eh} (N_{xx} + N_{yy}) \Big]$$

$$+ \frac{\nu}{5Eh} \Big[L_4 + L_2 Q_{xz} - L_1 Q_{yz} - 6 L_3 \Big] - \frac{\omega_{x,y}}{2E} \Big\{ \frac{12}{5h} L_5 + L_3 + 2P_3^+ + \frac{8}{5} L_6$$

$$- \frac{69h}{8\times 105}\left[\omega_{x,y} Q_{xz,x} - \omega_{x,x} Q_{xz,y} - \omega_{y,y} Q_{yz,x} + \omega_{y,x} Q_{yz,y}\right]\Big\} \quad ,$$

$$C_{Q_{xzx}} = \frac{11h^2}{240}\omega_{x,y}\left[\bar{w}' + \frac{\nu}{Eh}(N_{xx}+N_{yy})\right] - \frac{\omega_{x,y}}{2E}\Big\{\frac{23h}{280}L_5 + \frac{23h^2}{420}L_6$$

$$+ \frac{11h^2}{480}P_3^+ + \frac{77h^2}{16\times 105}L_3\Big\} - \frac{\omega_{x,y}}{2E}\cdot\frac{179h^3}{16\times 24\times 105}\left[\omega_{x,y} Q_{xz,x}\right.$$

$$\left.- \omega_{x,x} Q_{xz,y} - \omega_{y,y} Q_{yz,x} + \omega_{y,x} Q_{yz,y}\right] \quad ,$$

$$C_{Q_{xzy}} = \frac{11h^2}{240}\omega_{x,x}\left[\bar{w}' + \frac{\nu}{Eh}(N_{xx}+N_{yy})\right] - \frac{\omega_{x,x}}{2E}\left[\frac{23h}{280}L_5 + \frac{23h^2}{420}L_6\right.$$

$$\left.+ \frac{11h^2}{480}P_3^+ + \frac{77h^2}{16\times 105}L_3\right] + \frac{\omega_{x,x}}{2E}\cdot\frac{179h^3}{16\times 24\times 105}\left[\omega_{x,y} Q_{xz,x}\right.$$

$$\left.- \omega_{x,x} Q_{xz,y} - \omega_{y,y} Q_{yz,x} + \omega_{y,x} Q_{yz,y}\right] \quad ,$$

$$C_{Q_{xz}} = \left(\overset{\circ}{\gamma}_{xz} + \frac{h^2}{40}\bar{w}_x''\right) + \frac{h^2}{60}\left(\omega_{x,y}\bar{w}_x' - \omega_{x,x}\bar{w}_y'\right) + \omega_y\left[\bar{w}' + \frac{\nu}{Eh}(N_{xx} +\right.$$

$$\left.N_{yy})\right] + \frac{h^2}{60}L_2\left[\bar{w}'' + \frac{\nu}{E}\cdot\frac{12}{h^3}(M_{xx}+M_{yy})\right] - \frac{1}{2E}\Big\{$$

$$L_2\frac{h}{105}\left[L_4 + L_2 Q_{xz} - L_1 Q_{yz}\right] + \omega_y\left[\frac{12}{5h}L_5 + L_3 + 2P_3^+ + \frac{8}{5}L_6\right]$$

$$+ \frac{23h}{280}\omega_y\left[\omega_{x,y} Q_{xz,x} - \omega_{x,x} Q_{xz,y} - \omega_{y,y} Q_{yz,x} + \omega_{y,x} Q_{yz,y}\right]\Big\}$$

$$- \frac{(1+\nu)}{E}\Big\{\frac{12}{5h} Q_{xz} + \frac{h}{105}\Big[\omega_{x,y}\left(\omega_{x,y} Q_{xz} - \omega_{y,y} Q_{yz}\right)$$

$$+ \omega_{x,x}\left(\omega_{x,x} Q_{xz} - \omega_{y,x} Q_{yz}\right)\Big]\Big\} \quad ,$$

$$C_{Q_{yzx}} = \frac{11h^2}{240}\omega_{y,y}\left[\bar{w}' + \frac{\nu}{Eh}(N_{xx}+N_{yy})\right] - \frac{\omega_{y,y}}{2E}\Big\{\frac{23h}{280}L_5 + \frac{23h^2}{420}L_6$$

$$+ \frac{11h^2}{480}P_3^+ + \frac{77h^2}{16\times 105}L_3\Big\} - \frac{\omega_{y,y}}{2E}\cdot\frac{179h^3}{16\times 24\times 105}\left[\omega_{x,y} Q_{xz,x}\right.$$

$$\left.- \omega_{x,x} Q_{xz,y} - \omega_{y,y} Q_{yz,x} + \omega_{y,x} Q_{yz,y}\right] \quad ,$$

$$
\begin{aligned}
C_{Q_{yzy}} = {} & \frac{11h^2}{240}\omega_{y,x}\left[\bar{w}' + \frac{\nu}{Eh}(N_{xx}+N_{yy})\right] - \frac{\omega_{y,x}}{2E}\left[\frac{23h}{280}L_5 + \frac{23h^2}{420}L_6 \right. \\
& \left. + \frac{11h^2}{480}p_3^{+} + \frac{11h^2}{240}L_3\right] - \frac{\omega_{y,x}}{2E}\cdot\frac{179h^3}{16\times 24\times 105}\left[\omega_{x,y}Q_{xz,x}\right. \\
& \left. - \omega_{x,x}Q_{xz,y} - \omega_{y,y}Q_{yz,x} + \omega_{y,x}Q_{yz,y}\right] ,
\end{aligned}
$$

$$
\begin{aligned}
C_{Q_{yz}} = {} & \left(\overset{\circ}{\gamma}_{yz} + \frac{h^2}{40}\bar{w}_y''\right) + \frac{h^2}{60}\left(\omega_{y,x}\bar{w}_y' - \omega_{y,y}\bar{w}_x'\right) \\
& - \omega_x\left[\bar{w}' + \frac{\nu}{Eh}(N_{xx}+N_{yy})\right] - \frac{h^2}{60}L_1\left[\bar{w}'' + \frac{\nu}{E}\frac{12}{h^3}(M_{xx}+M_{yy})\right] \\
& - \frac{1}{2E}\left\{\frac{h}{105}L_1\left[L_1Q_{yz} - L_2Q_{xz} - L_4\right] - \omega_x\left[\frac{12}{5h}L_5 + L_3 + 2p_3^{+} + \frac{8}{5}L_6\right]\right. \\
& \left. - \frac{23h}{280}\omega_x\left[\omega_{x,y}Q_{xz,x} - \omega_{x,x}Q_{xz,y} - \omega_{y,y}Q_{yz,x} + \omega_{y,x}Q_{yz,y}\right]\right\} \\
& - \frac{(1+\nu)}{E}\left\{\frac{12}{5h}Q_{yz} + \frac{h}{105}\left[\omega_{y,y}\left(\omega_{y,y}Q_{yz} - \omega_{x,y}Q_{xz}\right)\right.\right. \\
& \left.\left. + \omega_{y,x}\left(\omega_{y,x}Q_{yz} - \omega_{x,x}Q_{xz}\right)\right]\right\} ,
\end{aligned}
$$

$$
\begin{aligned}
C_{\omega_{yx}} = {} & \left[\frac{11h^2}{240}Q_{yz,y} - M_{xx}\right]\left[\bar{w}' + \frac{\nu}{Eh}(N_{xx}+N_{yy}) - \frac{1}{2E}L_3\right] \\
& - \frac{1}{2E}\frac{h^2}{280}\left[\frac{179h}{144}Q_{yz,y} - \frac{23}{h}M_{xx}\right]\left[\omega_{y,x}Q_{yz,y} + \omega_{x,y}Q_{xz,x}\right. \\
& \left. - \omega_{x,x}Q_{xz,y} - \omega_{y,y}Q_{yz,x}\right] - \frac{1}{2E}\frac{23h}{140}Q_{yz,y}\left(\frac{L_5}{2} + \frac{L_6}{3}\right) \\
& + \frac{1}{2E}\cdot\frac{24}{5h}M_{xx}\left(\frac{L_5}{2} + h\cdot\frac{L_6}{3}\right) + \left[M_{xy} + \frac{h^2}{60}\bar{w}_y' Q_{yz}\right] \\
& - \frac{p_3^{+}}{2E}\left[\frac{11h^2}{480}Q_{yz,y} - 2M_{xx}\right] - \frac{(1+\nu)}{E}\cdot\frac{h}{105}Q_{yz}\left[\omega_{y,x}Q_{yz} - \omega_{x,x}Q_{xz}\right] \\
& - \left[\omega_x Q_{yz} + N_{xx}\right]\left\{\frac{h^2}{60}\left[\bar{w}'' + \frac{\nu}{E}\frac{12}{h^3}(M_{xx}+M_{yy})\right] + \frac{1}{2E}\left[\frac{h}{105}\left(\right.\right.\right. \\
& \left.\left.\left. L_1Q_{yz} - L_2Q_{xz} - L_4\right) + \frac{3h}{70}L_3\right]\right\} ,
\end{aligned}
$$

$$C_{\omega yy} = \frac{1}{2E}\frac{h^2}{280}\left[\frac{179h}{144} Q_{yz,x} + \frac{23}{h} M_{xy}\right]\left[\omega_{y,x} Q_{yz,y} + \omega_{x,y} Q_{xz,x} - \omega_{x,x} Q_{xz,y} - \omega_{y,y} Q_{yz,x}\right] + \frac{1}{2E}\frac{23h}{140} Q_{yz,x}\left(\frac{L_5}{2} + \frac{L_6}{3}\right)$$

$$+ \frac{1}{2E}\frac{24}{5h} M_{xy}\left(\frac{L_5}{2} + h\cdot\frac{L_6}{3}\right) + \left[M_{yy} - \frac{h^2}{60}\bar{w}'_x Q_{yz}\right]$$

$$-\left[\frac{11h^2}{240} Q_{yz,x} + M_{xy}\right]\left[\bar{w}' + \frac{\nu}{Eh}(N_{xx} + N_{yy}) - \frac{1}{2E} L_3\right]$$

$$+ \frac{p_3^+}{2E}\left[\frac{11h^2}{480} Q_{yz,x} + 2M_{xy}\right] - \frac{(1+\nu)}{E}\cdot\frac{h}{105} Q_{yz}\left[\omega_{y,y} Q_{yz} - \omega_{x,y} Q_{xz}\right]$$

$$-\left[\omega_y Q_{yz} + N_{xy}\right]\left\{\frac{h^2}{60}\left[\bar{w}'' + \frac{\nu}{E}\frac{12}{h^3}(M_{xx} + M_{yy})\right] + \frac{1}{2E}\left[\frac{h}{105}\left(L_1 Q_{yz} - L_2 Q_{xz} - L_4\right) + \frac{3h}{70} L_3\right]\right\} ,$$

$$C_{\omega y} = \frac{1}{2E}\frac{23h}{280}\left[(M_{xx,x} + M_{xy,y}) - Q_{xz}\right]\left[\omega_{y,x} Q_{yz,y} + \omega_{x,y} Q_{xz,x} - \omega_{x,x} Q_{xz,y} - \omega_{y,y} Q_{yz,x}\right] + \left[Q_{xz} - (M_{xx,x} + M_{xy,y})\right]\left[\bar{w}'\right.$$

$$\left. + \frac{\nu}{Eh}(N_{xx} + N_{yy}) - \frac{1}{2E}\left(\frac{12}{5h} L_5 + L_3 + 2p_3^+ + \frac{8}{5} L_6\right)\right]$$

$$-\left[\omega_{y,y} Q_{yz} - \omega_{x,y} Q_{xz} + (N_{xx,x} + N_{xy,y})\right]\left\{\frac{h^2}{60}\left[\bar{w}'' + \frac{\nu}{E}\frac{12}{h^3}(M_{xx} + M_{yy})\right]\right.$$

$$\left. + \frac{1}{2E}\left[\frac{h}{105}\left(L_1 Q_{yz} - L_2 Q_{xz} - L_4\right) + \frac{3h}{70} L_3\right]\right\} + Q_{yz} ,$$

$$C_{\omega xx} = \left[M_{xy} - \frac{11h^2}{240} Q_{xz,y}\right]\left[\bar{w}' + \frac{\nu}{Eh}(N_{xx} + N_{yy}) - \frac{1}{2E} L_3\right]$$

$$+ \frac{1}{2E}\frac{h^2}{280}\left[\frac{179h}{144} Q_{xz,y} - \frac{23}{h} M_{xy}\right]\left[\omega_{y,x} Q_{yz,y} + \omega_{x,y} Q_{xz,x} - \omega_{x,x} Q_{xz,y} - \omega_{y,y} Q_{yz,x}\right] + \frac{1}{2E}\frac{23h}{140} Q_{xz,y}\left(\frac{L_5}{2} + \frac{L_6}{3}\right)$$

$$- \frac{1}{2E}\frac{24}{5h} M_{xy}\left(\frac{L_5}{2} + h\frac{L_6}{3}\right) + \left[M_{xx} - \frac{h^2}{60}\bar{w}'_y Q_{xz}\right]$$

$$+ \frac{p_3^+}{2E}\left[\frac{11h^2}{480} Q_{xz,y} - 2M_{xy}\right] - \frac{(1+\nu)}{E}\frac{h}{105} Q_{xz}\left[\omega_{x,x} Q_{xz} - \omega_{y,x} Q_{yz}\right]$$

$$+ \left[\omega_x Q_{yz} + N_{xx}\right]\left\{\frac{h^2}{60}\left[\bar{w}'' + \frac{\nu}{E}\frac{12}{h^3}(M_{xx}+M_{yy})\right] + \frac{1}{2E}\left[\frac{h}{105}\left(L_1 Q_{yz} - L_2 Q_{xz} - L_4\right) + \frac{3h}{70}L_3\right]\right\},$$

$$C_{\omega xy} = \left[\frac{11h^2}{240}Q_{xz,x} + M_{yy}\right]\left[\bar{w}' + \frac{\nu}{Eh}(N_{xx}+N_{yy}) - \frac{1}{2E}L_3\right]$$

$$-\frac{1}{2E}\frac{h^2}{280}\left[\frac{179h}{144}Q_{xz,x} + \frac{23}{h}M_{yy}\right]\left[\omega_{y,x}Q_{yz,y} + \omega_{x,y}Q_{xz,x} - \omega_{x,x}Q_{xz,y} - \omega_{y,y}Q_{yz,x}\right] - \frac{1}{2E}\frac{23h}{140}Q_{xz,x}\left[\frac{L_5}{2} + \frac{L_6}{3}\right]$$

$$-\frac{1}{2E}\frac{24}{5h}M_{yy}\left[\frac{L_5}{2} + h\cdot\frac{L_6}{3}\right] + \left[M_{xy} + \frac{h^2}{60}\bar{w}_x' Q_{xz}\right]$$

$$-\frac{P_3^+}{2E}\left[\frac{11h^2}{480}Q_{xz,x} + 2M_{yy}\right] - \frac{(1+\nu)}{E}\frac{h}{105}Q_{xz}\left(\omega_{x,y}Q_{xz} - \omega_{y,y}Q_{yz}\right)$$

$$+\left[\omega_y Q_{xz} + N_{yy}\right]\left\{\frac{h^2}{60}\left[\bar{w}'' + \frac{\nu}{E}\frac{12}{h^3}(M_{xx}+M_{yy})\right] + \frac{1}{2E}\left[\frac{h}{105}\left(L_1 Q_{yz} - L_2 Q_{xz} - L_4\right) + \frac{3h}{70}L_3\right]\right\},$$

$$C_{\omega x} = \frac{1}{2E}\frac{23h}{280}\left[Q_{yz} - (M_{xy,x} + M_{yy,y})\right]\left[\omega_{y,x}Q_{yz,y} + \omega_{x,y}Q_{xz,x} - \omega_{x,x}Q_{xz,y} - \omega_{y,y}Q_{yz,x}\right] + \left[(M_{xy,x} + M_{yy,y}) - Q_{yz}\right]\left[\bar{w}'\right.$$

$$\left. + \frac{\nu}{Eh}(N_{xx}+N_{yy}) - \frac{1}{2E}\left(\frac{12}{5h}L_5 + L_3 + 2P_3^+ + \frac{8}{5}L_6\right)\right]$$

$$+\left\{\frac{h^2}{60}\left[\bar{w}'' + \frac{\nu}{E}\frac{12}{h^3}(M_{xx}+M_{yy})\right] + \frac{1}{2E}\left[\frac{h}{105}\left(L_1 Q_{yz} - L_2 Q_{xz} - L_4\right) + \frac{3h}{70}L_3\right]\right.$$

$$\left.\right\}\left[\omega_{x,x}Q_{xz} - \omega_{y,x}Q_{yz} + (N_{xy,x} + N_{yy,y})\right] + Q_{xz} \qquad (E1)$$

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