PARAMETRIC VIBRATION OF BEAM-COLUMNS

by

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ABSTRACT

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The purpose of this thesis is to investigate the parametric vibration of a beam-column including the effects of shearing stress and transverse and rotary inertia.

The effect of shear on the dynamic regions of stability and instability of the beam-column is determined. The theory yields the classical Mathieu-type differential equation. The stable and unstable solutions of the equation are investigated using the Hill-determinant method. The effect of shear stress on the stability of the solutions of the equations is determined mathematically and is pictured graphically. A comparison is made with the classical solution for the special case when shear and rotary inertia are neglected.

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LIST OF NOTATIONS

SYMBOL	DEFINITION
A	Cross sectional area
E	Young 's modulus of elasticity
G	Shear modulus
I	Moment of inertia
L	Length of a beam,
P	Compressive axial force
Q	Moment of area
T	Kinetic energy
V	Potential energy
b	Width of a beam
k	Shape factor for strain energy of shear, see equation A. For rectangular cross section, k = 1.20.
q	Transverse distributed load
r	Radius of gyration
t	Time
x,y	Rectangular coordinates
ρ	Mass density
ν	Poisson 's ratio
β	Slope due to shear
ω	Natural frequency of vibration of a beam
Ω	Natural frequency of vibration of a beam-column

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CHAPTER I

INTRODUCTION

The problem of parametric vibration of a beam-column leads to the classical Mathieu-type differential equations. The exact solutions of such equations are in general not possible. However, the stability characteristics of these equations are well known and are reviewed by Stoker.[5] The regions of stability and instability of the parametric vibration of the beam-column due to flexural stress is investigated by Bolotin.[1]

In this thesis, in addition to flexural stress, the effects of shear stress and rotary inertia on the dynamic motions of beam-columns are investigated. The equations of motion are derived using the Hamilton's principle as considerd by Langhaar.[2]

CHAPTER II

METHOD OF ANALYSIS

2.1 Assumptions and Co-ordinate System

The beam-column herein considered is restricted to the case of a simple support end conditions subjected to the load conditions as shown in Fig. 1. below.

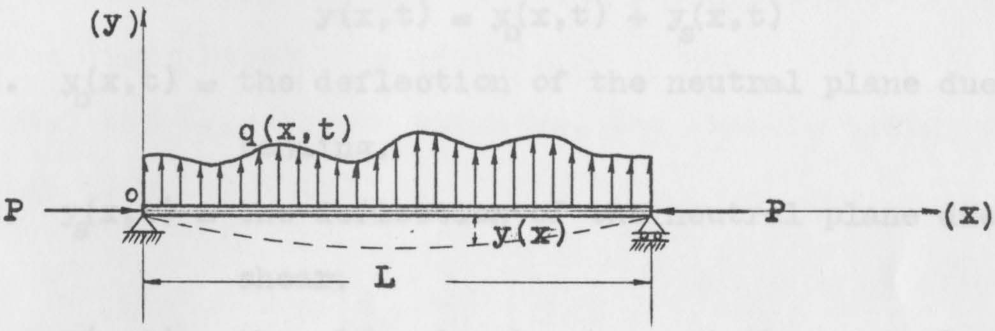

Fig. 1 Load Condition of a Beam-Column

In addition, the following assumptions are introduced:

1. The beam-column is made of a perfectly elastic material.
2. The beam-column is originally perfectly straight.
3. The axial loads are applied along the centroidal axis of the beam-column.

The co-ordinate system refered to throughout this paper will follow that is shown in Fig. 1.

2.2 Potential and Kinetic Energy of the Beam-Column

The total potential energy of a beam-column is the sum of the internal strain energy plus the increase in potential energy of the external loads. The internal strain energy is that caused by the effects of internal bending, axial, and shearing stresses.

Considering the following geometric definitions:

1. $y(x,t)$ = the total deflection of the neutral plane of the beam-column, where

$$y(x,t) = y_b(x,t) + y_s(x,t)$$

2. $y_b(x,t)$ = the deflection of the neutral plane due to bending.
3. $y_s(x,t)$ = the deflection of the neutral plane due to shear.
4. $\beta(x,t)$ = the slope due to shear at the neutral plane (i.e. $\beta = y_x$).
5. k = a geometrical parameter defined by the equation

$$k = \frac{A}{I^2}\int \frac{Q^2\,dy}{b}$$

where A is the cross-sectional area, b is the width of the cross section at ordinate y, Q is the moment of the area above the line with ordinate y about the x-axis, and I is the moment of inertia of the cross section about the x-axis. For rectangular cross section, $k = 1.20$.

Then, the total potential energy of the beam-column (see Fig. 1.) is

$$V = \int_0^L \left[\tfrac{1}{2}EI(y_{xx}-\beta_x)^2 + \tfrac{1}{2}\tfrac{GA}{K}\beta^2 - \tfrac{1}{2}P(y_x)^2 - q(x,t)y \right] dx, \tag{1}$$

where $y = y(x,t)$ and $\beta = \beta(x,t)$.

The first term in the integrand of equation (1) is the internal strain energy due to bending; the second term is the internal strain energy due to shear; the third term is the external potential energy due to the axial compressive force P; and the fourth term is the external potential energy of the transverse distributed load $q(x,t)$.

When the beam-column vibrates, the kinetic energy of the system is written as

$$T = \int_0^L \left[\tfrac{1}{2}\rho A y_t^2 + \tfrac{1}{2}\rho I(y_{xt}-\beta_t)^2 \right] dx. \tag{2}$$

The first term in the integrand of equation (2) is the kinetic energy of translational motion and the second term, the kinetic energy of rotational motion.

The total energy of the beam-column is

$$L = T - V \text{, or}$$

$$= \int_0^L \Big[\tfrac{1}{2}\rho A y_t^2 + \tfrac{1}{2}\rho I(y_{xt}-\beta_t)^2 - \tfrac{1}{2}EI(y_{xx}-\beta_x)^2 - \tfrac{1}{2}\tfrac{GA}{K}\beta^2 +$$

$$\tfrac{1}{2}P(y_x)^2 + q(x,t)y \Big] dx. \tag{3}$$

2.3 Hamilton's Principle

Hamilton's principle is stated as follows:[2] "Among all motions that will carry a conservative system from a given initial configuration to a given final configuration in a given time interval (t_0, t_1), that which actually occurs provides a stationary value to the integral A", where

$$A \equiv \int_{t=t_0}^{t=t_1} L\,dt\,, \qquad \text{and} \quad L = T - V.$$

Thus, Hamilton's principle takes the following mathematical form:

$$\delta A = \delta \int_{t=t_0}^{t=t_1} L\,dt = 0\,. \tag{4}$$

Proceeding with the variational operation defined by equation (4) and integrating the necessary terms by parts, equations (3) and (4) yield

$$\int_{t=t_0}^{t=t_1}\int_0^L \Big[\Big\{ -EI(y_{xxx} - \beta_{xx}) - \frac{GA}{K}\beta + \rho I(y_{xtt} - \beta_{tt}) \Big\} \delta\beta \, +$$

$$\Big\{ -EI(y_{xxxx} - \beta_{xxx}) - P(y_{xx}) - \rho A y_{tt} + \rho I(y_{xxtt} - \beta_{xtt}) + q(x,t) \Big\} \delta y \Big] dx\,dt \, +$$

$$\int_{t=t_0}^{t=t_1} \Big| \Big\{ EI(y_{xxx} - \beta_{xx}) + P(y_x) - \rho I(y_{xtt} - \beta_{tt}) \Big\} \delta y \, +$$

$$\Big\{ -EI(y_{xx} - \beta_x) \Big\} \delta(y_x - \beta) \Big|_0^L dt \, +$$

$$\int_0^L \Big| (\rho A y_t)\delta y + \Big\{ \rho I(y_{xt} - \beta_t) \Big\} \delta(y_x - \beta) \Big|_{t=t_0}^{t=t_1} dx = 0\,. \tag{5}$$

The differential equations of motion of the beam-column are obtained by setting the coefficients of the terms $\delta\beta$ and δy in the first integral of equation (5) equal to zero. Thus,

$$\left.\begin{aligned} EI(y_{xxx}-\beta_{xx})+\frac{GA}{K}\beta-\rho I(y_{xtt}-\beta_{tt}) &= 0\text{, and}\\ EI(y_{xxxx}-\beta_{xxx})+P(y_x)+\rho A y_{tt}-\rho I(y_{xxtt}-\beta_{xtt})+q(x,t) &= 0\,. \end{aligned}\right\} \quad (6)$$

The boundary conditions are obtained by noting the terms of the second integral of equation (5) and take the following forms:

@ x = 0, or x = L

either

$$EI(y_{xxx}-\beta_{xx})+P(y_x)-\rho I(y_{xtt}-\beta_{tt}) = 0 \quad (7.a)$$

or

$$y = 0 \text{, and} \quad (7.b)$$

either

$$EI(y_{xx}-\beta_x) = 0 \quad (7.c)$$

or

$$(y_x-\beta) = 0\,. \quad (7.d)$$

Rearranging equations (6.a) and (6.b) into matrix form yields

$$\begin{bmatrix} -EI\left(\frac{\partial^2}{\partial x^2}\right)+\frac{GA}{K}+\rho I\left(\frac{\partial^2}{\partial t^2}\right) & EI\left(\frac{\partial^3}{\partial x^3}\right)-\rho I\left(\frac{\partial^3}{\partial x\partial t^2}\right) \\ EI\left(\frac{\partial^3}{\partial x^3}\right)-\rho I\left(\frac{\partial^3}{\partial x\partial t^2}\right) & -EI\left(\frac{\partial^4}{\partial x^4}\right)-P\left(\frac{\partial^2}{\partial x^2}\right)-\rho A\left(\frac{\partial^2}{\partial t^2}\right)+\rho I\left(\frac{\partial^4}{\partial x^2\partial t^2}\right) \end{bmatrix}\begin{bmatrix}\beta(x,t)\\ y(x,t)\end{bmatrix}=\begin{bmatrix}0\\ q(x,t)\end{bmatrix}. \quad (8)$$

Equations (8), together with the boundary conditions from equations (7) define the complete boundary value problem of a beam-column including the effects of shear stress and rotary inertia.

2.4 The Effect of Shear on the Critical Static Buckling Load

The critical static buckling load of the shearing beam-column is determined directly from equation (8). The terms involving transverse and rotary inertia are neglected together with the transverse load. Hence, the matrix equation (8) reduces to the form

$$\begin{bmatrix} -EI\left(\frac{d^2}{dx^2}\right)+\frac{GA}{K} & EI\left(\frac{d^3}{dx^3}\right) \\ EI\left(\frac{d^3}{dx^3}\right) & -EI\left(\frac{d^4}{dx^4}\right)-P\left(\frac{d^2}{dx^2}\right) \end{bmatrix} \begin{bmatrix} \beta(x) \\ y(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{9}$$

where $y = y(x)$ and $\beta = \beta(x)$ only.

The linear differential operator which uncouples equation (9) is determined by noting the following determinant

$$\Delta_s = \begin{vmatrix} -EI\left(\frac{d^2}{dx^2}\right)+\frac{GA}{K} & EI\left(\frac{d^3}{dx^3}\right) \\ EI\left(\frac{d^3}{dx^3}\right) & -EI\left(\frac{d^4}{dx^4}\right)-P\left(\frac{d^2}{dx^2}\right) \end{vmatrix}, \tag{10}$$

which simpliflies to the form

$$\Delta_s = -EI\left(\frac{GA}{K}-P\right)\left(\frac{d^4}{dx^4}\right) - \frac{GA}{K}P\left(\frac{d^2}{dx^2}\right) . \tag{11}$$

Thus, the differential equation of transverse displacement of the beam-column is written as

$$\Delta_s y(x) = 0 \quad . \tag{12}$$

The boundary conditions for the case of a simply supported beam-column are

$$\left.\begin{aligned} y(x)\Big|_{x=0}^{x=L} &= 0 \quad , \text{ and} \\ EI\left[y_{xx}(x) - \beta_x(x)\right]\Big|_{x=0}^{x=L} &= 0 \quad . \end{aligned}\right\} \tag{13}$$

The solutions that satisfy exactly the above boundary conditions take the forms

$$\left.\begin{aligned} \beta(x) &= B\frac{n\pi}{L}\cos\left(\frac{n\pi}{L}\right)x \quad , \text{ and} \\ y(x) &= B\sin\left(\frac{n\pi}{L}\right)x \quad . \end{aligned}\right\} \tag{14}$$

Substituting equation (14) into equation (12) yields

$$\left(\frac{n\pi}{L}\right)^4 - \frac{P}{EI\left(1-\frac{KP}{GA}\right)}\left(\frac{n\pi}{L}\right)^2 = 0 \, . \tag{15}$$

Equation (15) yields the value of the critical load as

$$P_{cr_n} = \frac{P_{e_n}}{\left(1+\frac{KP_{e_n}}{GA}\right)} \quad , \tag{16}$$

where P_{cr_n} is the critical load of the beam-column when the effect of shear is included and P_{e_n} is the critical buckling load when shear stress is neglected (i.e., $P_{e_n} = EI(\frac{n\pi}{L})^2$).

Thus, if the effect of shear is encountered, the critical load is decreased. This condition is examined by Timoshenko.[6] When shear effect is neglected, the critical load becomes

$$P_{cr_n} = P_{e_n} \, .$$

A plot of the relation between P_{cr_n} and $\frac{GA}{k}$ as given in equation (16) is shown in Fig. 2.

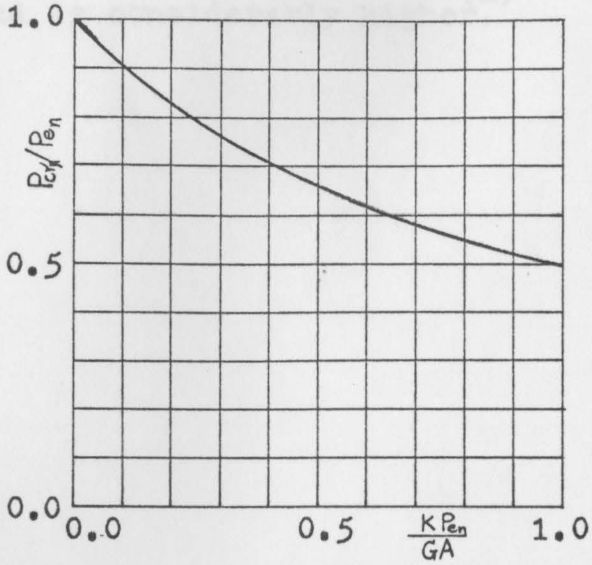

Fig. 2 Graphical Relationship between $\frac{P_{cr_n}}{P_{e_n}}$ and $\frac{kP_{e_n}}{GA}$

Furthermore, both P_{cr_n} and P_{e_n} are expressed in terms of the slenderness ratio $\frac{L}{r}$ as follows:

$$\left.\begin{aligned} \frac{P_{cr_n}}{\pi^2 AE} &= \frac{n^2 C_1}{\left(\frac{L}{r}\right)^2} \quad \text{and} \\ \frac{P_{e_n}}{\pi^2 AE} &= \frac{n^2}{\left(\frac{L}{r}\right)^2}\ , \end{aligned}\right\} \tag{17}$$

where
$$C_1 = \frac{1}{1+\frac{\gamma}{(L/r)^2}}\ , \tag{18}$$

$$G = \frac{E}{2(1+\nu)}\ ,$$

and
$$\gamma = 2kn^2\pi^2(1+\nu)\ . \tag{19}$$

Letting $n = 1$, $k = 1.2$, and $\nu = 0.3$, the relations between P_{cr_n}, P_{e_n} and $\frac{L}{r}$ are represented by the curves as shown in Fig. 3.

For the values of $60 < \frac{L}{r} < 120$, the inclusion of shear stress reduces the critical buckling load by an average in equation (16) is shown in Fig. 2.

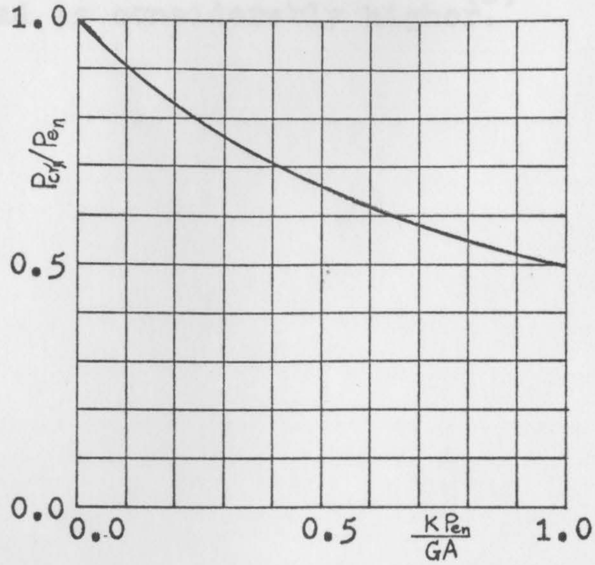

Fig. 2 Graphical Relationship between $\frac{P_{cr_n}}{P_{e_n}}$ and $\frac{kP_{e_n}}{GA}$

Furthermore, both P_{cr_n} and P_{e_n} are expressed in terms of the slenderness ratio $\frac{L}{r}$ as follows:

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where

$$C_1 = \frac{1}{1 + \frac{\gamma}{(L/r)^2}} \,, \tag{18}$$

$$G = \frac{E}{2(1+\nu)} \,,$$

and

$$\gamma = 2kn^2\pi^2(1+\nu) \,. \tag{19}$$

Letting n = 1, k = 1.2, and $\nu = 0.3$, the relations between P_{cr_n}, P_{e_n} and $\frac{L}{r}$ are represented by the curves as shown in Fig. 3.

For the values of $60 < \frac{L}{r} < 120$, the inclusion of shear stress reduces the critical buckling load by an average value 0.5 % for typical rolled sections. Timoshenko shows that for built-up sections the effect of shear on the critical buckling load is considerably higher.[6]

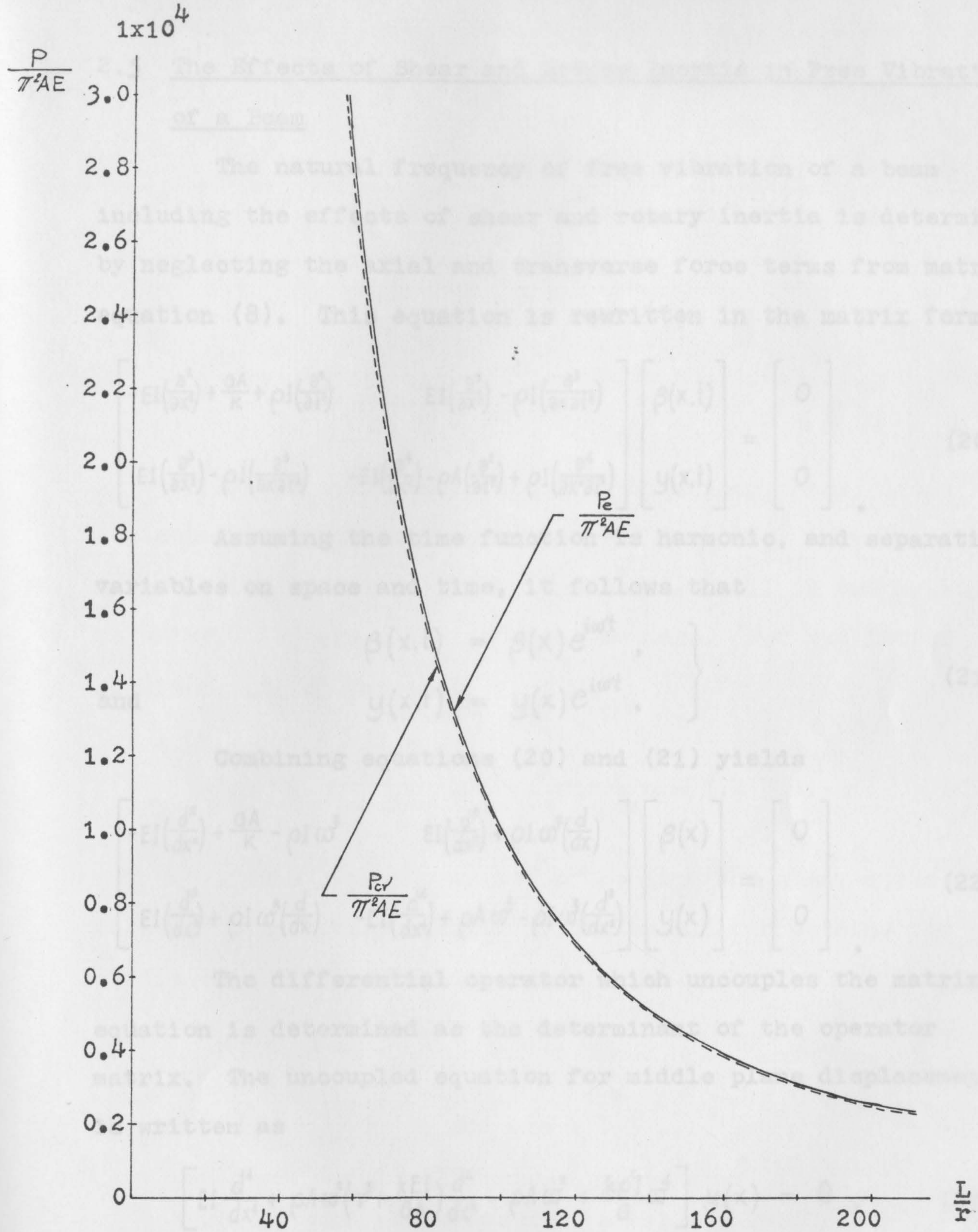

Fig. 3 Graphical Relationship between P_{e_n}, P_{cr_n}, and $\frac{L}{r}$

2.5 The Effects of Shear and Rotary Inertia in Free Vibration of a Beam

The natural frequency of free vibration of a beam including the effects of shear and rotary inertia is determined by neglecting the axial and transverse force terms from matrix equation (8). This equation is rewritten in the matrix form

$$\begin{bmatrix} -EI\left(\frac{\partial^2}{\partial x^2}\right)+\frac{GA}{K}+\rho I\left(\frac{\partial^2}{\partial t^2}\right) & EI\left(\frac{\partial^3}{\partial x^3}\right)-\rho I\left(\frac{\partial^3}{\partial x\,\partial t^2}\right) \\ EI\left(\frac{\partial^3}{\partial x^3}\right)-\rho I\left(\frac{\partial^3}{\partial x\,\partial t^2}\right) & -EI\left(\frac{\partial^4}{\partial x^4}\right)-\rho A\left(\frac{\partial^2}{\partial t^2}\right)+\rho I\left(\frac{\partial^4}{\partial x^2\partial t^2}\right) \end{bmatrix} \begin{bmatrix} \beta(x,t) \\ y(x,t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \qquad (20)$$

Assuming the time function is harmonic, and separating variables on space and time, it follows that

$$\left.\begin{aligned} \beta(x,t) &= \beta(x)e^{i\omega t}, \\ \text{and} \qquad y(x,t) &= y(x)e^{i\omega t}. \end{aligned}\right\} \qquad (21)$$

Combining equations (20) and (21) yields

$$\begin{bmatrix} -EI\left(\frac{d^2}{dx^2}\right)+\frac{GA}{K}-\rho I\omega^2 & EI\left(\frac{d^3}{dx^3}\right)+\rho I\omega^2\left(\frac{d}{dx}\right) \\ EI\left(\frac{d^3}{dx^3}\right)+\rho I\omega^2\left(\frac{d}{dx}\right) & -EI\left(\frac{d^4}{dx^4}\right)+\rho A\omega^2-\rho I\omega^2\left(\frac{d^2}{dx^2}\right) \end{bmatrix} \begin{bmatrix} \beta(x) \\ y(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \qquad (22)$$

The differential operator which uncouples the matrix equation is determined as the determinant of the operator matrix. The uncoupled equation for middle plane displacement is written as

$$\left[EI\frac{d^4}{dx^4}+\rho A\omega^2\left(r^2+\frac{kEI}{GA}\right)\frac{d^2}{dx^2}-\rho A\omega^2+\frac{k\rho^2 I}{G}\omega^4\right]y(x) = 0 . \qquad (23)$$

The term $\frac{k\rho^2 I}{G}\omega^4$ is significant for high frequency vibration.[3,7] Thus, for low modal frequencies, this term is neglected, and equation (23) becomes

$$\left[EI\frac{d^4}{dx^4} + \rho A\omega^2\left(r^2+\frac{kEI}{GA}\right)\frac{d^2}{dx^2} - \rho A\omega^2 \right] y(x) = 0 \,. \tag{24}$$

For the case of a simply supported-beam, equations (13) and (14) when substituted into equation (24) yield the natural frequency of free vibration as

$$\omega_n^2 = \frac{\frac{EI}{\rho A}\left(\frac{n\pi}{L}\right)^4}{1+\left(r^2+\frac{kEI}{GA}\right)\left(\frac{n\pi}{L}\right)^2} \text{ , where } n = 1,2,3,\ldots \,. \tag{25}$$

If the combined effect of bending and shear stress is neglected in comparison to unity, the term $EI\frac{k}{GA}$ is eliminated. Also, if the effect of rotary inertia is small in comparison to unity, the term r^2 is set equal to zero. For the latter conditon, it follows that

$$\omega_{G_n}^2 = \frac{\frac{EI}{\rho A}\left(\frac{n\pi}{L}\right)^4}{1+\frac{kEI}{GA}\left(\frac{n\pi}{L}\right)^2} \text{ , where } n = 1,2,3,\ldots \,. \tag{26}$$

If the combined effect of bending and shear stress together with rotary inertia is neglected, one obtains the condition

$$\omega_{G_n}^2 = \omega_{o_n}^2 \,,$$

where $$\omega_{o_n}^2 = \frac{EI}{\rho A}\left(\frac{n\pi}{L}\right)^4 \text{ , and } n = 1,2,3,\ldots \,. \tag{27}$$

Thus, when the effects of shear and rotary inertia are included, the natural frequency of free vibration of the beam is reduced in all mode shapes.

2.6 The Effects of Shear and Rotary Inertia in Free Vibration of a Beam-Column

For the general case of free vibration of a beam-column, the transverse force is neglected in equation (8) and the following matrix equation holds:

$$\begin{bmatrix} -EI\left(\frac{\partial^2}{\partial x^2}\right)+\frac{GA}{K}+\rho I\left(\frac{\partial^2}{\partial t^2}\right) & EI\left(\frac{\partial^3}{\partial x^3}\right)-\rho I\left(\frac{\partial^3}{\partial x \partial t^2}\right) \\ EI\left(\frac{\partial^3}{\partial x^3}\right)-\rho I\left(\frac{\partial^3}{\partial x \partial t^2}\right) & -EI\left(\frac{\partial^4}{\partial x^4}\right)-P\left(\frac{\partial^2}{\partial x^2}\right)-\rho A\left(\frac{\partial^2}{\partial t^2}\right)+\rho I\left(\frac{\partial^4}{\partial x^2 \partial t^2}\right) \end{bmatrix} \begin{bmatrix} \beta(x,t) \\ y(x,t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \tag{28}$$

Similarly, using the separation of variables technique of harmonic time and space variables, it follows that

$$\left.\begin{aligned} \beta(x,t) &= \beta(x)e^{i\Omega t}, \\ \text{and} \quad y(x,t) &= y(x)e^{i\Omega t}. \end{aligned}\right\} \tag{29}$$

Combining equations (28) and (29) yields

$$\begin{bmatrix} -EI\left(\frac{d^2}{dx^2}\right)+\frac{GA}{K}-\rho I\Omega^2 & EI\left(\frac{d^3}{dx^3}\right)+\rho I\Omega^2\left(\frac{d}{dx}\right) \\ EI\left(\frac{d^3}{dx^3}\right)+\rho I\Omega^2\left(\frac{d}{dx}\right) & -EI\left(\frac{d^4}{dx^4}\right)-P\left(\frac{d^2}{dx^2}\right)+\rho A\Omega^2-\rho I\Omega^2\left(\frac{d^2}{dx^2}\right) \end{bmatrix} \begin{bmatrix} \beta(x) \\ y(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \tag{30}$$

Proceeding in the same manner as in the previous sections, the uncoupled equation of midplane displacement is written as

$$\left[EI\left(1-\frac{kP}{GA}\right)\frac{d^4}{dt^4}+\rho A\Omega^2\left\{r^2\left(1-\frac{kP}{GA}\right)+\frac{kEI}{GA}\right\}\frac{d^2}{dt^2}+P\frac{d^2}{dt^2}-\rho A\Omega^2+\frac{k\rho^2 I}{G}\Omega^4\right]y(x)=0 . \tag{31}$$

Neglecting the term $\frac{k\rho^2 I}{G}\Omega^4$, using equations (13) and (14), it follows that

$$\Omega_n^2 = \frac{\frac{EI}{\rho A}\left(\frac{n\pi}{L}\right)^4\left(1-\frac{P}{P_{cr_n}}\right)}{1+\left\{r^2\left(1-\frac{kP}{GA}\right)+\frac{kEI}{GA}\right\}\left(\frac{n\pi}{L}\right)^2}, \text{ where } n = 1,2,3,.. \tag{32}$$

Thus, it is seen that inclusion of shear and rotary inertia reduces the natural frequency. Also, the effect of axial compressive force reduces the natural frequency in all mode shapes.

If the effect of axial force is neglected, the natural frequency reduces to the form given in equation (25) (i.e., $\Omega_n^2 = \omega_n^2$). Neglecting the combination term $r^2\frac{kP}{GA}$ in comparison to unity and also the effect of rotational inertia (i.e., r^2 is small in comparison to unity), it follows that

$$\Omega_{G_n}^2 = \frac{\omega_{0_n}^2\left(1 - \frac{P}{P_{cr_n}}\right)}{1 + \frac{kEI}{GA}\left(\frac{n\pi}{L}\right)^2}\text{, where } n = 1,2,3,\ldots \quad . \qquad (33)$$

For convenience, noting equations (26) and (27), equation (33) is rewritten as

$$\Omega_{G_n}^2 = \omega_{G_n}^2\left(1 - \frac{P}{P_{cr_n}}\right)\text{, where } n = 1,2,3,\ldots \quad . \qquad (34)$$

In addition, if the term $EI\frac{k}{GA}\left(\frac{n\pi}{L}\right)^2$ is small in comparison to unity, that is the effect of shear is neglected, equations (33) and (34) reduce to the form

$$\Omega_{0_n}^2 = \omega_{0_n}^2\left(1 - \frac{P}{P_{e_n}}\right)\text{, where } n = 1,2,3,\ldots \quad . \qquad (35)$$

This result has been investigated by Timoshenko.[7]

2.7 Parametric Vibration of a Beam-Column

To investigate the parametric vibration of the beam-column, the solutions of equation (28) are assumed to take the forms

$$\left.\begin{aligned} \beta(x,t) &= \frac{n\pi}{L} B\cos\left(\frac{n\pi}{L}x\right) f_n(t) \quad \text{, and} \\ y(x,t) &= B\sin\left(\frac{n\pi}{L}x\right) g_n(t) \quad , \end{aligned}\right\} \tag{36}$$

where $f_n(t)$ and $g_n(t)$ are unknown functions of time. Equation (36) satisfies the boundary conditions of equation (14) for the case of a simply supported beam-column.

Combining equations (28) and (36), one obtains

$$\begin{bmatrix} -EI\left(\frac{n\pi}{L}\right)^4 - \frac{GA}{K}\left(\frac{n\pi}{L}\right)^2 - \rho I\left(\frac{n\pi}{L}\right)^2\frac{d^2}{dt^2} & EI\left(\frac{n\pi}{L}\right)^4 + \rho I\left(\frac{n\pi}{L}\right)^2\frac{d^2}{dt^2} \\ EI\left(\frac{n\pi}{L}\right)^4 + \rho I\left(\frac{n\pi}{L}\right)^2\frac{d^2}{dt^2} & -EI\left(\frac{n\pi}{L}\right)^4 + P\left(\frac{n\pi}{L}\right)^2 - \rho A\frac{d^2}{dt^2} - \rho I\left(\frac{n\pi}{L}\right)^2\frac{d^2}{dt^2} \end{bmatrix} \begin{bmatrix} f_n(t) \\ g_n(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \tag{37}$$

Similarly, utilizing the same technique as in the previous sections, the uncoupled equation of parametric vibration is written as

$$\left[\frac{k\rho I}{GA}\frac{d^4}{dt^4} + \left\{1 + \left[r^2\left(1-\frac{kP}{GA}\right) + \frac{kEI}{GA}\right]\left(\frac{n\pi}{L}\right)^2\right\}\frac{d^2}{dt^2} + \frac{1}{\rho A}\left\{EI\left(1-\frac{kP}{GA}\right)\left(\frac{n\pi}{L}\right)^4 - P\left(\frac{n\pi}{L}\right)^2\right\}\right] g_n(t) = 0 . \tag{38}$$

Substituting equations (16), (27), and (32) into equation (38), yields

$$\left[\frac{k\rho I}{GA}\frac{d^4}{dt^4} + \frac{\omega_{on}^2}{\Omega_n^2}\left(1-\frac{P}{P_{cr_n}}\right)\frac{d^2}{dt^2} + \omega_{o_n}^2\left(1-\frac{P}{P_{cr_n}}\right)\right] g_n(t) = 0 . \tag{39}$$

Consider the case where the axial force P is of the form

$$P = P_o + P_t\cos\theta t \, , \tag{40}$$

that is P consists of a constant term P_o and a periodically varying term $P_t \cos \theta t$ which has an amplitude of P_t and period of $\frac{2\pi}{\theta}$ (see Fig. 4).

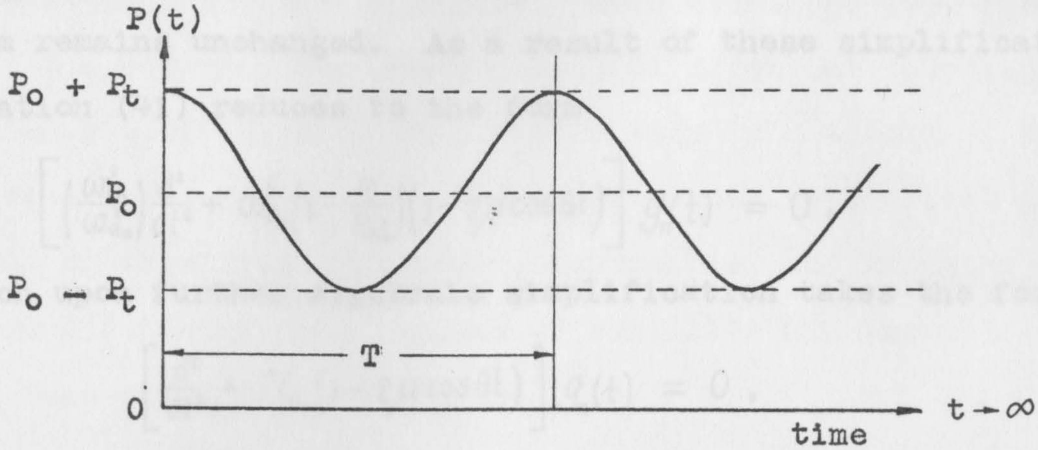

Fig. 4 Graphical Representation of the Periodic Force P

Substituting equation (40) into equation (39), the following modified equation is obtained:

$$\left[\frac{k\rho I}{GA}\frac{d^4}{dt^4}+\frac{\omega_{o_n}^2}{\Omega_n^2}\left(1-\frac{P_o}{P_{cr_n}}\right)(1-2\mu\cos\theta t)\frac{d^2}{dt^2}+\omega_{o_n}^2(1-2\mu\cos\theta t)\right]g_n(t) = 0\,, \tag{41}$$

where

$$2\mu = \frac{P_t}{P_{cr_n}-P_o}\;. \tag{42}$$

Equation (41) is simplified neglecting certain higher order terms. The product term of shear and rotational inertia (i.e. the term $\frac{k\rho I}{GA}$) is neglected. The combined term of shear, rotational inertia, and axial force (i.e. the term $r^2\frac{kP}{GA}$) is neglected in comparison to unity. Further, for lower mode frequencies, the effect of rotational inertia (i.e. the term r^2) is neglected in comparison to unity. The above approximations simplify equation (41) in the following ways. Firstly, the 4^{th} order derivative term is eliminated. Secondly, since the term $r^2\frac{kP}{GA}$ is neglected, this condition implies that $P_0 = P_t = \mu = 0$ in the coefficient of the 2^{nd} order derivative term. Thirdly, the coefficient of the 0^{th} order derivative term remains unchanged. As a result of these simplifications, equation (41) reduces to the form

$$\left[\left(\frac{\omega_{0n}^2}{\omega_{Gn}^2}\right)\frac{d^2}{dt^2} + \omega_{0n}^2\left(1-\frac{P_0}{P_{cr_n}}\right)\left(1-2\mu\cos\theta t\right)\right] g_n(t) = 0 ,$$

which upon further algebraic simplification takes the form

$$\left[\frac{d^2}{dt^2} + \Omega_{Gn}^2\left(1-2\mu\cos\theta t\right)\right] g_n(t) = 0 , \tag{43}$$

where

$$\Omega_{Gn}^2 = \omega_{Gn}^2\left(1-\frac{P_0}{P_{cr_n}}\right) , \tag{44}$$

and

$$2\mu = \frac{P_t}{P_{cr_n} - P_0} . \tag{45}$$

In addition, if the effect of shear $(\frac{k}{GA})$ is also neglected, equation (43) reduces further to the classical form

$$\left[\frac{d^2}{dt^2} + \Omega_{0n}^2\left(1-2\mu_0\cos\theta t\right)\right] g_n(t) = 0 , \tag{46}$$

where

$$\Omega_{0n}^2 = \omega_{0n}^2\left(1-\frac{P_0}{P_{en}}\right) , \tag{47}$$

and

$$2\mu_0 = \frac{P_t}{P_{en} - P_0} , \tag{48}$$

as given by Bolotin.[1]

2.8 The Effect of Bending Stress on the Region of Stability

Equation (46) is the classical form as formulated by Bolotin[1] and reviewed by Stoker[5]. Both equations (43) and (46) take the forms of the Mathieu type differential equation with periodic coefficient. They differ only in the form of the constants which appear in the coefficients. the exact solutions of this type equation is impossible, but by using Floquet's theory, the regions of stability and instability of the beam-column are determined[1,5].

Using Floquet's theory, it follows that:"The regions of unbounded increasing solutions are separated from the regions of bounded solutions by periodic solutions with periods of T and 2T. In other words, two solutions of identical periods bound the region of instability and two solutions of different periods bound the region of stability".[1]

In investigating the effect of bending stress on the region of instability, equation (46) is considered.

To determine the regions of instability bounded by the periodic solutions with a period T, the solution of the equation (46) for any arbitary value of n is assumed to take the form

$$g(t) = b_0 + \sum_{k=2,4,6,\ldots}^{\infty} \left(a_k \sin\frac{k\theta t}{2} + b_k \cos\frac{k\theta t}{2} \right). \tag{49}$$

Substituting equation (49) into equation (46) and expanding the series, the result is set into the following two matrices:

$$\begin{bmatrix} 1-\frac{\theta^2}{\Omega_o^2} & -\mu_o & 0 & 0 & 0 & \cdots \\ -\mu_o & 1-\frac{4\theta^2}{\Omega_o^2} & -\mu_o & 0 & 0 & \cdots \\ 0 & -\mu_o & 1-\frac{16\theta^2}{\Omega_o^2} & -\mu_o & 0 & \cdots \\ 0 & 0 & -\mu_o & 1-\frac{36\theta^2}{\Omega_o^2} & -\mu_o & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} a_2 \\ a_4 \\ a_6 \\ a_8 \\ \cdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cdots \end{bmatrix}, \tag{50.a}$$

and

$$\begin{bmatrix} 1 & -\mu_o & 0 & 0 & 0 & \cdots \\ -2\mu_o & 1-\frac{\theta^2}{\Omega_o^2} & -\mu_o & 0 & 0 & \cdots \\ 0 & -\mu_o & 1-\frac{4\theta^2}{\Omega_o^2} & -\mu_o & 0 & \cdots \\ 0 & 0 & -\mu_o & 1-\frac{16\theta^2}{\Omega_o^2} & -\mu_o & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} b_0 \\ b_2 \\ b_4 \\ b_6 \\ \cdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{50.b}$$

Next, to determine the regions of instability bounded by the periodic solutions with the period 2T, the periodic solution is assumed to be

$$g(t) = \sum_{k=1,3,5,\ldots}^{\infty} \left(a_k \sin\frac{k\theta t}{2} + b_k \cos\frac{k\theta t}{2} \right). \tag{51}$$

Similarly, substituting equation (51) into equation (46) and expanding the series, the resulting matrices are:

$$\begin{bmatrix} 1-\frac{\theta^2}{4\Omega_0^2}+\mu_0 & -\mu_0 & 0 & 0 & 0 & \cdots \\ -\mu_0 & 1-\frac{9\theta^2}{4\Omega_0^2} & -\mu_0 & 0 & 0 & \cdots \\ 0 & -\mu_0 & 1-\frac{25\theta^2}{4\Omega_0^2} & -\mu_0 & 0 & \cdots \\ 0 & 0 & -\mu_0 & 1-\frac{49\theta^2}{4\Omega_0^2} & -\mu_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} a_1 \\ a_3 \\ a_5 \\ a_7 \\ \cdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cdots \end{bmatrix}, \qquad (52.a)$$

and

$$\begin{bmatrix} 1-\frac{\theta^2}{4\Omega_0^2}-\mu_0 & -\mu_0 & 0 & 0 & 0 & \cdots \\ -\mu_0 & 1-\frac{9\theta^2}{4\Omega_0^2} & -\mu_0 & 0 & 0 & \cdots \\ 0 & -\mu_0 & 1-\frac{25\theta^2}{4\Omega_0^2} & -\mu_0 & 0 & \cdots \\ 0 & 0 & -\mu_0 & 1-\frac{49\theta^2}{4\Omega_0^2} & -\mu_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} b_1 \\ b_3 \\ b_5 \\ b_7 \\ \cdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cdots \end{bmatrix}. \qquad (52.b)$$

For nontrivial solutions of a_K and b_K, each determinant of the coefficients of a_K and b_K in equations (50.a),(50.b), (52.a), and (52.b) must be zero. From these equations the regions of stability and instability of the beam-column are determined by the relations between θ and Ω_0. The higher the order of the determinant, the higher the accuracy of boundary curves of the regions of stability and instability.

For the first approximation, in region 2T

$$1 - \frac{\theta^2}{4\Omega_o^2} + \mu_o = 0 \ ,$$

and
$$1 - \frac{\theta^2}{4\Omega_o^2} - \mu_o = 0 \ ,$$

or
$$\frac{\theta^2}{4\Omega_o^2} = 1 \pm \mu_o \ ,$$

$$\frac{\theta}{2\Omega_o} = (1 \pm \mu_o)^{1/2} . \tag{53}$$

If $\mu_o = 0$,

$$\theta = 2\Omega_o \ . \tag{54}$$

Equation (54) is interpretted physically by the condition that a small periodic-varying force induces violent lateral vibration of the beam-column if its frequency is twice the lateral frequency of vibration of the beam-column under constant axial force P_o.

The second approximation is given by the second order determinants as

$$\Delta_{A_T} = \begin{vmatrix} 1 - \frac{\theta^2}{\Omega_o^2} & -\mu_o \\ -\mu_o & 1 - \frac{4\theta^2}{\Omega_o^2} \end{vmatrix} = 0 \ , \tag{55.a}$$

$$\Delta_{B_T} = \begin{vmatrix} 1 & -\mu_o \\ -2\mu_o & 1 - \frac{\theta^2}{\Omega_o^2} \end{vmatrix} = 0 \ , \tag{55.b}$$

$$\Delta_{A_{2T}} = \begin{vmatrix} 1-\frac{\theta^2}{4\Omega_0^2}+\mu_0 & -\mu_0 \\ -\mu_0 & 1-\frac{9\theta^2}{4\Omega_0^2} \end{vmatrix} = 0\,, \tag{55.c}$$

and

$$\Delta_{B_{2T}} = \begin{vmatrix} 1-\frac{\theta^2}{4\Omega_0^2}-\mu_0 & -\mu_0 \\ -\mu_0 & 1-\frac{9\theta^2}{4\Omega_0^2} \end{vmatrix} = 0\,. \tag{55.d}$$

Ploting the relation between $\frac{\theta}{2\Omega_0}$ and μ_0 in equations (55.a), (55.b), (55.c), and (55.d), the second approximation regions of stability and instability of the beam-column are shown in Fig. 5 where the crosshatched areas indicate regions of instability. The graphical interpretation is given in Bilotin.[1] The numerical results are tabulated in Table 1.

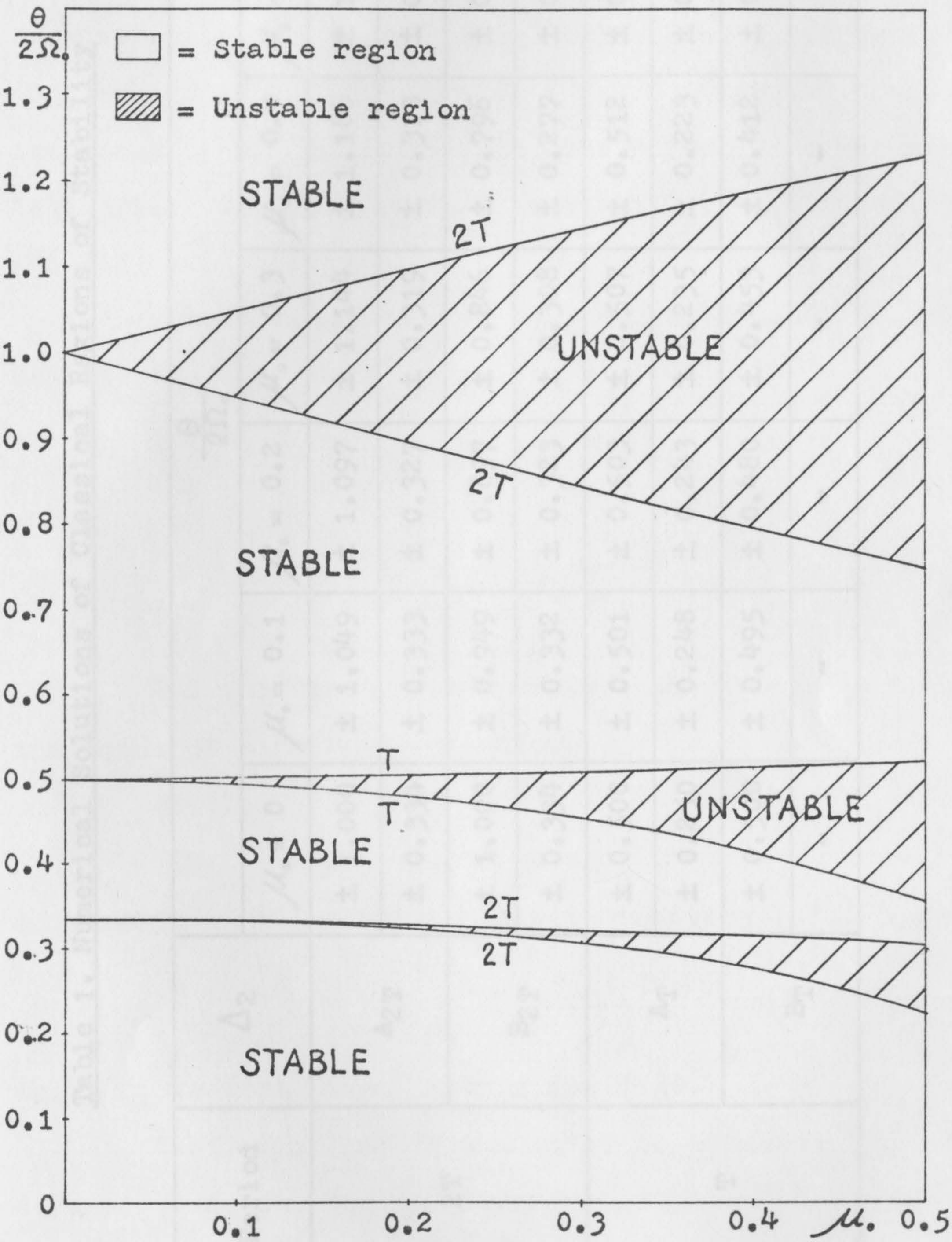

Fig. 5 Classical Regions of Stability

Neglecting the Effect of Shear Stress

Table 1. Numerical Solutions of Classical Regions of Stability

Period	Δ_2	$\frac{\theta}{2\Omega_0}$					
		$\mu_0 = 0$	$\mu_0 = 0.1$	$\mu_0 = 0.2$	$\mu_0 = 0.3$	$\mu_0 = 0.4$	$\mu_0 = 0.5$
2T	A_{2T}	$\pm$ 1.000	$\pm$ 1.049	$\pm$ 1.097	$\pm$ 1.144	$\pm$ 1.188	$\pm$ 1.230
		$\pm$ 0.334	$\pm$ 0.333	$\pm$ 0.327	$\pm$ 0.319	$\pm$ 0.313	$\pm$ 0.302
	B_{2T}	$\pm$ 1.000	$\pm$ 0.949	$\pm$ 0.897	$\pm$ 0.846	$\pm$ 0.796	$\pm$ 0.749
		$\pm$ 0.334	$\pm$ 0.332	$\pm$ 0.323	$\pm$ 0.308	$\pm$ 0.277	$\pm$ 0.224
T	A_T	$\pm$ 0.500	$\pm$ 0.501	$\pm$ 0.503	$\pm$ 0.507	$\pm$ 0.512	$\pm$ 0.518
		$\pm$ 0.250	$\pm$ 0.248	$\pm$ 0.243	$\pm$ 0.235	$\pm$ 0.223	$\pm$ 0.208
	B_T	$\pm$ 0.500	$\pm$ 0.495	$\pm$ 0.480	$\pm$ 0.453	$\pm$ 0.412	$\pm$ 0.354
		-	-	-	-	-	-

2.9 The Combined Effect of Bending Stress and Shear Stress on the Region of Stability

The influence of the shear on the natural frequency of a vibrating beam is approximately three times greater than that of the rotational inertia for a rectangular cross - sectional beam.[3] To investigate the effect of shear stress on the regions of stability and instability, equations (43), (44), and (45) are considered.

Noting equation (44) together with equation (47), it follows that

$$\Omega_{G_n}^2 = \alpha \Omega_{o_n}^2 \;, \tag{56}$$

where

$$\alpha = \frac{P_{cr_n} - P_o}{P_{e_n} - P_o} \;. \tag{57}$$

For the special case where shear stress is neglected, the value of α is equal to unity.

Setting

$$P_o = pP_{e_n} \;, \tag{58}$$

and combining this definition with equation (57), one obtains

$$\alpha = \frac{C_1 - p}{1 - p} \;, \quad (0 < \alpha < 1) \tag{59}$$

where C_1 is that given by equation (18).

Similarly, the term μ given in equation (45) is defined using the term μ_o given in equation (48) in the following form:

$$\mu = C_2 \mu_o \;, \tag{60}$$

where

$$C_2 = \frac{1 + \frac{\gamma}{(L/r)^2}}{1 - \frac{\gamma}{(L/r)^2}\left(\frac{p}{1-p}\right)}, \qquad (1 < C_2 < \infty)$$

or

$$C_2 = \frac{1}{C_1}\left[\frac{1}{1 - \frac{\gamma}{(L/r)^2}\left(\frac{p}{1-p}\right)}\right]. \tag{61}$$

The relashionships among α, p, and $\frac{L}{r}$ are represented geometrically by the curves shown in Fig. 6. The relation - ships among C_2, p, and $\frac{L}{r}$ are represented geometrically by the curves shown in Fig. 7.

The determinant equations defining the stability and instability zones including the effect of shear take the modified form for the first approximation,

$$\frac{\theta}{2\Omega_0} = \alpha(1 \pm C_2\mu_0)^{1/2}. \tag{62}$$

For the second order approximations the determinant forms are written as:

$$\left.\begin{aligned}
&\Delta_{A_T} = \begin{vmatrix} 1 - \frac{\theta^2}{\alpha\Omega_0^2} & -C_2\mu_0 \\ -C_2\mu_0 & 1 - \frac{4\theta^2}{\alpha\Omega_0^2} \end{vmatrix} = 0, \qquad \Delta_{B_T} = \begin{vmatrix} 1 & -C_2\mu_0 \\ -2C_2\mu_0 & 1 - \frac{\theta^2}{\alpha\Omega_0^2} \end{vmatrix} = 0, \\
&\Delta_{A_{2T}} = \begin{vmatrix} 1 - \frac{\theta^2}{4\alpha\Omega_0^2} + C_2\mu_0 & -C_2\mu_0 \\ -C_2\mu_0 & 1 - \frac{9\theta^2}{4\alpha\Omega_0^2} \end{vmatrix} = 0, \text{ and } \Delta_{B_{2T}} = \begin{vmatrix} 1 - \frac{\theta^2}{4\alpha\Omega_0^2} - C_2\mu_0 & -C_2\mu_0 \\ -C_2\mu_0 & 1 - \frac{9\theta^2}{4\alpha\Omega_0^2} \end{vmatrix} = 0.
\end{aligned}\right\} \tag{63}$$

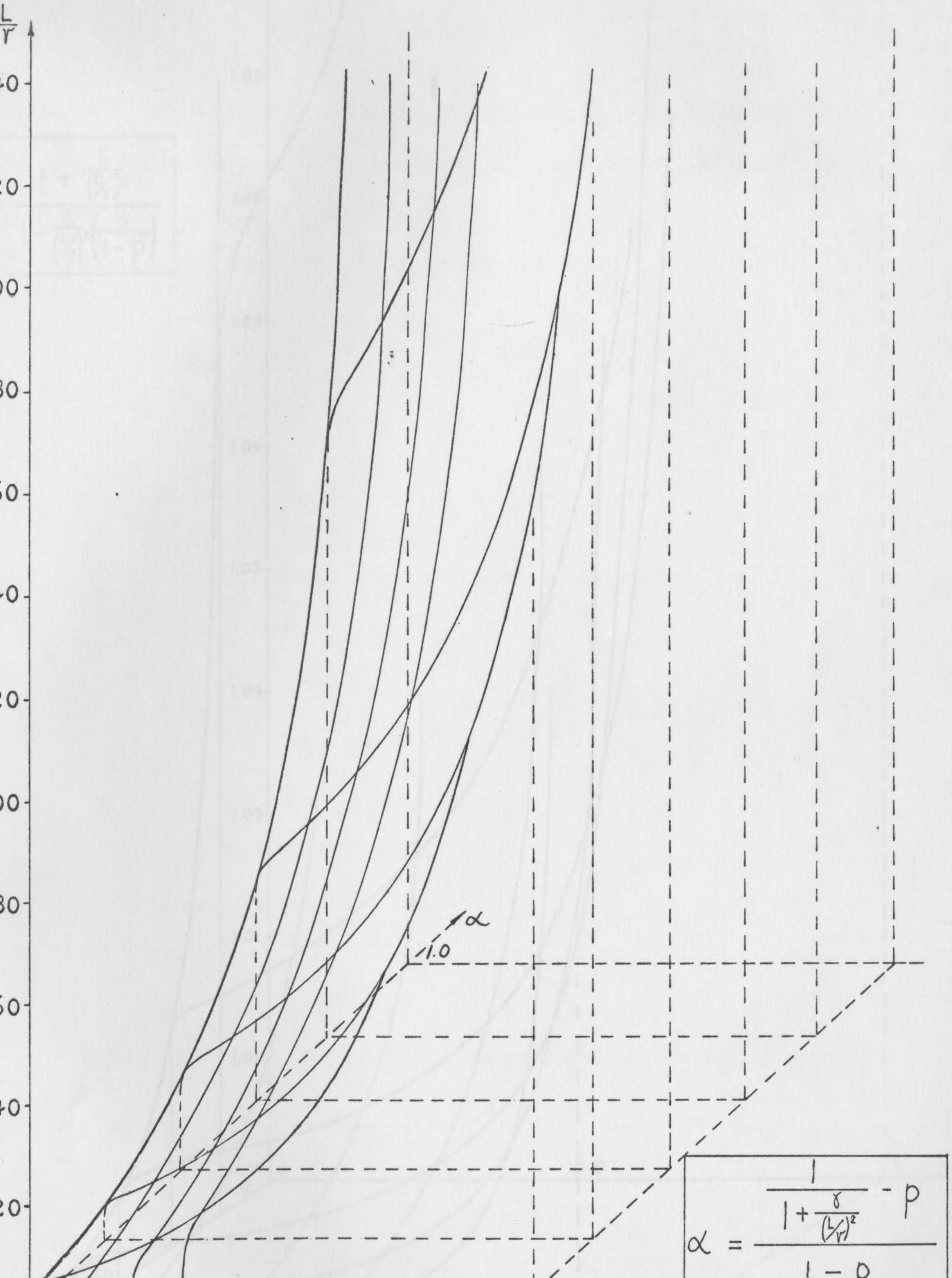

Fig. 6 Geometrical Relationships among $\frac{L}{r}$, p, and α

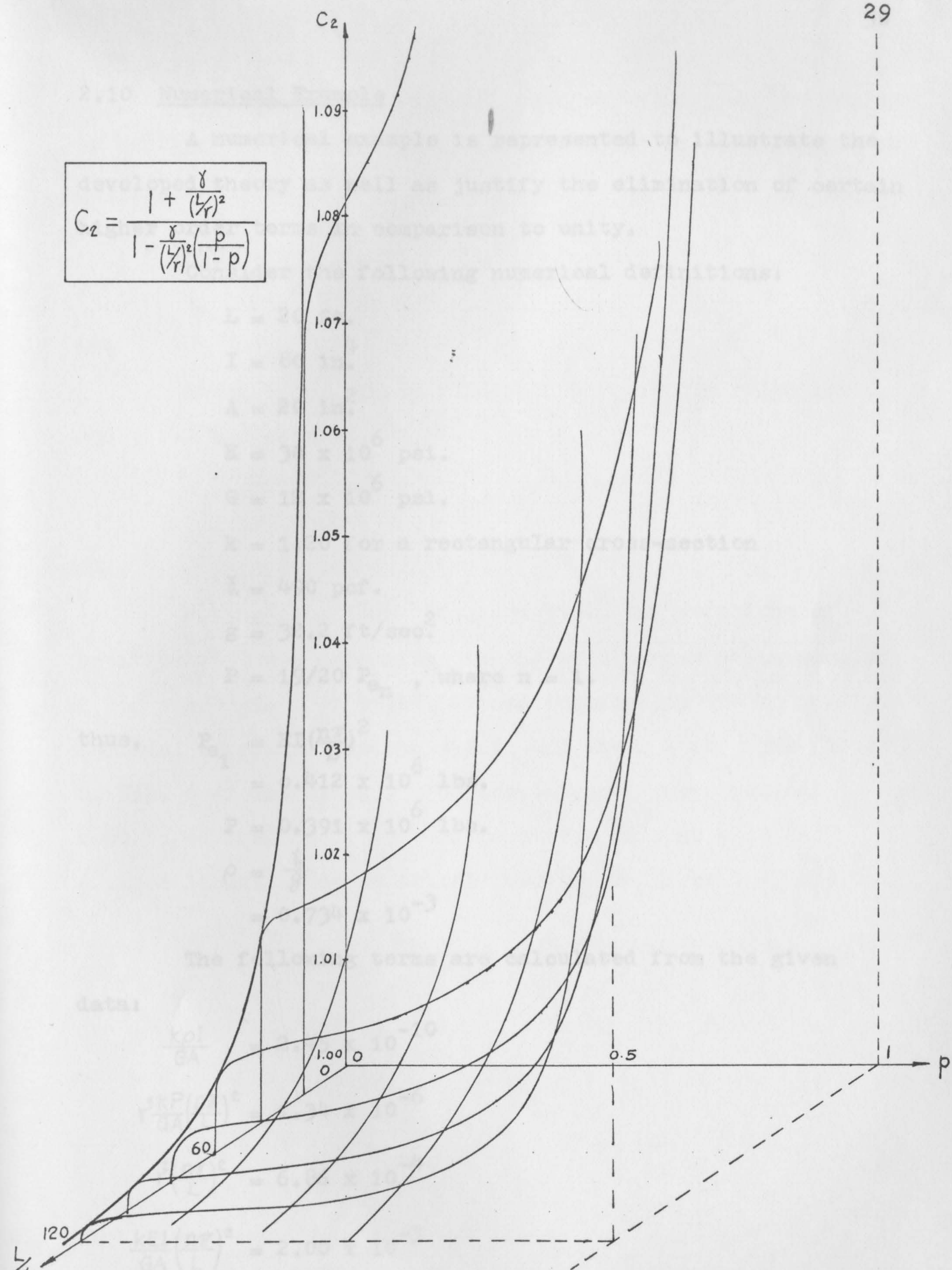

Fig. 7 Geometrical Relationships among $\frac{L}{r}$, p, and C_2

2.10 Numerical Example

A numerical example is represented to illustrate the developed theory as well as justify the elimination of certain higher order terms in comparison to unity.

Consider the following numerical definitions:

$L = 20$ ft.

$I = 80$ in.^{4}

$A = 20$ in.^{2}

$E = 30 \times 10^6$ psi.

$G = 12 \times 10^6$ psi.

$k = 1.20$ for a rectangular cross-section

$\gamma_1 = 490$ pcf.

$g = 32.2$ ft/sec.^{2}

$P = 19/20\ P_{e_n}$, where $n = 1$.

thus, $P_{e_1} = EI(\frac{n\pi}{L})^2$

$= 0.412 \times 10^6$ lbs.

$P = 0.391 \times 10^6$ lbs.

$\rho = \frac{\gamma_1}{g}$

$= 0.734 \times 10^{-3}$

The following terms are calculated from the given data:

$$\frac{k\rho I}{GA} = 2.45 \times 10^{-10}$$

$$r^2\frac{kP}{GA}\left(\frac{n\pi}{L}\right)^2 = 1.34 \times 10^{-6}$$

$$r^2\left(\frac{n\pi}{L}\right)^2 = 6.86 \times 10^{-4}$$

$$\frac{kEI}{GA}\left(\frac{n\pi}{L}\right)^2 = 2.00 \times 10^{-3}$$

In order to investigate the shear effect of the regions of stability and instability, numerical values are assigned as follows:

$$\left.\begin{aligned} \nu &= 0.30, \\ p &= 19/20, \\ \text{and} \qquad \frac{L}{r} &= 120. \end{aligned}\right\} \tag{64}$$

Substituting these values into equations (18), (59), and (61), one obtains

$$\left.\begin{aligned} C_1 &= 0.9979 \\ \alpha &= 0.9573 \\ C_2 &= 1.0446 \;. \end{aligned}\right\} \tag{65}$$

Noting equations (56), and (60), the solutions of equation (63) are determined in the same manner as in section 2.8. The regions of stability and instability due to the combined effect of bending stress and shear stress are plotted in Fig. 8 (solid lines). For convenience, the classical results of the effect of bending stress only is also shown (dotted lines). The numerical results are given in Table 2.

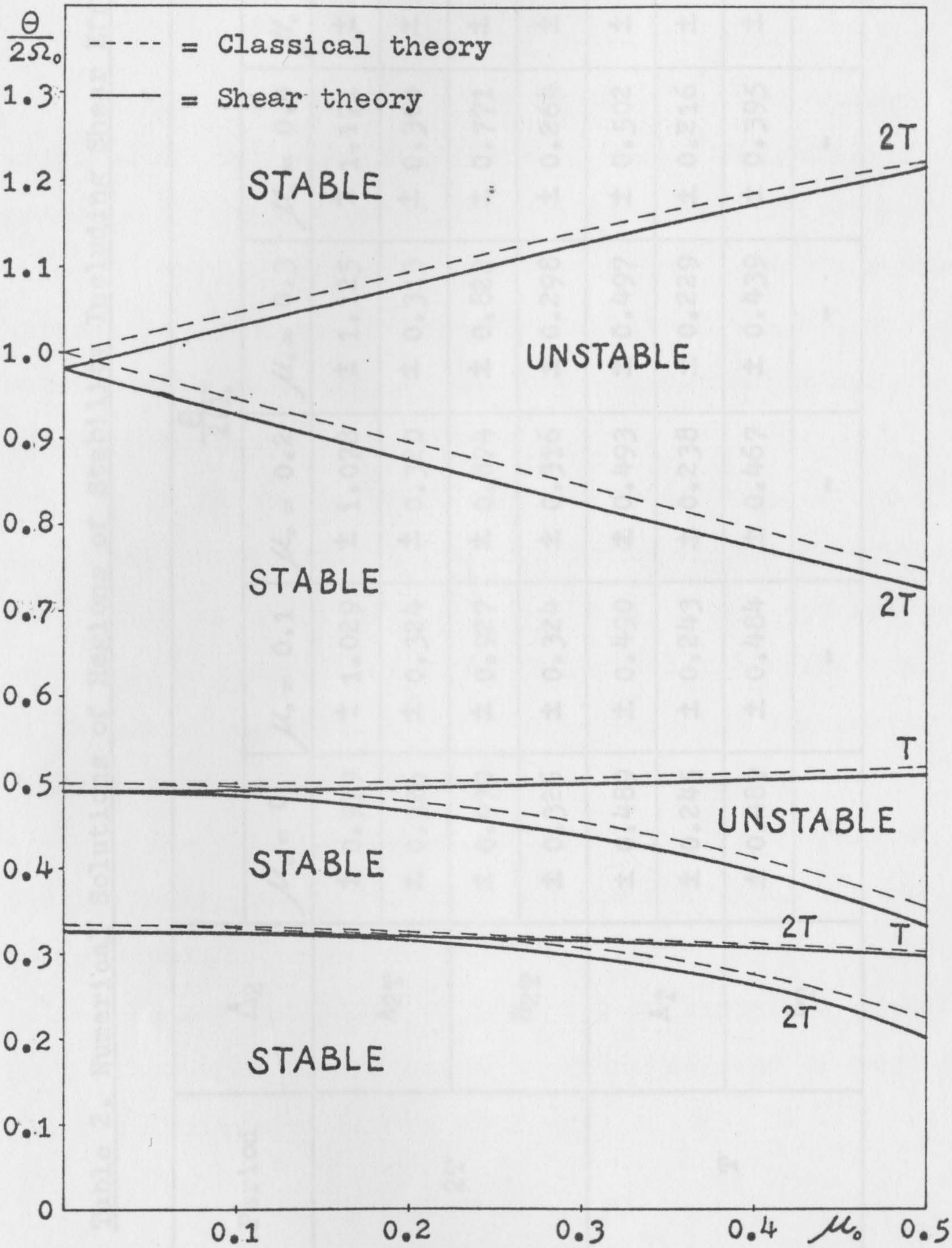

Fig. 8 Stability Regions Including Shear Effects

Table 2. Numerical Solutions of Regions of Stability Including Shear Effects

Period	Δ_2	$\frac{\theta}{2\Omega_0}$					
		$\mu_0 = 0$	$\mu_0 = 0.1$	$\mu_0 = 0.2$	$\mu_0 = 0.3$	$\mu_0 = 0.4$	$\mu_0 = 0.5$
2T	A_{2T}	$\pm$ 0.979	$\pm$ 1.029	$\pm$ 1.078	$\pm$ 1.125	$\pm$ 1.171	$\pm$ 1.216
		$\pm$ 0.326	$\pm$ 0.324	$\pm$ 0.320	$\pm$ 0.313	$\pm$ 0.304	$\pm$ 0.293
	B_{2T}	$\pm$ 0.979	$\pm$ 0.927	$\pm$ 0.874	$\pm$ 0.822	$\pm$ 0.771	$\pm$ 0.724
		$\pm$ 0.326	$\pm$ 0.324	$\pm$ 0.316	$\pm$ 0.298	$\pm$ 0.264	$\pm$ 0.199
T	A_T	$\pm$ 0.489	$\pm$ 0.490	$\pm$ 0.493	$\pm$ 0.497	$\pm$ 0.502	$\pm$ 0.509
		$\pm$ 0.245	$\pm$ 0.243	$\pm$ 0.238	$\pm$ 0.229	$\pm$ 0.216	$\pm$ 0.200
	B_T	$\pm$ 0.489	$\pm$ 0.484	$\pm$ 0.467	$\pm$ 0.439	$\pm$ 0.395	$\pm$ 0.330
		-	-	-	-	-	-

CHAPTER III

DISCUSSIONS

For the parametric vibration of a beam-column, the addition of the combined effect of shear and rotary inertia produces a fourth-order differential equation for the time function which has periodic coefficient. The necessary mathematical theory to investigate the stable and unstable solutions to such type of equation is currently not available in the mathematical literature.

If these additional higher order terms are neglected in comparison to unity, that is, terms formed by coupling between shear and rotary inertia, and if the rotary inertia terms are neglected for low frequency modal vibrations, the differential equation for the time function reduces to a general type Mathieu equation.

The elimination of certain terms in comparison to unity is completely justified by the numerical example as represented.

The formulation of the problem in matrix-operator form allows for an efficient and unique mean of uncoupling the resulting equation of motions.

CHAPTER IV

CONCLUSIONS

The following conclusions are determined from the resulting theoretical analysis:

1. If the combined effect of shear and rotary inertia is included for all modal frequencies in the parametric vibration problem, the theory yields a fourth-order differential equation whose solutions and stability characteristics are not available.

2. If the effect of shear is included in the parametric vibration problem, the ratio of the longitudinal frequency to the lateral frequency at which resonance occurs is reduced. Thus, the system becomes unstable for a lower frequency condition.

3. The size of the instability zones increase as the effect of shear becomes important, since inclusion of shear deformation reduces the geometrical and physical constraints on the system.

4. The natural frequency of free vibration of a beam-column is reduced if the effects of shear, rotary inertia, and axial force are included.

5. For rectangular section or for typical rolled sections, the effect of shear is small. However, for built-up sections or short-stubby sections, the effect of shear stress must be considered. The theory may be directly applied for this condition.

APPENDIX A

APPENDIX A

As a complement to the equations developed using the minimum potential theorem, the following nonlinear beam-theory using the variational theorem of E. Reissner[2] is formulated as a double check of the equations of motion.

A.1 Stress Resultants and Stress Couples

A parallelopiped element of the beam with dimensions dx, dy, and dz is subjected to the forces and couples $N_{yy}dx$, $V_{yz}dx$, and $M_{yy}dx$ respectively. (See Fig. A.1)

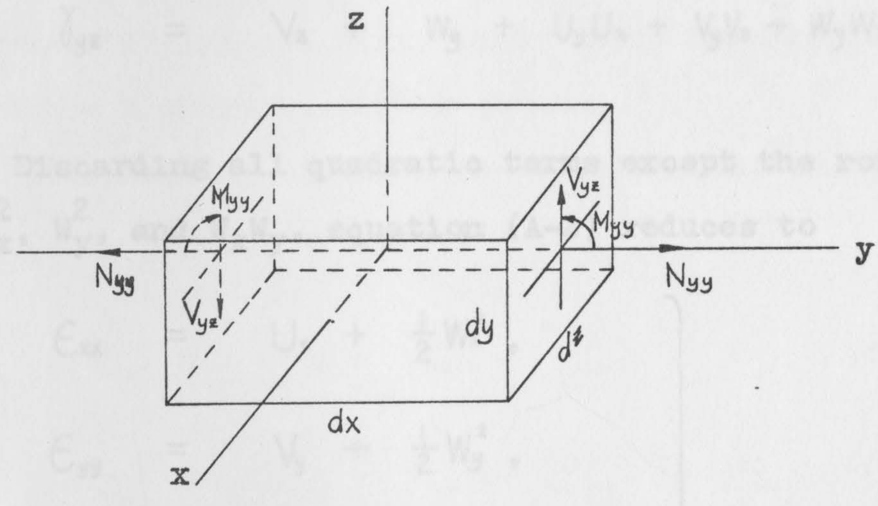

Fig. A.1 Stresses on an Element of a Beam

The tractions N_{yy}, V_{yz}, and the intensity of bending moment M_{yy} are expressed in terms of stress by means of the statical relations

$$\left.\begin{aligned} N_{yy} &= \int_{-h/2}^{h/2} \sigma_{yy}\, dz \,, \\ V_{yz} &= \int_{-h/2}^{h/2} \tau_{yz}\, dz \,, \\ \text{and} \qquad M_{yy} &= \int_{-h/2}^{h/2} z\, \sigma_{yy}\, dz \,. \end{aligned}\right\} \qquad \text{(A-1)}$$

A.2 Strain-Displacement Relations

The equations of nonlinear strain components are:

$$\left.\begin{aligned}
\epsilon_{xx} &= U_x + \tfrac{1}{2}\left(U_x^2 + V_x^2 + W_x^2\right), \\
\epsilon_{yy} &= V_y + \tfrac{1}{2}\left(U_y^2 + V_y^2 + W_y^2\right), \\
\epsilon_{zz} &= W_z + \tfrac{1}{2}\left(U_z^2 + V_z^2 + W_z^2\right), \\
\gamma_{xy} &= U_y + V_x + U_x U_y + V_x V_y + W_x W_y, \\
\gamma_{xz} &= U_z + W_x + U_x U_z + V_x V_z + W_x W_z, \\
\text{and} \quad \gamma_{yz} &= V_z + W_y + U_y U_z + V_y V_z + W_y W_z.
\end{aligned}\right\} \quad \text{(A-2)}$$

Discarding all quadratic terms except the rotation terms W_x^2, W_y^2, and $W_x W_y$, equation (A-2) reduces to

$$\left.\begin{aligned}
\epsilon_{xx} &= U_x + \tfrac{1}{2} W_x^2, \\
\epsilon_{yy} &= V_y + \tfrac{1}{2} W_y^2, \\
\epsilon_{zz} &= W_z, \\
\gamma_{xy} &= U_y + V_x + W_x W_y, \\
\gamma_{xz} &= U_z + W_x, \\
\text{and} \quad \gamma_{yz} &= V_z + W_y.
\end{aligned}\right\} \quad \text{(A-3)}$$

To obtain the appropriate stress-strain relation, the following approximate equations are assumed:

$$\left.\begin{aligned} U(x,y,z) &= \bar{U}(x,y) + z\phi(x,y) , \\ V(x,y,z) &= \bar{V}(x,y) + z\psi(x,y) , \\ \text{and} \quad W(x,y,z) &= \bar{W}(x,y) + z\hat{W}(x,y) + \tfrac{1}{2}z^2\hat{\hat{W}}(x,y) , \end{aligned}\right\} \tag{A-4}$$

where U(x,y,z), V(x,y,z), and W(x,y,z) are the displacements of an arbitary point (x,y,z) in the beam; $\bar{U}(x,y)$, $\bar{V}(x,y)$, and $\bar{W}(x,y)$ are the displacements of the corresponding point on the middle plane; $\phi(x,y)$ and $\psi(x,y)$ are the changes of slope of the normal to the middle plane along the x and y coordinate lines respectively; and $\hat{W}(x,y)$ and $\hat{\hat{W}}(x,y)$ are the contributions to the transverse normal strain.

Substituting equation (A-4) into equation (A-3) one obtains

$$\left.\begin{aligned} \epsilon_{xx} &= \bar{\epsilon}_{xx} + zK_x + z^2C_x + z^3S_x + z^4T_x , \\ \epsilon_{yy} &= \bar{\epsilon}_{yy} + zK_y + z^2C_y + z^3S_y + z^4T_y , \\ \epsilon_{zz} &= \hat{W} + z\hat{\hat{W}} , \\ \gamma_{xy} &= \bar{\gamma}_{xy} + zD_{xy} + z^2E_{xy} + z^3F_{xy} + z^4H_{xy} , \\ \gamma_{xz} &= \bar{\gamma}_{xz} + z\hat{W}_x + \tfrac{1}{2}z^2\hat{\hat{W}}_x , \\ \text{and} \quad \gamma_{yz} &= \bar{\gamma}_{yz} + z\hat{W}_y + \tfrac{1}{2}z^2\hat{\hat{W}}_y . \end{aligned}\right\} \tag{A-5}$$

where

$$\left.\begin{aligned}
\bar{\epsilon}_{xx} &= \bar{U}_x + \tfrac{1}{2}\bar{W}_x^{\,2}\,, \\
K_x &= \phi_x + \bar{W}_x\hat{W}_x\,, \\
C_x &= \tfrac{1}{2}\left(\bar{W}_x\hat{\hat{W}}_x + \hat{W}_x^{2}\right), \\
S_x &= \tfrac{1}{2}\hat{W}_x\hat{\hat{W}}_x\,, \\
T_x &= \tfrac{1}{8}\hat{\hat{W}}_x^{2}\,, \\
\bar{\epsilon}_{yy} &= \bar{V}_y + \tfrac{1}{2}\bar{W}_y^{\,2}\,, \\
K_y &= \psi_y + \bar{W}_y\hat{W}_y\,, \\
C_y &= \tfrac{1}{2}\left(\bar{W}_y\hat{\hat{W}}_y + \hat{W}_y^{2}\right), \\
S_y &= \tfrac{1}{2}\hat{W}_y\hat{\hat{W}}_y\,, \\
T_y &= \tfrac{1}{8}\hat{\hat{W}}_y\,, \\
\bar{\gamma}_{xy} &= \bar{V}_x + \bar{U}_y + \bar{W}_x\bar{W}_y\,, \\
D_{xy} &= \phi_y + \psi_x + \bar{W}_x\hat{W}_y + \hat{W}_x\bar{W}_y\,, \\
E_{xy} &= \tfrac{1}{2}\bar{W}_x\hat{\hat{W}}_y + \hat{W}_x\hat{W}_y + \tfrac{1}{2}\hat{\hat{W}}_x\bar{W}_y\,, \\
F_{xy} &= \tfrac{1}{2}\left(\hat{W}_x\hat{\hat{W}}_y + \hat{\hat{W}}_x\hat{W}_y\right), \\
H_{xy} &= \tfrac{1}{4}\hat{\hat{W}}_x\hat{\hat{W}}_y\,, \\
\bar{\gamma}_{xz} &= \phi + \bar{W}_x\,, \\
\text{and}\qquad \bar{\gamma}_{yz} &= \psi + \bar{W}_y\,.
\end{aligned}\right\}\qquad \text{(A-6)}$$

A.3 The Components of Stress

With equation (A-1), the following components of stress are assumed:

$$
\left.
\begin{aligned}
\sigma_{yy} &= \frac{N_{yy}}{h} + \frac{12 z M_{yy}}{h^3}, \\
\sigma_{xx} &= 0, \\
\tau_{xy} &= 0, \\
\text{and} \quad \tau_{xz} &= 0.
\end{aligned}
\right\} \qquad (A\text{-}7)
$$

The other components of stress, i.e., σ_{zz} and τ_{yz} are determined by the direct solution of the three stress-equilibrium equations which are:

$$
\left.
\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho X_B &= 0, \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho Y_B &= 0, \\
\text{and} \quad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho Z_B &= 0.
\end{aligned}
\right\} \qquad (A\text{-}8)
$$

With the assumptions of equation (A-7), equation (A-8) reduces to

$$
\left.
\begin{aligned}
\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho Y_B &= 0, \\
\text{and} \quad \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho Z_B &= 0.
\end{aligned}
\right\} \qquad (A\text{-}9)
$$

A.4 Force and Moment Equilibrium Conditions

Neglecting the body force and integrating equation (A-9) with respect to z yields the traction force-equilibrium equations as follows:

$$\left.\begin{aligned} \frac{\partial}{\partial y}\left(N_{yy}\right) &= 0, \\ \frac{\partial}{\partial y}\left(V_{yz}\right) + p_z &= 0. \end{aligned}\right\} \qquad \text{(A-10)}$$

and

The moment-equilibrium equation obtained by multiplying the first equation of equation (A-9) by z and then integrating through the thickness takes the form

$$\frac{\partial}{\partial y}\left(M_{yy}\right) - V_{yz} = 0. \qquad \text{(A-11)}$$

A.5 Determination the Stresses τ_{yz} and σ_{zz}

The stress component τ_{yz} is obtained by substituting equation(A-7) into the first equation of equation (A-9), noting the force and moment relations in equations(A-10) and (A-11), and applying the boundary condition for τ_{yz} at $z = \pm\frac{h}{2}$. This results in the following stress distribution:

$$\tau_{yz} = \frac{3V_{yz}}{2h}\left[1 - \left(\frac{z}{h/2}\right)\right]. \qquad \text{(A-12)}$$

The same procedure is performed to the second equation of equation (A-9) to determine σ_{zz}, and there results,

$$\sigma_{zz} = \frac{p_z}{2}\left[1 + \frac{z}{h/2}\left\{\frac{3}{2} - \frac{1}{2}\left(\frac{z}{h/2}\right)^2\right\}\right]. \qquad \text{(A-13)}$$

A.6 <u>Reissner 's Variational Theorem</u>

Reissner 's theorem states that: "The equilibrium state of a body is such that $\delta I = 0$ for arbitary variations of $U, V, W, \sigma_{xx}, \sigma_{yy}, \ldots\ldots, \tau_{yz}$. The condition that $\delta I = 0$ ensures the following:

(1) The satisfaction of the differential equations of equilibrium,

(2) the stress-displacement relations, and

(3) the boundary conditions."

Reissner 's variational theorem of three dimensional elasticity is written in the form

$$\delta I = \delta \int_{t_1}^{t_2} \Bigg[\iiint_V \Big[\left(\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{zz}\epsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz} \right)$$

$$- \frac{1}{2E} \Big\{ \left(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 \right) - 2\nu \left(\sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz} \right)$$

$$+ 2(1+\nu)\left(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2 \right) \Big\} \Big] \, dx\,dy\,dz$$

$$- \frac{\rho}{2} \iiint_V \left[U_t^2 + V_t^2 + W_t^2 \right] dx\,dy\,dz$$

$$- \iint_S \left\{ \left(p_1^+ U^+ + p_2^+ V^+ + p_3^+ W^+ \right) + \left(p_1^- U^- + p_2^- V^- + p_3^- W^- \right) \right\} dx\,dy \Bigg] \, dt = 0. \qquad \text{(A-14)}$$

where the first term in the integrand represents twice the internal strain energy, the second term - the complementary energy, the third term - the kinetic energy, and the last term - the work done by the external forces on the upper and lower surfaces of the beam.

In this particular case, equation (A-14) reduces to

$$\delta I = \delta \int_{t_1}^{t_2} \Bigg[\iiint_V \Big[\Big\{ \sigma_{yy}\epsilon_{yy} + \sigma_{zz}\epsilon_{zz} + \tau_{yz}\gamma_{yz} \Big\} - \frac{1}{2E}\Big\{ (\sigma_{yy}^2 + \sigma_{zz}^2)$$

$$- 2\nu(\sigma_{yy}\sigma_{zz}) + 2(1+\nu)(\tau_{yz}^2) \Big\} \Big] dx\,dy\,dz$$

$$- \frac{\rho}{2} \iiint_V \Big[V_t^2 + W_t^2 \Big] dx\,dy\,dz$$

$$- \iint_S \Big\{ p_z W^+ \Big\} dx\,dy\,dz \Bigg] dt = 0. \qquad \text{(A-15)}$$

Substituting equations (A-5), (A-4), (A-7), (A-12), and (A-13) into equation (A-15) yields,

$$
\begin{aligned}
\delta I = \delta \int_{t_1}^{t_2} \Bigg[\iiint_V \Bigg[\Bigg\{ & \left(\frac{N_{yy}}{h} + \frac{12 M_{yy} z}{h^3} \right) \left(\bar{\epsilon}_{yy} + z K_y + z^2 C_y + z^3 S_y + z^4 T_y \right) + \\
& p_z \left(\frac{1}{2} + \frac{3z}{2} - \frac{2z^3}{h^3} \right) \left(\hat{W} + z \hat{\hat{W}} \right) + \frac{3 V_{yz}}{2h} \left[1 - \left(\frac{z}{h/2} \right)^2 \right] \left(\bar{\gamma}_{yz} + z \hat{W}_y + \frac{1}{2} z^2 \hat{\hat{W}}_y \right) \Bigg\} \\
& - \frac{1}{2E} \Bigg\{ \left(\frac{N_{yy}}{h} + \frac{12 M_{yy} z}{h^3} \right)^2 + p_z^2 \left(\frac{1}{2} + \frac{3z}{2h} - \frac{2z^3}{h^3} \right)^2 - 2 \nu p_z \Bigg(\frac{N_{yy}}{h} + \\
& \frac{12 M_{yy} z}{h^3} \Bigg) \left(\frac{1}{2} + \frac{3z}{2h} - \frac{2z^3}{h^3} \right) + 2(1+\nu) \left(\frac{3 V_{yz}}{2h} \right)^2 \left[1 - \left(\frac{z}{h/2} \right)^2 \right]^2 \Bigg\} \Bigg] dx\,dy\,dz \\
& - \frac{\rho}{2} \iiint_V \left[\left(\bar{V}_t + z \psi_t \right)^2 + \left(\bar{W}_t + z \hat{W}_t + \frac{z^2}{2} \hat{\hat{W}}_t \right)^2 \right] dx\,dy\,dz \\
& - \iint_S \left\{ p_z \left(\bar{W} + \frac{h}{2} \hat{W} + \frac{h^2}{8} \hat{\hat{W}} \right) \right\} dx\,dy \Bigg] dt = 0 . \qquad \text{(A-16)}
\end{aligned}
$$

Carrying out the integration in equation (A-16) with respect to z in the limits of $\pm\frac{h}{2}$ yields,

$$
\begin{aligned}
\delta\int_{t_1}^{t_2}\Bigg[\iint_S\Bigg[&\Big\{N_{yy}\Big(\bar{\epsilon}_{yy}+\frac{h^2}{12}C_y+\frac{h^4}{80}T_y\Big)+M_{yy}\Big(K_y+\frac{3h^2}{20}S_y\Big)+\Big(\frac{h}{2}\Big)p_z\hat{W}+\\
&\Big(\frac{h^2}{10}\Big)p_z\hat{\hat{W}}+V_{yz}\Big(\bar{\gamma}_{yz}+\frac{h^2}{40}\hat{\hat{W}}_y\Big)\Big\}-\frac{1}{2E}\Big\{\frac{N_{yy}^2}{h}+\frac{12M_{yy}^2}{h^3}+\frac{13h}{35}p_z-2\nu p_z\Big(\frac{1}{2}N_{yy}+\\
&\frac{6M_{yy}}{5h}\Big)+\frac{12(1+\nu)}{5h}V_{yz}^2\Big\}\Bigg]dx\,dy-\frac{\rho h}{2}\iint_S\Big\{\Big(\bar{V}_t^2+\frac{h^2}{12}\psi_t^2\Big)+\\
&\Big(\bar{W}_t^2+\frac{h^2}{12}\bar{W}_t\hat{\hat{W}}_t+\frac{h^2}{12}\hat{W}_t+\frac{h^4}{320}\hat{\hat{W}}_t^2\Big)\Big\}dx\,dy-\iint_S\Big\{p_z\Big(\bar{W}+\frac{h}{2}\hat{W}+\\
&\frac{h^2}{8}\hat{\hat{W}}\Big)\Big\}dx\,dy\Bigg]dt=0. \qquad \text{(A-17)}
\end{aligned}
$$

Substituting the values of ϵ_{yy}, C_y, T_y, K_y, S_y, and γ_{yz}, into equation (A-17) yields,

$$
\begin{aligned}
\delta\int_{t_1}^{t_2}\Bigg[\iint_S\Bigg[&\Big\{N_{yy}\Big(\bar{V}_y+\frac{1}{2}\bar{W}_y^2+\frac{h^2}{24}\bar{W}_y\hat{\hat{W}}_y+\frac{h^2}{24}\hat{W}_y^2+\frac{h^4}{640}\hat{\hat{W}}_y^2\Big)+\\
&M_{yy}\Big(\psi_y+\bar{W}_y\hat{\hat{W}}_y+\frac{3h^2}{40}\hat{W}_y\hat{\hat{W}}_y\Big)+\Big(\frac{h}{2}\Big)p_z\hat{W}+\Big(\frac{h^2}{10}\Big)p_z\hat{\hat{W}}+\\
&V_{yz}\Big(\psi+\bar{W}_y+\frac{h^2}{40}\hat{\hat{W}}_y\Big)\Big\}-\frac{1}{2Eh}\Big\{N_{yy}^2+\frac{12M_{yy}^2}{h^2}+\frac{13h^2}{35}p_z^2-\\
&\nu h p_z\Big(N_{yy}+\frac{12M_{yy}}{5h}\Big)+\frac{12}{5}(1+\nu)V_{yz}^2\Big\}\Bigg]dx\,dy-\\
&\frac{\rho h}{2}\iint_S\Big\{\Big(\bar{V}_t^2+\frac{h^2}{12}\psi_t^2\Big)+\Big(\bar{W}_t^2+\frac{h^2}{12}\bar{W}_t\hat{\hat{W}}_t+\frac{h^2}{12}\hat{W}_t^2+\frac{h^4}{340}\hat{\hat{W}}_t^2\Big)\Big\}dx\,dy-\\
&\iint_S\Big\{p_z\Big(\bar{W}+\frac{h}{2}\hat{W}+\frac{h^2}{8}\hat{\hat{W}}\Big)\Big\}dx\,dy\Bigg]dt=0. \qquad \text{(A-18)}
\end{aligned}
$$

Integrating by parts and performing the variational operation, equation (A-18) becomes

$$\int_{t_1}^{t_2}\Bigg[\iint_S \Bigg[\left\{-\frac{\partial}{\partial y}(N_{yy}) + \rho h \overline{V}_{tt}\right\}\delta\overline{V} + \left\{-\frac{\partial}{\partial y}(M_{yy}) + V_{yz} + \frac{\rho h^3}{12}\psi_{tt}\right\}\delta\psi +$$

$$\left\{-\frac{\partial}{\partial y}\left[N_{yy}\left(\overline{W}_y + \frac{h^2}{24}\hat{\hat{W}}_y\right) + M_{yy}\hat{W}_y + V_{yz}\right] + \rho h\left(\overline{W}_{tt} + \frac{h^2}{24}\hat{\hat{W}}_{tt}\right) - p_z\right\}\delta\overline{W} +$$

$$\left\{-\frac{\partial}{\partial y}\left[N_{yy}\left(\frac{h^2}{12}\hat{W}_y\right) + M_{yy}\left(\overline{W}_y + \frac{3h^2}{40}\hat{\hat{W}}_y\right)\right] + \frac{\rho h^3}{12}\hat{W}_{tt}\right\}\delta\hat{W} +$$

$$\left\{-\frac{\partial}{\partial y}\left[N_{yy}\left(\frac{h^2}{24}\overline{W}_y + \frac{h^4}{320}\hat{\hat{W}}_y\right) + M_{yy}\left(\frac{3h^2}{40}\hat{W}_y\right) + \frac{h^2}{40}V_{yz}\right] - \frac{h^2}{40}p_z + \frac{\rho h^3}{24}\left(\overline{W}_{tt} + \frac{3h^2}{40}\hat{\hat{W}}_{tt}\right)\right\}\delta\hat{\hat{W}}\Bigg]dx\,dy +$$

$$\iint_S\Bigg[\left\{\left(\overline{V}_y + \frac{1}{2}\overline{W}_y^2 + \frac{h^2}{24}\overline{W}_y\hat{\hat{W}}_y + \frac{h^2}{24}\hat{W}_y^2 + \frac{h^4}{640}\hat{\hat{W}}_y^2\right) - \frac{1}{Eh}\left(N_{yy} - \frac{\nu h}{2}p_z\right)\right\}\delta N_{yy} +$$

$$\left\{\left(\psi_t + \overline{W}_y\hat{W}_y + \frac{3h^2}{40}\hat{W}_y\hat{\hat{W}}_y\right) - \frac{12}{Eh^3}\left(M_{yy} - \frac{\nu h^2}{10}p_z\right)\right\}\delta M_{yy} +$$

$$\left\{\left(\psi + \overline{W}_y + \frac{h^2}{40}\hat{\hat{W}}_y\right) - \frac{1}{Eh}\left[\frac{12(1+\nu)}{5}V_{yz}\right]\right\}\delta V_{yz} +$$

$$\left\{-\overline{W} - \frac{h^2}{40}\hat{\hat{W}} - \frac{1}{Eh}\left[\frac{13h^2}{35}p_z - \frac{\nu h}{2}\left(N_{yy} + \frac{12M_{yy}}{5h}\right)\right]\right\}\delta p_z\Bigg]dx\,dy\Bigg]dt = 0\,. \quad \text{(A-19)}$$

The boundary conditions take the following form:

$$\int_{t_1}^{t_2}\Bigg[\iint\Bigg|(N_{yy})\delta\bar{V} + (M_{yy})\delta\psi + \left\{N_{yy}(\bar{W}_y + \frac{h^2}{24}\hat{\hat{W}}_y) + M_{yy}\hat{W}_y + V_{yz}\right\}\delta\bar{W} +$$

$$\left\{N_{yy}(\frac{h^2}{12}\hat{W}_y) + M_y(\bar{W}_y + \frac{3h^2}{40}\hat{\hat{W}}_y)\right\}\delta\hat{W} +$$

$$\left\{N_{yy}(\frac{h^2}{24}\bar{W}_y + \frac{h^4}{320}\hat{\hat{W}}_y) + M_{yy}(\frac{3h^2}{40}\hat{W}_y) + \frac{h^2}{40}V_{yz}\right\}\delta\hat{\hat{W}}\,\Bigg|_{-y/2}^{y/2} dx\Bigg] dt -$$

$$\iint_S\Bigg|(\rho h\bar{V}_t)\delta\bar{V} + (\frac{\rho h^3}{12}\psi_t)\delta\psi + (\rho h\bar{W}_t + \frac{\rho h^3}{24}\hat{\hat{W}}_t)\delta\bar{W} + (\frac{\rho h^3}{12}\hat{W}_t)\delta\hat{W} +$$

$$(\frac{\rho h^3}{24}\bar{W}_t + \frac{\rho h^5}{320}\hat{\hat{W}}_t)\delta\hat{\hat{W}}\Bigg|_{t_1}^{t_2} dx\,dy = 0. \qquad \text{(A-20)}$$

2.7 Equations of Equilibrium

Since δI vanishes for arbitary variations of $\delta\bar{V}$, $\delta\psi$, $\delta W, \ldots$, and $\delta\hat{\hat{W}}$, it follows that their coefficients are all equal to zero. This condition yields the equilibrium equations below. Also, noting that p_z is $q(y,t)$, one obtains

$$\left.\begin{aligned}
\frac{\partial}{\partial y}(N_{yy}) &= \rho h\bar{V}_{tt}\,, \\
-\frac{\partial}{\partial y}(M_{yy}) + V_{yz} &= -\frac{\rho h^3}{12}\psi_{tt}\,, \\
\frac{\partial}{\partial y}\left\{N_{yy}(\bar{W}_y + \frac{h^2}{24}\hat{\hat{W}}_y) + M_{yy}(\hat{W}_y) + V_{yz}\right\} &= \rho h(\bar{W}_{tt} + \frac{h^2}{24}\hat{\hat{W}}_{tt}) - q(y,t), \\
\frac{\partial}{\partial y}\left\{N_{yy}(\frac{h^2}{12}\hat{W}_y) + M_{yy}(\hat{W}_y + \frac{3h^2}{40}\hat{\hat{W}}_y)\right\} &= \frac{\rho h^3}{12}\hat{W}_{tt}\,, \text{ and} \qquad \text{(A-21)} \\
\frac{\partial}{\partial y}\left\{N_{yy}(\frac{h^2}{24}\bar{W}_y + \frac{h^4}{320}\hat{\hat{W}}_y) + M_{yy}(\frac{3h^2}{40}\hat{W}_y) + \frac{h^2}{40}V_{yz}\right\} &= \frac{\rho h^3}{24}(\bar{W}_{tt} + \frac{3h^2}{40}\hat{\hat{W}}_{tt}) - \frac{h^2}{40}q(y,t).
\end{aligned}\right\}$$

A.8 Stress-Displacement Relations

Following the same argument and setting the coefficients of the functions δN_{yy}, δM_{yy}, and δV_{yz} equal to zero, the stress-displacement relations become

$$\left.\begin{aligned} \bar{V}_y + \frac{1}{2}\bar{W}_y^2 + \frac{h^2}{24}\left(\bar{W}_y\hat{\hat{W}}_y + \hat{W}_y^2\right) + \frac{h^2}{640}\hat{\hat{W}}_y^2 &= \frac{1}{Eh}\left(N_{yy} - \frac{1}{2}\nu h q(y,t)\right), \\ \psi_y + \bar{W}_y\hat{W}_y + \frac{3h^2}{40}\hat{W}_y\hat{\hat{W}}_y &= \frac{12}{Eh^3}\left(M_{yy} - \frac{\nu h^2 q}{10}\right), \\ \text{and} \qquad \psi + \bar{W}_y + \frac{h^2}{40}\hat{\hat{W}}_y &= \frac{1}{Eh}\left\{\frac{12(1+\nu)}{5}V_{yz}\right\}. \end{aligned}\right\} \qquad \text{(A-22)}$$

Integrating each equation of equation (A-22) through the thickness of the beam, the stresses are expressed in terms of displacement as:

$$\left.\begin{aligned} N_y &= EA\left\{\bar{V}_y + \frac{1}{2}\bar{W}_y^2 + \frac{h^2}{24}\left(\bar{W}_y\hat{\hat{W}}_y + \hat{W}_y^2\right) + \frac{h^2}{640}\hat{\hat{W}}_y^2\right\}, \\ M_y &= EI\left(\psi_y + \bar{W}_y\hat{W}_y + \frac{3h^2}{40}\hat{W}_y\hat{\hat{W}}_y\right), \text{ and} \\ V_{yz} &= \frac{GA}{k}\left(\psi + \bar{W}_y + \frac{h^2}{40}\hat{\hat{W}}_y\right), \end{aligned}\right\} \qquad \text{(A-23)}$$

where

$$\left.\begin{aligned} A &= bh, \\ I &= \frac{bh^3}{12}, \\ G &= \frac{E}{2(1+\nu)}, \\ \text{and} \qquad k &= \frac{6}{5}. \end{aligned}\right\} \qquad \text{(A-24)}$$

Similarly, the equations of equilibrium take the following modified form:

$$\left.\begin{aligned}
\frac{\partial}{\partial y}(N_y) &= \rho A \bar{V}_{tt}\,, \\
\frac{\partial}{\partial y}(M_y) + V_{yz} &= -\rho I \psi_{tt}\,, \\
\frac{\partial}{\partial y}\left\{N_y\left(\bar{W}_y + \frac{h^2}{24}\hat{\hat{W}}_y\right) + M_y\left(\hat{W}_y\right) + V_{yz}\right\} &= \rho A\left(\bar{W}_{tt} + \frac{h^2}{40}\hat{\hat{W}}_{tt}\right) - q(y,t)\,, \\
\frac{\partial}{\partial y}\left\{N_y\left(\frac{h^2}{12}\hat{W}_y\right) + M_y\left(\bar{W}_y + \frac{3h^2}{40}\hat{\hat{W}}_y\right)\right\} &= \rho I \hat{W}_{tt}\,, \quad \text{and} \\
\frac{\partial}{\partial y}\left\{N_y\left(\frac{h^2}{24}\bar{W}_y + \frac{h^4}{320}\hat{\hat{W}}_y\right) + M_y\left(\frac{3h^2}{40}\hat{W}_y\right) + \frac{h^2}{40}V_{yz}\right\} &= \frac{1}{2}\rho I\left(\bar{W}_{tt} + \frac{3h^2}{40}\right) - \frac{h^2}{40}q(y,t)\,.
\end{aligned}\right\} \qquad \text{(A-25)}$$

Neglecting all the quadratic terms, equations (A-23) reduce to

$$\left.\begin{aligned}
N_y &= EA\left(\bar{V}_y\right)\,, \\
M_y &= EI\left(\psi_y\right)\,, \\
\text{and} \quad V_{yz} &= \frac{GA}{k}\left(\psi + \bar{W}_y + \frac{h^2}{40}\hat{\hat{W}}_y\right)\,.
\end{aligned}\right\} \qquad \text{(A-26)}$$

Substituting equation (A-26) into equation (A-25) and arranging the resulting equation in matrix form yields

$$\begin{bmatrix} A_{11} & 0 & 0 & 0 & 0 \\ 0 & A_{22} & A_{23} & 0 & A_{25} \\ 0 & A_{32} & A_{33} & A_{34} & A_{35} \\ 0 & 0 & A_{43} & A_{44} & A_{45} \\ 0 & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix} \begin{bmatrix} \bar{V} \\ \psi \\ \bar{W} \\ \hat{W} \\ \hat{\hat{W}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -q(y,t) \\ 0 \\ -\frac{h^2}{40}q(y,t) \end{bmatrix}. \qquad \text{(A-27)}$$

where

$$\left.\begin{aligned}
A_{11} &= EA\left(\frac{\partial^2}{\partial y^2}\right) - \rho A\left(\frac{\partial^2}{\partial t^2}\right) , \\
A_{22} &= -EI\left(\frac{\partial^2}{\partial y^2}\right) + \frac{GA}{K} + \rho I\left(\frac{\partial^2}{\partial t^2}\right) , \\
A_{33} &= \left(N_y + \frac{GA}{K}\right)\left(\frac{\partial^2}{\partial y^2}\right) + \left(\frac{\partial N_y}{\partial y}\right)\left(\frac{\partial}{\partial y}\right) - \rho A\left(\frac{\partial^2}{\partial t^2}\right) , \\
A_{44} &= \frac{h^2}{24}\left\{ N_y\left(\frac{\partial^2}{\partial y^2}\right) + \left(\frac{\partial N_y}{\partial y}\right)\left(\frac{\partial}{\partial y}\right)\right\} - \rho I\left(\frac{\partial^2}{\partial t^2}\right) , \\
A_{55} &= \left(\frac{h^4}{320}N_y + \frac{h^4}{1600}\frac{GA}{K}\right)\left(\frac{\partial^2}{\partial y^2}\right) + \frac{h^4}{1920}\left(\frac{\partial M_y}{\partial y}\right)\left(\frac{\partial}{\partial y}\right) - \frac{3h^2}{80}\left(\frac{\partial^2}{\partial t^2}\right) , \\
A_{23} = A_{32} &= \frac{GA}{K}\left(\frac{\partial}{\partial y}\right) , \\
A_{25} = A_{52} &= \frac{h^2}{40}\frac{GA}{k}\left(\frac{\partial}{\partial y}\right) , \\
A_{34} = A_{43} &= M_y\left(\frac{\partial^2}{\partial y^2}\right) + \left(\frac{\partial M_y}{\partial y}\right)\left(\frac{\partial}{\partial y}\right) , \\
A_{35} = A_{53} &= \left(\frac{h^2}{24}N_y + \frac{h^2}{40}\frac{GA}{K}\right)\left(\frac{\partial^2}{\partial y^2}\right) + \frac{h^2}{24}\left(\frac{\partial M_y}{\partial y}\right)\left(\frac{\partial}{\partial y}\right) - \frac{\rho I}{2}\left(\frac{\partial^2}{\partial t^2}\right) , \\
\text{and} \quad A_{45} = A_{54} &= \frac{3h^2}{40}M_y\left(\frac{\partial^2}{\partial y^2}\right) + \frac{3h^2}{40}\left(\frac{\partial M_y}{\partial y}\right)\left(\frac{\partial}{\partial y}\right) .
\end{aligned}\right\} \tag{A-28}$$

A.9 Special Case of the Beam-Column

In order that stability condition can be investigated and also corresponding to the notation used in literature, the term N is replaced by $-P$. Furthermore, if the effects of transverse normal strain and axial deformation are neglected (i.e. $\bar{V} = \hat{W} = \hat{\hat{W}} = 0$), while referring to the coordinate system in Fig. 1, equations (A-27) and (A-28) reduce to the matrix form

$$\begin{bmatrix} -EI\left(\frac{\partial^2}{\partial x^2}\right) - \frac{GA}{K} - \rho I\left(\frac{\partial^2}{\partial t^2}\right) & \frac{GA}{K}\left(\frac{\partial}{\partial x}\right) \\ \frac{GA}{K}\left(\frac{\partial}{\partial x}\right) & \left(-P + \frac{GA}{K}\right)\left(\frac{\partial^2}{\partial x^2}\right) - \rho A\left(\frac{\partial^2}{\partial t^2}\right) \end{bmatrix} \begin{bmatrix} \psi \\ W \end{bmatrix} = \begin{bmatrix} 0 \\ -q(x,t) \end{bmatrix} . \tag{A-29}$$

To correlate this result with the theory derived by using the minimum potential energy theorem, the function ψ is replaced by the following identity:

$$\psi = \beta - \frac{\partial W}{\partial x} , \tag{A-30}$$

where β is the slope due to shear at the midplane.

Combining equations (A-29) and (A-30) gives

$$\begin{bmatrix} -EI\left(\frac{\partial^2}{\partial x^2}\right) - \frac{GA}{K} - \rho I\left(\frac{\partial^2}{\partial t^2}\right) & EI\left(\frac{\partial^3}{\partial x^3}\right) - \rho I\left(\frac{\partial^3}{\partial x \partial t^2}\right) \\ EI\left(\frac{\partial^3}{\partial x^3}\right) - \rho I\left(\frac{\partial^3}{\partial x \partial t^2}\right) & -EI\left(\frac{\partial^4}{\partial x^4}\right) - P\left(\frac{\partial^2}{\partial x^2}\right) - \rho A\left(\frac{\partial^2}{\partial t^2}\right) - \rho I\left(\frac{\partial^4}{\partial y^2 \partial t^2}\right) \end{bmatrix} \begin{bmatrix} \beta \\ W \end{bmatrix} = \begin{bmatrix} 0 \\ -q(x,t) \end{bmatrix} . \tag{A-31}$$

Comparison of equation (A-31) with equation (8) shows that they are identical. Thus, Reissner 's Variational Theorem yields the same mathematical result.

BIBLIOGRAPHY

1. Bolotin, V.V., "Dynamic Stability of Elastic Systems", Holden-Day, San Francisco, 1964.
2. Langhaar, H.L., "Energy Methods in Applied Mechanics", John Wiley, New York, 1962.
3. Nowacki, W., "Dynamics of Elastic System", John Wiley, New York, 1963.
4. Reissner, E., "On a Variational Theorem for Finite Elastic Deformations", J. Math. and Phys., Volume 32, 1953.
5. Stoker, J.J., "Nonlinear Vibration", Interscience Publishers, New York, 1950.
6. Timoshenko, S.P., and J.M. Gere, "Theory of Elastic Stability", McGraw-Hill, New York, 1961.
7. Timoshenko, S.P., "Vibration Probrems in Engineering", Van Nostrand Reinhold, New York, 1955.