OPTIMUM STRUCTURAL DESIGN, BEAM-COLUMNS

by

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Submitted in Partial Fulfillment of the Requirements
for the Degree of
Master of Science
in the
Civil Engineering
Program

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YOUNGSTOWN STATE UNIVERSITY

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ABSTRACT

OPTIMUM STRUCTURAL DESIGN

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The purpose of this thesis was to derive the equations for optimum design of slender beam-columns by analytical methods. Design charts for simply supported steel beam-columns have been constructed from the equations that were derived. Techniques which lead to fully stressed solutions were employed. The algebraic techniques which are tedious and difficult to handle are presented along with design charts and optimum design equations. From these equations and design charts, it is relatively easy to design a specific form of beam-column to optimum state without using trial and error techniques.

ACKNOWLEDGEMENTS

The author wished to express his sincere gratitude to Dr. Jack D. Bakos, Jr., Assistant Professor of Civil Engineering, Youngstown State University, for his invaluable assistance in the development of this thesis.

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LIST OF SYMBOLS

A	cross sectional area
c	effective column length ratio or distance from neutral axis to outter edge
D	tube diameter
E	modulus of elasticity
e	eccentricity
FC_i	failure constraint
h	web depth
I	moment of inertia
K, k_p	buckling coefficient
k_1	ratio of flange thickness to web thickness
L	length
M	moment
P	concentrated load
r	radius of gyration
s_A	design stress
s_E	Euler buckling stress
s_L	local buckling stress
s_y	yield strength
t	thickness
ψ	slack variable

BACKGROUND

Optimum structural design is the process of determining the best configuration (froms and proportions) over other possible choices which are acceptable under the applicable constraints (limitations & restrictions). Form is the shape and relative arrangements of the component elements while proportions are the size of components.(1)* Proportions are also called "design variables".(2)

Leonard Spunt in <u>Optimum Structural Design,</u>(1) derived the optimum design process into five major steps as following:

Phase 1- <u>Recognition of Environments</u>

The loads acting on the structure and the purpose of the structure must be known since these limit the shape of the structure. For the beam-columns, the factors which are of enviromental concern are:

P , load on the structure

e , essentricity of the load which causes moment on beam-column

L , length of span

c , factor which depends on end conditions

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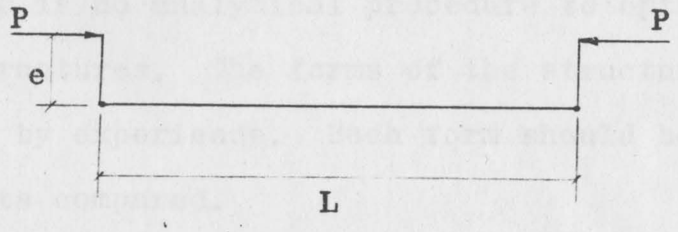
*Numbers in parenthesis indicate reference cites.



$$E_s = \begin{Bmatrix} P \\ L \\ e \\ c \end{Bmatrix}$$

Fig. B.1

These factors are called environment factors (E_s).

Phase 2- Establishment of Criteria

The goal in optimizing such as minimum weight or minimum area is represented by a function. This funtion is called "merit function".(1) For example in designing a minimum cross section area of a circular tube column, the merit function is:

$$A = \pi D t \qquad \text{--------*}$$

where

A is cross section area

D is diameter of tube

t is thickness of wall

This function is also called "objective function".(2)

Phase 3- Specification of Form

These is no analytical procedure to optimize the forms of the structures. The forms of the structure have to be predicted by experience. Each form should be optimized and the results compared.

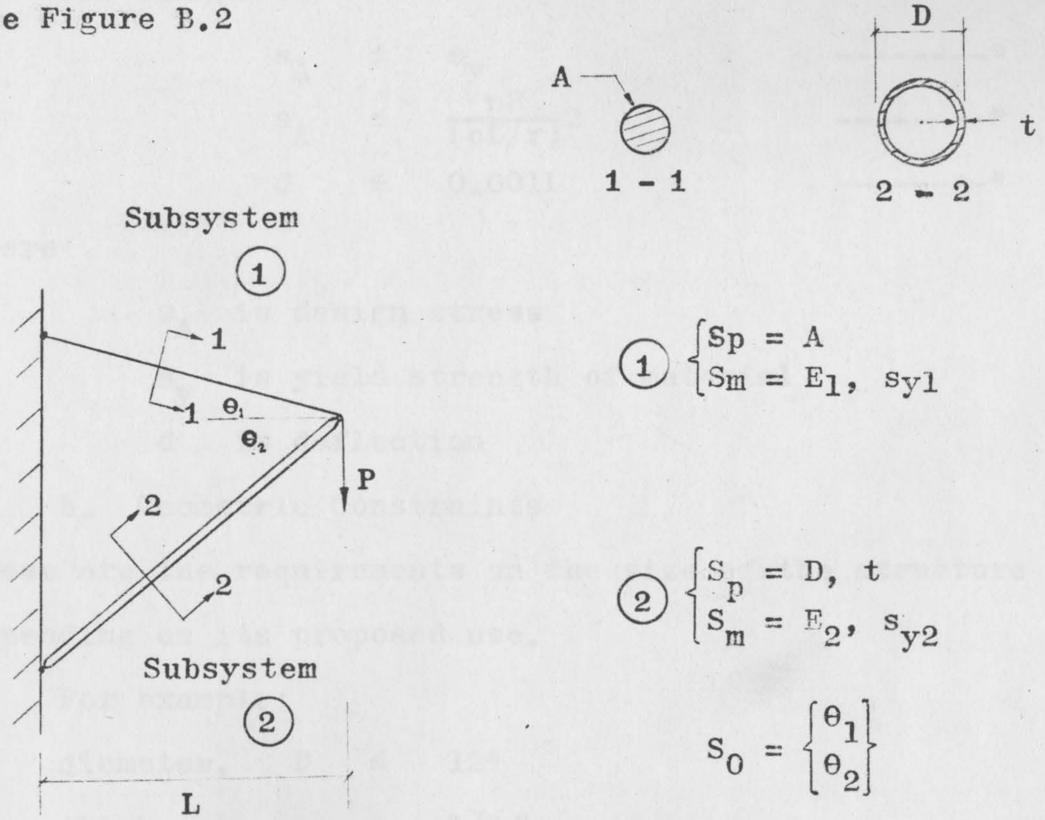
When the form has been specified, the system variables are defined. There are three types of system variables:

S_p proportion variables

S_0 orientation variables

S_m material variables

See Figure B.2



① $\begin{cases} Sp = A \\ Sm = E_1, \; s_{y1} \end{cases}$

② $\begin{cases} S_p = D, \; t \\ S_m = E_2, \; s_{y2} \end{cases}$

$$S_0 = \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

Fig. B.2

Phase 4- <u>Recognition of Constraints</u>

Since it is required to find the minimum area of a given section, the question is; *what is the least area that can be used?* The answer is; *when it does not fail and it satisfies the geometric requirements.* The constraints can be classified as two types.

a. Failure Constraints

These constraints are the limitations and restrictions that prevent the structure from excessive stressing and deflection and buckling.

For example:

$$s_A \leq s_y \qquad \text{--------*}$$

$$s_A \leq \frac{nE}{(cL/r)^2} \qquad \text{--------*}$$

$$d \leq 0.001L \qquad \text{--------*}$$

Where ,

s_A is design stress

s_y is yield strength of material

d is deflection

b. Geometric Constraints

These are the requirements on the size of the structure depending on its proposed use.

For example:

diameter, $D \leq 12''$

thickness, $t \geq 1/2''$

height, $H \geq 24''$

Phase 5- Optimization

The techniques of optimization can be classified broadly as analytical and numerical.

Numerical Methods

These methods employ the concept of mathematical programming and the use of systematic numerical algorithms. The work that will be presented here employs only analytical methods.

Analytical Methods

An analytical methods employs algebraic techniques and the following:

a. Slack Variable

Slack variables are the parameters which are less than or equal to unity ($\Psi \leq 1$). By employing slack variables, the failure constraints which are written in inequality forms can be changed to equality equations and the nature of inequality still remains.

For example:

$$s_A \leq s_y \qquad \text{--------*}$$

$$s_A = \Psi_y s_y \qquad \text{--------*}$$

where

$$\Psi_y \leq 1$$

It will be seen that these two equations are the same.

SMD)

be stated as follow:

failure modes which

neously under the action

ints cannot be explicitly

to the difficulty in

e to optimize by

technique.

Design Space

Design space is not a part of analytical methods, but it is a tool that assists in understanding the techniques used in optimization. The n-dimension space represents all possible design points, and the failure constraints can be visualized as a hypersurfaces which divide the design space into acceptable and unacceptable regions. Another type of hypersurface is the family of constant value contours of the merit function. (see Figure B.3)

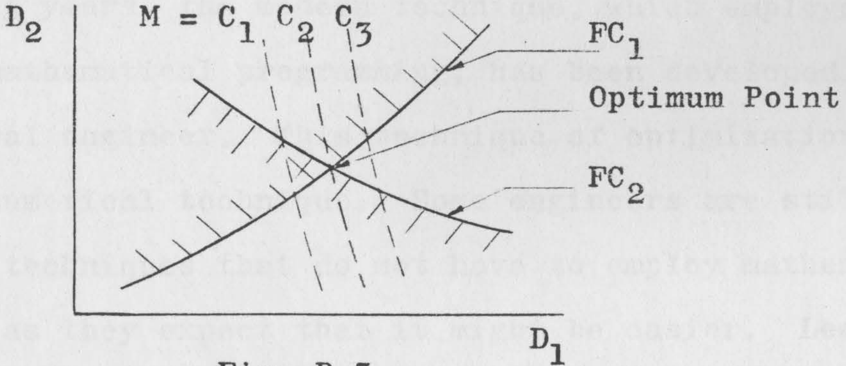


Fig. B.3

The unacceptable regions are crosshatched in Figure B.3 and where

D_1 , D_2 , are design variables.

FC_1, FC_2, are failure constraints.

M is a merit function, shown by dashed curves for each constant value.

INTRODUCTION

Most of the engineering structures today use trial and error procedure in their design project. Time saving economic designs have been a long desired goal. The best economical design that satisfies all the limitations without a lengthy trial process is desired by designers. This is the way that optimum design aspect arises. With the aid of the electrical computer, the problem of designing for an optimum state has become possible.

In recent years, the modern technique, which employs the concept of mathematical programming, has been developed by the structural engineer. This technique of optimization is called the numerical technique. Some engineers are still looking for techniques that do not have to employ mathematical programming as they expect that it might be easier. Leonard Spunt(1) has derived the equations for optimum design for beams and columns by using analytical methods. The backgro~~o~~und of this work has been taken from his book, "Optimum Structural Design" and extended to beam-columns.

The work that is presented here derives the equations that determine the best proportions for the specific form of beam-column. With these proportions, the beam-column will be ~~the~~ designed optimumly.

The shapes of the cross-sections of the beam-columns which will be considered in this study are as follows:

1. Thin Circular Tube
2. H - Section
3. Thin Rectangular Tube

Only uniform cross sections will be considered.

CHAPTER I

CIRCULAR TUBE SECTION

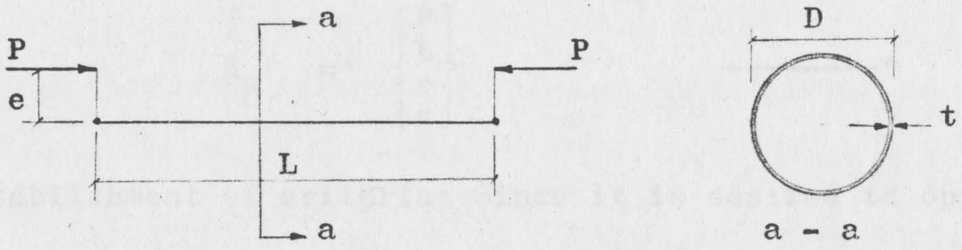


Fig. 1.1

For thin wall circular tube section shown in Figure 1.1 ;

$$A = \pi D t \qquad \text{---------(1.1)}$$

$$I = \frac{\pi D^3 t}{8} \qquad \text{---------(1.2)}$$

$$r^2 = \frac{I}{A}$$

$$= \frac{D^2}{8} \qquad \text{---------(1.3)}$$

$$M = Pe \qquad \text{---------(1.4)}$$

$$s_A = \frac{Mc}{I} + \frac{P}{A}$$

$$= \frac{4Pe + DP}{\pi D^2 t} \qquad \text{---------(1.5)}$$

The phases of optimum design applied to the circulár tube are as follows;

Recognition of environment; The following is applicable

$$E_s = \begin{Bmatrix} P \\ L \\ e \\ c \end{Bmatrix} \qquad \text{--------*}$$

Establishment of criteria; Since it is desired to optimize the area of cross-section, the merit function is given by

$$A = \pi Dt \qquad \text{--------*}$$

Specification of form; For the circular tube section considered here;

$$S_p = \begin{Bmatrix} D \\ t \end{Bmatrix} \qquad \text{--------*}$$

$$S_m = \begin{Bmatrix} E \\ s_y \end{Bmatrix} \qquad \text{--------*}$$

Recognition of constraints; Only failure constraints will be considered here. These constraints will prevent the structure from over stress and buckling.

FC_1 , design stress can not be more than yield strength;

$$s_A \leq s_y \qquad \text{--------*}$$

FC_2 , from Euler Buckling formula;(1)(4)

$$s_A \leq \frac{\pi^2 E}{(cL/r)^2}$$

$$\leq \frac{\pi^2 E D^2}{8c^2 L^2} \qquad \text{--------*}$$

FC_3 , local buckling;$^{(1)(4)}$

$$s_A \leq KE(t/D) \qquad \text{---------*}$$

where

FC_i are failure constraints

c is factor which depends on end conditions

K is buckling coefficient

For optimization, the failure constraints are rewritten into equality equations by employing slack variables.

$$FC_1 , \quad s_A = \psi_y s_y \qquad \text{---------(1.6)}$$

where $\psi_y \leq 1$

$$FC_2 , \quad s_A = \frac{\psi_E \pi^2 ED^2}{8c^2L^2} \qquad \text{---------(1.7)}$$

where $\psi_E \leq 1$

$$FC_3 , \quad s_A = \psi_L KE(t/D) \qquad \text{---------(1.8)}$$

where $\psi_L \leq 1$

It is impossible to optimize the cross-section area directly because of the difficulty in algebraic techniques. Indirect method is presented here by optimizing the design stress.

$$s_A \longrightarrow \text{Maximum.}$$

By equating Equations (1.6) and (1.7);

$$\psi_y s_y = \frac{\psi_E \pi^2 E D^2}{8c^2 L^2}$$

$$D^2 = \frac{8\psi_y s_y c^2 L^2}{\psi_E \pi^2 E}$$

$$D = \left(\frac{8\psi_y s_y}{\psi_E E}\right)^{\frac{1}{2}} \frac{cL}{\pi} \quad \text{---------(1.9)}$$

By equating Equation (1.6) and (1.8);

$$\psi_y s_y = \psi_L KE(t/D)$$

$$t = \frac{\psi_y s_y D}{\psi_L KE} \quad \text{---------(1.10)}$$

Substituting the value of D from equation (1.9) into Equation (1.10);

$$t = \frac{\psi_y s_y}{\psi_L KE} \left(\frac{8\psi_y s_y}{\psi_E E}\right)^{\frac{1}{2}} \frac{cL}{\pi} \quad \text{--(1.11)}$$

By equating Equations (1.6) and (1.5);

$$\psi_y s_y = \frac{4Pe + DP}{\pi D^2 t} \quad \text{---------(1.12)}$$

Substituting the values of t and D from Equation (1.11) and (1.9) into Equation (1.12);

$$\Psi_y s_y = \frac{4Pe + \left(\frac{8\Psi_y s_y}{\Psi_E E}\right)^{\frac{1}{2}} \frac{cL}{\pi} P}{\frac{\pi 8 \Psi_y s_y c^2 L^2}{\Psi_E \pi^2 E} \quad \frac{\Psi_y s_y}{\Psi_L KE} \quad \left(\frac{8\Psi_y s_y}{\Psi_E E}\right)^{\frac{1}{2}} \quad \frac{cL}{\pi}}$$

$$\Psi_y^{7/2} = \frac{4Pe\Psi_E^{3/2}\pi^2\Psi_L KE^{5/2}}{8^{3/2}c^3L^3s_y^{7/2}} + \frac{\Psi_y^{\frac{1}{2}}\pi\Psi_E\Psi_L E^2KP}{8s_y^3c^2L^2} \quad (1.13)$$

Since there are two proportional variables, t and D, only two failure constraints can occur simultaneously. Letting FC_2 and FC_3 be the failure constraints that will occur simultaneously;

$$\Psi_E \;\&\; \Psi_L = 1 \quad \text{--------}(1.14)$$

Substituting the values of Ψ_E and Ψ_L from Equation (1.14) into Equation (1.13) and rearranging;

$$\Psi_y^{7/2} - \frac{\Psi_y^{\frac{1}{2}}\pi E^2K(P/L)^2}{8s_y^3c^2} - \frac{4(P/L^2)(e/L)\pi^2KE^{5/2}}{8^{3/2}c^3s_y^{1/2}} = 0 \quad \text{------}(1.15)$$

For a simply supported beam-column;

$$c = 1$$

From Timoshenko;(4)

$$K = 0.4$$

Using AISI 1025 steel;

$$E = 30 \times 10^6 \text{ psi.}$$

$$s_y = 36{,}000 \text{ psi.}$$

Substituting these values into Equation (1.15);

$$\psi_y^{7/2} - 3.04(P/L^2)\psi_y^{\frac{1}{2}} - 388(P/L^2)(e/L) = 0 \text{ ----(1.16)}$$

Solutions of Equation (1.16) were obtained by running the computer programs (see Appendix A). For $\psi_y = 1$, the relation between P/L^2 and e/L is shown in Figure 1.2 and Figure 1.3 . The relation between ψ_y and P/L^2 for e/L equal to 0.00 , 0.01 , 0.05 and 0.10 are shown in Figure 1.4 .

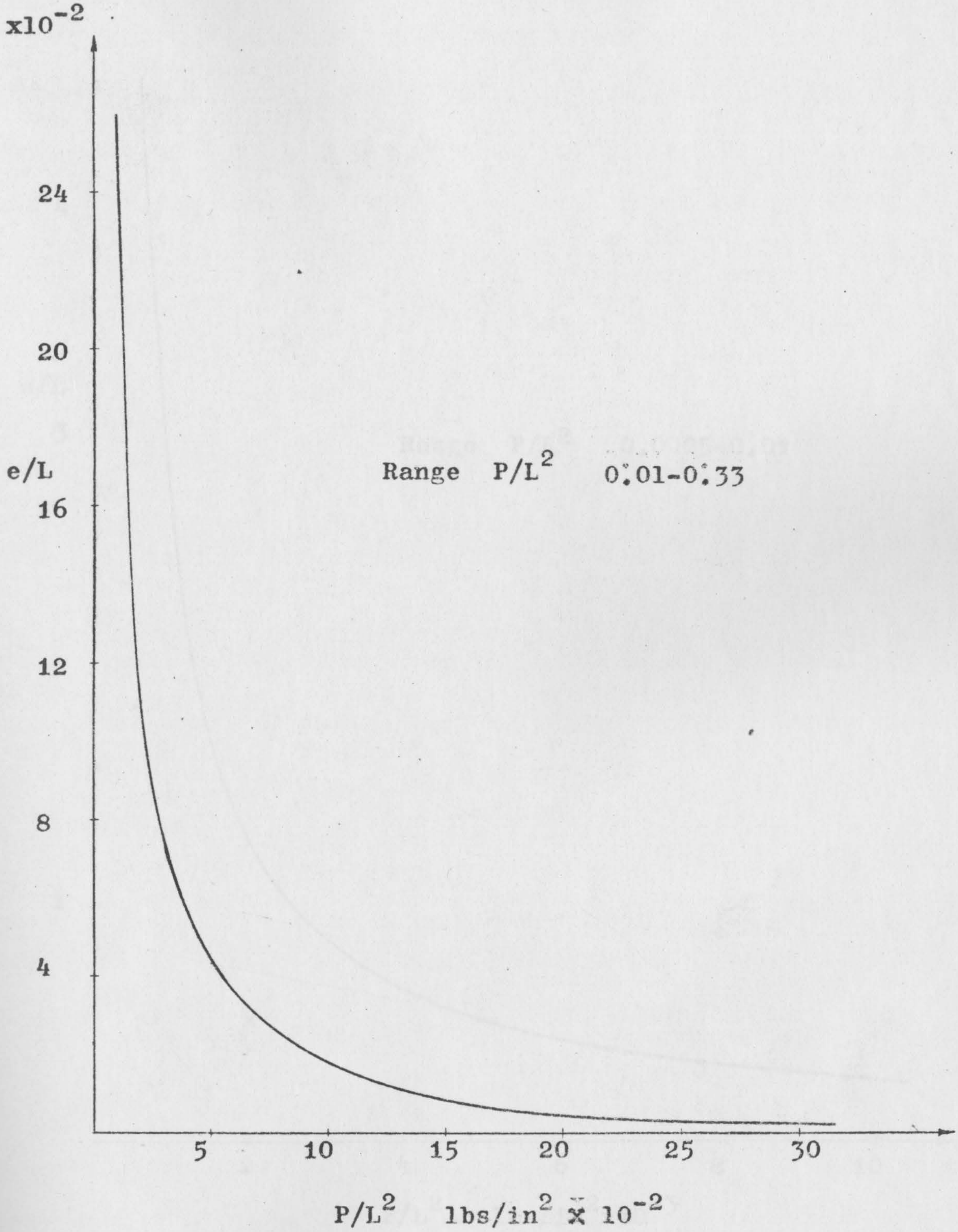


Figure 1.2 e/L & P/L^2 Relationship for $\psi_y = 1$

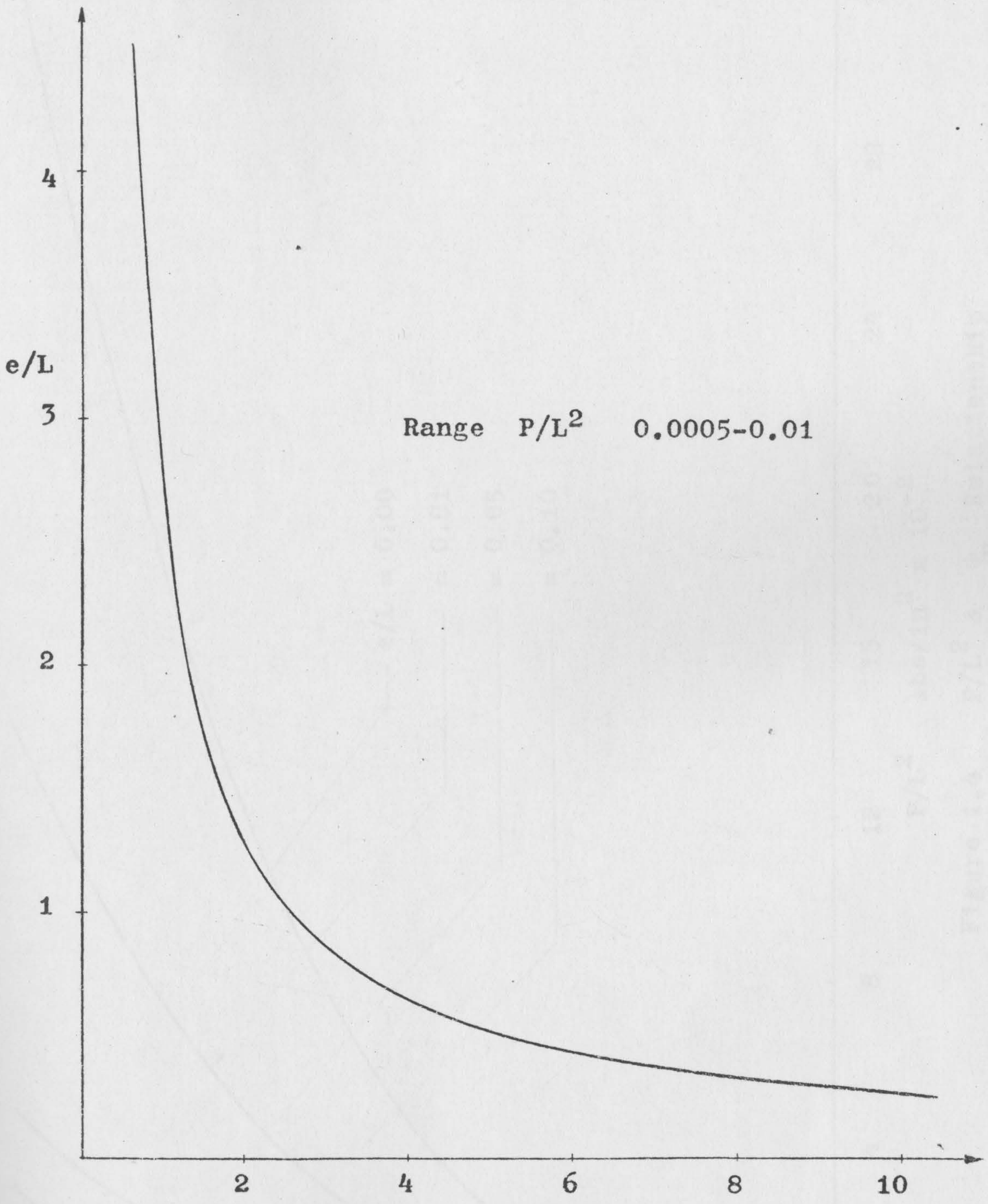


Figure 1.3 e/L & P/L^2 Relationship for $\psi_y = 1$

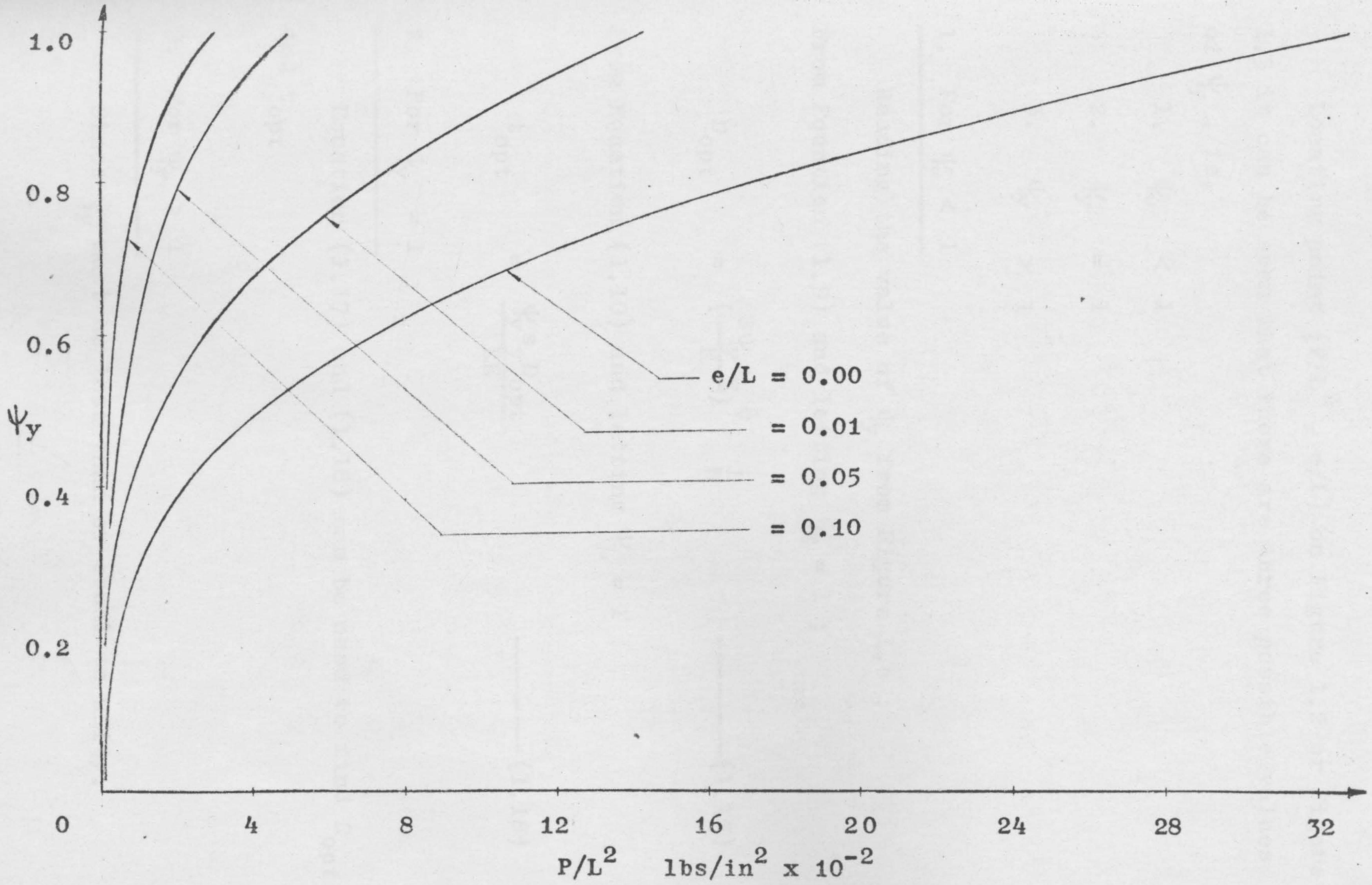


Figure 1.4 P/L^2 & ψ_y Relationship

Locating point $(P/L^2 , e/L)$ on Figure 1.2 or Figure 1.3 it can be seen that there are three possible values of ψ_y , ie, i.e.

1. $\psi_y < 1$
2. $\psi_y = 1$
3. $\psi_y > 1$

1. For $\psi_y < 1$

Reading the value of ψ_y from Figure 1.4 ;

From Equation (1.9) and letting $\psi_E = 1$;

$$D_{opt} = \left(\frac{8\psi_y s_y}{E}\right)^{\frac{1}{2}} \frac{L}{\pi} \qquad \text{---------(1.17)}$$

From Equation (1.10) and letting $\psi_L = 1$;

$$t_{opt} = \frac{\psi_y s_y D_{opt}}{KE} \qquad \text{---------(1.18)}$$

2. For $\psi_y = 1$

Equation (1.17) and (1.18) can be used to find D_{opt} and t_{opt}

3. For $\psi_y > 1$

? — Since ψ_y must be less than or equal to unity.

Letting $\Psi_y = 1$ --------*

Therefore, the values of Ψ_E and Ψ_L cannot be equal to unity at the same time because there are only two proportional variables (D & t).

Letting $\Psi_L = 1$ --------*

Since $\Psi_y = 1$, from Equation (1.6)

$$s_A = s_y \qquad \text{--------*}$$

From Equation (1.8);

$$s_y = KE(t/D) \qquad \text{--------(1.19)}$$

From Equation (1.5);

$$s_y = \frac{4Pe + DP}{\pi D^2 t} \qquad \text{--------(1.20)}$$

Solving Equations (1.19) and (1.20) for D_{opt} and t_{opt} . Result of these two equations were obtained by running the computer programs (see Appendix A) and shown in Figure 1.5 .

Note: In this case, if $\Psi_E = 1$, the optimum solution will not be obtained.

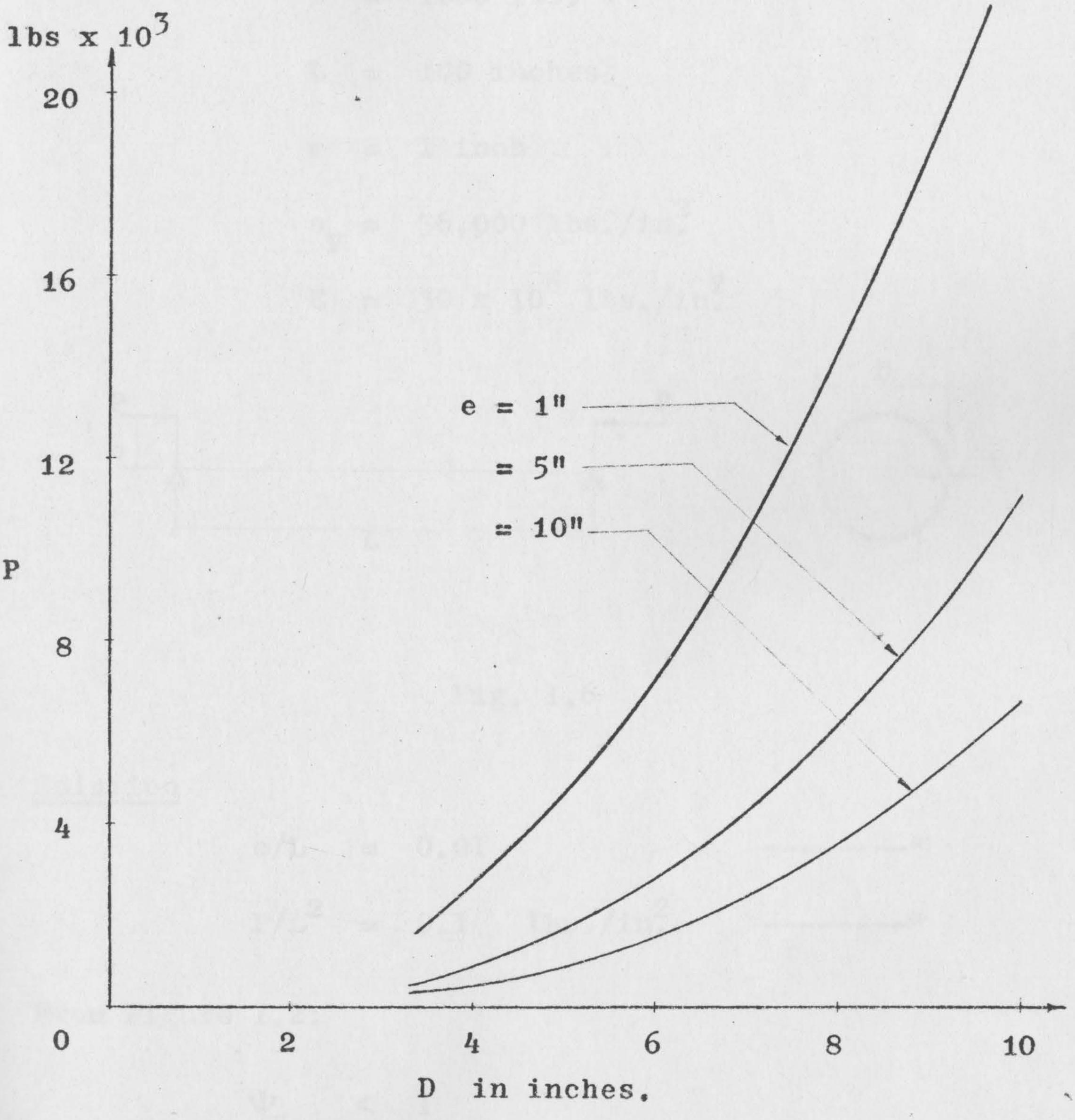


Figure 1.5 Result of Equations (1.19) & (1.20)

Example 1.1 Design a simply supported circular tube beam-column for the following environment factors and material variables :

$$P = 1000 \text{ lbs.}$$

$$L = 100 \text{ inches}$$

$$e = 1 \text{ inch}$$

$$s_y = 36{,}000 \text{ lbs./in.}^2$$

$$E = 30 \times 10^6 \text{ lbs./in.}^2$$

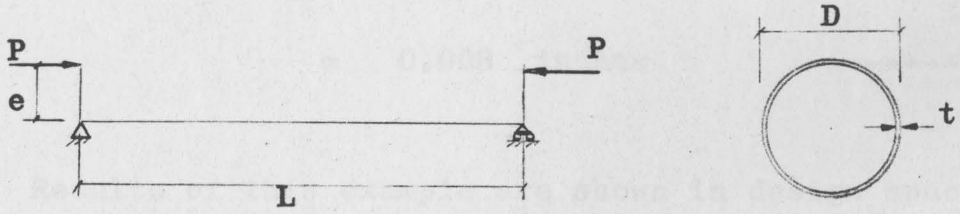


Fig. 1.6

Solution

$$e/L = 0.01 \qquad \text{--------*}$$

$$P/L^2 = 0.1 \quad \text{lbs./in.}^2 \qquad \text{--------*}$$

From Figure 1.2;

$$\psi_y < 1$$

From Figure 1.4;

$$\psi_y = 0.892$$

From Equation (1.17);

$$D_{opt} = \left(\frac{8 \times 0.892 \times 36,000}{30 \times 10^6} \right)^{\frac{1}{2}} \frac{100}{3.14}$$

$$= 2.95 \text{ inches} \qquad \text{--------*}$$

From Equation (1.18);

$$t_{opt} = \frac{0.892 \times 36,000}{0.4 \times 30 \times 10^6} \times 2.95$$

$$= 0.008 \text{ inches} \qquad \text{--------*}$$

Results of this example are shown in design space in Figure 1.7 .

$$\Psi_y = 0.892$$

$$FC_2 \quad D \geq 2.95 \quad .....(1)$$

$$FC_3 \quad t/D \geq 0.0027 \quad .....(2)$$

$$\frac{10}{\pi D t} + \frac{40}{\pi D^2 t} = 321 \quad .....(3)$$

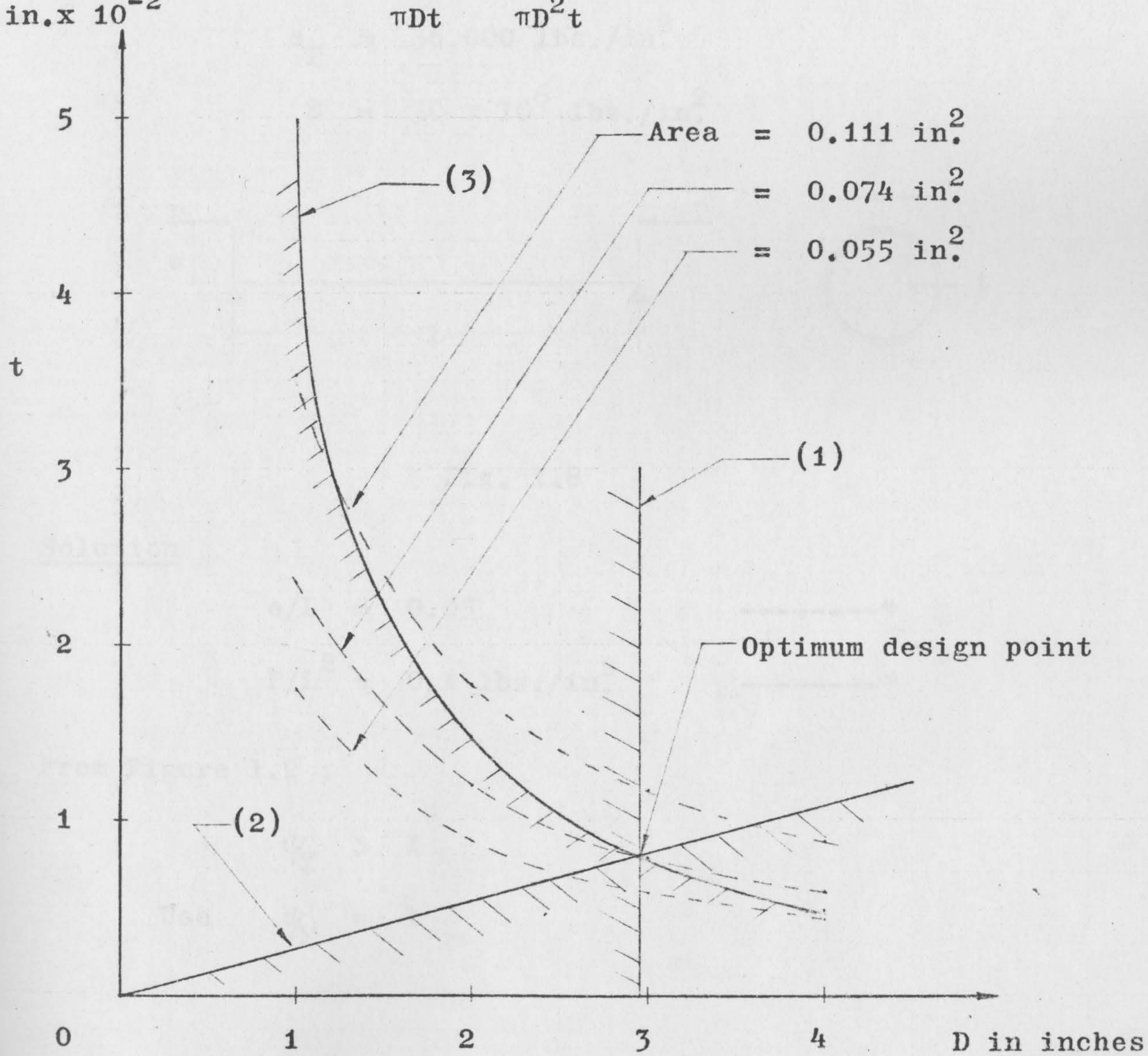


Figure 1.7 Design space of Example 1.1 , unacceptable regions are shown crosshatched.

Example 1.2 Design a simply supported circular beam-column for the following environment factors and material variables:

$$P = 1000 \text{ lbs.}$$

$$L = 100 \text{ inches}$$

$$e = 5 \text{ inches}$$

$$s_y = 36{,}000 \text{ lbs./in.}^2$$

$$E = 30 \times 10^6 \text{ lbs./in.}^2$$

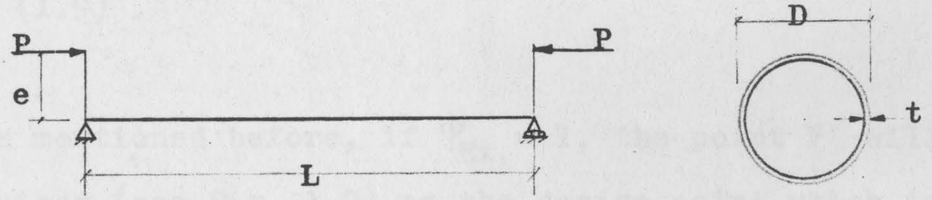


Fig. 1.8

Solution

$$e/L = 0.05 \qquad \text{---------*}$$

$$P/L^2 = 0.1 \text{ lbs./in.}^2 \qquad \text{---------*}$$

From Figure 1.2 ;

$$\psi_y > 1$$

$$\text{Use} \quad \psi_y = 1$$

From Figure 1.5 ;

$$D_{opt} = 4.15 \text{ inches} \qquad \text{--------}*$$

From Equation (1.19) ;

$$t_{opt} = 0.0125 \text{ inches} \qquad \text{--------}*$$

Results of this example are shown in design space in Figure (1.9) .

As mentioned before, if $\Psi_{Ex} = 1$, the point P' will be obtained (see Fig. 1.9) as the design point which is not the optimum design point.

The contraints that intersect at the optimum design point are called "active constraints".(3) The other constraints are called "passive constraints".

In Fig. 1.9 FC_3 is an active constraint and FC_2 is a passive constraint. FC_1 is also an active constraint since $\Psi_y = 1$.

$\psi_y = 1$

$FC_1 \quad s_A = 36{,}000 \text{ lbs./in.}^2$ .....(1)

$FC_2 \quad D \geq 3.12 \text{ in.}$ .....(2)

$FC_3 \quad t/D \geq 0.003$ .....(3)

$\frac{1}{\pi Dt} + \frac{20}{\pi D^2 t} = 36$ .....(4)

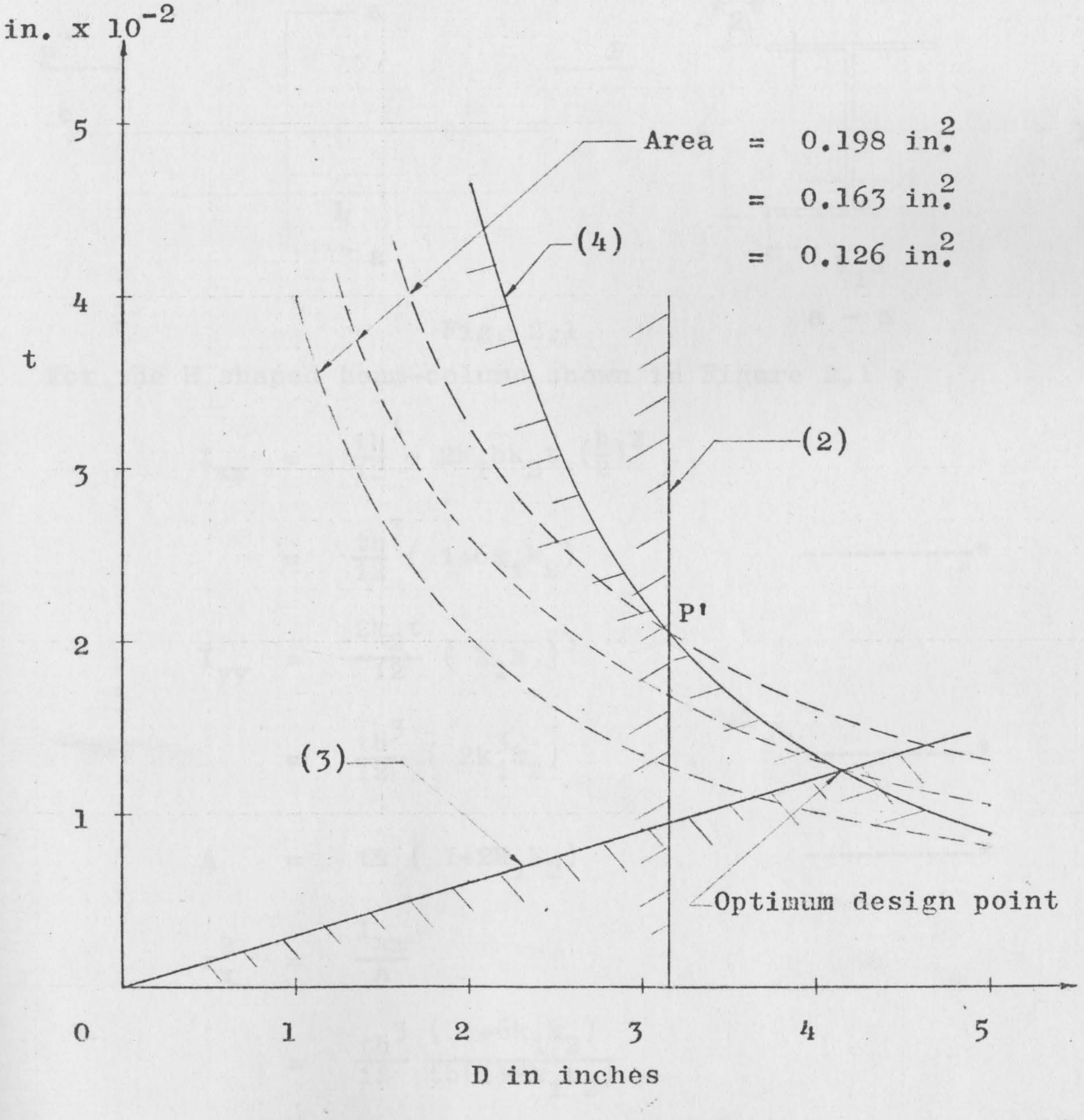
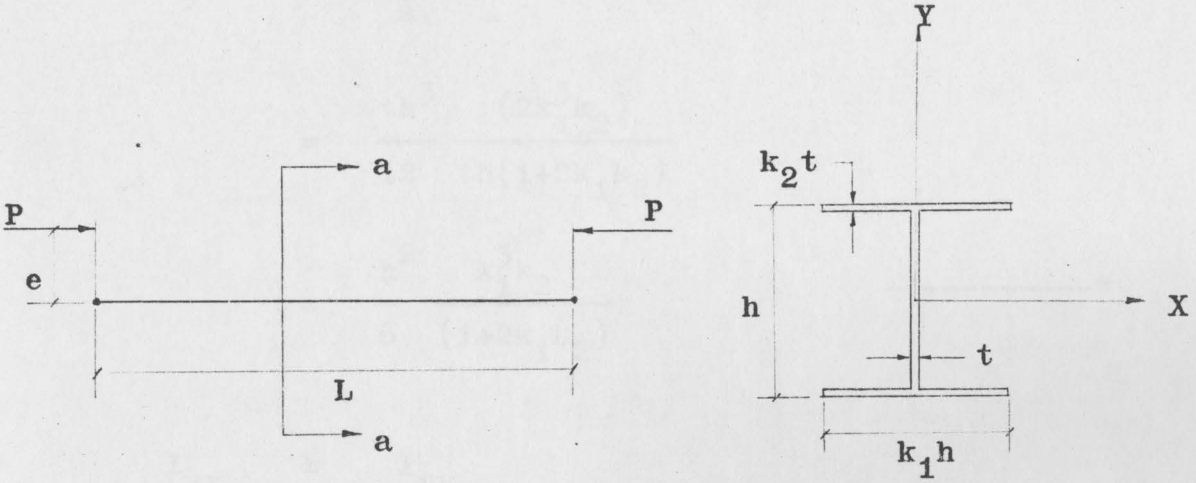


Figure 1.9 Design space of Example 1.2 , unacceptable regions are shown crosshatched.

CHAPTER II

H - SECTION



a - a

Fig. 2.1

For the H shaped beam-column shown in Figure 2.1 ;

$$I_{xx} = \frac{th^3}{12} + 2k_1hk_2t\ (\tfrac{h}{2})^2$$

$$= \frac{th^3}{12}(\ 1+6k_1k_2) \qquad \text{----------*}$$

$$I_{yy} = \frac{2k_2t}{12}(\ k_1h\ )^3$$

$$= \frac{th^3}{12}(\ 2k_1^3k_2) \qquad \text{----------*}$$

$$A = th\ (\ 1+2k_1k_2) \qquad \text{----------*}$$

$$r_x^2 = \frac{I_{xx}}{A}$$

$$= \frac{th^3}{12}\,\frac{(\ 1+6k_1k_2)}{th(1+2k_1k_2)}$$

$$= \frac{h^2}{12} \frac{(1+6k_1k_2)}{(1+2k_1k_2)} \qquad \text{-----------*}$$

$$r_y^2 = \frac{I_{yy}}{A}$$

$$= \frac{th^3}{12} \frac{(2k_1^3k_2)}{th(1+2k_1k_2)}$$

$$= \frac{h^2}{6} \frac{k_1^3k_2}{(1+2k_1k_2)} \qquad \text{-----------*}$$

$$I_{xx} \geq I_{yy}$$

$$1+6k_1k_2 \geq 2k_1^3k_2 \qquad \text{-----------*}$$

$$M = P.e \qquad \text{-----------*}$$

The failure constraints applied to the H-section shown in Figure 2.1 as follows :

FC_1, local buc$\overset{l}{\wedge}$king in flange;(1)(4)

$$s_A \leq s_{Lf}$$

$$\leq 0.385\ E\ \left(\frac{k_2t}{k_1h/2} \right)^2$$

$$= \Psi_{Lf}\ 0.385\ E\ \left(\frac{k_2t}{k_1h/2} \right)^2 \qquad \text{---------(2.1)}$$

where

s_A is design stress

s_{Lf} is local buckling failure stress in flange

k_1, k_2, t and h are proportional variables shown in Figure 2.1

ψ_{Lf} is slack variable

FC_2, local buckling in web;(4)(5)

$$\begin{aligned} s_A &\leq s_{Lw} \\ &\leq k_p E (t/h)^2 \\ &= \psi_{Lw} k_p E (t/h)^2 \end{aligned} \text{------------(2.2)}$$

where

s_{Lw} is local buckling failure stress in web

k_p is buckling coefficient

FC_3, Euler buckling in bending axis;(1)(4)

$$\begin{aligned} s_A &\leq s_{Ex} \\ &\leq \frac{\pi^2 E}{c^2 L^2} \frac{h^2}{12} \frac{(1+6k_1k_2)}{(1+2k_1k_2)} \\ &= \psi_{Ex} \frac{\pi^2 E}{c^2 L^2} \frac{h^2}{12} \frac{(1+6k_1k_2)}{(1+2k_1k_2)} \end{aligned} \text{----------(2.3)}$$

where

s_{Ex} is buckling stress in bending axis

c is factor which depends on end conditions

FC_4, Euler buckling in lateral direction;$^{(1)(4)}$

$$s_{A1} \leq s_{Ey}$$

$$\leq \frac{\pi^2 E}{c^2 L^2} \frac{h^2}{6} \frac{k_1^3 k_2}{(1+2k_1 k_2)} \qquad \text{----------(2.4)}$$

where

s_{A1} is compressive stress due to axial load only

s_{Ey} is buckling stress in lateral direction

FC_5, design stress can not be more than yield strength;

$$s_A \leq s_y$$

$$= \psi_y s_y \qquad \text{---------(2.5)}$$

? For linear materials in which the applied stress is not more than yield strength. The formula for combined axial and bending stress is;

$$s_A = \frac{P}{A} + \frac{M.c}{I_{xx}}$$

$$= \frac{P}{th(1+2k_1k_2)} + \frac{6\ Pe}{th^2(1+6k_1k_2)} \qquad \text{----(2.6)}$$

Using the same procedure used in Chapter I, the method proceeds indirectly by optimizing the design stress.

By equating Equations (2.1) and (2.2);

$$\psi_{Lw} k_p \; E(t/h)^2 = \psi_{Lf} \; 0.358 \; E \left(\frac{k_2 t}{k_1 h/2} \right)^2$$

Letting $\psi_{Lw} = \psi_{Lf}$ by S.M.D.

$$k_2 = k_1 \left(\frac{k_p}{1.54} \right)^{\frac{1}{2}} \qquad \text{----------(2.7)}$$

By equating Equations(2.5) and (2.6);

$$\psi_y s_y = \frac{P}{th(1+2k_1 k_2)} + \frac{6Pe}{th^2(1+6k_1 k_2)} \qquad \text{--------(2.8)}$$

Multiplying Equation (2.2) by Equation (2.3);

$$s_A^2 = \psi_{Lw} \; k_p \; E \; (t/h)^2 \; \psi_{Ex} \frac{\pi^2 E}{c^2 L^2} \; \frac{h^2}{12} \; \frac{(1+6k_1 k_2)}{(1+2k_1 k_2)} \qquad \text{--(2.9)}$$

Substituting the value of s_A from Equation (2.5) into Equation (2.9);

$$\psi_y^2 s_y^2 = \psi_{Lw} \; \psi_{Ex} \; k_p \frac{t^2 \pi^2 E^2}{12c^2 L^2} \; \frac{(1+6k_1 k_2)}{(1+2k_1 k_2)}$$

$$t^2 = \frac{12c^2 L^2 \psi_y^2 s_y^2}{\psi_{Lw} \psi_{Ex} k_p \pi^2 E^2} \; \frac{(1+2k_1 k_2)}{(1+6k_1 k_2)}$$

$$t = \frac{12^{\frac{1}{2}} cL \psi_y s_y}{(\psi_{Lw}\psi_{Ex})^{\frac{1}{2}} k_p^{\frac{1}{2}} \pi E} \frac{(1+2k_1k_2)^{\frac{1}{2}}}{(1+6k_1k_2)^{\frac{1}{2}}} \text{------(2.10)}$$

By equating Equations (2.3) and (2.5);

$$\psi_y s_y = \psi_{Ex} \frac{\pi^2 E}{c^2 L^2} \frac{h^2}{12} \frac{(1+6k_1k_2)}{(1+2k_1k_2)}$$

$$h = \frac{\psi_y^{\frac{1}{2}} s_y^{\frac{1}{2}} 12^{\frac{1}{2}} cL}{\psi_{Ex}^{\frac{1}{2}} \pi E^{\frac{1}{2}}} \frac{(1+2k_1k_2)^{\frac{1}{2}}}{(1+6k_1k_2)^{\frac{1}{2}}} \text{---------(2.11)}$$

Substituting the values of t and h from Equations (2.10) and (2.11) into Equation (2.8) ;

$$\psi_y s_y = \frac{P}{\frac{12^{\frac{1}{2}} cL \psi_y s_y}{(\psi_{Lw}\psi_{Ex})^{\frac{1}{2}} k_p^{\frac{1}{2}} \pi E} \frac{(1+2k_1k_2)^{\frac{1}{2}}}{(1+6k_1k_2)^{\frac{1}{2}}} \frac{\psi_y^{\frac{1}{2}} s_y^{\frac{1}{2}}\, 12^{\frac{1}{2}} cL}{\psi_{Ex}^{\frac{1}{2}} \pi E^{\frac{1}{2}}} \frac{(1+2k_1k_2)^{\frac{1}{2}}}{(1+6k_1k_2)^{\frac{1}{2}}} (1+2k_1k_2)}$$

$$+ \frac{6Pe}{\frac{12^{\frac{1}{2}} cL \psi_y s_y}{(\psi_{Lw}\psi_{Ex})^{\frac{1}{2}} k_p^{\frac{1}{2}} \pi E} \frac{(1+2k_1k_2)^{\frac{1}{2}}}{(1+6k_1k_2)^{\frac{1}{2}}} \frac{\psi_y s_y\; 12c^2L^2}{\psi_{Ex}\; \pi^2 E} \frac{(1+2k_1k_2)}{(1+6k_1k_2)} (L+6k_1k_2)}$$

Simplifying the above equation and letting $\psi_{Lw} = \psi_{Ex} = 1$ and for the simple support end condition, $c = 1$. Thus;

$$\frac{P}{L^2} = \frac{12^{3/2}\psi_y^3 s_y^3 (1+2k_1k_2)^2}{k_p^{\frac{1}{2}}\left[\pi^2E^{3/2}(1+6k_1k_2)12^{\frac{1}{2}}\ \psi_y^{\frac{1}{2}}s_y^{\frac{1}{2}} + 6\ \frac{e}{L}\ \pi^3E^2(1+6k_1k_2)^{\frac{1}{2}}(1+2k_1k_2)^{\frac{1}{2}}\right]} \qquad \text{----------(2.12)}$$

From Equation (2.12) and letting $\psi_y = 1$;

$$\frac{P}{L^2} = \frac{12^{3/2}\ s_y^3\ (1+2k_1k_2)^2}{k_p^{\frac{1}{2}}\left[\pi^2E^{3/2}(1+6k_1k_2)12^{\frac{1}{2}}s_y^{\frac{1}{2}}+6\ \frac{e}{L}\ \pi^3E^2\ (1+6k_1k_2)^{\frac{1}{2}}(1+2k_1k_2)^{\frac{1}{2}}\right]} \qquad \text{----------(2.13)}$$

From <u>BUCKLING STRENGTH of METAL STRUCTURES</u> by Bleich:(5)

$$\left.\begin{array}{llll} e/L = 0.00, & & k_p & = 3.62 \\ e/L = 0.01, & & k_p & = 5.47 \\ e/L = 0.05, & & k_p & = 7.54 \\ e/L = 0.10, & & k_p & = 10.30 \\ e/L > 0.10, & & k_p & = 21.7 \end{array}\right\} \text{Approximate Values}$$

From Equation (2.7) ;

$$k_2 = k_1 \left(k_p/1.54 \right)^{\frac{1}{2}} \quad \text{-----*}$$

$e/L = 0.00,\qquad k_2 = 1.53\ k_1$

$e/L = 0.01,\qquad k_2 = 1.89\ k_1$

$e/L = 0.05,\qquad k_2 = 2.22\ k_1$

$e/L = 0.10,\qquad k_2 = 2.59\ k_1$

$e/L > 0.10,\qquad k_2 = 3.76\ k_1$

Using AISI 1025 steel ;

$$E = 30 \times 10^6 \text{ psi.}$$

$$s_y = 36{,}000 \text{ psi.}$$

Substituting the above values into Equation (2.13), the relation between P/L^2 and k_1 are shown in Figures 2.2 - 2.6 for each value of e/L . (see computer program in Appendix A)

The next step is to check whether FC_4 is satisfied or not.

From Equation (2.4);

$$s_{A1} \leq \frac{\pi^2 E}{c^2 L^2} \; \frac{h^2}{6} \; \frac{k_1^3 k_2}{(1+2k_1k_2)}$$

$$\frac{P}{th(1+2k_1k_2)} = \psi_{Ey} \; \frac{\pi^2 E}{c^2 L^2} \; \frac{h^2}{6} \; \frac{k_1^3\ k_2}{(1+2k_1k_2)}$$

$$P = \psi_{Ey} \; \frac{\pi^2 E}{c^2 L^2} \; \frac{th^3}{6} \; k_1^3 k_2 \quad \text{---------}(2.14)$$

Substituting the values of t and h from Equations (2.10) and (2.11) into Equation (2.14) and simplifying:

$$\frac{P}{L^2} = \frac{24\psi_{Ey}c^2\psi_y^{5/2}s_y^{5/2}}{\psi_{Lw}^{\frac{1}{2}}\psi_{Ex}^2k_p^{\frac{1}{2}}\pi^2E^{3/2}} \quad \frac{(1+2k_1k_2)^2\ k_1^3k_2}{(1+6k_1k_2)^2} \qquad \text{---(2.15)}$$

From Equation (2.15) and letting $\psi_{Lw} = \psi_{Ex} = \psi_{Ey} = 1$ and for the simple support end condition, c = 1. Thus;

$$\frac{P}{L^2} = \frac{24\ \psi_y^{5/2}s_y^{5/2}}{k_p^{\frac{1}{2}}\ \pi^2\ E^{3/2}} \quad \frac{(1+2k_1k_2)^2}{(1+6k_1k_2)^2}\ k_1^3k_2 \qquad \text{------(2.16)}$$

From Equation (2.16) and with $\psi_y = 1$;

$$\frac{P}{L^2} = \frac{24\ s_y^{5/2}}{k_p^{\frac{1}{2}}\pi^2E^{3/2}} \quad \frac{(1+2k_1k_2)^2}{(1+6k_1k_2)^2}\ k_1^3k_2 \qquad \text{------(2.17)}$$

Equation (2.17) is satisfied for FC_1, FC_2, FC_3, FC_4, and FC_5, but it is not satisfied for Equation (2.8). The relation between P/L^2 and k_1 of Equation (2.17) are shown in Figures 2.2 - 2.6 . The computer program is shown in Appendix A .

Since $\qquad I_{xx} \geqq I_{yy}$

$$1 + 6k_1k_2 \geqq 2k_1^3k_2 \qquad \text{-------*}$$

For e/L = 0.00 ;

$$1 + 9.18k_1^2 \geqq 3.06k_1^4$$

$$k_1 \leqq 1.77 \qquad \text{--------*}$$

For e/L = 0.01 ;

$$1 + 11.34k_1^2 \geqq 3.72k_1^4$$

$$k_1 \leqq 1.74 \qquad \text{--------*}$$

For e/L = 0.05 ;

$$1 + 13.32k_1^2 \geqq 4.44k_1^4$$

$$k_1 \leqq 1.74 \qquad \text{--------*}$$

For e/L = 0.10 ;

$$1 + 15.54k_1^2 \geqq 5.18k_1^4$$

$$k_1 \leqq 1.74 \qquad \text{--------*}$$

For e/L = 0.20 ;

$$1 + 22.56k_1^2 \geqq 7.52k_1^4$$

$$k_1 \leqq 1.74 \qquad \text{--------*}$$

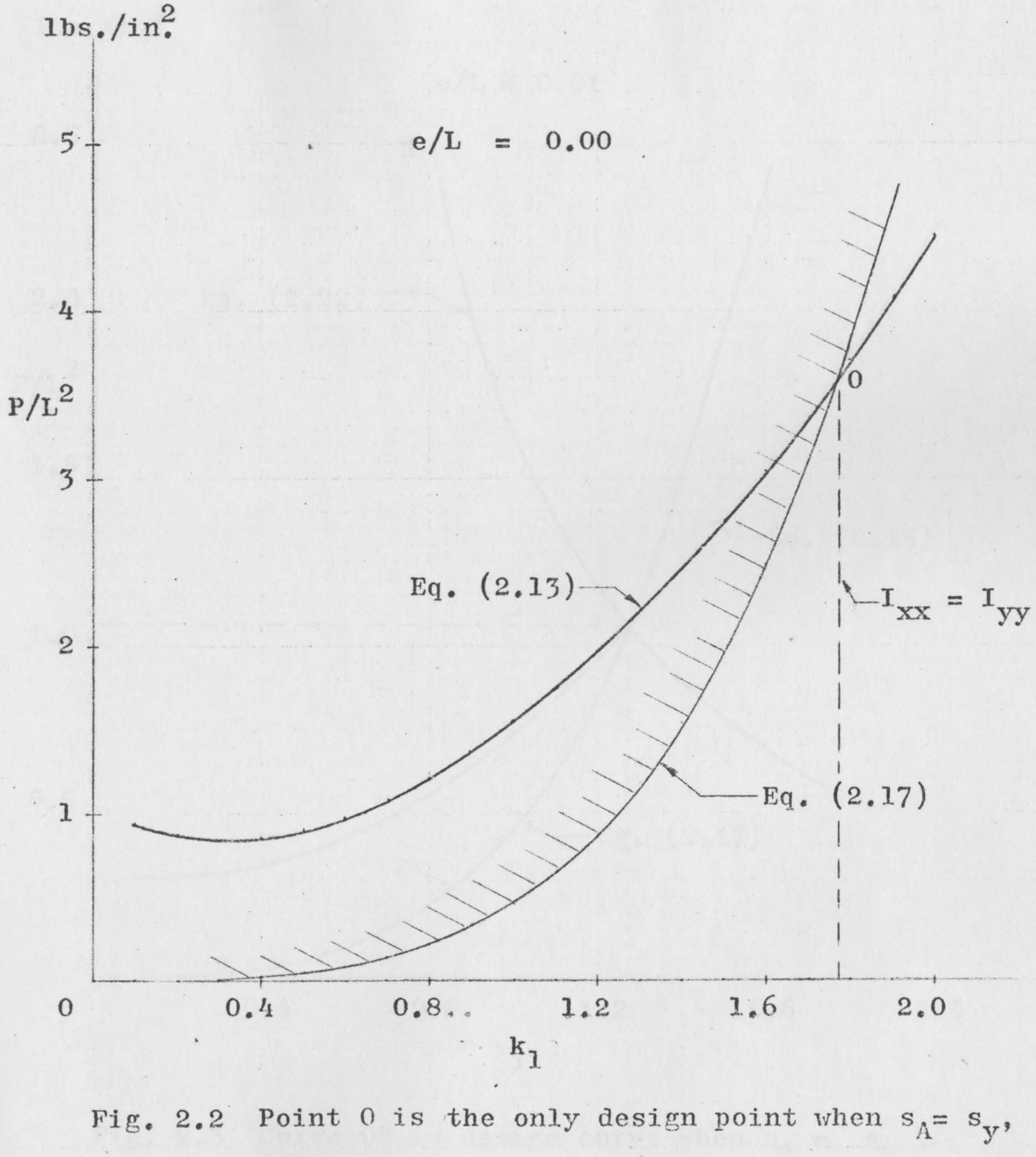


Fig. 2.2 Point O is the only design point when $s_A = s_y$, region which violates FC_4 is shown by crosshatching.

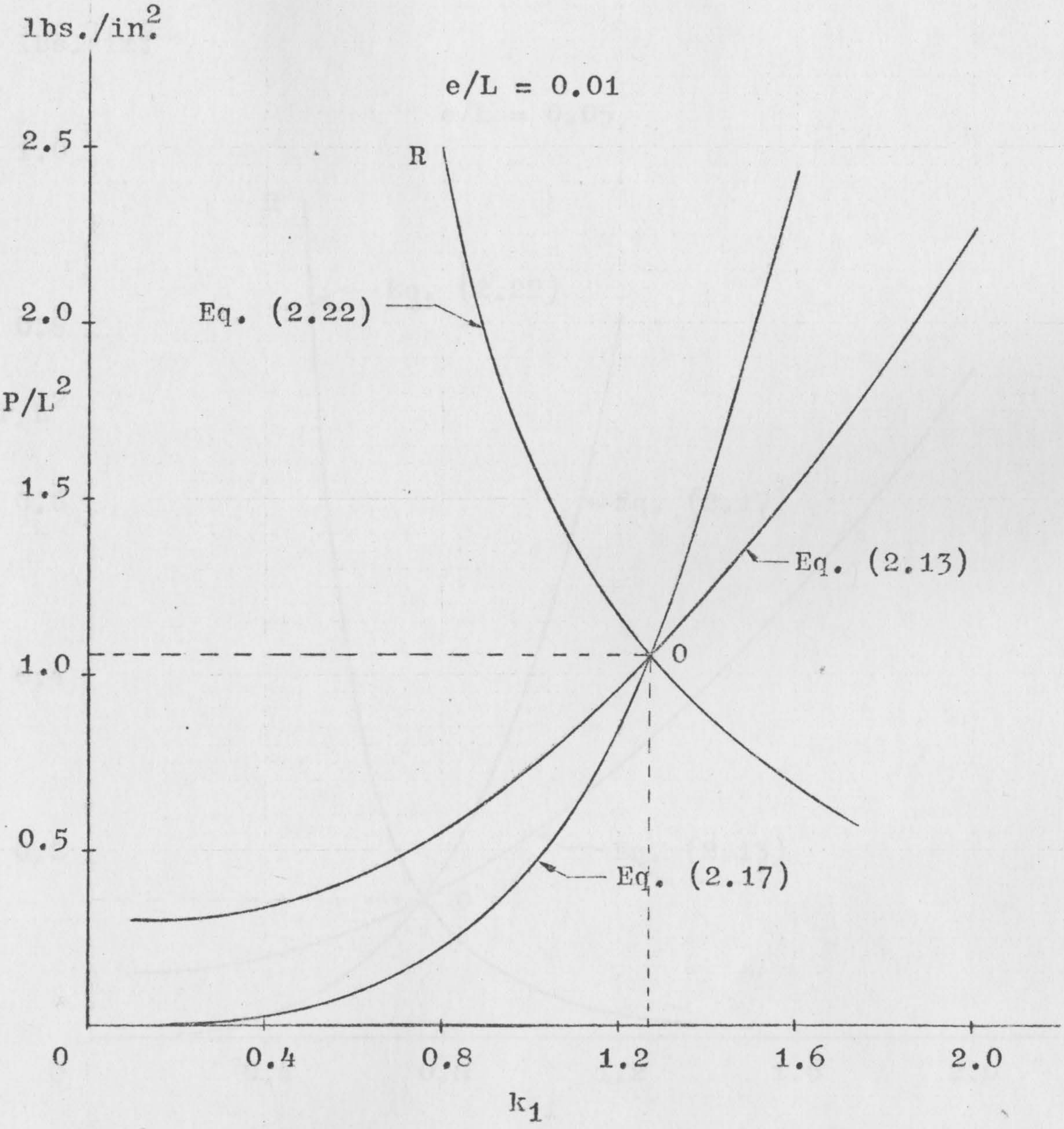


Fig. 2.3 Curve OR is design curve when $s_A = s_y$.

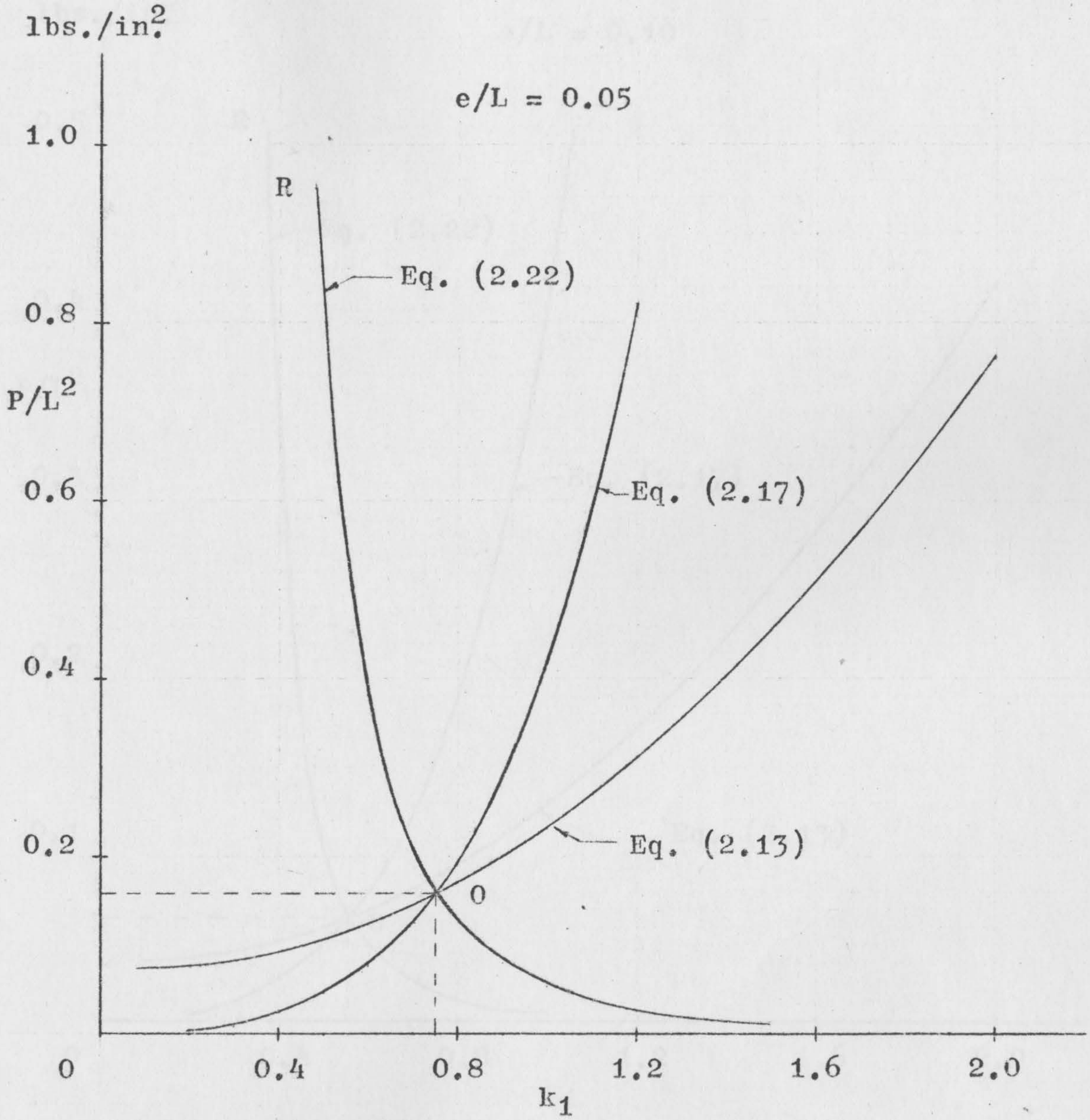


Fig. 2.4 Curve OR is design curve when $s_A = s_y$.

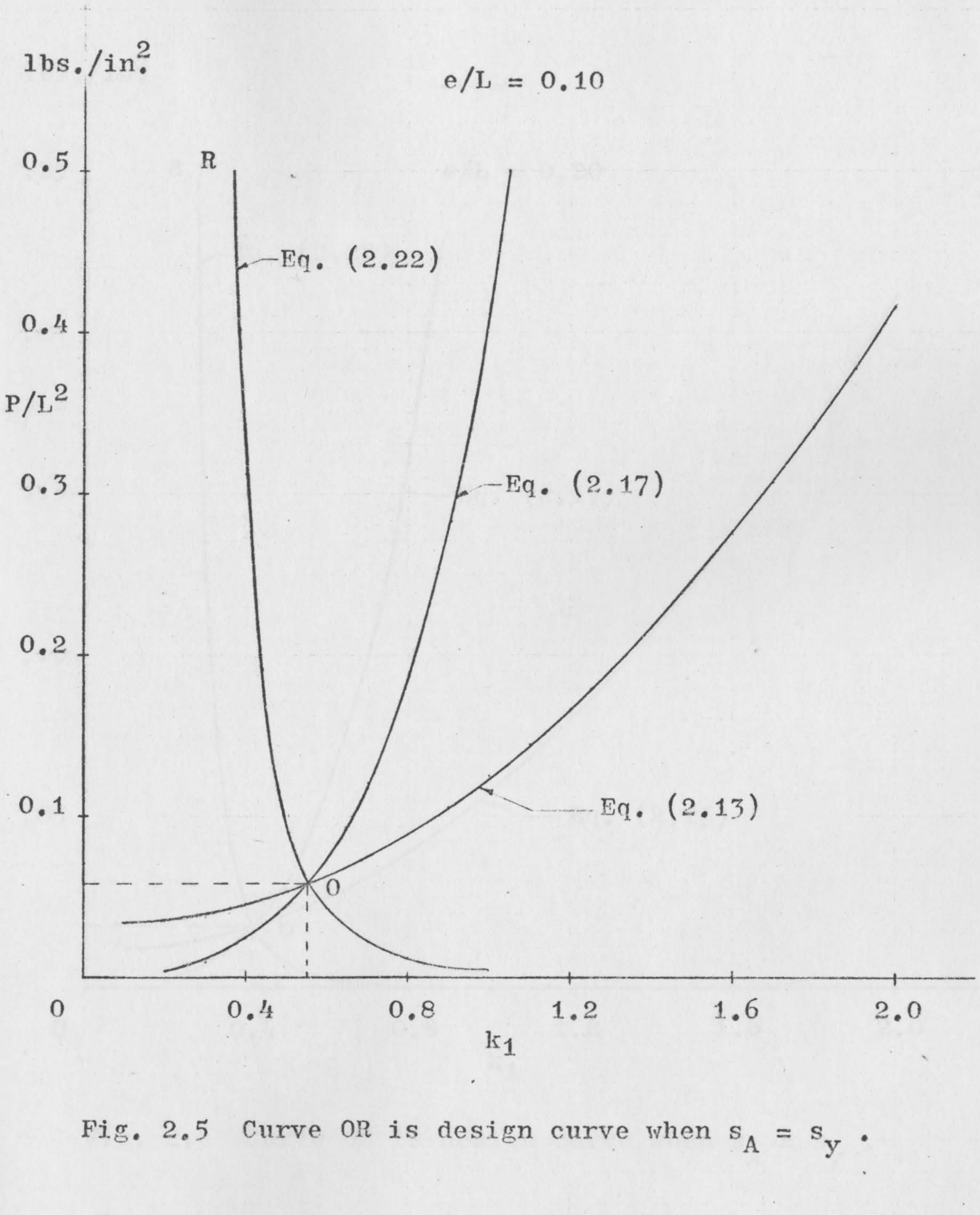


Fig. 2.5 Curve OR is design curve when $s_A = s_y$.

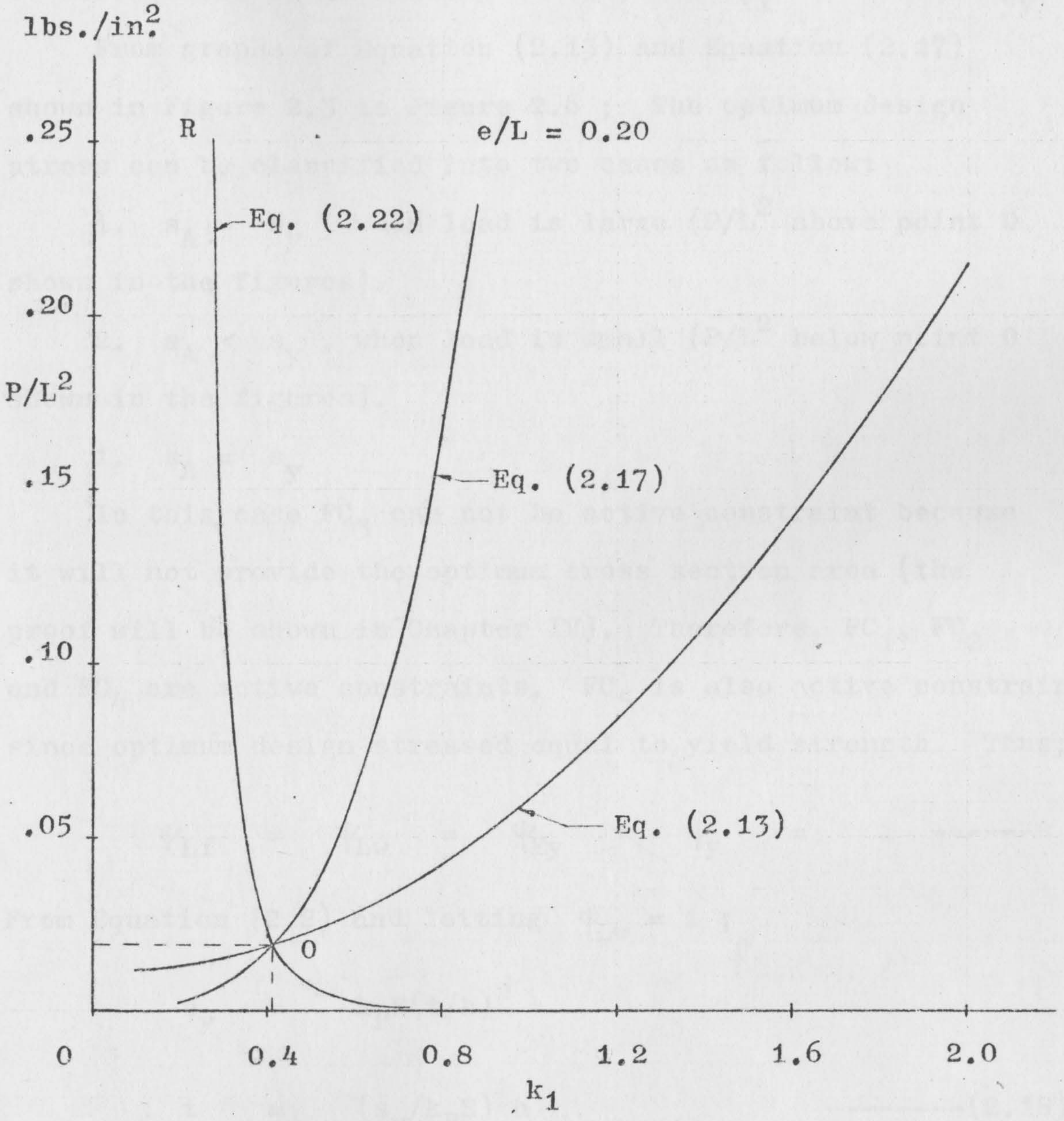


Fig. 2.6 Curve OR is design curve when $s_A = s_y$.

In Figure 2.2 , point O is the only design point and hence, k_1 equal to 1.77 , because in this case, e is equal to zero (there is no bending moment) and I_{xx} must equal I_{yy}.

From graphs of Equation (2.13) and Equation (2.17) shown in Figure 2.3 to Figure 2.6 ; The optimum design stress can be classified into two cases as follow:

1. $s_A = s_y$, when load is large (P/L^2 above point O shown in the figures).

2. $s_A < s_y$, when load is small (P/L^2 below point O shown in the figures).

1. $s_A = s_y$

In this case FC_3 can not be active constraint because it will not provide the optimum cross section area (the proof will be shown in Chapter IV). Therefore, FC_1, FC_2 and FC_4 are active constraints. FC_5 is also active constraint since optimum design stressed equal to yield strength. Thus;

$$\Psi_{Lf} = \Psi_{Lw} = \Psi_{Ey} = \Psi_y = 1 \quad \text{------*}$$

From Equation (2.2) and letting $\Psi_{Lw} = 1$;

$$s_y = k_p E (t/h)^2$$

$$t = (s_y/k_p E)^{\frac{1}{2}} h \qquad \text{--------(2.18)}$$

From Equation (2.14) and letting $\psi_{Ey} = 1$ and for simple support end condition, $c = 1$. Thus;

$$P = \frac{\pi^2 E}{L^2} \frac{th^3}{6} k_1^3 k_2 \qquad \text{----------(2.19)}$$

Substituting value of t from Equation (2.18) into Equation (2.19) and rearranging ;

$$h^4 = \frac{6PL^2 k_p^{\frac{1}{2}}}{\pi^2 s_y^{\frac{1}{2}} E^{\frac{1}{2}} k_1^3 k_2}$$

$$h = \left(\frac{6PL^2 k_p^{\frac{1}{2}}}{\pi^2 s_y^{\frac{1}{2}} E^{\frac{1}{2}} k_1^3 k_2} \right)^{\frac{1}{4}} \qquad \text{----------(2.20)}$$

Substituting value of h from Equation (2.20) into Equation (2.18) ;

$$t = (s_y/k_p E)^{\frac{1}{2}} \left(\frac{6PL^2 k_p^{\frac{1}{2}}}{\pi^2 s_y^{\frac{1}{2}} E^{\frac{1}{2}} k_1^3 k_2} \right)^{\frac{1}{4}} \qquad \text{------(2.21)}$$

Substituting values of t and h from Equations (2.20) and (2.21) into Equation (2.6) and simplifying;

$$s_y = \frac{(P/L^2)^{\frac{1}{2}} k_p^{\frac{1}{4}} E^{3/4} \pi k_1^{3/2} k_2^{\frac{1}{2}}}{6^{\frac{1}{2}} s_y^{\frac{1}{4}} (1+2k_1 k_2)} + \frac{6^{\frac{1}{4}} (P/L^2)^{\frac{1}{4}} (e/L) k_p^{1/8} E^{7/8} \pi^{3/2} k_1^{9/4} k_2^{3/4}}{s_y^{1/8} (1+6k_1 k_2)} \qquad \text{----------(2.22)}$$

The relation between P/L^2 and k_1 in Equation (2.22) was obtained by writing a computer program and are shown in Figure 2.3 to Figure 2.6 . For designing, the relation between P/L^2 and k_1 of Equation (2.22) is plotted in more accurate scale shown in Figure 2.7 to Figure 2.10 .

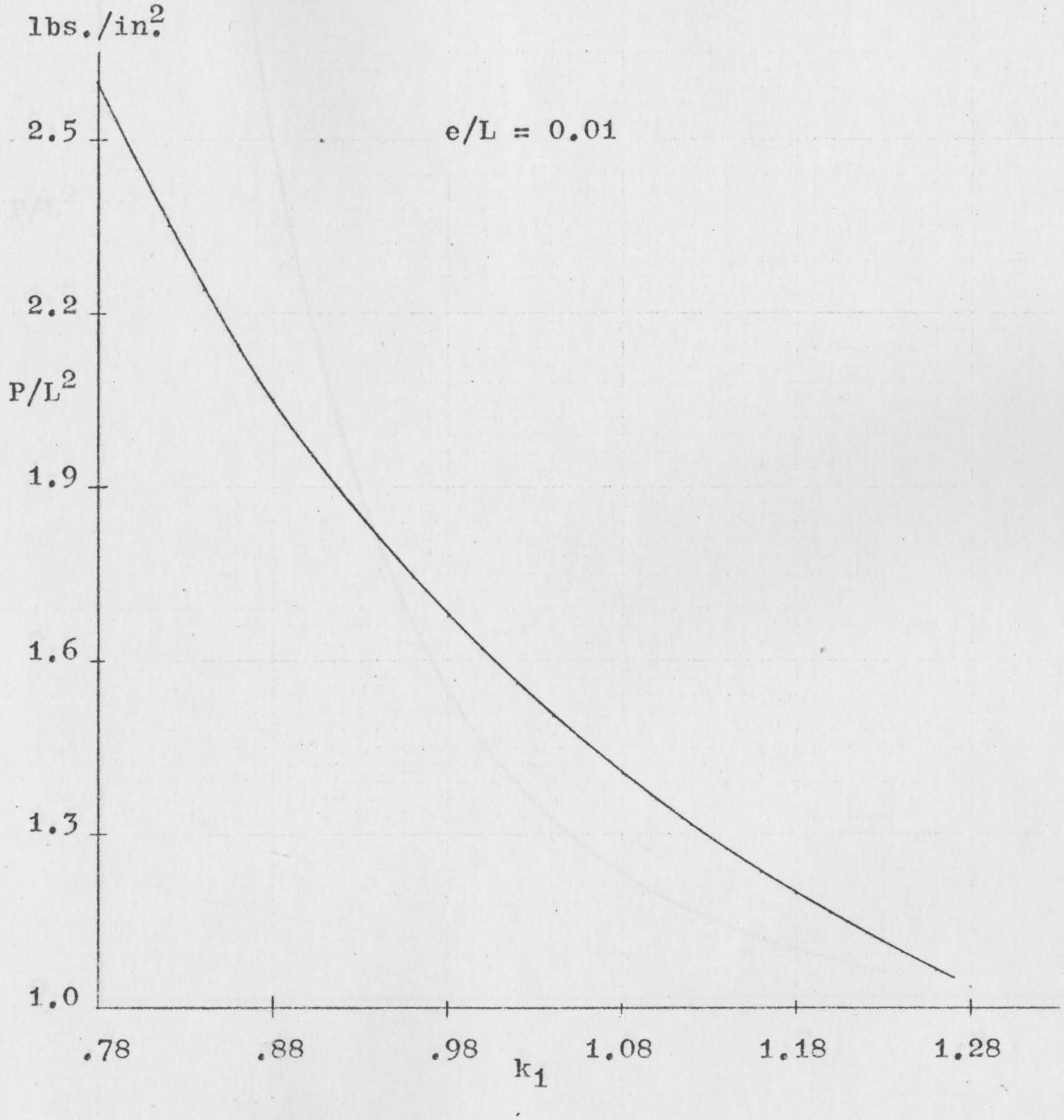


Fig. 2.7 Relation between P/L^2 and k_1 in Equation (2.22) when $s_A = s_y$.

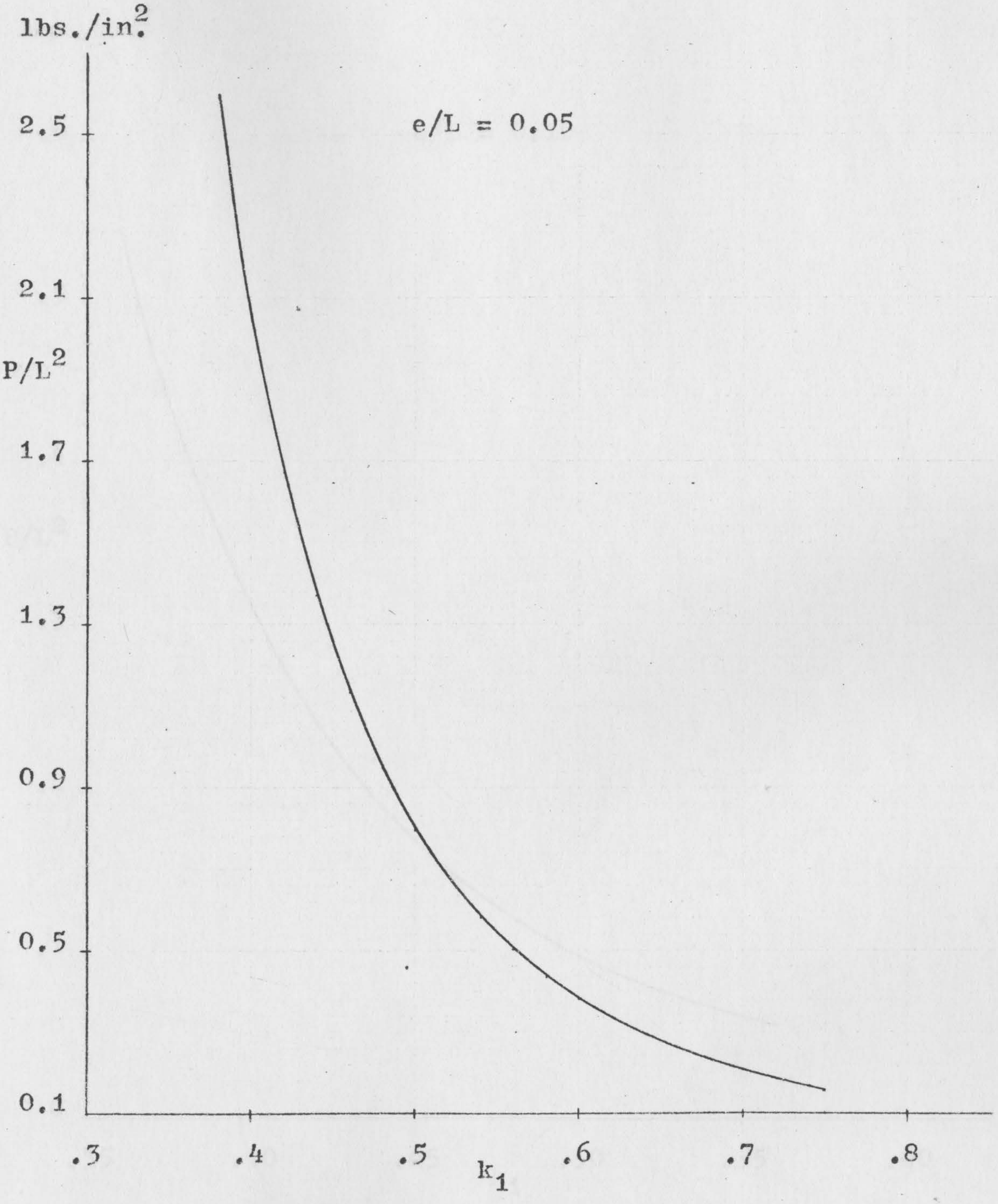


Fig. 2.8 Relation between P/L^2 and k_1 in Equation (2.22) when $s_A = s_y$.

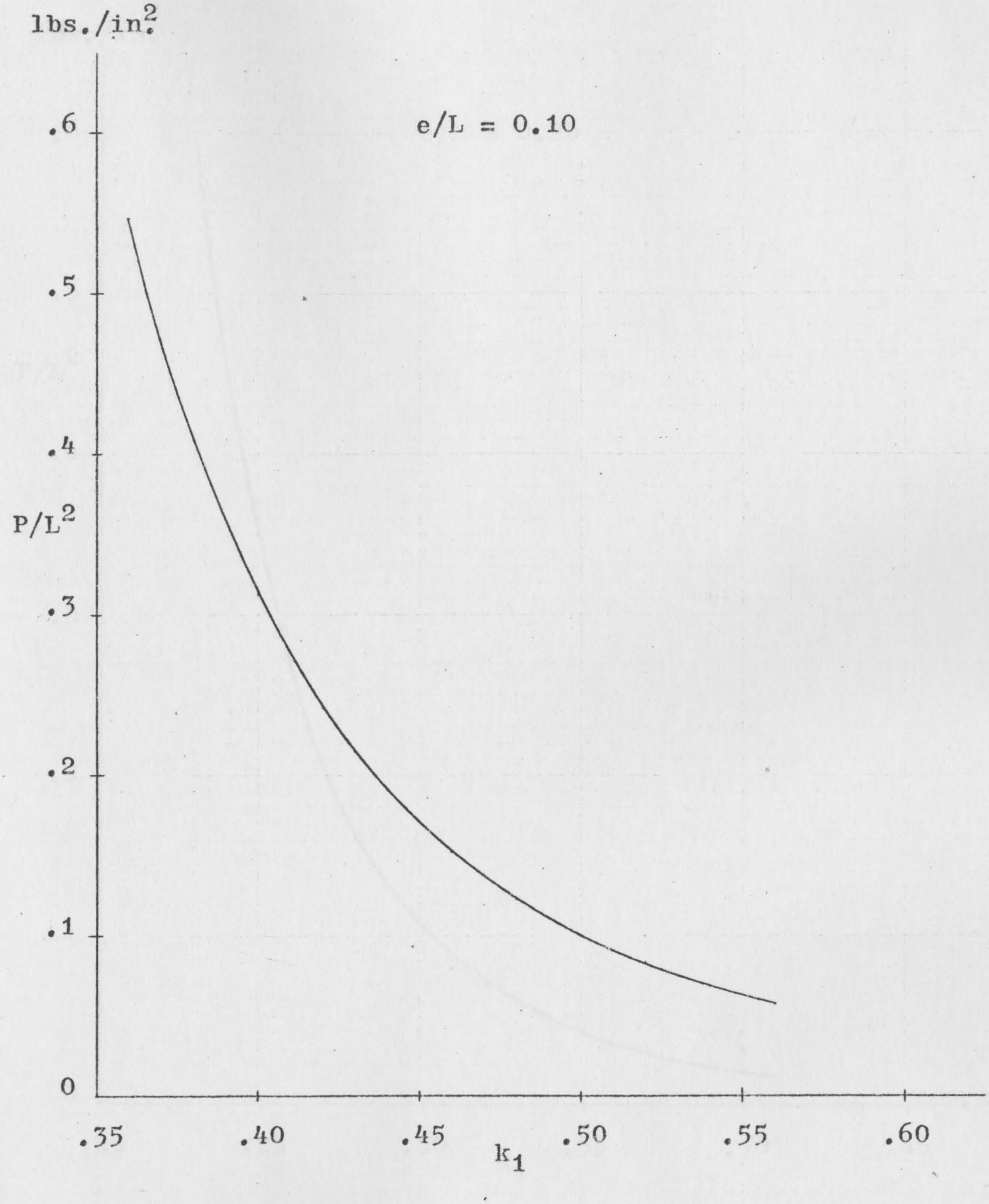


Fig. 2.9 Relation between P/L^2 and k_1 in Equation (2.22) when $s_A = s_y$.

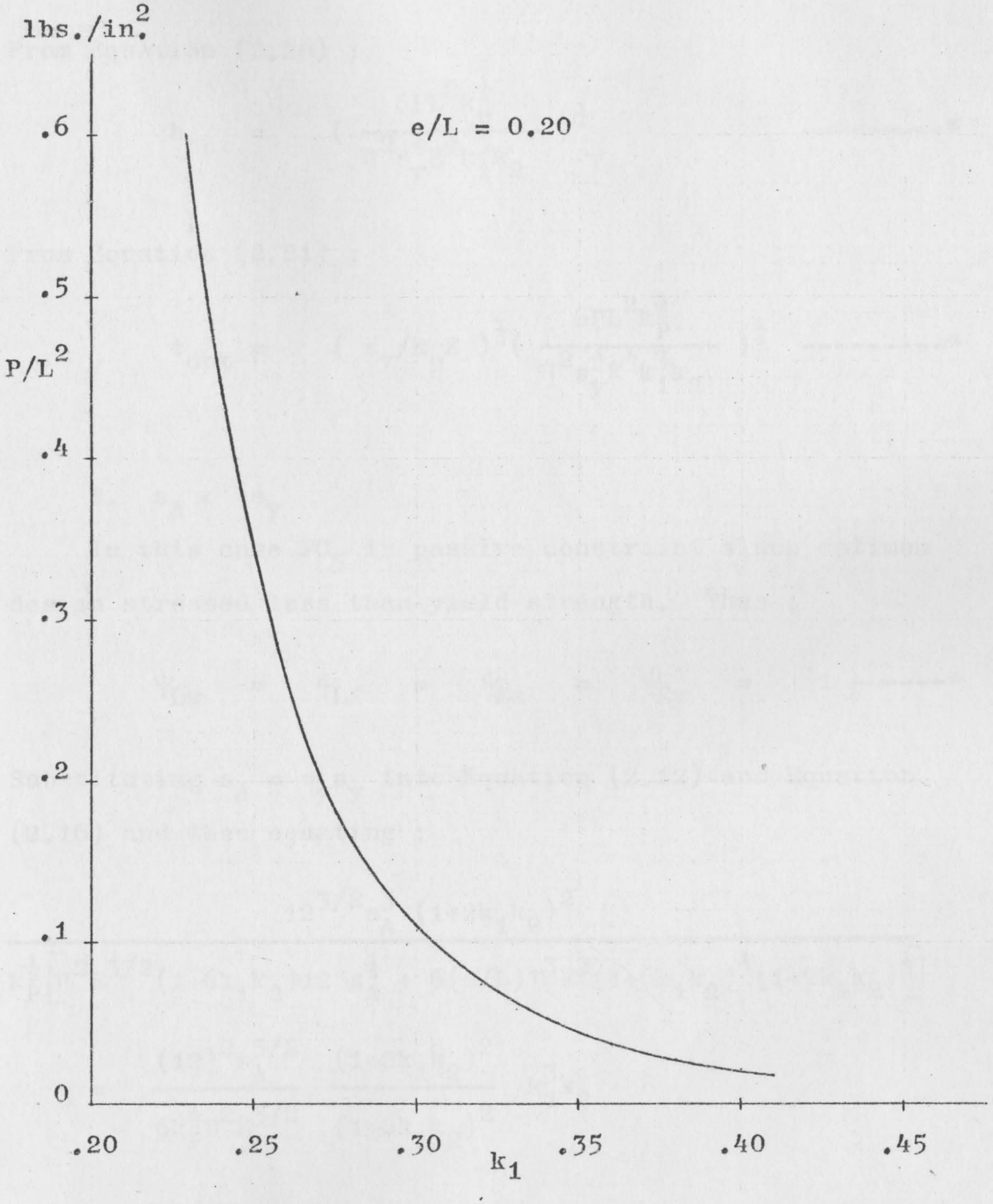


Fig. 2.10 Relation between P/L^2 and k_1 in Equation (2.22) when $s_A = s_y$.

From Equation (2.20) ;

$$h_{opt} = \left(\frac{6PL^2k_p^{\frac{1}{2}}}{\pi^2 s_y^{\frac{1}{2}} E^{\frac{1}{2}} k_1^3 k_2} \right)^{\frac{1}{4}} \quad \text{----------*}$$

From Equation (2.21) ;

$$t_{opt} = \left(s_y/k_pE \right)^{\frac{1}{2}} \left(\frac{6PL^2k_p^{\frac{1}{2}}}{\pi^2 s_y^{\frac{1}{2}} E^{\frac{1}{2}} k_1^3 k_2} \right)^{\frac{1}{4}} \quad \text{----------*}$$

2. $s_A < s_y$

In this case FC_5 is passive constraint since optimum design stressed less than yield strength. Thus ;

$$\psi_{Lw} = \psi_{Lf} = \psi_{Ex} = \psi_{Ey} = 1 \quad \text{------*}$$

Substituting $s_A = \psi_y s_y$ into Equation (2.12) and Equation (2.16) and then equating ;

$$\frac{12^{3/2} s_A^3 \, (1+2k_1k_2)^2}{k_p^{\frac{1}{2}} \left[\pi^2 E^{3/2} (1+6k_1k_2) 12^{\frac{1}{2}} s_A^{\frac{1}{2}} + 6(e/L) \pi^3 E^2 (1+6k_1k_2)^{\frac{1}{2}} (1+2k_1k_2)^{\frac{1}{2}} \right]}$$

$$= \frac{(12)^2 s_A^{5/2}}{6k_p^{\frac{1}{2}} \pi^2 E^{3/2}} \; \frac{(1+2k_1k_2)^2}{(1+6k_1k_2)^2} \; k_1^3 k_2$$

Simplifying the above equation ;

$$s_A = \left[\frac{12^{\frac{1}{2}} k_1 k_2 (e/L) \pi E^{\frac{1}{2}} \, (1+2k_1k_2)^{\frac{1}{2}}}{(1+6k_1k_2)^{3/2} - 2k_1^3 k_2 (1+6k_1k_2)^{\frac{1}{2}}} \right]^2 \quad \text{----------(2.23)}$$

The relation between s_A and k_1 in Equation (2.23) is shown in Figure 2.11 . Substituting the values of s_A and k_1 from Equation (2.23) into Equation (2.12), or Equation (2.16), can yield the relationship between P/L^2, s_A, and k_1.

The relation between P/L^2 and s_A is shown in Figures 2.12 and 2.13 . The relation between P/L^2 and k_1 is shown in Figure 2.14 to Figure 2.17 .

From Equation (2.10) ;

$$t_{opt} = \frac{12^{\frac{1}{2}} L s_A}{k_P^{\frac{1}{2}} \pi E} \; \frac{(1+2k_1 k_2)^{\frac{1}{2}}}{(1+6k_1 k_2)^{\frac{1}{2}}} \qquad \text{------------}(2.24)$$

From Equation (2.11) ;

$$h_{opt} = \frac{12^{\frac{1}{2}} s_A^{\frac{1}{2}} L}{\pi E^{\frac{1}{2}}} \; \frac{(1+2k_1 k_2)^{\frac{1}{2}}}{(1+6k_1 k_2)^{\frac{1}{2}}} \qquad \text{------------}(2.25)$$

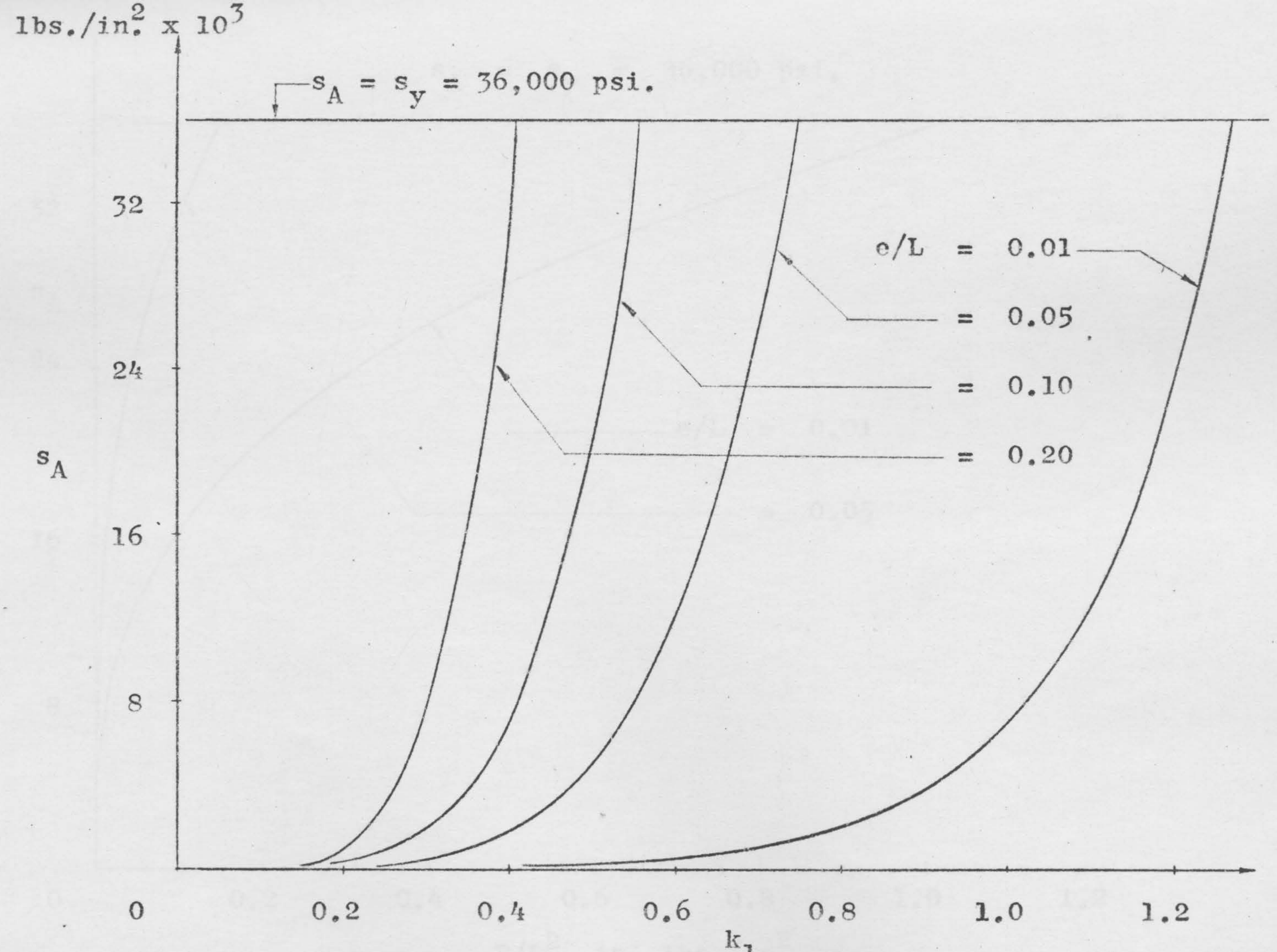


Fig. 2.11 Relation between s_A and k_1 in Equation (2.19) .

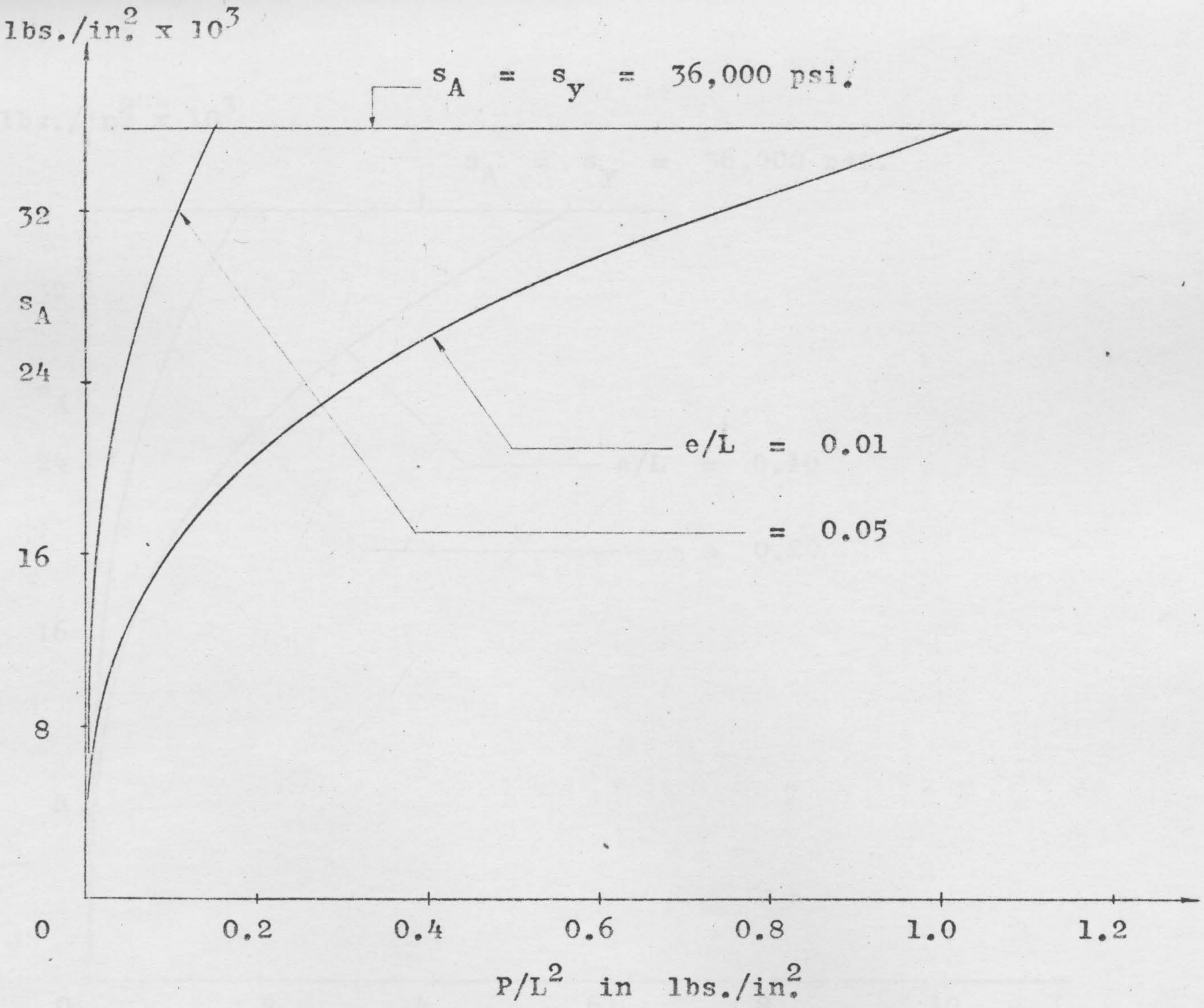


Fig. 2.12 Relation between P/L^2 and Optimum Design Stress.

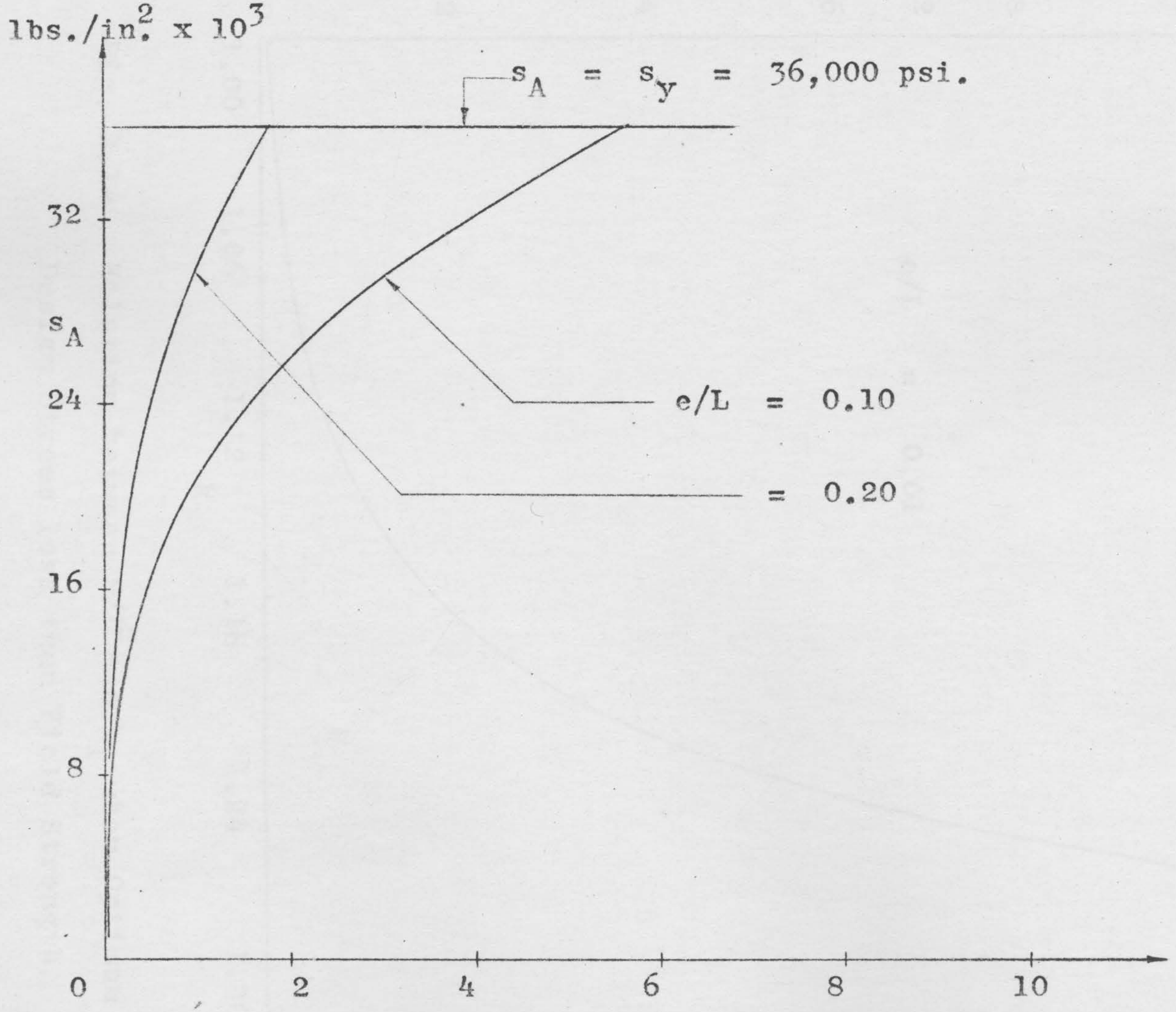


Fig. 2.13 Relation between P/L^2 and Optimum Design Stress.

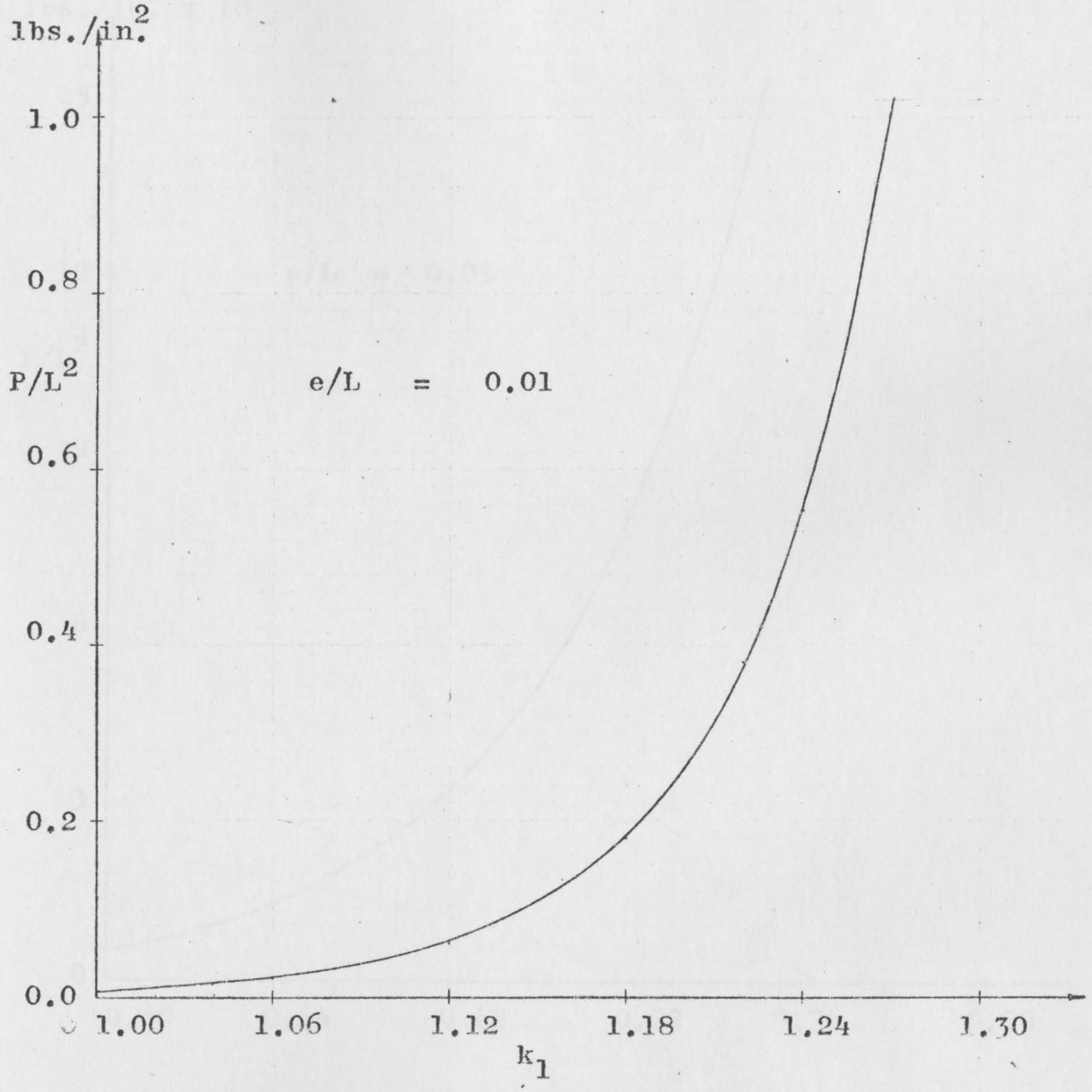


Fig. 2.14 Relation between P/L^2 and k_1 when Optimum Design Stress less than Yield Strength.

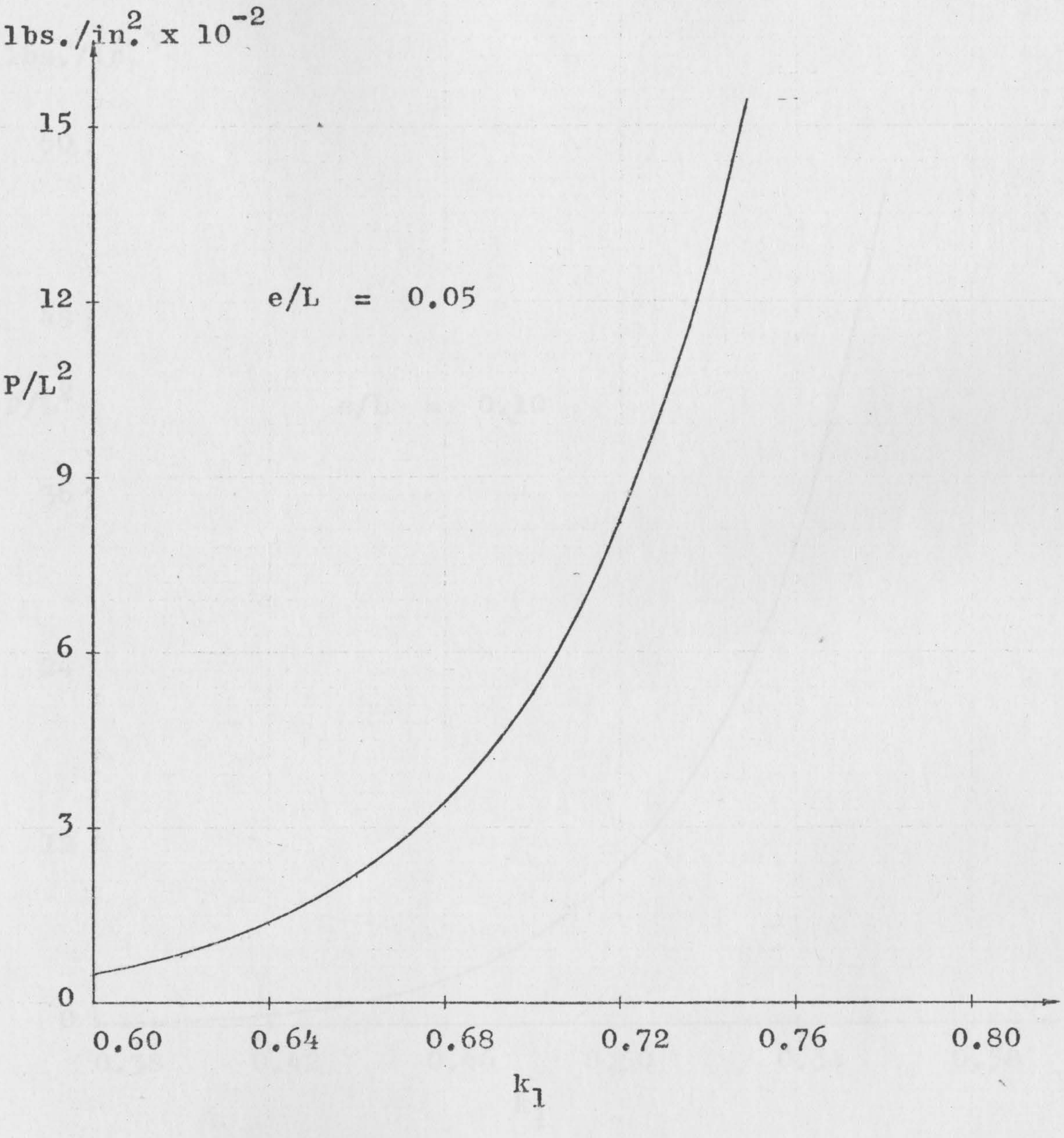


Fig. 2.15 Relation between P/L^2 and k_1 when Optimum Design Stress less than Yield Strength.

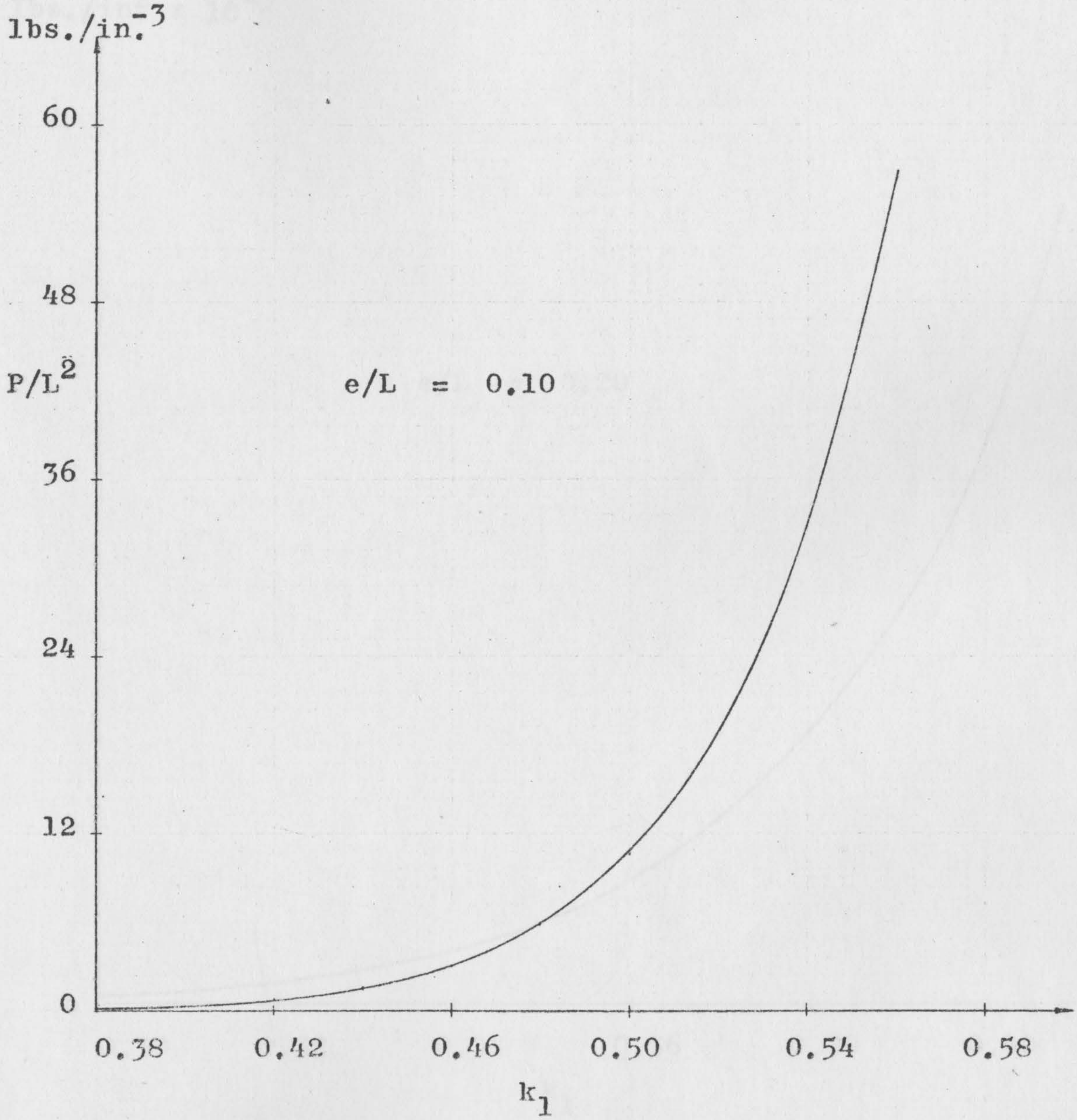


Fig. 2.16 Relation between P/L^2 and k_1 when Optimum Design Stress less than Yield Strength.

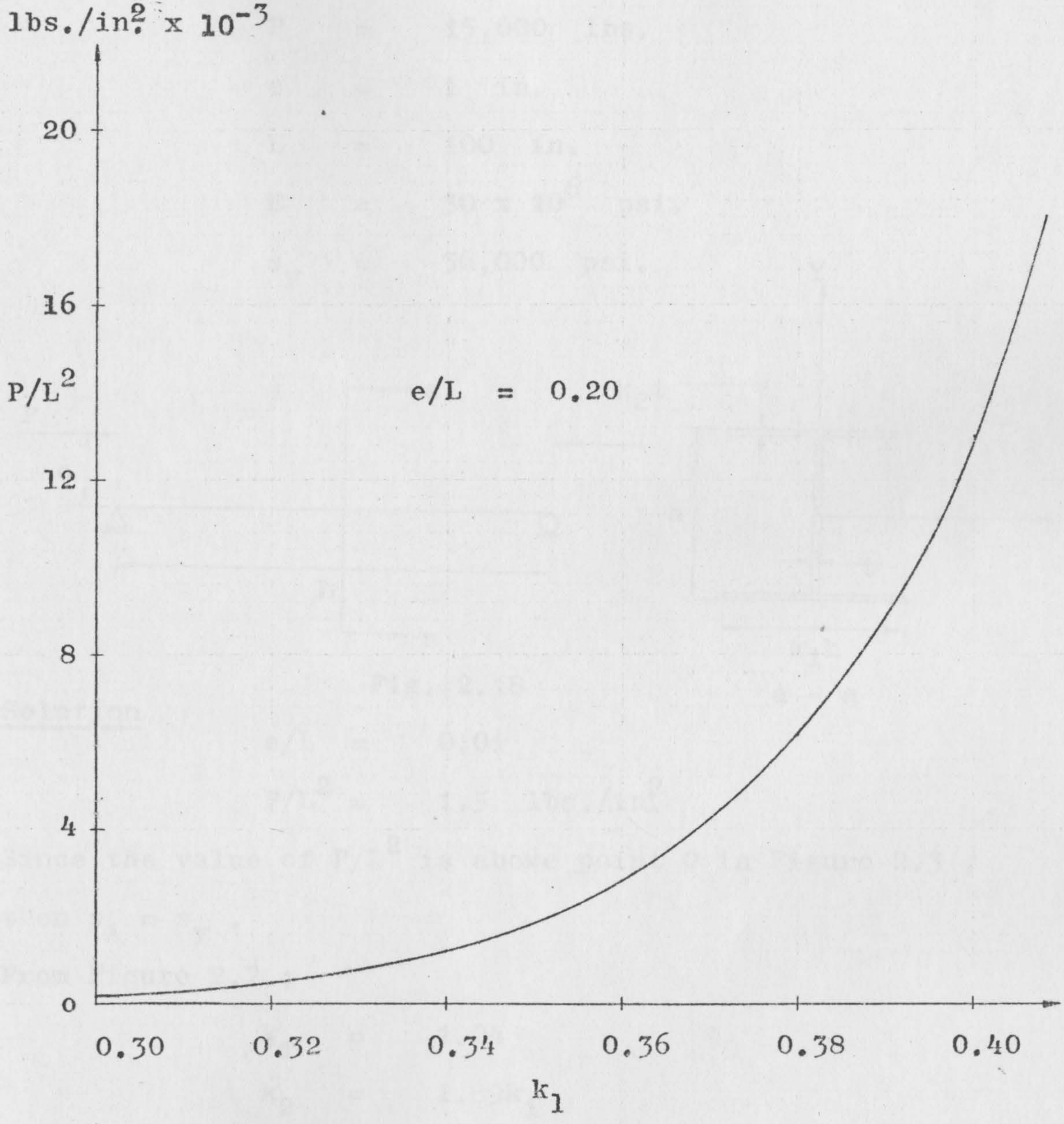


Fig. 2.17 Relation between P/L^2 and k_1 when Optimum Design Stress less than Yield Strength.

Example 2.1 Design a simply supported H-section beam-column as shown in Figure 2.18 for the following environment factors and material variables:

$$P = 15{,}000 \text{ lbs.}$$

$$e = 1 \text{ in.}$$

$$L = 100 \text{ in.}$$

$$E = 30 \times 10^6 \text{ psi.}$$

$$s_y = 36{,}000 \text{ psi.}$$

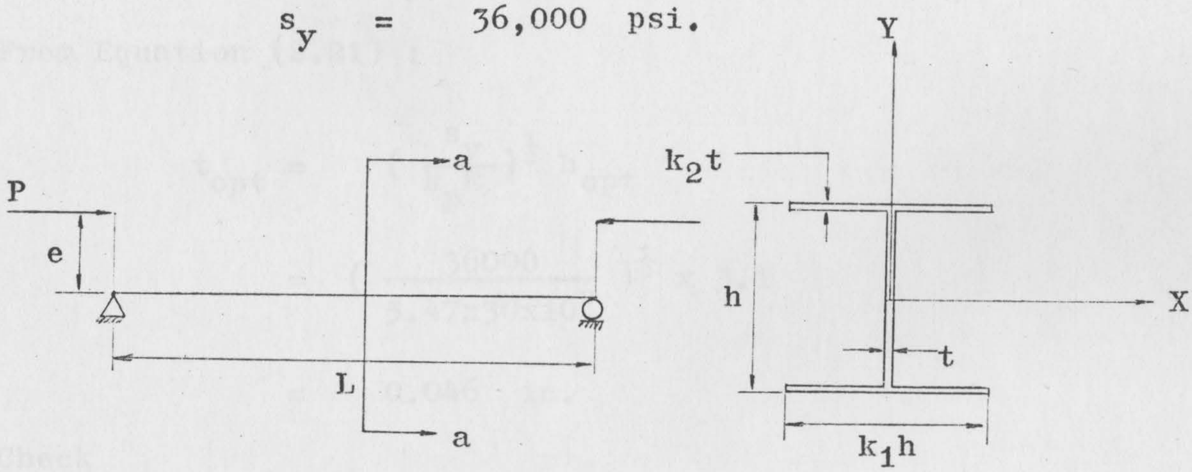


Fig. 2.18

a - a

Solution

$$e/L = 0.01$$

$$P/L^2 = 1.5 \text{ lbs./in.}^2$$

Since the value of P/L^2 is above point 0 in Figure 2.3 , then $s_A = s_y$.

From Figure 2.7 ;

$$k_1 = 1.04$$

$$k_2 = 1.89k_1$$

$$= 1.96$$

From Equation (2.20) ;

$$h_{opt} = \frac{(6P)^{\frac{1}{4}}L^{\frac{1}{2}}k_p^{1/8}}{\pi^{\frac{1}{2}}s_y^{1/8}k_1^{3/4}k_2^{\frac{1}{4}}E^{1/8}}$$

$$= \left(\frac{6x15000}{1.96} \right)^{\frac{1}{4}} \left(\frac{5.47}{36000x30x10^6} \right)^{1/8} \left(\frac{100}{3.14}\right)^{\frac{1}{2}} \frac{1}{1.04^{3/4}}$$

$$= 3.1 \text{ in.}$$

From Equation (2.21) ;

$$t_{opt} = \left(\frac{s_y}{k_p E} \right)^{\frac{1}{2}} h_{opt}$$

$$= \left(\frac{36000}{5.47x30x10^6} \right)^{\frac{1}{2}} x \ 3.1$$

$$= 0.046 \text{ in.}$$

Check

From Equation (2.6) ;

$$s_A = \frac{15000}{0.725} + \frac{90000}{(.046)(3.1)^2(1+6x1.04x1.96)}$$

$$= 20600 + 15400$$

$$= 36000 \text{ psi.}$$ --------------O.K.

From Equation (2.3) ;

$$s_{Ex} = \frac{3.14^2x\ 30x\ 10^6}{10000} x \frac{3.1^2}{12} x \frac{13.21}{5.07}$$

$$= 61500 \text{ psi. } 36000 \text{ psi.}$$ ---------O.K.

From Equation (2.4) ;

$$s_{Ey} = \frac{3.14^2 x\ 30\ x\ 10^6}{10000}\ x\ \frac{3.1^2}{6}\ x\ \frac{1.04^3 x\ 1.96}{5.07}$$

$$= 20600 \text{ psi.}$$

$$= P/A \qquad \text{-----------O.K.}$$

From Equation (2.1) ;

$$s_{Lf} = 0.385\ x\ 30\ x\ 10^6 x\ \left(\frac{1.96\ x\ .046}{1.04\ x\ 1.55} \right)^2$$

$$= 36000 \text{ psi.} \qquad \text{-----------O.K.}$$

$$A = .046\ x\ 3.1\ (1+2x1.04x1.96)$$

$$= 0.725 \text{ in.}^2 \qquad \text{------------*}$$

From Figure 4.2 ;

$$A/L^2 = 0.725\ x\ 10^{-4}$$

$$A = 0.725 \text{ in.}^2 \qquad \text{-----------O.K.}$$

<u>Example 2.2</u> Design a simply supported H-section beam-column as shown in Figure 2.18 for the following environment factors and material variables :

$$P = 1000 \text{ lbs.}$$

$$e = 5 \text{ in.}$$

$$L = 100 \text{ in.}$$

$$E = 30 \text{ x } 10^{6} \text{ psi.}$$

$$s_{y} = 36000 \text{ psi.}$$

<u>Solution</u>

$$e/L = 0.05$$

$$P/L^{2} = 0.1 \text{ lbs./in}^{2}.$$

From Figure 2.4 ;

$$s_{A} \qquad s_{y}$$

From Figure 2.12 ;

$$s_{A_{opt}} = 31300 \text{ psi.}$$

From Figure 2.15 ;

$$k_{1} = 0.73$$

$$k_{2} = 2.22 \, k_{1}$$

$$= 1.61$$

From Equation (2.24) ;

$$t_{opt} = \frac{12^{\frac{1}{2}} \text{ x } 100 \text{ x } 31300}{1.54^{\frac{1}{2}} \text{x } 3.14 \text{ x } 30 \text{ x } 10^{6}} \; \frac{(\,1 + 2\text{x}.73\text{x}1.61)^{\frac{1}{2}}}{(\,1 + 6\text{x}.73\text{x } 1.61)^{\frac{1}{2}}}$$

$$= 0.027 \text{ in.} \qquad \text{-----------*}$$

From Equation (2.25) ;

$$h_{opt} = \left(\frac{12 \times 31300}{30 \times 10^6} \right)^{\frac{1}{2}} \times \frac{100}{3.14} \times \left(\frac{1 + 2 \times .73 \times 1.61}{1 + 6 \times .73 \times 1.61} \right)^{\frac{1}{2}}$$

$$= 2.3 \text{ in.} \qquad \text{----------*}$$

<u>Check</u>

$$A_{opt} = 0.027 \times 2.3 \times (1 + 2 \times .73 \times 1.61)$$

$$= 0.208 \text{ in}^2. \qquad \text{----------*}$$

From Figure 4.3 ;

$$A/L^2 = 0.207 \times 10^{-4}$$

$$A = 0.207 \text{ in}^2. \qquad \text{--------O.K.}$$

CHAPTER III

RECTANGULAR SECTION

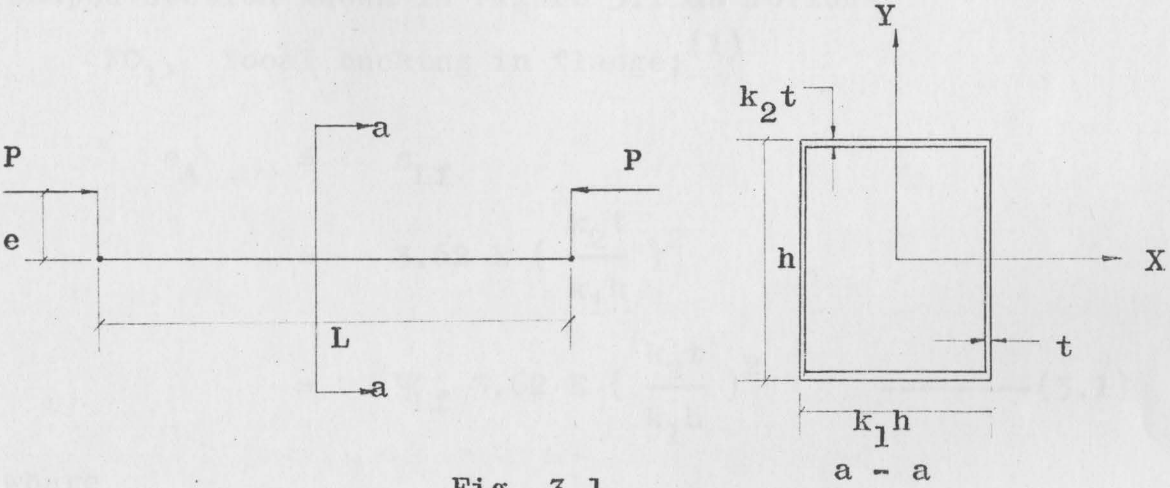


Fig. 3.1

For the rectangular shaped beam-column shown in Figure 3.1 ;

$$I_{xx} = \frac{th^3}{6} (1 + 3k_1k_2) \qquad ----------*$$

$$I_{yy} = \frac{th^3}{6} (k_1^3k_2 + 3k_1^2) \qquad ----------*$$

$$A = 2th (1 + k_1k_2) \qquad ----------*$$

$$r_x^2 = \frac{h^2}{12} \frac{(1 + 3k_1k_2)}{(1 + k_1k_2)} \qquad ----------*$$

$$r_y^2 = \frac{h^2}{12} \frac{(k_1^3k_2 + 3k_1^2)}{(1 + k_1k_2)} \qquad ----------*$$

$$I_{xx} \geqq I_{yy}$$

$$1 + 3k_1k_2 \geqq k_1^3k_2 + 3k_1^2 \quad \text{---------*}$$

$$M = Pe \quad \text{---------*}$$

The failure constraints applied to the rectangular shaped section shown in Figure 3.1 as follows:

FC_1, local bucking in flange;$^{(1)}$

$$s_A \leqq s_{Lf}$$

$$\leqq 3.62\ E\ \left(\frac{k_2 t}{k_1 h}\right)^2$$

$$= \psi_{Lf}\ 3.62\ E\ \left(\frac{k_2 t}{k_1 h}\right)^2 \quad \text{---------(3.1)}$$

where

s_A is design stress

s_{Lf} is local buckling failure stress in flange

k_1, k_2, t and h are proportional variables shown in Figure 3.1

ψ_{Lf} is slack variable

FC_2, local buckling in web;$^{(4)(5)}$

$$s_A \leqq s_{Lw}$$

$$\leqq k_p E\ (t/h)^2$$

$$= \psi_{Lw}\ k_p E\ (t/h)^2 \quad \text{---------(3.2)}$$

where

s_{Lw} is local buckling failure stress in web

k_p is buckling coefficient

FC_3, Euler buckling in bending axis;(4)(5)

$$s_A \leq s_{Ex}$$

$$\leq \frac{\pi^2 E}{c^2 L^2} \; \frac{h^2}{12} \; \frac{(1+3k_1k_2)}{(1+k_1k_2)}$$

$$= \Psi_{Ex} \frac{\pi^2 E}{c^2 L^2} \; \frac{h^2}{12} \; \frac{(1+3k_1k_2)}{(1+k_1k_2)} \text{ ---------(3.3)}$$

where

s_{Ex} is buckling stress in bending axis

c is factor which depends on end conditions

FC_4, Euler buckling in lateral direction;(1)(4)

$$s_{A1} \leq s_{Ey}$$

$$\leq \frac{\pi^2 E}{c^2 L^2} \; \frac{h^2}{12} \; \frac{(k_1^3 k_2 + 3k_1^2)}{(1+k_1k_2)} \text{ ---------(3.4)}$$

where

s_{A1} is compressive stress due to axial load only

s_{Ey} is buckling stress in lateral direction

FC_5, design stress can not be more than yield strength;

$$s_A \leq s_y$$

$$= \Psi_y s_y \qquad \text{----------(3.5)}$$

For linear materials which the applied stress in not more than yield strength, the formula for combined axial and bending stress is;

$$s_A = \frac{Mc}{I_{xx}} + \frac{P}{A}$$

$$= \frac{3Pe}{th^2(1+3k_1k_2)} + \frac{P}{2th(1+k_1k_2)} \qquad \text{---(3.6)}$$

By the same procedure that has been used previously, the procedure proceeds indirectly by optimizing the design stress.

FC_1, FC_2, FC_3, and FC_5 are active constraints, since there are only 4 proportional variables t, h, k_1, k_2 . Hence, FC_4 is a passive constraints.

By equating Equations (3.1) and (3.2) ;

$$\Psi_{Lf}\ 3.62\ E\left(\frac{k_2 t}{k_1 h} \right)^2 = \Psi_{Lw}\ k_p E(\ t/h\ )^2$$

Letting $\Psi_{Lf} = \Psi_{Lw}$ by S.M.D.

$$k_2 = k_1 \left(\frac{k_p}{3.62} \right)^{\frac{1}{2}} \qquad \text{-------(3.7)}$$

By equating Equations (3.5) & (3.6);

$$\psi_y s_y = \frac{3Pe}{th^2(1+3k_1k_2)} + \frac{P}{2th(1+k_1k_2)} \qquad \text{----------(3.8)}$$

Multiplying Equation (3.2) by Equation (3.3);

$$s_A^2 = \psi_{Lw}\, k_p E(t/h)^2\, \psi_{Ex}\, \frac{\pi^2 E}{c^2L^2}\, \frac{h^2}{12}\, \frac{(1+3k_1k_2)}{(1+k_1k_2)}$$

From Equation (3.5);

$$\psi_y^2 s_y^2 = \psi_{Lw}\psi_{Ex}\, \frac{k_p E^2 \pi^2}{12c^2L^2}\, \frac{(1+3k_1k_2)}{(1+k_1k_2)}\, t^2$$

$$t^2 = \frac{12c^2L^2\psi_y^2 s_y^2}{\psi_{Lw}\psi_{Ex}k_p\pi^2E^2}\, \frac{(1+k_1k_2)}{(1+3k_1k_2)} \qquad \text{----------(3.9)}$$

$$t = \frac{12^{\frac{1}{2}}cL\ \psi_y s_y}{\psi_{Lw}^{\frac{1}{2}}\psi_{Ex}^{\frac{1}{2}}k_p^{\frac{1}{2}}\pi E}\, \frac{(1+k_1k_2)^{\frac{1}{2}}}{(1+3k_1k_2)^{\frac{1}{2}}} \qquad \text{----------(3.10)}$$

By equating Equations (3.3) & (3.5);

$$\psi_y s_y = \psi_{Ex}\, \frac{\pi^2 E}{c^2L^2}\, \frac{h^2}{12}\, \frac{(1+3k_1k_2)}{(1+k_1k_2)}$$

$$h^2 = \frac{12c^2L^2\ \psi_y s_y}{\psi_{Ex}\pi^2E}\, \frac{(1+k_1k_2)}{(1+3k_1k_2)} \qquad \text{----------(3.11)}$$

$$h = \frac{12^{\frac{1}{2}}\ cL\ \psi_y^{\frac{1}{2}} s_y^{\frac{1}{2}}}{\psi_{Ex}^{\frac{1}{2}}\ \pi E^{\frac{1}{2}}}\, \frac{(1+k_1k_2)^{\frac{1}{2}}}{(1+3k_1k_2)^{\frac{1}{2}}} \qquad \text{----------(3.12)}$$

Substituting the value of t and h from Equations (3.10) and (3.12) into Equation (3.8);

$$\psi_y s_y = \frac{3\,Pe}{\dfrac{12^{\frac{1}{2}} cL\psi_y s_y}{\psi_{Lw}^{\frac{1}{2}}\psi_{Ex}^{\frac{1}{2}} k_p^{\frac{1}{2}}\pi E}\;\dfrac{(1+ k_1k_2)^{\frac{1}{2}}}{(1+3k_1k_2)^{\frac{1}{2}}}\;\dfrac{12c^2L^2\psi_y s_y}{\psi_{Ex}\pi^2 E}\;\dfrac{(1+ k_1k_2)}{(1+3k_1k_2)}\,(1+3k_1k_2)}$$

$$+\frac{P}{\dfrac{2(12)^{\frac{1}{2}} cL\psi_y s_y}{\psi_{Lw}^{\frac{1}{2}}\psi_{Ex}^{\frac{1}{2}} k_p^{\frac{1}{2}}\,\pi E}\;\dfrac{(1+ k_1k_2)^{\frac{1}{2}}}{(1+3k_1k_2)^{\frac{1}{2}}}\;\dfrac{12^{\frac{1}{2}} cL\psi_y^{\frac{1}{2}} s_y^{\frac{1}{2}}}{\psi_{Ex}^{\frac{1}{2}}\;\pi E^{\frac{1}{2}}}\;\dfrac{(1+ k_1k_2)^{\frac{1}{2}}}{(1+3k_1k_2)^{\frac{1}{2}}}\,(1+k_1k_2)}$$

Simplifing the above equation and letting $\psi_{Lw} = \psi_{Ex} = 1$ and for the simple support end condition, $c = 1$. Thus;

$$P/L^2 = \frac{24\psi_y^3 s_y^3 (1+k_1k_2)^2}{k_p^{\frac{1}{2}}\left[3^{\frac{1}{2}}(e/L)\pi^3 E^2 (1+3k_1k_2)^{\frac{1}{2}}(1+k_1k_2)^{\frac{1}{2}} + \pi^2 E^{3/2}(1+3k_1k_2)\psi_y^{\frac{1}{2}}\; s_y^{\frac{1}{2}}\right]}$$

---------(3.13)

Letting ψ_y in Equation (3.13) be equal to unity;

$$P/L^2 = \frac{24 s_y^3 (1+k_1k_2)^2}{k_p^{\frac{1}{2}}\left[3^{\frac{1}{2}}(e/L)\pi^3 E^2 (1+3k_1k_2)^{\frac{1}{2}}(1+k_1k_2)^{\frac{1}{2}} + \pi^2 E^{3/2}(1+3k_1k_2) s_y^{\frac{1}{2}}\right]}$$

---------(3.14)

From BUCKLING STRENGTH of METAL STRUCTURES by Bleich(5) ;

$$\begin{array}{ll} e/L = 0.00, & k_p = 3.62 \end{array}$$

$$\left.\begin{array}{ll} e/L = 0.01, & k_p = 7.54 \\ e/L = 0.05, & k_p = 10.30 \\ e/L = 0.10, & k_p = 14.10 \\ e/L > 0.10, & k_p = 21.70 \end{array}\right\} \text{Approximate Values}$$

From Equation (3.7) ;

$$k_2 = k_1 \; (k_p/3.62)^{\frac{1}{2}} \qquad \text{-------}*$$

$$\begin{array}{ll} e/L = 0.00, & k_2 = k_1 \\ e/L = 0.01, & k_2 = 1.45\,k_1 \\ e/L = 0.05, & k_2 = 1.71\,k_1 \\ e/L = 0.10, & k_2 = 2.00\,k_1 \\ e/L > 0.10, & k_2 = 2.49\,k_1 \end{array}$$

Using AISI 1025 steel ;

$$E = 30 \times 10^6 \text{ psi.}$$

$$s_y = 36{,}000 \text{ psi.}$$

Substituting the above value into Equation (3.14), the relation between P/L^2 and k_1 is shown in Figures 3.2 to 3.6 for each value of e/L .

These procedures are the same as in Chapter 2. The next step is to check whether FC_4 is satisfied or not.

From Equation (3.4) ;

$$s_{A1} \leq \frac{\pi^2 E}{c^2 L^2} \quad \frac{h^2}{12} \quad \frac{(k_1^3 k_2 + 3k_1^2)}{(1+k_1 k_2)}$$

$$\frac{P}{2th(1+k_1 k_2)} = \psi_{Ey} \quad \frac{\pi^2 E}{c^2 L^2} \quad \frac{h^2}{12} \quad \frac{(k_1^3 k_2 + 3k_1^2)}{(1+k_1 k_2)} \qquad \text{---(3.15)}$$

where $\psi_{Ey} \leq 1$

Substituting the values of t and h from Equation (3.10) and Equation (3.12) into Equation (3.15); Simplifying the equation, letting $\psi_{Lw} = \psi_{Ex} = \psi_{Ey} = 1$, and for the simple support end condition, $c = 1$. Thus ;

$$\frac{P}{L^2} = \frac{24\psi_y^{5/2} s_y^{5/2} (k_1^3 k_2 + 3k_1^2)(1+k_1 k_2)^2}{k_p^{\frac{1}{2}} \pi^2 E^{3/2} (1+3k_1 k_2)^2} \qquad \text{---(3.16)}$$

Letting ψ_y in Equation (3.16) be equal to unity ;

$$\frac{P}{L^2} = \frac{24 s_y^{5/2} (k_1^3 k_2 + 3k_1^2)(1+k_1 k_2)^2}{k_p^{\frac{1}{2}} \pi^2 \; {}^{3/2} (1+3k_1 k_2)^2} \qquad \text{---(3.17)}$$

As in Chapter 2 , Equation (3.17) does not satisfied Equation (3.8). The relation between P/L^2 and k_1 of Equation (3.17) is shown in Figures 3.2 to 3.6 .

Since

$$I_{xx} \geqq I_{yy}$$

$$1 + 3k_1k_2 \geqq k_1^3k_2 + 3k_1^2 \qquad \text{------*}$$

For e/L = 0.00 ;

$$1 + 3k_1^2 \geqq k_1^4 + 3k_1^2$$

$$k_1 \leqq 1 \qquad \text{------*}$$

For e/L = 0.01 ;

$$1 + 4.35k_1^2 \geqq 1.45k_1^4 + 3k_1^2$$

$$k_1 \leqq 1.18 \qquad \text{------*}$$

For e/L = 0.05 ;

$$1 + 5.13k_1^2 \geqq 1.71k_1^4 + 3k_1^2$$

$$k_1 \leqq 1.26 \qquad \text{------*}$$

For e/L = 0.10 ;

$$1 + 6k_1^2 \geqq 2k_1^4 + 3k_1^2$$

$$k_1 \leqq 1.32 \qquad \text{-------*}$$

For e/L = 0.02 ;

$$1 + 7.47k_1^2 \geqq 2.49k_1^4 + 3k_1^2$$

$$k_1 \leqq 1.40 \qquad \text{--------*}$$

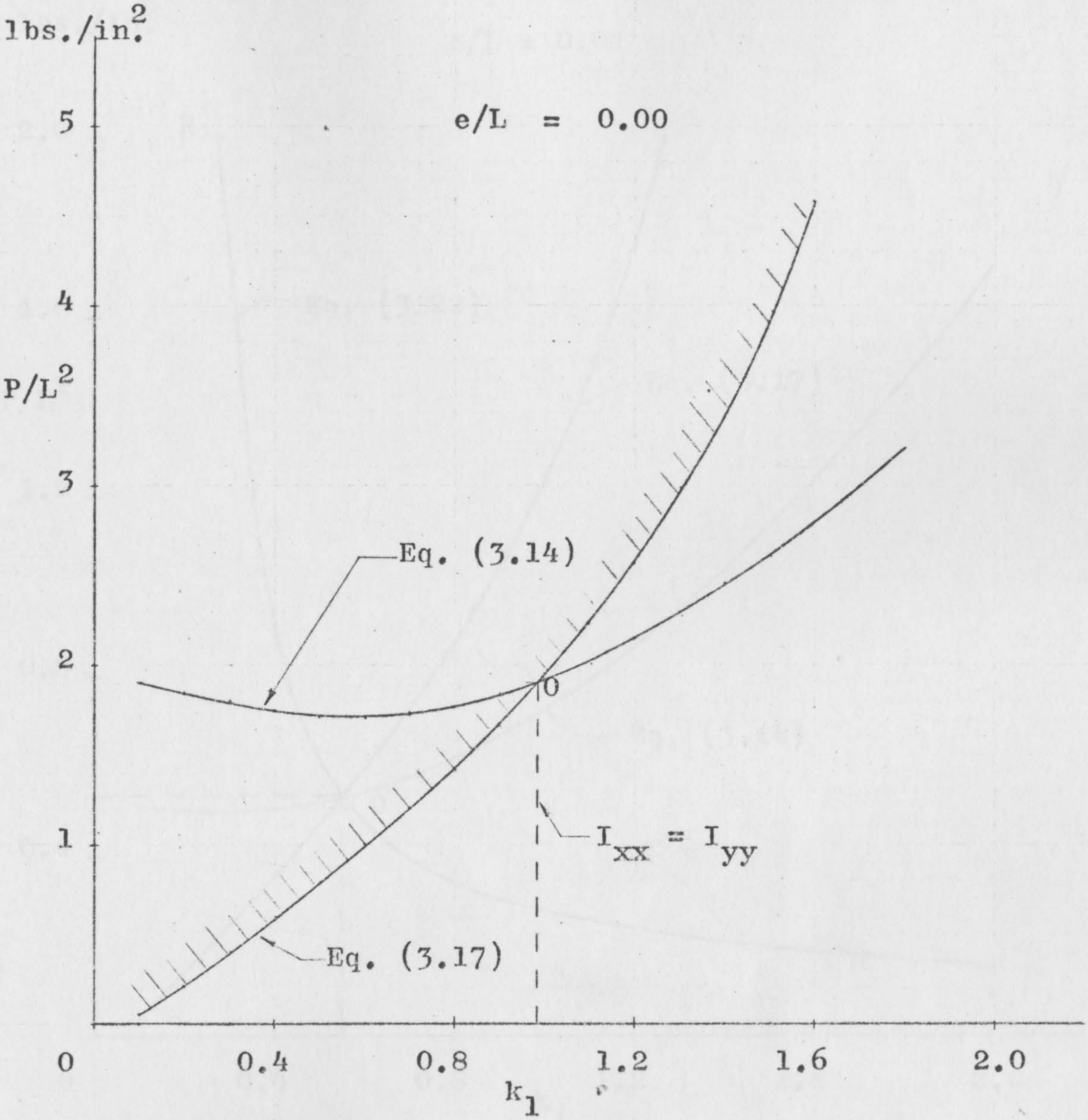


Fig. 3.2 Point O is the only design point when $s_A = s_y$, region which violates FC_4 is shown by crosshatching.

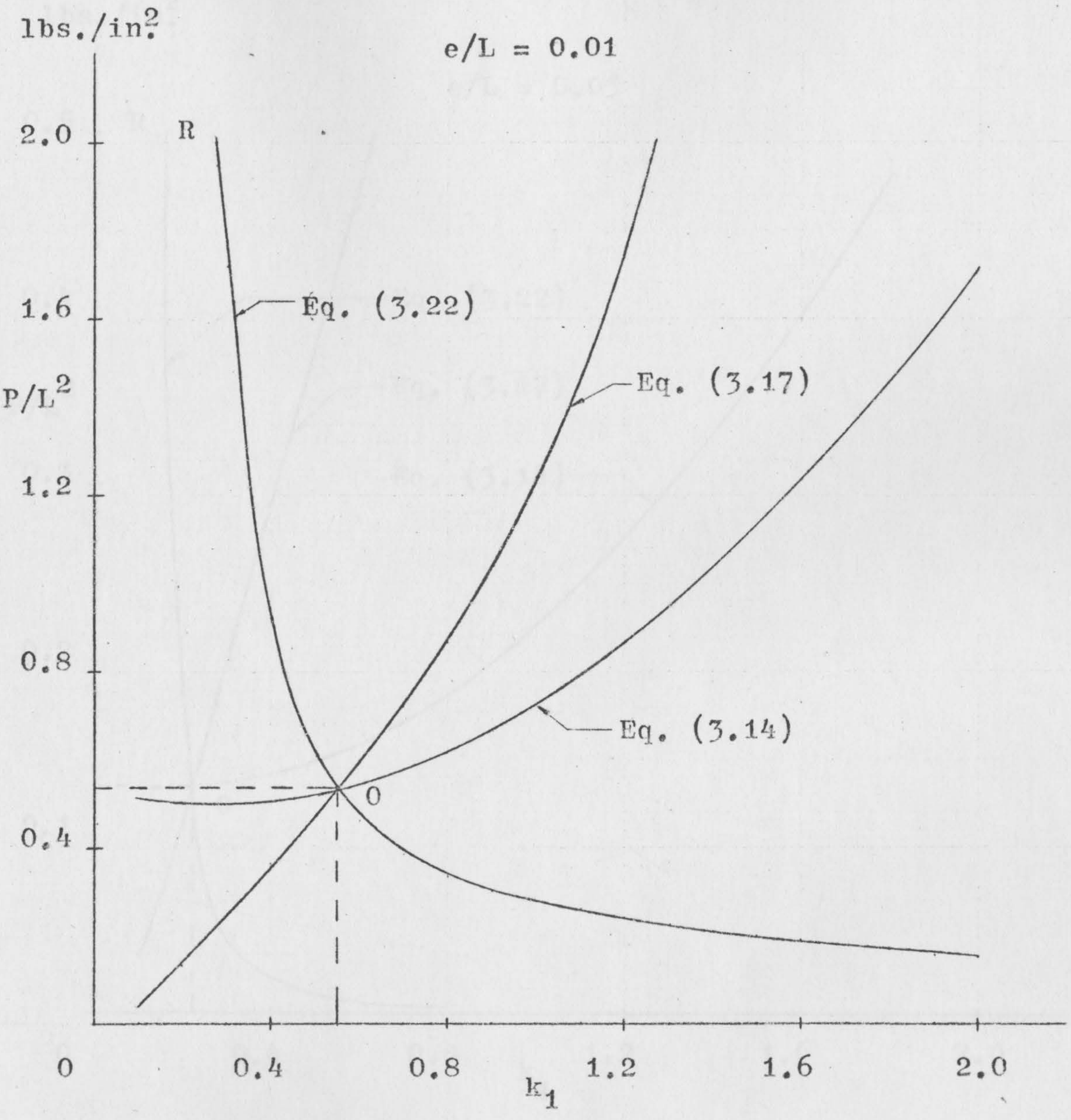


Fig. 3.3 Curve OR is design curve when $s_A = s_y$.

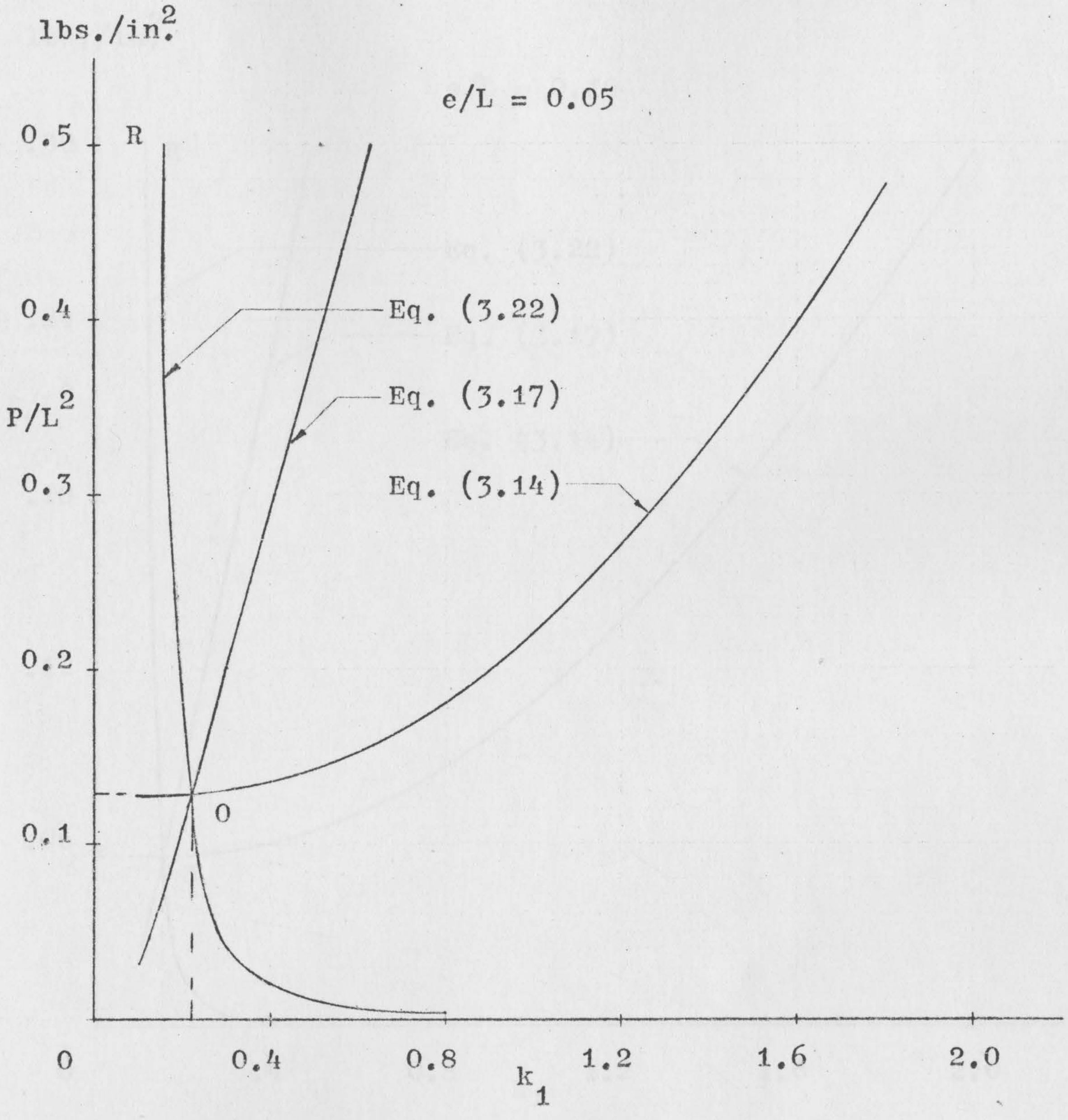


Fig. 3.4 Curve OR is design curve when $s_A = s_y$.

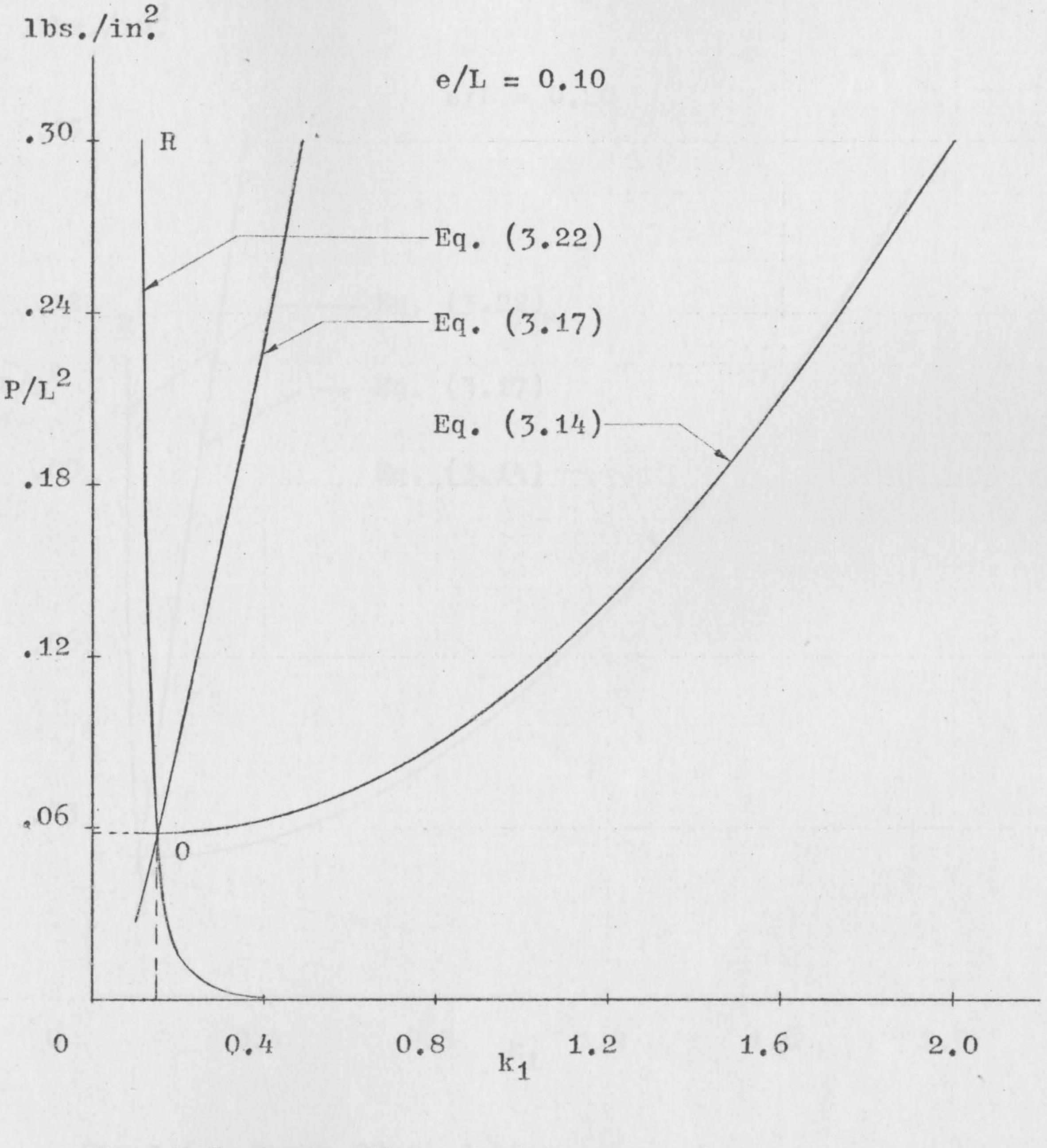


Fig. 3.5 Curve OR is design curve when $s_A = s_y$.

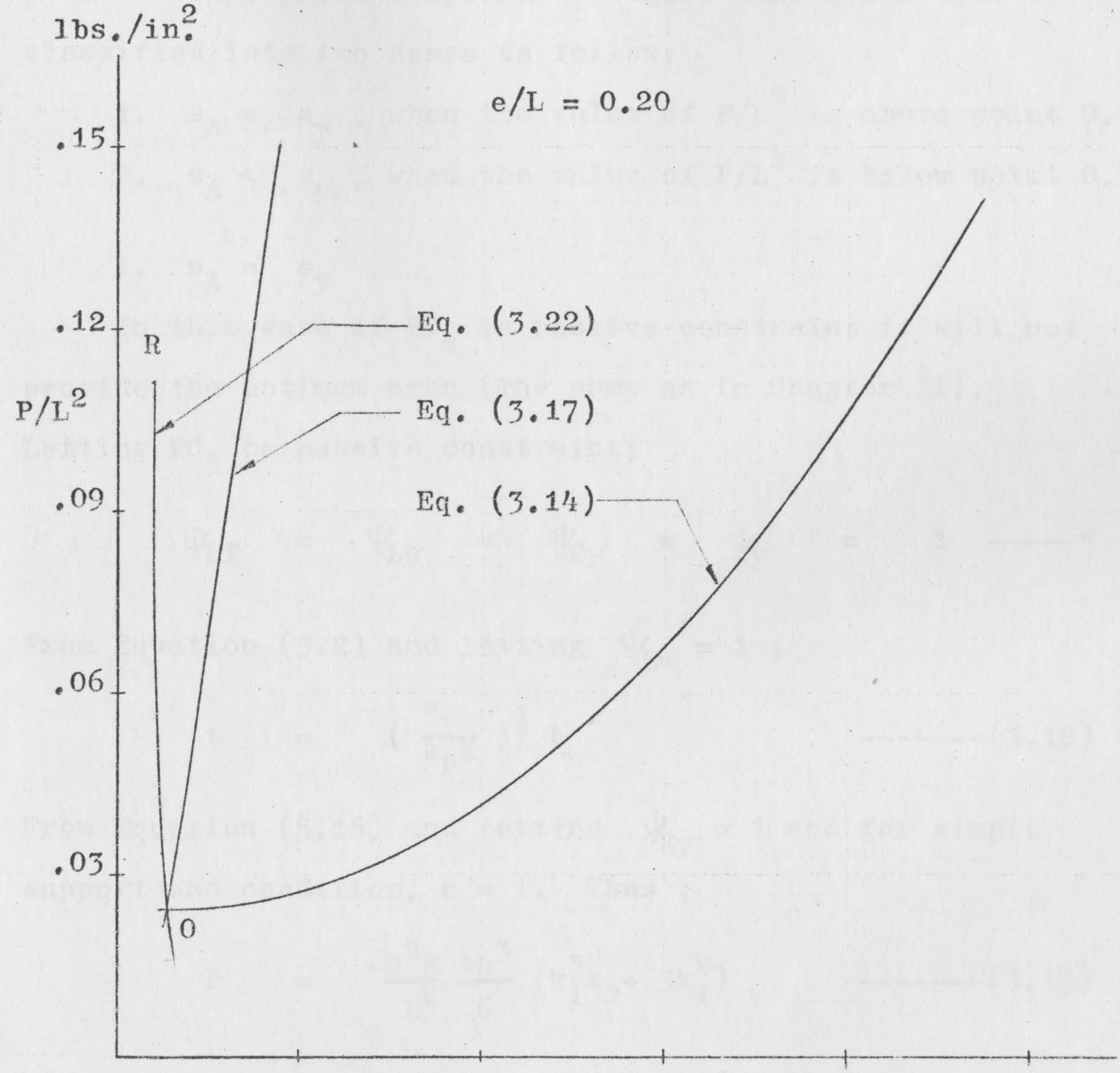


Fig.3.6 Curve OR is design curve when $s_A = s_y$.

The relation between P/L^2 and k_1 shown in Figures 3.2 to 3.6 is similar to Figures 2.2 to 2.6 respectively. From these graphs the optimum design stressed can be classified into two cases as follow:

1. $s_A = s_y$, when the value of P/L^2 is above point O.
2. $s_A < s_y$, when the value of P/L^2 is below point O.

1. $s_A = s_y$

In this case if FC_4 is passive constraint it will not provide the optimum area (The same as in Chapter II). Letting FC_3 be passive constraint;

$$\Psi_{Lf} = \Psi_{Lw} = \Psi_{Ey} = \Psi_y = 1 \quad \text{-----*}$$

From Equation (3.2) and letting $\Psi_{Lw} = 1$;

$$t = \left(\frac{s_y}{k_p E} \right)^{\frac{1}{2}} h \qquad \text{-------(3.18)}$$

From Equation (3.15) and letting $\Psi_{Ey} = 1$ and for simple support end condition, c = 1. Thus ;

$$P = \frac{\pi^2 E}{L^2} \frac{th^3}{6} (k_1^3 k_2 + 3k_1^2) \qquad \text{-------(3.19)}$$

Substituting value of t from Equation (3.18) into Equation (3.19) and simplifying;

$$h = \left[\frac{6PL^2 k_p^{\frac{1}{2}} s_y^{\frac{1}{2}}}{\pi^2 E^{\frac{1}{2}} (k_1^3 k_2 + 3k_1^2)} \right]^{\frac{1}{4}} \qquad \text{-------(3.20)}$$

Substituting value of h from Equation (3.20) into Equation (3.18)

$$t = \left[\frac{6PL^2 k_P^{\frac{1}{2}} s_y^{\frac{1}{2}}}{\pi^2 E\ (k_1^3 k_2 + 3k_1^2)}\right]^{\frac{1}{4}} (s_y/k_P E)^{\frac{1}{2}} \text{ ------(3.21)}$$

Substituting values of t and h from Equations (3.20) and (3.21) into Equation (3.6) and simplifying;

$$s_y = \frac{(P/L^2)^{\frac{1}{2}} E^{3/4} (k_1^3 k_2 + 3k_1^2)^{\frac{1}{2}} \pi k_P^{\frac{1}{4}}}{24^{\frac{1}{2}} s_y^{\frac{1}{4}}\ (1 + k_1 k_2)}$$

$$+ \frac{3(P/L^2)^{\frac{1}{4}} (e/L) E^{7/8} (k\ k\ +3k\ )^{3/4} \pi^{3/2} k_P^{1/8}}{6^{3/4} s_y^{1/8} (1 + 3k_1 k_2)}$$

$$\text{----------(3.22)}$$

The relation between P/L^2 and k_1 in Equation (3.22) was obtained by writing a computer program (See Appendix A) and are shown in Figures 3.3 to 3.6 . For designing the relation between P/L^2 and k_1 is plotted on more accurate scale shown in Figures 3.7 to 3.10 .

From Equation (3.20) ;

$$h_{opt} = \left[\frac{6PL^2 k_P^{\frac{1}{2}} s_y^{\frac{1}{2}}}{\pi^2 E^{\frac{1}{2}} (k_1^3 k_2 + 3k_1^2)}\right]^{\frac{1}{4}} \text{ ------------*}$$

From Equation (3.21) ;

$$t_{opt} = (s_y/k_P E)^{\frac{1}{2}}\ h_{opt} \text{ ------------*}$$

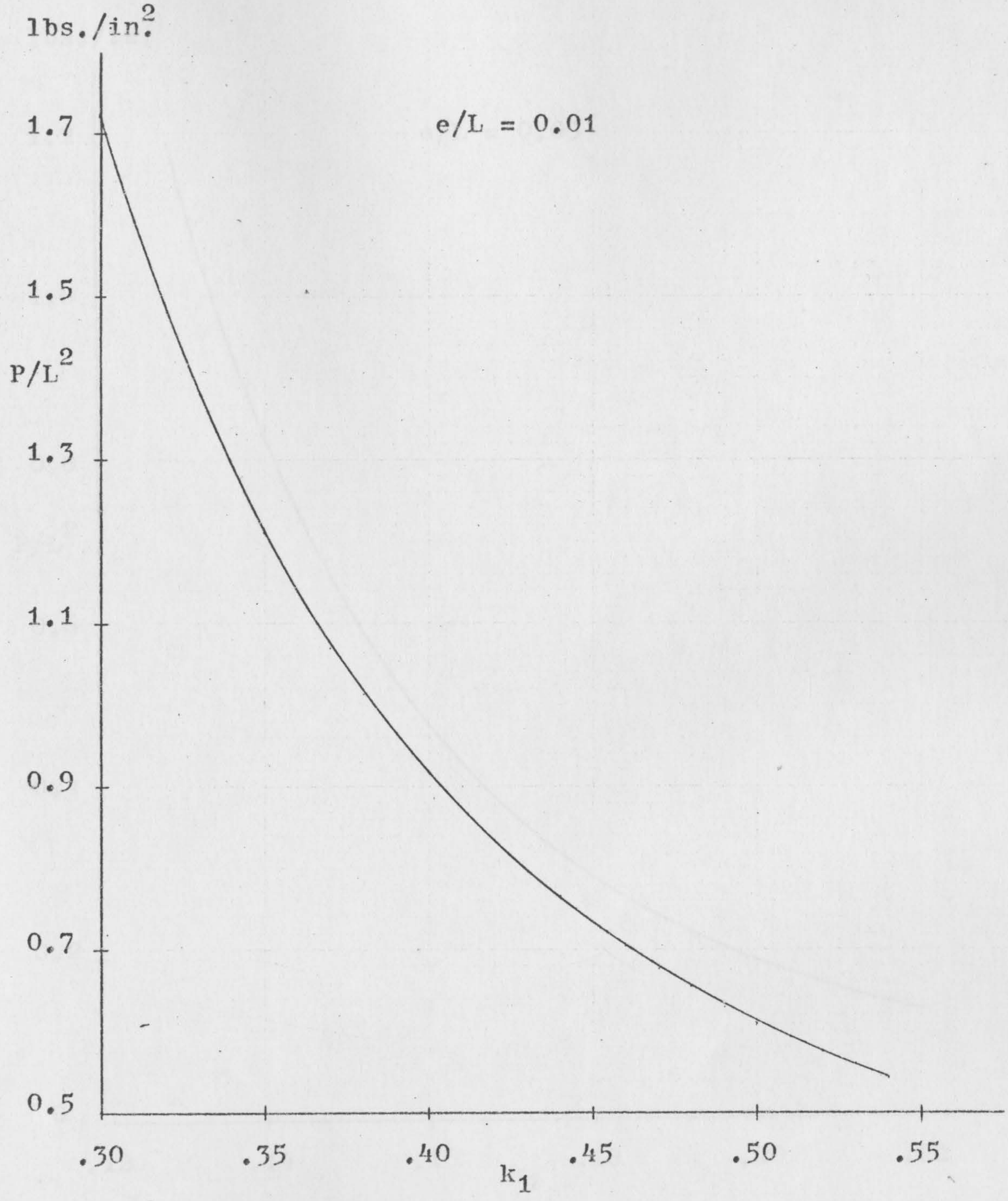


Fig. 3.7 Relation between P/L^2 and k_1 in Equation (3.22) when $s_A = s_y$.

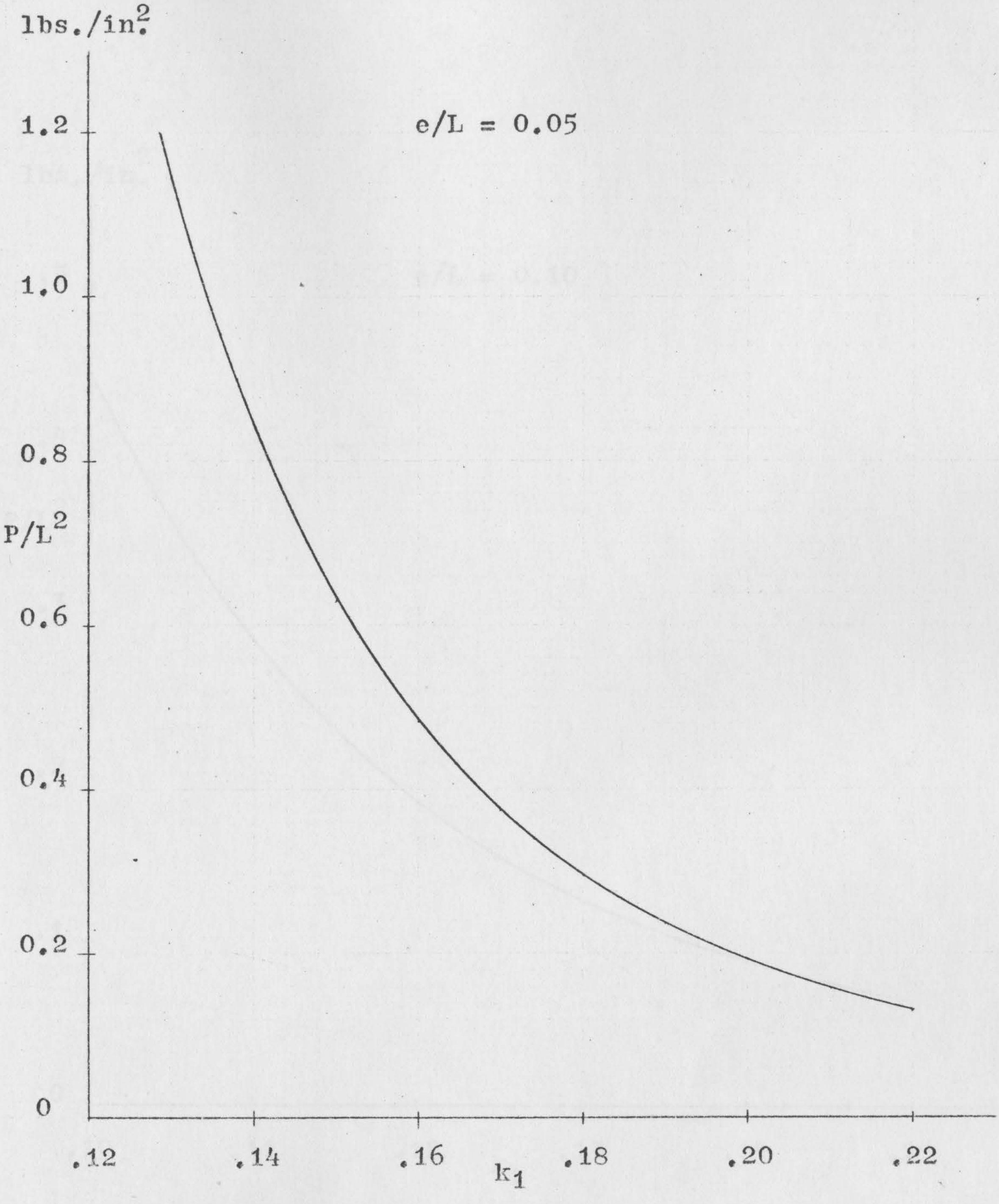


Fig. 3.8 Relation between P/L^2 and k_1 in Equation (3.22) when $s_A = s_y$.

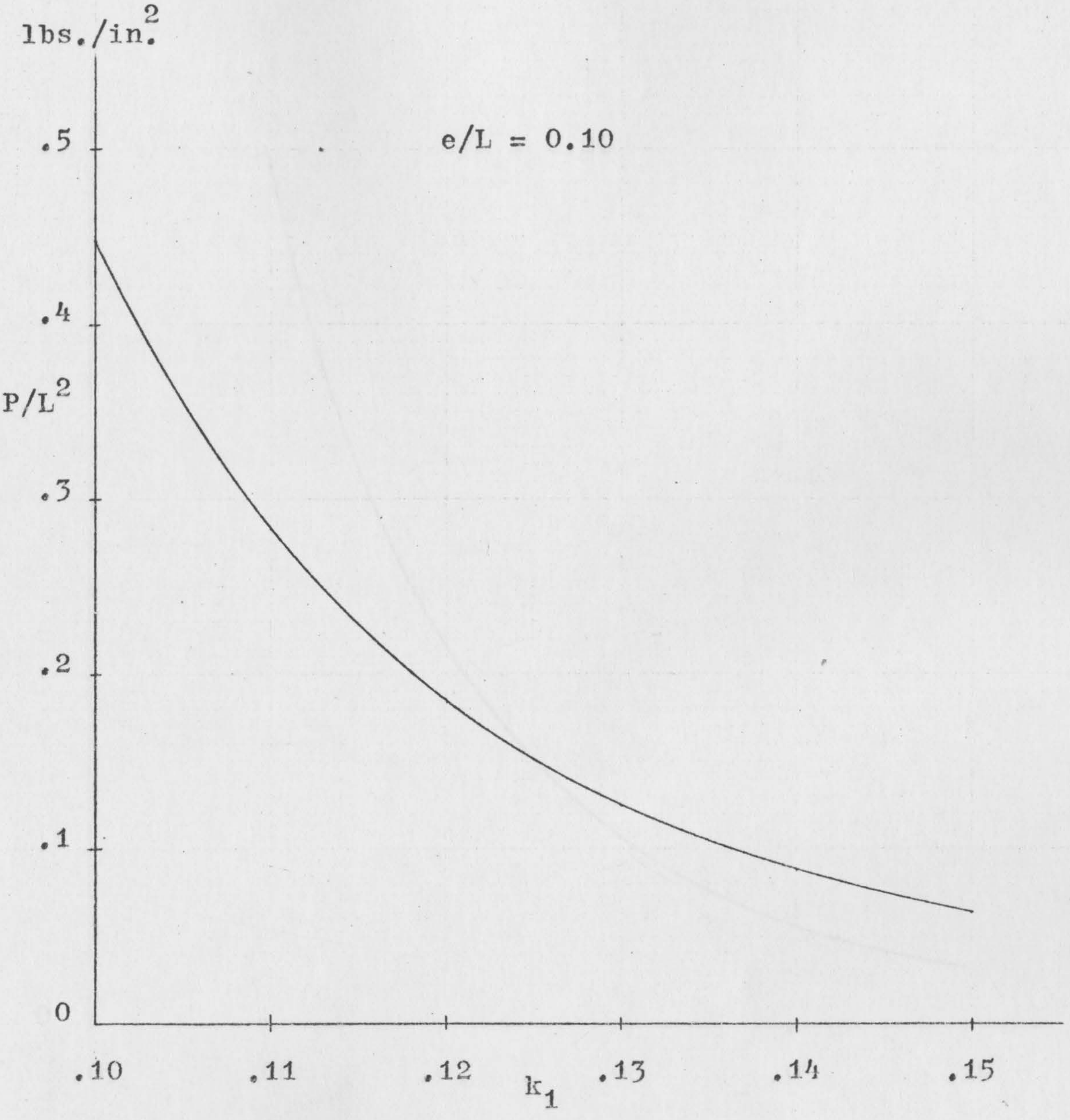


Fig. 3.9 Relation between P/L^2 and k_1 in Equation (3.22) when $s_A = s_y$.

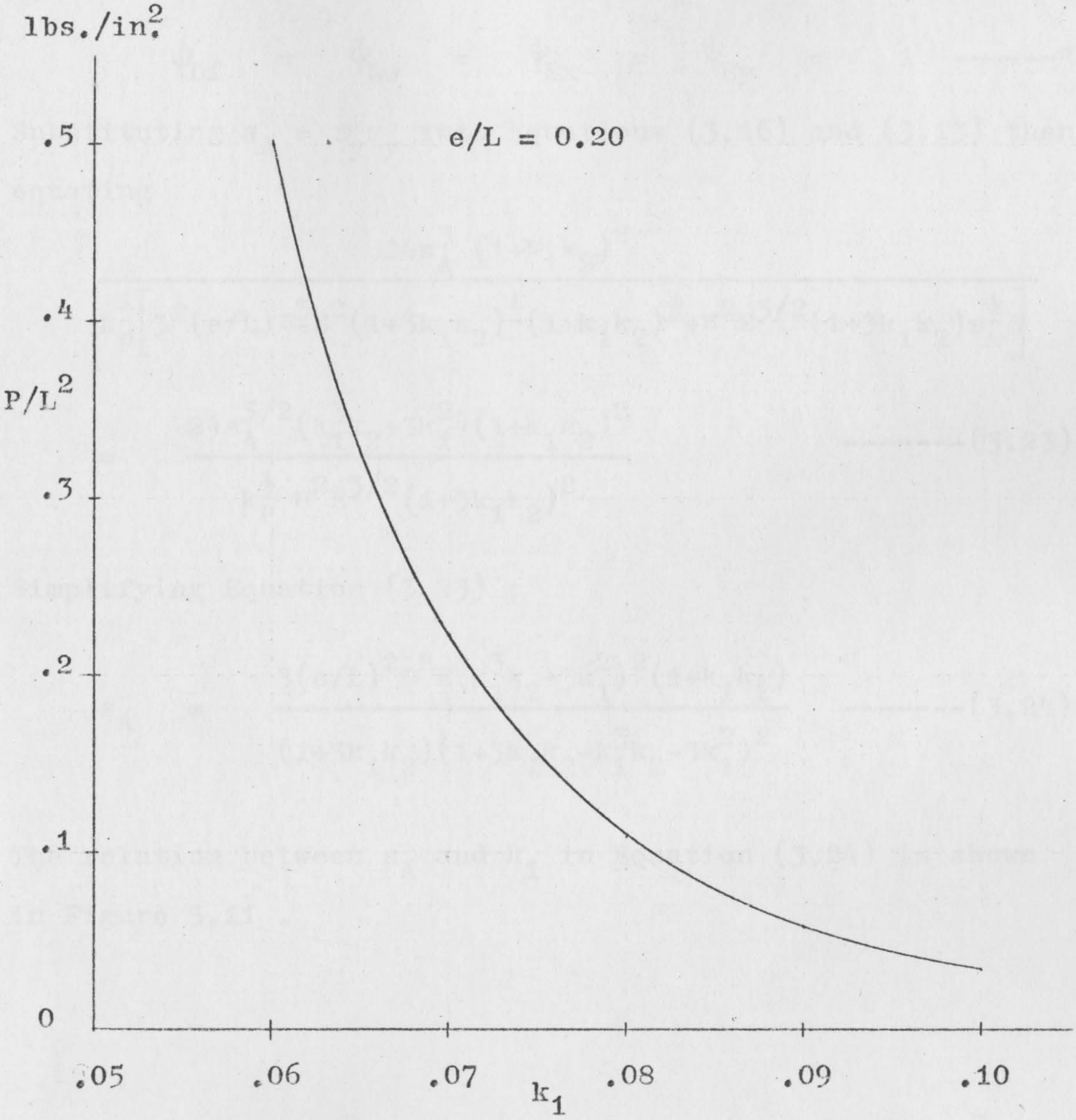


Fig. 3.10 Relation between P/L^2 and k_1 in Equation (3.22) when $s_A = s_y$.

2. $s_A < s_y$

In this case FC_5 is passive constraint, the other constraints are active constraints ;

$$\psi_{Lf} = \psi_{Lw} = \psi_{Ex} = \psi_{Ey} = 1 \quad ------*$$

Substituting $s_A = \psi_y s_y$ into Equations (3.16) and (3.13) then equating

$$\frac{24 s_A^3 \; (1+k_1 k_2)^2}{k_p^{\frac{1}{2}}\left[3^{\frac{1}{2}}(e/L)\pi^3 E^2 (1+3k_1 k_2)^{\frac{1}{2}}(1+k_1 k_2)^{\frac{1}{2}}+\pi^2 E^{3/2}(1+3k_1 k_2) s_A^{\frac{1}{2}}\right]}$$

$$= \frac{24 s_A^{5/2}(k_1^3 k_2+3k_1^2)(1+k_1 k_2)^2}{k_p^{\frac{1}{2}} \; \pi^2 E^{3/2}(1+3k_1 k_2)^2} \quad --------(3.23)$$

Simplifying Equation (3.23) ;

$$s_A = \frac{3(e/L)^2 \pi^2 E (k_1^3 k_2+3k_1^2)^2 (1+k_1 k_2)}{(1+3k_1 k_2)(1+3k_1 k_2-k_1^3 k_2-3k_1^2)^2} \quad -------(3.24)$$

The relation between s_A and k_1 in Equation (3.24) is shown in Figure 3.11 .

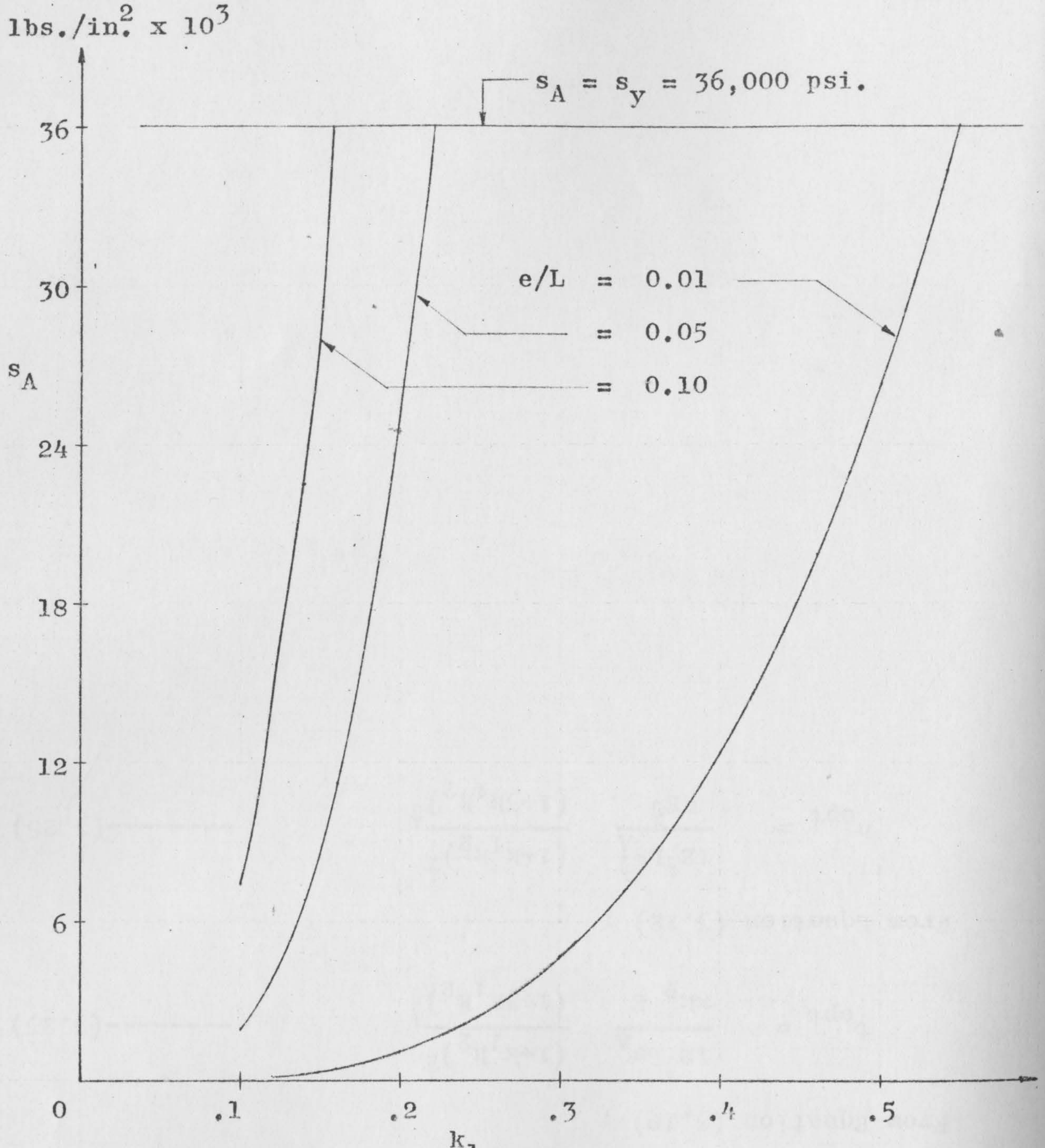


Fig. 3.11 Relation between s_A and k_1 in Equation (3.24) .

Substituting the values of s_A and k_1 from Equation (3.24) into Equation (3.13) or Equation (3.16), will yield the relationship between P/L^2, s_A and k_1 . The relation between k_1, P/L^2 and s_A is shown in Figures 3.12 to 3.16 .

From Equation (3.10) ;

$$t_{opt} = \frac{12^{\frac{1}{2}} L s_A}{\pi k_P^{\frac{1}{2}} E} \frac{(1+k_1 k_2)^{\frac{1}{2}}}{(1+3k_1 k_2)^{\frac{1}{2}}} \quad \text{--------}(3.25)$$

From Equation (3.12) ;

$$h_{opt} = \frac{12^{\frac{1}{2}} L s_A^{\frac{1}{2}}}{\pi E^{\frac{1}{2}}} \frac{(1+k_1 k_2)^{\frac{1}{2}}}{(1+3k_1 k_2)^{\frac{1}{2}}} \quad \text{--------}(3.26)$$

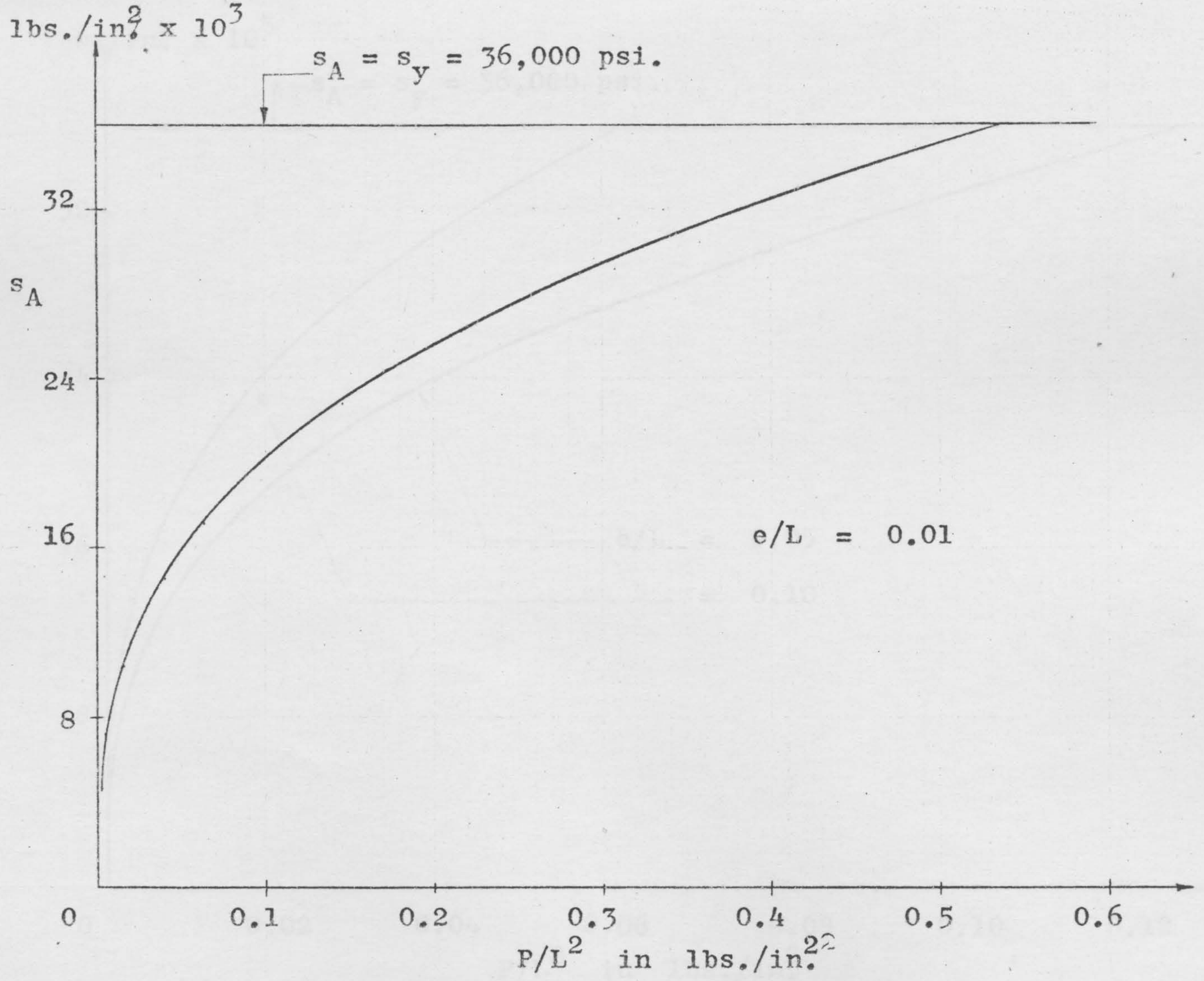


Fig. 3.12 Relation between P/L^2 and Optimum Design Stress.

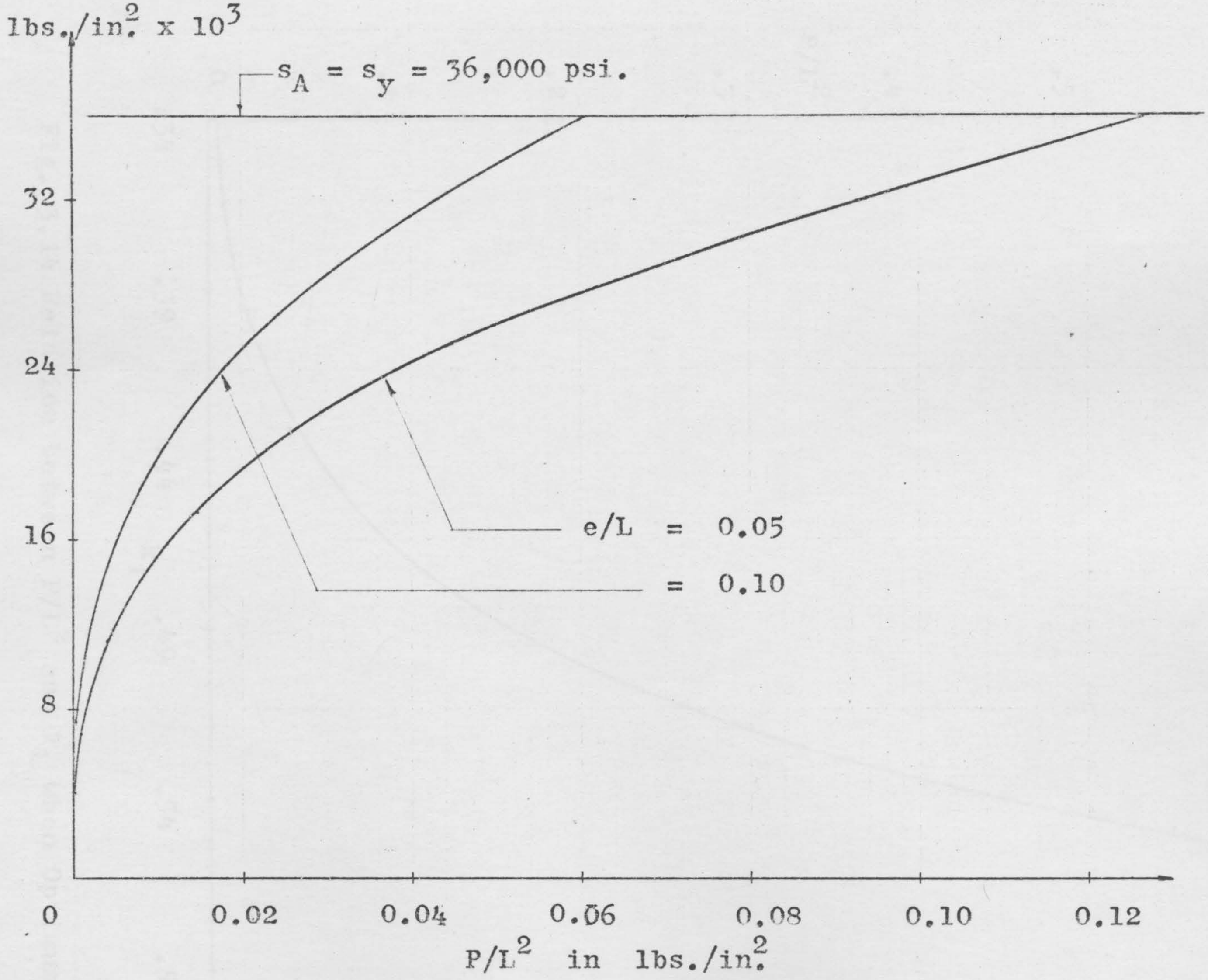


Fig. 3.13 Relation between P/L^2 and Optimum Design Stress.

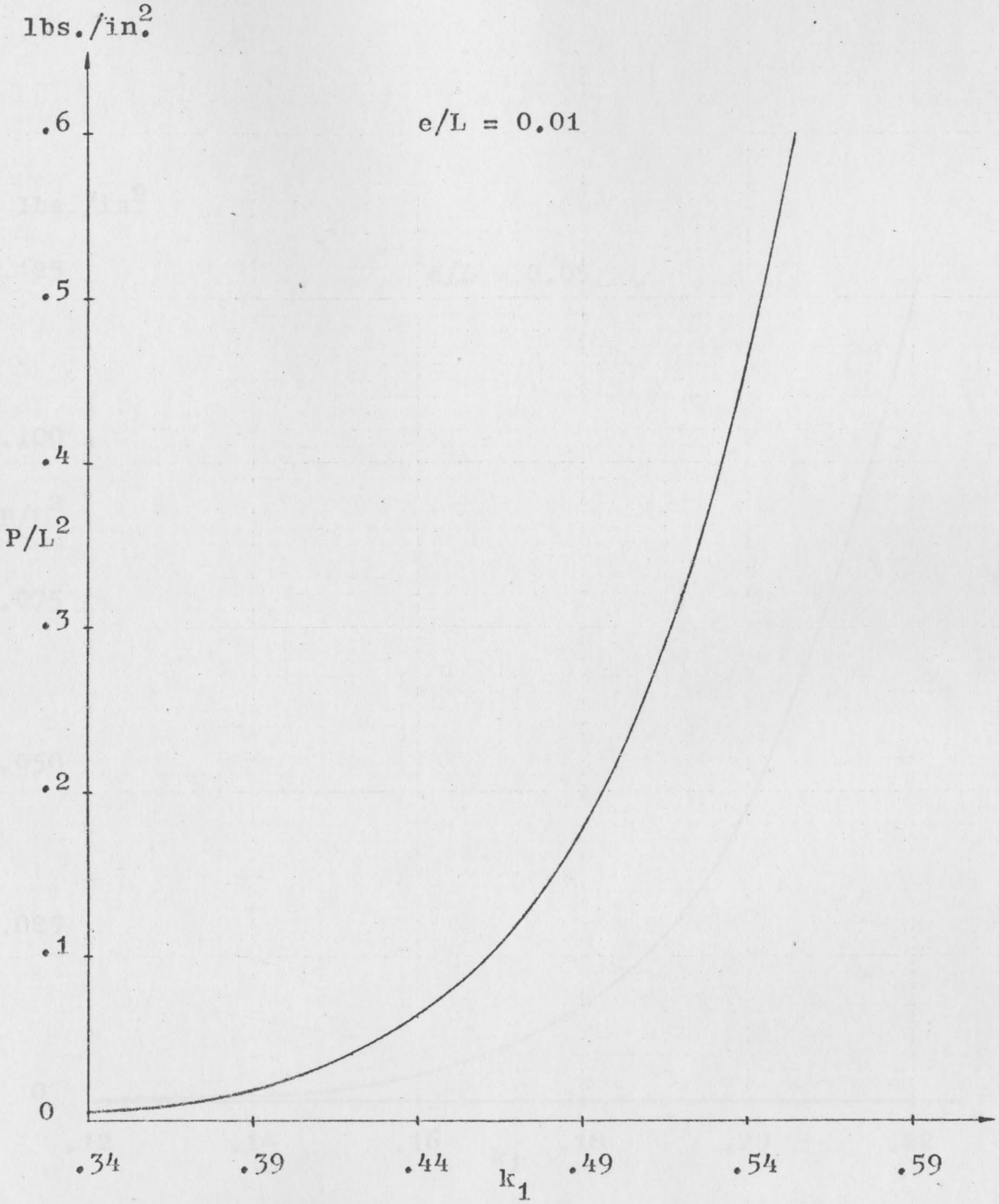


Fig. 3.14 Relation between P/L^2 and k_1 when Optimum Design Stress less than Yield Strength.

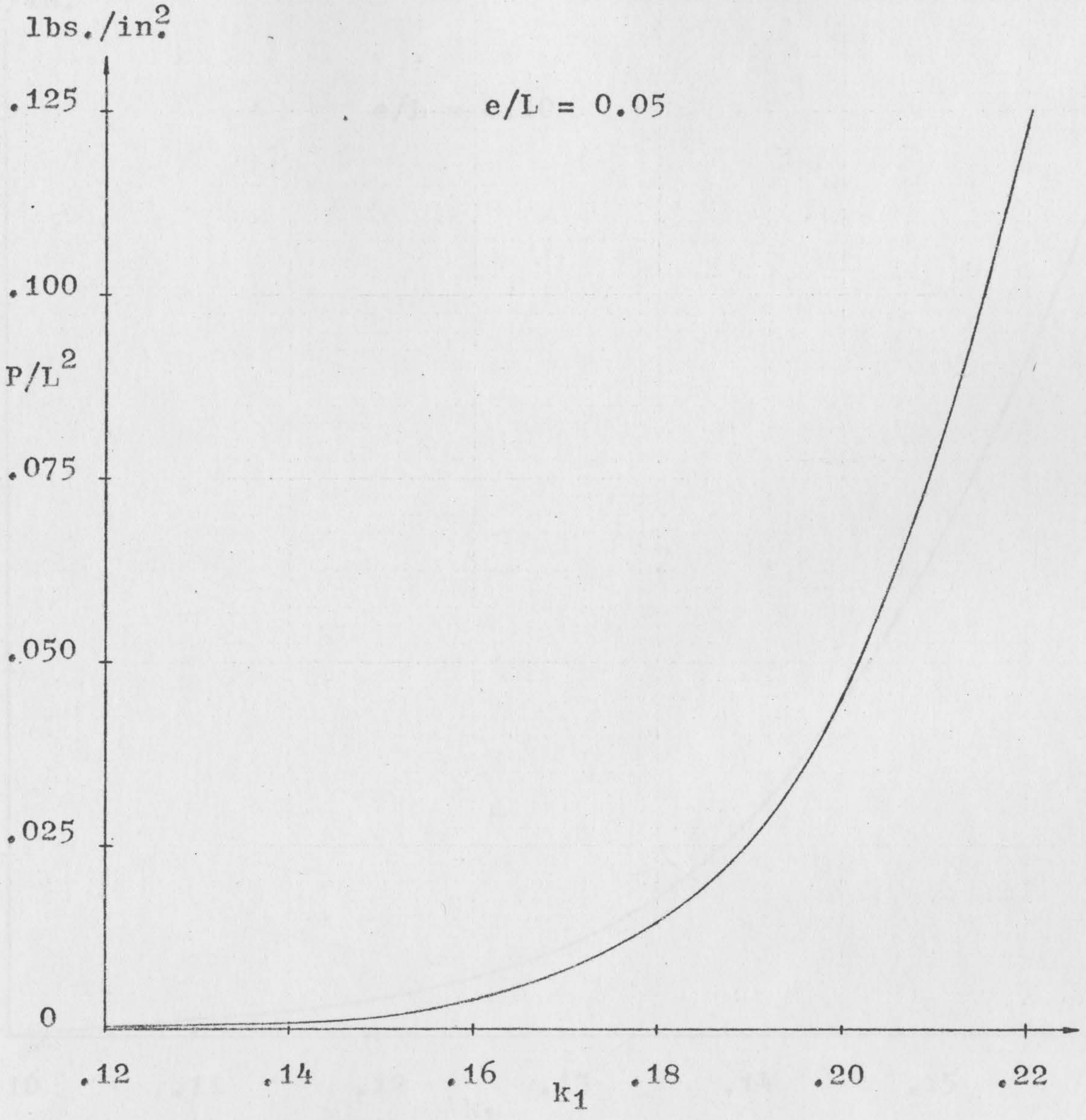


Fig. 3.15 Relation between P/L^2 and k_1 when Optimum Design Stress less than Yield Strength.

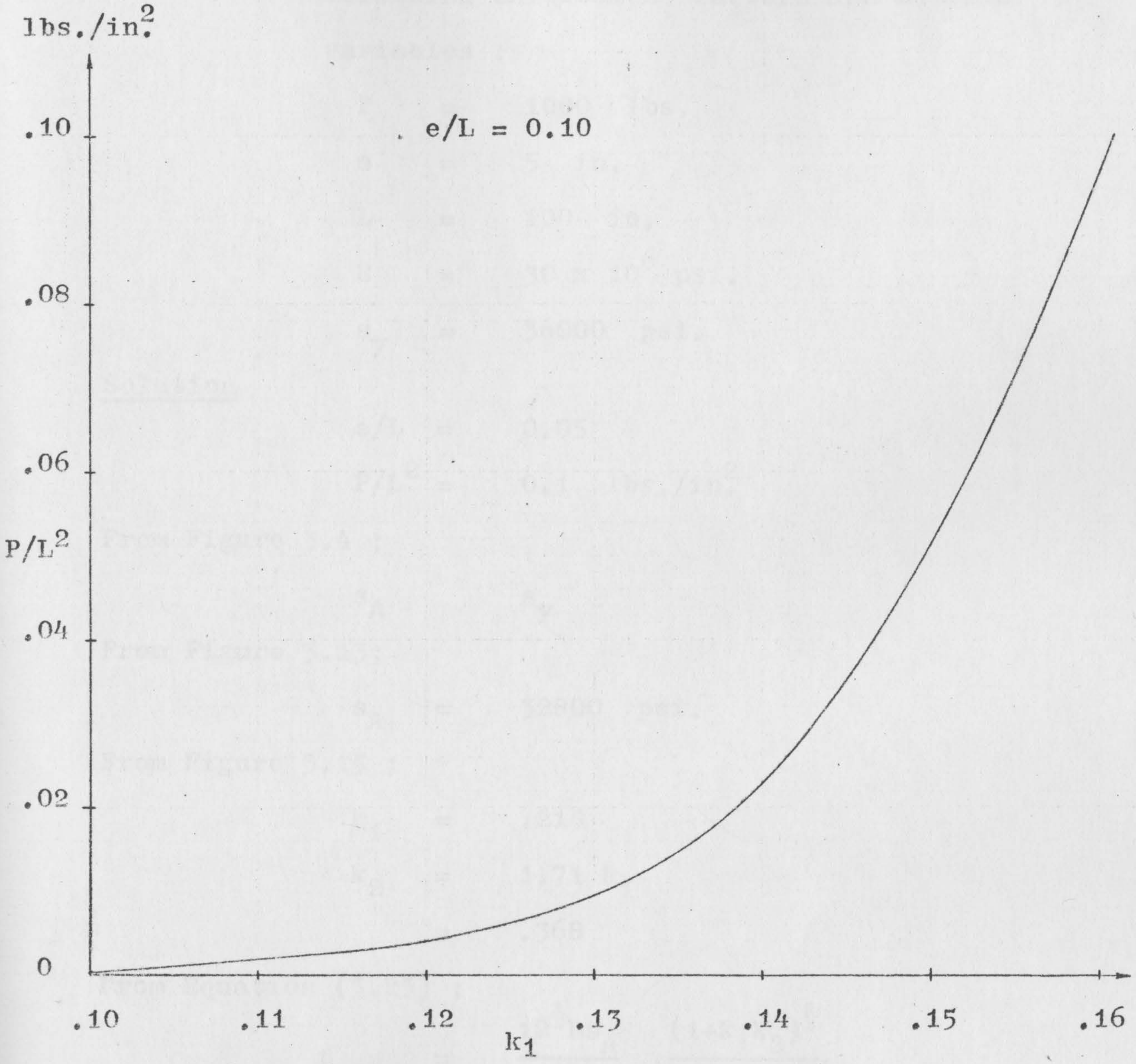


Fig. 3.16 Relation between P/L and k when Optimum Design Stress less than Yield Strength.

Example 3.1 Design a simply supported rectangular tube beam-column as shown in Figure 3.1 for the following environment factors and material variables :

$P = 1000$ lbs.

$e = 5$ in.

$L = 100$ in.

$E = 30 \times 10^6$ psi.

$s_y = 36000$ psi.

Solution

$e/L = 0.05$

$P/L^2 = 0.1$ lbs./in.^{2}

From Figure 3.4 ;

$s_A \quad s_y$

From Figure 3.13;

$s_A = 32800$ psi.

From Figure 3.15 ;

$k_1 = .215$

$k_2 = 1.71\ k_1$

$= .368$

From Equation (3.25) ;

$$t_{opt} = \frac{12^{\frac{1}{2}} L s_A}{\pi k_p^{\frac{1}{2}} E} \frac{(1+k_1k_2)^{\frac{1}{2}}}{(1+3k_1k_2)^{\frac{1}{2}}}$$

$$= \left(\frac{12}{10.3}\right)^{\frac{1}{2}} \frac{32800x100}{3.14x30x10^6} \left(\frac{1+0.215x0.368}{1+3x.215x.368}\right)^{\frac{1}{2}}$$

$$= .0352 \text{ in.}$$

From Equation (3.26) ;

$$h_{opt} = \frac{12^{\frac{1}{2}} L s_A^{\frac{1}{2}}}{\pi E^{\frac{1}{2}}} \frac{(1+k_1k_2)^{\frac{1}{2}}}{(1+3k_1k_2)^{\frac{1}{2}}}$$

$$= \left(\frac{12x32800}{30x10^6}\right)^{\frac{1}{2}} \frac{100}{3.14} \left(\frac{1+0.215x0.368}{1+3x.215x.368}\right)^{\frac{1}{2}}$$

$$= 3.42 \text{ in.}$$

Check

From Figure 3.1 ;

$$A = 2th(1+k_1k_2)$$

$$= 2x0.0352x3.42(1+0.215x0.368)$$

$$= 0.262 \text{ in}^2.$$

From Figure 4.2 ;

$$A/L^2 = 0.260x10^{-4}$$

$$A = 0.260 \text{ in}^2.$$ ---------O.K.

From Equation (3.1) ;

$$s_{Lf} = 3.62E\left(\frac{k_2t}{k_1h}\right)^2$$ ----------*

$$= 3.62x30x10^6 \left(\frac{0.368x0.0352}{0.215x3.42}\right)^2$$

$$= 33000 \text{ psi.}$$ ---------O.K.

From Equation (3.3) and for simple support end condition, c = 1;

$$s_{Ex} = \frac{\pi^2E}{L^2} \frac{h^2}{12} \frac{(1+3k_1k_2)}{(1+k_1k_2)}$$ ----------*

$$= \frac{(3.14x3.42)^2x30x10^6}{10^4x12} \left(\frac{1+0.237}{1+0.079}\right)$$

$$= 33000 \text{ psi.}$$ ---------O.K.

From Equation (3.4) and for simple support end condition,

c = 1 ;

$$s_{Ey} = \frac{\pi^2 E}{L^2}\,\frac{h^2}{12}\,\frac{(k_1^3k_2+3k_1^2)}{(1+k_1k_2)} \qquad \text{-----------*}$$

$$= \frac{(3.14\text{x}3.42)^2}{10^4\text{x}12}\,\text{x}30\text{x}10^6\,\frac{(.215^3\text{x}.368+3\text{x}.215^2)}{1.079}$$

$$= 3780 \text{ psi.}$$

$$P/A = \frac{1000}{.262}$$

$$= 3800 \text{ psi.} \qquad \text{-----------O.K.}$$

From Equation (3.6) ;

$$s_A = \frac{3Pe}{th^2(1+3k_1k_2)} + \frac{P}{2th(1+k_1k_2)}$$

$$= \frac{3\text{x}1000\text{x}5}{0.0354\text{x}3.42^2(1.237)} + 3800$$

$$= 29200 + 3800$$

$$= 33000 \text{ psi.} \qquad \text{----------O.K.}$$

CHAPTER IV

COMPARISON OF RESULTS

Circular Tube Section

From Figure 1.1 ;

$$A = \pi Dt \qquad \text{--------(4.1)}$$

When the optimum design stress is less than the yield strength, D_{opt} and t_{opt} are obtained from Equations (1.17) and (1.18).

Substituting D_{opt} and t_{opt} into Equation (4.1) ;

$$A_{opt} = \pi(\psi_y s_y/KE)(8\psi_y s_y/E)\frac{L^2}{\pi^2}$$

$$\left(\frac{A}{L^2}\right)_{opt} = \frac{8}{\pi K}\left(\frac{\psi_y s_y}{E}\right)^2 \qquad \text{--------(4.2)}$$

Using the relation between ψ_y and P/L^2 from Equation (1.16) as data in running the computer program* for Equation (4.2), the relation between P/L^2 and A/L^2 were obtained and are shown in Figure (4.1) to Figure (4.4).

When the optimum design stress equals the yield strength, D_{opt} and t_{opt} are obtained from Equations (1.19) and (1.20).

From Equation (1.19) ;

$$t = \frac{s_y D}{KE} \qquad \text{--------(4.3)}$$

From Equation (1.20) ;

$$s_y = \frac{4Pe + DP}{\pi D^2 t} \qquad \text{----------*}$$

---

*See computer program in Appendix A

Substituting the value of t from Equation (4.3) into this equation and rearranging ;

$$P/L^2 \quad = \quad \frac{\pi s_y^2 \ (D/L)^3}{KE(4e/L + D/L)} \qquad \text{----------}(4.4)$$

Substituting the value of t from Equation (4.3) into Equation (4.1);

$$A \quad = \quad \frac{\pi s_y D^2}{KE}$$

$$A/L^2 \quad = \quad \frac{\pi s_y (D/L)^2}{KE} \qquad \text{----------}(4.5)$$

From Equations (4.4) and (4.5) the relation between P/L^2 , D/L and A/L^2 is obtained by running the computer program. The relation between P/L^2 and A/L^2 is shown in Figure (4.1) to Figure (4.4) .

H - Section

From Figure 2.1 ;

$$A \quad = \quad th(1+2k_1k_2) \qquad \text{----------}(4.6)$$

When the optimum design stress is less than the yield strenght, t_{opt} and h_{opt} are obtained from Equations (2.24) and(2.25) . Substituting values of t_{opt} and h_{opt} into Equation (4.6) and simplifying ;

$$(A/L^2)_{opt} \quad = \quad \frac{12 \ s_A^{3/2}}{k_p^{\frac{1}{2}} \ \pi^2 E^{3/2}} \frac{(1+2k_1k_2)^2}{(1+6k_1k_2)} \qquad \text{-------}(4.7)$$

Using the relation between P/L^2 and k_1 shown in Figures 2.14 to 2.16 as data in running the computer program for Equation

(4.7) the relation between P/L^2 and A/L^2 were obtained and are shown in Figures 4.1 to 4.4 .

When the optimum design stress equals the yield strength, t_{opt} and h_{opt} are obtained from Equations (2.20) and (2.21). Substituting values of t_{opt} and h_{opt} into Equation (4.6) and simplifying ;

$$(A/L^2)_{opt} = \frac{6^{\frac{1}{2}}(P/L^2)^{\frac{1}{2}}(s_y/k_p)^{\frac{1}{4}}(1+2k_1k_2)}{\pi E^{3/4}\, k_1^{3/2}\, k_2^{1/2}} \text{ --------(4.8)}$$

The relation between P/L^2 and k_1 from Equation (2.22) was used to obtain* relation between A/L^2 and P/L^2 in Equation (4.8). The results are shown in Figure 4.1 to Figure 4.4 .

In this case if FC_3 is the active constraint and FC_4 is the passive constraint the relation between P/L^2 and A/L^2 will be obtained as follows:

From Equation (2.10) and letting $\psi_y = \Psi_{Ex} = \Psi_{Lw} = 1$ and for simple support end condition, $c = 1$. Thus ;

$$t = \frac{12^{\frac{1}{2}} L s_y}{k_p^{\frac{1}{2}}\, \pi E} \; \frac{(1+2k_1k_2)^{\frac{1}{2}}}{(1+6k_1k_2)^{\frac{1}{2}}} \text{ ---------(4.9)}$$

From Equation (2.11) ;

$$h = \frac{12^{\frac{1}{2}} s_y^{\frac{1}{2}} L}{\pi E^{\frac{1}{2}}} \; \frac{(1+2k_1k_2)^{\frac{1}{2}}}{(1+6k_1k_2)^{\frac{1}{2}}} \text{ --------(4.10)}$$

Substituting values of t and h from Equations (4.9) and (4.10) into Equation (4.6) and simplifying ;

---

*See computer program in Appendix A

$$(A/L^2) = \frac{12s_y^{3/2}}{k_p^{\frac{1}{2}}\pi^2E^{3/2}} \frac{(1+2k_1k_2)^2}{(1+6k_1k_2)} \text{---------}(4.11)$$

Using the relation between P/L^2 and k_1 from Equation (2.13) to obtain the relation between A/L^2 and P/L^2 in Equation (4.11), the relations between P/L^2 and A/L^2 in Equations (4.8) and (4.11) are plotted on same figure for comparision, Figure 4.5 and Figure 4.6 for e/L equal to 0.01 and 0.05 . It will be seen that if FC_3 is an active constraint, optimum area will not be obtained.

Rectangular Tube Section

From Figure 3.1 ;

$$A = 2th(1+k_1k_2) \text{--------}(4.12)$$

When optimum design stressed is less than yield strength, t_{opt} and h_{opt} are obtained from Equations (3.25) and (3.26). Substituting values of t_{opt} and h_{opt} into Equation (4.12) and simplifying ;

$$(A/L^2)_{opt} = \frac{24s_A^{3/2}}{k_p^{\frac{1}{2}}\pi^2E^{3/2}} \frac{(1+k_1k_2)^2}{(1+3k_1k_2)} \text{--------}(4.13)$$

Using the relation between P/L^2 and k_1 shown in Figures 3.14 to 3.16 as data in running the computer program for Equation (4.13) the relation between P/L^2 and A/L^2 was obtained and is shown in Figures 4.1 to 4.4 .

When the optimum stress equals the yield strength, t_{opt} and h_{opt} are obtained from Equations (3.20) and (3.21)

Then, substituting values of t_{opt} and h_{opt} into Equation (4.12) and simplifying ;

$$(A/L^2)_{opt} = \frac{24^{\frac{1}{2}}(P/L\ )^{\frac{1}{2}}(s_y/k_p)^{\frac{1}{4}}(1+k_1k_2)}{\pi E^{3/4}(k_1^3k_2+3k_1^2)^{1/2}} \quad \text{-----}(4.14)$$

Using the relation between P/L^2 amd k_1 from Equation (3.22) for running the computer program (see Appendix A), the relation was obtained between P/L^2 and A/L^2 in Equation (4.11) and is shown in Figure 4.1 to 4.4 .

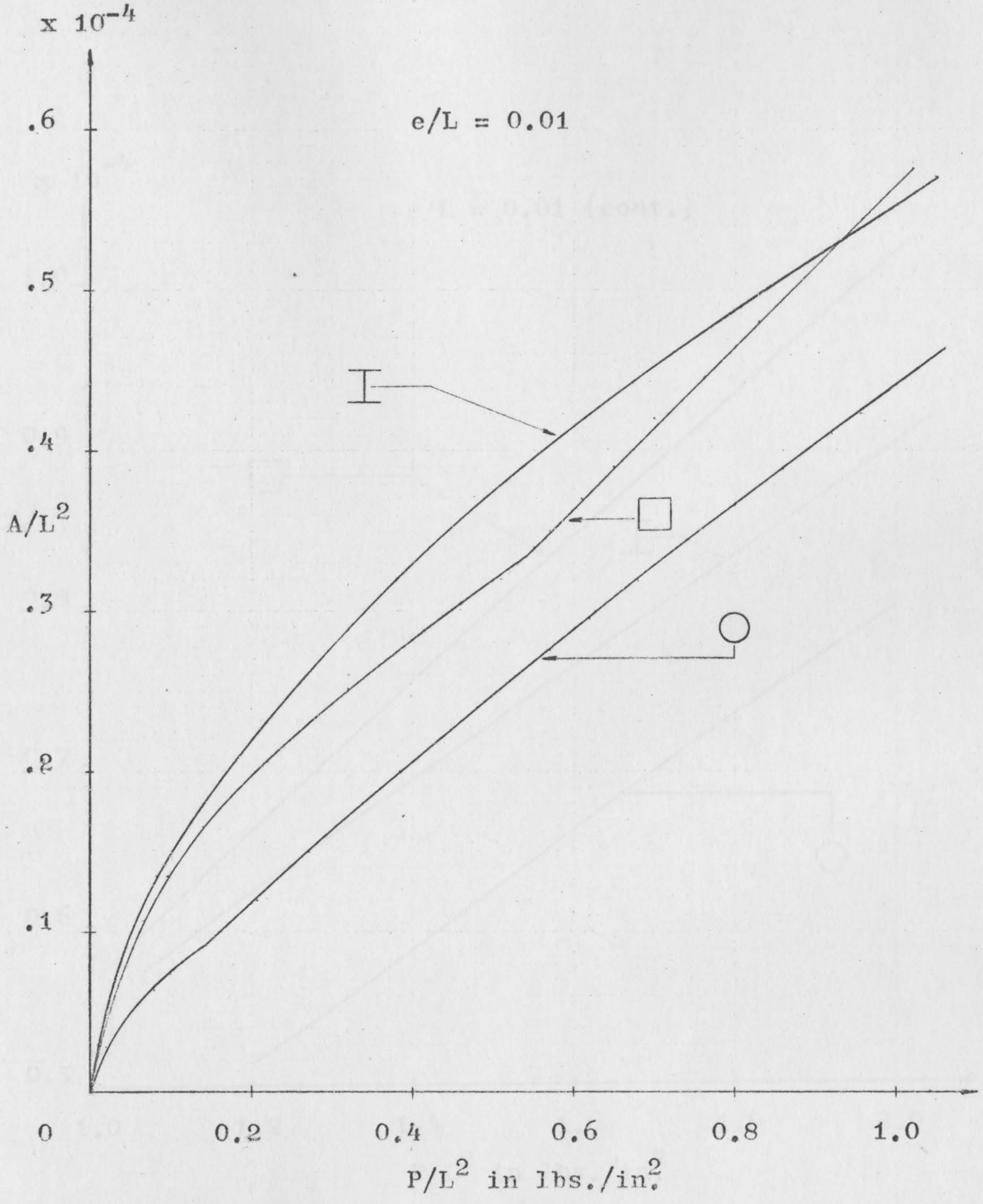


Figure 4.1 Form comparison of cross sectional area

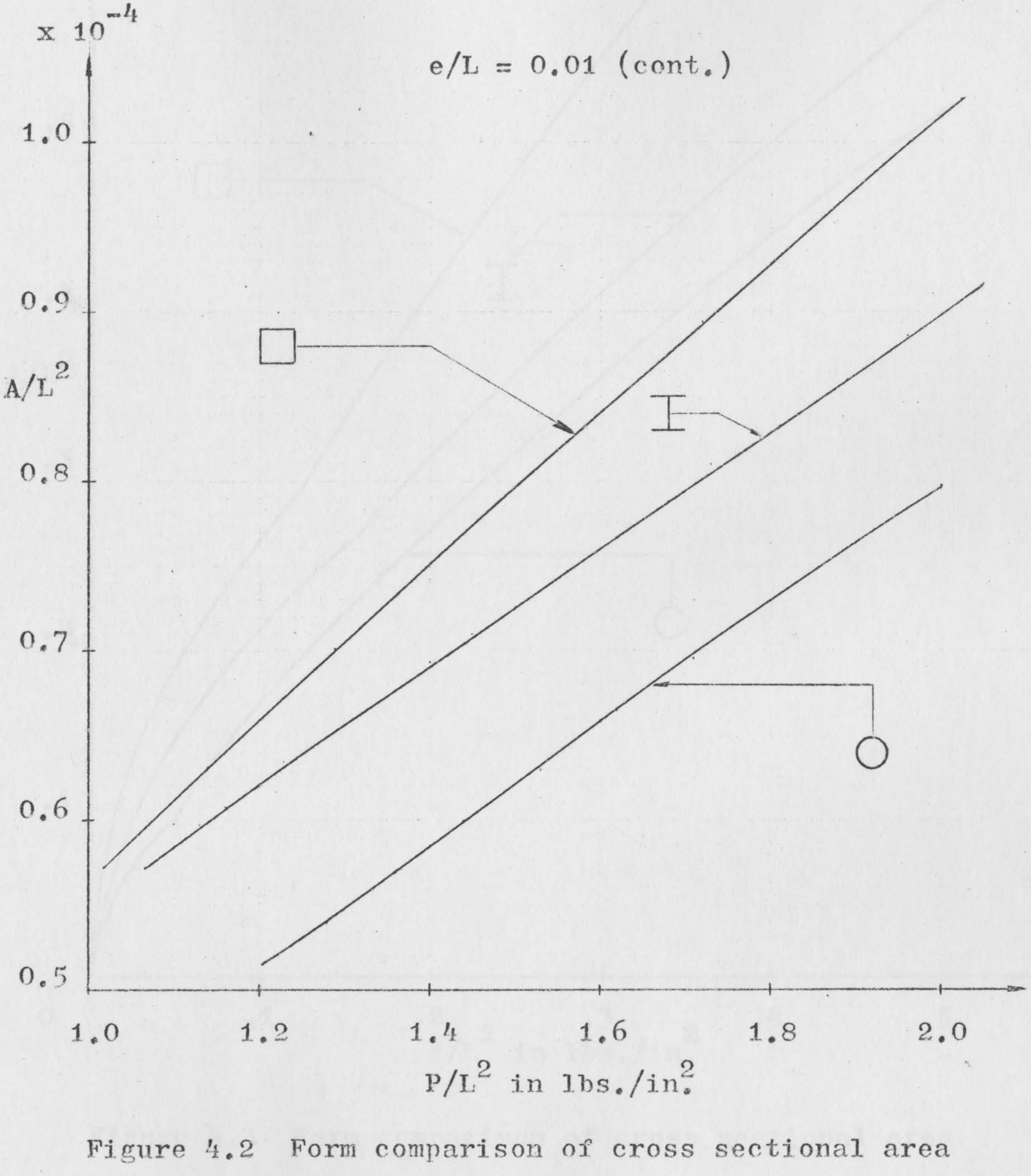


Figure 4.2 Form comparison of cross sectional area

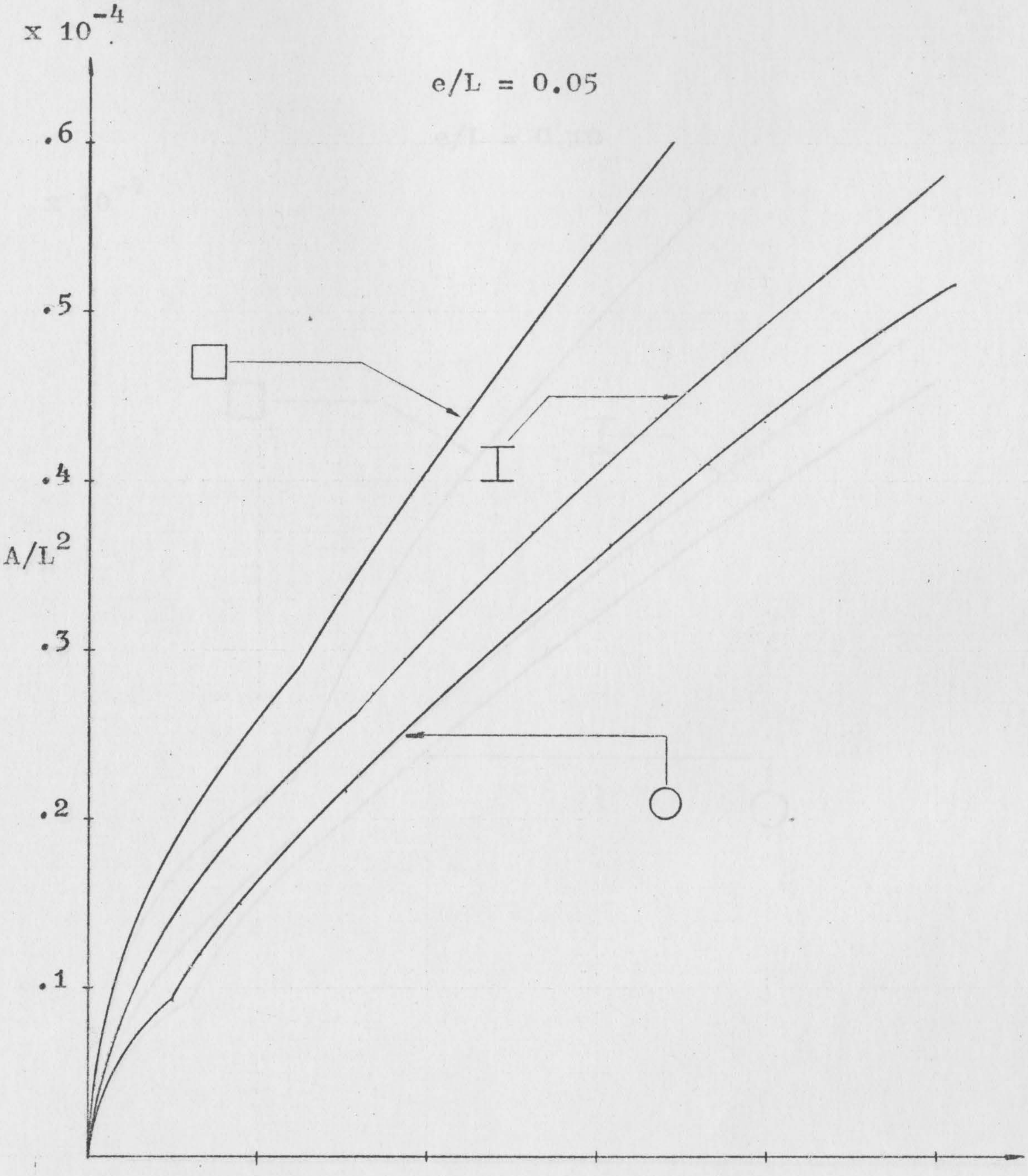


Figure 4.3 Form comparison of cross sectional area

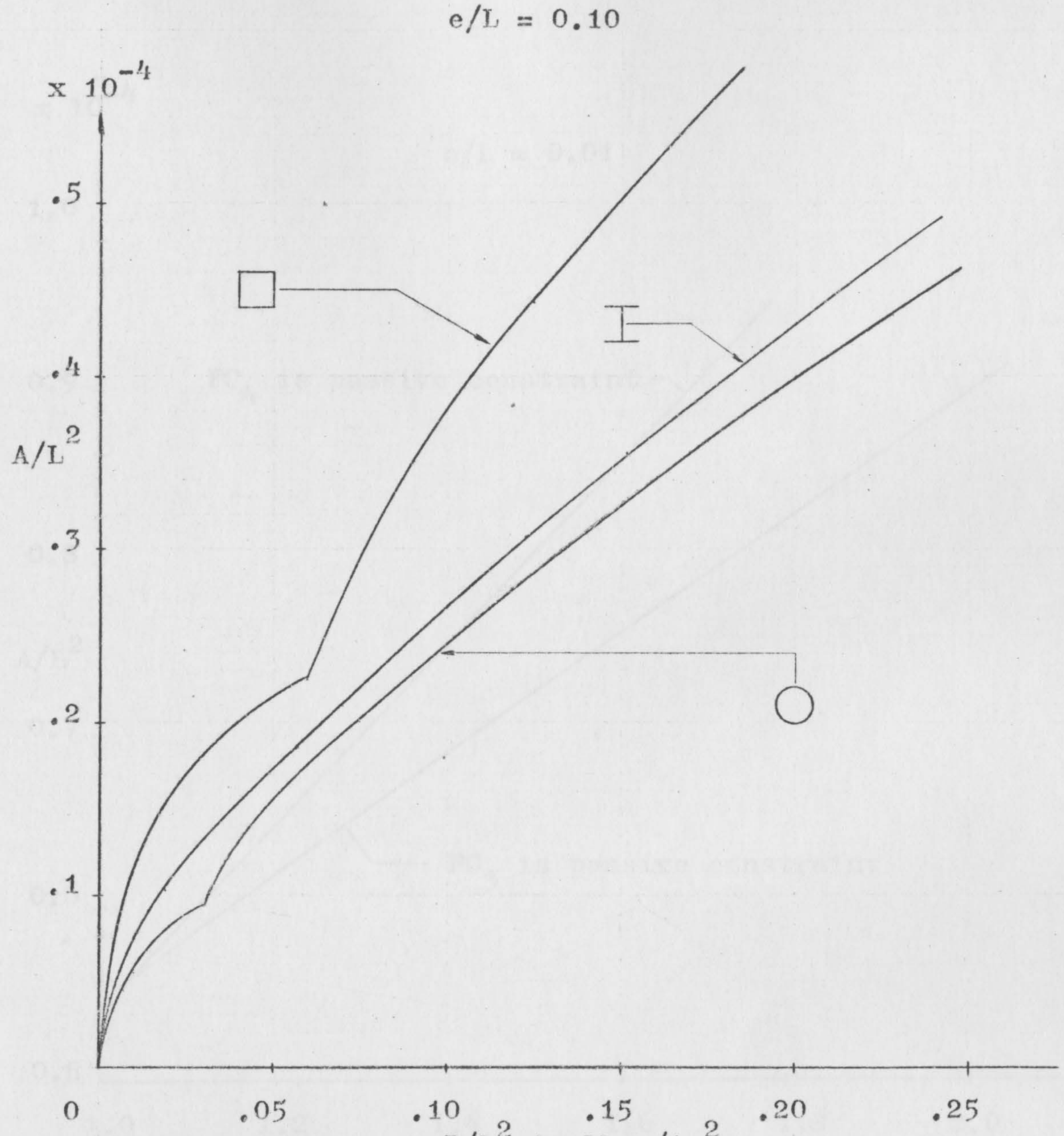


Figure 4.4 Form comparison of cross sectional area

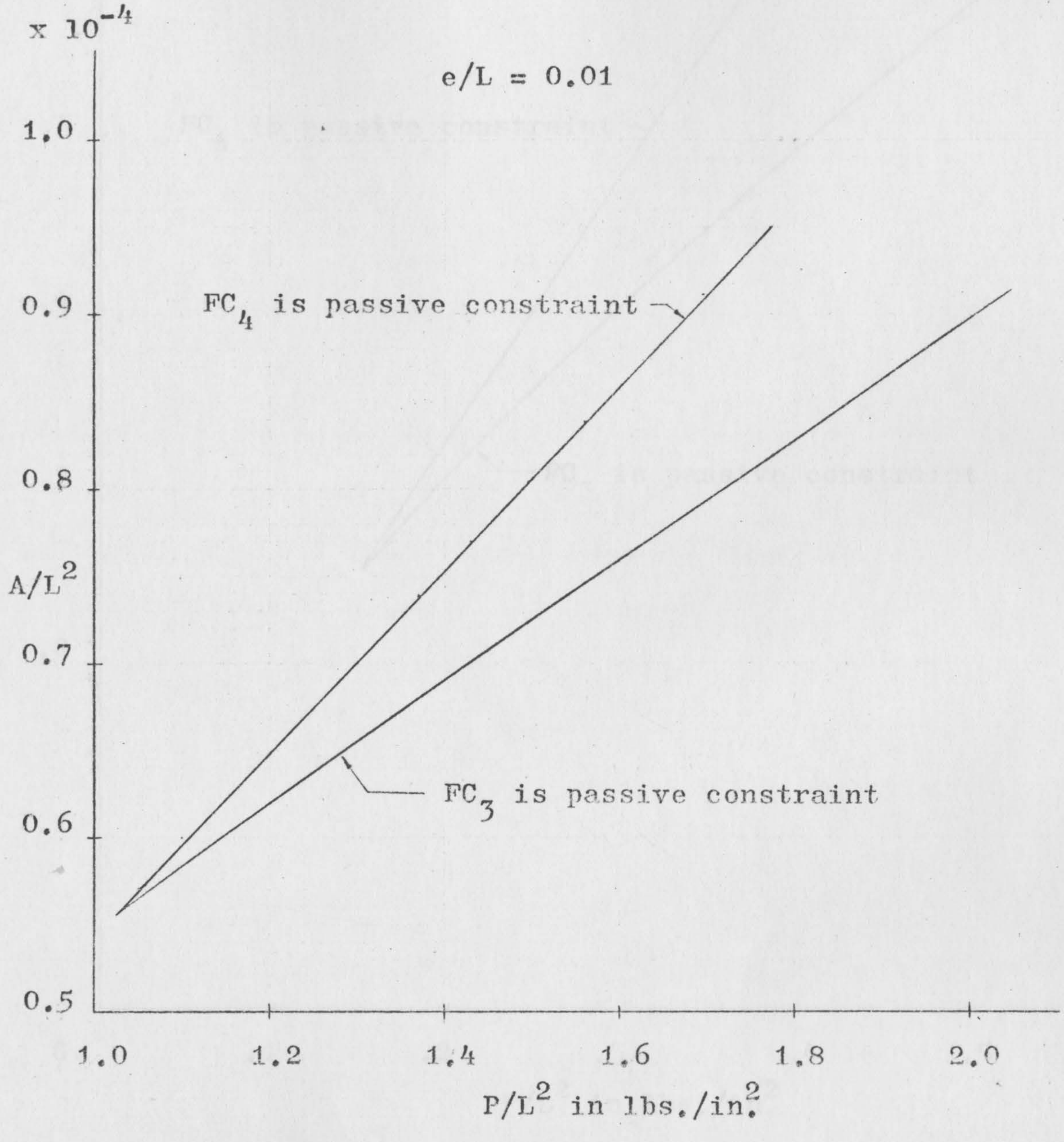


Figure 4.5 The proof that FC_3 is active constraint optimum cross sectional area will not be obtained.

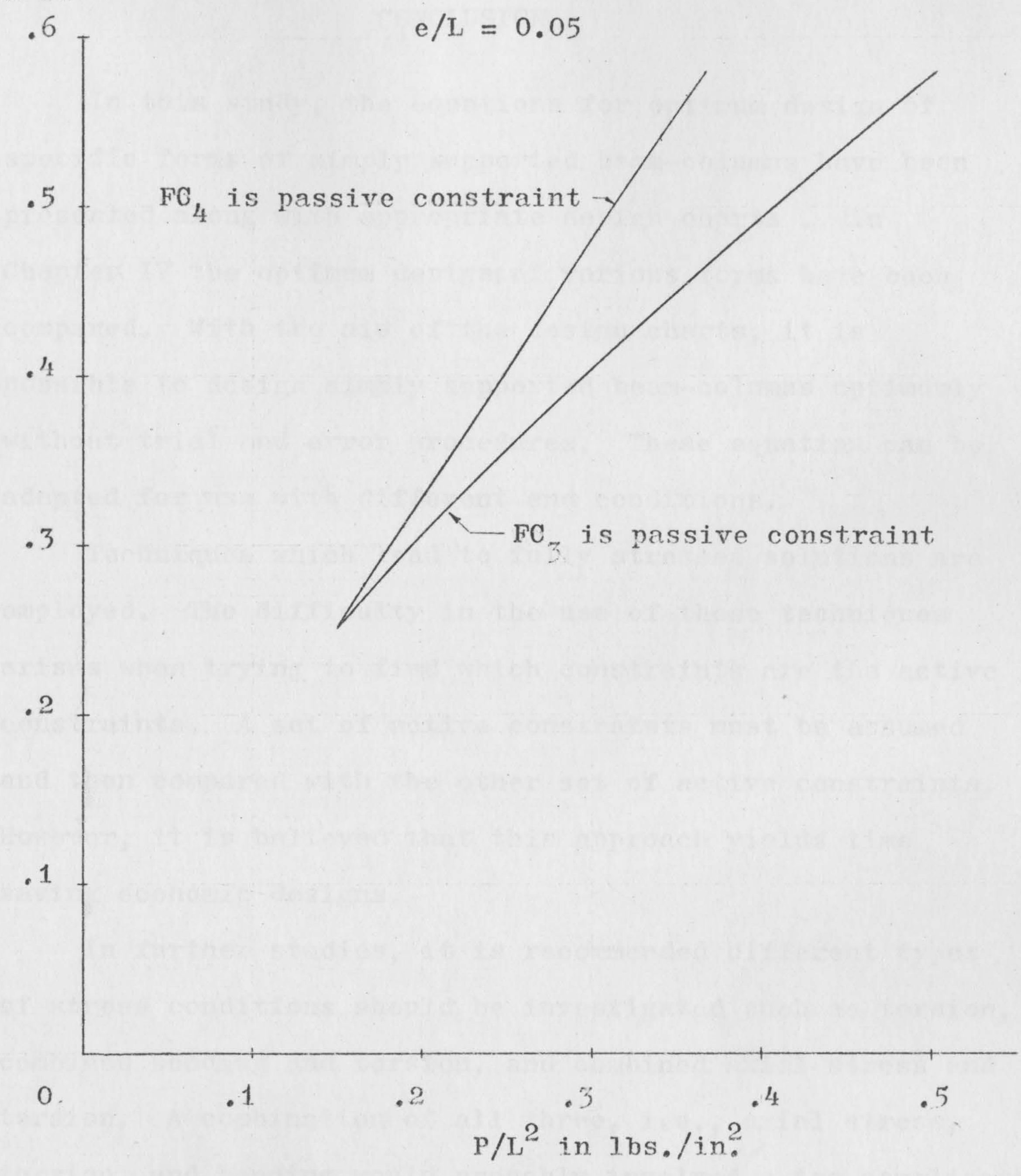


Figure 4.6 The proof that FC_3 is active constraint optimum cross sectional area will not be obtained.

CHAPTER V

CONCLUSION

In this study, the equations for optimum design of specific forms of simply supported beam-columns have been presented along with appropriate design charts . In Chapter IV the optimum designs of various forms have been compared. With the aid of the design charts, it is possible to design simply supported beam-columns optimumly without trial and error procedures. These equation can be adapted for use with different end conditions.

Techniques which lead to fully stressed solutions are employed. The difficulty in the use of these techniques arises when trying to find which constraints are the active constraints. A set of active constraints must be assumed and then compared with the other set of active constraints. However, it is believed that this approach yields time saving economic designs.

In further studies, it is recommended different types of stress conditions should be investigated such as torsion, combined bending and torsion, and combined axial stress and torsion. A combination of all three, i.e., axial stress, torsion, and bending would probably involve~~d~~ a too complicated procedure and other optimum techniques would need to be employed.

APPENDIX A

FORTAN COMPUTER PROGRAM

The symbols use in the computer program are as following:

AL2 = A/L^2

C, PL2 = P/L^2

CK1 = k_1

CK2 = k_2/k_1

CKP = k_p

DL = D/L

S = e/L

SA = s_A

X = ψ

```
TRAN IV G LEVEL  20                    MAIN                    DATE = 72091           1

          C       OPTIMUM STRUCTURAL DESIGN
          C       CIRCULAR SECTION S & C RELATIONSHIP   FOR X=1
                  DO 20 K=1,35,1
                  P=K
                  C=0.0005*P
                  S=(1.0-3.04*C)/388.0/C
            100   FORMAT(2F15.5)
             20   WRITE(6,100)C,S
                  STOP
                  END
```

```
FORTRAN IV G LEVEL  20                     MAIN                    DATE = 72091

               C      OPTIMUM STRUCTURAL DESIGN
               C      CIRCULAR SECTION X & C RELATIONSHIP WHEN S=0
                      DO 20 K=1,35,1
                      P=K
                      C=0.01*P
                      X=(3.04*C)**0.3333
                 100  FORMAT(2F15.5)
                 20   WRITE(6,100)C,X
                      STOP
                      END
```

```
RTRAN IV G LEVEL  20                    MAIN                DATE = 72091          1

           C      OPTIMUM STRUCTURAL DESIGN
           C      CIRCULAR SECTION X & C RELATIONSHIP   FOR  S=.01
            100   FORMAT(3F15.5)
                  C=0.01
            101   DO 20 K=1,100,1
                  P=K
                  X=0.01*P
                  A=X**3.5-3.04*C*X**0.5-3.88*C
            20    WRITE(6,100)C,X,A
                  C=C+0.01
                  IF(C-0.35)101,102,102
            102   STOP
                  END
```

```
C       OPTIMUM STRUCTURAL DESIGN
C       CIRCULAR SECTION X & C RELATIONSHIP   FOR  S=.05
  100   FORMAT(3F15.5)
        C=0.01
  101   DO 20 K=1,100,1
        P=K
        X=0.01*P
        A=X**3.5-3.04*C*X**0.5-19.4*C
  20    WRITE(6,100)C,X,A
        C=C+0.01
        IF(C-0.35)101,102,102
  102   STOP
        END
```

```
TRAN IV G LEVEL  20                  MAIN                 DATE = 72091

          C          OPTIMUM STRUCTURAL DESIGN
          C          CIRCULAR SECTION X & C RELATIONSHIP    FOR  S=.1
            100    FORMAT(3F15.5)
                   C=0.01
            101    DO 20 K=1,100,1
                   P=K
                   X=0.01*P
                   A=X**3.5-3.04*C*X**0.5-38.8*C
             20    WRITE(6,100)C,X,A
                   C=C+0.01
                   IF(C-0.35)101,102,102
            102    STOP
                   END
```

```
G LEVEL  20                         MAIN                    DATE = 72230             20/03/14

C         RELATION BETWEEN  AREA &PL2 WHEN SA < SY
C         OPTIMUM STRUCTURAL DESIGN  CIRCULAR  TUBE SECTION
  100     FORMAT(5F15.5)
  101     READ(5,100,END=99)S,PL2,X
          AL2=8.0*(X*36000.0)**2.0/(0.4*30000000.0**2.0*3.14)*10000.0
          WRITE(6,100)S,PL2,AL2
          GO TO 101
   99     STOP
          END
```

```
C      OPTIMUM STRUCTURAL DESIGN  CIRCULAR   TUBE SECTION
C      RELATION BETWEEN  AREA &PL2 WHEN SA = SY
  100  FORMAT(5F15.5)
  101  READ(5,100,END=99)S
       DO 20 K=10,200,2
       P=K
       DL=0.001*P
       PL2=36000.0**2.0*3.14*DL**3.0/(0.4*30000000.0)/(4.0*S+DL)
       AL2=3.14*DL**2.0*36000.0/(0.4*30000000.0)*10000.0
   20  WRITE(6,100)S,DL,PL2,AL2
       GO TO 101
   99  STOP
       END
```

```
G LEVEL  20                     MAIN                  DATE = 72234          19/16/35

 C      OPTIMUM STRUCTURAL DESIGN  H-SECTION
 C      C & K1 RELATION FOR PSI-Y=1           P/L**2=C        ,K1=CK1
 C      RELATION BETWEEN  P/L**2 & K1 IN EQUATION (2.13)
  100   FORMAT(5F15.5)
  101   READ(5,100,END=99)S,CKP,CK2
        DO 20 K=10,200,1
        P=K
        CK1=0.01*P
        C=12.0**1.5*36000.0**3.0*(1.0+2.0*CK1**2.0*CK2)**2.0/CKP**0.5/(3.1
       14**2.0*30000000.0**1.5*(1.0+6.0*CK1**2.0*CK2)*12.0**0.5*36000.0**0
       2.5+6.0*S*3.14**3.0*30000000.0**2.0*(1.0+5.0*CK1**2.0*CK2)**0.5*(1.
       30+2.0*CK1**2.0*CK2)**0.5)
   20   WRITE(6,100)S,CK1,CK2,C
        GO TO 101
   99   STOP
        END
```

```
G LEVEL  20                    MAIN                  DATE = 72234              19/25/35

C      OPTIMUM STRUCTURAL DESIGN  H-SECTION
C      C & K1 RELATION FOR ALL PSI=1          P/L**2=C         K1=CK1
C      RELATION BETWEEN P/L**2 & K1 IN EQUATION (2.17)
 100   FORMAT(5F15.5)
 101   READ(5,100,END=99)S,CKP,CK2
       DO 20 K=10,200,1
       P=K
       CK1=0.01*P
       C=12.0**2.0*36000.0**2.5*(1.0+2.0*CK1**2.0*CK2)**2.0*CK1**4.0*CK2/
      1(CKP**0.5*3.14**2.0*30000000.0**1.5*(1.0+6.0*CK1**2.0*CK2)**2.0)
      2/6.0
 20    WRITE(6,100)S,CK1,CK2,C
       GO TO 101
 99    STOP
       END
```

```
C     OPTIMUM STRUCTURAL DESIGN H-SECTION
C     RELATION BETWEEN PL2 & K1 WHEN  FC3 IS PASSIVE CONSTRAINT
 100  FORMAT(5F15.5)
 101  READ(5,100,END=99)S,CKP,CK2
      DO 20 K=10,200,2
      D=K
      CK1=0.01*D
      A=3.14*CK1**2.0*CK2**0.5*30000000.0**0.75/(1.0+2.0*CK1**2.0*CK2)/
     16.0**0.5
      B=6.0*S*3.14**1.5*CK1**3.0*CK2**0.75*30000000.0**0.875/(1.0+6.0*
     1CK1**2.0*CK2)/(6.0**0.75*CKP**0.125)*36000.0**0.125
      C=36000.0**1.25/CKP**0.25
      X1=((B**2.0+4.0*A*C)**0.5-B)/2.0/A
      PL2=X1**4.0
 20   WRITE(6,100)S,CK1,X1,PL2
      GO TO 101
 99   STOP
      END
```

```
G LEVEL  20                    MAIN                 DATE = 72234             10/16/57

 C        OPTIMUM STRUCTURAL DESIGN  H-SECTION
 C        SA & K1 RELATION     K1=CK1               SA=DESIGN STRESS
  100     FORMAT(5F15.5)
  101     READ(5,100,END=99)S,CKP,CK2
          DO 20 K=5,128,1
          P=K
          CK1=0.01*P
          SA=(12.0**0.5*CK1**4.0*CK2*1.0*S*3.14*30000000.0**0.5*(1.0+2.0*CK1
         1**2.0*CK2)**0.5/((1.0+6.0*CK1**2.0*CK2)**1.5-02.0*CK1**4.0*CK2*(1.
         20+6.0*CK1**2.0*CK2)**0.5))**2.0
   20     WRITE(6,100)S,CK1,CK2,SA
          GO TO 101
   99     STOP
          END
```

```
LEVEL  20                    MAIN                    DATE = 72234                19/19/15

C      OPTIMUM STRUCTURAL DESIGN  H-SECTION
C      RELATION BETWEEN  P/L**2 & SA & K1
 100   FORMAT(5F15.5)
 101   READ(5,100,END=99)S,CKP,CK2
       DO 20 K=5,128,1
       P=K
       CK1=0.01*P
       SA=(12.0**0.5*CK1**4.0*CK2*1.0*S*3.14*30000000.0**0.5*(1.0+2.0*CK1
      1**2.0*CK2)**0.5/((1.0+6.0*CK1**2.0*CK2)**1.5-02.0*CK1**4.0*CK2*(1.
      20+6.0*CK1**2.0*CK2)**0.5))**2.0
       C=12.0**1.5*SA**3.0*(1.0+2.0*CK1**2.0*CK2)**2.0/CKP**0.5/(3.14**2.
      10*30000000.0**1.5*(1.0+6.0*CK1**2.0*CK2)*12.0**0.5*SA**0.5+6.0*S*
      23.14**3.0*30000000.0**2.0*(1.0+6.0*CK1**2.0*CK2)**0.5*(1.0+2.0*CK1
      3**2.0*CK2)**0.5)
 20    WRITE(6,100)S,CK1,SA,C
       GO TO 101
 99    STOP
       END
```

```
C       OPTIMUM STRUCTURAL DESIGN  H-SECTION
C       RELATION BETWEEN  AREA & K1 WHEN SA < SY
 100    FORMAT(5F15.5)
 101    READ(5,100,END=99)S,CKP,CK2,CK1,SA
        AL2=12.0*SA**1.5*(1.0+2.0*CK1**2.0*CK2)**2.0/(CKP**0.5*3.14**2.0*3
      10000000.0**1.5)/(1.0+6.0*CK1**2.0*CK2)*10000.0
        WRITE(6,100)S,CK1,SA,AL2
        GO TO 101
 99     STOP
        END
```

```
C        OPTIMUM STRUCTURAL DESIGN H-SECTION
C        RELATION BETWEEN PL2 & K1 WHEN  FC2 IS PASSIVE CONSTRAINT
C        RELATION BETWEEN  AREA &PL2 WHEN SA = SY
 100     FORMAT(5F15.5)
 101     READ(5,100,END=99)S,CKP,CK2
         DO 20 K=10,200,2
         D=K
         CK1=0.01*D
         A=3.14*CK1**2.0*CK2**0.5*30000000.0**0.75/(1.0+2.0*CK1**2.0*CK2)/
        16.0**0.5
         B=6.0*S*3.14**1.5*CK1**3.0*CK2**0.75*30000000.0**0.875/(1.0+6.0*
        1CK1**2.0*CK2)/(6.0**0.75*CKP**0.125)*36000.0**0.125
         C=36000.0**1.25/CKP**0.25
         X1=((B**2.0+4.0*A*C)**0.5-B)/2.0/A
         PL2=X1**4.0
         AL2=(6.0*PL2)**0.5*(36000.0/CKP)**0.25*(1.0+2.0*CK1**2.0*CK2)/(3.1
        14*30000000.0**0.75*CK1**2.0*CK2**0.5)*10000.0
 20      WRITE(6,100)S,CK1,X1,PL2,AL2
         GO TO 101
 99      STOP
         END
```

```
EL   20                      MAIN                     DATE = 72234              19/21/35

    OPTIMUM STRUCTURAL DESIGN RECTANGULAR TUBE SECTION
    C & K1 RELATION FOR PSI-Y=1              P/L**2=C          K1=CK1
    RELATION BETWEEN  P/L**2 & K1 IN EQUATION (3.14)
    FORMAT(5F15.5)
1   READ(5,100,END=99)S,CKP,CK2
    DO 20 K=10,200,1
    P=K
    CK1=0.01*P
    C=12.0**1.5*36000.0**3.0*(1.0+CK1**2.0*CK2)**2.0/CKP**0.5/(3.0*S*3
   1.14**3.0*30000000.0**2.0*(1.0+3.0*CK1**2.0*CK2)**0.5*(1.0+CK1**2.0
   2*CK2)**.5+3.14**2.0*30000000.0**1.5*(1.0+3.0*CK1**2.0*CK2)*36000.0
   3**0.5*3.0**0.5)
    WRITE(6,100)S,CK1,CK2,C
    GO TO 101
    STOP
    END
```

```
LEVEL  20                    MAIN                    DATE = 72234              19/44/22

C      OPTIMUM STRUCTURAL DESIGN RECTANGULAR TUBE SECTION
C      RELATION BETWEEN C,& CK1 FOR PSIEY=1    C=P/L**2    CK1=K1
C      RELATION BETWEEN P/L**2 & K1 IN EQUATION (3.17)
 100   FORMAT(5E15.5)
 101   READ(5,100,END=99)S,CKP,CK2
       DO 20 K=10,200,1
       P=K
       CK1=0.01*P
       C=24.0*36000.0**2.5*(CK1**4.0*CK2+3.0*CK1**2.0)*(1.0+CK1**2.0*CK2)
      1**2.0/(CKP**0.5*3.14**2.0*30000000.0**1.5*(1.0+3.0*CK1**2.0*CK2)**
      22.0)
 20    WRITE(6,100)S,CK1,CK2,C
       GO TO 101
 99    STOP
       END
```

```
C     OPTIMUM STRUCTURAL DESIGN RECTANGULAR TUBE SECTION
C     RELATION BETWEEN PL2 & K1 WHEN  FC3 IS PASSIVE CONSTRAINT
100   FORMAT(5F15.5)
101   READ(5,100,END=99)S,CKP,CK2
      DO 20 K=10,200,2
      D=K
      CK1=0.01*D
      A=3.14*CKP**0.125*30000000.0**0.75*(CK1**4.0*CK2+3.0*CK1**2.0)**0.
     15/(2.0*6.0**0.5*36000.0**0.25)/(1.0+CK1**2.0*CK2)
      B=3.14**1.5*3.0*S*30000000.0**0.875*(CK1**4.0*CK2+3.0*CK1**2.0)**
     10.75/(6.0**0.75*36000.0**0.125)/(1.0+3.0*CK1**2.0*CK2)
      C=36000.0/CKP**0.125
      X1=((B**2.0+4.0*A*C)**0.5-B)/2.0/A
      PL2=X1**4.0
20    WRITE(6,100)S,CK1,X1,PL2
      GO TO 101
99    STOP
      END
```

```
VEL  20                    MAIN                   DATE = 72234           19/56/18

      OPTIMUT STRUCTURAL DESIGN RECTANGULAR TUBE SECTION
      SA & K1 RELATION     K1=CK1             SA=DESIGN STRESS
00    FORMAT(5F15.5)
01    READ(5,100,END=99)S,CKP,CK2
      DO 20 K=5,99,1
      P=K
      CK1=0.01*P
      SA=3.0*S**2.0*3.14**2.0*30000000.0*(CK1**4.0*CK2+3.0*CK1**2.0)**2.
     10*(1.0+CK1**2.0*CK2)/(1.0+3.0*CK1**2.0*CK2)/(1.0+3.0*CK1**2.0*CK2-
     2CK1**4.0*CK2-3.0*CK1**2.0)**2.0
0     WRITE(6,100)S,CK1,CK2,SA
      GO TO 101
9     STOP
      END
```

```
EVEL  20                        MAIN                  DATE = 72234              19/58/20

      OPTIMUM STRUCTURAL DESIGN RECTANGULAR TUBE SECTION
      RELATION BETWEEN  P/L**2 & SA & K1
100   FORMAT(5F19.5)
101   READ(5,100,END=99)S,CKP,CK2
      DO 20 K=5,49,1
      P=K
      CK1=0.01*P
      SA=3.0*S**2.0*3.14**2.0*30000000.0*(CK1**4.0*CK2+3.0*CK1**2.0)**2.
     10*(1.0+CK1**2.0*CK2)/(1.0+3.0*CK1**2.0*CK2)/(1.0+3.0*CK1**2.0*CK2-
     2CK1**4.0*CK2-3.0*CK1**2.0)**2.0
      C=24.0*SA**3.0*(1.0+CK1**2.0*CK2)**2.0/CKP**0.5/(3.0**0.5*S*3.14**
     13.0*30000000.0**2.0*(1.0+3.0*CK1**2.0*CK2)**0.5*(1.0+CK1**2.0*CK2)
     2**0.5+3.14**2.0*30000000.0**1.5*(1.0+3.0*CK1**2.0*CK2)*SA**0.5)
20    WRITE(6,100)S,CK1,SA,C
      GO TO 101
99    STOP
      END
```

```
  C      OPTIMUM STRUCTURAL DESIGN RECTANGULAR TUBE SECTION
  C      RELATION BETWEEN  AREA & K1 WHEN SA < SY
  100    FORMAT(5F15.5)
  101    READ(5,100,END=99)S,CKP,CK2,CK1,SA
         AL2=24.0*(SA/30000000.0)**1.5*(1.0+CK1**2.0*CK2)**2.0/(CKP**0.5*3.
        114**2.0)/(1.0+3.0*CK1**2.0*CK2)*10000.0
         WRITE(6,100)S,CK1,SA,AL2
         GO TO 101
  99     STOP
         END
```

```
EVEL  20                    MAIN                    DATE = 72234             17/59/46

     OPTIMUM STRUCTURAL DESIGN RECTANGULAR TUBE SECTION
     RELATION BETWEEN PL2 & K1 WHEN  FC3 IS PASSIVE CONSTRAINT
     RELATION BETWEEN  AREA &PL2 WHEN SA = SY
100  FORMAT(5F15.5)
101  READ(5,100,END=99)S,CKP,CK2
     DO 20 K=10,200,2
     D=K
     CK1=0.005*D
     A=3.14*CKP**0.125*30000000.0**0.75*(CK1**4.0*CK2+3.0*CK1**2.0)**0.
    15/(2.0*6.0**0.5*36000.0**0.25)/(1.0+CK1**2.0*CK2)
     B=3.14**1.5*3.0*S*30000000.0**0.875*(CK1**4.0*CK2+3.0*CK1**2.0)**
    10.75/(6.0**0.75*36000.0**0.125)/(1.0+3.0*CK1**2.0*CK2)
     C=36000.0/CKP**0.125
     X1=((B**2.0+4.0*A*C)**0.5-B)/2.0/A
     PL2=X1**4.0
     AL2=(6.0*PL2)**0.5*(36000.0/CKP)**0.25*(1.0+CK1**2.0*CK2)/3.14/300
    100000.0**0.75/(CK1**4.0*CK2+3.0*CK1**2.0)**0.5*2.0*10000.0
20   WRITE(6,100)S,CK1,X1,PL2,AL2
     GO TO 101
99   STOP
     END
```

```
C       OPTIMUM STRUCTURAL DESIGN  H-SECTION
C       RELATION BETWEEN  AL2 & PL2  WHEN FC3 IS ACTIVE CONSTRAINT
  100   FORMAT(5F15.5)
  101   READ(5,100,END=99)S,CKP,CK2
        DO 20 K=10,200,1
        P=K
        CK1=0.01*P
        C=12.0**1.5*36000.0**3.0*(1.0+2.0*CK1**2.0*CK2)**2.0/CKP**0.5/(3.1
       14**2.0*30000000.0**1.5*(1.0+6.0*CK1**2.0*CK2)*12.0**0.5*36000.0**0
       2.5+6.0*S*3.14**3.0*30000000.0**2.0*(1.0+6.0*CK1**2.0*CK2)**0.5*(1.
       30+2.0*CK1**2.0*CK2)**0.5)
        SA=36000.0
        AL2=12.0*SA**1.5*(1.0+2.0*CK1**2.0*CK2)**2.0/(CKP**0.5*3.14**2.0*3
       10000000.0**1.5)/(1.0+6.0*CK1**2.0*CK2)*10000.0
   20   WRITE(6,100)S,CK1,CK2,C,AL2
        GO TO 101
   99   STOP
        END
```

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