THE STATIC BUCKLING OF CONTINUOUS STRUCTURAL SYSTEMS

by

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ABSTRACT

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The purpose of this thesis is to formulate a matrix-type solution to determine the critical buckling loads of continuous columns and simple, planar, orthogonal, portal frames. Both the stiffness method and flexbility method are utilized and the efficiency of each is investigated.

A variety of boundary conditions are employed including simple supports, fixed supports, and partially restrained supports.

The modal vectors of deformation associated with each critical buckling load are determined. These modal vectors are combined into a general modal matrix for which the orthogonality conditions are formulated.

The dynamic stability approach to the problem is derived for the purpose of future consideration.

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Great appreciation is given to my dear parents for supporting me during my studies.

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LIST OF NOTATIONS

SYMBOL	DEFINITION
a	$= \frac{4\psi-\phi}{6}$
E	Young 's modulus of elasticity
$[F]$	Flexibility matrix
I	Moment of inertia
$[K]$	Stiffness matrix
k	$= \left(\frac{P}{EI}\right)^{\frac{1}{2}}$
kt	Torsional spring
L	Length of member
M	Bending moment
P	Load
$[U]$	Modal matrix
u	$= \frac{kL}{2}$
Δ	Sidesway displacement
θ	Joint rotation
ϕ	$= \frac{3}{2u}\left(\frac{1}{2u} - \frac{1}{\tan 2u}\right)$
ψ	$= \frac{3}{u}\left(\frac{1}{\sin 2u} - \frac{1}{2u}\right)$
$[\Lambda]$	Diagonal matrix
∞	Infinity
ρ	Mass density per unit volume
Ω	Natural frequency of free vibration of the beam-column

SUBSCRIPTS

b	Beam
cr	Critical buckling load
f	Flexibility matrix
k	Stiffness matrix
m	Member
r	Reduced form
s	Continuous structural systems
t	Transcendental functions

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CHAPTER I

INTRODUCTION

The elastic stability problem of continuous columns and simple, planar, orthogonal, portal frames have been investigated in recent year using many different methods.

Timoshenko[3] determined the critical buckling loads of continuous structural systems for various support conditions using classical scalar methods.

Galambos[1] determined the critical buckling loads of continuous portal frames with simple supports and fixed supports by the slope-deflection method.

Gregory[2] determined the critical buckling loads of continuous structural systems utilizing the matrix stiffness method for the special cases of simple supports and fixed supports.

The purpose of this thesis is to determine the critical buckling loads of continuous columns and simple, planar, orthogonal, portal frames by both the matrix stiffness method and the matrix flexibility method. In addition, the modal matrix is determined for each method and the resulting orthogonality conditions are considered. Partially restrained supports are included, which mathematically are easily converted to either a fixed support or a simple support.

1.1 Derivation of Basic Flexibility and Stiffness Matrices

The basic differential equation of a column subjected to both bending stress and axial compressive force P is given by Timoshenko[3] as

$$y''''(x) + \frac{P}{EI}y''(x) = 0 \tag{1}$$

where E, I, and P are assumed constant.

The above equation is based upon the following five assumptions;

1. The undeformed member is initially straight.
2. The column is made of perfectly elastic material.
3. The slope of the deformed member is very small compared to unity.
4. The axial loads are applied along the centroidal axis of the column.
5. The effect of shear stress is neglected.

Defining $\frac{P}{EI} = k^2$, the general solution of equation (1) is given as

$$y(x) = A\cos kx + B\sin kx + Cx + D, \tag{2}$$

where the constants A, B, C, and D are determined directly from the boundary conditions of the column.

The sign convention used throughout this work is that given by Timoshenko (3) and is illustrated in Fig. I

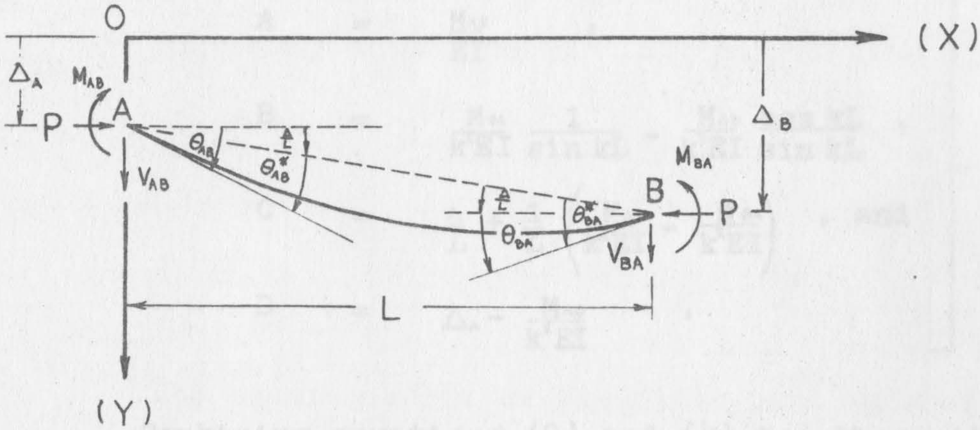


Fig. I Sign Convention for the Column Including Sidesway

The column must satisfy the following six boundary conditions:

$$\left.\begin{aligned}
y(0) &= +\Delta_A \,, \\
y(L) &= +\Delta_B \,, \\
EIy''(0) &= -M_{AB} \,, \\
EIy''(L) &= -M_{BA} \,, \\
y'(0) &= +\theta_{AB} = \theta_{AB}^* + \frac{\Delta}{L}, \text{ and} \\
y'(L) &= -\theta_{BA} = -(\theta_{BA}^* - \frac{\Delta}{L}) = \frac{\Delta}{L} - \theta_{BA}^* ,
\end{aligned}\right\} \quad (3)$$

where $\Delta = \Delta_B - \Delta_A$.

The constants A, B, C and D of equation (2) determined by using the first boundary conditions, given in equation (3), become

$$\left.\begin{aligned} A &= \frac{M_{AB}}{EI}\,, \\ B &= \frac{M_{BA}}{k^2EI}\,\frac{1}{\sin kL} - \frac{M_{AB}}{k^2EI}\,\frac{\cos kL}{\sin kL}\,, \\ C &= \frac{\Delta}{L} + \frac{1}{L}\left(\frac{M_{AB}}{k^2EI} - \frac{M_{BA}}{k^2EI}\right), \text{ and} \\ D &= \Delta_A - \frac{M_{AB}}{k^2EI}\,. \end{aligned}\right\} \qquad (4)$$

Combining equations (2) and (4) together with the last two boundary conditions given in equation (3), it follows that,

$$\theta^*_{AB} = \frac{M_{AB}}{k\,EI}\left(\frac{1}{kL} - \frac{1}{\tan kL}\right) + \frac{M_{BA}}{k\,EI}\left(\frac{1}{\sin kL} - \frac{1}{kL}\right) \qquad (5)$$

and

$$\theta^*_{BA} = \frac{M_{AB}}{k\,EI}\left(\frac{1}{\sin kL} - \frac{1}{kL}\right) + \frac{M_{BA}}{k\,EI}\left(\frac{1}{kL} - \frac{1}{\tan kL}\right). \qquad (6)$$

For convenience the following definitions are introduced:

$$\left.\begin{aligned} 2u &= kL\,, \\ \psi(u) &= \frac{3}{2u}\left(\frac{1}{2u} - \frac{1}{\tan 2u}\right), \\ \text{and}\quad \phi(u) &= \frac{3}{u}\left(\frac{1}{\sin 2u} - \frac{1}{2u}\right). \end{aligned}\right\} \qquad (7)$$

It follows that equations (5) and (6) reduce to the scalar form

$$\theta^*_{AB} = \frac{M_{AB}L}{3EI}\psi(u) + \frac{M_{BA}L}{6EI}\phi(u) \ ,$$

and
$$\theta^*_{BA} = \frac{M_{AB}L}{6EI}\phi(u) + \frac{M_{BA}L}{3EI}\psi(u) \ . \tag{8}$$

Arranging equations (8) into matrix form yields

$$\begin{bmatrix} \theta^*_{AB} \\ \theta^*_{BA} \end{bmatrix} = \begin{bmatrix} \frac{\psi(u)}{3}\frac{L}{EI} & \frac{\phi(u)}{6}\frac{L}{EI} \\ \frac{\phi(u)}{6}\frac{L}{EI} & \frac{\psi(u)}{3}\frac{L}{EI} \end{bmatrix} \begin{bmatrix} M_{AB} \\ M_{BA} \end{bmatrix} . \tag{9}$$

The square matrix on the right hand side of equation (9) is the flexibility matrix for a single member. Equation (9) is rewritten in the following symbolic matrix form

$$\left\{\theta^*\right\} = \left[F_m\right]\left\{m\right\} . \tag{10}$$

Premultiplying equation (10) by $\left[F_m\right]^{-1}$, it follows that

$$\left\{m\right\} = \left[F_m\right]^{-1}\left\{\theta^*\right\} \equiv \left[K_m\right]\left\{\theta^*\right\} . \tag{11}$$

The matrix $\left[K_m\right]$ is defined as the stiffness matrix for a single member.

Noting equations (9) and (11), one obtains

$$\left[K_m\right] \equiv \left[F_m\right]^{-1} = \begin{bmatrix} \frac{2\psi}{a}\frac{EI}{L} & -\frac{\phi}{a}\frac{EI}{L} \\ -\frac{\phi}{a}\frac{EI}{L} & \frac{2\psi}{a}\frac{EI}{L} \end{bmatrix} , \tag{12}$$

where $\quad a = \dfrac{4\psi^2 - \phi^2}{6} \neq 0 .$

Equation (11) is rewritten in component matrix form as

$$\begin{bmatrix} M_{AB} \\ \\ M_{BA} \end{bmatrix} = \begin{bmatrix} \dfrac{2\psi}{a}\dfrac{EI}{L} & -\dfrac{\phi}{a}\dfrac{EI}{L} \\ \\ -\dfrac{\phi}{a}\dfrac{EI}{L} & \dfrac{2\psi}{a}\dfrac{EI}{L} \end{bmatrix} \begin{bmatrix} \theta^*_{AB} \\ \\ \theta^*_{BA} \end{bmatrix} . \qquad (13)$$

The stiffness and flexibility matrices are reduced to the special case of a beam with zero axial load by noting the conditions that

$$\lim_{2u \to 0} \psi(u) = 1 , \text{ and}$$

$$\lim_{2u \to 0} \phi(u) = 1 .$$

The resulting stiffness and flexibility matrices become

$$\left[F_{mb}\right] = \begin{bmatrix} \dfrac{L}{3EI} & \dfrac{L}{6EI} \\ \\ \dfrac{L}{6EI} & \dfrac{L}{3EI} \end{bmatrix} \qquad (14)$$

and

$$\left[K_{mb}\right] = \begin{bmatrix} \dfrac{4EI}{L} & -2\dfrac{EI}{L} \\ \\ -2\dfrac{EI}{L} & \dfrac{4EI}{L} \end{bmatrix} .$$

1.2 General Stiffness and Flexibility Matrices for the System

In the case of continuous structural system, the stiffness matrix for each member will be $[K_{m_1}]$, $[K_{m_2}]$, $[K_{m_3}]$ .... $[K_{mn}]$. Combining the stiffness matrices for the entire system, it follows that

$$[K_s] = \begin{bmatrix} [K_{m_1}] & 0 & 0 & . & . & 0 \\ 0 & [K_{m_2}] & 0 & . & . & 0 \\ 0 & 0 & [K_{m_3}] & . & . & 0 \\ . & . & & . & & . \\ . & . & & & & . \\ 0 & 0 & 0 & . & . & [K_{mn}] \end{bmatrix}, \qquad (15)$$

where $[K_s]$ is a $(2n \times 2n)$ banded diagonal matrix with n equal to the number of members in the system.

If follows from equation (11) that

$$\{m_s\} = [K_s]\{\theta_s^*\}, \qquad (16)$$

where $\{m_s\}$ and $\{\theta_s^*\}$ are of size $(2n \times 1)$.

Similarly, the flexibility matrix for continuous structural systems is

$$\left[F_s\right] = \begin{bmatrix} \left[F_{m_1}\right] & 0 & 0 & \cdot & \cdot & 0 \\ 0 & \left[F_{m_2}\right] & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \left[F_{m_3}\right] & \cdot & \cdot & 0 \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & & & & & \\ 0 & 0 & 0 & \cdot & \cdot & \left[F_{mn}\right] \end{bmatrix}, \tag{17}$$

with equation (10) yielding

$$\left\{\theta_s^*\right\} = \left[F_s\right]\left\{m_s\right\} \quad . \tag{18}$$

After applying the boundary conditions for moments and rotations at the ends of each member, and combining the resulting equations. The form of the stiffness matrix $\left[K_s\right]$ reduces to $\left[K_{sr}\right]$, and $\left[F_s\right]$ reduces to the form $\left[F_{sr}\right]$. Then from equations (16) and (18), one obtains

$$\left.\begin{aligned} \left[K_{sr}\right]\left\{\theta_{sr}^*\right\} &= \left\{0\right\} \text{ and} \\ \left[F_{sr}\right]\left\{m_{sr}\right\} &= \left\{0\right\} \quad . \end{aligned}\right\} \tag{19}$$

For non-trivial solutions of $\left\{\theta_{sr}^*\right\}$ and $\left\{m_{sr}\right\}$, the determinant of $\left[K_{sr}\right]$ and $\left[F_{sr}\right]$ must be zero. The determinant yields transcendental equations, the roots of which produce the critical buckling loads for the system. The buckled mode shapes (i.e. the relative end rotations) and the ratio of bending moments at the ends of members are found for each critical buckling load utilizing equation (19).

1.3 Modal Vector and Orthogonality Conditions

The modal vectors of deformation associated with each critical buckling load are formulated. These modal vectors are combined into a general modal matrix for which the orthogonality conditions are determined.

The scalar components of the modal vectors for the stiffness method are the ratio of end rotations, while the components of the modal vectors for the flexibility method are the ratio of end moments of the members.

Defining the modal matrix associated with the ratios of joint rotations as $\left[U_k\right]$, and the modal matrix associated with the ratios of joint moments as $\left[U_f\right]$, one defines the following orthogonality conditions:

$$\left.\begin{aligned} \left[U_k\right]^T \left[K_{sr}\right] \left[U_k\right] &= \left[K_{sr}^*\right] \text{, and} \\ \left[U_f\right]^T \left[F_{sr}\right] \left[U_f\right] &= \left[F_{sr}^*\right] \text{,} \end{aligned}\right\} \quad (20)$$

where $\left[K_{sr}^*\right]$ and $\left[F_{sr}^*\right]$ are symmetric matrices with components which define the individual transcendental functions that yield the critical buckling loads when equated to zero.

If the determinant of either the stiffness or flexibility matrix is determined as a product of a set of functions in the form

$$\det \left[K_{sr}\right] = \alpha_1(u)\ \alpha_2(u) \ldots\ldots\ \alpha_n(u)\ ,$$

$$\text{or} \quad \det \left[F_{sr}\right] = \beta_1(u)\ \beta_2(u) \ldots\ldots\ \beta_m(u)\ ,$$

it follows that, the matrices $\left[K^*_{sr}\right]$ and $\left[F^*_{sr}\right]$ reduce to a diagonal form given as

$$\left.\begin{array}{l} \left[K^*_{sr}\right] = \left[\Lambda_{k_t}\right] = \begin{bmatrix} \alpha_1(u) & 0 & \cdot\cdot & 0 \\ 0 & \alpha_2(u) & \cdot\cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot\cdot & \alpha_n(u) \end{bmatrix} \\ \text{or} \quad \left[F^*_{sr}\right] = \left[\Lambda_{f_t}\right] = \begin{bmatrix} \beta_1(u) & 0 & \cdot\cdot & 0 \\ 0 & \beta_2(u) & \cdot\cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot\cdot & \beta_m(u) \end{bmatrix} \end{array}\right\} \quad (21)$$

Equation (20) transforms the matrices $\left[K_{sr}\right]$ and $\left[F_{sr}\right]$ into a diagonal matrices which are produced in canonical function form as illustrated in equation (21) .

If the determinant of either $\left[K_{sr}\right]$ or $\left[F_{sr}\right]$ cannot be reduced to the product of functions as given above, the matrices $\left[K^*_{sr}\right]$ and $\left[F^*_{sr}\right]$ will not be diagonal as shown in equation (21). However, the matrices $\left[K^*_{sr}\right]$ and $\left[F^*_{sr}\right]$, when evaluated at the individual values of critical buckling, transform to diagonal matrices.

For convenience, the diagonal matrix $\left[\Lambda_{cr}\right]$ is defined, where contains the critical buckling loads defined by equations (21) and takes the form

$$\left[\Lambda_{cr}\right] = \begin{bmatrix} P_{1cr} & 0 & \cdot & \cdot & 0 \\ 0 & P_{2cr} & \cdot & \cdot & 0 \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & P_{jcr} \end{bmatrix} . \qquad (22)$$

The values of P_{jcr} , $j = 1, 2, \ldots\ldots n$, appear in order of increasing magnitude.

CHAPTER II

SOLUTIONS OF CONTINUOUS COLUMN PROBLEM

2.1 Simply-Supported Four-Span Column

Consider the column ABCDE subjected to the axial compressive loads P at both ends as shown in Fig. 2 below.

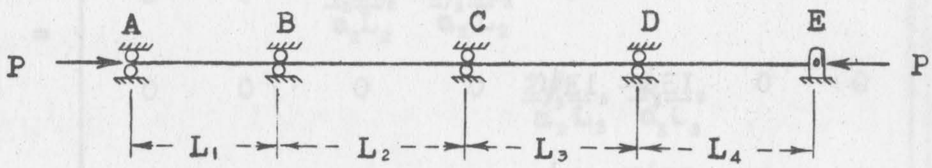


Fig. 2 Simply-Supported Four-Span Column

The eight boundary conditions are

$$\left.\begin{aligned}
M_{AB} &= 0 , \\
M_{BA} &= M_{BC} = M_B , \\
M_{CB} &= M_{CD} = M_C , \\
M_{DC} &= M_{DE} = M_D , \\
M_{ED} &= 0 , \\
\theta_{BA} &= -\theta_{BC} = \theta_B , \\
\theta_{CB} &= -\theta_{CD} = \theta_C , \text{ and} \\
\theta_{DC} &= -\theta_{DE} = \theta_D .
\end{aligned}\right] \quad (23)$$

2.1A Stiffness Method

The stiffness matix for the system is constructed as follow;

$$\begin{bmatrix} M_{AB} \\ M_{BA} \\ M_{BC} \\ M_{CB} \\ M_{CD} \\ M_{DC} \\ M_{DE} \\ M_{ED} \end{bmatrix} = \begin{bmatrix} \frac{2\psi_1 EI_1}{a_1 L_1} & -\frac{\phi_1 EI_1}{a_1 L_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\phi_1 EI_1}{a_1 L_1} & \frac{2\psi_1 EI_1}{a_1 L_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2\psi_2 EI_2}{a_2 L_2} & -\frac{\phi_2 EI_2}{a_2 L_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\phi_2 EI_2}{a_2 L_2} & \frac{2\psi_2 EI_2}{a_2 L_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2\psi_3 EI_3}{a_3 L_3} & -\frac{\phi_3 EI_3}{a_3 L_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\phi_3 EI_3}{a_3 L_3} & \frac{2\psi_3 EI_3}{a_3 L_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2\psi_4 EI_4}{a_4 L_4} & -\frac{\phi_4 EI_4}{a_4 L_4} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\phi_4 EI_4}{a_4 L_4} & \frac{2\psi_4 EI_4}{a_4 L_4} \end{bmatrix} \begin{bmatrix} \theta_{AB} \\ \theta_{BA} \\ \theta_{BC} \\ \theta_{CB} \\ \theta_{CD} \\ \theta_{DC} \\ \theta_{DE} \\ \theta_{ED} \end{bmatrix}. \qquad (24)$$

Applying the boundary conditions given above, one obtains the reduced form of equation (24) as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2\psi_1 EI_1}{a_1 L_1} & -\frac{\phi_1 EI_1}{a_1 L_1} & 0 & 0 & 0 \\ -\frac{\phi_1 EI_1}{a_1 L_1} & \frac{2\psi_1 EI_1}{a_1 L_1}+\frac{2\psi_2 EI_2}{a_2 L_2} & \frac{\phi_2 EI_2}{a_2 L_2} & 0 & 0 \\ 0 & \frac{\phi_2 EI_2}{a_2 L_2} & \frac{2\psi_2 EI_2}{a_2 L_2}+\frac{2\psi_3 EI_3}{a_3 L_3} & \frac{\phi_3 EI_3}{a_3 L_3} & 0 \\ 0 & 0 & \frac{\phi_3 EI_3}{a_3 L_3} & \frac{2\psi_3 EI_3}{a_3 L_3}+\frac{2\psi_4 EI_4}{a_4 L_4} & \frac{\phi_4 EI_4}{a_4 L_4} \\ 0 & 0 & 0 & \frac{\phi_4 EI_4}{a_4 L_4} & \frac{2\psi_4 EI_4}{a_4 L_4} \end{bmatrix} \begin{bmatrix} \theta_{AB} \\ \theta_{B} \\ \theta_{C} \\ \theta_{D} \\ \theta_{ED} \end{bmatrix}. \qquad (25)$$

The determinant of the reduced stiffness matrix yields

$$24\frac{I_1}{a_1L_1}\frac{I_4}{a_4L_4}E^5\Big[\Big(\psi_2\frac{I_1}{L_1}+\psi_1\frac{I_2}{L_2}\Big)\Big(\frac{2\psi_3}{a_2}\;\frac{\psi_4}{a_3}\frac{I_2}{L_2}\frac{I_3}{L_3}+\frac{3}{a_2}\frac{I_2}{L_2}\frac{I_4}{L_4}\Big)+$$

$$\Big(\psi_4\frac{I_3}{L_3}+\psi_3\frac{I_4}{L_4}\Big)\Big(\frac{2\psi_1}{a_2}\;\frac{\psi_2}{a_3}\frac{I_2}{L_2}\frac{I_3}{L_3}+\frac{3}{a_3}\frac{I_1}{L_1}\frac{I_3}{L_3}\Big)\Big] = 0 \quad . \qquad (26)$$

For the special case where $L_1 = L_2 = L_3 = L_4 = L$ and $I_1 = I_2 = I_3 = I_4 = I$, equation (26) becomes

$$8\times 6^5\Big(\frac{EI}{L}\Big)^5\Big[-\frac{\psi(2\sqrt{2}\,\psi-\phi)}{(2\psi+\phi)^4}\,\frac{(2\sqrt{2}\,\psi+\phi)}{(2\psi-\phi)^4}\Big] = 0 \quad . \qquad (27)$$

Noting equation (27), the following **five transcendental equations hold:**

$$\frac{1}{2\psi+\phi} = 2\sqrt{2}\,\psi-\phi = \psi = 2\sqrt{2}\,\psi+\phi = \frac{1}{2\psi-\phi} = 0 \quad . \qquad (28)$$

Applying each function of equation (28) to equation (25), the following modal matrix $\left[U_k\right]$ is obtained:

$$\left[U_k\right] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & -1 \\ -1 & 0 & 1 & 0 & -1 \\ 1 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & -1 \\ -1 & 1 & -1 & 1 & -1 \end{bmatrix} . \qquad (29)$$

Applying each function of equation (28) to equation (24), and noting the associated equation (29), the following modal matrix is obtained:

$$\left[U_f\right] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{30}$$

The ratio of bending moments of the first and the last transcendental functions are zero.

The orthogonality conditions of the mode shapes defined for the stiffness method become

$$\left[U_k\right]^T \left[K_{sr}\right]\left[U_k\right]$$

$$= \left[\Lambda_{k_t}\right]$$

$$= \frac{EI}{L} \begin{bmatrix} 48\left(\frac{1}{2\psi+\phi}\right) & 0 & 0 & 0 & 0 \\ 0 & \frac{2\sqrt{2}}{a}(2\sqrt{2}\,\psi-\phi) & 0 & 0 & 0 \\ 0 & 0 & \frac{8}{a}\psi & 0 & 0 \\ 0 & 0 & 0 & \frac{2\sqrt{2}}{a}(2\sqrt{2}\,\psi+\phi) & 0 \\ 0 & 0 & 0 & 0 & 48\left(\frac{1}{2\psi-\phi}\right) \end{bmatrix}. \tag{31}$$

The diagonal terms of the $\left[\Lambda_{k_t}\right]$ matrix define the individual transcendental functions which yield the following critical buckling loads

$$
\left[\Lambda_{cr}\right] = \begin{bmatrix} \frac{(n\pi)^2 EI}{L^2}, n=1,3,5.. & 0 & 0 & 0 & 0 \\ 0 & \frac{12.816EI}{L^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{20.187EI}{L^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{29.703EI}{L^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{(n\pi)^2 EI}{L^2}, n=2,4,6.. \end{bmatrix}. \qquad (32)
$$

The associated mode shapes are shown in Fig. (3)

2.1B Flexibility Method

Equation (18) takes the form

$$\begin{bmatrix} \theta_{AB} \\ \theta_{BA} \\ \theta_{BC} \\ \theta_{CB} \\ \theta_{CD} \\ \theta_{DC} \\ \theta_{DE} \\ \theta_{ED} \end{bmatrix} = \begin{bmatrix} 2\psi_1\frac{L_1}{6EI_1} & \phi_1\frac{L_1}{6EI_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \phi_1\frac{L_1}{6EI_1} & 2\psi_1\frac{L_1}{6EI_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\psi_2\frac{L_2}{6EI_2} & \phi_2\frac{L_2}{6EI_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi_2\frac{L_2}{6EI_2} & 2\psi_2\frac{L_2}{6EI_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\psi_3\frac{L_3}{6EI_3} & \phi_3\frac{L_3}{6EI_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_3\frac{L_3}{6EI_3} & 2\psi_3\frac{L_3}{6EI_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\psi_4\frac{L_4}{6EI_4} & \phi_4\frac{L_4}{6EI_4} \\ 0 & 0 & 0 & 0 & 0 & 0 & \phi_4\frac{L_4}{6EI_4} & 2\psi_4\frac{L_4}{6EI_4} \end{bmatrix} \begin{bmatrix} M_{AB} \\ M_{BA} \\ M_{BC} \\ M_{CB} \\ M_{CD} \\ M_{DC} \\ M_{DE} \\ M_{ED} \end{bmatrix} . \quad (33)$$

Applying the conditions of equation (23), one obtains

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\psi_1\frac{L_1}{6EI_1}+2\psi_2\frac{L_2}{6EI_2} & \phi_2\frac{L_2}{6EI_2} & 0 \\ \phi_2\frac{L_2}{6EI_2} & 2\psi_2\frac{L_2}{6EI_2}+2\psi_3\frac{L_3}{6EI_3} & \phi_3\frac{L_3}{6EI_3} \\ 0 & \phi_3\frac{L_3}{6EI_3} & 2\psi_3\frac{L_3}{6EI_3}+2\psi_4\frac{L_4}{6EI_4} \end{bmatrix} \begin{bmatrix} M_B \\ M_C \\ M_D \end{bmatrix} . \quad (34)$$

The determinant of the matrix $\left[F_{sr}\right]$ yields

$$2\left(\frac{1}{6E}\right)^3\left[\left\{\psi_1\frac{L_1}{I_1}+\psi_2\frac{L_2}{I_2}\right\}\left\{4\left(\psi_2\frac{L_2}{I_2}+\psi_3\frac{L_3}{I_3}\right)\left(\psi_3\frac{L_3}{I_3}+\psi_4\frac{L_4}{I_4}\right)-\left(\phi_3\frac{L_3}{I_3}\right)^2\right\}-\left\{\psi_3\frac{L_3}{I_3}+\psi_4\frac{L_4}{I_4}\right\}\left(\phi_2\frac{L_2}{I_2}\right)^2\right]$$

$$= 0 \quad . \qquad (35)$$

For the special case where $L_1 = L_2 = L_3 = L_4 = L$ and $I_1 = I_2 = I_3 = I_4 = I$, equation (35) reduces to the form

$$8\left(\frac{L}{6EI}\right)^3\left[\psi(2\sqrt{2}\psi-\phi)(2\sqrt{2}\psi+\phi)\right] = 0 \quad . \qquad (36)$$

Noting equation (36), the following three transcendental equations hold:

$$2\sqrt{2}\psi-\phi = \psi = 2\sqrt{2}\psi+\phi = 0 \quad . \qquad (37)$$

The modal matrix $\left[U_f\right]$ for flexibility becomes

$$\left[U_f\right] = \begin{bmatrix} 1 & 1 & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 1 & -1 & 1 \end{bmatrix} \qquad (38)$$

The modal matrix $[U_k]$ is exactly equal to that given by equation (30), except the first and the fifth columns cannot be obtained.

The orthogonality conditions become

$$[U_f]^T [F_{sr}][U_f] = [\Lambda_{ft}]$$

$$= 4\sqrt{2}\frac{L}{6EI}\begin{bmatrix} 2\sqrt{2}\psi-\phi & 0 & 0 \\ 0 & \sqrt{2}\psi & 0 \\ 0 & 0 & 2\sqrt{2}\psi+\phi \end{bmatrix}. \qquad (39)$$

The associated matrix $[\Lambda_{ft}]$ containing the critical buckling loads takes the form

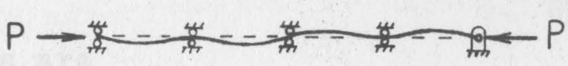
$$[\Lambda_{cr}] = \begin{bmatrix} \frac{12.816EI}{L^2} & 0 & 0 \\ 0 & \frac{20.187EI}{L^2} & 0 \\ 0 & 0 & \frac{29.703EI}{L^2} \end{bmatrix}. \qquad (40)$$

It is seen that one does not obtain the minimum value of $P_{cr} = \frac{(n\pi)^2 EI}{L^2}$, n = 1, 3, 5 ... and the maximum value of $P_{cr} = \frac{(n\pi)^2 EI}{L^2}$, n = 2, 4, 6 ... as obtained using the stiffness method, since the components of the reduced moment vectors $\{m_{sr}\}$ are zero (see equation 34) and as a result equation (35) is not valid. The resulting mode shapes are shown in Fig. (3).

$$P1_{cr} = \frac{(n\pi)^2 EI}{L^2}, \; n = 1,3,5 \, ..$$

$$P2_{cr} = \frac{12.816EI}{L^2}$$

$$P3_{cr} = \frac{20.187EI}{L^2}$$



$$P4_{cr} = \frac{29.703EI}{L^2}$$

$$P5_{cr} = \frac{(n\pi)^2 EI}{L^2}, \; n = 2,4,6..$$

Fig. 3 The Five Possible Mode Shapes of Simply-Supported Four-Span Column.

2.2 Fixed-Supported Four-Span Column

Consider the column ABCDE subjected to the axial compressive loads P at both ends (see Fig. 4).

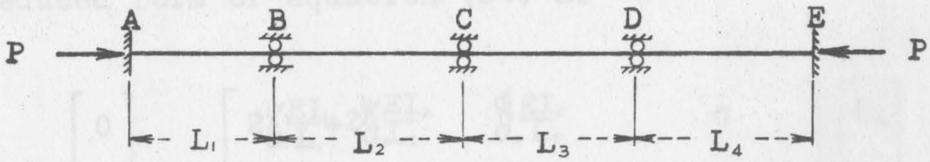


Fig. 4 Fixed-Supported Four-Span Column

The eight boundary conditions are

$$
\left.\begin{array}{lllll}
M_{BA} & = & M_{BC} & = & M_B\,, \\
M_{CB} & = & M_{CD} & = & M_C\,, \\
M_{DC} & = & M_{DE} & = & M_D\,, \\
\theta_{AB} & = & 0\,, & & \\
\theta_{BA} & = & -\theta_{BC} & = & \theta_B\,, \\
\theta_{CB} & = & -\theta_{CD} & = & \theta_C\,, \\
\theta_{DC} & = & -\theta_{DE} & = & \theta_D\,, \text{ and} \\
\theta_{ED} & = & 0\,. & &
\end{array}\right] \qquad (41)
$$

2.2A Stiffness Method

The stiffness matrix for the system is constructed in the same form as equation (24).

Applying the boundary conditions given above, one obtains the reduced form of equation (24) as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\frac{\psi_1 EI_1}{a_1 L_1}+2\frac{\psi_2 EI_2}{a_2 L_2} & \frac{\phi_2 EI_2}{a_2 L_2} & 0 \\ \frac{\phi_2 EI_2}{a_2 L_2} & 2\frac{\psi_2 EI_2}{a_2 L_2}+2\frac{\psi_3 EI_3}{a_3 L_3} & \frac{\phi_3 EI_3}{a_3 L_3} \\ 0 & \frac{\phi_3 EI_3}{a_3 L_3} & 2\frac{\psi_3 EI_3}{a_3 L_3}+2\frac{\psi_4 EI_4}{a_4 L_4} \end{bmatrix} \begin{bmatrix} \Theta_B \\ \Theta_C \\ \Theta_D \end{bmatrix}. \quad (42)$$

The determinant of the reduced stiffness matrix yields

$$2E^3\left[\left\{\frac{\psi_1 I_1}{a_1 L_1}+\frac{\psi_2 I_2}{a_2 L_2}\right\}\left\{4\left(\frac{\psi_2 I_2}{a_2 L_2}+\frac{\psi_3 I_3}{a_3 L_3}\right)\left(\frac{\psi_3 I_3}{a_3 L_3}+\frac{\psi_4 I_4}{a_4 L_4}\right)-\left(\frac{\phi_3 I_3}{a_3 L_3}\right)^2\right\}-\left\{\frac{\psi_3 I_3}{a_3 L_3}+\frac{\psi_4 I_4}{a_4 L_4}\right\}\left\{\frac{\phi_2 I_2}{a_2 L_2}\right\}^2\right] = 0 . \quad (43)$$

For the special case where $L_1 = L_2 = L_3 = L_4 = L$ and $I_1 = I_2 = I_3 = I_4 = I$, equation (43) becomes

$$8\left(\frac{EI}{L}\right)^3\left[\frac{\psi(2\sqrt{2}\psi-\phi)}{(2\psi+\phi)^3}\frac{(2\sqrt{2}\psi+\phi)}{(2\psi-\phi)^3}\right] = 0 . \quad (44)$$

Noting equation (44), the following five transcendental equations hold:

$$2\sqrt{2}\psi-\phi = \psi = 2\sqrt{2}\psi+\phi = \frac{1}{2\psi+\phi} = \frac{1}{2\psi-\phi} = 0. \quad (45)$$

Applying each function of equation (45) to equation (42), and the following modal matrix $\left[U_k\right]$ is obtained:

$$\left[U_k\right] = \begin{bmatrix} 1 & 1 & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 1 & -1 & 1 \end{bmatrix}. \tag{46}$$

It should be noted that the functions $\frac{1}{2\psi+\phi} = \frac{1}{2\psi-\phi} = 0$ yield the condition $\left\{\Theta_{sr}\right\} = 0$, hence the special case of equation (43) does not hold. As a result the stiffness modal matrix reduces to a (3×3) matrix.

Noting equations (24) and (46), the modal matrix $\left[U_f\right]$ for flexibility becomes

$$\left[U_f\right] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{\sqrt{2}} & 1 \\ -1 & 0 & 1 \\ 0 & -\frac{1}{\sqrt{2}} & 1 \\ 1 & -1 & 1 \end{bmatrix}. \tag{47}$$

The orthogonality conditions of the mode shapes defined for the stiffness method become

$$[U_k]^T [K_{sr}][U_k] = [\Lambda_{kt}]$$

$$= 4\sqrt{2}\frac{EI}{La}\begin{bmatrix} 2\sqrt{2}\psi-\phi & 0 & 0 \\ 0 & \sqrt{2}\psi & 0 \\ 0 & 0 & 2\sqrt{2}\psi+\phi \end{bmatrix} . \quad (48)$$

where $\frac{1}{a} \neq 0$ for the conditions $2\sqrt{2}\psi-\phi = \sqrt{2}\psi = 2\sqrt{2}\psi+\phi = 0.$ The diagonal terms of the matrix $[\Lambda_{kt}]$ define the individual transcendental functions which yield the following critical buckling loads:

$$[\Lambda_{cr}] = \begin{bmatrix} \frac{12.816EI}{L^2} & 0 & 0 \\ 0 & \frac{20.187EI}{L^2} & 0 \\ 0 & 0 & \frac{29.703EI}{L^2} \end{bmatrix} . \quad (49)$$

The associated mode shapes are shown in Fig. (5).

Applying each function of equation (52) to equation (50), the following modal matrix $\left[U_f\right]$ is obtained:

$$\left[U_f\right] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 1 & -1 \\ 0 & -1 & 0 & 1 & 1 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \end{bmatrix} . \qquad (53)$$

Noting equations (33) and (53), the following modal matrix $\left[U_k\right]$ for stiffness becomes

$$\left[U_k\right] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ -\sqrt{2} & 0 & \sqrt{2} & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} . \qquad (54)$$

The orthogonality conditions become

$$[U_f]^T [F_{sr}][U_f]$$

$$= [\Lambda_{ft}]$$

$$= 2\sqrt{2}\frac{L}{6EI}\begin{bmatrix} 2\sqrt{2}\psi-\phi & 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{2}\psi & 0 & 0 & 0 \\ 0 & 0 & 2\sqrt{2}\psi+\phi & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{2}(2\psi+\phi) & 0 \\ 0 & 0 & 0 & 0 & 2\sqrt{2}(2\psi-\phi) \end{bmatrix}. \qquad (55)$$

The diagonal terms of the matrix $[\Lambda_{ft}]$ define the individual transcendental functions which yield the following critical buckling loads:

$$[\Lambda_{cr}]$$

$$= \begin{bmatrix} \frac{12.816EI}{L^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{20.187EI}{L^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{29.703EI}{L^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{(2n\pi)^2EI}{L^2},\ n=1,2,3\ \ldots & 0 \\ 0 & 0 & 0 & 0 & \frac{80.748EI}{L^2} \end{bmatrix}. \qquad (56)$$

The resulting mode shapes are shown in Fig. (5)

It is seen that one does not obtain the last two critical buckling loads by using the stiffness method, since the components of the reduced rotation vectors $\left\{\Theta_{sr}\right\}$ are zero (see equation (42)), and as a result equation (43) does not hold.

$$P_{1cr} = \frac{12.816EI}{L^2}$$

$$P_{2cr} = \frac{20.187EI}{L^2}$$

$$P_{3cr} = \frac{29.703EI}{L^2}$$

$$P_{4cr} = \frac{(2n\pi)^2EI}{L^2}, n=1,2,3\ldots$$

$$P_{5cr} = \frac{80.748EI}{L^2}$$

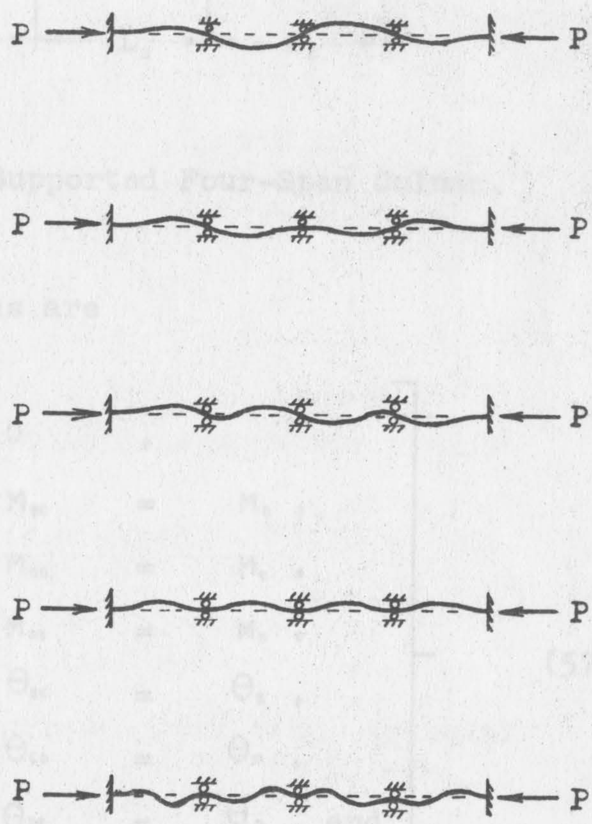


Fig. 5 The Five Possible Mode Shapes of Fixed-Supported Four-Span Column

2.3 Fixed, Simply-Supported Four-Span Column

Consider the column ABCDE subjected to the axial compressive loads P at both ends with a simple support on the extreme left end and a fixed support on the right end (see Fig. 6).

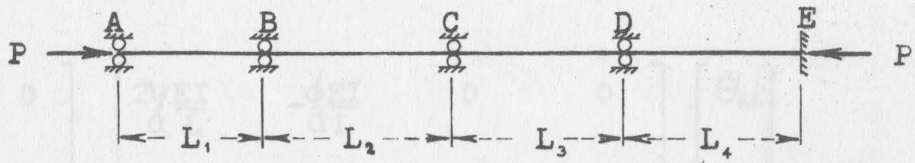


Fig. 6 Fixed, Simply-Supported Four-Span Column.

The eight boundary conditions are

$$\left.\begin{aligned}
M_{AB} &= 0 \,, \\
M_{BA} &= M_{BC} = M_B \,, \\
M_{CB} &= M_{CD} = M_C \,, \\
M_{DC} &= M_{DE} = M_D \,, \\
\theta_{BA} &= -\theta_{BC} = \theta_B \,, \\
\theta_{CB} &= -\theta_{CD} = \theta_C \,, \\
\theta_{DC} &= -\theta_{DE} = \theta_D \,, \text{ and} \\
\theta_{ED} &= 0 \,.
\end{aligned}\right\} \qquad (57)$$

2.3A Stiffness Method

The stiffness matrix for the system is constructed in the same form as equation (24).

For the special case where $L_1 = L_2 = L_3 = L_4 = L$ and $I_1 = I_2 = I_3 = I_4 = I$, applying the boundary conditions given above, one obtains the reduced form of equation (24) as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2\psi EI}{d\,L} & -\frac{\phi EI}{dL} & 0 & 0 \\ -\frac{\phi EI}{d\,L} & \frac{4\psi EI}{dL} & \frac{\phi EI}{d\,L} & 0 \\ 0 & \frac{\phi EI}{d\,L} & \frac{4\psi EI}{d\,L} & \frac{\phi EI}{d\,L} \\ 0 & 0 & \frac{\phi EI}{d\,L} & \frac{4\psi EI}{d\,L} \end{bmatrix} \begin{bmatrix} \theta_{AB} \\ \theta_B \\ \theta_C \\ \theta_D \end{bmatrix} . \tag{58}$$

The determinant of the reduced stiffness matrix yields

$$\left(6\frac{EI}{L}\right)^4 \left[\frac{(d_1\psi+\phi)(d_1\psi-\phi)(d_2\psi+\phi)(d_2\psi-\phi)}{(2\psi+\phi)^4(2\psi-\phi)^4} \right] = 0, \tag{59}$$

where $d_1 = 2(4+2\sqrt{2})^{\frac{1}{2}}$ and $d_2 = 2(4-2\sqrt{2})^{\frac{1}{2}}$.

Noting equation (59) , the following six transcendental equations hold:

$$d_2\psi-\phi = d_1\psi-\phi = d_1\psi+\phi = d_2\psi+\phi = \frac{1}{2\psi+\phi} = \frac{1}{2\psi-\phi} . \tag{60}$$

Applying each function of equation (60) to equation (58), the following modal matrix $\left[U_k\right]$ is obtained:

$$\left[U_k\right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{2}{d_2} & \frac{2}{d_1} & -\frac{2}{d_1} & -\frac{2}{d_2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{d_1} & -\frac{2}{d_2} & \frac{2}{d_2} & -\frac{2}{d_1} \end{bmatrix}. \qquad (61)$$

It should be noted that the functions $\frac{1}{2\psi+\phi} = \frac{1}{2\psi-\phi} = 0$ yield the condition $\left\{\Theta_{sr}\right\} = 0$, hence equation (59) does not hold. As a result the stiffness modal matrix reduces to a (4 × 4) matrix.

Noting equations (24) and (61), the modal matrix $\left[U_f\right]$ for flexibility becomes

$$\left[U_f\right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -\frac{4}{d_2} & -\frac{4}{d_1} & \frac{4}{d_1} & \frac{4}{d_2} \\ \sqrt{2}+1 & -(\sqrt{2}-1) & -(\sqrt{2}-1) & \sqrt{2}+1 \\ -(\frac{\sqrt{2}+1}{2})d_2 & (\frac{\sqrt{2}-1}{2})d_1 & -(\frac{\sqrt{2}-1}{2})d_1 & (\frac{\sqrt{2}+1}{2})d_2 \end{bmatrix}. \qquad (62)$$

The orthogonality conditions of the mode shapes defined for the stiffness method become

$$\left[U_k\right]^T \left[K_{sr}\right]\left[U_k\right]$$

$$= \left[\Lambda_{k_t}\right]$$

$$= \frac{EI}{La}\begin{bmatrix} 3.656(d_2\psi-\phi) & 0 & 0 & 0 \\ 0 & 1.532(d_1\psi-\phi) & 0 & 0 \\ 0 & 0 & 1.532(d_1\psi+\phi) & 0 \\ 0 & 0 & 0 & 3.656(d_2\psi+\phi) \end{bmatrix}, \qquad (63)$$

where $\frac{1}{a} \neq 0$ for the conditions $d_2\psi-\phi = d_1\psi-\phi = d_1\psi+\phi = d_2\psi+\phi = 0$. The diagonal terms of the matrix $\left[\Lambda_{k_t}\right]$ define the individual transcendental functions which yield the following critical buckling loads:

$$\left[\Lambda_{cr}\right] = \begin{bmatrix} \frac{10.628EI}{L^2} & 0 & 0 & 0 \\ 0 & \frac{16.080EI}{L^2} & 0 & 0 \\ 0 & 0 & \frac{24.900EI}{L^2} & 0 \\ 0 & 0 & 0 & \frac{34.81EI}{L^2} \end{bmatrix}. \qquad (64)$$

The associated mode shapes are shown in Fig. 7 .

2.3B Flexibility Method

The flexibility matrix for the system is given by equation (33). For the special case where $L_1 = L_2 = L_3 = L_4 = L$ and $I_1 = I_2 = I_3 = I_4 = I$, applying the boundary conditions of equation (57), one obtains

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4\psi\frac{L}{6EI} & \phi\frac{L}{6EI} & 0 & 0 \\ \phi\frac{L}{6EI} & 4\psi\frac{L}{6EI} & \phi\frac{L}{6EI} & 0 \\ 0 & \phi\frac{L}{6EI} & 4\psi\frac{L}{6EI} & \phi\frac{L}{6EI} \\ 0 & 0 & \phi\frac{L}{6EI} & 2\psi\frac{L}{6EI} \end{bmatrix} \begin{bmatrix} M_B \\ M_C \\ M_D \\ M_{ED} \end{bmatrix} . \tag{65}$$

The determinant of the matrix $\left[F_{sr}\right]$ yields

$$\left(\frac{L}{6EI}\right)^4 \left[(d_1\psi+\phi)(d_1\psi-\phi)(d_2\psi+\phi)(d_2\psi-\phi)\right] = 0 . \tag{66}$$

Noting equation (66) , the following four transcendental equations hold:

$$d_2\psi-\phi = d_1\psi-\phi = d_1\psi+\phi = d_2\psi+\phi = 0. \tag{67}$$

Applying each function of equation (67) to equation (65), the modal matrix $\left[U_f\right]$ for flexibility is obtained exactly the same as that given by equation (62).

Noting equations (33) and (62), the modal matrix $\left[U_k\right]$ for stiffness is exactly equal to that given by equation (61).

The orthogonality conditions become

$$\left[U_f\right]^T\left[F_{sr}\right]\left[U_f\right]$$

$$= \left[\Lambda_{f_t}\right]$$

$$= \frac{L}{6EI}\begin{bmatrix} 24.99(d_2\psi-\phi) & 0 & 0 & 0 \\ 0 & 1.786(d_1\psi-\phi) & 0 & 0 \\ 0 & 0 & 1.786(d_1\psi+\phi) & 0 \\ 0 & 0 & 0 & 24.99(d_2\psi+\phi) \end{bmatrix} \quad (68)$$

The associated matrix $\left[\Lambda_{cr}\right]$ containing the critical buckling loads is exactly the same as that given by equation (64).

The resulting mode shapes are shown in Fig. 7 .

It is seen that both the stiffness method and flexibility method obtain the same four critical buckling loads and associated mode shapes.

For convenience and simplicity, the analysis of three-span and two-span continuous columns, subject to similar boundary conditions as presented above, is summarized in APPENDIX A . All critical buckling loads, mode shapes, stiffness matrices, flexibility matrices, and orthogonality conditions are presented in a compact tabular form.

$$P_{1cr} = \frac{10.628EI}{L^2}$$

$$P_{2cr} = \frac{16.080EI}{L^2}$$

$$P_{3cr} = \frac{24.90EI}{L^2}$$

$$P_{4cr} = \frac{34.81EI}{L^2}$$

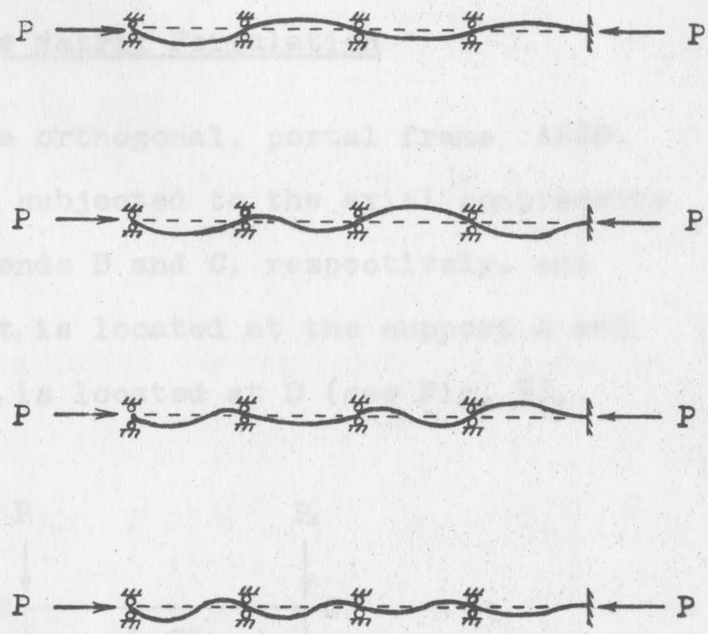


Fig. 7 The Four Possible Mode Shapes of Fixed, Simply Supported Four-Span Column

CHAPTER III

SOLUTIONS OF THE ORTHOGONAL PORTAL FRAME PROBLEM

3.1 General Stiffness Matrix Formulation

Consider the orthogonal, portal frame ABCD, columns AB and CD are subjected to the axial compressive loads P_1 and P_2 at the ends B and C, respectively, and a torsional spring kt_1 is located at the support A and a torsional spring kt_2 is located at D (see Fig. 8).

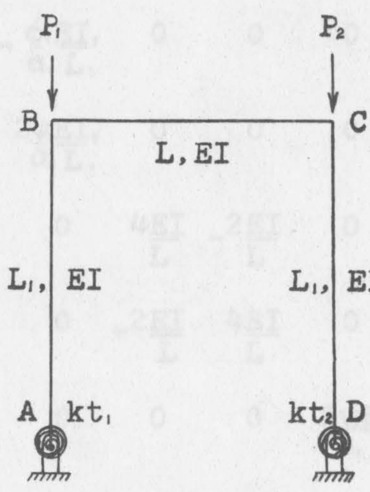


Fig. 8 Orthogonal Portal Frame

The seven boundary conditions are

$$
\left.
\begin{aligned}
M_{AB} &= kt_1\Theta_{AB} = kt_1\left(\frac{\Delta}{L}+\Theta^*_{AB}\right),\\
M_{BA} &= M_{BC} = M_B,\\
M_{CB} &= -M_{CD} = M_C,\\
M_{DC} &= kt_2\Theta_{DC} = kt_2\left(\frac{\Delta}{L}+\Theta^*_{DC}\right),\\
M_{BA} &+ M_{CD} - M_{AB} - M_{DC} - (P_1+P_2)\Delta = 0,\\
-\Theta_{BC} &= \Theta_{BA} = \left(\Theta^*_{BA}-\frac{\Delta}{L}\right), \text{ and}\\
\Theta_{CB} &= \Theta_{CD} = \left(\Theta^*_{CD}-\frac{\Delta}{L}\right).
\end{aligned}
\right\} \quad (69)
$$

The stiffness matrix for the system is constructed as follow;

$$
\begin{bmatrix} M_{AB} \\ M_{BA} \\ M_{BC} \\ M_{CB} \\ M_{CD} \\ M_{DC} \end{bmatrix}
=
\begin{bmatrix}
\frac{2\psi_1 EI_1}{a_1 L_1} & -\frac{\phi_1 EI_1}{a_1 L_1} & 0 & 0 & 0 & 0 \\
-\frac{\phi_1 EI_1}{a_1 L_1} & \frac{2\psi_1 EI_1}{a_1 L_1} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{4EI}{L} & -\frac{2EI}{L} & 0 & 0 \\
0 & 0 & -\frac{2EI}{L} & \frac{4EI}{L} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{2\psi_2 EI_1}{a_2 L_1} & -\frac{\phi_2 EI_1}{a_2 L_1} \\
0 & 0 & 0 & 0 & -\frac{\phi_2 EI_1}{a_2 L_1} & \frac{2\psi_2 EI_1}{a_2 L_1}
\end{bmatrix}
\begin{bmatrix} \Theta^*_{AB} \\ \Theta^*_{BA} \\ \Theta_{BC} \\ \Theta_{CB} \\ \Theta^*_{CD} \\ \Theta^*_{DC} \end{bmatrix} . \quad (70)
$$

For simplicity and convenience the stiffness method is utilized to solve frame problems, since, in general, the vector $\{\Theta_{sr}\} \neq 0$ for the usual modal shapes defined at critical loading.

Applying the boundary conditions given above, one obtains the reduced form of equation (70) as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} \Theta_{BC} \\ \Theta_{CB} \\ \frac{\Delta}{L_1} \end{bmatrix}, \tag{71}$$

where

$$k_{11} = \frac{2\psi_1}{a_1}\frac{EI_1}{L_1} + \frac{4EI}{L} + \frac{\left(\frac{\phi_1}{a_1}\frac{EI_1}{L_1}\right)^2}{kt_1 - \frac{2\psi_1}{a_1}\frac{EI_1}{L_1}},$$

$$k_{22} = \frac{2\psi_2}{a_2}\frac{EI_1}{L_1} + \frac{4EI}{L} + \frac{\left(\frac{\phi_2}{a_2}\frac{EI_1}{L_1}\right)^2}{kt_2 - \frac{2\psi_2}{a_2}\frac{EI_1}{L_1}},$$

$$k_{33} = \left\{\left(\frac{2\psi_1}{a_1} + \frac{\phi_1}{a_1}\right)\left(\frac{kt_1 + \frac{\phi}{a_1}\frac{EI_1}{L_1}}{kt_1 - \frac{2\psi}{a_1}\frac{EI_1}{L_1}} + 1\right) + \left(\frac{2\psi_2}{a_2} + \frac{\phi_2}{a_2}\right)\left(\frac{kt_2 + \frac{\phi}{a_2}\frac{EI_1}{L_1}}{kt_2 - \frac{2\psi}{a_1}\frac{EI_1}{L_1}} + 1\right) - 4(u_1^2 + u_2^2)\right\}\frac{EI_1}{L_1}, \tag{72}$$

$$k_{12} = k_{21} = -2\frac{EI}{L},$$

$$k_{13} = k_{31} = -\left\{\frac{2\psi_1}{a_1} + \frac{\phi_1}{a_1}\left(\frac{kt_1 + \frac{\phi_1}{a_1}\frac{EI_1}{L_1}}{kt_1 - \frac{2\psi_1}{a_1}\frac{EI_1}{L_1}}\right)\right\}\frac{EI_1}{L_1}, \text{ and}$$

$$k_{23} = k_{32} = \left\{\frac{2\psi_2}{a_2} + \frac{\phi_2}{a_2}\left(\frac{kt_2 + \frac{\phi_2}{a_2}\frac{EI_1}{L_1}}{kt_2 - \frac{2\psi_2}{a_2}\frac{EI_1}{L_1}}\right)\right\}\frac{EI_1}{L_1}.$$

For the special case where $P_1 = P_2 = P$, $kt_1 = kt_2 = kt$, $L_1 = L$ and $I_1 = I$, the determinant of the reduced stiffness matrix yields

$$D_1 D_2 = 0 \quad . \tag{73}$$

where,

$$D_1 = \left\{ \frac{2\psi}{a} + \frac{\left(\frac{\phi}{a}\right)^2 \frac{EI}{L}}{kt - \frac{2\psi}{a}\frac{EI}{L}} + 6 \right\} \left\{ 2\left(\frac{2\psi}{a} + \frac{\phi}{a}\right)\left(\frac{kt + \frac{\phi}{a}\frac{EI}{L}}{kt + 2\frac{\psi}{a}\frac{EI}{L}} + 1\right) - 8u^2 \right\}$$

$$-2\left\{ \frac{2\psi}{a} + \frac{\phi}{a}\left(\frac{kt + \frac{\phi}{a}\frac{EI}{L}}{kt - \frac{2\psi}{a}\frac{EI}{L}}\right) \right\} \text{ and} \tag{74}$$

$$D_2 = \frac{2\psi}{a} + \frac{\left(\frac{\phi}{a}\right)^2 \frac{EI}{L}}{kt - \frac{2\psi}{a}\frac{EI}{L}} + 2 \quad .$$

Noting equation (73) and (74), the following two transcendental equations hold:

$$D_1 = D_2 = 0 \quad . \tag{75}$$

Applying each function of equation (75) to equation (71) noting equation (74), and the following modal matrix $\left[U_k\right]$ is obtained:

$$[U_k] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ C_1 & 0 \end{bmatrix} , \qquad (76)$$

where

$$C_1 = \frac{\dfrac{2\psi}{a} + \dfrac{\left(\frac{\phi}{a}\right)^2 \frac{EI}{L}}{kt - \frac{2\psi}{a}\frac{EI}{L}} + 6}{\dfrac{2\psi}{a} + \dfrac{\phi}{a}\left(\dfrac{kt + \frac{\phi}{a}\frac{EI}{L}}{kt - 2\frac{\psi}{a}\frac{EI}{L}}\right)} . \qquad (77)$$

The orthogonality conditions of the mode shapes defined for the stiffness method become

$$[U_k]^T [K_{sr}][U_k] = [\Lambda_{kt}]$$

$$= \begin{bmatrix} \dfrac{C_1}{C_2} D_1 & 0 \\ 0 & 2D_1 \end{bmatrix} , \qquad (78)$$

where

$$C_2 = \frac{2\psi}{a} + \frac{\phi}{a}\left(\frac{kt + \frac{\phi}{a}\frac{EI}{L}}{kt - \frac{2\psi}{a}\frac{EI}{L}}\right) , \text{ and} \qquad (79)$$

$C_1 \neq 0$, $\frac{1}{C_2} \neq 0$ for the condition $D_1 = 0$.

3.2 Simply Supported-Frame

Consider the orthogonal, portal frame ABCD, in Fig. 8 in the case of simple supports, that is $kt_1 = kt_2 = 0$ and $M_{AB} = M_{DC} = 0$ (see Fig. 9).

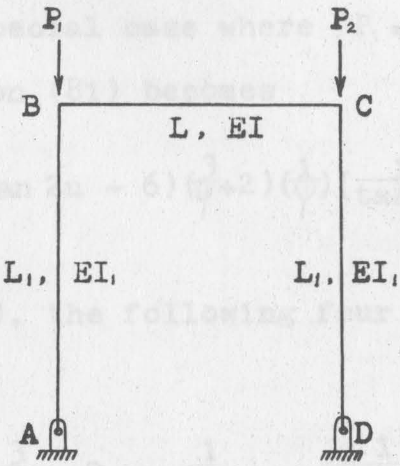


Fig. 9 Simply Supported-Orthogonal, Portal Frame

Equation (71) reduces to the form

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3EI_1}{\psi_1 L_1} + \frac{4EI}{L} & -2\frac{EI}{L} & -\frac{3EI_1}{\psi_1 L_1} \\ -2\frac{EI}{L} & \frac{3EI_1}{\psi_2 L_1} + \frac{4EI}{L} & \frac{3EI}{\psi_2 L_1} \\ -\frac{3EI_1}{\psi_1 L_1} & \frac{3EI_1}{\psi_2 L_1} & \left\{\frac{3}{\psi_1} + \frac{3}{\psi_2} - 4\left(u_1^2 + u_2^2\right)\right\}\frac{EI_1}{L_1} \end{bmatrix} \begin{bmatrix} \Theta_{BC} \\ \Theta_{CB} \\ \frac{\Delta}{L_1} \end{bmatrix} . \qquad (80)$$

The determinant of the reduced stiffness matrix yields

$$\frac{12}{\psi_1\psi_2}\frac{EI_1}{L_1}\left[3\left\{(\psi_1+\psi_2)\left(\frac{EI}{L}\right)^2 + 3\frac{EI}{L}\frac{EI_1}{L_1}\right\} - (u_1^2+u_2^2)\left\{4\psi_1\psi_2\left(\frac{EI}{L}\right)^2 + 4(\psi_1+\psi_2)\frac{EI}{L}\frac{EI_1}{L_1} + 3\left(\frac{EI_1}{L_1}\right)^2\right\}\right] = 0. \qquad (81)$$

For the special case where $P_1 = P_2 = P$, $L_1 = L$ and $I_1 = I$, equation (81) becomes

$$-12u\left(\frac{EI}{L}\right)^3(2u\tan 2u - 6)\left(\frac{3}{\psi}+2\right)\left(\frac{1}{\psi}\right)\left(\frac{1}{\tan 2u}\right) = 0. \qquad (82)$$

Noting equation (82), the following four transcendental equations hold:

$$2u\tan 2u - 6 = \frac{3}{\psi} + 2 = \frac{1}{\psi} = \frac{1}{\tan 2u} = 0 \quad . \qquad (83)$$

It should be noted that the functions $\frac{1}{\psi} = \frac{1}{\tan 2u} = 0$ yield the condition $\left\{\Theta_{sr}\right\} = 0$, hence equation (83) does not hold.

The modal matrix $\left[U_k\right]$ of equation (76) becomes

$$\left[U_k\right] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ d_3 & 0 \end{bmatrix}, \qquad (84)$$

where $d_3 = \left| 2\psi + 1 \right|_{\psi = 1.1473} = 3.2946$.

Noting equations (70) and (84), the modal matrix $\left[U_f\right]$ for flexibility becomes

$$\left[U_f\right] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} . \tag{85}$$

The orthogonality conditions of the mode shapes defined for the stiffness method take the form

$$\left[U_k\right]^T \left[K_{sr}\right]\left[U_k\right] = \left[\Lambda_{kt}\right] = \frac{EI}{L}\begin{bmatrix} C_3(2u\tan 2u - 6) & 0 \\ 0 & 2\,\dfrac{3}{\psi}+2 \end{bmatrix}, \tag{86}$$

where $C_3 = -4u(2\psi + 1)\dfrac{1}{\tan 2u} \neq 0$ for the condition $2u\tan 2u - 6 = 0$.

The diagonal terms of the matrix $\left[\Lambda_{kt}\right]$ define the individual transcendental functions which yield the following critical buckling loads:

$$\left[\Lambda_{cr}\right] = \begin{bmatrix} \dfrac{1.823EI}{L^2} & 0 \\ 0 & \dfrac{12.888EI}{L^2} \end{bmatrix} . \tag{87}$$

The associated mode shapes are shown in Fig. 12 .

3.2A Simply Supported-Frame Neglecting Sidesway

In the case where sideway is neglected, $\Delta = 0$, hence $\Theta^*_{AB} = \Theta_{AB}$, $\Theta^*_{BA} = \Theta_{BA}$, $\Theta^*_{CD} = \Theta_{CD}$, $\Theta^*_{DC} = \Theta_{DC}$,

equation (80) reduces to the form

$$\begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{\psi_1}\frac{EI_1}{L_1} + \frac{4EI}{L} & -2\frac{EI}{L} \\ \\ -2\frac{EI}{L} & \frac{3}{\psi_2}\frac{EI_1}{L_1} + 4\frac{EI}{L} \end{bmatrix} \begin{bmatrix} \Theta_{BC} \\ \\ \Theta_{CB} \end{bmatrix} . \tag{88}$$

For the special case where $P_1 = P_2 = P$, $L_1 = L$ and $I_1 = I$, the determinant of the reduced stiffness matrix yields

$$3\left(\frac{EI}{L}\right)^2\left(\frac{3}{\psi} + 2\right)\left(\frac{1}{\psi} + 2\right) = 0 . \tag{89}$$

Noting equation (89), the following two transcendental equations hold:

$$\left(\frac{3}{\psi} + 2\right) = \left(\frac{1}{\psi} + 2\right) = 0 . \tag{90}$$

Applying each function of equation (90) to equation (88), and the following modal matrix $\left[U_k\right]$ is obtained:

$$[U_k] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} . \tag{91}$$

Noting equations (70) and (91), the modal matrix $[U_f]$ for flexibility becomes

$$[U_f] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} . \tag{92}$$

The orthogonality conditions of the mode shapes defined for the stiffness method reduces to

$$\begin{aligned} [U_k]^T [K_{sr}][U_k] &= [\Lambda_{kt}] \\ &= 2\frac{EI}{L}\begin{bmatrix} \frac{3}{\psi} + 2 & 0 \\ 0 & 3\left(\frac{1}{\psi} + 2\right) \end{bmatrix} . \end{aligned} \tag{93}$$

The diagonal terms of the matrix $[\Lambda_{kt}]$ define the individual transcendental functions which yield the following critical buckling loads:

$$[\Lambda_{cr}] = \begin{bmatrix} \frac{12.888EI}{L^2} & 0 \\ 0 & \frac{15.80EI}{L^2} \end{bmatrix} . \tag{94}$$

The resulting mode shapes are shown in Fig. 12.

It is seen that the lower critical buckling load neglecting sidesway is the same as the higher one of including sidesway-case. Hence, the lowest critical buckling load occurs when sidesway is present.

3.3 Fixed Supported-Frame

Consider the orthogonal, portal frame ABCD in Fig. 8 in the case of fixed supports, that is $kt_1 = kt_2 = \infty$, and $\theta_{AB} = \theta_{DC} = 0$ (see Fig. 10).

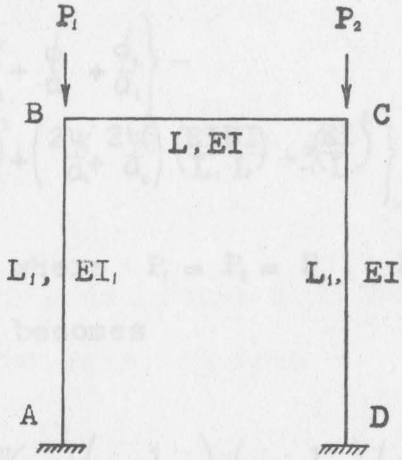


Fig. 10 Fixed Supported-Orthogonal, Portal Frame

Equation (71) reduces to the form

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix} 2\frac{\psi_1}{a_1}\frac{EI_1}{L_1}+4\frac{EI}{L} & -2\frac{EI}{L} & -\left(2\frac{\psi_1}{a_1}+\frac{\phi_1}{a_1}\right)\frac{EI_1}{L_1} \\ -2\frac{EI}{L} & 2\frac{\psi}{a}\frac{EI}{L}+4\frac{EI}{L} & \left(2\frac{\psi_2}{a_2}+\frac{\phi_2}{a_2}\right)\frac{EI_1}{L_1} \\ -\left(2\frac{\psi_1}{a_1}+\frac{\phi_1}{a_1}\right)\frac{EI_1}{L_1} & \left(2\frac{\psi_1}{a_2}+\frac{\phi_2}{a_2}\right)\frac{EI_1}{L_1} & k_{33} \end{bmatrix} \begin{bmatrix}\theta_{BC}\\ \theta_{CB}\\ \frac{\Delta}{L_1}\end{bmatrix}, \qquad (95)$$

where $$k_{33} = \left\{ 2\left(\frac{2\psi_1}{a_1}+\frac{\phi_1}{a_1}\right) + 2\left(\frac{2\psi_2}{a_2}+\frac{\phi_2}{a_2}\right) - 4(u_1^2+u_2^2)\right\}\frac{EI_1}{L_1}. \qquad (96)$$

The determinant of the reduced stiffness matrix yields

$$\begin{aligned}\frac{EI_1}{L_1}\Bigg[&\left(\frac{EI_1}{L_1}\right)^2\left\{8\left(\frac{\psi_1}{a_1}\right)^2\frac{\psi_2}{a_2}+8\frac{\psi_1}{a_1}\left(\frac{\psi_2}{a_2}\right)^2-2\frac{\psi_1}{a_1}\left(\frac{\phi_2}{a_2}\right)^2-2\frac{\psi_2}{a_2}\left(\frac{\phi_1}{a_1}\right)^2\right\}+\\ &4\left(\frac{EI_1}{L_1}\frac{EI}{L}\right)\left\{4\left(\frac{\psi_1}{a_1}\right)^2+20\frac{\psi_1\psi_2}{a_1a_2}+4\left(\frac{\psi_2}{a_2}\right)^2-\left(\frac{\phi_1}{a_1}\right)^2-\left(\frac{\phi_2}{a_2}\right)^2+\frac{\phi_1}{a_1}\frac{\phi_2}{a_2}+6\frac{\psi_1\phi_2}{a_1a_2}+\frac{\psi_2}{a_2}\frac{\phi_1}{a_1}\right\}+\\ &24\left(\frac{EI}{L}\right)^2\left\{2\frac{\psi_1}{a_1}+2\frac{\psi_2}{a_2}+\frac{\phi_1}{a_1}+\frac{\phi_2}{a_2}\right\}-\\ &16(u_1^2+u_2^2)\left\{\frac{\psi_1}{a_1}\frac{\psi_2}{a_2}\left(\frac{EI_1}{L_1}\right)^2+\left(\frac{2\psi_1}{a_1}+\frac{2\psi_2}{a_2}\right)\left(\frac{EI_1}{L_1}\frac{EI}{L}\right)+3\left(\frac{EI}{L}\right)^2\right\}\Bigg] = 0. \qquad (97)\end{aligned}$$

For the special case where $P_1 = P_2 = P$, $L_1 = L$ and $I_1 = I$, equation (97) becomes

$$-\frac{6^5}{u}\left(\frac{EI}{L}\right)^3(2u+6\tan 2u)\left(\frac{6\psi}{4\psi-\phi}+1\right)\left(\frac{1}{\tan 2u}\right)\left(u+\frac{1}{\tan u}\right)\left(\frac{1}{2\psi+\phi}\right)\left(\frac{1}{2\psi-\phi}\right)=0. \qquad (98)$$

It should be noted that the functions $\frac{1}{\tan 2u} = u+\frac{1}{\tan u}$

$\frac{1}{2\psi+\phi} = \frac{1}{2\psi-\phi} = 0$ yield the condition $\left\{\Theta_{sr}\right\} = 0$,

hence, equation (98) does not hold.

The modal matrix $\left[U_k\right]$ of equation (76) takes the form

$$\left[U_k\right] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ d_4 & 0 \end{bmatrix}, \qquad (99)$$

where $d_4 = \left.\frac{2\psi}{2\psi+\phi}+2\psi-\phi\right|_{\substack{\psi = 2.838 \\ \phi = 4.528}} = 1.704.$

Noting equations (70) and (99), the modal matrix $\left[U_f\right]$ for flexibility becomes

$$\left[U_f\right] = \begin{bmatrix} 1 & 1 \\ -0.911 & 0.415 \\ 0.911 & 0.415 \\ 1 & -1 \end{bmatrix} . \qquad (100)$$

The orthogonality conditions of the mode shapes defined for the stiffness method take the form

$$\left[U_k\right]^T\left[K_{sr}\right]\left[U_k\right] = \left[\Lambda_{kt}\right] = \frac{EI}{L}\begin{bmatrix} C_4(2u+6\tan 2u) & 0 \\ 0 & 4\left(\frac{6}{4\psi^2-\phi^2}+1\right) \end{bmatrix} , \qquad (101)$$

where $C_4 = -\frac{3\times 6^2}{u}\left(\frac{2\psi}{a}+6\right)\left(\frac{1}{\tan 2u}\right)\left(u+\frac{1}{\tan u}\right)\left(\frac{1}{2\psi+\phi}\right)^2 \neq 0$ for the condition $2u + 6\tan 2u = 0$.

The diagonal terms of the matrix $\left[\Lambda_{kt}\right]$ define the individual transcendental functions which yield the following critical buckling loads:

$$\left[\Lambda_{cr}\right] = \begin{bmatrix} \frac{7.371EI}{L^2} & 0 \\ 0 & \frac{25.20EI}{L^2} \end{bmatrix} . \tag{102}$$

The associated mode shapes are shown in Fig. 12 .

3.3A Fixed Supported-Frame Neglecting Sidesway

In the case where sidesway is neglected, $\Delta = 0$, thus $\Theta^*_{AB} = \Theta_{AB}$, $\Theta^*_{BA} = \Theta_{BA}$, $\Theta^*_{CD} = \Theta_{CD}$, $\Theta^*_{DC} = \Theta_{DC}$ and equation (95) reduces to the form

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2\psi_1 EI_1}{a_1 L_1} + 4\frac{EI}{L} & -2\frac{EI}{L} \\ -2\frac{EI}{L} & \frac{2\psi_2 EI_2}{a_2 L_2} + 4\frac{EI}{L} \end{bmatrix} \begin{bmatrix} \Theta_{BC} \\ \Theta_{CB} \end{bmatrix} \tag{103}$$

For the special case where $P_1 = P_2 = P$, $L_1 = L$ and $I_1 = I$, the determinant of the reduced stiffness matrix yields

$$12\left(\frac{EI}{L}\right)^2 \left(\frac{6\psi}{4\psi^2-\phi^2}+1\right)\left(\frac{2\psi}{4\psi^2-\phi^2}+1\right) = 0 . \tag{104}$$

Noting equation (104) , the following two trancendental equations hold:

$$\left(\frac{6\psi}{4\psi^2-\phi^2}+1\right) = \left(\frac{2\psi}{4\psi^2-\phi^2}+1\right) = 0 . \tag{105}$$

Applying each function of equation (105) to equation (103), the following modal matrix $\left[U_k\right]$ is obtained:

$$\left[U_k\right] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} . \qquad (106)$$

Noting equations (70) and (106), the modal matrix $\left[U_f\right]$ for flexibility becomes

$$\left[U_f\right] = \begin{bmatrix} 1 & 1 \\ 0.415 & 0.76 \\ 0.415 & -0.76 \\ -1 & 1 \end{bmatrix} . \qquad (107)$$

The orthogonality conditions of the shapes defined for the stiffness method become

$$\left[U_k\right]^T \left[K_{sr}\right]\left[U_k\right]$$

$$= \left[\Lambda_{k_t}\right]$$

$$= 4\frac{EI}{L} \begin{bmatrix} \frac{6\psi}{4\psi^2-\phi^2}+1 & 0 \\ 0 & 3\left(\frac{2\psi}{4\psi^2-\phi^2}+1\right) \end{bmatrix} . \qquad (108)$$

The diagonal terms of the matrix $\left[\Lambda_{kt}\right]$ define the individual transcendental functions which yield the following critical buckling loads:

$$\left[\Lambda_{cr}\right] = \begin{bmatrix} \frac{25.20EI}{L^2} & 0 \\ 0 & \frac{30.526EI}{L^2} \end{bmatrix} . \qquad (109)$$

The resulting mode shapes are shown in Fig. 12.

It is seen that the lowest critical buckling load occurs for the case where sidesway is present.

3.4 Fixed, Simply Supported-Frame

Consider the orthogonal, portal frame ABCD in Fig.8 for the case of a simple support on the left and a fixed support on the right end, that is $kt_1 = 0$, $kt_2 = \infty$, $M_{AB} = 0$ and $\Theta_{DC} = 0$ (see Fig. 11).

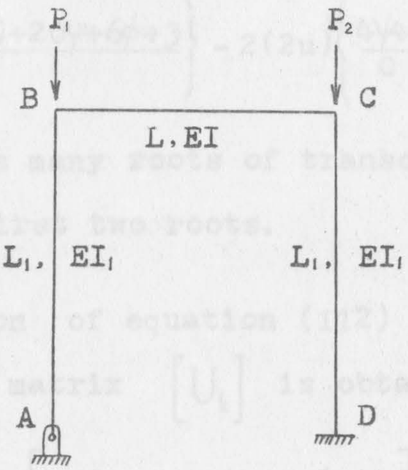


Fig. 11 Fixed, Simply Supported-Orthogonal, Portal Frame

Equation (71) reduces to the form

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3EI_1}{\psi_1 L_1} + \frac{4EI}{L} & -\frac{2EI}{L} & -\frac{3EI_1}{\psi_1 L_1} \\ -\frac{2EI}{L} & \frac{2\psi_2 EI_1}{a_2 L_1} + \frac{4EI}{L} & \left(\frac{2\psi_2}{a_2} + \frac{\phi_2}{a_2}\right)\frac{EI_1}{L_1} \\ -\frac{3EI_1}{\psi_1 L_1} & \left(\frac{2\psi_2}{a_2} + \frac{\phi_2}{a_2}\right)\frac{EI_1}{L_1} & \left\{\frac{3}{\psi_1} + 2\left(\frac{2\psi_2}{a_2} + \frac{\phi_2}{a_2}\right) - 4\,(u_1^2 + u_2^2)\right\}\frac{EI_1}{L_1} \end{bmatrix} \begin{bmatrix} \Theta_{BC} \\ \Theta_{CB} \\ \frac{\Delta}{L_1} \end{bmatrix} \quad (110)$$

The determinant of the reduced stiffness matrix yields

$$2\frac{EI_1}{L_1}\left[\frac{a}{\psi_1 a_2}\left(\frac{EI_1}{L_1}\right)^2+\frac{6}{a_2}\left(2+\frac{8\psi_2}{\psi_1}+\frac{3\phi_2}{\psi_1}\right)\left(\frac{EI_1EI}{L_1L}\right)+18\left(\frac{1}{\psi_1}+\frac{4}{2\psi_2-\phi_2}\right)\left(\frac{EI}{L}\right)^2-\right.$$

$$\left.4(u_1^2+u_2^2)\left\{\frac{3\psi_2}{\psi_1 a_2}\left(\frac{EI_1}{L_1}\right)^2+2\frac{EI_1EI}{L_1L}\left(\frac{3}{\psi_1}+\frac{2\psi_2}{a_2}\right)+6\left(\frac{EI}{L}\right)^2\right\}\right] = 0 \quad . \qquad (111)$$

For the special case where $P_1 = P_2 = P$, $L_1 = L$, and $I_1 = I$ equation (111) becomes

$$2\left(\frac{EI}{L}\right)^3\left[3\left\{\frac{(6\psi-\phi)(2\psi+\phi)+20\psi+6\phi+3}{\psi a}\right\}-2(2u)^2\left\{\frac{4\psi+3}{a}+\frac{6(\psi+1)}{\psi}\right\}\right]=0. \qquad (112)$$

This equation yields many roots of transcendental function, one considers the first two roots.

Applying the function of equation (112) to equation (110), the following modal matrix $[U_k]$ is obtained:

$$[U_k] = \begin{bmatrix} 1 & 1 \\ -8.316 & 0.20409 \\ 11.425 & 0.20830 \end{bmatrix} . \qquad (113)$$

Noting equations (70) and (113), the modal matrix $[U_f]$ for flexibility becomes

$$[U_f] = \begin{bmatrix} 1 & 1 \\ -1.709 & -0.3295 \\ -2.194 & -0.4139 \end{bmatrix} . \qquad (114)$$

The orthogonality conditions of the mode shapes defined for the stiffness method become

$$[U_k]^T [K_{sr}][U_k] = [K^*_{sr}]$$

$$= \frac{EI}{L}\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}, \qquad (115)$$

where $e_{11} = \dfrac{326.04}{\psi} + 280.39\dfrac{\psi}{a} + 71.041\dfrac{\phi}{a} + 313.89 - 1044.25\,u,$

$$e_{22} = \frac{1.8804}{\psi} + 0.4269\frac{\psi}{a} + 0.1718\frac{\phi}{a} + 3.3502 - 0.3471\,u, \qquad (116)$$

and $e_{12} = e_{21}$

$$= \frac{-24.760}{\psi} + 7.3240\frac{\psi}{a} + 5.3592\frac{\phi}{a} + 13.435 - 19.0386u$$

Equating $e_{11} = 0$, it follows that, $2u = 2.103$, $\psi = 1.5184$, $\phi = 1.954$, $a = 0.9006$. For these values, one obtains

$$[\Lambda_{kt}] = \begin{bmatrix} 0 & 0 \\ 0 & -15.3690 \end{bmatrix}, \qquad (117)$$

where it is noted that $e_{12} = 0$.

Equating $e_{22} = 0$, it follows that, $2u = 3.8765$, $\psi = -0.66125$, $\phi = -2.71710$, and $a = -0.93894$.

For these values, one obtains

$$\left[\Lambda_{kt}\right] = \begin{bmatrix} -3699.1671 & 0 \\ 0 & 0 \end{bmatrix}, \qquad (118)$$

with the condition that $e_{12} = 0$.

The associated critical load matrix takes the form

$$\left[\Lambda_{cr}\right] = \begin{bmatrix} \frac{4.423EI}{L^2} & 0 \\ 0 & \frac{15.027EI}{L^2} \end{bmatrix} \qquad (119)$$

The resulting mode shapes are shown in Fig. 12 .

3.4A <u>Fixed, Simply Supported-Frame Neglecting Sidesway</u>

In the case where sidesway is neglected, $\Delta = 0$, hence $\theta^*_{AB} = \theta_{AB}$, $\theta^*_{BA} = \theta_{BA}$, $\theta^*_{CD} = \theta_{CD}$, $\theta^*_{DC} = \theta_{DC}$ and equation (110) to the form

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3EI}{\psi L} + \frac{4EI}{L} & -\frac{2EI}{L} \\ -\frac{2EI}{L} & \frac{2\psi}{a}\frac{EI}{L} + \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} \theta_{BC} \\ \theta_{CD} \end{bmatrix} . \qquad (120)$$

For the special case where $P_1 = P_2 = P$, $L_1 = L$ and $I_1 = I$,

the determinant of the reduced stiffness matrix yields

$$12\left(\frac{EI}{L}\right)^2\left[\frac{\psi(2\psi+3)(2\psi+1)+\phi^2(\psi+1)}{\psi(2\psi+\phi)(2\psi-\phi)}\right] = 0 \ . \qquad (121)$$

Noting equation (121) , the following transcendental equations hold:

$$\psi(2\psi+3)(2\psi+1)-\phi(\psi+1) = \frac{1}{\psi} = \frac{1}{2\psi+\phi} = \frac{1}{2\psi-\phi} = 0 \ . \qquad (122)$$

Applying each function of equation (122) to equation (120), the following modal matrix $[U_k]$ is obtained:

$$[U_k] = \begin{bmatrix} 1 & 1 \\ 0.36 & 5.01 \end{bmatrix} \ . \qquad (123)$$

It should be noted that the functions $\frac{1}{\psi} = \frac{1}{2\psi+\phi} = \frac{1}{2\psi-\phi} = 0$ yield the condition $\{\theta_{sr}\} = 0$, hence equation (117) does not hold. The modal matrix $[U_k]$ is obtained from the first transcendental function.

Noting equations (70) and (123) , the modal matrix $[U_f]$ for flexibility becomes

$$[U_f] = \begin{bmatrix} 1 & 1 \\ -0.175 & -2.973 \\ -0.304 & 4.938 \end{bmatrix} \ . \qquad (124)$$

The orthogonality conditions of the mode shapes defined for the stiffness method take the form

$$[U_k]^T [K_{sr}] [U_k] = [K^*_{sr}] = \frac{EI}{L} \begin{bmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{bmatrix} , \qquad (125)$$

where $\eta_{11} = \frac{3}{\psi} + 0.259\frac{\psi}{a} + 3.078$,

$$\eta_{22} = \frac{3}{\psi} + 50.2\frac{\psi}{a} + 84.36 \text{ , and} \qquad (126)$$

$\eta_{12} = \eta_{21} = \frac{3}{\psi} + 3.61\frac{\psi}{a} + 0.474$.

Equating $\eta_{11} = 0$, it follows that, $2u = 3.76$, $\psi = -0.9192$, $\phi = -3.1973$, and $a = -1.141$.

For these values, one obtains

$$[\Lambda_{kt}] = \begin{bmatrix} 0 & 0 \\ 0 & 121.538 \end{bmatrix} , \qquad (127)$$

where it is noted that $\eta_{12} = 0$.

Equating $\eta_{22} = 0$, it follows that, $2u = 5.28$, $\psi = 0.4732$, $\phi = -1.5718$, and $a = -0.2624$.

For these values, one obtains

$$[\Lambda_{kt}] = \begin{bmatrix} 8.9507 & 0 \\ 0 & 0 \end{bmatrix} , \qquad (128)$$

with the condition that $\eta_{12} = 0$.

The associated critical load matrix takes the form

$$\left[\Lambda_{cr}\right] = \begin{bmatrix} \frac{14.138EI}{L^2} & 0 \\ 0 & \frac{27.878EI}{L^2} \end{bmatrix} \tag{129}$$

The associated mode shapes are shown in Fig. 12 .

It is seen that the lowest critical buckling load occurs for the case where sidesway is present and the next higher critical buckling load occurs for the case when sidesway is neglected.

Simple Supports Fixed Supports Fixed, Simple Supports

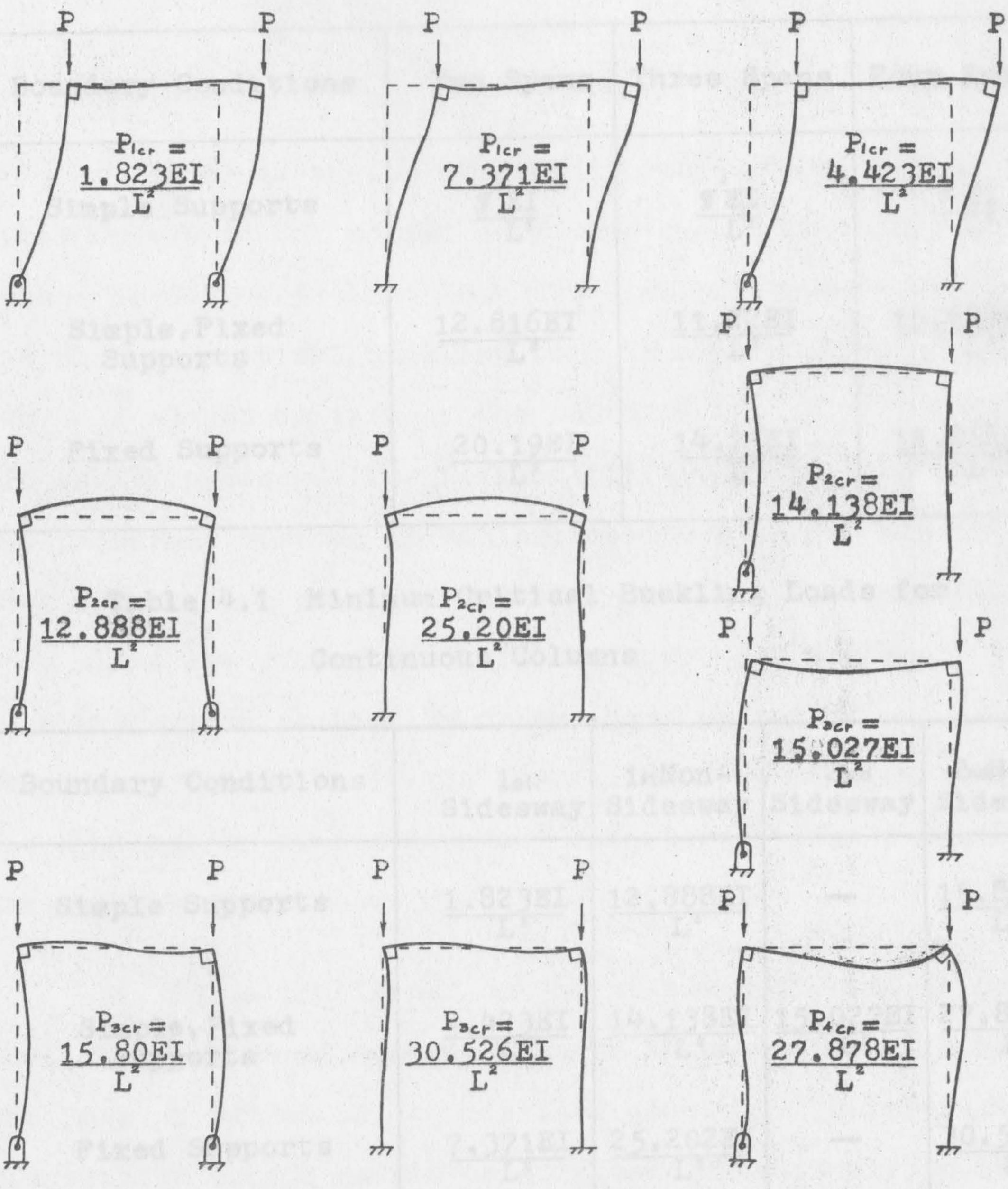


Fig. 12 The Possible Mode Shapes of the Orthogonal Portal Frame

CHAPTER IV

TABULAR RESULTS OF MINIMUM CRITICAL BUCKLING LOADS

Boundary Conditions	Two Spans	Three Spans	Four Spans
Simple Supports	$\frac{\pi^2 EI}{L^2}$	$\frac{\pi^2 EI}{L^2}$	$\frac{\pi^2 EI}{L^2}$
Simple, Fixed Supports	$\frac{12.816EI}{L^2}$	$\frac{11.22EI}{L^2}$	$\frac{10.628EI}{L^2}$
Fixed Supports	$\frac{20.19EI}{L^2}$	$\frac{14.75EI}{L^2}$	$\frac{12.816EI}{L^2}$

Table 4.1 Minimum Critical Buckling Loads for Continuous Columns

Boundary Conditions	1st Sidesway	1st Non-Sidesway	2nd Sidesway	2nd Non-Sidesway
Simple Supports	$\frac{1.823EI}{L^2}$	$\frac{12.888EI}{L^2}$	—	$\frac{15.80EI}{L^2}$
Simple, Fixed Supports	$\frac{4.423EI}{L^2}$	$\frac{14.138EI}{L^2}$	$\frac{15.027EI}{L^2}$	$\frac{27.878EI}{L^2}$
Fixed Supports	$\frac{7.371EI}{L^2}$	$\frac{25.202EI}{L^2}$	—	$\frac{30.526EI}{L^2}$

Table 4.2 Critical Buckling Loads for the Orthogonal Portal Frames

CHAPTER V

DISCUSSION AND CONCLUSIONS

5.1 Discussion

For the static stability problem, considering the axial forces in the column only, the use of the stiffness method is more efficient than the flexibility method, since the minimum critical buckling load is always obtained and the mode shapes defined by the ratio of the joint rotations are easily produced. The flexibility method is also useful, but possesses certain irregularities. One is not assured that the minimum critical buckling load and the associated mode shape are produced. It is, however, not necessary to make the system statically determinant by removal of certain redundant forces, as in the general bending problem since the application of the boundary conditions yield a set of homogeneous equations.

For a continuous column simply supported at both ends, the ratios of bending moments at each support takes the form $\frac{0}{0}$ which is an undefined quantity. Therefore, the flexibility method cannot be utilized conveniently to determine critical buckling loads.

For a continuous column fixed at both ends, the ratios of rotations at each support is equal to $\frac{0}{0}$. Hence, the use of stiffness method is mathematically restricted.

In the case of a continuous column with one end simply supported and the other is fixed, the ratio of bending moments or the ratio of the joint rotations are always a defined values. Thus, both the stiffness method and the flexibility method are equally convenient to use.

For the continuous column-problem, the lowest critical buckling load is always determined by the stiffness method. If the flexibility method is used, the lowest critical buckling load may or may not be determined.

For a orthogonal portal frame, the ratio of joint rotations and the ratio of bending moments are always defined. The lowest critical buckling load occurs for the case where sidesway is present, regardless of the type of boundary conditions. The use of the stiffness method is more efficient, since the lowest critical buckling load and corresponding mode shape are always produced.

5.2 Conclusions

Generally, the stiffness method is more complete, convenient and useful to solve the structural stability problem.

For the simply-supported two, three and four-span continuous columns of equal span lengths, one obtains the same lowest critical buckling load $P_{cr} = \frac{\pi^2 EI}{L^2}$ for each case.

The value of the lowest critical buckling loads for a continuous column fixed at both ends and having equal span lengths are determined for the two, three, and four-span geometry respectively as $P_{cr_2} = \frac{20.19EI}{L^2}$, $P_{cr_3} = \frac{14.75EI}{L^2}$, and $P_{cr_4} = \frac{12.816EI}{L^2}$. It is seen that as the number of equal length spans increase, the lowest critical load decreases.

For the case of a continuous columns fixed at one end and simply-supported at the other, the minimum critical buckling loads for the two, three, and four-span conditions become $P_{cr_2} = \frac{12.816EI}{L^2}$, $P_{cr_3} = \frac{11.22EI}{L^2}$, and $P_{cr_4} = \frac{10.628EI}{L^2}$. Hence, as the number of equal length spans increase, the value of the minimum critical buckling load decreases.

As the degree of fixity of the continuous column increase, the value of the lowest critical buckling load increases.

For the orthogonal portal frames, the value of the lowest critical buckling load always occurs for the case where sidesway is present. The next higher critical buckling load usually occurs as the first non-sidesway mode. The lowest critical buckling loads for the cases of simple supports at both ends, simple-fixed supports and fixed supports at both ends are respectively, $P_{1cr} = \frac{1.823EI}{L^2}$, $P_{2cr} = \frac{4.423EI}{L^2}$, and $P_{3cr} = \frac{7.371EI}{L^2}$.

For a simply supported column, the value of the Euler load is given as $P_e = \frac{\pi^2 EI}{L^2}$. For the simply supported frame, the lowest value of P_{cr} is only 18.47 % of the Euler buckling load. That is, the critical buckling load is reduce by 81.53 %.

For the case of fixed-simply supported frame, the lowest value of P_{cr} is 44.81 % of the Euler buckling load.

For the case of fixed-fixed supported frame, the lowest value of P_{cr} is 74.66 % of the Euler buckling load.

APPENDIX A

SUMMARY OF THE THREE AND TWO SPAN-CONTINUOUS COLUMN

A. 1 <u>Three-Span Column</u>

Stiffness Matrix

$$\begin{bmatrix} M_{AB} \\ M_{BA} \\ M_{BC} \\ M_{CB} \\ M_{CD} \\ M_{DC} \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} \frac{2\psi}{\alpha} & -\frac{\phi}{\alpha} & 0 & 0 & 0 & 0 \\ -\frac{\phi}{\alpha} & \frac{2\psi}{\alpha} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2\psi}{\alpha} & -\frac{\phi}{\alpha} & 0 & 0 \\ 0 & 0 & -\frac{\phi}{\alpha} & \frac{2\psi}{\alpha} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2\psi}{\alpha} & -\frac{\phi}{\alpha} \\ 0 & 0 & 0 & 0 & -\frac{\phi}{\alpha} & \frac{2\psi}{\alpha} \end{bmatrix} \begin{bmatrix} \Theta_{AB} \\ \Theta_{BA} \\ \Theta_{BC} \\ \Theta_{CB} \\ \Theta_{CD} \\ \Theta_{DC} \end{bmatrix}.$$

Flexibility Matrix

$$\begin{bmatrix} \Theta_{AB} \\ \Theta_{BA} \\ \Theta_{BC} \\ \Theta_{CB} \\ \Theta_{CD} \\ \Theta_{DC} \end{bmatrix} = \frac{L}{6EI} \begin{bmatrix} 2\psi & \phi & 0 & 0 & 0 & 0 \\ \phi & 2\psi & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\psi & \phi & 0 & 0 \\ 0 & 0 & \phi & 2\psi & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\psi & \phi \\ 0 & 0 & 0 & 0 & \phi & 2\psi \end{bmatrix} \begin{bmatrix} M_{AB} \\ M_{BA} \\ M_{BC} \\ M_{CB} \\ M_{CD} \\ M_{DC} \end{bmatrix}.$$

Six boundary conditions

Simple Supports

$$M_{AB} = 0 ,$$

$$M_{BA} = M_{BC} = M_B ,$$

$$M_{CB} = M_{CD} = M_C ,$$

$$M_{DC} = 0 ,$$

$$\Theta_{BA} = -\Theta_{BC} = \Theta_B \text{ ,and}$$

$$\Theta_{CB} = -\Theta_{CD} = \Theta_C .$$

Fixed Supports

$$M_{BA} = M_{BC} = M_B ,$$

$$M_{CB} = M_{CD} = M_C ,$$

$$\Theta_{AB} = 0 ,$$

$$\Theta_{BA} = -\Theta_{BC} = \Theta_B ,$$

$$\Theta_{CB} = -\Theta_{CD} = \Theta_C \text{ ,and}$$

$$\Theta_{DC} = 0 .$$

Fixed, Simple Supports

$$M_{AB} = 0 ,$$

$$M_{BA} = M_{BC} = M_B ,$$

$$M_{CB} = M_{CD} = M_C ,$$

$$\Theta_{BA} = -\Theta_{BC} = \Theta_B ,$$

$$\Theta_{CB} = -\Theta_{CD} = \Theta_C \text{ ,and}$$

$$\Theta_{DC} = 0 .$$

Stiffness Method

Simple Supports — Fixed Supports — Fixed, Simple Supports

Applying the boundary conditions, one obtains

Simple Supports

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} \frac{2\psi}{a} & -\frac{\phi}{a} & 0 & 0 \\ -\frac{\phi}{a} & \frac{4\psi}{a} & \frac{\phi}{a} & 0 \\ 0 & \frac{\phi}{a} & \frac{4\psi}{a} & \frac{\phi}{a} \\ 0 & 0 & \frac{\phi}{a} & \frac{2\psi}{a} \end{bmatrix} \begin{bmatrix} \theta_{AB} \\ \theta_B \\ \theta_C \\ \theta_{DC} \end{bmatrix} .$$

Fixed Supports

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} \frac{4\psi}{a} & \frac{\phi}{a} \\ \frac{\phi}{a} & \frac{4\psi}{a} \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} .$$

Fixed, Simple Supports

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} \frac{2\psi}{a} & -\frac{\phi}{a} & 0 \\ -\frac{\phi}{a} & \frac{4\psi}{a} & \frac{\phi}{a} \\ 0 & \frac{\phi}{a} & \frac{4\psi}{a} \end{bmatrix} \begin{bmatrix} \theta_{AB} \\ \theta_B \\ \theta_C \end{bmatrix} .$$

The determinant of the reduced stiffness matrix yields

$$\frac{(4\psi-\phi)(4\psi+\phi)}{(2\psi+\phi)(2\psi-\phi)} = 0 .$$

$$(4\psi-\phi)(4\psi+\phi) = 0 .$$

$$\psi(4\psi-\sqrt{3})(4\psi+\sqrt{3}) = 0 .$$

Simple Supports

The transcendental equations become:

$$\frac{1}{2\psi+\phi} = 0 \;.$$

$$4\psi-\phi = 0 \;,$$

$$4\psi+\phi = 0 \;,\text{and}$$

$$\frac{1}{2\psi-\phi} = 0 \;.$$

The modal matrix $\left[U_k\right]$ takes the form

$$\left[U_k\right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} & -\frac{1}{2} & -1 \\ -1 & \frac{1}{2} & \frac{1}{2} & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} .$$

Fixed Supports

$$4\psi-\phi = 0 \;,\text{and}$$

$$4\psi+\phi = 0 \;.$$

$$\left[U_k\right] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} .$$

Fixed, Simple Supports

$$4\psi-\sqrt{3} = 0 \;,$$

$$\psi = 0 \;,\text{and}$$

$$4\psi+\sqrt{3} = 0 .$$

$$\left[U_k\right] = \begin{bmatrix} 1 & 1 & 1 \\ \frac{\sqrt{3}}{2} & 0 & -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix} .$$

Simple Supports | Fixed Supports | Fixed, Simple Supports

The modal matrix $[U_f]$ becomes

$$[U_f] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad [U_f] = \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \\ 1 & -1 \end{bmatrix}. \qquad [U_f] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ -\sqrt{3} & 0 & \sqrt{3} \\ 2 & -1 & 2 \end{bmatrix}.$$

The orthogonality conditions are $[\Lambda_{kt}] = [U_k]^T [K_{sr}][U_k]$, or

$$[\Lambda_{kt}] = \begin{bmatrix} \frac{1}{2\psi+\phi} & 0 & 0 & 0 \\ 0 & 4\psi-\phi & 0 & 0 \\ 0 & 0 & 4\psi+\phi & 0 \\ 0 & 0 & 0 & \frac{1}{2\psi-\phi} \end{bmatrix}. \qquad [\Lambda_{kt}] = \begin{bmatrix} 4\psi-\phi & 0 \\ 0 & 4\psi+\phi \end{bmatrix}. \qquad [\Lambda_{kt}] = \begin{bmatrix} 4\psi-\sqrt{3} & 0 & 0 \\ 0 & \psi & 0 \\ 0 & 0 & 4\psi+\sqrt{3} \end{bmatrix}.$$

Simple Supports　　　　Fixed Supports　　　　Fixed, Simple Supports

The critical buckling loads reduce to

Simple Supports:

$$\left[\Lambda_{cr}\right] = \begin{bmatrix} \frac{(n\pi)^2 EI}{L^2}, n=1,3,5.. & 0 & 0 & 0 \\ 0 & \frac{14.75EI}{L^2} & 0 & 0 \\ 0 & 0 & \frac{26.52EI}{L^2} & 0 \\ 0 & 0 & 0 & \frac{(n\pi)^2 EI}{L^2}, n=2,4,6.. \end{bmatrix}$$

Fixed Supports:

$$\left[\Lambda_{cr}\right] = \begin{bmatrix} \frac{14.75EI}{L^2} & 0 \\ 0 & \frac{26.52EI}{L^2} \end{bmatrix}$$

Fixed, Simple Supports:

$$\left[\Lambda_{cr}\right] = \begin{bmatrix} \frac{11.22EI}{L^2} & 0 & 0 \\ 0 & \frac{20.19EI}{L^2} & 0 \\ 0 & 0 & \frac{32.89EI}{L^2} \end{bmatrix}$$

<u>Flexibility Method</u>

Simple Supports   Fixed Supports   Fixed, Simple Supports

Applying the boundary conditions, one obtains

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{L}{6EI} \begin{bmatrix} 4\psi & \phi \\ \phi & 4\psi \end{bmatrix} \begin{bmatrix} M_B \\ M_C \end{bmatrix} .$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{L}{6EI} \begin{bmatrix} 2\psi & \phi & 0 & 0 \\ \phi & 4\psi & \phi & 0 \\ 0 & 0 & 4\psi & \phi \\ 0 & 0 & \phi & 2\psi \end{bmatrix} \begin{bmatrix} M_{AB} \\ M_B \\ M_C \\ M_{DC} \end{bmatrix} .$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \frac{L}{6EI} \begin{bmatrix} 4\psi & \phi & 0 \\ \phi & 4\psi & \phi \\ 0 & \phi & 2\psi \end{bmatrix} \begin{bmatrix} M_B \\ M_C \\ M_{DC} \end{bmatrix} .$$

The determinant of the reduced flexibility matrix yields

$$(4\psi-\phi)(4\psi+\phi) = 0 .$$

$$(4\psi-\phi)(4\psi+\phi)(2\psi+\phi)(2\psi-\phi) = 0 .$$

$$\psi(4\psi-\sqrt{3})(4\psi+\sqrt{3}) = 0 .$$

Simple Supports	Fixed Supports	Fixed, Simple Supports

The transcendental equations become

Simple Supports

$$4\psi - \phi = 0 \text{ , and}$$
$$4\psi + \phi = 0 \text{ .}$$

Fixed Supports

$$4\psi - \phi = 0 \text{ ,}$$
$$4\psi + \phi = 0 \text{ ,}$$
$$2\psi + \phi = 0 \text{ , and}$$
$$2\psi - \varphi = 0 \text{ .}$$

Fixed, Simple Supports

$$4\psi - \sqrt{3} = 0 \text{ ,}$$
$$\psi = 0 \text{ , and}$$
$$4\psi + \sqrt{3} = 0 \text{ .}$$

The modal matrix $[U_f]$ takes the form

Simple Supports

$$[U_f] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} .$$

Fixed Supports

$$[U_f] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -\frac{1}{2} & \frac{1}{2} & 1 & -1 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} .$$

Fixed, Simple Supports

$$[U_f] = \begin{bmatrix} 1 & 1 & 1 \\ -\sqrt{3} & 0 & \sqrt{3} \\ 2 & -1 & 2 \end{bmatrix} .$$

Simple Supports | Fixed Supports | Fixed, Simple Supports

The modal matrix $\left[U_k\right]$ becomes

$$\left[U_k\right] = \begin{bmatrix} 1 & 1 \\ 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \\ 1 & -1 \end{bmatrix} . \qquad \left[U_k\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \qquad \left[U_k\right] = \begin{bmatrix} 1 & 1 & 1 \\ \frac{\sqrt{3}}{2} & 0 & -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} .$$

The orthogonality conditions are $\left[\Lambda_{f_t}\right] = \left[U_f\right]^T \left[F_{sr}\right]\left[U_f\right]$, or

$$\left[\Lambda_{f_t}\right] = \begin{bmatrix} 4\psi-\phi & 0 \\ 0 & 4\psi+\phi \end{bmatrix} . \qquad \left[\Lambda_{f_t}\right] = \begin{bmatrix} 4\psi-\phi & 0 & 0 & 0 \\ 0 & 4\psi+\phi & 0 & 0 \\ 0 & 0 & 2\psi+\phi & 0 \\ 0 & 0 & 0 & 2\psi-\phi \end{bmatrix} . \qquad \left[\Lambda_{f_t}\right] = \begin{bmatrix} 4\psi-\sqrt{3} & 0 & 0 \\ 0 & \psi & 0 \\ 0 & 0 & 4\psi+\sqrt{3} \end{bmatrix} .$$

Simple Supports Fixed Supports Fixed, Simple Supports

The critical buckling loads reduce to

$$[\Lambda_{cr}] = \begin{bmatrix} \frac{14.75EI}{L^2} & 0 \\ 0 & \frac{26.52EI}{L^2} \end{bmatrix}.$$

$$[\Lambda_{cr}] = \begin{bmatrix} \frac{11.22EI}{L^2} & 0 & 0 \\ 0 & \frac{20.19EI}{L^2} & 0 \\ 0 & 0 & \frac{32.89EI}{L^2} \end{bmatrix}.$$

$$[\Lambda_{cr}] = \begin{bmatrix} \frac{14.75EI}{L^2} & 0 & 0 & 0 \\ 0 & \frac{26.52EI}{L^2} & 0 & 0 \\ 0 & 0 & \frac{(2n\pi)^2EI}{L^2}, n=1,2,3.. & 0 \\ 0 & 0 & 0 & \frac{80.748EI}{L^2} \end{bmatrix}.$$

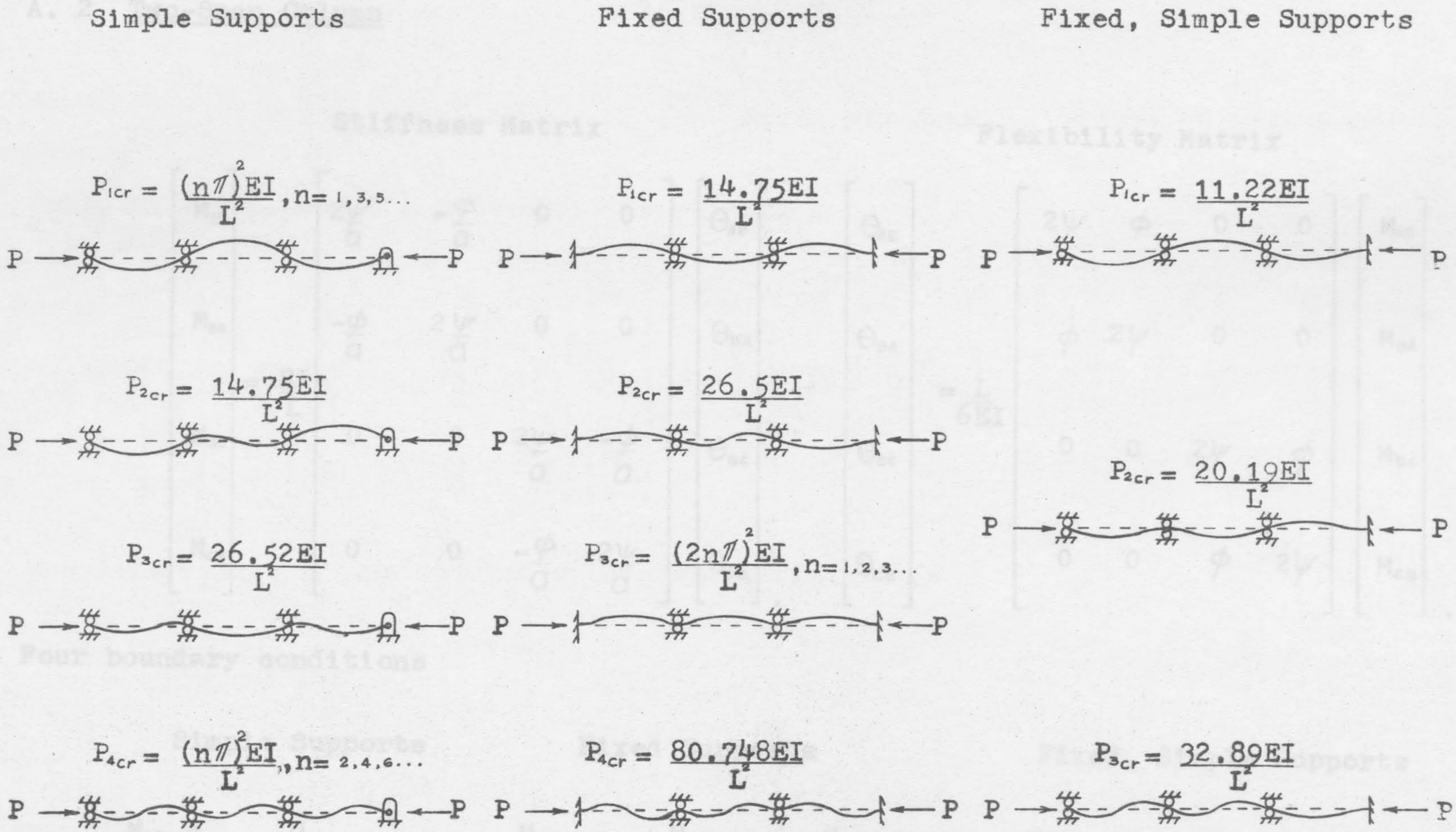


Fig. A.1 The Possible Mode Shapes of Three-Span Column

A. 2 Two-Span Column

Stiffness Matrix

$$\begin{bmatrix} M_{AB} \\ M_{BA} \\ M_{BC} \\ M_{CB} \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} \frac{2\psi}{a} & -\frac{\phi}{a} & 0 & 0 \\ -\frac{\phi}{a} & \frac{2\psi}{a} & 0 & 0 \\ 0 & 0 & \frac{2\psi}{a} & -\frac{\phi}{a} \\ 0 & 0 & -\frac{\phi}{a} & \frac{2\psi}{a} \end{bmatrix} \begin{bmatrix} \theta_{AB} \\ \theta_{BA} \\ \theta_{BC} \\ \theta_{CB} \end{bmatrix} .$$

Flexibility Matrix

$$\begin{bmatrix} \theta_{AB} \\ \theta_{BA} \\ \theta_{BC} \\ \theta_{CB} \end{bmatrix} = \frac{L}{6EI} \begin{bmatrix} 2\psi & \phi & 0 & 0 \\ \phi & 2\psi & 0 & 0 \\ 0 & 0 & 2\psi & \phi \\ 0 & 0 & \phi & 2\psi \end{bmatrix} \begin{bmatrix} M_{AB} \\ M_{BA} \\ M_{BC} \\ M_{CB} \end{bmatrix} .$$

Four boundary conditions

Simple Supports

$M_{AB} = 0$,

$M_{BA} = M_{BC} = M_B$,

$M_{CB} = 0$, and

$\theta_{BA} = -\theta_{BC} = \theta_B$.

Fixed Supports

$M_{BA} = M_{BC} = M_B$,

$\theta_{AB} = 0$,

$\theta_{BA} = -\theta_{BC} = \theta_B$, and

$\theta_{CB} = 0$.

Fixed, Simple Supports

$M_{AB} = 0$,

$M_{BA} = M_{BC} = M_B$,

$\theta_{BA} = -\theta_{BC} = \theta_B$, and

$\theta_{CB} = 0$.

Stiffness Method

Simple Supports	Fixed Supports	Fixed, Simple Supports

Applying the boundary conditions, one obtains

Simple Supports

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} \frac{2\psi}{a} & -\frac{\phi}{a} & 0 \\ -\frac{\phi}{a} & \frac{4\psi}{a} & \frac{\phi}{a} \\ 0 & \frac{\phi}{a} & \frac{2\psi}{a} \end{bmatrix} \begin{bmatrix} \theta_{AB} \\ \theta_{B} \\ \theta_{CB} \end{bmatrix} .$$

Fixed Supports

$$\begin{bmatrix} 0 \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} \frac{4\psi}{a} \end{bmatrix} \begin{bmatrix} \theta_{B} \end{bmatrix} .$$

Fixed, Simple Supports

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} \frac{2\psi}{a} & -\frac{\phi}{a} \\ -\frac{\phi}{a} & \frac{4\psi}{a} \end{bmatrix} \begin{bmatrix} \theta_{AB} \\ \theta_{B} \end{bmatrix} .$$

The determinant of the reduced stiffness matrix yields

Simple Supports

$$\frac{\psi}{(2\psi+\phi)(2\psi-\phi)} = 0 .$$

Fixed Supports

$$\psi = 0 .$$

Fixed, Simple Supports

$$(2\sqrt{2}\psi-\phi)(2\sqrt{2}\psi+\phi) = 0 .$$

Simple Supports	Fixed Supports	Fixed, Simple Supports

The transcendental equations become

Simple Supports:

$$\frac{1}{2\psi+\phi} = 0 \,,$$

$$\psi = 0 \,, \text{and}$$

$$\frac{1}{2\psi-\phi} = 0 \,.$$

Fixed Supports:

$$\psi = 0 \,.$$

Fixed, Simple Supports:

$$2\sqrt{2}\psi-\phi = 0 \,, \text{and}$$

$$2\sqrt{2}\psi+\phi = 0 \,.$$

The modal matrix $\left[U_k\right]$ takes the form

Simple Supports:

$$\left[U_k\right] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & -1 \end{bmatrix} .$$

Fixed Supports:

$$\left[U_k\right] = \left[\,1\,\right] .$$

Fixed, Simple Supports:

$$\left[U_k\right] = \begin{bmatrix} 1 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} .$$

Simple Supports	Fixed Supports	Fixed, Simple Supports

The modal matrix $\left[U_f\right]$ becomes

Simple Supports

$$\left[U_f\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Fixed Supports

$$\left[U_f\right] = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} .$$

Fixed, Simple Supports

$$\left[U_f\right] = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ -\sqrt{2} & \sqrt{2} \end{bmatrix} .$$

The orthogonality conditions are $\left[\Lambda_{kt}\right] = \left[U_k\right]^T \left[K_{sr}\right]\left[U_k\right]$, or

Simple Supports

$$\left[\Lambda_{kt}\right] = \begin{bmatrix} \frac{1}{2\psi+\phi} & 0 & 0 \\ 0 & \psi & 0 \\ 0 & 0 & \frac{1}{2\psi-\phi} \end{bmatrix} .$$

Fixed Supports

$$\left[\Lambda_{kt}\right] = \left[\psi\right] .$$

Fixed, Simple Supports

$$\left[\Lambda_{kt}\right] = \begin{bmatrix} 2\sqrt{2}\psi-\phi & 0 \\ 0 & 2\sqrt{2}\psi+\phi \end{bmatrix} .$$

The critical buckling loads reduce to

Simple Supports

$$\left[\Lambda_{cr}\right] = \begin{bmatrix} \frac{(n\pi)^2 EI}{L^2}, n=1,3,5.. & 0 & 0 \\ 0 & \frac{20.19EI}{L^2} & 0 \\ 0 & 0 & \frac{(n\pi)^2 EI}{L^2}, n=2,4,6.. \end{bmatrix}.$$

Fixed Supports

$$\left[\Lambda_{cr}\right] = \left[\frac{20.19EI}{L^2}\right].$$

Fixed, Simple Supports

$$\left[\Lambda_{cr}\right] = \begin{bmatrix} \frac{12.816EI}{L^2} & 0 \\ 0 & \frac{29.703EI}{L^2} \end{bmatrix}.$$

Flexibility Method

Simple Supports — Fixed Supports — Fixed, Simple Supports

Applying the boundary conditions, one obtains

Simple Supports:

$$\begin{bmatrix} 0 \end{bmatrix} = \frac{L}{6EI}\begin{bmatrix} 4\psi \end{bmatrix}\begin{bmatrix} M_B \end{bmatrix}.$$

Fixed Supports:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \frac{L}{6EI}\begin{bmatrix} 2\psi & \phi & 0 \\ \phi & 4\psi & \phi \\ 0 & \phi & 2\psi \end{bmatrix}\begin{bmatrix} M_{AB} \\ M_B \\ M_{CB} \end{bmatrix}.$$

Fixed, Simple Supports:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{L}{6EI}\begin{bmatrix} 4\psi & \phi \\ \phi & 2\psi \end{bmatrix}\begin{bmatrix} M_B \\ M_{CB} \end{bmatrix}.$$

The determinant of the reduced flexibility matrix yields

$$\psi = 0 .$$

$$\psi(2\psi+\phi)(2\psi-\phi) = 0 .$$

$$(2\sqrt{2}\psi-\phi)(2\sqrt{2}\psi+\phi) = 0 .$$

Simple Supports | Fixed Supports | Fixed, Simple Supports

The transcendental equations become

Simple Supports

$$\psi = 0 \, .$$

Fixed Supports

$$\psi = 0 \, ,$$

$$2\psi + \phi = 0 \, , \text{ and}$$

$$2\psi - \phi = 0 \, .$$

Fixed, Simple Supports

$$2\sqrt{2}\psi - \phi = 0 \, , \text{ and}$$

$$2\sqrt{2}\psi + \phi = 0 \, .$$

The modal matrix $\left[U_f\right]$ takes the form

Simple Supports

$$\left[U_f\right] = \left[1\right] .$$

Fixed Supports

$$\left[U_f\right] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} .$$

Fixed, Simple Supports

$$\left[U_f\right] = \begin{bmatrix} 1 & 1 \\ -\sqrt{2} & \sqrt{2} \end{bmatrix} .$$

Simple Supports | Fixed Supports | Fixed, Simple Supports

The modal matrix $\left[U_k\right]$ becomes

$$\left[U_k\right] = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}. \qquad \left[U_k\right] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad \left[U_k\right] = \begin{bmatrix} 1 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \end{bmatrix}.$$

The orthogonality conditions are $\left[\Lambda_{f_t}\right] = \left[U_f\right]^T \left[F_{sr}\right]\left[U_f\right]$, or

$$\left[\Lambda_{f_t}\right] = \left[\psi\right]. \qquad \left[\Lambda_{f_t}\right] = \begin{bmatrix} \psi & 0 & 0 \\ 0 & 2\psi+\phi & 0 \\ 0 & 0 & 2\psi-\phi \end{bmatrix}. \qquad \left[\Lambda_{f_t}\right] = \begin{bmatrix} 2\sqrt{2}\psi-\phi & 0 \\ 0 & 2\sqrt{2}\psi+\phi \end{bmatrix}.$$

Simple Supports Fixed Supports Fixed, Simple Supports

The critical buckling loads reduce to

$$\left[\Lambda_{cr}\right] = \left[\frac{20.19EI}{L^2}\right].$$

$$\left[\Lambda_{cr}\right] = \begin{bmatrix} \frac{20.19EI}{L^2} & & 0 \\ 0 & \frac{(2n\pi)^2 EI}{L^2}, n=1,2,3.. & 0 \\ 0 & 0 & \frac{80.748EI}{L^2} \end{bmatrix}.$$

$$\left[\Lambda_{cr}\right] = \begin{bmatrix} \frac{12.816EI}{L^2} & 0 \\ 0 & \frac{29.703EI}{L^2} \end{bmatrix}.$$

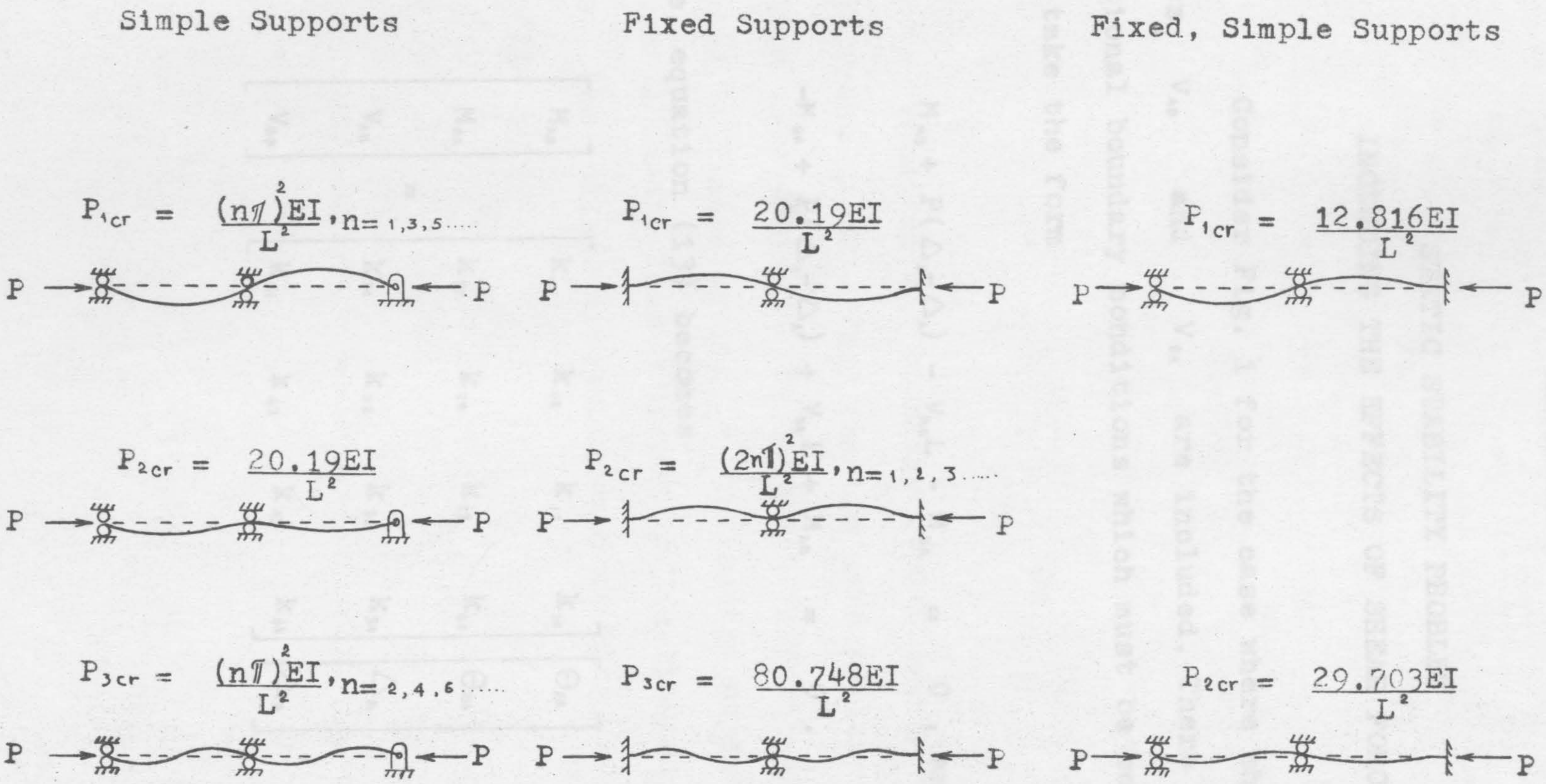


Fig. A.2 The Possible Mode Shapes of Two Span-Column

APPENDIX B

STATIC STABILITY PROBLEM
INCLUDING THE EFFECTS OF SHEAR FORCE

Consider Fig. 1 for the case where shearing forces V_{AB} and V_{BA} are included. There are two additional boundary conditions which must be considered; they take the form

$$\left.\begin{aligned} M_{AB} + P(\Delta_B - \Delta_A) - V_{AB}L - M_{BA} &= 0 \text{ , and} \\ -M_{BA} + P(\Delta_B - \Delta_A) + V_{BA}L + M_{AB} &= 0 \text{ .} \end{aligned}\right] \quad \text{(B-1)}$$

Hence, equation (13) becomes

$$\begin{bmatrix} M_{AB} \\ M_{BA} \\ V_{AB} \\ V_{BA} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} \Theta_{AB} \\ \Theta_{BA} \\ \Delta_A \\ \Delta_B \end{bmatrix} , \quad \text{(B-2)}$$

$$\left.\begin{aligned}
\text{where}\quad k_{11} &= k_{22} = \frac{2\psi}{a}\frac{EI}{L} ,\\
k_{12} &= k_{21} = -\frac{\phi}{a}\frac{EI}{L} ,\\
k_{13} &= k_{31} = -k_{14} = -k_{41}\\
-k_{23} &= -k_{32} = k_{24} = k_{42}\\
&= \left(\frac{2\psi+\phi}{a}\right)\frac{EI}{L^2} , \quad \text{and}\\
k_{33} &= -k_{34} = -k_{43} = k_{44}\\
&= \left\{2\,\frac{(2\psi+\phi)}{a} - (2u)^2\right\}\frac{EI}{L^3} .
\end{aligned}\right\}\quad (B\text{-}3)$$

Consider the boundary conditions of the orthogonal, portal frame ABCD . They are the same as those given in equation (69) , except two linear springs k_1 and k_2 are included. A linear spring k_1 is located horizontally at support A and a linear spring k_2 is located horizontally at D , two rollers are positioned at both A and D (see Fig. B.1) .

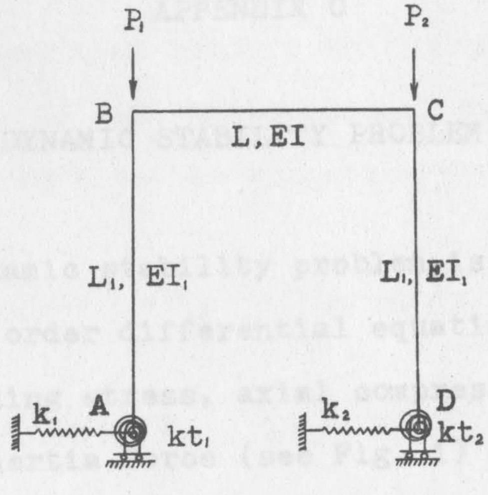


Fig. B.1 Orthogonal Portal Frame
Including Linear Springs

The eight boundary conditions are

$$
\left.
\begin{aligned}
M_{AB} &= kt_1 \Theta_{AB} , \\
M_{BA} &= M_{BC} = M_B , \\
M_{CB} &= -M_{CD} = M_C , \\
M_{DC} &= kt_2 \Theta_{DC} , \\
-\Theta_{BC} &= \Theta_{BA} , \\
\Theta_{CB} &= \Theta_{CD} , \\
EI\, y_1'''(0) + P_1 y_1'(0) &= -V_{AB}(0) = -k_1 \Delta_A , \text{ and} \\
EI\, y_2'''(0) + P_2 y_2'(0) &= -V_{DC}(0) = -k_2 \Delta_D .
\end{aligned}
\right] \qquad \text{(B-4)}
$$

APPENDIX C

DYNAMIC STABILITY PROBLEM

The dynamic stability problem is formulated from the basic fourth order differential equation of a column subjected to bending stress, axial compressive forces P and transverse inertia force (see Fig. 1) as

$$y_x^{iv}(x,t) + \frac{P}{EI} y_x''(x,t) + \frac{\rho A}{EI} y_t''(x,t) = 0 , \qquad \text{(C-1)}$$

where E , I , P , A , and ρ are assumed constant.

The above equation is based upon the same five assumptions given in Chapter I , the general solution of equation (C-1) becomes

$$y(x) = A \cosh \gamma x + B \sinh \gamma x + C \cos \delta x + D \sin \delta x , \qquad \text{(C-2)}$$

where

$$\left.\begin{aligned} \gamma &= \left[-\frac{k^2}{2} + \left\{ \left(\frac{k^2}{2}\right)^2 + \lambda^4 \right\}^{\frac{1}{2}} \right]^{\frac{1}{2}} , \\ \delta &= \left[\frac{k^2}{2} + \left\{ \left(\frac{k^2}{2}\right)^2 + \lambda^4 \right\}^{\frac{1}{2}} \right]^{\frac{1}{2}} , \\ \lambda^4 &= \frac{\rho P A \Omega^2}{EI} , \text{ and} \end{aligned}\right] \qquad \text{(C-3)}$$

Ω is the natural frequency of free vibration of the beam-column.

The column must satisfy the six boundary conditions given in equation (3) and two additional transverse shearing force-conditions for the free vibration problem given as

$$\left.\begin{aligned} M_{AB} + P(\triangle_B - \triangle_A) - V_{BA}L - M_{BA} + \rho A \Omega^2 \int_0^L xy(x)\,dx &= 0 \quad , \\ \text{and} \quad V_{AB} + V_{BA} - \rho A \Omega^2 \int_0^L y(x)\,dx &= 0 \quad . \end{aligned}\right] \quad (C\text{-}4)$$

The constants A, B, C and D of equation (C-2), determinined directly by using the first four boundary conditions given in equation (3), become

$$\left.\begin{aligned} A &= \frac{1}{\gamma^2+\delta^2}\left(\delta^2\triangle_A - \frac{M_{AB}}{EI}\right) \quad , \\ B &= \frac{1}{(\gamma^2+\delta^2)\sinh\gamma L}\left\{\delta^2\triangle_B - \frac{M_{BA}}{EI} + \left(\frac{M_{AB}}{EI} - \delta^2\triangle_A\right)\cosh\gamma L\right\} \\ C &= \frac{1}{\gamma^2+\delta^2}\left(\gamma^2\triangle_A + \frac{M_{AB}}{EI}\right) \quad , \quad \text{and} \\ D &= \frac{1}{(\gamma^2+\delta^2)\sin\delta L}\left\{\gamma^2\triangle_B + \frac{M_{BA}}{EI} - \left(\frac{M_{AB}}{EI} + \gamma^2\triangle_A\right)\cos\delta L\right\} \end{aligned}\right] \quad (C\text{-}5)$$

Combining equations (C-2) and (C-5) together with the last two boundary conditions given in equation (3) and equation (C-4), it follows that,

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & 1 & 0 \\ a_{41} & a_{42} & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{M_{AB}}{EI} \\ \frac{M_{BA}}{EI} \\ \frac{V_{AB}L}{EI} \\ \frac{V_{BA}L}{EI} \end{bmatrix} = \begin{bmatrix} 1 & 0 & b_{13} & b_{14} \\ 0 & 1 & b_{23} & b_{24} \\ 0 & 0 & b_{33} & b_{34} \\ 0 & 0 & b_{43} & b_{44} \end{bmatrix} \begin{bmatrix} \Theta_{AB} \\ \Theta_{BA} \\ \Delta_{A} \\ \Delta_{B} \end{bmatrix}, \qquad \text{(C-6)}$$

where

$$a_{11} = a_{22} = \frac{\gamma \cosh \gamma L}{(\gamma^2+\delta^2)\sinh \gamma L} - \frac{\delta \cos \delta L}{(\gamma^2+\delta^2)\sin \delta L},$$

$$a_{12} = a_{21} = -\frac{\gamma}{(\gamma^2+\delta^2)\sinh \gamma L} + \frac{\delta}{(\gamma^2+\delta^2)\sin \delta L},$$

$$a_{31} = a_{42} = 1 + \frac{(\gamma\delta)^2}{\gamma^2+\delta^2}\left(\frac{L}{\gamma}\frac{\cosh \gamma L}{\sinh \gamma L} - \frac{1}{\gamma^2} + \frac{L}{\delta}\frac{\cos \delta L}{\sin \delta L} - \frac{1}{\delta^2}\right),$$

$$a_{32} = a_{41} = -1 - \frac{(\gamma\delta)^2}{\gamma^2+\delta^2}\left(\frac{L}{\gamma}\frac{1}{\sinh \gamma L} - \frac{1}{\gamma^2} + \frac{L}{\delta}\frac{1}{\sin \delta L} - \frac{1}{\delta^2}\right),$$

$$b_{13} = b_{24} = \frac{\gamma\delta^2 \cosh \gamma L}{(\gamma^2+\delta^2)\sinh \gamma L} + \frac{\gamma^2\delta \cos \delta L}{(\gamma^2+\delta^2)\sin \delta L},$$

$$b_{14} = b_{23} = -\frac{\gamma\delta^2}{(\gamma^2+\delta^2)\sinh \gamma L} - \frac{\gamma^2\delta}{(\gamma^2+\delta^2)\sin \delta L},$$

$$b_{33} = b_{44} = \left(\frac{2u}{L}\right)^2 + \frac{(\gamma\delta)^2}{\gamma^2+\delta^2}\left\{\delta^2\left(\frac{L}{\gamma}\frac{\cosh \gamma L}{\sinh \gamma L} - \frac{1}{\gamma^2}\right) - \gamma^2\left(\frac{L}{\delta}\frac{\cos \delta L}{\sin \delta L} - \frac{1}{\delta^2}\right)\right\},$$

and

$$b_{34} = b_{43} = -\left(\frac{2u}{L}\right)^2 - \frac{(\gamma\delta)^2}{\gamma^2+\delta^2}\left\{\delta^2\left(\frac{L}{\gamma}\frac{1}{\sinh \gamma L} - \frac{1}{\gamma^2}\right) - \gamma^2\left(\frac{L}{\delta}\frac{1}{\sin \delta L} - \frac{1}{\delta^2}\right)\right\}. \qquad \text{(C-7)}$$

Equation (C-6) is rewritten in the following symbolic matrix form

$$A\{m\} = B\{\Theta\} \quad . \qquad \text{(C-8)}$$

Premultiplying equation (C-8) by A^{-1}, it follows that

$$\{m\} = A^{-1}B\{\Theta\} \equiv [K_{Dm}]\{\Theta\} . \qquad \text{(C-9)}$$

The matrix $[K_{Dm}]$ is defined as the stiffness matrix for a single member.

Noting equation (C-6) and (C-9), one obtains the component matrix form as

$$\begin{bmatrix} M_{AB} \\ M_{BA} \\ V_{AB} \\ V_{BA} \end{bmatrix} = \frac{1}{a_{11}^2 - a_{21}^2} \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} \Theta_{AB} \\ \Theta_{BA} \\ \Delta_A \\ \Delta_B \end{bmatrix} , \qquad \text{(C-10)}$$

where

$$\left.\begin{aligned}
k_{11} &= k_{22} = a_{11}EI , \\
k_{12} &= k_{21} = -a_{12}EI , \\
k_{13} &= k_{24} = (a_{11}b_{13} - a_{12}b_{14})EI , \\
k_{14} &= k_{23} = (a_{11}b_{14} - a_{12}b_{13})EI , \\
k_{31} &= k_{42} = (a_{12}a_{32} - a_{11}a_{31})\frac{EI}{L} , \\
k_{32} &= k_{41} = (a_{12}a_{31} - a_{11}a_{32})\frac{EI}{L} , \\
k_{33} &= k_{44} = \Big\{ b_{13}(a_{11}a_{31} - a_{12}a_{32}) + b_{14}(a_{11}a_{32} - a_{12}a_{31}) \\
&\qquad + b_{33}(a_{11}^2 - a_{12}^2) \Big\} \frac{EI}{L} , \quad \text{and} \\
k_{34} &= k_{43} = \Big\{ b_{14}(a_{11}a_{31} - a_{12}a_{32}) + b_{13}(a_{11}a_{32} - a_{12}a_{31}) \\
&\qquad + b_{34}(a_{11}^2 - a_{12}^2) \Big\} \frac{EI}{L} .
\end{aligned}\right\} \qquad \text{(C-11)}$$

Algebraic simplification of equation (C-11) yields a symmetric stiffness matrix in equation (C-10).

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