MOTION OF SEMI-DEFINITE SYSTEMS

INCLUDING THE EFFECT OF AXIAL FORCES

by

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Submitted in Partial Fulfillment of the Requirements

for the Degree of

Master of Science in Engineering

in the

Civil Engineering

Program

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Dean of the Graduate School Date

YOUNGSTOWN STATE UNIVERSITY

February, 1974

ABSTRACT

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The purpose of this thesis is to determine a general closed-form solution of a discrete linear dynamic system having n degrees of freedom. The solution includes the effect of axial force as well as rigid body motion. This class of dynamic system which represent a large group of practical engineering problems are called "semi-definite systems."

The solution is given in a compact matrix form which eliminates the necessity of a series-type solution. The matrix solution is developed in Duhamel's integral form which allows for the application of any type of time-varying external forcing function.

ACKNOWLEDGMENT

The author wishes to thank gratefully his advisor, Dr. Paul X. Bellini for his valuable time, guidance and excellent assistance in completion and success of this project.

Credit is also due to the review committee, Dr. Michael K. Householder and Prof. Jack Ritter for their valuable time and interest in this work.

I must appreciate my wife Usha for her sacrifice and patience.

I would also like to thank Miss Lucy Shaheen who has typed the entire work.

TABLE OF CONTENTS

LIST OF SYMBOLS

SYMBOL		DEFINITION
$x_j(t)$		Generalized displacements
$x_j(t)$		Generalized velocity
j	=	$1,2,\ldots,n$
p		Scalar parameter
$f_j(t)$		External time varying forces
T		Total kinetic energy
V_e		Total external potential energy
V_i		Total internal potential energy
Ω		Natural frequency of free vibration including axial forces
$[M]$		Partitioned mass matrix
$[K]$		Partitioned stiffness matrix
$[P]$		Partitioned stability matrix
$[U]$		Partitioned eigenvector matrix
		Diagonal partitioned matrices
$[A]$		Diagonal matrix with terms $\cos \Omega_j t$
$[B]$		Diagonal matrix with terms $\sin \Omega_j t$
$[E]$		Diagonal matrix with terms $\sin \Omega_j (t-\tau)$
$[F]$		Diagonal matrix with terms $\cos \Omega_j (t-\tau)$

LIST OF FIGURES

LIST OF TABLES

CHAPTER I

INTRODUCTION

The static stability problem of lumped-mass system including the effect of axial force is considered by Timoshenko, [9] where the inertial terms are neglected and general solution is given in algebraic form. The matrix formulation of static stability problem is considered by Rubinstein. [8]

The effect of inertial forces and axial forces on these systems is considered by Newmark and Rosenblueth. [7] The general equation of motion for this special case are formulated in matrix form. The solution of these formulated equations is given by Bellini [1] using model analysis. The effect of simultaneous diagonalization of the mass, stiffness and axial force matrices is also considered.

Herein, the effect of rigid body motions is considered together with axial-type forces. The effect of damping forces is neglected for the mathematical models considered. A formal closed-form matrix-type solution is presented for arbitrary external time-varying forces.

CHAPTER II

General Formulation of Problem

Consider a linear dynamic system with n degrees of freedom, where the motion of the system is described by m relative generalized displacements $X_j(t)$, $j=1,2,\ldots m$ and $(n-m)$ absolute generalized displacements $X_j(t)$, $j=m+1,\ldots n$ relative to a fixed inertial frame. In symbolic partitioned column matrix form, these displacements are defined as

$$\{X(t)\} = \left\{\begin{array}{c}\{X_1(t)\}\\ \hdashline \{X_2(t)\}\end{array}\right\}$$

where $\{X_1(t)\}$ is the matrix of relative displacements and $\{X_2(t)\}$ is the matrix of absolute displacements.

In a similar manner the generalized velocity components are written in partitioned column matrix form as:

$$\{\dot{X}(t)\} = \left\{\begin{array}{c}\{\dot{X}_1(t)\}\\ \hdashline \{\dot{X}_2(t)\}\end{array}\right\}$$

Hence, the kinetic energy is defined by the following equation

$$T = \frac{1}{2}\{\dot{X}\}^T[M]\{\dot{X}\} \tag{1a}$$

Where $[M]$ defines the partitioned mass matrix as follows:

$$\left[\begin{array}{c:c} \begin{bmatrix} m_{11} & m_{12} \cdots m_{1m} \\ \vdots & \\ m_{m1} & m_{m2} \cdots m_{mm} \end{bmatrix} & \begin{bmatrix} m_{1,m+1} \cdots & m_{1,n} \\ \vdots & \\ m_{m,m+1} \cdots & m_{mn} \end{bmatrix} \\ \hdashline \begin{bmatrix} m_{m+1,1} \cdots & m_{m+1,m} \\ \vdots & \\ m_{n,1} \cdots & m_{n,m} \end{bmatrix} & \begin{bmatrix} m_{m+1,1} \cdots & m_{m+1,n} \\ \vdots & \\ m_{n,m+1} \cdots & m_{nn} \end{bmatrix} \end{array}\right]$$

$$= \left[\begin{array}{c:c} [M_{11}] & [M_{12}] \\ \hdashline [M_{12}] & [M_{22}] \end{array}\right]$$

$[M11]$ and $[M22]$ are always symmetric square matrices and $[M21]$ is a transpose of $[M12]$. The matrix $[M12]$ is a square matrix only if $(n-m) = m$.

The total internal potential energy V_i is expressed in terms of the generalized displacements in the following form:

$$V_i = \frac{1}{2}\{x\}^T[K]\{x\} \tag{1b}$$

Where $[K]$ is defined as the stiffness matrix possessing similar partitioned characteristics as the mass matrix; the matrix is defined as follows:

$$[K] = \left[\begin{array}{c:c} [K_{11}] & [K_{12}] \\ \hdashline [K_{12}] & [K_{22}] \end{array}\right]$$

The total external potential energy is comprised of two parts: the part due to axial conservative forces and the part due to non-conservative time-varying forces. The total external potential energy V_e is then written in the following form:

$$V_e = \frac{1}{2}\{x\}^T[P]\{x\} + \{f(t)\}^T\{x\} \tag{1c}$$

where $[P]$ defines the partitioned stability matrix as:

$$[P] = \left[\begin{array}{c|c} [P_{11}] & [P_{12}] \\ \hline [P_{12}] & [P_{22}] \end{array}\right]$$

Using the matrix quadratic forms given by equations (1a) through (1c), the following set of differential equations of motion are obtained in matrix form using the Lagrange equation approach [7]:

$$\begin{bmatrix} [M_{11}] & [M_{12}] \\ [M_{12}] & [M_{22}] \end{bmatrix} \begin{Bmatrix} \{\ddot{x}_1\} \\ \{\ddot{x}_2\} \end{Bmatrix} + \begin{bmatrix} [K_{11}] - p[P_{11}] & [K_{12}] - p[P_{12}] \\ [K_{12}] - p[P_{12}] & [K_{22}] - p[P_{22}] \end{bmatrix} \begin{Bmatrix} \{x_1\} \\ \{x_2\} \end{Bmatrix} = \begin{Bmatrix} \{f_1(t)\} \\ \{f_2(t)\} \end{Bmatrix} \tag{2}$$

The (n-m) absolute displacements are directly associated with the (n-m) rigid body motions. The semi-definite dynamic system which results, produces a series of mathematical simplifications. In general, the following matrix definitions are produced:

$$\left.\begin{aligned} [K_{12}] &= [K_{21}]^T = [0] \\ [P_{12}] &= [P_{21}]^T = [0] \\ [K_{22}] &= [0] \\ [P_{22}] &= [0] \end{aligned}\right] \qquad (3)$$

and the matrix $[M22]$ is diagonal.

Noting the above conditions equation (2) reduces to the form:

$$\begin{bmatrix} [M_{11}] & [M_{12}] \\ [M_{12}] & [M_{22}] \end{bmatrix} \begin{Bmatrix} \{\ddot{X}_1\} \\ \{\ddot{X}_2\} \end{Bmatrix} + \begin{bmatrix} [K_{11}] - p[P_{11}] & [0] \\ [0] & [0] \end{bmatrix} \begin{Bmatrix} \{X_1\} \\ \{X_2\} \end{Bmatrix} = \begin{Bmatrix} \{f_1(t)\} \\ \{f_2(t)\} \end{Bmatrix} \qquad (4)$$

CHAPTER III

Free Vibration Problem

The free vibration problem including the effect of axial force is given by the partitioned matrix equation

$$\begin{bmatrix} [M_{11}] & [M_{12}] \\ [M_{12}] & [M_{22}] \end{bmatrix} \begin{Bmatrix} \{\ddot{X}_1\} \\ \{\ddot{X}_2\} \end{Bmatrix} + \begin{bmatrix} [K_{11}] - p[P_{11}] & [0] \\ [0] & [0] \end{bmatrix} \begin{Bmatrix} \{X_1\} \\ \{X_2\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix} \tag{5}$$

Solution of the Free Vibration Problem

Referring to equation (5), the general solution is assumed to take the form

$$\begin{Bmatrix} \{X_1\} \\ \{X_2\} \end{Bmatrix} = e^{i\Omega t} \begin{Bmatrix} \{U_1\} \\ \{U_2\} \end{Bmatrix} \tag{6}$$

Where Ω is defined as natural frequency of free vibration and the partitioned column matrix in $\{U\}$ is defined as the associated partitioned eigenvector matrix. Substituting equation (6) into equation (5) yields:

$$\left[\begin{bmatrix} [-\Omega^2 M_{11}] & [-\Omega^2 M_{12}] \\ [-\Omega^2 M_{12}] & [-\Omega^2 M_{22}] \end{bmatrix} + \begin{bmatrix} [K_{11}] - p[P_{11}] & [0] \\ [0] & [0] \end{bmatrix} \right] \begin{Bmatrix} \{U_1\} \\ \{U_2\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix} \tag{7}$$

which for the non-trivial solution of the eigenvector $\begin{Bmatrix}\{U_1\}\\\{U_2\}\end{Bmatrix}$ require that,

$$\det\left[\begin{bmatrix}[K_{11}]-p[P_{11}] & [0]\\ [0] & [0]\end{bmatrix}-\Omega^2\begin{bmatrix}[M_{11}] & [M_{12}]\\ [M_{12}] & [M_{22}]\end{bmatrix}\right]=\begin{Bmatrix}\{0\}\\\{0\}\end{Bmatrix} \tag{8}$$

Equation (7) and (8) define the generalized eigenvalue-eigenvector problem in partitioned matrix form. Equation (8) yields j values of the parameter Ω_j^2, $j = 1,2,\ldots m, m+1,\ldots n$. The values of the natural frequencies associated with the rigid body motions are equal to zero, i.e., $\Omega_j^2 = 0$, $j = (m+1),\ldots n$. Corresponding to each value of Ω_j^2, equation (7) yields a single partitioned eigenvector $\{U\}_j$, $j=1,2,\ldots m,m+1,\ldots n$.

The set of n eigenvectors are combined into a single partitioned eigenvector matrix $[U]$ which is defined as:

$$[U]=\begin{bmatrix}[U_{11}] & [U_{12}]\\ [U_{21}] & [U_{22}]\end{bmatrix}$$

The form of equation (7) requires that the matrix $[U_{12}]=[0]$ and $[U_{22}]$ is a diagonal matrix with arbitrary terms. For convenience, the $[U_{22}]$ matrix is taken as the identity matrix $[I]$. Thus, the eigenvector matrix $[U]$ reduces to the form

$$[U] = \begin{bmatrix} [U_{11}] & [0] \\ [U_{21}] & [I] \end{bmatrix}$$

The following two orthogonality conditions result from the generalized eigenvalue-eigonvector form of equation (7):

$$\begin{bmatrix} [U_{11}]^T & [U_{21}]^T \\ [0] & [I] \end{bmatrix} \begin{bmatrix} [M_{11}] & [M_{12}] \\ [M_{12}] & [M_{22}] \end{bmatrix} \begin{bmatrix} [U_{11}] & [0] \\ [U_{21}] & [I] \end{bmatrix} = \begin{bmatrix} [\Lambda_{m11}] & [0] \\ [0] & [\Lambda_{m22}] \end{bmatrix} \tag{9a}$$

and

$$\begin{bmatrix} [U_{11}]^T & [U_{21}]^T \\ [0] & [I] \end{bmatrix} \begin{bmatrix} [K_{11}] - p[P_{11}] & [0] \\ [0] & [0] \end{bmatrix} \begin{bmatrix} [U_{11}] & [0] \\ [U_{21}] & [I] \end{bmatrix} = \begin{bmatrix} [\Lambda_{K11}] & [0] \\ [0] & [\Lambda_{K22}] \end{bmatrix} \tag{9b}$$

The right hand side of equation (9a) and (9b) are diagonal partitioned matrices. Referring to equation (7) it follows that,

$$\begin{bmatrix} [K_{11}] - p[P_{11}] & [0] \\ [0] & [0] \end{bmatrix} \begin{bmatrix} [U_{11}] & [0] \\ [U_{21}] & [I] \end{bmatrix} = \begin{bmatrix} [M_{11}] & [M_{12}] \\ [M_{12}] & [M_{22}] \end{bmatrix} \begin{bmatrix} [U_{11}] & [0] \\ [U_{21}] & [I] \end{bmatrix} \begin{bmatrix} [\Lambda_{\Omega 11}] & [0] \\ [0] & [\Lambda_{\Omega 22}] \end{bmatrix} \tag{10}$$

where $\begin{bmatrix} [\Lambda_{\Omega 11}] & [0] \\ [0] & [\Lambda_{\Omega 22}] \end{bmatrix}$ is a diagonal partitioned matrix with terms Ω_j^2, j=1,2,...m and where $[\Lambda_{\Omega 22}] = [0]$ since $\Omega_j^2 = 0$, j=m+1,...n. Premultiplying equation (10) by $\begin{bmatrix} [U_{11}] & [0] \\ [U_{21}] & [I] \end{bmatrix}^T$ and noting equations (9a) and (9b), one obtains

$$\begin{bmatrix} [\Lambda_{k11}] & [0] \\ [0] & [0] \end{bmatrix} = \begin{bmatrix} [\Lambda_{m11}] & [0] \\ [0] & [\Lambda_{m22}] \end{bmatrix} \begin{bmatrix} [\Lambda_{\Omega 11}] & [0] \\ [0] & [0] \end{bmatrix} \tag{11a}$$

or in simplified form

$$[\Lambda_{k11}] = [\Lambda_{m11}][\Lambda_{\Omega 11}] \tag{11b}$$

CHAPTER IV

General Solution of the Forced Vibration Problem Including the Effect of Axial Force

Referring to equation (4), making the substitution

$$\begin{Bmatrix} \{x_1\} \\ \{x_2\} \end{Bmatrix} = \begin{bmatrix} [U_{11}] & [0] \\ [U_{21}] & [I] \end{bmatrix} \begin{Bmatrix} \{y_1\} \\ \{y_2\} \end{Bmatrix}, \tag{12}$$

premultiplying by $\begin{bmatrix} [U_{11}]^T & [U_{21}]^T \\ [0] & [I] \end{bmatrix}$ and noting equations (9a) and (9b), it follows that

$$\begin{bmatrix} [\Lambda m_{11}] & [\ 0\ ] \\ [\ 0\ ] & [\Lambda m_{22}] \end{bmatrix} \begin{Bmatrix} \{\ddot{y}_1\} \\ \{\ddot{y}_2\} \end{Bmatrix} + \begin{bmatrix} [\Lambda k_{11}] & [0] \\ [\ 0\ ] & [0] \end{bmatrix} \begin{Bmatrix} \{y_1\} \\ \{y_2\} \end{Bmatrix} = \begin{bmatrix} [U_{11}]^T & [U_{21}]^T \\ [\ 0\ ] & [I] \end{bmatrix} \begin{Bmatrix} \{f_1(t)\} \\ \{f_2(t)\} \end{Bmatrix} \tag{13}$$

Premultiplying the equation (13) by $\begin{bmatrix} [\Lambda m_{11}]^{-1} & [0] \\ [0] & [\Lambda m_{22}]^{-1} \end{bmatrix}$ and substituting the equation (11a), one obtains

$$\begin{bmatrix} [I] & [0] \\ [0] & [I] \end{bmatrix} \begin{Bmatrix} \{\ddot{y}_1\} \\ \{\ddot{y}_2\} \end{Bmatrix} + \begin{bmatrix} [\Lambda_{\Omega 11}] & [0] \\ [\ 0\ ] & [0] \end{bmatrix} \begin{Bmatrix} \{y_1\} \\ \{y_2\} \end{Bmatrix} = \begin{Bmatrix} \{g_1(t)\} \\ \{g_2(t)\} \end{Bmatrix} \tag{14}$$

where

$$\begin{Bmatrix} \{g_1(t)\} \\ \{g_2(t)\} \end{Bmatrix} = \begin{bmatrix} [\Lambda m_{11}]^{-1} & [\ 0\ ] \\ [\ 0\ ] & [\Lambda m_{22}]^{-1} \end{bmatrix} \begin{bmatrix} [U_{11}]^T & [U_{21}]^T \\ [0] & [\ I\ ] \end{bmatrix} \begin{Bmatrix} \{f_1(t)\} \\ \{f_2(t)\} \end{Bmatrix} \tag{14a}$$

The form of equation (14) represents the total uncoupling of equations of motion. Using Lagrange variation of parameters, the solution of equation (14) in partitioned matrix form becomes

$$
\begin{aligned}
\begin{Bmatrix} \{y_1\} \\ \{y_2\} \end{Bmatrix} &= \begin{bmatrix} [A] & [0] \\ [0] & [I] \end{bmatrix} \begin{Bmatrix} \{a_1\} \\ \{a_2\} \end{Bmatrix} + \begin{bmatrix} [\Lambda_{\Omega 11}]^{-1/2} & [0] \\ [0] & [0] \end{bmatrix} \begin{bmatrix} [B] & [0] \\ [0] & t[I] \end{bmatrix} \begin{Bmatrix} \{b_1\} \\ \{b_2\} \end{Bmatrix} \\
&\quad + \int_{\tau=0}^{\tau=t} \begin{bmatrix} [E] & [0] \\ [0] & (t-\tau)[I] \end{bmatrix} \begin{Bmatrix} \{g_1(\tau)\} \\ \{g_2(\tau)\} \end{Bmatrix} d\tau
\end{aligned}
\tag{15}
$$

where $[A]$, $[B]$ and $[E]$ are diagonal matrices with terms $\mathrm{Cos}\,\Omega_j t$, $\mathrm{Sin}\,\Omega_j t$, and $\mathrm{Sin}\,\Omega_j(t-\tau)$ respectively. Substituting equations (15) into equation (12) yields

$$
\begin{aligned}
\begin{Bmatrix} \{x_1\} \\ \{x_2\} \end{Bmatrix} &= \begin{bmatrix} [U_{11}] & [0] \\ [U_{21}] & [I] \end{bmatrix} \begin{bmatrix} [A] & [0] \\ [0] & [I] \end{bmatrix} \begin{Bmatrix} \{a_1\} \\ \{a_2\} \end{Bmatrix} + \begin{bmatrix} [U_{11}] & [0] \\ [U_{21}] & [I] \end{bmatrix} \begin{bmatrix} [\Lambda_{\Omega 11}]^{-1/2} & [0] \\ [0] & [0] \end{bmatrix} \begin{bmatrix} [B] & [0] \\ [0] & t[I] \end{bmatrix} \begin{Bmatrix} \{b_1\} \\ \{b_2\} \end{Bmatrix} \\
&\quad + \begin{bmatrix} [U_{11}] & [0] \\ [U_{21}] & [I] \end{bmatrix} \begin{bmatrix} [\Lambda_{\Omega 11}]^{-1/2} & [0] \\ [0] & [0] \end{bmatrix} \int_{\tau=0}^{\tau=t} \begin{bmatrix} [E] & [0] \\ [0] & (t-\tau)[I] \end{bmatrix} \begin{bmatrix} [\Lambda_{m11}]^{-1} & [0] \\ [0] & [\Lambda_{m22}]^{-1} \end{bmatrix} \begin{bmatrix} [U_{11}]^T & [U_{21}]^T \\ [0] & [I] \end{bmatrix} \begin{Bmatrix} \{f_1(\tau)\} \\ \{f_2(\tau)\} \end{Bmatrix} d\tau
\end{aligned}
\tag{16}
$$

Similarly,

$$
\begin{aligned}
\begin{Bmatrix} \{\dot{x}_1\} \\ \{\dot{x}_2\} \end{Bmatrix} &= -\begin{bmatrix} [U_{11}] & [0] \\ [U_{21}] & [I] \end{bmatrix} \begin{bmatrix} [\Lambda_{\Omega 11}]^{-1/2}[B] & [0] \\ [0] & [0] \end{bmatrix} \begin{Bmatrix} \{a_1\} \\ \{a_2\} \end{Bmatrix} + \begin{bmatrix} [U_{11}] & [0] \\ [U_{21}] & [I] \end{bmatrix} \begin{bmatrix} [A] & [0] \\ [0] & [I] \end{bmatrix} \begin{Bmatrix} \{b_1\} \\ \{b_2\} \end{Bmatrix} \\
&\quad + \begin{bmatrix} [U_{11}] & [0] \\ [U_{21}] & [I] \end{bmatrix} \int_{\tau=0}^{\tau=t} \begin{bmatrix} [F] & [0] \\ [0] & [I] \end{bmatrix} \begin{bmatrix} [\Lambda_{m11}]^{-1} & [0] \\ [0] & [\Lambda_{m22}]^{-1} \end{bmatrix} \begin{bmatrix} [U_{11}]^T & [U_{21}]^T \\ [0] & [I] \end{bmatrix} \begin{Bmatrix} \{f_1(\tau)\} \\ \{f_2(\tau)\} \end{Bmatrix} d\tau
\end{aligned}
\tag{17}
$$

where, [F] is a diagonal matrix with terms $\cos\Omega_j(t-\tau)$.

Using the following prescribed initial conditions

$$\begin{aligned}&(i)\ @\ t=0,\quad \{x(t)\} = \{x(0)\} \quad \text{and}\\ &(ii)\ @\ t=0,\quad \{\dot{x}(t)\} = \{\dot{x}(0)\},\end{aligned} \tag{18}$$

it follows that,

$$\begin{Bmatrix}\{x_1(0)\}\\ \{x_2(0)\}\end{Bmatrix} = \begin{bmatrix}[U_{11}] & [0]\\ [U_{21}] & [I]\end{bmatrix}\begin{Bmatrix}\{a_1\}\\ \{a_2\}\end{Bmatrix}, \quad \text{and}$$

$$\begin{Bmatrix}\{\dot{x}_1(0)\}\\ \{x_2(0)\}\end{Bmatrix} = \begin{bmatrix}[U_{11}] & [0]\\ [U_{21}] & [I]\end{bmatrix}\begin{Bmatrix}\{b_1\}\\ \{b_2\}\end{Bmatrix}$$

Noting equation (9a), one obtains

$$\begin{Bmatrix}\{a_1\}\\ \{a_2\}\end{Bmatrix} = \begin{bmatrix}[\Lambda_{m11}]^{-1} & [0]\\ [0] & [\Lambda_{m22}]^{-1}\end{bmatrix}\begin{bmatrix}[U_{11}]^T & [U_{21}]^T\\ [0] & [I]\end{bmatrix}\begin{bmatrix}[M_{11}] & [M_{12}]\\ [M_{12}] & [M_{22}]\end{bmatrix}\begin{Bmatrix}\{x_1(0)\}\\ \{x_2(0)\}\end{Bmatrix} \tag{19a}$$

$$\text{and} \quad \begin{Bmatrix}\{b_1\}\\ \{b_2\}\end{Bmatrix} = \begin{bmatrix}[\Lambda_{m11}]^{-1} & [0]\\ [0] & [\Lambda_{m22}]^{-1}\end{bmatrix}\begin{bmatrix}[U_{11}]^T & [U_{21}]^T\\ [0] & [I]\end{bmatrix}\begin{bmatrix}[M_{11}] & [M_{12}]\\ [M_{12}] & [M_{22}]\end{bmatrix}\begin{Bmatrix}\{\dot{x}_1(0)\}\\ \{\dot{x}_2(0)\}\end{Bmatrix} \tag{19b}$$

The general solution of equation (4) then takes the following form:

$$\begin{Bmatrix}\{x_1\}\\ \{x_2\}\end{Bmatrix} = \begin{bmatrix}[U_{11}] & [0]\\ [U_{21}] & [I]\end{bmatrix}\begin{bmatrix}[A] & [0]\\ [0] & [I]\end{bmatrix}\begin{bmatrix}[\Lambda_{m11}]^{-1} & [\,0\,]\\ [\,0\,] & [\Lambda_{m22}]^{-1}\end{bmatrix}\begin{bmatrix}[U_{11}]^T & [U_{21}]^T\\ [0] & [I]\end{bmatrix}\begin{bmatrix}[M_{11}] & [M_{12}]\\ [M_{12}] & [M_{22}]\end{bmatrix}\begin{Bmatrix}\{x_1(0)\}\\ \{x_2(0)\}\end{Bmatrix}$$

$$+ \begin{bmatrix}[U_{11}] & [0]\\ [U_{21}] & [I]\end{bmatrix}\begin{bmatrix}[\Lambda_{\Omega 11}]^{-1/2} & [0]\\ [\,0\,] & [I]\end{bmatrix}\begin{bmatrix}[B] & [0]\\ [0] & t[I]\end{bmatrix}\begin{bmatrix}[\Lambda_{m11}]^{-1} & [\,0\,]\\ [\,0\,] & [\Lambda_{m22}]^{-1}\end{bmatrix}\begin{bmatrix}[U_{11}]^T & [U_{21}]^T\\ [0] & [I]\end{bmatrix}\begin{bmatrix}[M_{11}] & [M_{12}]\\ [M_{12}] & [M_{22}]\end{bmatrix}\begin{Bmatrix}\{\dot{x}_1(0)\}\\ \{x_2(0)\}\end{Bmatrix}$$

$$+ \begin{bmatrix}[U_{11}] & [0]\\ [U_{21}] & [I]\end{bmatrix}\begin{bmatrix}[\Lambda_{\Omega 11}]^{-1/2} & |0|\\ [\,0\,] & [I]\end{bmatrix}\begin{bmatrix}[\Lambda_{m11}]^{-1} & [\,0\,]\\ [\,0\,] & [\Lambda_{m22}]^{-1}\end{bmatrix}\int_{\tau=0}^{\tau=t}\begin{bmatrix}[E] & [0]\\ [0] & (t-\tau)[I]\end{bmatrix}\begin{bmatrix}[U_{11}]^T & [U_{21}]^T\\ [0] & [I]\end{bmatrix}\begin{Bmatrix}\{f_1(\tau)\}\\ \{f_2(\tau)\}\end{Bmatrix}d\tau \qquad (20)$$

The simplification of the above equation is given in Appendix I. Equation (20) is investigated for the special cases of externally applied force i.e. $\begin{Bmatrix}\{f_1(t)\}\\ \{f_2(t)\}\end{Bmatrix}$.

CASE: i

Taking the initial conditions as zero,

$$\begin{Bmatrix}\{x_1(0)\}\\ \{x_2(0)\}\end{Bmatrix} = \begin{Bmatrix}\{\dot{x}_1(0)\}\\ \{\dot{x}_2(0)\}\end{Bmatrix} = \begin{Bmatrix}\{0\}\\ \{0\}\end{Bmatrix} \quad \text{and}$$

$$\begin{Bmatrix}\{f_1(t)\}\\ \{f_2(t)\}\end{Bmatrix} = \begin{Bmatrix}\{f_1(0)\}\\ \{f_2(0)\}\end{Bmatrix},$$

where the external forces are assumed as constants, equation (20) reduces to the form

$$\begin{Bmatrix}\{x_1\}\\\{x_2\}\end{Bmatrix}=\begin{bmatrix}[U_{11}] & [0]\\ [U_{21}] & [I]\end{bmatrix}\begin{bmatrix}[\Lambda_{\Omega 11}]^{-1/2} & [0]\\ [0] & [I]\end{bmatrix}\begin{bmatrix}[\Lambda_{m11}]^{-1} & [0]\\ [0] & [\Lambda_{m22}]^{-1}\end{bmatrix}\begin{bmatrix}[\Lambda_{\Omega 11}]^{-1/2}[I-A] & [0]\\ [0] & \frac{t^2}{2}[I]\end{bmatrix}\begin{bmatrix}[U_{11}]^T & [U_{21}]^T\\ [0] & [I]\end{bmatrix}\begin{Bmatrix}\{f_1(0)\}\\\{f_2(0)\}\end{Bmatrix} \tag{21}$$

CASE ii

Assuming the initial conditions as zero and the externally applied forces as harmonic variation of time in the form

$$\{f(\tau)\} = \begin{Bmatrix}\begin{Bmatrix}f_1 \sin\alpha_1\tau\\ f_2 \sin\alpha_2\tau\\ \vdots\\ f_m \sin\alpha_m\tau\end{Bmatrix}\\ \hdashline \begin{Bmatrix}f_{m+1}\sin\alpha_{m+1}\tau\\ \vdots\\ f_n \sin\alpha_n\tau\end{Bmatrix}\end{Bmatrix} \tag{22}$$

$$= \begin{Bmatrix}\{f_1(\tau)\}\\ \{f_2(\tau)\}\end{Bmatrix}$$

it follows that, for steady state motion only equation (20) reduces to the form

$$\begin{Bmatrix}\{x_1\}\\\{x_2\}\end{Bmatrix}=\begin{bmatrix}[U_{11}] & [0]\\ [U_{21}] & [I]\end{bmatrix}\begin{bmatrix}[\Lambda_{\Omega 11}]^{-1/2} & [0]\\ [0] & [0]\end{bmatrix}\begin{bmatrix}[\Lambda_{m11}]^{-1} & [0]\\ [0] & [\Lambda_{m22}]^{-1}\end{bmatrix}\begin{bmatrix}[\hat{U}_{11}] & [\hat{U}_{12}]\\ [\hat{U}_{21}] & [\hat{U}_{22}]\end{bmatrix}\begin{Bmatrix}\{f_1(0)\}\\\{f_2(0)\}\end{Bmatrix} \tag{23}$$

where

$$\left[\hat{U}_{11}\right] = \begin{bmatrix} U_{11}\dfrac{\Omega_1}{\Omega_1^2-\alpha_1^2}\sin\alpha_1 t & \cdots & U_{m1}\dfrac{\Omega_1}{\Omega_1^2-\alpha_m^2}\sin\alpha_m t \\ \vdots & & \vdots \\ U_{1m}\dfrac{\Omega_m}{\Omega_m^2-\alpha_1^2}\sin\alpha_1 t & \cdots & U_{mm}\dfrac{\Omega_m}{\Omega_m^2-\alpha_m^2}\sin\alpha_m t \end{bmatrix}$$

$$\left[\hat{U}_{22}\right] = \begin{bmatrix} \dfrac{t}{\alpha_{m+1}}-\dfrac{f_{m+1}}{\alpha_{m+1}^2}\sin\alpha_{m+1} t & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \dfrac{t}{\alpha_n}-\dfrac{f_n}{\alpha_n^2}\sin\alpha_n t \end{bmatrix}$$

$$\left[\hat{U}_{21}\right] = \left[0\right]$$

and

$$\left[\hat{U}_{12}\right] = \begin{bmatrix} U_{m+1,1}\dfrac{\Omega_1}{\Omega_1^2-\alpha_{m+1}^2}\sin\alpha_{m+1} t & \cdots & U_{n1}\dfrac{\Omega_1}{\Omega_1^2-\alpha_n^2}\sin\alpha_n t \\ \vdots & & \vdots \\ U_{m+1,m}\dfrac{\Omega_m}{\Omega_m^2-\alpha_{m+1}^2}\sin\alpha_{m+1} t & \cdots & U_{n,m}\dfrac{\Omega_m}{\Omega_m^2-\alpha_n^2}\sin\alpha_n t \end{bmatrix}$$

If any of the impressed frequencies $\alpha_1, \alpha_2, \ldots \alpha_n$ is equal to any of the natural frequencies, $\Omega_1, \Omega_2, \ldots \Omega_n$ then the resulting motion is unstable, that is, at least one of generalized displacements $x_j(t)$ takes on an infinite value.

CHAPTER V

Discussion

The use of the matrix type form for the equations of motion is proven more efficient than the series or algebraic type form. Its efficiency arises due to the fact the matrix type solution is easily programmed for computer use.

Since the solution is given in Duhamel's integral form, it is applicable for any type of time varying external forcing functions. In this particular thesis, constant external and harmonic time varying forces are considered as special cases where steady-state motion is considered.

The solution obtained is based on the existence of a simplified mathematical model of a complex dynamics problem. It is not the intention of this thesis to develop directly a design procedure to convert a physical dynamics problem into a mathematical model as illustrated in this thesis. This ability is obtained only by considerable experience both in the design office and under actual field conditions.

The basic matrix computations utilized in the general solution involved typical matrix addition and multiplication. The formal matrix type solution presented in this thesis requires inversion of diagonal matrices only which is extremely important for large scale system, since the general matrix inversion process requires a large amount of memory core in the computer.

To better understand the theory, a numerical example of forced vibration problem is solved, the solution of which resulted in a complex algebraic form. Hence, numerical values are assigned to the physical parameters m, L, k and k_t. Then upon varying the axial load P from 0 to reasonable positive values, behavior of Ω^2, the square of the natural frequency of vibration, is tabulated and graph is plotted.

CHAPTER VI

Conclusion

For the dynamic system considered the values of the square of the natural frequencies of free vibration decrease as the axial force increases. In addition, the inclusion of rigid body motion produce a condition where some of the square of the natural frequencies are equal to zero. The remaining square of the natural frequencies are decreased towards zero as the axial load is increased. Thus, for the semi-definite system considered the square of the frequency equal to zero is obtainable either by the existence of rigid body motion or by the increase of the axial load to a value equal to the minimum critical buckling load.

In general, semi-definite systems produce lower values of square of the natural frequencies of free vibration than the ordinary systems. This is evident by the graphical interpretation of fig. II-2, where the ordinary dynamic system as well as the corresponding semi-definite system are considered simultaneously.

In general, rigid body motions produce more complicated conditions of mathematical analysis. Some reduction in complexities are realized (i.e., certain values of natural frequencies are zero), however, the eigenvector problem becomes much more complex from a condition of physical understanding.

It is uniquely shown in this analysis, that the introduction of partitioned matrix form not only allows for a simplified mathematical approach but also yields a simplified physical interpretation of the resulting mathematical constraints.

APPENDIX I

The simplification of equation (20) is as follows:

$$\{x_1(t)\} = \Big[[U_{11}][A][\Lambda_{m11}]^{-1}[U_{11}]^T[M_{11}] + [U_{11}][A][\Lambda_{m11}]^{-1}[U_{21}]^T[M_{12}]\Big]\{x_1(0)\}$$

$$+\Big[[U_{11}][A][\Lambda_{m11}]^{-1}[U_{11}]^T[M_{12}] + [U_{11}][A][\Lambda_{m11}]^{-1}[U_{21}]^T[M_{22}]\Big]\{x_2(0)\}$$

$$+\Big[[U_{11}][\Lambda_{\Omega 11}]^{-1/2}[B][\Lambda_{m11}]^{-1}[U_{11}]^T[M_{11}] + [U_{11}][\Lambda_{\Omega 11}]^{-1/2}[B][\Lambda_{m11}]^{-1}[U_{21}]^T[M_{12}]\Big]\{\dot{x}_1(0)\}$$

$$+\Big[[U_{11}][\Lambda_{\Omega 11}]^{-1/2}[B][\Lambda_{m11}]^{-1}[U_{11}]^T[M_{12}] + [U_{11}][\Lambda_{\Omega 11}]^{-1/2}[B][\Lambda_{m11}]^{-1}[U_{21}]^T[M_{12}]\Big]\{\dot{x}_2(0)\}$$

$$+\int_{\tau=0}^{\tau=t}[U_{11}][\Lambda_{\Omega 11}]^{-1/2}[\Lambda_{m11}]^{-1}[E]\{g(\tau)\}\,d\tau \qquad \text{(I-1)}$$

$$\{x_2(t)\} = \Big[[U_{21}][A][\Lambda_{m11}]^{-1}[U_{11}]^T[M_{11}] + [U_{21}][A][\Lambda_{m11}]^{-1}[U_{21}]^T[M_{12}]$$

$$+[I][\Lambda_{m22}]^{-1}[I]^T[M_{12}]\Big]\{x_1(0)\}$$

$$+\Big[[U_{21}][A][\Lambda_{m11}]^{-1}[U_{11}]^T[M_{12}] + [U_{21}][A][\Lambda_{m11}]^{-1}[U_{21}]^T[M_{22}]$$

$$+[I][\Lambda_{m22}]^{-1}[I]^T[M_{22}]\Big]\{x_2(0)\} \qquad \text{(I-2)}$$

It should be noted carefully that the displacements defined by the components of $\{x_1(t)\}$ are relative displacements only by the definition given in Chapter II. If the absolute displacements of these latter displacements are desired, they are computed by proper scaler addition of the individual components associated with vectors $\{x_1(t)\}$ and $\{x_2(t)\}$.

APPENDIX II

Numerical Example of the Forced Vibration Problem

II-1 Mathematical Model

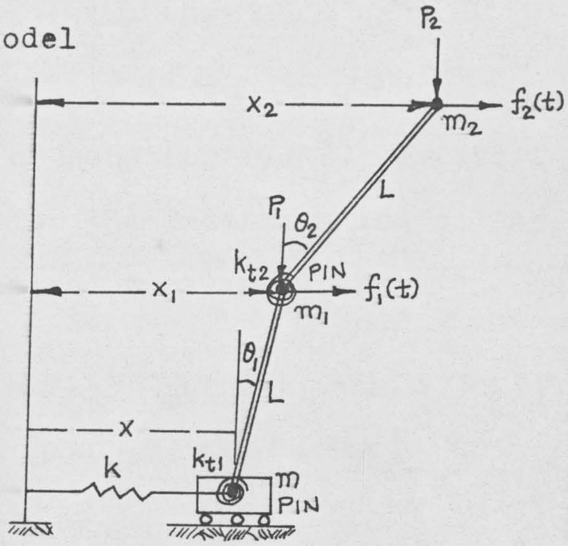

Fig. II-1

For the mathematical model shown above, assuming $m_1 = m_2 = m$, $k_{t1} = k_{t2} = k_t$, $P_1 = p$ and $P_2 = \beta p$, following matrices are obtained:

$$[M] = \begin{bmatrix} 2mL^2 & mL^2 & 2mL \\ mL^2 & mL^2 & mL \\ 2mL & mL & 3m \end{bmatrix}, \qquad [K] = \begin{bmatrix} 2k_t & -k_t & 0 \\ -k_t & k_t & 0 \\ 0 & 0 & k \end{bmatrix} \quad \text{and}$$

$$[P] = \begin{bmatrix} (1+\beta)L & 0 & 0 \\ 0 & \beta L & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{II-1}$$

II-2 Solution of the Free Vibration Problem

Noting equations (5), (6), (7) and (8) the free vibration problem yields the following determinant:

$$\begin{vmatrix} -2mL^2\Omega^2+2k_t-3PL & -mL^2\Omega^2-k_t & -2mL\Omega^2 \\ -mL^2\Omega^2-k_t & -mL^2\Omega^2+k_t-2PL & -mL\Omega^2 \\ -2mL\Omega^2 & -mL\Omega^2 & -3m\Omega^2+k \end{vmatrix} = 0 \qquad \text{(II-2)}$$

The simplification of equation (II-2) results in the following algebraic equation for the determination of the natural frequencies and critical buckling loads:

$$-\Omega^6 + \frac{\Omega^4}{m}(8k_t+k-10P) + \frac{\Omega^2}{m^2}(-6kk_t-3k^2+7Pk+21Pk_t-18P^2) + \frac{1}{m^3}(kk_t^2-7Pkk_t+6P^2k) = 0 \qquad \text{(II-3)}$$

CASE i $\quad k \neq 0$

The solution of equation (II-3) in the form given is algebraically complex. Hence, numerical values are assigned to the parameters m, L, k and k_t. In addition, the parameter P is varied over the range o to reasonable positive values. The results are tabulated in tables (T-1) and (T-2). A graphical solution of the tabular results is shown in figure (II-2).

The form of equation (7) gives three tensor invariants which are defined as follows:

$$\begin{aligned} \Omega_1^2 + \Omega_2^2 + \Omega_3^2 &= -a \\ \Omega_1^2\Omega_2^2 + \Omega_2^2\Omega_2^2 + \Omega_3^2\Omega_1^2 &= b \\ \Omega_1^2\Omega_2^2\Omega_3^2 &= -c \end{aligned} \qquad \text{(II-4)}$$

where

$$a = \frac{1}{m}(8k_t + k - 10P)$$

$$b = \frac{1}{m^2}(-6kk_t - 3k_t^2 - 7Pk + 21Pk_t - 18P^2)$$

and
$$c = \frac{1}{m^3}(kk_t^2 - 7Pkk_t + 6P^2k)$$

relative to equation (II-3)

For each variation of physical parameters mentioned above, the equations (II-4) are checked to insure that the implied equalities are satisfied. These results are tabulated in tables (T-1) and (T-2).

In general, the free vibration problem is satisfied by a set of natural frequencies which are all positive real values for the condition of stable oscillations about the equilibrium configuration. Since the value of $(P_{CR})_{min}$ is unknown apriori, a value of P is assumed initially and the square of the natural frequencies are calculated. If the resultant frequencies are all positive, one is assured that the assumed value of P is less than $(P_{CR})_{min}$. The value of $(P_{CR})_{min}$ is therefore obtained by a simple inspection of the signs of the square of the natural frequencies. This is uniquely apparent in tables (T-1) and (T-2). Observation of the condition under which the value of one of the square of the natural frequencies is identically zero yields the condition from which the value of $(P_{CR})_{min}$ is obtained. This can be seen in tables (T-1) and (T-2) in 3rd and 7th/8th lines where $(P_{CR})_{min} = (Pcr)_1 = 0.167$ and $(Pcr)_2 = 1.0$ are lower and higher critical buckling loads, respectively. It should also be noted that as the value of the load P increases, all the values of the square of the frequency decrease. Furthermore, the graphical solution in fig. (II-2) shows that one of the square of the natural frequencies remains positive and asymtotic to the horizontal axis. The remaining two square of the frequencies are intially positive, decrease to a value of zero and then assume negative values for any increase in the load P.

The reader should note at this point that this particular case involves no rigid body motion. However, the analysis is performed so that a comparision may be made with the problem which includes rigid body motion. This problem appears in the next section.

CASE ii $k = o$

Equation (II-3) reduces to the following form:

$$\Omega^6 - \frac{\Omega^4}{m}(8k_t - 10P) - \frac{\Omega^2}{m^2}(-3k_t^2 + 21Pk_t - 18P^2) = 0 \qquad \text{(II-3)}$$

Noting equations (5) to (8), (9a), (9b) and (11a) the following results in matrix form are obtained:

$$[\Lambda_\Omega] = \left[\begin{array}{cc|c} \frac{1}{m}(4k_t - 5P + \psi) & 0 & 0 \\ 0 & \frac{1}{m}(4k_t - 5P - \psi) & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

$$[U_d] = \left[\begin{array}{cc|c} \delta\gamma & \alpha\beta_1 & 0 \\ \delta\beta_2 & \alpha\beta_2 & 0 \\ \hline 1 & 1 & 1 \end{array}\right]$$

$$[\Lambda_m] = \left[\begin{array}{cc|c} -m\delta[\gamma + \delta\beta_2\gamma + 2\beta_2] & 0 & 0 \\ 0 & -m\alpha[\beta_1 + \alpha\beta_1\beta_2 + 2\beta_2] & 0 \\ \hline 0 & 0 & 3a^2m \end{array}\right]$$

and

$$[\Lambda_{kp}] = \left[\begin{array}{cc|c} \delta^2[\gamma\{\gamma A - 2\beta_2 k_t\} + \beta_2^2 C] & 0 & 0 \\ 0 & \alpha^2[\beta_1\{\beta_1 A - 2\beta_2 k_t\} + \beta_2^2 C] & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \qquad \text{(II-5)}$$

where

$$\alpha = \frac{-4k_t + 5P + \psi}{(-k_t + P - \psi)(-7k_t + 4P + \psi) - 4k_t\psi}$$

$$\delta = \frac{-4k_t - 5P - \psi}{(-k_t + P + \psi)(-7k_t + 4P - \psi) + 4k_t\psi}$$

$$\begin{aligned} \beta_1 &= -k_t + P + \psi \\ \gamma &= -k_t + P - \psi \\ \beta_2 &= 4k_t - 3P \\ \psi &= \sqrt{13k_t^2 - 19k_tP + 7P^2} \\ A &= (2k_t - 3P) \\ C &= (k_t - 2P) \end{aligned} \qquad \text{(II-6)}$$

The inclusion of one rigid body motion produces a condition where one of the natural frequencies is zero. In addition, it is obvious that the components of the $[U_{12}]$ matrix are zero. Also, the $[U_{22}]$ matrix is identically the unit matrix $[I]$, in addition, the components of matrix $[\Lambda_{k22}]$ are identically zero.

To determine the minimum critical value of the stability force, the values of the two frequencies Ω_1, and Ω_2 are equated to zero,

$$\text{i.e.} \qquad \frac{1}{m}(4k_t - 5P + \psi) = 0$$
$$\text{and} \qquad \frac{1}{m}(4k_t - 5P - \psi) = 0 \tag{II-7}$$

The two above equations yield

$$(P_{CR})_{min.} = (P_{CR})_1 = \frac{k_t}{6L}$$

and $$(P_{CR})_2 = \frac{k_t}{L}\,. \tag{II-8}$$

Numerical results of the above equations are tabulated in table II-3 and a graphical solution is shown in Fig. (II-2) where specific numerical values of the physical parameters are chosen therein.

APPENDIX III

Justification of [U] Matrix

The form of equation (7) requires that the matrix $[U_{12}] = o$ and $[U_{22}]$ is a diagonal matrix with arbitrary terms. For convenience, the $[U_{22}]$ matrix is taken as the identity matrix $[I]$. These conditions are further justified by considering a special case where two rigid body displacements are considered. The mathematical model is shown in figure below.

Mathematical Model
Two Rigid Body Motions

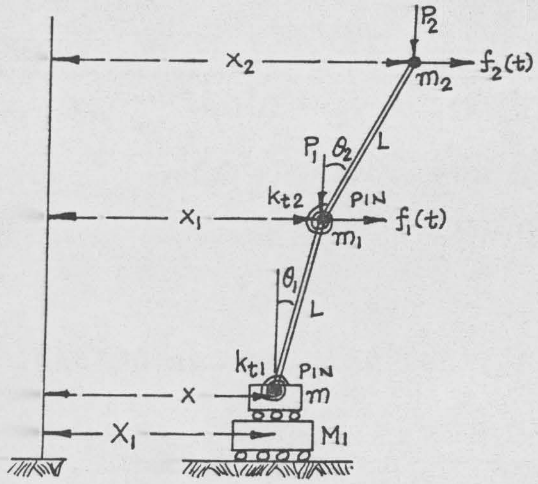

Fig. 3

From the above model, one obtains

$$[M] = \begin{bmatrix} 2mL^2 & mL^2 & 2mL & o \\ mL^2 & mL^2 & mL & o \\ 2mL & mL & 3m & o \\ o & o & o & M_1 \end{bmatrix}$$

$$[K] = \begin{bmatrix} 2k_t & -k_t & 0 & 0 \\ -k_t & k_t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(III-1a)}$$

$$[P] = P \begin{bmatrix} 3L & 0 & 0 & 0 \\ 0 & 2L & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Using equations (5) to (7), it follows that

$$\begin{bmatrix} -2mL^2\Omega^2 + 2k_t - 3PL & -mL^2\Omega^2 - k_t & -2mL\Omega^2 & 0 \\ -mL^2\Omega^2 - k_t & -mL^2\Omega^2 + k_t + 2PL & -mL\Omega^2 & 0 \\ -2mL\Omega^2 & -mL\Omega^2 & -3m\Omega^2 & 0 \\ 0 & 0 & 0 & -M_1\Omega^2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(III-1)}$$

Using equation (8) and solving for $(\Omega)^2$, one obtains

$$(\Omega_1)^2 = (\Omega_2)^2 = 0$$

$$(\Omega_3)^2 = \frac{1}{m}(4k_t - 5P - \psi)$$

$$(\Omega_4)^2 = \frac{1}{m}(4k_t - 5P + \psi)$$

where $\psi = \sqrt{13k_t^2 - 19Pk_t + 7P^2}$

Substituting $(\Omega_1^2) = (\Omega_2^2) = 0$ in equation (II-1) it follows that

$$(2k_t - 3PL)U_1 - k_t U_2 + 0.U_3 + 0.U_4 = 0 \qquad \text{(III-2a)}$$

$$0.U_1 + (k_t + 2PL)U_2 + 0.U_3 + 0.U_4 = 0 \qquad \text{(III-2b)}$$

$$0.U_1 + 0.U_2 + 0.U_3 + 0.U_4 = 0 \qquad \text{(III-2c)}$$

$$0.U_1 + 0.U_2 + 0.U_3 + 0.U_4 = 0 \qquad \text{(III-2d)}$$

From equation (III-2a) one obtains,

$$U_2 = \frac{2k_t - 3PL}{k_t} U_1 \qquad \text{(III-3)}$$

and from equation (III-2b),

$$U_2 = \frac{k_t}{k_t - 2PL} U_1 \qquad \text{(III-4)}$$

Equations (III-3) and (III-4) for $P < (P_{CR})_1$, are true only when:

$$U_1 = U_2 = 0 \qquad \text{(III-5)}$$

i.e. $U_{12} = 0$.

From equations (III-2c) and (III-2d), it follows that

$$0.U_3 + 0.U_4 = 0$$

Hence, U_3 and U_4 can have any arbitrary values.

$$\text{i.e. Let} \quad [U_{22}] = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

Using orthogonality conditions in equations (9a) and (9b), one obtains,

$$\begin{bmatrix} C_{11}^2 M_{11} + C_{21}^2 M_{22} & C_{11}C_{12}M_{11} + C_{21}C_{22}M_{22} \\ C_{11}C_{12}M_{11} + C_{21}C_{22}M_{22} & C_{22}^2 M_{22} \end{bmatrix} \qquad \text{(III-6)}$$

where $C_{11}C_{12}M_{11} + C_{21}C_{22}M_{22} = 0$. Hence it follows that

$$C_{12} = -\left(\frac{M_{22}}{M_{11}}\right) . C_{21} \left(\frac{C_{22}}{C_{11}}\right) . \qquad \text{(III-7)}$$

Since $\frac{C_{22}}{C_{11}}$ is arbitrary and $\frac{M_{22}}{M_{11}}$ is any positive value, the equality will hold for the condition where

$$C_{21} = C_{12} = 0 \tag{III-8}$$

Thus, the matrix $[U_{22}]$ is a diagonal matrix with arbitrary terms. For convenience, the $[U_{22}]$ matrix is taken as the identity matrix $[I]$. Hence the $[U]$ matrix reduces to the following form,

$$[U] = \begin{bmatrix} [U_{11}] & [0] \\ [U_{21}] & [I] \end{bmatrix}$$

DYNAMIC STABILITY CASE

$m = k_t = L = k = 1$

Value of P	Natural Frequencies			Tensor Invariants					
	Ω_1^2	Ω_2^2	Ω_3^2	$-a$		b		$-c$	
				Calculated	Actual	Calculated	Actual	Calculated	Actual
.0	0.127	1.000	7.872	8.999	9.000	8.998	9.000	1	1
0.1	.061	0.829	7.111	8.001	8.000	6.380	6.380	0.36	0.36
0.167	0	9.733	6.597	7.33	7.33	4.836	4.826	0	0
0.2	-0.037	0.689	6.347	6.999	7.000	4.113	4.120	-0.162	-0.16
0.25	-0.099	0.633	5.966	6.5	6.5	3.123	3.125	-0.374	-0.375
0.50	-0.527	0.465	4.061	3.999	4.000	-0.497	0.500	-0.995	-1
0.75	-1.057	0.379	2.177	1.499	1.500	-1.877	-1.875	-0.872	-0.875
1.00	-1.618	0	0.618	-1	-1	-0.999	-1	-0.999	-1.000
1.25	-2.193	-1.735	0.427	-3.501	-3.500	2.217	-2.215	+1.625	+1.625
1.50	-3.632	-2.766	0.399	-5.999	-6	7.493	7.500	+4.008	+4.000
1.75	-5.534	-3.35	0.384	-8.5	-8.5	15.128	15.125	7.119	7.125
2.00	-7.443	-3.935	0.376	-11.002	-11.000	25.000	+25.000	11.012	11.000

TABLE T-1

DYNAMIC STABILITY CASE

$m = k_t = L = 1, \; k = 2$

Value of P	Natural Frequencies			Tensor Invariants					
	Ω_1^2	Ω_2^2	Ω_3^2	$-a$		b		$-c$	
				Calculated	Actual	Calculated	Actual	Calculated	Actual
0.0	0.148	1.650	8.200	9.998	10	14.978	15	2.002	2
0.1	+0.064	1.490	7.445	8.999	9	11.66	11.68	0.71	0.72
0.167	0	1.393	6.940	8.333	8.333	9.667	9.667	0	0
0.20	0.034	1.346	6.689	8.001	8	8.73	8.75	-0.31	-0.32
0.50	-0.448	0.998	4.450	5	5	2	2	-1.990	-2
0.75	-0.902	+0.724	2.676	2.498	2.5	-1.129	-1.125	-1.75	-1.75
1.00	-1.414	0	+1.414	0	0	1.999	-2	0	0
1.25	-1.986	-1.544	1.030	-2.5	-2.5	-0.570	-0.625	-3.16	+3.25
1.50	-3.473	-2.463	0.934	-5.002	-5	+3.010	+3.000	+7.989	+8.000
1.75	-5.337	-3.041	0.877	-7.501	-7.5	8.882	+8.875	+14.234	+14.25
2.00	-7.226	-3.616	0.841	-10.001	-10	17.010	+17.000	-21.974	+22.000

TABLE T-2

DYNAMIC STABILITY CASE

$m = k_t = L = 1, \; k = 0$

Value of P	ψ	(Natural Frequencies)^2 Ω_1^2	Ω_2^2	Ω_3^2
0.0	3.605	0.395	0	7.605
0.1	3.342	0.158	0	6.804
0.167	3.165	0	0	6.330
0.25	2.947	-0.197	0	5.697
0.50	2.291	-0.791	0	3.791
0.75	1.640	-1.390	0	1.890
1.00	1.000	-2.000	0	0
1.25	0.433	-2.683	0	-1.817
1.50	0.500	-4.000	0	-3.000
1.75	1.090	-5.840	0	-3.660
2.00	1.732	-7.732	0	-4.268

TABLE T-3

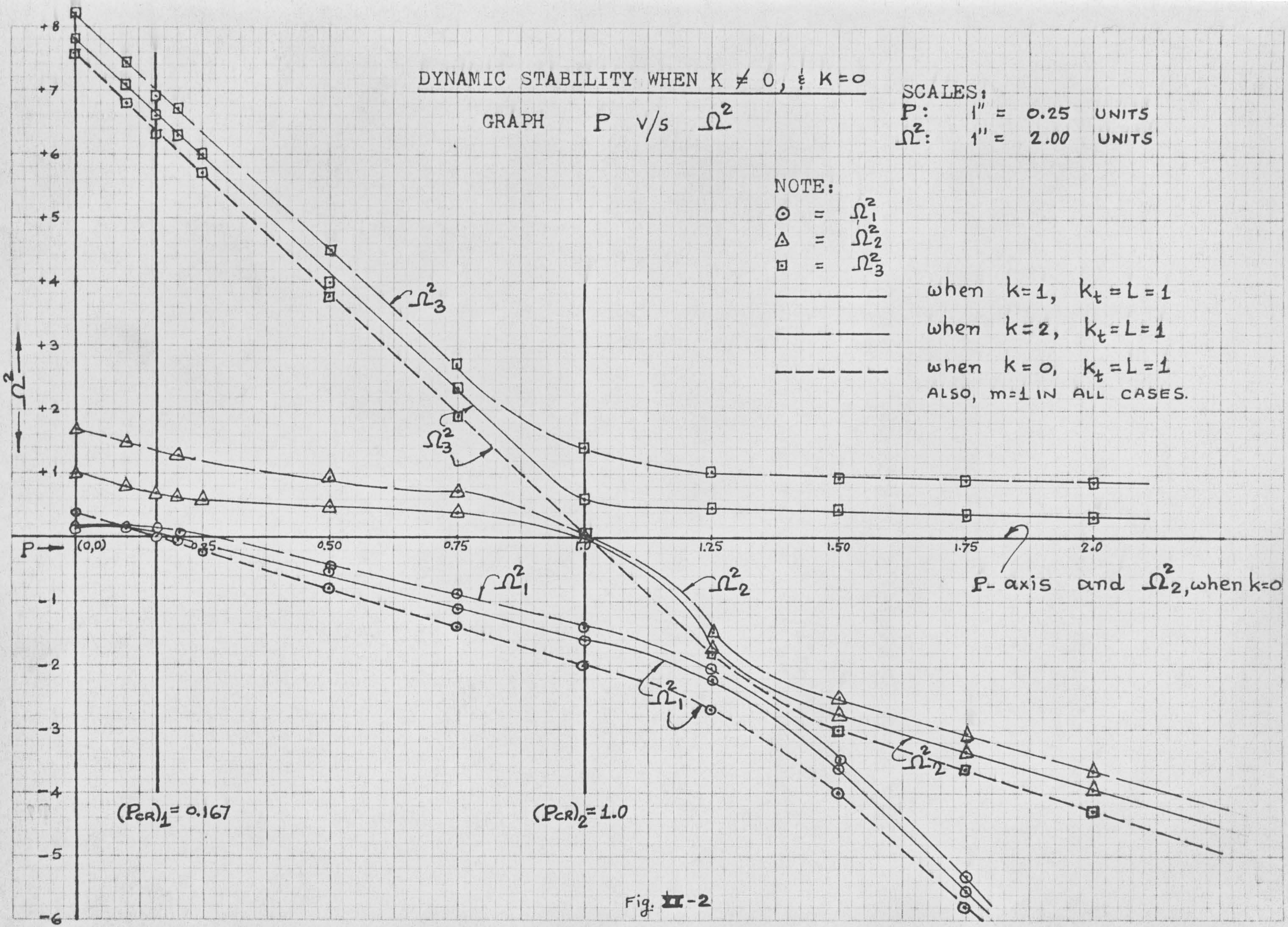
DYNAMIC STABILITY WHEN K ≠ 0, & K=0
GRAPH P V/S Ω^2
SCALES:
P: 1" = 0.25 UNITS
Ω^2: 1" = 2.00 UNITS
NOTE:
⊙ = Ω_1^2
△ = Ω_2^2
⊡ = Ω_3^2
when k=1, k_t = L = 1
when k=2, k_t = L = 1
when k=0, k_t = L = 1
ALSO, m=1 IN ALL CASES.
Ω_3^2
Ω_2^2
Ω_1^2
P-axis and Ω_2^2, when k=0
$(P_{CR})_1$ = 0.167
$(P_{CR})_2$ = 1.0
P
Ω^2
(0,0)
.25
0.50
0.75
1.0
1.25
1.50
1.75
2.0
+8
+7
+6
+5
+4
+3
+2
+1
-1
-2
-3
-4
-5
-6

Fig. II-2

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