AN OPTIMAL NONLINEAR SYSTEM IDENTIFIER BASED ON QUASILINEARIZATION

by

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Submitted in Partial Fulfillment of the Requirements

for the Degree of

Master of Science

in the

Electrical Engineering

Program

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YOUNGSTOWN STATE UNIVERSITY

June, 1974

ABSTRACT

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Identification techniques which use quasilinearization as a basis are among the fastest converging, however, they often suffer from having a narrow region of convergence when applied to nonlinear systems. An identification process, based on quasilinearization, will be presented which has a larger region of convergence than other, similar methods.

This identification process will be demonstrated and compared with similar methods in an example using the Van der Pol equation.

ACKNOWLEDGEMENTS

I am indebted to Dr. Charles Alexander for his assistance in the preparation of this thesis and to my parents for their encouragement.

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CHAPTER I

INTRODUCTION

Identification of Systems

In recent years, system identification has aroused an appreciable amount of interest. One reason for this is the needs of engineers in process industries to obtain a better knowledge of their plants. This knowledge of plant dynamics is necessary to exercise the proper control of the dynamic systems.

Knowledge of a system and its environments, which is required in the design of a proper control system is seldom available a priori. Identification, in the sense that it will be used in this paper, is the determination of the system dynamics.

L. A. Zadeh states "Identification is the determination, on the basis of input and output, of a system, within a specified class of systems, to which the system under test is equivalent."[1] The problem in identification is to determine a model whose output is identical, or nearly identical, to the output of the unknown system for all possible inputs to that unknown system. In

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[1] L. A. Zadeh, "From Circuit Theory to System Theory," Proc. IRE, L(1962), 856-865.

practice, a model is assumed to represent the actual system as closely as possible. The models used in modern control theory are, with few exceptions, parametric models in terms of state equations. The parameters are then adjusted on an iterative basis until the model is as nearly identical as possible to the actual system.

Figure 1 shows a basic block diagram of how an adjustible model identification process works. First, the form of the model is determined as accurately as possible. Then the output of the actual system is compared with the model response, under the same input conditions, resulting in an error signal. This error signal is used in the adjustment of the parameters of the model. This process is repeated until some function of the error signal is satisfied. In general, the function may assume any value. In this paper, the value will be chosen to be zero.

There are two categories of identifiers, linear and nonlinear system identifiers. The concern of this paper will be with nonlinear techniques of system identification.

Quasilinearization

There are many techniques for identifying unknown parameters of nonlinear systems. Some of the fastest converging use quasilinearization as a basis. These techniques result in quadratic convergence when they converge. Unfortunately, for many nonlinear systems, the region of convergence is small.

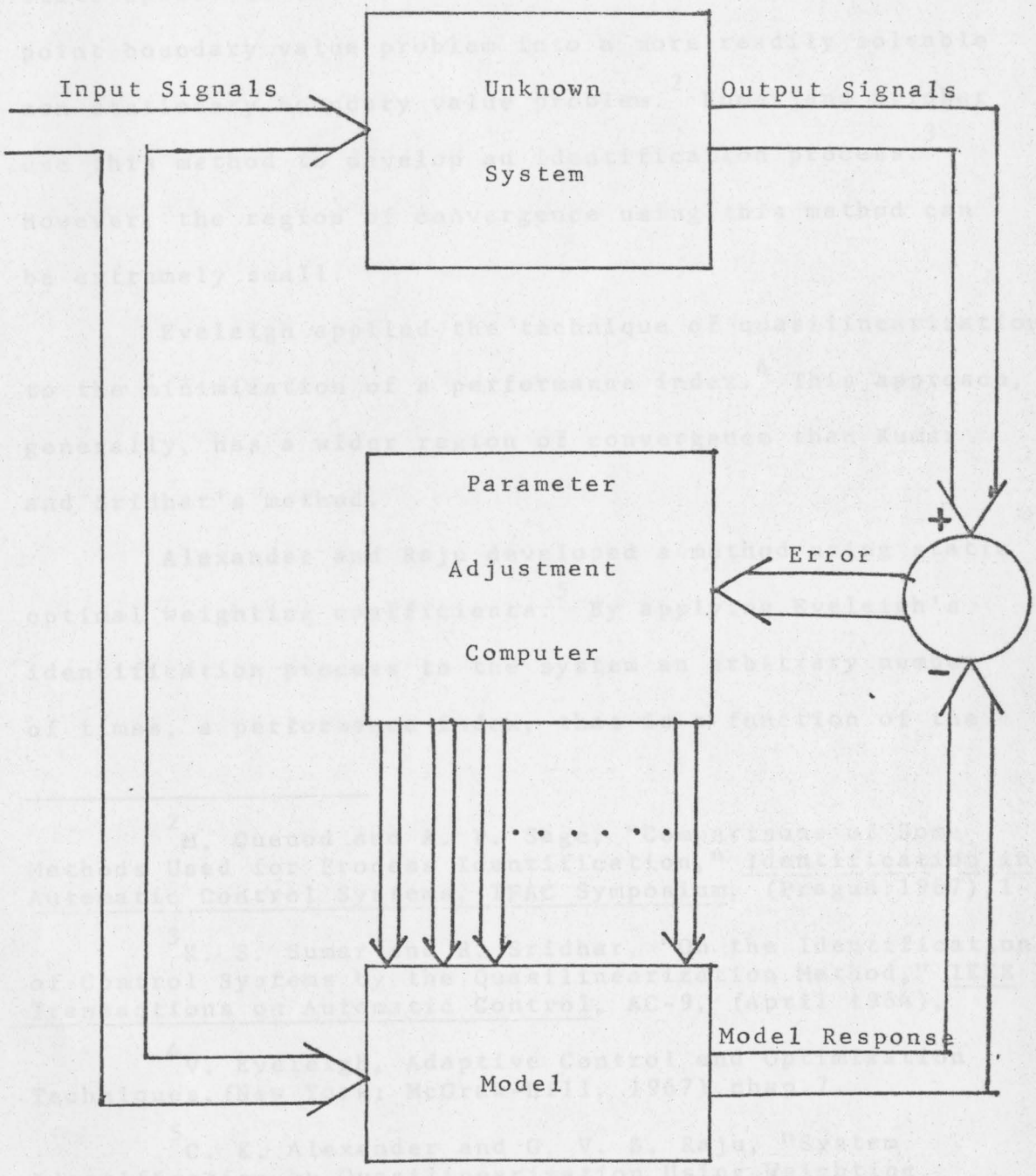

Fig.1.--Adjustable model identification process

Quasilinearization is the small signal linearization of the system response about a nominal path through the state space. It can be used to transfer a nonlinear multi-point boundary value problem into a more readily solvable non-stationary boundary value problem.[2] Kumar and Sridhar use this method to develop an identification process.[3] However, the region of convergence using this method can be extremely small.

Eveleigh applied the technique of quasilinearization to the minimization of a performance index.[4] This approach, generally, has a wider region of convergence than Kumar and Sridhar's method.

Alexander and Raju developed a method using static optimal weighting coefficients.[5] By applying Eveleigh's identification process to the system an arbitrary number of times, a performance index, that is a function of the

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[2] M. Cuenod and A. P. Sage, "Comparisons of Some Methods Used for Process Identification," Identification in Automatic Control Systems, IFAC Symposium, (Prague:1967),1-14.

[3] K. S. Sumar and R. Sridhar, "On the Identification of Control Systems by the Quasilinearization Method," IEEE Transactions on Automatic Control, AC-9, (April 1964),

[4] V. Eveleigh, Adaptive Control and Optimization Techniques,(New York: McGraw-Hill, 1967),chap.7.

[5] C. K. Alexander and G. V. S. Raju, "System Identification by Quasilinearization Using Weighting Coefficients," Proc. Eighth Annual Allerton Conference on Circuit and System Theory, (Urbana,Ill:1970),131-140.

difference between the actual system unknowns and the computed approximations of these unknowns, is formulated. This performance index is then minimized with respect to weighting coefficients resulting in an optimal set of weighting coefficients. Use of these optimal values results in a new identification process called "The Method of Optimal Static Weighting Coefficients." This process converges faster and has a larger region of convergence than either Kumar and Sridhar's or Eveleigh's methods.

By expanding on this method, a new technique called "Modified Dynamic Weighting Coefficients and Sampling Interval" is developed.[6] Again, Eveleigh's identification process is employed to determine a performance index. This performance index is then minimized with respect to both the weighting coefficients and sampling time interval after each iteration. Thus, a set of optimal weighting coefficients and sampling interval is determined for each iteration. The advantage of this technique over the others is a still larger region of convergence.

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[6] Victor Hanna and Charles Alexander, "System Identification Using Quasilinearization and Dynamic-Optimal Weighting Coefficients," Proc. Fifth Modeling and Simulation Conference, (Pittsburgh, Pa. : 1974).

CHAPTER II

QUASILINEARIZATION

Quasilinearization is the method of approximating the solution trajectories of a vector set of differential equations by another, nearby, set of solution trajectories and a linear approximation of the difference between the two trajectories.

Let $\underline{x}^*(t)$ represent the set of solution trajectories of a system response to the input control, $\underline{u}^*(t)$. Let the nearby solution trajectories be formed by the response of the system equations shown in equation (1).

$$\dot{\underline{x}} = f(\underline{x}, \underline{u}, t) \tag{1}$$

where:

an underlined term denotes a vector or array and

$\underline{x}$ = Nx1 state vector

$\underline{u}$ = Mx1 control vector

t = independent variable

Assume that f and its derivatives, relative to $\underline{x}$ and $\underline{u}$ are continuous functions of $\underline{x}$ and $\underline{u}$. Also assume that $\underline{x}^*(t)$ and $\underline{u}^*(t)$ are known over the interval $[t_o, t_f]$.

Expand $\dot{\underline{x}}^*(t)$ in a Taylor series about $\underline{x}(t)$ as shown in equation (2).

$$\dot{x}^*_i = \dot{x}_i + \sum_{j=1}^{N} \frac{\partial f_i}{\partial x_j} \delta x_j + \sum_{k=1}^{M} \frac{\partial f_i}{\partial u_k} \delta u_k + 0(e) \quad ; \quad i=1,2,\ldots N \qquad (2)$$

where:

$$\delta x_j = x^*_j - x_j$$

$$\delta u_k = u^*_k - u_k$$

$0(e)$ = higher ordered terms

The higher ordered terms in equation (2) can be neglected if $\underline{x}^*(t)$ is sufficiently near $\underline{x}(t)$. Let $\delta\underline{\dot{x}}^* = \underline{\dot{x}}^* - \underline{\dot{x}}$. Neglecting the higher ordered terms, equation (2) reduces to equation (3).

$$\delta\dot{x}_i = \sum_{j=1}^{N} \frac{\partial f_i}{\partial x_j} \delta x_j + \sum_{k=1}^{M} \frac{\partial f_i}{\partial u_k} \delta u_k \quad ; \quad i = 1,2,\ldots N \qquad (3)$$

or

$$\delta\underline{\dot{x}} = \underline{f}\,\delta\underline{x} + \underline{g}\,\delta\underline{u} \qquad (4)$$

where:

$$\underline{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_N}{\partial x_1} & \frac{\partial f_N}{\partial x_2} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix}^*$$

$$\underline{g}_{\underline{u}} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_M} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_N}{\partial u_M} & \frac{\partial f_N}{\partial u_M} & \cdots & \frac{\partial f_N}{\partial u_M} \end{bmatrix}^* \qquad (6)$$

The " * " in equations (5) and (6) denote that $\underline{f}$ and $\underline{g}$ are evaluated along the nominal solution trajectories, $\underline{x}^*(t)$.

The state equations in (4) are linear equations which are easily solved even though their coefficients are generally time dependent.

How well the solution of equations (4) and (5) approximates $\underline{x}^*(t)$ depends on the degree of nonlinearity of the system equations and the value of $\underline{x}^*(t) - \underline{x}(t)$.

Kumar and sridhar applied the technique of quasilinearization directly to boundary value problems to obtain solutions.[7] However, Alexander has shown that the region of convergence for this method can be seriously restricted.[8]

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[7] K. S. Kumar and R. Sridhar, op cit.

[8] C. K. Alexander, "System Identification,"(Ph.D dissertation, Ohio University, 1971), 61-67.

CHAPTER III

EVELEIGH'S METHOD[9]

Eveleigh uses quasilinearization and a weighted integral squared error performance index, where the error is the measure of the difference between the actual system response and the model response.

A specific form of equation is assumed to describe the system under study. The system is observed over a given time interval and all available inputs and outputs over that interval are stored in memory for future reference. The observed inputs are then applied to the assumed model in the computer and the model responses are compared to those of the actual system using a performance index, p, to measure difference. The model equations are linearized relative to small changes in model parameters. The model parameters are then adjusted in an effort to reduce p. The procedure is repeated, if necessary, until p is sufficiently near minimum to justify stopping. The resulting coefficients represent the best available system approximation. A block diagram of the process is shown in Figure 2.

The problem is to choose the model parameters, $\underline{a}$, and initial states, $\underline{x}^o$, that minimize p. The resulting

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[9] V. Eveleigh, op. cit.

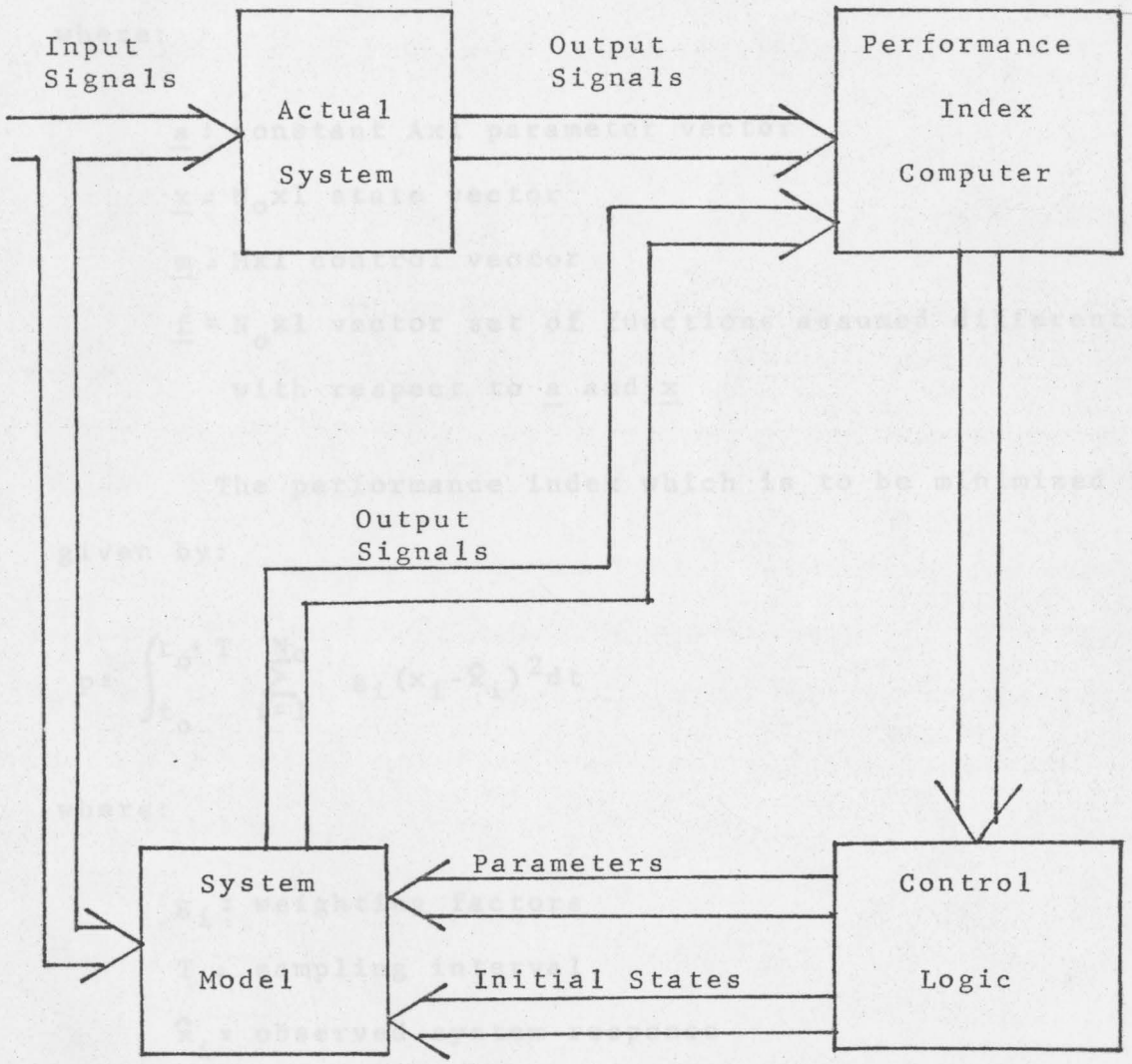

Fig. 2.--Block diagram of quasilinear model

parameter vector $\underline{a}$ provides the desired system identification. The form of the assumed model is:

$$\dot{\underline{x}} = \underline{f}(\underline{a}, \underline{x}, \underline{m}) \quad ; \quad \underline{x}(t_o) = \underline{x}^o \tag{7}$$

where:

$\underline{a}$ = constant Ax1 parameter vector

$\underline{x}$ = N_ox1 state vector

$\underline{m}$ = Mx1 control vector

$\underline{f}$ = N_ox1 vector set of functions assumed differentiable with respect to $\underline{a}$ and $\underline{x}$

The performance index which is to be minimized is given by:

$$p = \int_{t_o}^{t_o + T} \sum_{i=1}^{N_o} g_i (x_i - \hat{x}_i)^2 dt \tag{8}$$

where:

g_i = weighting factors

T = sampling interval

$\hat{x}_i$ = observed system response

x_i = model response

The g_i's corresponding to unobservable states are zero. The sampling interval, T, is chosen such that the time variations of the system unknowns are negligible over the interval. It must be large enough, however, to provide

adequate information from which the desired unknowns can be obtained with acceptable accuracy.

Since the values of $\underline{x}^o$ and $\underline{a}$ which minimize p are desired, $\underline{x}^o$ and $\underline{a}$ may be treated alike. Thus, the state equations can be augmented in the following way:

$$\begin{aligned} x_{N_o+i} &= a_i \\ \dot{x}_{N_o+i} &= a_i = 0 \end{aligned} \quad ; \quad i = 1, 2, \ldots A \tag{9}$$

Thus, $\underline{x}^o = \underline{x}(t_o)$ becomes an Nx1 vector where $N = A + N_o$ and equation (7) becomes:

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{m}) \quad ; \quad \underline{x}(t_o) = \underline{x}^o \tag{10}$$

The problem now is to find $\underline{x}^o$ such that the model response is equal to the actual system response. The first step is to guess an $\underline{x}^o$. Then equation (10) is linearized about the nominal solution trajectory. These linearized equations are:

$$\delta\dot{x}_i = \sum_{j=1}^{N} \frac{\partial f_i}{\partial x_j} \delta x_j \quad ; \quad 1 \leq i \leq N \tag{11}$$

where:

δx_j = changes in x_j from the nominal solution trajectory

$\delta\dot{x}_i$ = incremental changes in the time derivative of x_i

$\frac{\partial f_i}{\partial x_j}$ = time varying coefficients evaluated along the nominal solution trajectory using equation (10)

$\frac{\partial f_i}{\partial x_j}$ forms the Jacobian, $\underline{J}$, and when evaluated along the nominal solution path, can be represented by an NxN matrix $\underline{F}$ given in the following equation.

$$\underline{F} = \underline{J}\left[\underline{x}(\underline{x}^o)\right] \tag{12}$$

Equation (11) can now be rewritten as:

$$\delta\dot{\underline{x}} = \underline{F}\ \delta\underline{x} \tag{13}$$

Dependence of $\delta\underline{x}(t)$ upon $\delta\underline{x}^o$ is given by:

$$\delta\underline{x}(t) = \underline{\phi}(t, t_o)\ \delta\underline{x}^o \tag{14}$$

where $\underline{\phi}(t, t_o)$ is the fundamental solution matrix obtained by solving the following equation:

$$\dot{\underline{\phi}}(t, t_o) = \underline{F}\ \underline{\phi}(t, t_o) \quad ; \quad \underline{\phi}(t_o, t_o) = \underline{I} \tag{15}$$

For given values of $\underline{x}^o$, the particular change $\delta\underline{x}^o$ which minimizes equation (8) must be determined.

The model response to initial conditions $\underline{x}^o + \delta\underline{x}^o$ is given by:

$$\underline{x}(\underline{x}^o + \delta\underline{x}^o) \cong \underline{x}(\underline{x}^o) + \underline{\phi}\ \delta\underline{x}^o \tag{16}$$

or

$$x_i(\underline{x}^o + \delta\underline{x}^o) \cong x_i(\underline{x}^o) + \sum_{j=1}^{N} \delta x_j^o\ \phi_{ij} \tag{17}$$

By substituting equation (17) into equation (8), p may be expressed as:

$$P = \int_{t_o}^{t_o+T} \sum_{i=1}^{N} g_i \left[x_i(\underline{x}^o) + \sum_{j=1}^{N} \delta x_j^o \, \phi_{ij} - \hat{x}_i \right]^2 dt \qquad (18)$$

For minimum with respect to x_j, it is necessary that:

$$\frac{\partial P}{\partial \delta x_j} = 0 = \int_{t_o}^{t_o+T} 2 \sum_{i=1}^{N} \left\{ g_i \left[x_i(\underline{x}^o) + \sum_{k=1}^{N} \delta x_k^o \, \phi_{ik} - \hat{x}_i \right] \phi_{ij} \right\} dt \qquad (19)$$

$$j = 1, 2, \ldots . N$$

Since the values of g_i and T are assumed, all functions in the integrand of the above equation are known with the exception of δx_k^o. However, equation (19) provides N algebraic equations from which $\delta \underline{x}^o$ can be determined. Then, a new set of initial conditions for the model is formed from the previous set as shown in equation (20).

$$\underline{x}_{new}^o = \underline{x}_{old}^o + \delta \underline{x}^o \qquad (20)$$

The entire process is then repeated based on these new initial conditions. The iterative process is terminated when improvements ($\delta \underline{x}^o$) are negligible.

This process converges very rapidly when the guessed values are chosen close to the actual values, but may fail to converge if the initial model parameter choice is far from the true value, or if the system is highly nonlinear.

CHAPTER IV

METHOD OF OPTIMAL STATIC WEIGHTING COEFFICIENTS

In the performance index used in Eveleigh's method, the values of the weighting coefficients, $\underline{G}$, and the sampling time interval, T, were assumed known. However, it can be shown that, for certain values of $\underline{G}$ and T, the identification process has a larger region of convergence than the process does for other values of $\underline{G}$ and T.[10]

Alexander developed a method of identification that uses a performance index with optimal static weighting coefficients and sampling interval.[11] A performance index, $\underline{P}(\underline{G},T)$, is formed by squaring the difference between the system unknowns and the approximations of these unknowns obtained by applying Eveleigh's process an arbitrary number of times, K. $\underline{P}(\underline{G},T)$ is then minimized with respect to $\underline{G}$ and T to obtain $\underline{G}$ optimal and T optimal. These values of $\underline{G}$ and T are used in p to obtain the performance index p optimal. Quasilinearization is used to obtain a linearized approximation of the measured system response. The performance index, p optimal, is then minimized with respect to the linear approximation. The results are then used to adjust

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[10] C. K. Alexander, op cit., Ch. IV.

[11] C. K. Alexander, op cit., Ch. III.

the system model values. This yields an iterative process that can converge to the actual system unknowns.

For convenience, the performance index is given again in equation (21).

$$p = \int_{t_o}^{t_o+T} \sum_{i=1}^{N} g_i \left[x_i(\underline{x}^o) + \sum_{j=1}^{N} \delta x_j^o \, \phi_{ij} - \hat{x}_i(\underline{\hat{x}}^o) \right]^2 dt \qquad (21)$$

where:

T = sampling time interval

N = number of states in augmented state equations

g_i = ith component of the 1xN weighting coefficient vector $\underline{G}$

$\underline{x}^o$ = Nx1 vector representing initial conditions of the actual system

$x_i(x^o)$ = ith component of the Nx1 model response vector $\underline{x}(\underline{x}^o)$ with the initial conditions $\underline{x}^o$

$\hat{x}_i(\underline{\hat{x}}^o)$ = ith component of the Nx1 actual system response vector $\underline{\hat{x}}(\underline{\hat{x}}^o)$ with initial conditions $\underline{\hat{x}}^o$

ϕ_{ij} = ijth element of the NxN fundamental matrix $\underline{\phi}$

δx_j^o = jth component of the Nx1 perturbation vector $\delta \underline{x}^o$

Differentiating equation (21) with respect to x_j:

$$\frac{\partial p}{\partial \delta x_j} = 0 = \int_{t_o}^{t_o+T} \sum_{i=1}^{N} \left\{ g_i \left[x_i(\underline{x}^o) + \sum_{k=1}^{N} \delta x_k^o \, \phi_{ik} - \hat{x}_i(\underline{\hat{x}}^o) \right] \phi_{ij} \right\} dt \qquad (22)$$

$$j = 1, 2, \ldots N$$

$\delta \underline{x}^o$ is found by solving equation (22) in terms of T, $\underline{\hat{x}}^o$, $\underline{x}^o$, and $\underline{G}$. Equation (22) can be rewritten as;

$$\int_{t_o}^{t_o+T} \sum_{i=1}^{N} \left\{ g_i \left[\hat{x}_i(\underline{\hat{x}}^o) - x_i(\underline{x}^o) \right] \phi_{ij} \right\} dt = \int_{t_o}^{t_o+T} \sum_{i=1}^{N} g_i \left[\sum_{k=1}^{N} \delta x_k^o \, \phi_{ik} \, \phi_{ij} \right] dt \quad (23)$$

or

$$\sum_{i=1}^{N} g_i \int_{t_o}^{t_o+T} \left\{ \left[\hat{x}_i(\underline{\hat{x}}^o) - x_i(\underline{x}^o) \right] \phi_{ij} \right\} dt = \sum_{k=1}^{N} \sum_{i=1}^{N} \left\{ g_i \left[\int_{t_o}^{t_o+T} (\phi_{ik} \, \phi_{ij}) dt \right] \delta x_k^o \right\} \quad (24)$$

Let:

$$A_{jk} = \sum_{i=1}^{N} \int_{t_o}^{t_o+T} g_i (\phi_{ik} \, \phi_{ij}) dt \quad (25)$$

$$C_j = \sum_{i=1}^{N} \int_{t_o}^{t_o+T} g_i \left\{ \left[\hat{x}_i(\underline{\hat{x}}^o) - x_i(\underline{x}^o) \right] \phi_{ij} \right\} dt \quad (26)$$

$$\underline{G} = \left[1, g_2, g_3, \ldots . g_{N_o}, 0, 0, \ldots . 0 \right] \quad (27)$$

Therefore, equation (24) becomes:

$$\underline{C}(\underline{G}, \underline{x}^o, \underline{\hat{x}}^o, T) = \underline{A}(\underline{G}, \underline{x}^o, T) \, \delta \underline{x}^o \quad (28)$$

or:

$$\delta \underline{x}^o = \underline{A}^{-1} \underline{C} = \delta \underline{x}^o (\underline{G}, \underline{x}^o, \underline{\hat{x}}^o, T) \quad (29)$$

$\delta \underline{x}^o$ is found by solving equation (22) in terms of T, $\underline{\hat{x}}^o$, $\underline{x}^o$, and $\underline{G}$. Equation (22) can be rewritten as;

$$\int_{t_o}^{t_o+T} \sum_{i=1}^{N} \left\{ g_i \left[\hat{x}_i(\underline{\hat{x}}^o) - x_i(\underline{x}^o) \right] \phi_{ij} \right\} dt = \int_{t_o}^{t_o+T} \sum_{i=1}^{N} g_i \left[\sum_{k=1}^{N} \delta x_k^o \phi_{ik} \phi_{ij} \right] dt \quad (23)$$

or

$$\sum_{i=1}^{N} g_i \int_{t_o}^{t_o+T} \left\{ \left[\hat{x}_i(\underline{\hat{x}}^o) - x_i(\underline{x}^o) \right] \phi_{ij} \right\} dt = \sum_{k=1}^{N} \sum_{i=1}^{N} \left\{ g_i \left[\int_{t_o}^{t_o+T} (\phi_{ik} \phi_{ij}) dt \right] \delta x_k^o \right\} \quad (24)$$

Let:

$$A_{jk} = \sum_{i=1}^{N} \int_{t_o}^{t_o+T} g_i (\phi_{ik} \phi_{ij}) dt \quad (25)$$

$$C_j = \sum_{i=1}^{N} \int_{t_o}^{t_o+T} g_i \left\{ \left[\hat{x}_i(\underline{\hat{x}}^o) - x_i(\underline{x}^o) \right] \phi_{ij} \right\} dt \quad (26)$$

$$\underline{G} = \left[1, g_2, g_3, \ldots . g_{N_o}, 0, 0, \ldots . 0 \right] \quad (27)$$

Therefore, equation (24) becomes:

$$\underline{C}(\underline{G}, \underline{x}^o, \underline{\hat{x}}^o, T) = \underline{A}(\underline{G}, \underline{x}^o, T) \ \delta \underline{x}^o \quad (28)$$

or:

$$\delta \underline{x}^o = \underline{A}^{-1} \underline{C} = \delta \underline{x}^o (\underline{G}, \underline{x}^o, \underline{\hat{x}}^o, T) \quad (29)$$

Thus, for a given set of $\underline{x}^o$, $\underline{\hat{x}}^o$, $\underline{G}$, and T, the Kth+1 set of initial conditions is obtained from the Kth by the following equation:

$$\left[\underline{x}^o\right]^{K+1} = \left[\underline{x}^o\right]^K + \delta\underline{x}^o\left(\underline{G}, \left[\underline{x}^o\right]^K, \underline{\hat{x}}^o, T\right) \tag{30}$$

The problem now is to find the value of $\underline{G}$ and T which will result in the most rapid convergence for all $\underline{x}^o$ and $\underline{\hat{x}}^o$. From the initial design, it is possible to determine the limits of the values of $\underline{x}^o$ and $\underline{\hat{x}}^o$. The convergence should be the worst as $\underline{x}^o$ approaches one limit, and $\underline{\hat{x}}^o$ approaches the other. Therefore, the values of $\underline{G}$ and T that are optimal with respect to these limits can be assumed to be optimal, or near optimal for all intermediate values. $\underline{G}$ optimal and T optimal are obtained by using a least squares fit performance index, $\underline{P}(\underline{G}, T)$, given in equation (31).

$$\underline{P}(\underline{G}, T) = \sum_{i=1}^{M} \left(\left[\underline{x}^o\right]^K - \underline{\hat{x}}^o \right)_i^2 \tag{31}$$

where:

$M = 2^N$

i = the index of the ith extreme case of $\underline{x}^o$ and $\underline{\hat{x}}^o$

K = number of iterations before $\underline{P}(\underline{G}, T)$ is evaluated

The performance index is the sum of the squares of the measured differences between the model and actual system initial states after K iterations.

The M sets of $(\underline{x}^o, \underline{\hat{x}}^o)$ used in equation (31), are formed by setting each component of $\underline{x}^o$ equal to its upper limit or its lower limit, and the corresponding value of $\underline{\hat{x}}^o$ set equal to the opposite limit. This process is repeated for all possible combinations of $\underline{x}^o$ and $\underline{\hat{x}}^o$.

Minimizing equation (31) with respect to $\underline{G}$ and T simultaneously, results in an optimal set of g_i, and an optimal sampling interval. The use of $\underline{G}$ optimal and T optimal in the performance index, p, results in p optimal. Unknown system parameters are identified by minimizing p optimal using quasilinearization outlined in chapter three. The first step in identifying $\underline{\hat{x}}^o$ is to guess an $\underline{x}^o$. This value of $\underline{x}^o$ is augmented by $\delta\underline{x}^o$ which is determined by applying quasilinearization to the minimization of p optimal. This augmented value of $\underline{x}^o$ is then used as the guessed value, and the process is repeated on an iterative basis until the improvements $\delta\underline{x}^o$ are zero or near zero. The resulting values of $\underline{x}^o$ are the desired system parameters.

CHAPTER V

METHOD OF OPTIMAL DYNAMIC WEIGHTING COEFFICIENTS AND SAMPLING INTERVAL

For a large class of problems, methods of system identification which employ quasilinearization may suffer from the limitation of a narrow region of convergence. Alexander was able to expand the region of convergence by using the method of "Optimal Static Weighting Coefficients" presented in chapter four. The process developed in this chapter will be called the method of "Optimal Dynamic Weighting Coefficients and Sampling Interval." This method will further increase the region of convergence. This process determines a set of optimal weighting coefficients, $\underline{G}$, and sampling time interval, T, for each iteration.

The performance index is the one used by Eveleigh. It is given in equation (32).

$$p = \int_{t_o}^{t_o+T} \sum_{i=1}^{N} g_i \left[x_i(\underline{x}^o) + \sum_{j=1}^{N} \delta x_j \, \phi_{ij} - \hat{x}_i(\hat{\underline{x}}^o) \right]^2 dt \qquad (32)$$

For the first iteration, $\delta \underline{x}^o$ can be determined, in terms of $\underline{G}$ and T by following the procedure set forth in chapter four (equations 28-36). The following result is obtained:

$$\delta \underline{x}^o(\underline{G}, \underline{x}^o, \hat{\underline{x}}^o, T) = \underline{A}^{-1} \, \underline{C} \qquad (33)$$

where:

$$A_{jk} = \sum_{i=1}^{N} \int_{t_o}^{t_o + T} g_i (\phi_{ik}\ \phi_{ij})\, dt$$

$$C_j = \sum_{i=1}^{N} \int_{t_o}^{t_o + T} \left\{ g_i \left[\hat{x}_i (\hat{\underline{x}}^o) - x_i (\underline{x}^o) \right] \phi_{ij} \right\} dt$$

This value of $\delta \underline{x}^o$ is used to form the next model initial conditions by means of the following equation:

$$(\underline{x}^o)^1 = \underline{x}^o + \delta \underline{x}^o \tag{34}$$

Substituting $(\underline{x}^o)^1$ into a least squares fit performance index, $\underline{P}(\underline{G}, T)$, results in equation (35).

$$\underline{P}(\underline{G}, T) = \sum_{i=1}^{M} \left[(\underline{x}^o)^1 - \hat{\underline{x}}^o \right]_i^2 \tag{35}$$

where:

$M = 2^N$

i = index of the ith extreme case of $\hat{\underline{x}}^o$ and $\underline{x}^o$

$\underline{P}(\underline{G}, T)$ is then optimized with respect to $\underline{G}$ and T to obtain $\underline{G}_1$ and T_1 optimal. These values are used to find p_1 optimal. This process is repeated for each iteration until p_K is zero or near zero. The values of the resulting parameters are the desired system identification.

An outline of the computational procedure is given in the following steps.

(1) $\underline{m}(t)$ and $\underline{x}(t)$ are recorded over T.

(2) The augmented model equations of the form:

$$\dot{\underline{x}} = f(\underline{x}, m)$$

are programmed into the computer.

(3) Starting conditions $\underline{x}^o$ as near the true values as possible are assumed.

(4) The model equations are linearized about the nominal solution path.

(5) The values of $\delta\underline{x}^o$ are determined.

(6) Using a least squares fit performance index, $\underline{G}_1$ optimal and T_1 optimal are obtained. These values are then used to find p_1 optimal.

(7) p_1 optimal is minimized by means of quasilinearization to determine parameter changes for the next iteration.

(8) The process is repeated, if necessary, until sucessive adjustments provide negligible improvements on p.

(9) The resulting model parameters are read out as the desired plant identification.

As will be shown by means of an example (the Van der Pol equation), this method of identification has a larger region of convergence than any of the other methods.

CHAPTER VI

EXAMPLE: THE VAN DER POL EQUATION

The method of Optimal Dynamic Weighting Coefficients and Sampling Interval is applied to the Van der Pol equation. The state equations for the Van der Pol equation are given as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -e(1-x_1^2)x_2 - ux_1 \end{aligned} \tag{36}$$

Let $x_3 = e$ and $x_4 = u$. This results in the following augmented equations:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_3(1-x_1^2)x_2 - x_4x_1 \\ \dot{x}_3 &= 0 \\ \dot{x}_4 &= 0 \end{aligned} \tag{37}$$

Solving for $\underline{J}$:

$$\underline{J} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2x_1x_2x_3 - x_4 & -x_3(1-x_1^2) & -(1-x_1)x_2 & -x_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{38}$$

$$\underline{F} = \underline{J}\ \underline{x}(\underline{x}^{o}) \tag{39}$$

$$\dot{\phi} = \underline{F}\ \phi \qquad ; \qquad \phi(0) = \underline{I} \tag{40}$$

The following initial guessed and actual values are used:

$$\underline{x}^{o} = \begin{bmatrix} 1.9 \\ -0.095 \\ 2.0 \\ 0.5 \end{bmatrix} \quad \text{and} \quad \hat{\underline{x}}^{o} = \begin{bmatrix} 2.0 \\ -0.055 \\ 3.65 \\ 1.17 \end{bmatrix} \tag{41}$$

Equations (37)-(41) lead to equation (42) for each iteration.

$$\int_0^T \sum_{i=1}^{4} g_i \left[\sum_{k=1}^{4} \phi_{ik}\, \phi_{ij}\, \delta x_k^{o} \right] dt =$$

$$\int_0^T \sum_{i=1}^{4} g_i \left[\hat{x}_i(\hat{\underline{x}}^{o}) - x_i(\underline{x}^{o}) \right] dt \tag{42}$$

$$j = 1, 2, 3, 4$$

The initial values and $\underline{\phi}$ are used together to evaluate equation (42).

Equation (42) can be rewritten as in equation (43).

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} \delta x_1^o \\ \delta x_2^o \\ \delta x_3^o \\ \delta x_4^o \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \tag{43}$$

where:

$$A_{jk} = \int_0^T \left[\phi_{1k}\, \phi_{1j} + \phi_{2k}\, \phi_{2j} \right] dt$$

$$C_j = \int_0^T \left\{ \left[\hat{x}_1(\hat{\underline{x}}^o) - x_1(\underline{x}^o) \right] \phi_{1j} + g_2 \left[\hat{x}_2(\hat{\underline{x}}^o) - x_2(\underline{x}^o) \right] \phi_{2j} \right\} dt$$

$\underline{P}(\underline{G}, T)$ is formed for each iteration.

$$\underline{P}(\underline{G}, T) = \sum_{j=1}^{4} (\underline{x}_j^o - \hat{\underline{x}}_j^o)^2 \tag{44}$$

Minimizing $\underline{P}(\underline{G}, T)$ with respect to $\underline{G}$ and T gives the following results for three iterations.

Iteration	$\underline{G}$ Optimal	T Optimal
1	[1, 6.5, 0, 0]	0.33sec.
2	[1, 1.2, 0, 0]	2.0sec.
3	[1, 5.0, 0, 0]	0.2sec.

(45)

Using $\underline{G}$ optimal and T optimal of iteration for iterations 4 and 5 results in convergence in five iterations. This is the same as results obtained using Alexander's

method. This is shown in Table 1.

When the parameter space is expanded, the method presented here results in convergence while the others may not. For example, let

$$\begin{aligned} 1.9 &\leq x_1^o \leq 2.0 \\ -0.095 &\leq x_2^o \leq -0.055 \\ 2.0 &\leq x_3^o \leq 3.81 \\ 0.5 &\leq x_4^o \leq 1.23 \end{aligned} \tag{46}$$

Using these values, convergence resulted in five iterations. However, Alexander's method could not converge for the expanded parameter space. This is shown in Table 2.

TABLE 1

RESULTS FOR ORIGINAL PARAMETER SPACE

Iteration	Alexander	Hanna
1	88.62	28.67
2	30.12	3.002
3	4.638	0.534
4	0.89×10^{-1}	0.15×10^{-1}
5	0.19×10^{-4}	0.16×10^{-4}

TABLE 2

RESULTS FOR EXPANDED PARAMETER SPACE

Iteration	Alexander	Hanna
1	342.85	38.19
2	1947.26	18,96
3	915.98	2.567
4	411.74	0.27×10^{-1}
5	1733.44	0.37×10^{-5}

CHAPTER VII

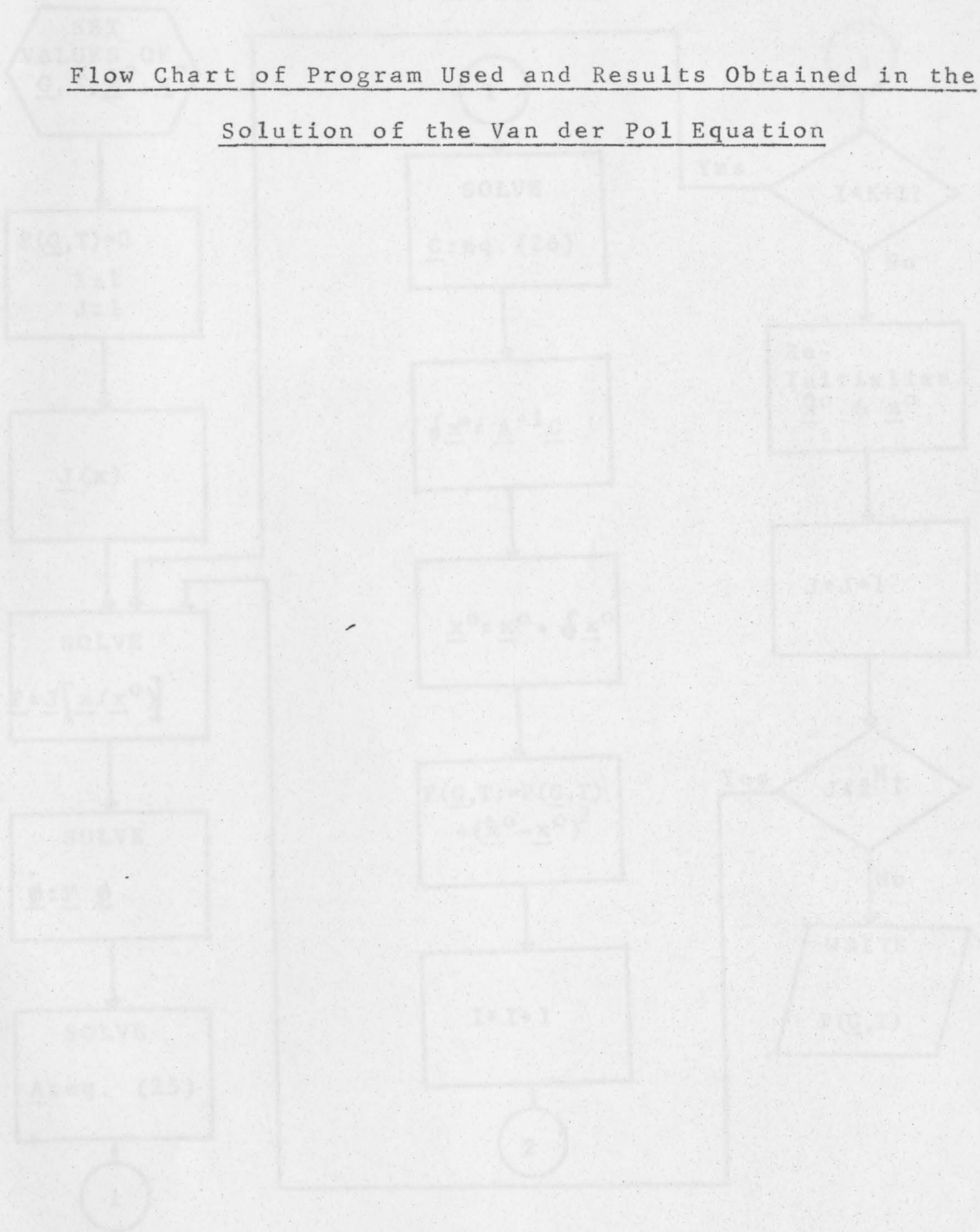
CONCLUSIONS

A method of identification for both linear and nonlinear systems was developed. This method of Optimal Dynamic Weighting Coefficients and Sampling Interval uses quasilinearization as a basis. It was shown that this process has a larger region of convergence than other methods which also employ quasilinearization.

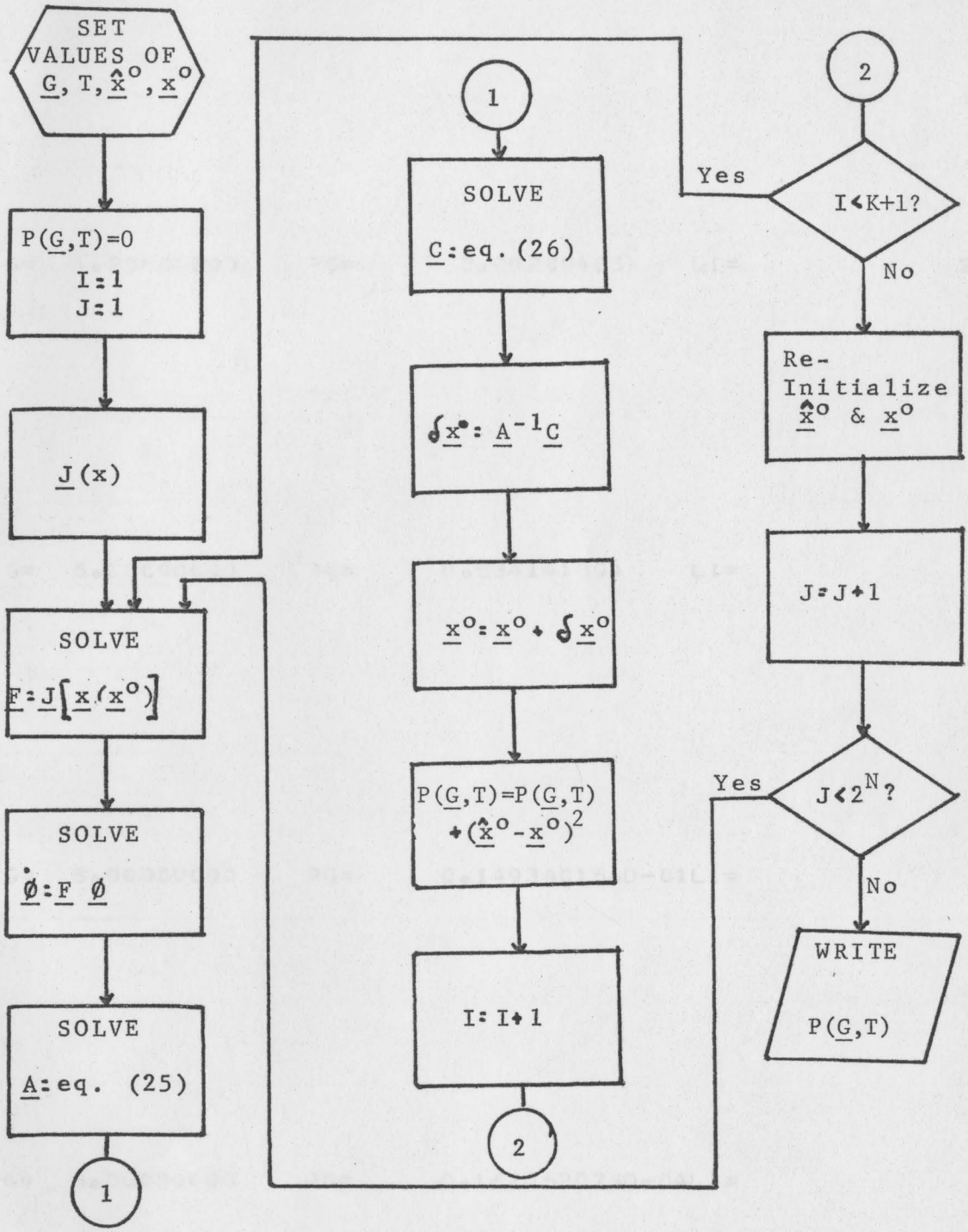
This method requires a large amount of computer time. However, since the result of this process is an identifier which allows identification over a large parameter space, the benefits outweigh this drawback.

APPENDIX A

Flow Chart of Program Used and Results Obtained in the Solution of the Van der Pol Equation



Flow Chart of Program Used



Results of Original Parameter Space

G= 6.5000000 PG= 28.5724683 LI= 33

G= 1.2000000 PG= 3.00289485 LI= 200

G= 5.0000000 PG= 0.534141894 LI= 20

G= 5.0000000 PG= 0.1493401540-01LI= 20

G= 5.0000000 PG= 0.1620620750-04LI= 20

Results of Expanded Parameter Space

G= 6.00000000 PG= 38.19255867 LI= 33

G= 0.400000000 PG= 18.9665978 LI= 42

G= 1.00000000 PG= 2.56781154 LI= 35

G= 1.00000000 PG= 0.275033488D-01LI= 35

G= 1.00000000 PG= 0.370849810D-05LI= 35

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