DYNAMIC STABILITY OF A BEAM-COLUMN

by

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ABSTRACT

DYNAMIC STABILITY OF BEAM-COLUMN

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The purpose of this thesis is to investigate the dynamic stability of a beam-column including the effects of shear and rotary inertia.

The nonlinear theory yields a coupled-set of linear partial differential equations of motion. The natural frequencies of free vibration are obtained for the case where shear and rotary inertia are included, as well as, the case of pure shear only. Comparisions are made with the natural frequencies given by the classical theory. Critical buckling loads are determined by the zero condition of natural frequency.

The critical buckling loads and natural frequencies, obtained by the Bernoulli-Euler theory, the Pure Shear theory, and Timoshenko theory, are analyzed and graphically compared.

The dynamic stiffness matrix which includes the effects of both shear and rotary inertia is formulated. The static stiffness matrix which includes the effect of shear is also obtained.

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LIST OF NOTATIONS

A = Area of cross-section

E = Young's modulus of Elasticity

G = Shear modulus

I = Moment inertia of cross-section

K = Shape factor for strain energy of shear

$= \frac{A}{I^2}\int \frac{Q^2(y)}{b}\,dy$ for rectangular cross-section $K = 1.2$

$Q(y)$ = First moment of area about neutral axis

L = Length of beam and beam-column

P = Axial compressive force

b = Width of beam and beam-column

$q(x,t)$ = Transverse load

r = Radius of gyration

$y(x,t)$ = Total defection

x, y = Rectangular co-ordinates

t = Time

$\phi(x,t)$ = Slope due to shear

ρ = Mass density

ω = Frequency of beam-radians/second

Ω = Frequency of beam-column-radians/second

$\beta = \frac{P}{P_{cr}}$

Δ = Defection

V = Shear force

$c = \sqrt{\frac{E}{\rho}}$

$n = n^{th}$ mode shape

$$\eta^2 = \frac{\rho A \Omega^2 \left\{ r^2 \left(1 - \frac{KP}{GA} + \frac{KE}{G} \right) \right\} + P}{EI \left(1 - \frac{KP}{GA} \right)}$$

$$\chi^2 = \frac{\rho A \Omega^2 \left(1 - \Omega^2 \frac{r^2 \rho K}{G} \right)}{EI \left(1 - \frac{KP}{GA} \right)}$$

$$\gamma = \frac{\eta}{\sqrt{2}} \sqrt{-1 + \sqrt{1 + \frac{4\chi^2}{\eta^2}}}$$

$$\delta = \frac{\eta}{\sqrt{2}} \sqrt{1 + \sqrt{1 + \frac{4\chi^2}{\eta^2}}}$$

$$\Gamma = \left(-EI\gamma^2 + \frac{GA}{K} - \rho I \Omega^2 \right)$$

$$\Lambda = \left(EI\gamma^2 + \rho I \Omega^2 \right)$$

$$\Pi = \left(-EI\delta^2 + \frac{GA}{K} - \rho I \Omega^2 \right)$$

$$\xi = \left(EI\delta^2 + \rho I \Omega^2 \right)$$

$$\Phi = \left(\gamma^2 + \frac{\Lambda \gamma^2}{\Gamma} \right)$$

$$\Psi = \left(\frac{\xi \delta^2}{\Pi} - \delta^2 \right)$$

$$\Theta = \delta - \frac{\xi \delta}{\Pi}$$

$$\zeta = \gamma + \frac{\Lambda \gamma}{\Gamma}$$

LIST OF SUBSCRIPTS

b = Bending stress

cr = Critical load

r = Rotary inertia

E = Euler theory

G = Shear stress theory

O = Bending stress theory

S = Pure shear theory

T = Timoshenko theory

n = n^{th} mode shape

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CHAPTER I
INTRODUCTION

The analytical determination of dynamic stability criteria of beam-columns leads to a set of coupled, linear partial differential equations of motion.

Timoshenko[1] determined the static buckling load of a beam-column including the effect of bending stress as well as shear stress.

Brunelle[2] considered the buckling loads of the beam-column, included bending and shear for a variety of boundary conditions, including simply-supported, fixed, and a combination of both.

Anderson[3] determined the natural frequency of a beam, including the effects of shear and rotary inertia. The forced vibration problem is solved using normal mode theory.

Langhaar[4] generated the equations of a beam using the minimum potential theorem.

In this thesis, the dynamic stability of free vibration of beam-column including the effects of shear and rotary inertia, is determined. Using Hamilton's principle. The equations of motion, the natural and forced boundary conditions are determined. Two special cases which are considered are simply-supported both ends and fixed-fixed boundary conditions both ends.

The stiffness matrix is formulated via the equations of motion of simply-supported beam-column. The special cases which are considered is as follows:

1) The effects of bending, shear, axial force, rotary inertia and transverse inertia.

2) Bending, shear, axial force and transverse inertia are included.

3) Bending, shear and axial force are included(i.e. the static case).

CHAPTER II

METHOD OF ANALYSIS

2-1 Assumptions

The assumptions for analysis of the beam-column, subject to axial load, are as follows:

1) The beam-column consists of a perfectly elastic material.

2) The beam-column is originally perfectly straight.

3) The axial loads are applied along the centroidal axis of the beam-column.

These assumptions are used for analysis throughout this thesis.

2-2 Hamilton's Principle

Hamilton's principle takes the following mathematical form:

$$\delta\hat{A} = \delta\int_{t=t_0}^{t=t_1} \hat{L}\,dt = 0\ , \qquad (2\text{-}1)$$

where $\hat{L} = \hat{T} - \hat{V}$, with $\hat{T}$ = kinetic energy

$\hat{V}$ = potential energy

and δ designates the first variational operation. This condition yields differential equations of motion[5] of a beam-column as

$$\begin{bmatrix} -EI\left(\frac{\partial^2}{\partial x^2}\right)+\frac{GA}{K}+\rho I\left(\frac{\partial^2}{\partial t^2}\right) & EI\left(\frac{\partial^3}{\partial x^3}\right)-\rho I\left(\frac{\partial^3}{\partial x\,\partial t^2}\right) \\ EI\left(\frac{\partial^3}{\partial x^3}\right)-\rho I\left(\frac{\partial^3}{\partial x\partial t^2}\right) & -EI\left(\frac{\partial^4}{\partial x^4}\right)-P\left(\frac{\partial^2}{\partial x^2}\right)-\rho A\left(\frac{\partial^2}{\partial t^2}\right)+\rho I\left(\frac{\partial^4}{\partial x^2\partial t^2}\right) \end{bmatrix} \begin{bmatrix} \phi(x,t) \\ y(x,t) \end{bmatrix} = \begin{bmatrix} 0 \\ q(x,t) \end{bmatrix} \quad (2\text{-}2)$$

where $y(x,t)$ is defined as total deflection,

$\phi(x,t)$ is slope due to shear,

and $K = \frac{A}{I^2}\int \frac{Q^2(y)}{b}\,dy$ with $Q(y)$ equal to the moment area of cross-section about the neutral axis (for rectangular cross-section $K = 1.2.$)

Also, the associated natural and force boundary conditions are prescribed:

@ $x = 0$ or $x = L$

either $V(x) = EI(y_{xxx} - \phi_{xx}) + Py_x - \rho I(y_{xtt} - \phi_{tt}) = 0$

or $y = 0,$

either $M(x) = -EI(y_{xx} - \phi_x) = 0$

or $(y_x - \phi) = 0.$

For a simply-supported beam-column the boundary conditions are

$$\left.\begin{aligned} y(o) &= 0, \\ y(L) &= 0, \\ M(0) &= 0, \text{ and} \\ M(L) &= 0, \end{aligned}\right\} \quad (2\text{-}3)$$

For fixed-fixed supports the boundary conditions are

$$\left.\begin{aligned} y(0) &= 0, \\ y(L) &= 0, \\ y_x(0) - \phi(0) &= 0, \text{ and} \\ y_x(L) - \phi(L) &= 0. \end{aligned}\right\} \quad (2\text{-}4)$$

For the special case of pure shear stress and transverse inertia equations(2-2) reduce to the single equation

$$\frac{GA}{K}\frac{\partial \phi}{\partial x} = P\frac{\partial^2 y}{\partial x^2} + \rho A\frac{\partial^2 y}{\partial t^2} \quad . \qquad (2\text{-}5)$$

2-3 <u>Free vibration of a beam-column.</u>

For the case of free vibration the transverse force in equation (2-2) is neglected, $q(x,t) = 0$. Assuming the time function is harmonic and separating variables on space and time in the form

$$\left.\begin{aligned} y(x,t) &= A\, e^{\lambda x} e^{i\Omega t} \quad \text{and} \\ \phi(x,t) &= B\, e^{\lambda x} e^{i\Omega t}, \end{aligned}\right\} \quad (2\text{-}6)$$

one obtains,

$$\begin{bmatrix} -EI\lambda^2 + \frac{GA}{K} - \rho I \Omega^2 & EI\lambda^3 + \rho I \Omega^2 \\ EI\lambda^3 + \rho I \Omega^2 & -EI\lambda^4 - P\lambda^2 + \rho A \Omega^2 - \rho I \lambda^2 \Omega^2 \end{bmatrix} \begin{bmatrix} B \\ A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (2\text{-}7)$$

For non-trival solutions of the parameters of A and B, the determinant coefficient matrix is set equal to zero and yields the following equation of natural frequency:

$$\left(\frac{\Omega_T}{\lambda c}\right)^2 = -\frac{1}{2}\left[1+\frac{G}{KE}-\frac{P}{AE}-\frac{1}{(\lambda r)^2}\frac{G}{KE}\right]\left[1 \pm \sqrt{1+\frac{4\left[\frac{P}{AE}-\frac{G}{KE}-\frac{1}{(\lambda c r)^2}\frac{PG}{\rho AKE}\right]}{\left[1+\frac{G}{KE}-\frac{P}{AE}-\frac{1}{(\lambda r)^2}\frac{G}{KE}\right]^2}}\right] \quad (2\text{-}8)$$

If the effect of axial force is neglected (i.e. P=0) equation (2-8) reduces to the form given in reference [3]

If the rotary inertia is neglected, (i.e. ρI is neglected), equation (2-8) becomes a single equation of the form

$$\left(\frac{\Omega_{Gn}}{\lambda c}\right)^2 = \frac{\frac{I}{A}\left[1-\frac{KP}{GA}\right]\lambda^2+\frac{P}{AE}}{1-\frac{KEI}{GA}\lambda^2} \quad . \quad (2\text{-}9)$$

For the case when shear is neglected, (i.e. $G=\infty$), as well as rotary inertia, the natural frequency becomes

$$\left(\frac{\Omega_{On}}{\lambda c}\right)^2 = \frac{I}{A}\lambda^2+\frac{P}{AE} \quad . \quad (2\text{-}10)$$

In the case of pure shear, the bending stress terms are neglected(i.c. terms containing EI are eliminated) together with rotary inertia terms; one obtains:

$$\left(\frac{\Omega_s}{\lambda c}\right)^2 = \frac{P}{AE}-\frac{G}{KE} . \quad (2\text{-}11)$$

For the special case of a beam, the axial force P, is equated to zero. The natural frequencies obtained by utilization of equations (2-8) to equation (2-11) become

$$\left[\frac{\omega_T}{\lambda c}\right]^2 = -\frac{1}{2}\left[1+\frac{G}{KE}-\frac{1}{(\lambda c)^2}\frac{G}{KE}\right]\left[1\pm\sqrt{1-\frac{4\frac{G}{KE}}{\left\{1+\frac{G}{KE}-\frac{1}{(\lambda c)^2}\frac{G}{KE}\right\}^2}}\right] \qquad (2\text{-}12)$$

for bending, shear, rotary and transverse inertia,

$$\left[\frac{\omega_{Gn}}{\lambda c}\right]^2 = \frac{\frac{I}{A}\lambda^2}{1-\frac{KEI}{GA}\lambda^2} \qquad (2\text{-}13)$$

for neglecting rotary inertia,

$$\left[\frac{\omega_{On}}{\lambda c}\right]^2 = \frac{I}{A}\lambda^2 \qquad (2\text{-}14)$$

for neglecting shear and rotary inertia, and

$$\left[\frac{\omega_S}{\lambda c}\right]^2 = -\frac{G}{KE} \qquad (2\text{-}15)$$

for neglecting bending and rotary inertia.

2-4 <u>Roots of the fourth order differential equation.</u>

For non-trival solution of equations (2-7). The determinant of the coefficient matrix is set equal to zero, and yields the following quartic equation:

$$\lambda^4 + \frac{\left[\rho A\Omega^2\left\{\left(1-\frac{KP}{GA}\right)r^2+\frac{KEI}{GA}\right\}+P\right]}{EI\left(1-\frac{KP}{GA}\right)}\lambda^2 - \frac{\left(\rho A\Omega^2-\frac{K\rho^2 I}{G}\Omega^4\right)}{EI\left(1-\frac{KP}{GA}\right)}\lambda^0 = 0\ . \qquad (2\text{-}16)$$

Equation (2-16) above yields four independent roots expressed as,

$$\left.\begin{aligned} \lambda_1 &= \gamma, \\ \lambda_2 &= -\gamma, \\ \lambda_3 &= i\delta, \quad \text{and} \\ \lambda_4 &= -i\delta, \end{aligned}\right\} \quad (2\text{-}17)$$

where

$$\gamma = \frac{\eta}{\sqrt{2}}\sqrt{-1+\sqrt{1+\frac{4\chi^2}{\eta^2}}},$$

$$\delta = \frac{\eta}{\sqrt{2}}\sqrt{1+\sqrt{1+\frac{4\chi^2}{\eta^2}}},$$

$$\eta = \frac{\rho A \Omega^2\left\{r^2\left(1-\frac{KP}{GA}+\frac{KE}{G}\right)\right\}+P}{EI\left(1-\frac{KP}{GA}\right)},$$

and

$$\chi = \frac{\rho A \Omega^2\left(1-\frac{r^2\rho K}{G}\Omega^2\right)}{EI\left(1-\frac{KP}{GA}\right)}.$$

These roots yield the solutions:

$$\left.\begin{aligned} y(x) &= A_1\cosh\gamma x + B_1\sinh\gamma x + C_1\cos\delta x + D_1\sin\delta x \\ \phi(x) &= A_2\cosh\gamma x + B_2\sinh\gamma x + C_2\cos\delta x + D_2\sin\delta x . \end{aligned}\right\} \quad (2\text{-}18)$$

Using the first equation of equations (2-7), together with equation (2-18), the constants in equation (2-18), are related as

$$\left.\begin{aligned} A_2 &= -\frac{\gamma\Lambda}{\Gamma}B_1, \\ B_2 &= \frac{\gamma\Lambda}{\Gamma}A_1, \\ C_2 &= \frac{\delta\xi}{\Pi}D_1, \quad \text{and} \\ D_2 &= -\frac{\delta\xi}{\Pi}C_1, \end{aligned}\right\} \quad (2\text{-}19)$$

where

$$\Lambda = (EI\gamma^2 + \rho I \Omega^2),$$

$$\Gamma = (-EI\gamma^2 + \frac{GA}{K} - \rho I \Omega^2),$$

$$\xi = (EI\delta^2 + \rho I \Omega^2), \text{ and}$$

$$\Pi = (-EI\delta^2 + \frac{GA}{K} - \rho I \Omega^2).$$

Hence, the constants of equations (2-18) reduce to four which allows for the application of only four boundary conditions, two at each end.

CHAPTER III

SIMPLY-SUPPORTED BEAM-COLUMN

3-1 The simply-supported beam-column and beam.

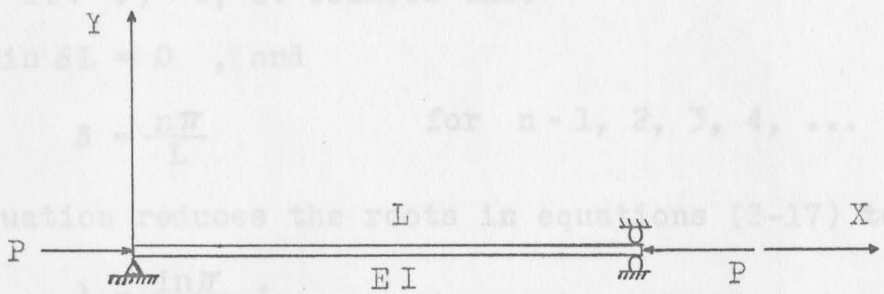


FIG. I Coordinate system of a beam-column.

For the simply-supported beam-column shown in Fig.I, applying the boundary conditions of equations (2-3) to equations (2-18), together with equations (2-19), it follows that

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ \cosh\gamma L & \sinh\gamma L & \cos\delta L & \sin\delta L \\ \Phi & 0 & \Psi & 0 \\ \Phi\cosh\gamma L & \Phi\sinh\gamma L & \Psi\cos\delta L & \Psi\sin\delta L \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad (3\text{-}1)$$

where

$$\Phi = \gamma^2 + \frac{\Lambda\gamma^2}{\Gamma} , \quad \text{and}$$

$$\Psi = \frac{\xi\delta^2}{\Pi} - \delta^2 .$$

For nontrival solution of this equation, the determinant of the coefficient matrix is set equal to zero and yields:

$$\sinh \gamma L \sin \delta L \, (\Psi^2 - 2\Psi\Phi + \Phi^2) = 0 \qquad (3\text{-}2)$$

Since the condition

$$\sinh \gamma L \neq 0 \quad ,$$

and $(\Psi^2 - 2\Psi\Phi + \Phi^2) \neq 0$, it follows that

$$\sin \delta L = 0 \quad \text{, and}$$

$$\delta = \frac{n\pi}{L} \qquad \text{for } n = 1, 2, 3, 4, \ldots \qquad (3\text{-}3)$$

This equation reduces the roots in equations (2-17) to

$$\lambda = \frac{in\pi}{L} \, . \qquad (3\text{-}4)$$

It follows from equation (3-1) that $A_1 = B_1 = C_1 = 0$ and D_1 is arbitrary; hence,

$$y(x,t) = D_1 \sin \frac{n\pi x}{L} \, e^{i\Omega t} \, ,$$

$$\text{and} \quad \phi(x,t) = C_2 \sin \frac{n\pi x}{L} \, e^{i\Omega t} \qquad (3\text{-}5)$$

where $C_2 = \frac{\delta \xi}{\Pi} D_1$.

Application of equation (3-4) with equation (2-8) to equation (2-15) yields the natural frequencies:

$$\left(\frac{\Omega_T L}{n\pi c}\right)^2 = \frac{1}{2}\left[1 + \frac{G}{KE} - \frac{P}{AE} + \left(\frac{L}{n\pi r}\right)^2 \frac{G}{KE}\right]\left[1 \pm \sqrt{1 + \frac{4\left[\frac{P}{AE} - \frac{G}{KE} + \left(\frac{L}{n\pi r}\right)^2 \frac{P}{AE}\frac{G}{KE}\right]}{\left[1 + \frac{G}{KE} - \frac{P}{AE} + \left(\frac{L}{n\pi r}\right)^2 \frac{G}{KE}\right]^2}}\,\right], \qquad (3\text{-}6)$$

$$\left(\frac{\Omega_{Gn} L}{n\pi c}\right)^2 = \frac{\left(1 - \frac{KP}{GA}\right)\left(\frac{n\pi r}{L}\right)^2 - \frac{P}{AE}}{1 + \frac{KE}{G}\left(\frac{n\pi r}{L}\right)^2} \, , \qquad (3\text{-}7)$$

$$\left(\frac{\Omega_{on}L}{n\pi c}\right)^2 = \left(\frac{n\pi r}{L}\right)^2 - \frac{P}{AE} \quad , \quad \text{and} \tag{3-8}$$

$$\left(\frac{\Omega_{S}L}{n\pi c}\right)^2 = \frac{G}{KE} - \frac{P}{AE} \quad , \tag{3-9}$$

and for the beam (i.e. P = 0) as

$$\left(\frac{\omega_T L}{n\pi c}\right)^2 = \frac{1}{2}\left[\left(1+\frac{G}{KE}\right)+\left(\frac{L}{n\pi r}\right)^2\frac{G}{KE}\right]\left[1 \pm \sqrt{1-\frac{4\frac{G}{KE}}{\left[\left(1+\frac{G}{KE}\right)+\left(\frac{L}{n\pi r}\right)^2\frac{G}{KE}\right]^2}}\,\right] \quad , \tag{3-10}$$

$$\left(\frac{\omega_{Gn}L}{n\pi c}\right)^2 = \frac{\left(\frac{n\pi r}{L}\right)^2}{1+\frac{KE}{G}\left(\frac{n\pi r}{L}\right)^2} \quad , \tag{3-11}$$

$$\left(\frac{\omega_{on}L}{n\pi c}\right)^2 = \left(\frac{n\pi r}{L}\right)^2 , \quad \text{and} \tag{3-12}$$

$$\left(\frac{\omega_{s}L}{n\pi c}\right)^2 = \frac{G}{KE} \quad . \tag{3-13}$$

3-2 Critical Buckling Loads of Simply-Supported Beam-Column

The static critical buckling loads of the beam-column are obtained by equating the natural frequencies of free vibration to zero. It follows from equations (3-6) and (3-7), with the condition of zero frequency that

$$P_{cr}=(P_n)_T= \frac{EI\left(\frac{n\pi}{L}\right)^2}{1+\frac{KE}{G}\left(\frac{n\pi r}{L}\right)^2} , \quad (3\text{-}14)$$

which is defined as the Timoshenko buckling load[1]. Performing a similar operation on equations (3-8) and (3-9), yields the Euler buckling load as

$$P_{cr}= (P_n)_E = EI\left(\frac{n\pi}{L}\right)^2 , \quad (3\text{-}15)$$

and the shear buckling load as

$$P_{cr} = P_s = \frac{GA}{K} . \quad (3\text{-}16)$$

Combining equations (3-14), equation (3-15), and equation (3-16), it follows that

$$\frac{1}{P_{nT}} = \frac{1}{P_{nE}} + \frac{1}{P_s} . \quad (3\text{-}17)$$

Using equation (3-14), equation (3-15), and equation (3-16), the natural frequencies in equation (3-6) through equation (3-9) are defined in terms of the critical buckling loads as

$$\left(\frac{\Omega_T L}{n\pi C}\right)^2 = \frac{1}{2}\left[\frac{P_s}{AE}\left(1+\frac{AE}{P_{nT}}-\frac{P}{P_s}\right)\right]\left[1 \pm \sqrt{1+\frac{4\frac{P_s}{AE}\left(\frac{P}{P_{nT}}-1\right)}{\left[\frac{P_s}{AE}\left(1+\frac{AE}{P_{nT}}-\frac{P}{P_s}\right)\right]^2}}\right] , \quad (3\text{-}18)$$

$$\left(\frac{\Omega_{Gn} L}{n\pi C}\right)^2 = \frac{P_{nT}}{AE}\left(1-\frac{P}{P_{nT}}\right) , \quad (3\text{-}19)$$

$$\left(\frac{\Omega_{on} L}{n\pi C}\right)^2 = \frac{P_{nE}}{AE}\left(1-\frac{P}{P_{nE}}\right) , \text{ and} \quad (3\text{-}20)$$

$$\left(\frac{\Omega_s L}{n\pi C}\right)^2 = \frac{P_s}{AE}\left(1-\frac{P}{P_s}\right) , \quad (3\text{-}21)$$

and for the beam (i.e. P = 0) as

$$\left(\frac{\omega_{Gn} L}{n\pi C}\right)^2 = \frac{P_{nT}}{AE} , \quad (3\text{-}22)$$

$$\left(\frac{\omega_{on} L}{n\pi C}\right)^2 = \frac{P_{nE}}{AE} , \text{ and} \quad (3\text{-}23)$$

$$\left(\frac{\omega_s L}{n\pi C}\right)^2 = \frac{P_s}{AE} . \quad (3\text{-}24)$$

Using equation (3-19) through equation (3-24), the natural frequency in equation (3-18) is defined as

$$\left(\frac{\Omega_T L}{n\pi C}\right)^2 = \frac{1}{2}\left[\left(\frac{\Omega_s L}{n\pi C}\right)^2 + \left(\frac{\omega_s}{\omega_{Gn}}\right)^2\right]\left[1 \pm \sqrt{1 - \frac{4\left(\frac{\omega_s}{\omega_{Gn}}\right)^2\left(\frac{\Omega_{Gn} L}{n\pi C}\right)^2}{\left[\left(\frac{\Omega_s L}{n\pi C}\right)^2 + \left(\frac{\omega_s}{\omega_{Gn}}\right)^2\right]^2}}\right]. \qquad (3\text{-}25)$$

CHAPTER IV

GRAPHICAL RESULTS FOR THE SIMPLY-SUPPORTED CASE

Equations (3-14) through (3-16) for critical buckling loads for the cases of combined bending and shear, bending only, and shear only are shown grapically in Figure 1.

Figure 2 represents a plot of the natural frequency equations of the beam with the combined shear and rotary inertia condition. The value of the parameter $\sqrt{\frac{G}{KE}} = 0.5774$ with $K = 1.2$, $G = 12 \times 10^6$, and $E = 30 \times 10^6$.

Equations (3-6) through (3-9) representing the natural frequencies of the beam-column are plotted in Figure 3 for various values of applied axial force. (i.e. $\frac{P_{cr}}{4} \leq P \leq 0.99 P_{cr}$) As the axial force P increases the lower set of frequencies is effected significantly, while the upper set is only minutely changed.

The approximate solution of the natural frequency of the beam is plotted in Figure 4. When the terms of equation (3-10) are expanded in power series form. This approximation yield valid results for the lower set of frequencies only.

Upon expansion of equation (3-6) in power series form, the approximate solution of the lower set of natural frequencies of the beam-column are plotted in Figure 5 for various values of applied axial force. (i.e. $\frac{P_{cr}}{4} \leq P \leq 0.99\ P_{cr}$)

The natural frequencies of the Timoshenko theory given by the exact solution of equation (3-6) are higher than the approximate values given by the series expansion of equation (3-6).

Figures 6 and 7 show the curves of the ratio of Timoshenko buckling load to Euler buckling load for a rectangular cross section for the values of $\frac{KE}{G}$ equal to 3.0 and 3.5, repectively.

Figures 8 and 9 represent the plots of the ratio of Timoshenko buckling load given by equation (3-14) to Euler buckling load in equation (3-15) verses L/r for the first three modes, and for the values of $\frac{KE}{G}$ equal to 3.0 and 3.5, respectively.

The natural frequencies of a vibrating beam or beam-column verses the L/r ratio are shown in Figure 10. The ratio of the natural frequency including rotary inertia to the frequency excluding rotary inertia is plotted against the L/r ratio with consideration of bending and transverse inertia. The value of axial force does not effect this ratio.

The natural frequencies of the vibrating beam-column are shown in Figures 11 and 12. The ratio of the natural frequencies including the effects of shear and transverse inertia to the natural frequencies given by the Bernoulli-Euler theory (see equation (3-19) and (3-20) is plotted verses L/r ratio. This frequency ratio is plotted for various values of applied axial force (i.e. $0 \leq P \leq P_{nT}$) with $\frac{KE}{G}$ equals to 3.0

and 3.5, respectively.

Figures 13 and 14 represent the ratio of the lower set of natural frequencies of a vibrating beam-column given by the Timoshenko theory to the Bernoulli-Euler theory (see equations (3-18) and (3-20)) verses L/r ratio for various values of applied axial force, (i.e. $0 \leq P \leq P_{nT}$), and for $\frac{KE}{G}$ equals to 3.0 and 3.5, respectively.

Figures 15 and 16 are the plots of the ratio of the lower set of the natural frequencies of a vibrating beam-column given by the Timoshenko theory (see equation 3-18) to the natural frequencies including shear and transverse inertia (see equation 3-19) for various values of applied axial force, (i.e. $0 \leq P \leq P_{nT}$) and $\frac{KE}{G}$ equals to 3.0 and 3.5, respectively.

The ratio of the upper set of the natural frequencies of a vibrating beam-column given by the Timoshenko theory (see equation 3-18)to the natural frequencies including the effects of shear and transverse inertia given in equation (3-19) is plotted in Figure 17 and 18 for various values of applied axial force, (i.e. $0 \leq P \leq P_{nT}$), and for $\frac{KE}{G}$ equals to 3.0 and 3.5, respectively.

Figures 19 and 20 represent the ratio of the upper set of the natural frequencies of a vibrating beam-column to the lower set of the natural frequencies given by the Timoshenko theory (see equation 3-18) with various values of applied axial force, (i.e. $0 \leq P \leq P_{cr}$) and for $\frac{KE}{G}$ equals to 3.0 and 3.5, respectively.

FIGURE 1 Critical Buckling Loads for the Beam-column.

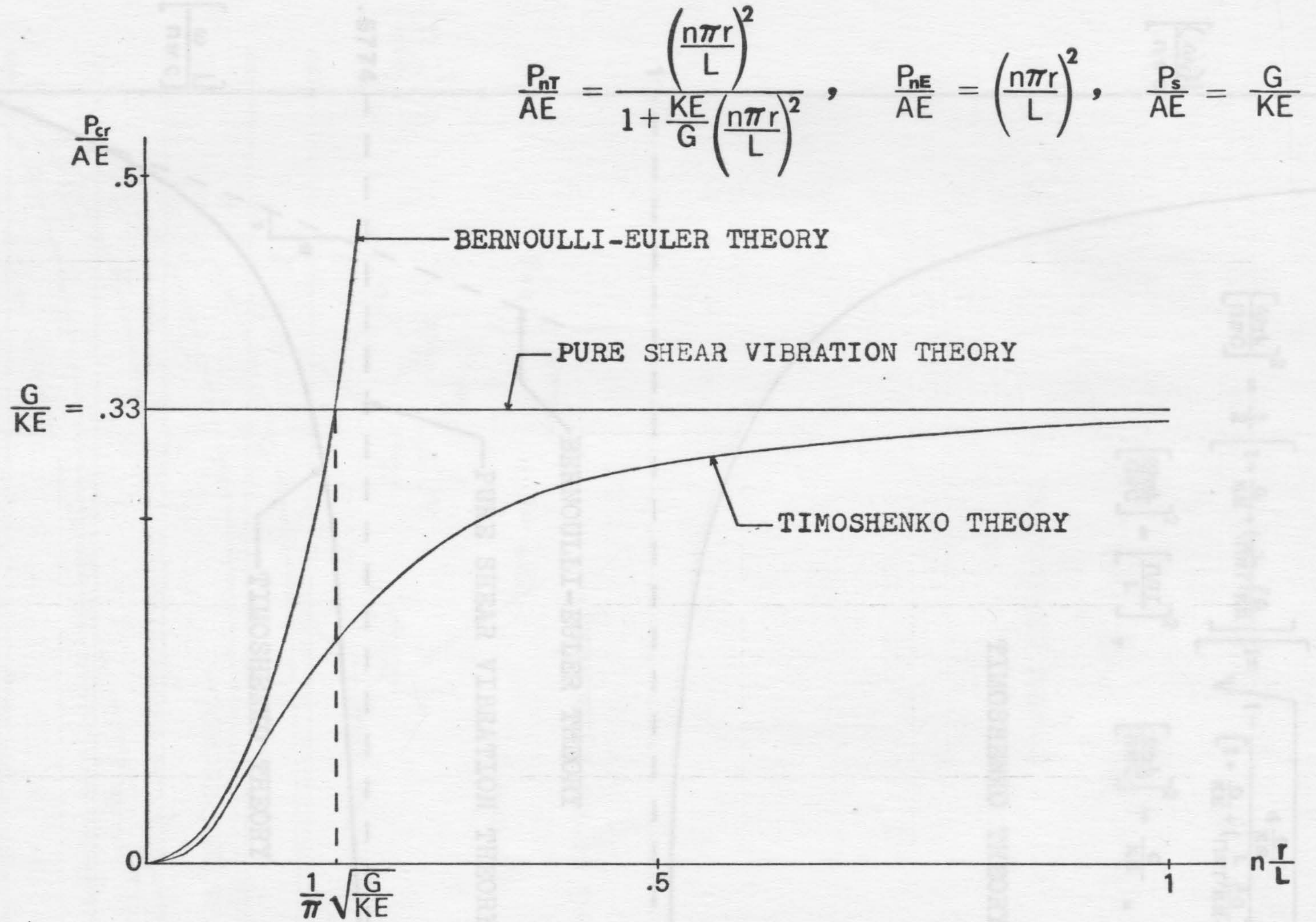


FIGURE 1 Critical Buckling Loads for the Beam-column.

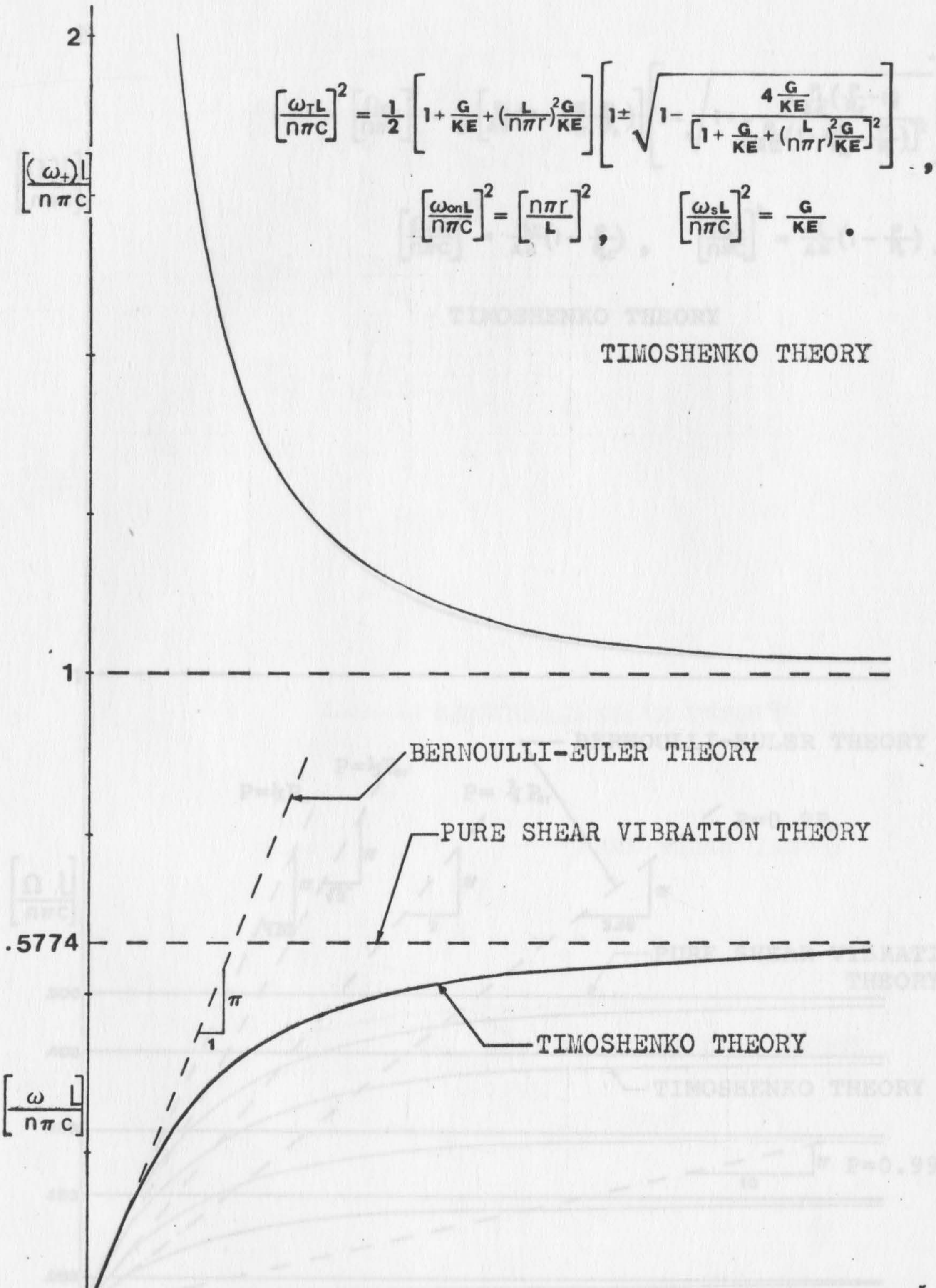


FIGURE 2 Natural Frequencies of Free Vibration of a Beam.

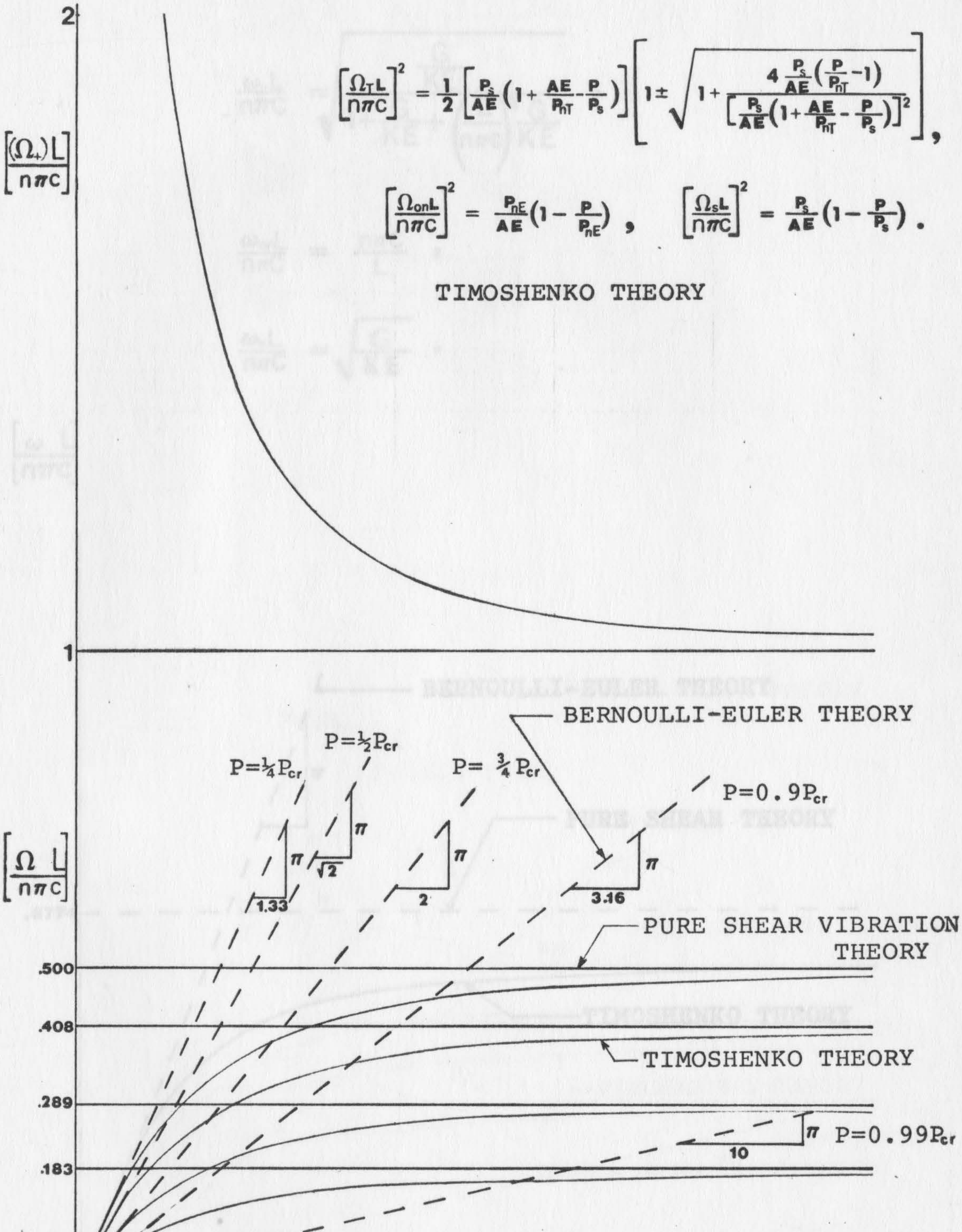


FIGURE 3 Natural Frequencies of a Beam-column for $\frac{1}{2} P_{cr} \leq P \leq 0.99P_{cr}$.

$$\frac{\omega_- L}{n\pi C} = \sqrt{\frac{\frac{G}{KE}}{1+\frac{G}{KE}+\left(\frac{L}{n\pi c}\right)^2\frac{G}{KE}}} \; ,$$

$$\frac{\omega_{on} L}{n\pi C} = \frac{n\pi C}{L} \; ,$$

$$\frac{\omega_s L}{n\pi C} = \sqrt{\frac{G}{KE}} \; .$$

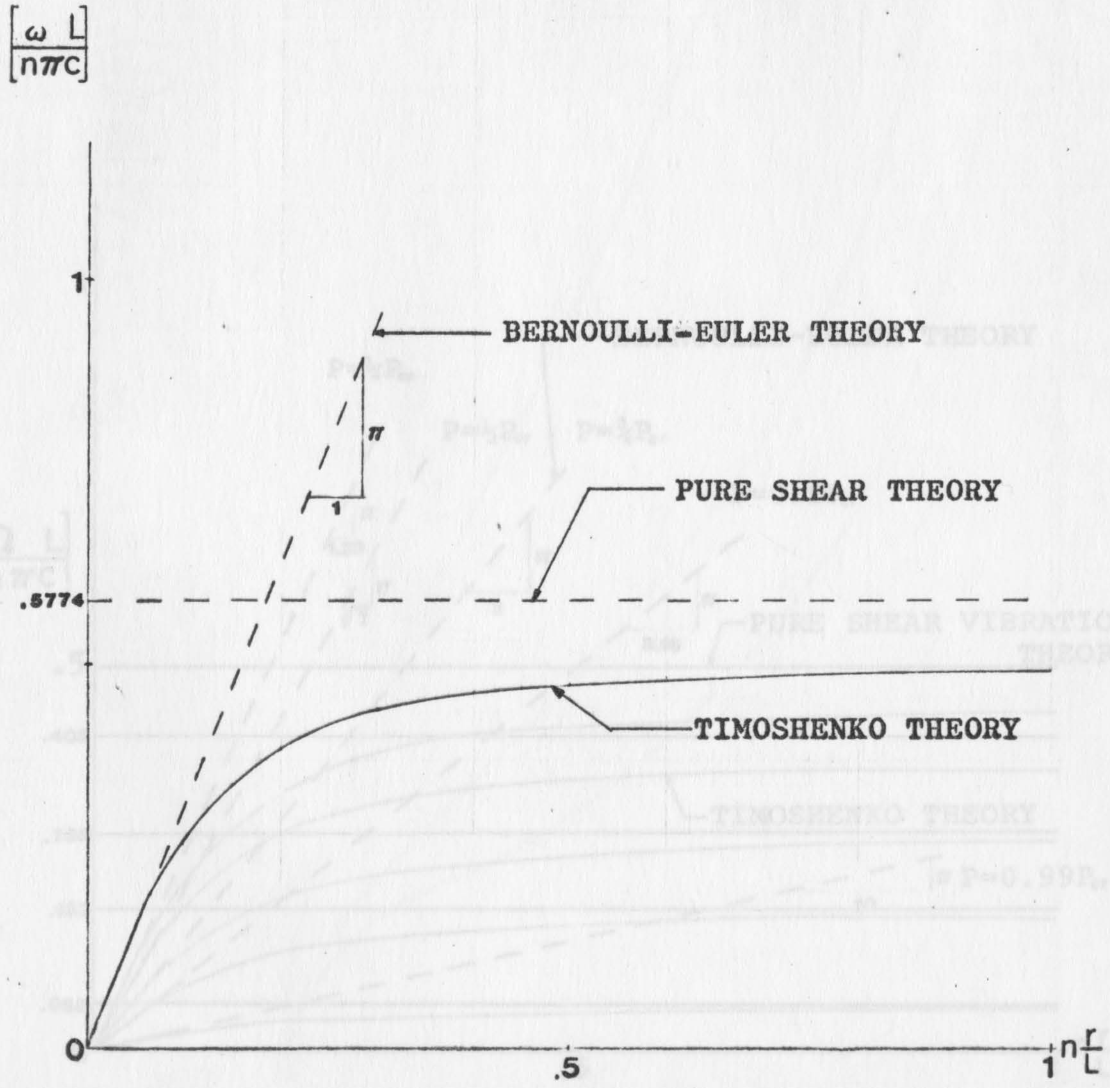


FIG.4 The approximate solution of Natural Frequencies of the Beam.

$$\frac{\Omega_T L}{n\pi C} = \sqrt{\frac{\frac{P_s}{AE}\left(1-\frac{P}{P_{nT}}\right)}{\frac{P_s}{AE}\left(1+\frac{AE}{P_{nT}}-\frac{P}{P_s}\right)}} \quad ,$$

$$\frac{\Omega_{on} L}{n\pi C} = \sqrt{\frac{P_{nE}}{AE}\left(1-\frac{P}{P_{nE}}\right)} \quad ,$$

$$\frac{\Omega_s L}{n\pi C} = \sqrt{\frac{P_s}{AE}\left(1-\frac{P}{P_s}\right)} \quad .$$

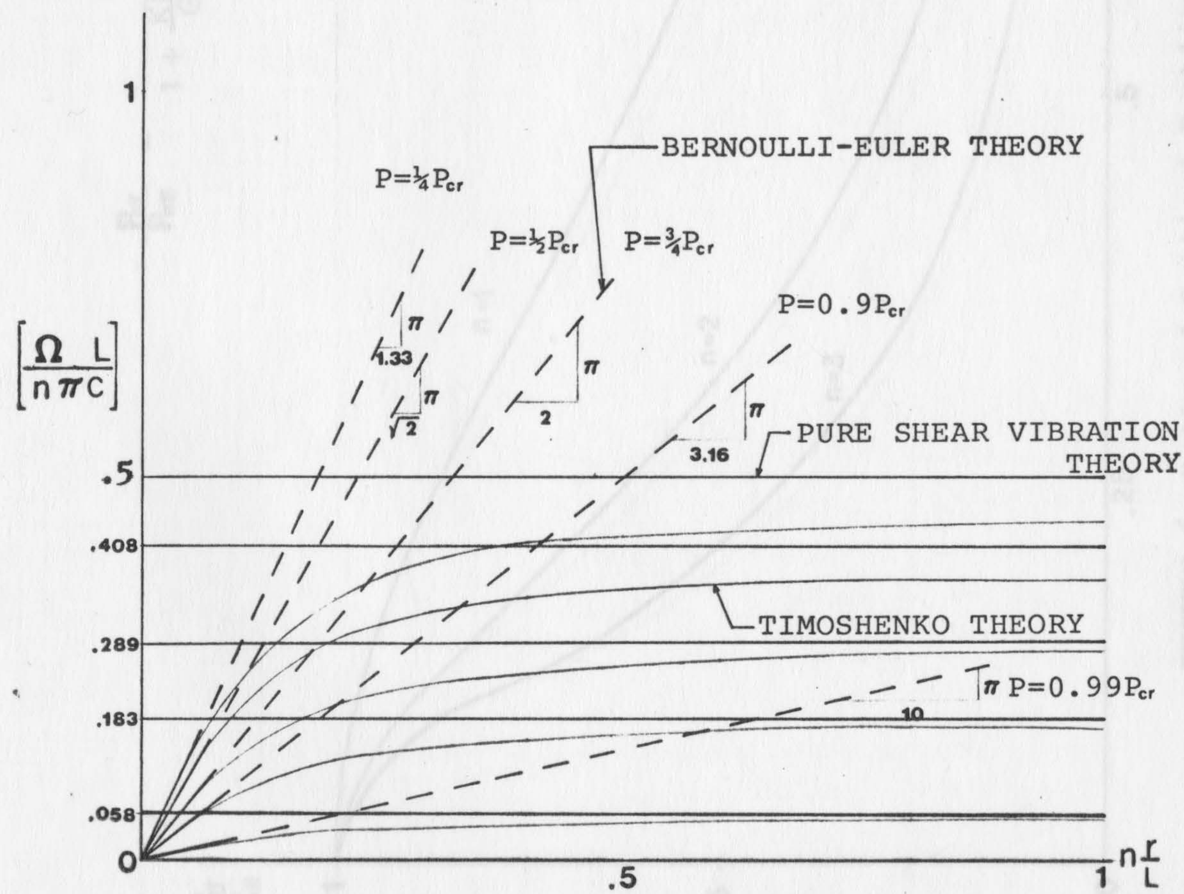


FIG. 5 The approximate solution of Natural Frequencies of the Beam-column for $\frac{1}{2} P_{cr} \leq P \leq 0.99 P_{cr}$.

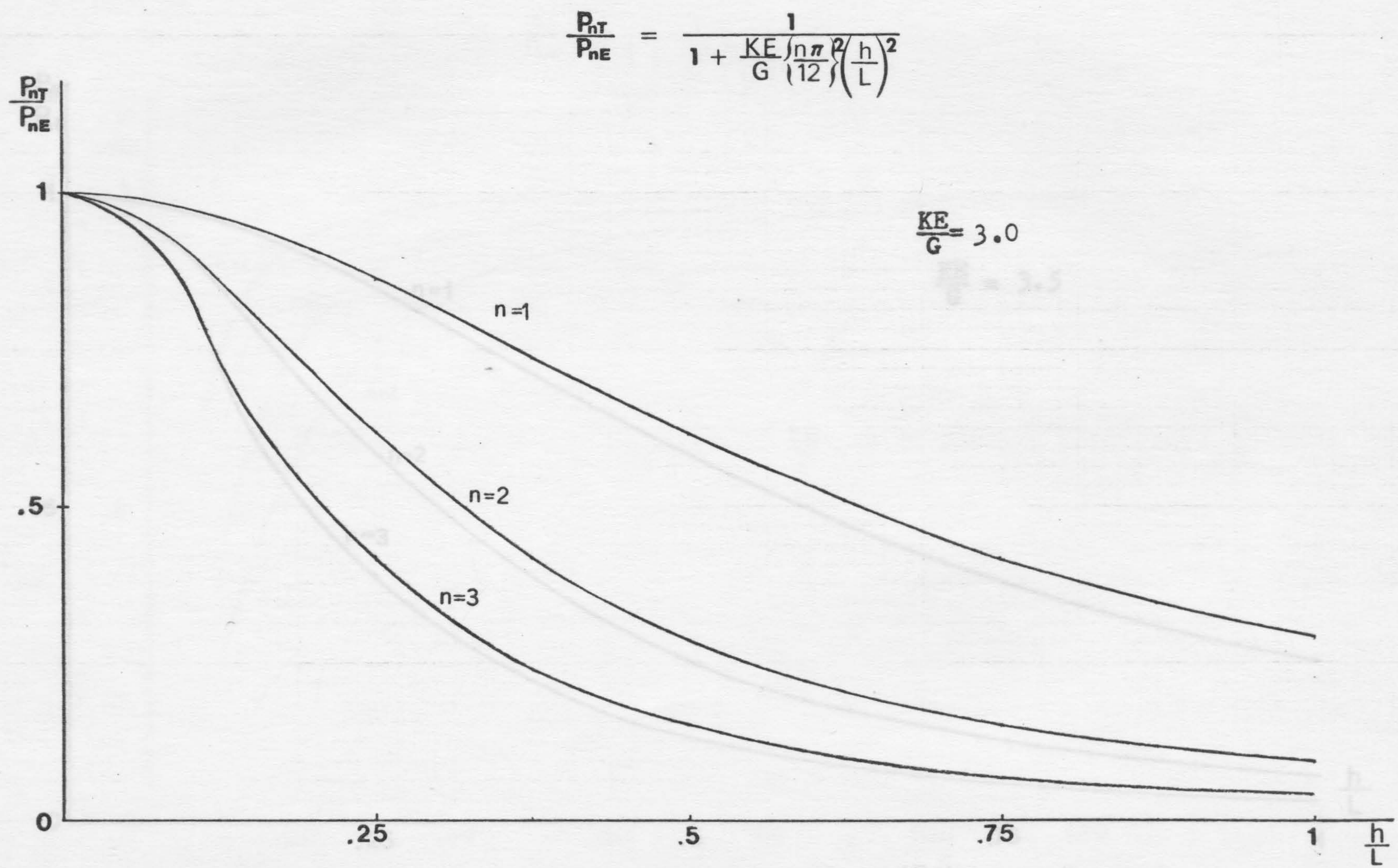


FIGURE 6 Ratio of Critical Buckling Loads $(\frac{P_{nE}}{P_{nT}})$ vs h/L ratio.

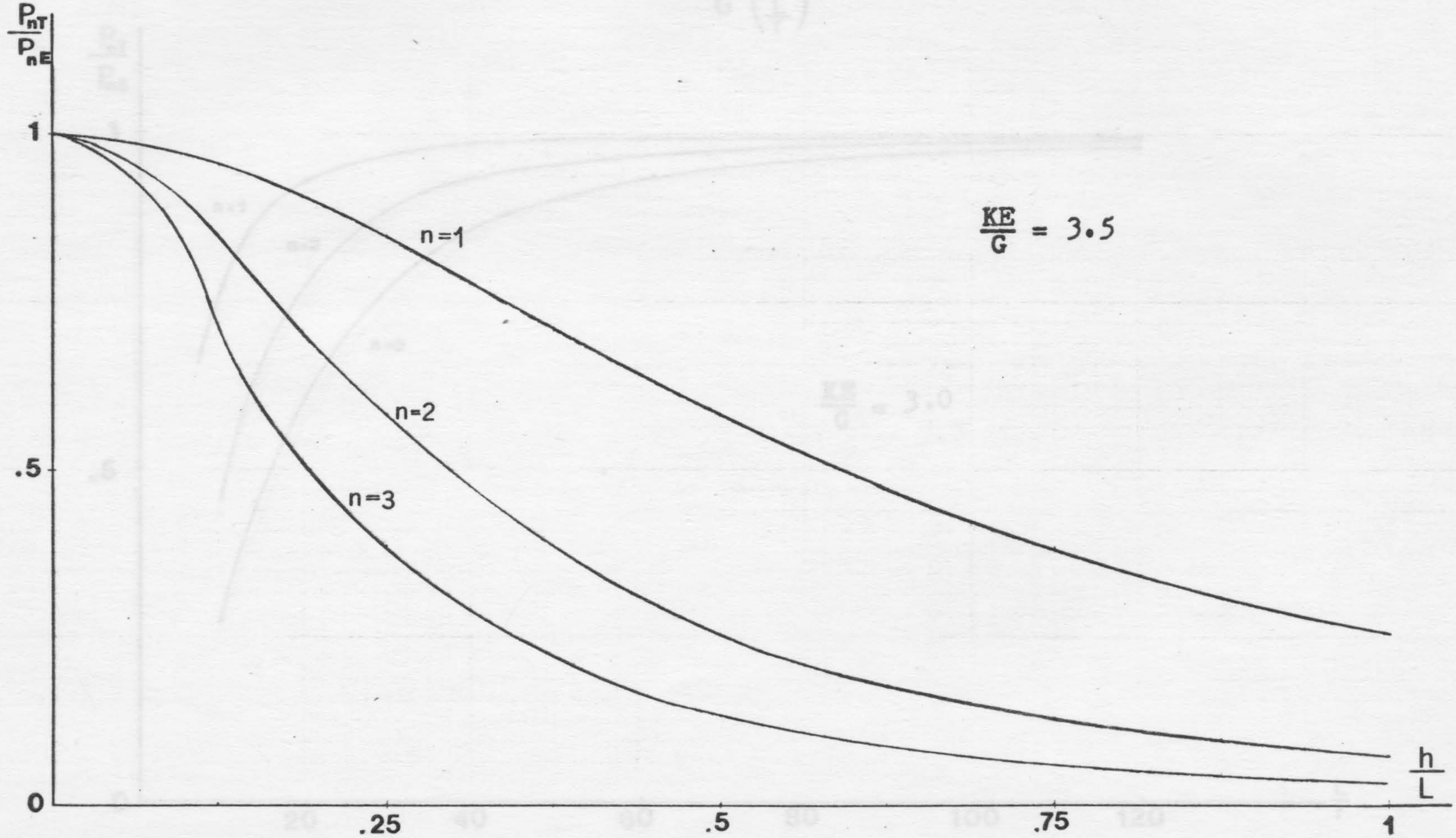


FIGURE 7 Ratio of Critical Buckling Loads ($\frac{P_{nE}}{P_{nT}}$) vs h/L ratio

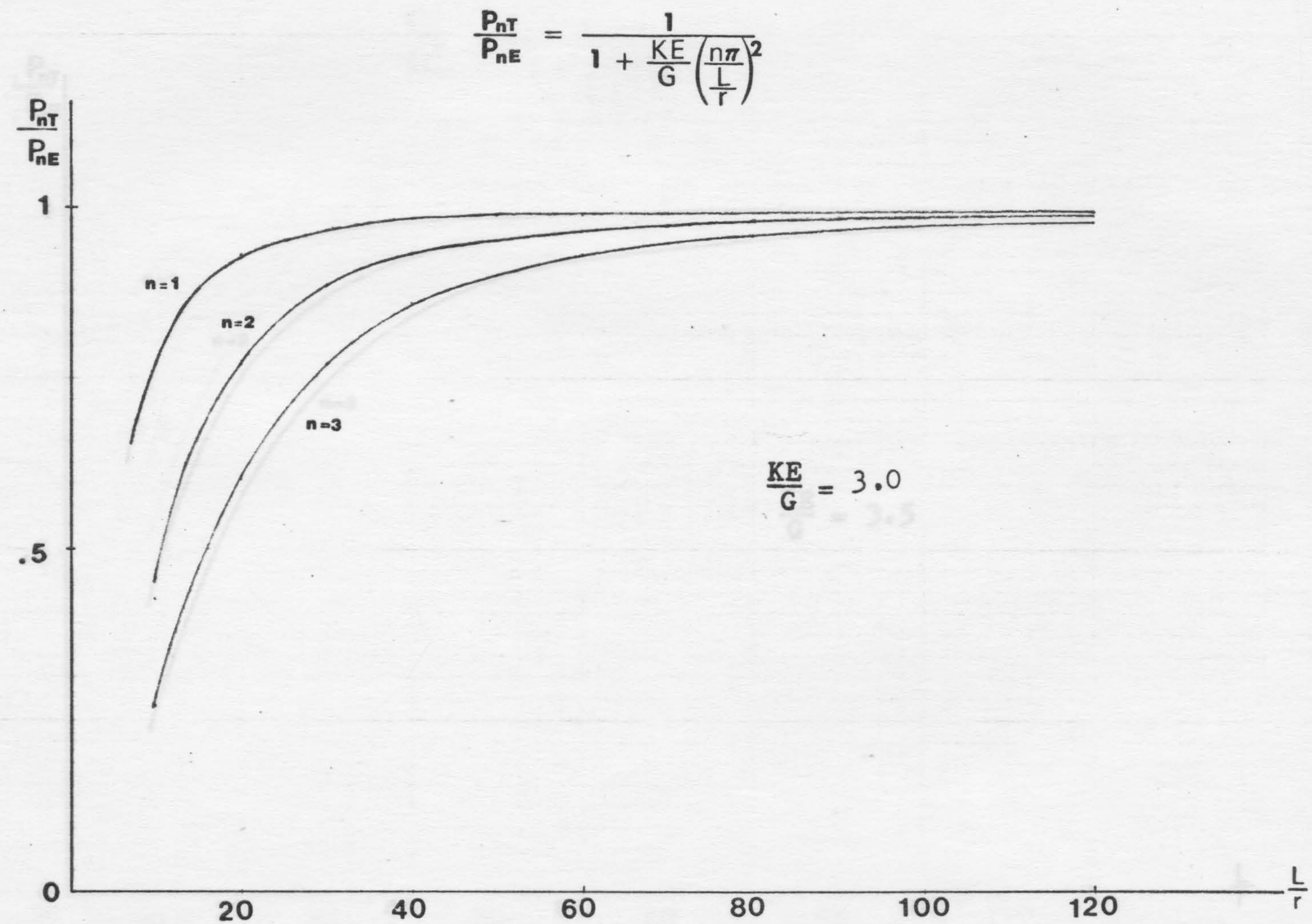


FIGURE 8 Ratio of Critical Buckling Loads $(\frac{P_{nE}}{P_{nT}})$ vs L/r ratio.

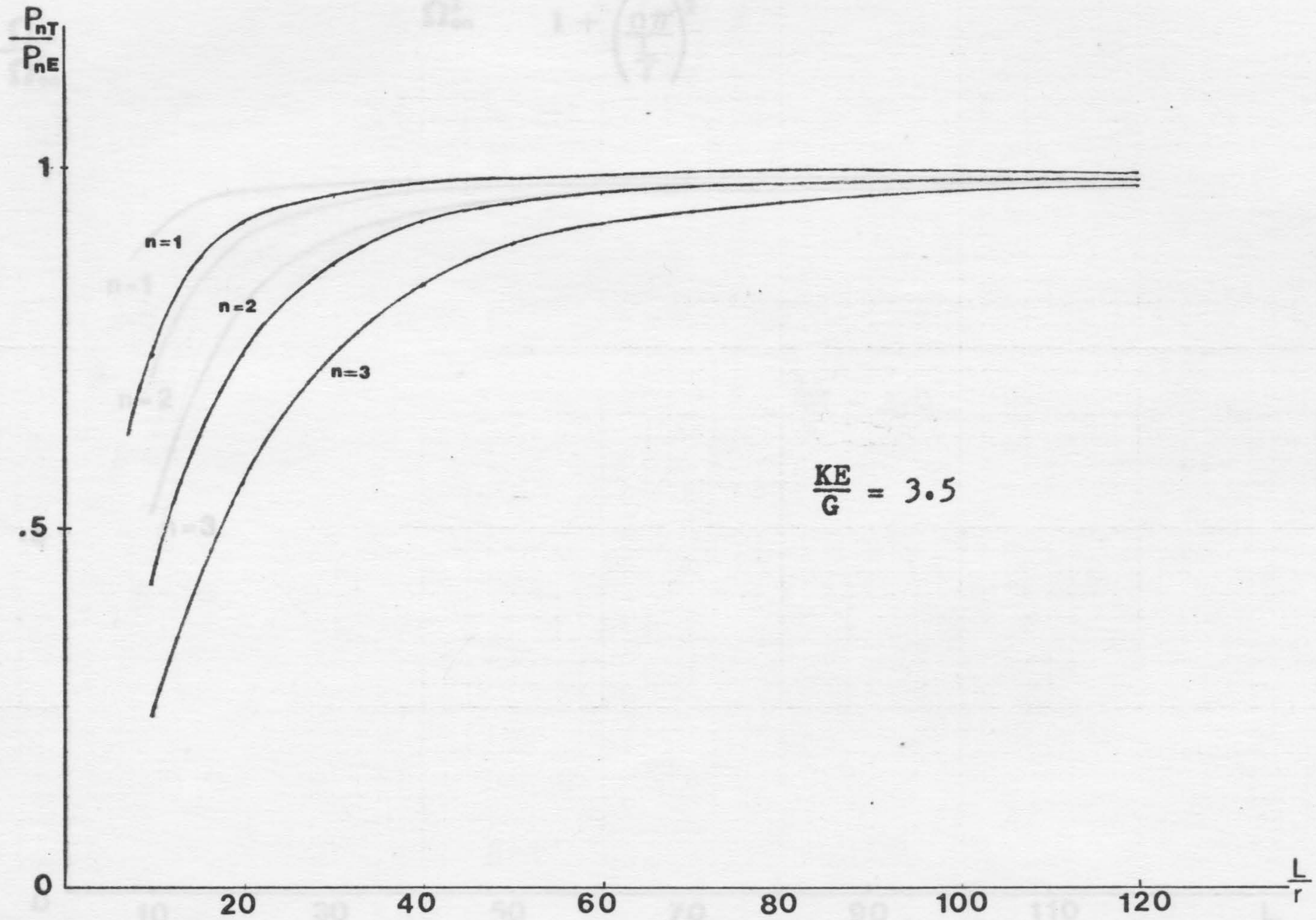


FIGURE 9 Ratio of Critical Buckling Loads ($\frac{P_{nE}}{P_{nT}}$) vs L/r ratio.

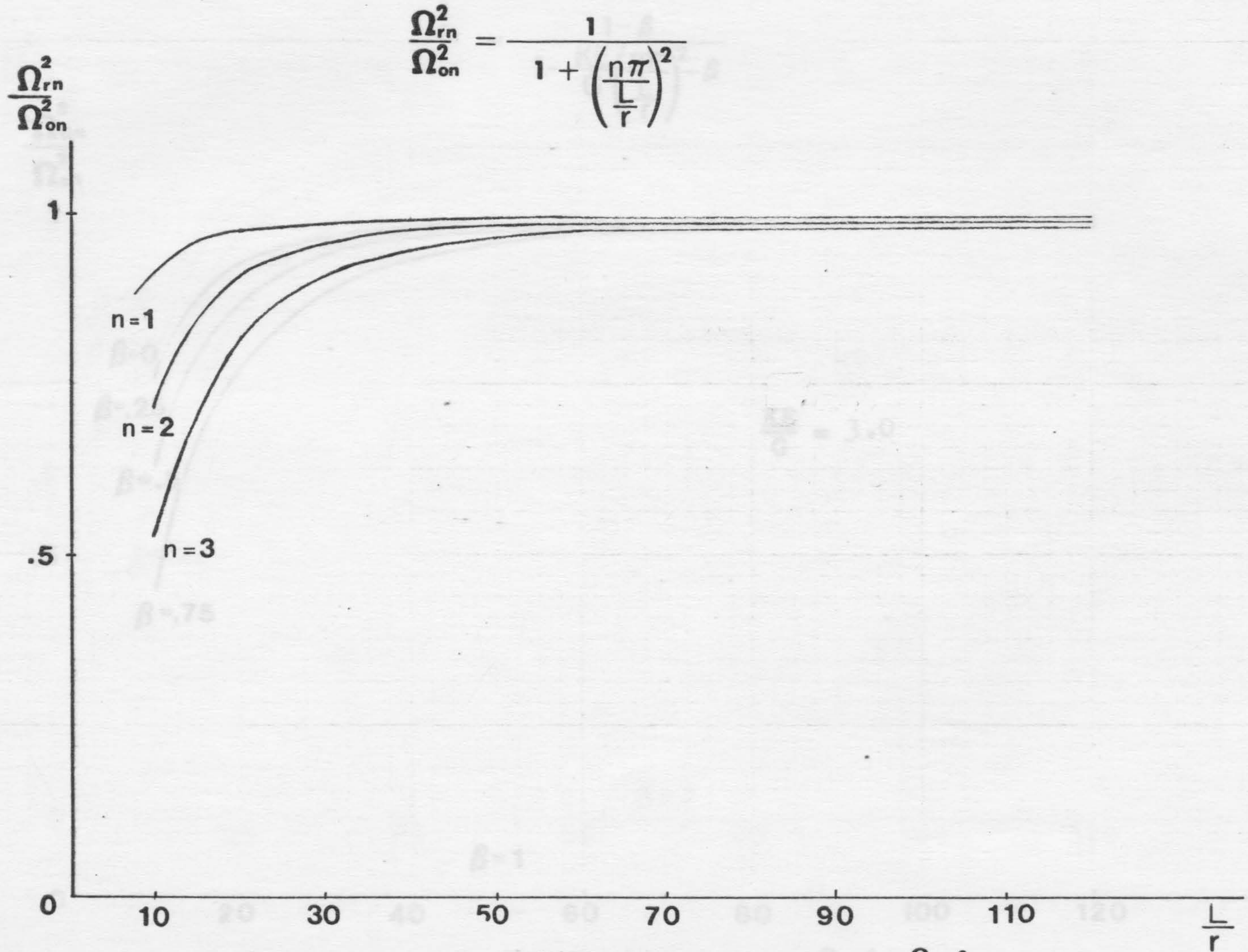


FIGURE 10 Ratio of Natural Frequencies $(\frac{\Omega_{rn}}{\Omega_{on}})^2$ vs L/r.

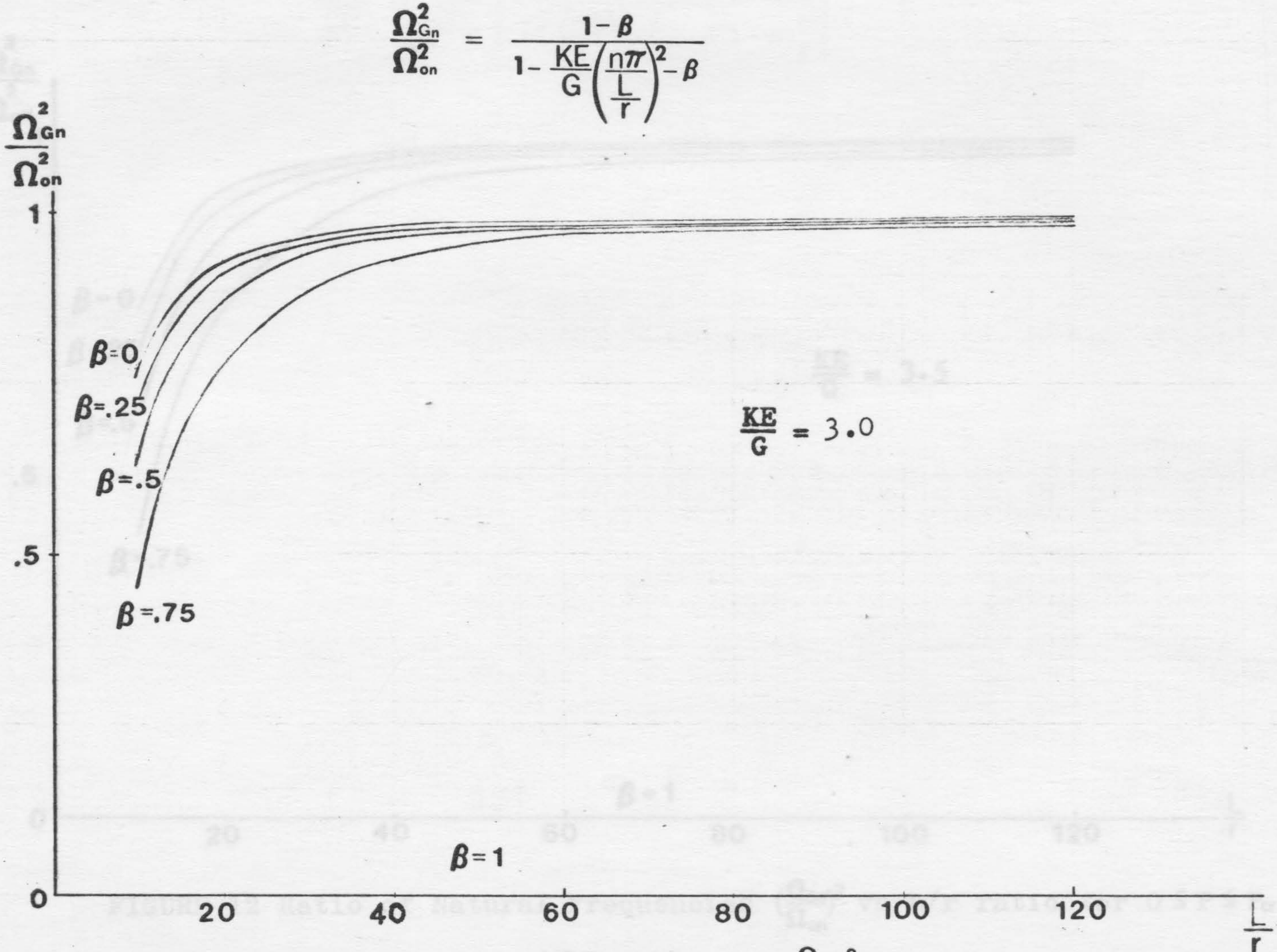


FIGURE 11 Ratio of Natural Frequencies $\left(\frac{\Omega_{Gn}}{\Omega_{on}}\right)^2$ vs L/r ratio for $0 \leq P \leq P_{cr}$.

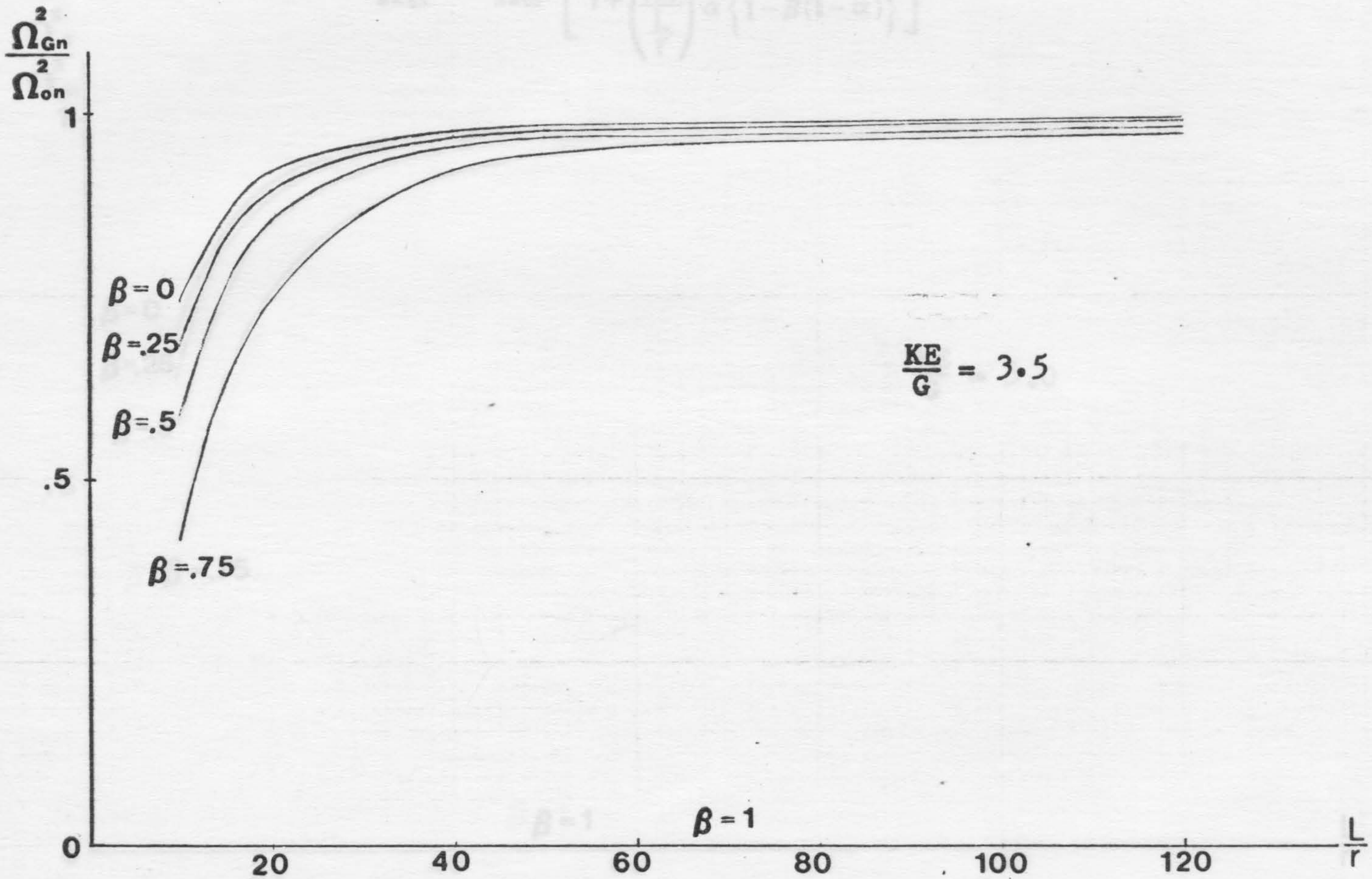


FIGURE 12 Ratio of Natural Frequencies $(\frac{\Omega_{Gn}}{\Omega_{on}})^2$ vs L/r ratio for $0 \leq P \leq P_{cr}$.

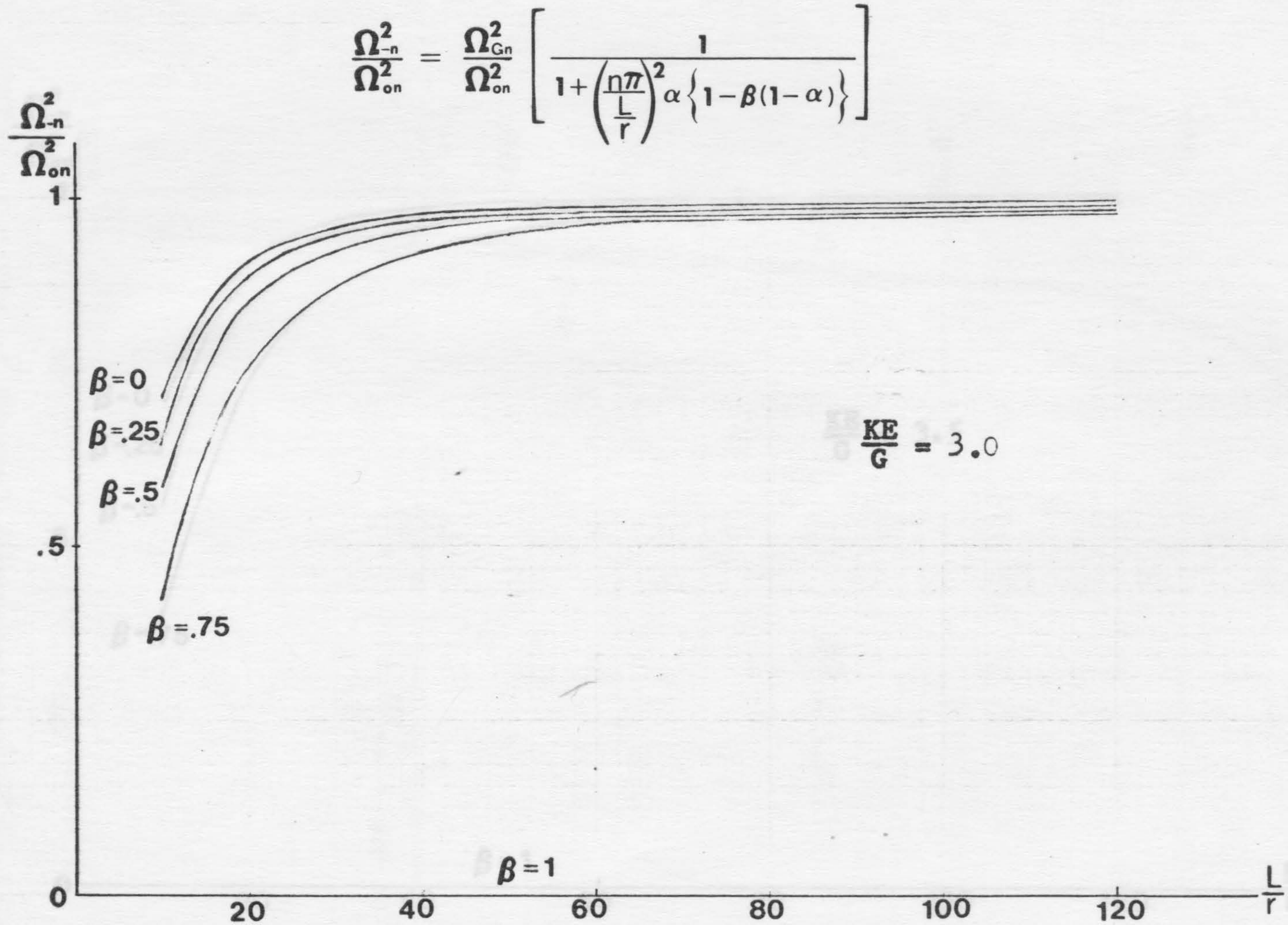


FIGURE 13 Ratio of Natural Frequencies $(\frac{\Omega_{-n}}{\Omega_{on}})^2$ vs L/r ratio for $0 \leq P \leq P_{cr}$.

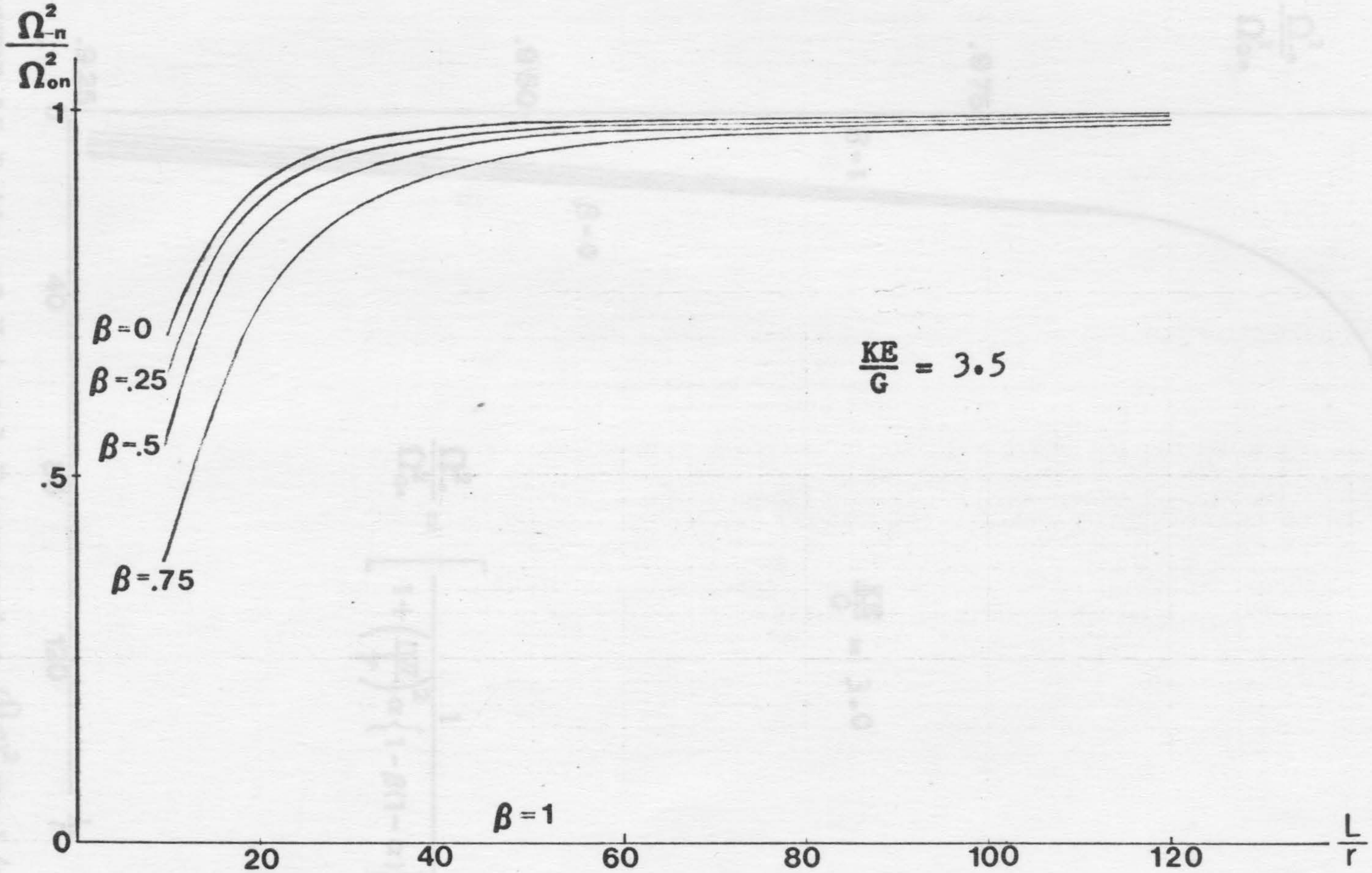


FIGURE 14 Ratio of Natural Frequencies $(\frac{\Omega_{-n}}{\Omega_{on}})^2$ vs L/r ratio for $0 \leq P \leq P_{cr}$.

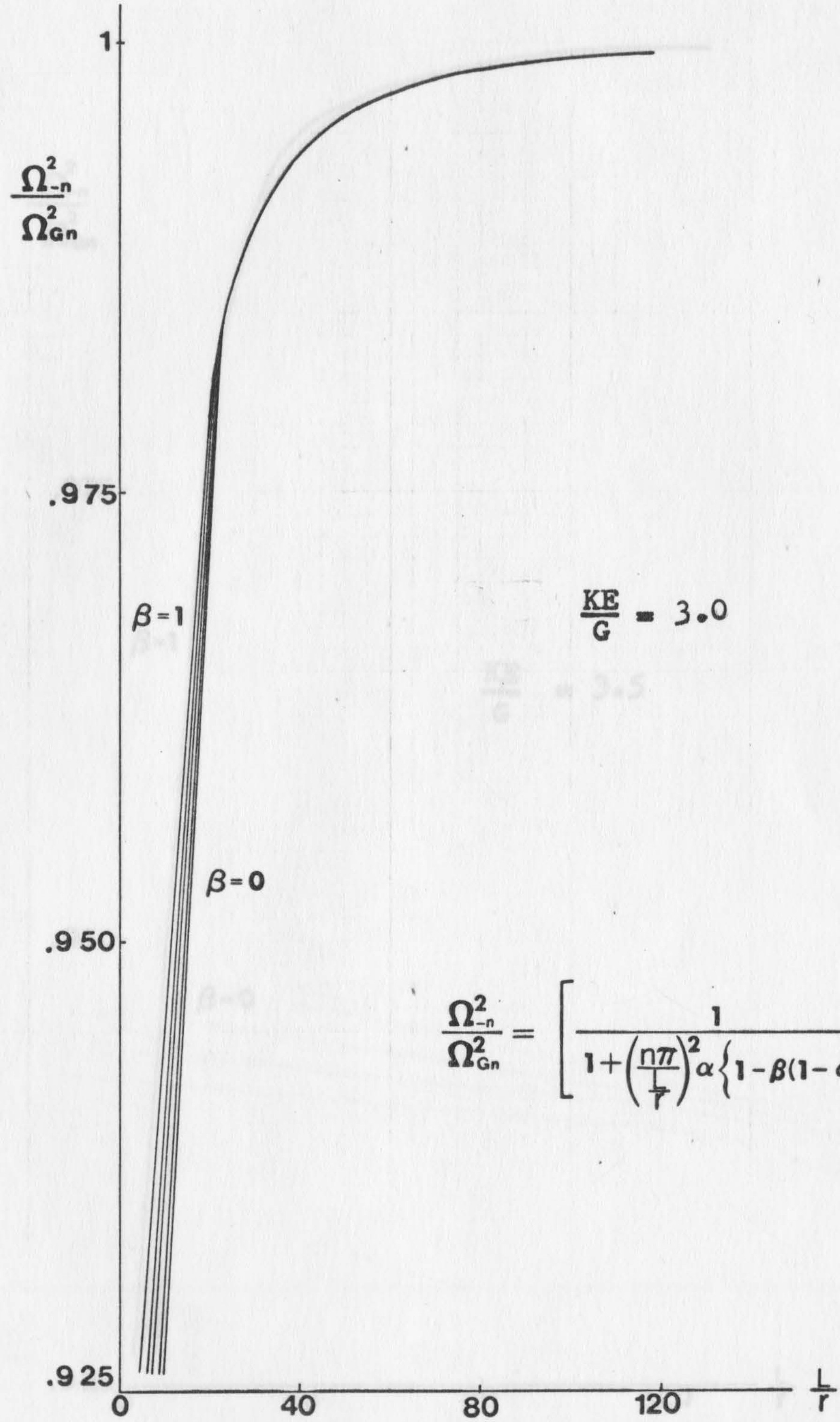


FIGURE 15 Ratio of Natural Frequencies $(\frac{\Omega_{-n}}{\Omega_{Gn}})^2$ vs L/r ratio for $0 \leq P \leq P_{cr}$.

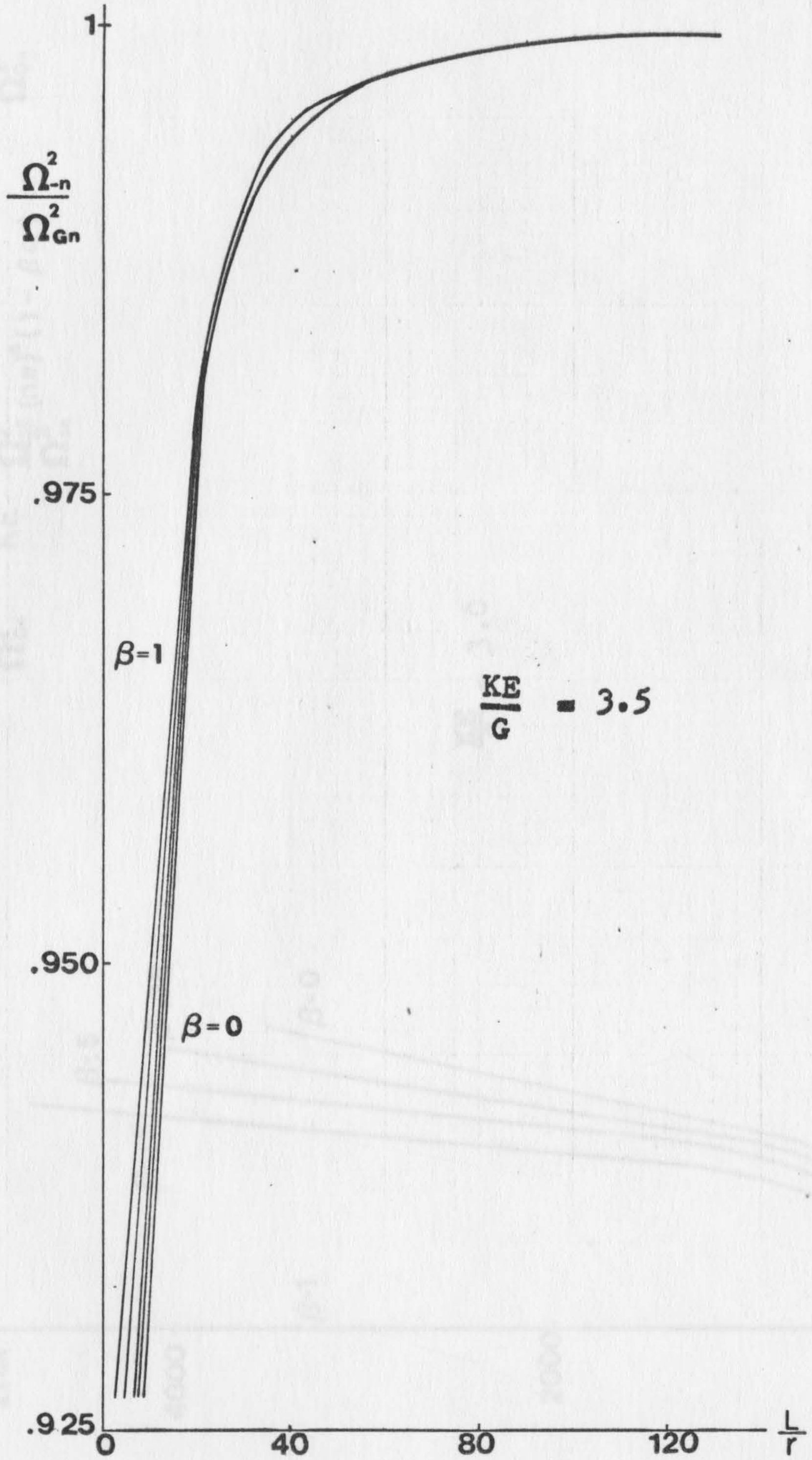


FIGURE 16 Ratio of Natural Frequencies $(\frac{\Omega_{-n}}{\Omega_{Gn}})^2$ vs L/r ratio for $0 \leq P \leq P_{cr}$.

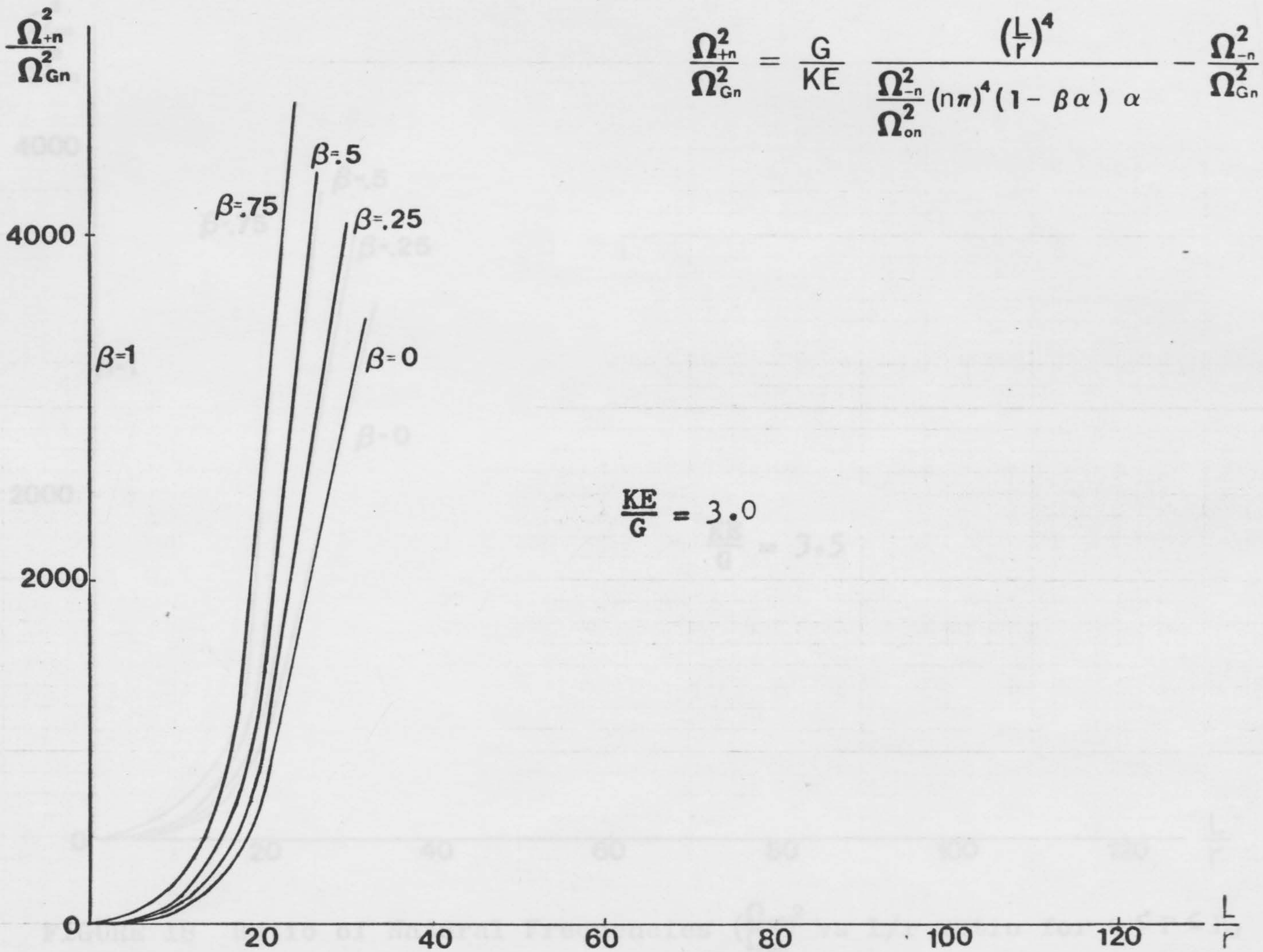


FIGURE 17 Ratio of Natural Frequencies $\left(\frac{\Omega_{+n}}{\Omega_{Gn}}\right)^2$ vs L/r ratio for $0 \leq P \leq P_{cr}$.

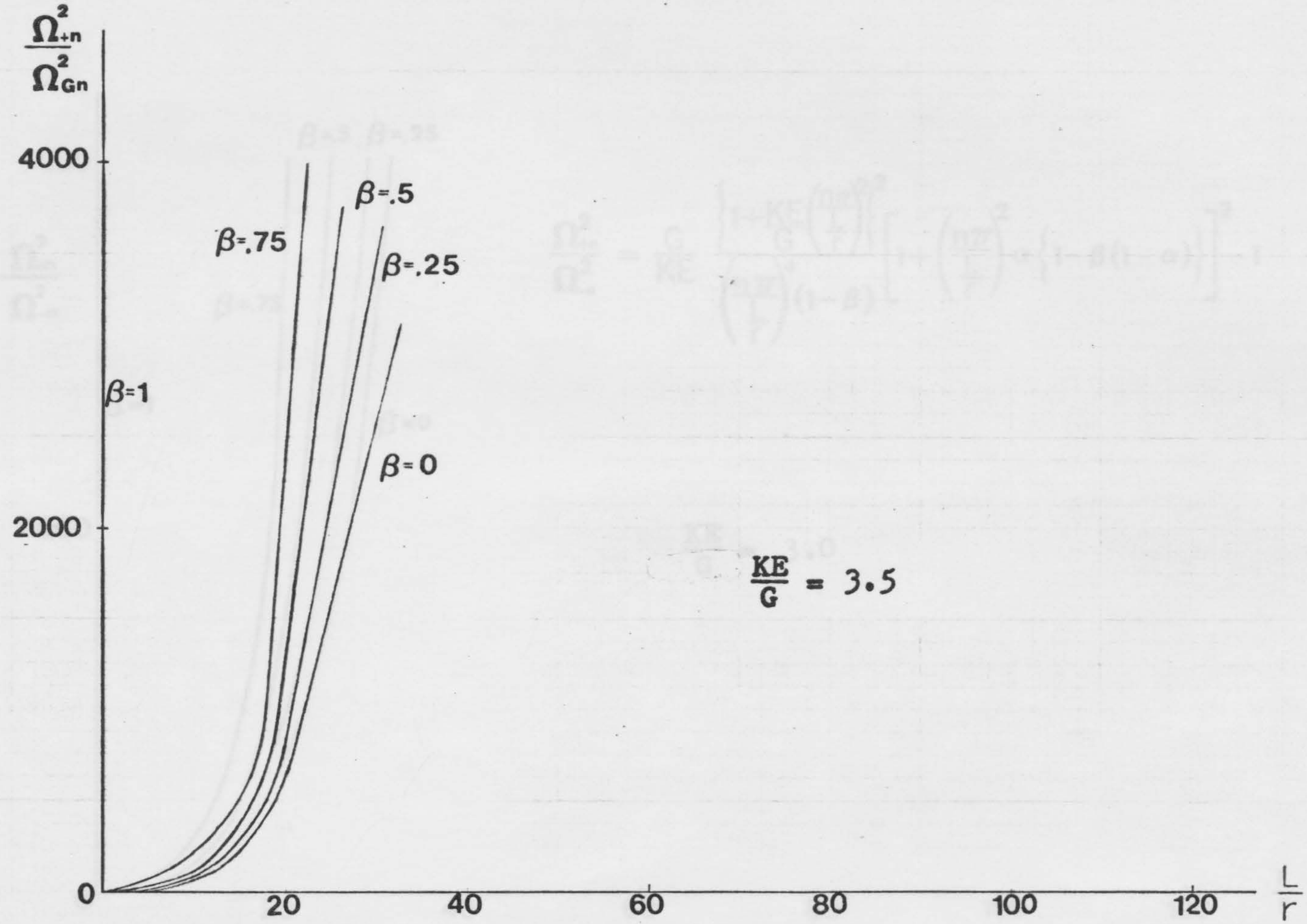
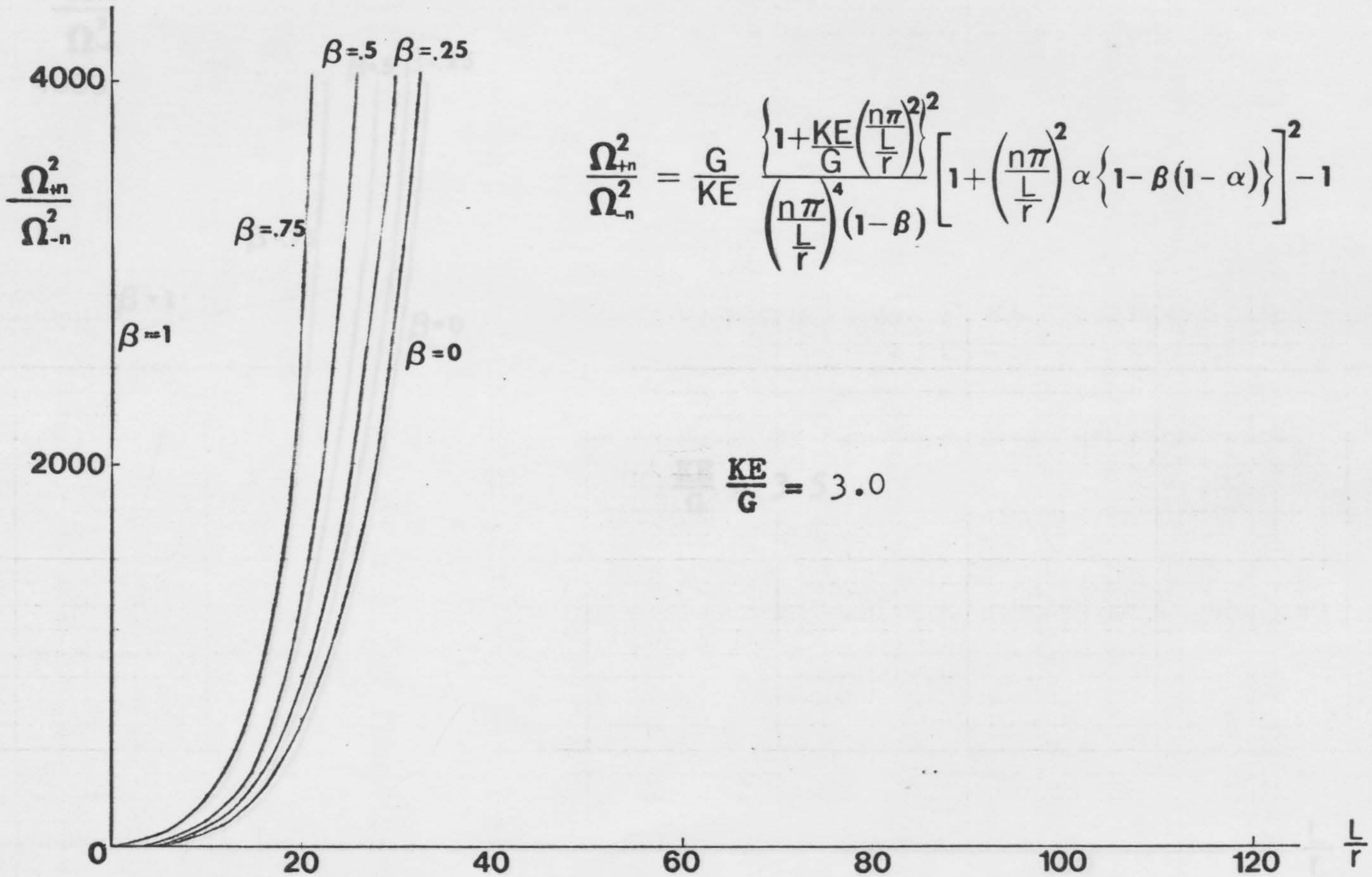


FIGURE 18 Ratio of Natural Frequencies $(\frac{\Omega_{+n}}{\Omega_{Gn}})^2$ vs L/r ratio for $0 \leq P \leq P_{cr}$.



$$\frac{\Omega^2_{+n}}{\Omega^2_{-n}} = \frac{G}{KE}\,\frac{\left\{1+\frac{KE}{G}\left(\frac{n\pi}{\frac{L}{r}}\right)^2\right\}^2}{\left(\frac{n\pi}{\frac{L}{r}}\right)^4(1-\beta)}\left[1+\left(\frac{n\pi}{\frac{L}{r}}\right)^2\alpha\left\{1-\beta(1-\alpha)\right\}\right]^2-1$$

FIGURE 19 Ratio of Natural Frequencies $\left(\frac{\Omega_{+n}}{\Omega_{-n}}\right)^2$ vs L/r ratio for $0 \le P \le P_{cr}$.

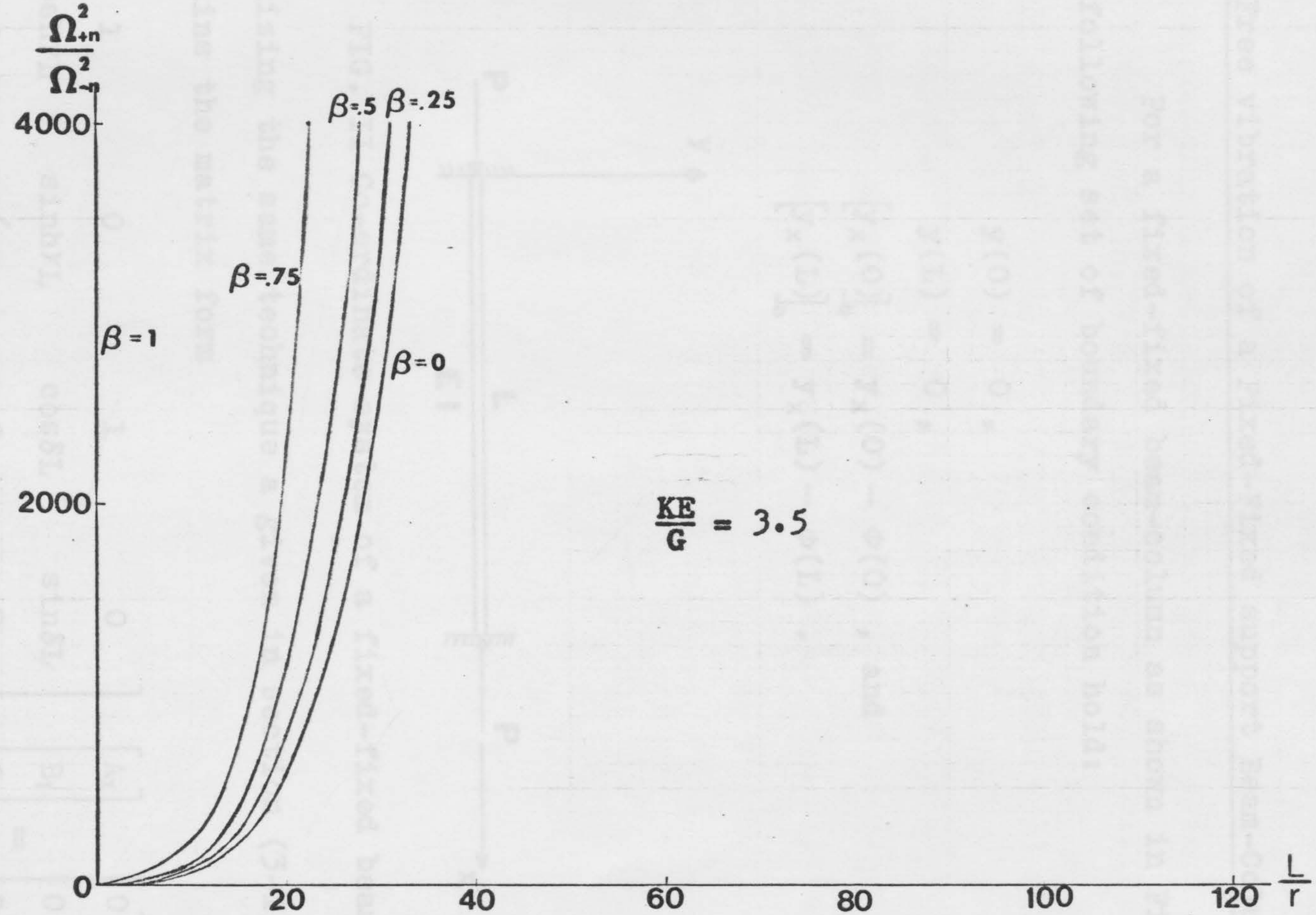


FIGURE 20 Ratio of Natural Frequencies $(\frac{\Omega_{+n}}{\Omega_{-n}})^2$ vs L/r ratio for $0 \leq P \leq P_{cr}$.

CHAPTER V

A FIXED-FIXED SUPPORT BEAM-COLUMN

5-1 Free vibration of a Fixed-Fixed support Beam-Column

For a fixed-fixed beam-column as shown in Fig. II, the following set of boundary condition hold:

$$\left.\begin{aligned} y(0) &= 0\ , \\ y(L) &= 0\ , \\ \left[y_x(0)\right]_b &= y_x(0) - \phi(0)\ ,\ \text{and} \\ \left[y_x(L)\right]_b &= y_x(L) - \phi(L)\ . \end{aligned}\right\} \quad (5\text{-}1)$$

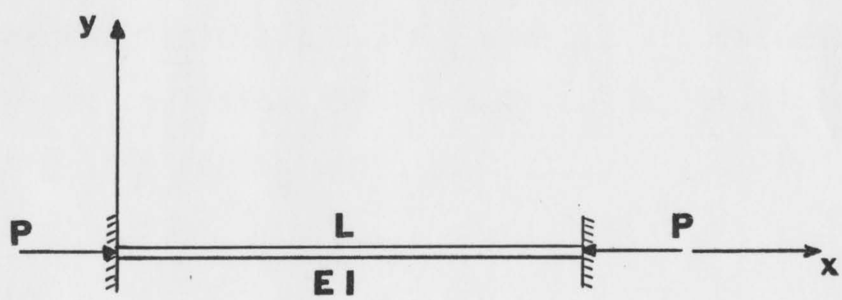


FIG. II Co-ordinate system of a fixed-fixed beam-column.

Utilizing the same technique a given in section (3-1), one obtains the matrix form

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ \cosh\gamma L & \sinh\gamma L & \cos\delta L & \sin\delta L \\ 0 & \zeta & 0 & \Theta \\ \zeta\sinh\gamma L & \zeta\cosh\gamma L & \Theta\sin\delta L & \Theta\cos\delta L \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} , \quad (5\text{-}2)$$

where $\zeta = \gamma + \frac{\Lambda\gamma}{\Gamma}$

and $\Theta = \delta - \frac{\xi\delta}{\Pi}$.

For a non-trival solution of constants, the determinant of the coefficient matrix is equated to zero, yielding

$$\sin\delta L \,\sinh\gamma L(\Theta-\zeta)(\Theta+\zeta)+2\,\Theta\zeta(\cos\delta L\,\cosh\gamma L - 1) = 0 \; . \quad (5\text{-}3)$$

The frequency of free vibration of a fixed-fixed beam-column is found by solving equation (5-3) using a numerical technique which may be efficiently programmed on a digital computer.

CHAPTER VI

DYNAMIC STIFFNESS MATRIX OF A BEAM-COLUMN

6-1 Stiffness matrix for free vibration of beam-column

From the fourth order differential equation (2-16) which includes shear and rotary inertia, the general solution of (2-16) is found in (2-18) as

$$\left.\begin{aligned} &y(x) = A_1 \cosh\gamma x + B_1 \sinh\gamma x + C_1 \cos\delta x + D_1 \sin\delta x \\ &\text{and} \\ &\phi(x) = \frac{\gamma\Lambda}{\Gamma} A_1 \sinh x - \frac{\gamma\Lambda}{\Gamma} B_1 \cosh x - \frac{\delta\xi}{\Pi} C_1 \sin x + \frac{\delta\xi}{\Pi} D_1 \cos x \;, \end{aligned}\right\} \quad (6\text{-}1)$$

where the constants A_1, B_1, C_1, and D_1, are determined from the boundary conditions of the beam-column that given by Timoshenko[1] and is illustrated in Fig. III.

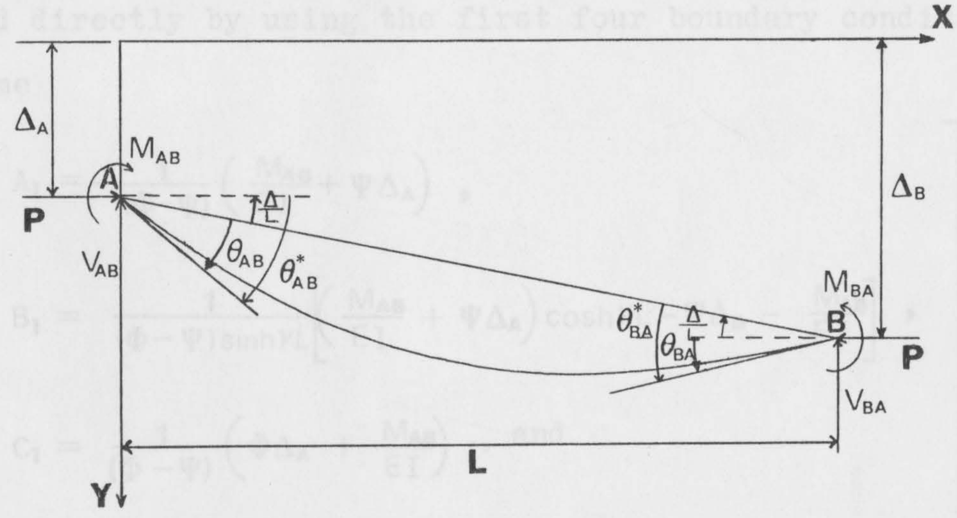


FIG. III. Sign-Convention of Beam-Column.

These boundary conditions are

$$\left.\begin{aligned}
y(0) &= \Delta_A \, , \\
y(L) &= \Delta_B \, , \\
EIy_{xx}(0) &= M_{AB} \, , \\
EIy_{xx}(L) &= M_{BA} \, , \\
y_X(0) &= \theta_{AB} \, , \\
y_X(L) &= -\theta_{BA} \, ,
\end{aligned}\right.$$

$$M_{AB} + P\Delta - V_{BA}\,L - M_{BA} + \rho A\Omega^2\int_0^L xy(x)dx - \rho I\int_0^L \frac{d^3y_b}{dx\,dt^2}dx = 0 \, , \qquad (6\text{-}2)$$

and

$$-V_{AB} - V_{BA} + \rho A\Omega^2\int_0^L y(x)dx \quad = \quad 0 \, ,$$

where $\Delta = \Delta_B - \Delta_A$.

The constants A_1, B_1, C_1, and D_1, of equation (2-18), determined directly by using the first four boundary conditions become

$$A_1 = -\frac{1}{(\Phi-\Psi)}\left(\frac{M_{AB}}{EI} + \Psi\Delta_A\right) \, ,$$

$$B_1 = \frac{1}{(\Phi-\Psi)\sinh\gamma L}\left[\left(\frac{M_{AB}}{EI} + \Psi\Delta_A\right)\cosh\gamma L - \Psi\Delta_B - \frac{M_{AB}}{EI}\right] \, , \qquad (6\text{-}3)$$

$$C_1 = \frac{1}{(\Phi-\Psi)}\left(\Phi\Delta_A + \frac{M_{AB}}{EI}\right) \, , \text{ and}$$

$$D_1 = \frac{1}{(\Phi-\Psi)\sin\delta L}\left[\Phi\Delta_B + \frac{M_{BA}}{EI} - \left(\Phi\Delta_A + \frac{M_{AB}}{EI}\right)\cos\delta L\right] .$$

Combining (6-1) and (6-3) together with the last four boundary conditions in equation (6-2), it follows that

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{M_{AB}}{EI} \\ \frac{M_{BA}}{EI} \\ \frac{V_{AB}\,L}{EI} \\ \frac{V_{BA}\,L}{EI} \end{bmatrix} = \begin{bmatrix} 1 & 0 & b_{13} & b_{14} \\ 0 & 1 & b_{23} & b_{24} \\ 0 & 0 & b_{33} & b_{34} \\ 0 & 0 & b_{43} & b_{44} \end{bmatrix} \begin{bmatrix} \theta_{AB} \\ \theta_{BA} \\ \Delta_A \\ \Delta_B \end{bmatrix}, \qquad (6\text{-}4)$$

where

$$a_{11} = a_{22} = \frac{\zeta}{(\Phi - \Psi)} \frac{\cosh\gamma L}{\sinh\gamma L} - \frac{\Theta}{(\Phi - \Psi)} \frac{\cos\delta L}{\sin\delta L},$$

$$= \frac{\hat{\zeta}}{\tanh\gamma L} - \frac{\hat{\Theta}}{\tan\delta L},$$

$$a_{21} = a_{12} = \frac{\hat{\Theta}}{\sin\delta L} - \frac{\hat{\zeta}}{\sinh\gamma L},$$

$$a_{31} = a_{42} = 1 + \frac{\rho A\Omega^2}{EI(\Phi-\Psi)}\left\{\frac{L}{\gamma}\frac{\cosh\gamma L}{\sinh\gamma L} - \frac{1}{\gamma^2} + \frac{1}{\delta}\frac{\cos\delta L}{\sin\delta L} - \frac{1}{\delta^2}\right\} + \frac{\rho I\Omega^2}{EI(\Phi-\Psi)}\left(\frac{\zeta}{\gamma} - \frac{\Theta}{\delta}\right),$$

$$= 1 + \frac{\rho A\Omega^2}{EI}\left(\hat{A} + \hat{B}\right) + \frac{\rho\Omega^2}{E}\left(\frac{\hat{\zeta}}{\gamma} - \frac{\hat{\Theta}}{\delta}\right),$$

$$a_{41} = a_{32} = -\left[1 + \frac{\rho A\Omega^2}{EI(\Phi-\Psi)}\left\{\frac{L}{\gamma}\frac{1}{\sinh\gamma L} - \frac{1}{\gamma^2} + \frac{L}{\delta}\frac{1}{\sin\delta L} - \frac{1}{\delta^2}\right\} + \frac{\rho I\Omega^2}{EI(\Phi-\Psi)}\left(\frac{\zeta}{\gamma} - \frac{\Theta}{\delta}\right)\right],$$

$$= -1 - \frac{\rho A\Omega^2}{EI}\left(\hat{C} + \hat{D}\right) - \frac{\rho\Omega^2}{E}\left(\frac{\hat{\zeta}}{\gamma} - \frac{\hat{\Theta}}{\delta}\right),$$

$$b_{13} = b_{24} = \frac{\hat{\Theta}\,\Phi}{\tan\delta L} - \frac{\hat{\zeta}\,\Psi}{\tanh\gamma L} ,$$

$$b_{14} = b_{23} = \frac{\hat{\zeta}\,\Psi}{\sinh\gamma L} - \frac{\hat{\Theta}\,\Phi}{\sin\delta L} ,$$

$$b_{33} = b_{44} = k^2 - \frac{\rho A\Omega^2}{EI}\left(\Psi\hat{A} + \Phi\hat{B}\right) + \frac{\rho\Omega^2}{E}\left(\frac{\Phi\hat{\Theta}}{\delta} - \frac{\Psi\hat{\zeta}}{\gamma}\right) , \text{ and}$$

$$b_{34} = b_{43} = -k^2 + \frac{\rho A\Omega^2}{EI}\left(\Psi\hat{C} + \Phi\hat{D}\right) - \frac{\rho\Omega^2}{E}\left(\frac{\Phi\hat{\Theta}}{\delta} - \frac{\Psi\hat{\zeta}}{\gamma}\right) .$$

Equation (6-4) is rewritten in the following symbolic matrix form

$$\left[A\right]\left\{m\right\} = \left[B\right]\left\{\boldsymbol{\theta}\right\} \qquad (6\text{-}5)$$

Premultiplying equation (6-5) by $\left[A\right]^{-1}$, one obtains

$$\left\{m\right\} = \left[A\right]^{-1}\left[B\right]\left\{\boldsymbol{\theta}\right\} = \left[K_T\right]\left\{\boldsymbol{\theta}\right\} \qquad (6\text{-}6)$$

The matrix $\left[K_T\right]$ is defined as the stiffness matrix for a single member, which includes the effects of both shear and rotary inertia. Equation (6-6) is written in component form as

$$\begin{bmatrix} M_{AB} \\ M_{BA} \\ V_{AB} \\ V_{BA} \end{bmatrix} = \frac{1}{a_{11}^2 - a_{12}^2} \begin{bmatrix} \hat{k}_{11} & \hat{k}_{12} & \hat{k}_{13} & \hat{k}_{14} \\ \hat{k}_{21} & \hat{k}_{22} & \hat{k}_{23} & \hat{k}_{24} \\ \hat{k}_{31} & \hat{k}_{32} & \hat{k}_{33} & \hat{k}_{34} \\ \hat{k}_{41} & \hat{k}_{42} & \hat{k}_{43} & \hat{k}_{44} \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta}_{AB} \\ \boldsymbol{\theta}_{BA} \\ \boldsymbol{\Delta}_A \\ \boldsymbol{\Delta}_B \end{bmatrix} , \qquad (6\text{-}7)$$

where $\hat{k}_{11} = \hat{k}_{22} = a_{11}EI$,

$$b_{13} = b_{24} = \frac{\hat{\Theta}\,\Phi}{\tan\delta L} - \frac{\hat{\zeta}\,\Psi}{\tanh\gamma L} \quad ,$$

$$b_{14} = b_{23} = \frac{\hat{\zeta}\,\Psi}{\sinh\gamma L} - \frac{\hat{\Theta}\,\Phi}{\sin\delta L} \quad ,$$

$$b_{33} = b_{44} = k^2 - \frac{\rho A \Omega^2}{EI}\left(\Psi\hat{A} + \Phi\hat{B}\right) + \frac{\rho\Omega^2}{E}\left(\frac{\Phi\hat{\Theta}}{\delta} - \frac{\Psi\hat{\zeta}}{\gamma}\right) \quad , \text{ and}$$

$$b_{34} = b_{43} = -k^2 + \frac{\rho A \Omega^2}{EI}\left(\Psi\hat{C} + \Phi\hat{D}\right) - \frac{\rho\Omega^2}{E}\left(\frac{\Phi\hat{\Theta}}{\delta} - \frac{\Psi\hat{\zeta}}{\gamma}\right) \quad .$$

Equation (6-4) is rewritten in the following symbolic matrix form

$$\left[A\right]\left\{m\right\} = \left[B\right]\left\{\theta\right\} \qquad (6\text{-}5)$$

Premultiplying equation (6-5) by $\left[A\right]^{-1}$, one obtains

$$\left\{m\right\} = \left[A\right]^{-1}\left[B\right]\left\{\theta\right\} = \left[K_T\right]\left\{\theta\right\} \qquad (6\text{-}6)$$

The matrix $\left[K_T\right]$ is defined as the stiffness matrix for a single member, which includes the effects of both shear and rotary inertia. Equation (6-6) is written in component form as

$$\begin{bmatrix} M_{AB} \\ M_{BA} \\ V_{AB} \\ V_{BA} \end{bmatrix} = \frac{1}{a_{11}^2 - a_{12}^2} \begin{bmatrix} \hat{k}_{11} & \hat{k}_{12} & \hat{k}_{13} & \hat{k}_{14} \\ \hat{k}_{21} & \hat{k}_{22} & \hat{k}_{23} & \hat{k}_{24} \\ \hat{k}_{31} & \hat{k}_{32} & \hat{k}_{33} & \hat{k}_{34} \\ \hat{k}_{41} & \hat{k}_{42} & \hat{k}_{43} & \hat{k}_{44} \end{bmatrix} \begin{bmatrix} \theta_{AB} \\ \theta_{BA} \\ \Delta_A \\ \Delta_B \end{bmatrix} , \qquad (6\text{-}7)$$

where $\hat{k}_{11} = \hat{k}_{22} = a_{11} EI$,

$$\hat{k}_{12} = \hat{k}_{21} = -a_{12}\, EI \,,$$

$$\hat{k}_{13} = \hat{k}_{24} = (a_{11}\, b_{13} - a_{12}\, b_{14})EI \,,$$

$$\hat{k}_{14} = \hat{k}_{23} = (a_{11}\, b_{14} - a_{12}\, b_{13})EI \,,$$

$$\hat{k}_{31} = \hat{k}_{42} = (a_{12}\, a_{32} - a_{11}\, a_{31})\frac{EI}{L} \,, \text{ and}$$

$$\hat{k}_{32} = \hat{k}_{41} = (a_{12}\, a_{31} - a_{11}\, a_{32})\frac{EI}{L} \,,$$

$$\hat{k}_{33} = \hat{k}_{44} = \left[b_{13}(a_{12}a_{32} - a_{11}a_{31}) + b_{14}(a_{12}a_{31} - a_{11}a_{32}) + b_{33}(a_{11}^2 - a_{12}^2)\right]\frac{EI}{L},$$

and

$$\hat{k}_{34} = \hat{k}_{43} = \left[b_{14}(a_{12}a_{32} - a_{11}a_{31}) + b_{13}(a_{12}a_{31} - a_{11}a_{32}) + b_{34}(a_{11}^2 - a_{12}^2)\right]\frac{EI}{L}.$$

6-2 Stiffness matrix of beam-column where shear stress and rotary inertia are neglected.

If shear and rotary inertia are neglected, the function/forms of equation (6-3) reduces to the following

$$\left.\begin{aligned} \Phi &= \gamma^2 \,, \\ \Psi &= -\delta^2 \,, \\ \zeta &= \gamma \,, \text{ and} \\ \Theta &= \delta \,. \end{aligned}\right] \qquad (6\text{-}8)$$

Equation (6-3) become :

$$\left.\begin{aligned}
A_1 &= \frac{1}{(\gamma^2 + \delta^2)}\left(\delta^2\Delta_A - \frac{M_{AB}}{EI}\right) , \\
B_1 &= \frac{1}{(\gamma^2 + \delta^2)\sinh\gamma L}\left[\delta^2\Delta_B - \frac{M_{BA}}{EI} + \left(\frac{M_{AB}}{EI} - \delta^2\Delta_A\right)\cosh\gamma L\right], \\
C_1 &= \frac{1}{(\gamma^2 + \delta^2)}\left(\gamma^2\Delta_A + \frac{M_{AB}}{EI}\right) , \text{ and} \\
D_1 &= \frac{1}{(\gamma^2 + \delta^2)\sin\delta L}\left[\gamma^2\Delta_B + \frac{M_{BA}}{EI} - \left(\frac{M_{AB}}{EI} + \gamma^2\Delta_A\right)\cos\delta L\right] .
\end{aligned}\right\} \quad (6\text{-}9)$$

Equation (6-4) reduces to

$$\begin{bmatrix} c_{11} & c_{12} & 0 & 0 \\ c_{21} & c_{22} & 0 & 0 \\ c_{31} & c_{32} & 0 & 0 \\ c_{41} & c_{42} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{M_{AB}}{EI} \\ \frac{M_{BA}}{EI} \\ \frac{V_{AB}\,L}{EI} \\ \frac{V_{BA}\,L}{EI} \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_{13} & d_{14} \\ 0 & 1 & d_{23} & d_{24} \\ 0 & 0 & d_{33} & d_{34} \\ 0 & 0 & d_{43} & d_{44} \end{bmatrix} \begin{bmatrix} \theta_{AB} \\ \theta_{BA} \\ \Delta_A \\ \Delta_B \end{bmatrix} , \quad (6\text{-}10)$$

where

$$c_{11} = c_{22} = \frac{1}{(\gamma^2 + \delta^2)}\left(\frac{\gamma}{\tanh\gamma L} - \frac{\delta}{\tan\delta L}\right) ,$$

$$c_{12} = c_{21} = \frac{1}{(\gamma^2 + \delta^2)}\left(\frac{\delta}{\sin\delta L} - \frac{\gamma}{\sinh\gamma L}\right) ,$$

$$c_{31} = c_{42} = 1 + \frac{(\gamma\delta)^2}{(\gamma^2 + \delta^2)}\left(\frac{L}{\gamma\tanh\gamma L} - \frac{1}{\gamma^2} + \frac{L}{\delta\tan\delta L} - \frac{1}{\delta^2}\right) ,$$

$$c_{32} = c_{41} = -1 - \frac{(\gamma\delta)^2}{(\gamma^2 + \delta^2)}\left(\frac{L}{\gamma\sinh\gamma L} - \frac{1}{\gamma^2} + \frac{L}{\delta\sin\delta L} - \frac{1}{\delta^2}\right) ,$$

$$d_{13} = d_{24} = \frac{(\gamma\delta)^2}{(\gamma^2+\delta^2)}\left(\frac{1}{\gamma\tanh\gamma L} + \frac{1}{\delta\tan\delta L}\right) ,$$

$$d_{14} = d_{23} = -\frac{(\gamma\delta)^2}{(\gamma^2+\delta^2)}\left(\frac{1}{\gamma\sinh\gamma L} + \frac{1}{\delta\sin\delta L}\right) ,$$

$$d_{33} = d_{44} = \left(\frac{2u}{L}\right)^2 + \frac{(\gamma\delta)^2}{(\gamma^2+\delta^2)}\left[\frac{\delta^2}{\gamma^2}\left(\frac{\gamma L}{\tanh\gamma L} - 1\right) - \frac{\gamma^2}{\delta^2}\left(\frac{\delta L}{\tan\delta L} - 1\right)\right] ,$$

and

$$d_{34} = d_{43} = -\left(\frac{2u}{L}\right)^2 - \frac{(\gamma\delta)^2}{(\gamma^2+\delta^2)}\left[\frac{\delta^2}{\gamma^2}\left(\frac{\gamma L}{\sinh\gamma L} - 1\right) - \frac{\gamma^2}{\delta^2}\left(\frac{\delta L}{\sin\delta L} - 1\right)\right] .$$

Utilizing similar techniques, the stiffness matrix $[K_E]$, for single member becomes

$$\begin{bmatrix} M_{AB} \\ M_{BA} \\ V_{AB} \\ V_{BA} \end{bmatrix} = \frac{1}{(c_{11}^2 + c_{12}^2)} \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} \theta_{AB} \\ \theta_{BA} \\ \Delta_A \\ \Delta_B \end{bmatrix} , \qquad (6\text{-}11)$$

where

$$k_{11} = k_{22} = c_{11}EI ,$$

$$k_{12} = k_{21} = -c_{12}EI ,$$

$$k_{13} = k_{24} = (c_{11}d_{13} - c_{12}d_{14})EI ,$$

$$k_{14} = k_{23} = (c_{11}d_{14} - c_{12}d_{13})EI ,$$

$$k_{31} = k_{42} = (c_{12}c_{32} - c_{11}c_{31})\frac{EI}{L},$$

$$k_{32} = k_{41} = (c_{12}c_{31} - c_{11}c_{32})\frac{EI}{L},$$

$$k_{33} = k_{44} = d_{13}(c_{12}c_{32} - c_{11}c_{31}) + d_{14}(c_{12}c_{31} - c_{11}c_{32}) + d_{33}(c_{11}^2 - c_{12}^2)\frac{EI}{L},$$

$$k_{34} = k_{43} = d_{14}(c_{12}c_{32} - c_{11}c_{31}) + d_{13}(c_{12}c_{31} - c_{11}c_{32}) + d_{34}(c_{11}^2 - c_{12}^2)\frac{EI}{L}.$$

6-3 <u>Stiffness matrix of beam-column when transverse inertia and rotary inertia are neglected.</u>

For the static beam-column transverse and rotary inertia are neglected. Equation (2-16) is reduces to the following

$$\left.\begin{aligned} y(x) &= A_1 \cos\delta x + B_1 \sin\delta x + C_1 x + D_1 \quad \text{and} \\ \phi(x) &= A_2 \cos\delta x + B_2 \sin\delta x + C_2 x + D_2 , \end{aligned}\right\} \qquad (6\text{-}12)$$

where $\delta = \sqrt{\dfrac{k^2}{1 + \dfrac{KP}{GA}}}$, and $k^2 = \dfrac{P}{EI}$.

utilized the same techniques as in section (6-1), the relation between the constants of y(x) and ϕ(x) in (6-12) become

$$\left.\begin{aligned} A_2 &= \frac{\xi}{\Pi}\delta B_1 , \\ B_2 &= -\frac{\xi}{\Pi}\delta A_1 , \\ C_2 &= 0 , \text{ and} \\ D_2 &= 0 , \end{aligned}\right] \qquad (6\text{-}13)$$

where $\xi = EI\delta^2$, and $\Pi = EI\delta^2 + \frac{GA}{K}$.

Combining the first four boundary conditions of equation (6-2) together with equation (6-12), and (6-13), the constants in y(x) become

$$\left.\begin{aligned}
A_1 &= -\frac{1}{\Psi}\frac{M_{AB}}{EI} ,\\
B_1 &= \frac{1}{\Psi}\frac{1}{\tan\delta L}\frac{M_{AB}}{EI} - \frac{1}{\Psi}\frac{1}{\sin\delta L}\frac{M_{BA}}{EI} ,\\
C_1 &= \frac{1}{L}\left(\Delta_B - \Delta_A + \frac{1}{\Psi}\frac{M_{BA}}{EI} - \frac{1}{\Psi}\frac{M_{AB}}{EI}\right) , \text{ and}\\
D_1 &= \Delta_A + \frac{1}{\Psi}\frac{M_{AB}}{EI} .
\end{aligned}\right] \qquad (6\text{-}14)$$

where $\Psi = \frac{\xi}{\Pi}\delta^2 - \delta^2$.

For the last four boundary conditions and equation (6-12), equation (6-13) and equation (6-14) yield upon rearrangement form,

$$\begin{bmatrix} e_{11} & e_{12} & 0 & 0 \\ e_{21} & e_{22} & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{M_{AB}}{EI} \\ \frac{M_{BA}}{EI} \\ \frac{V_{AB}\,L}{EI} \\ \frac{V_{BA}\,L}{EI} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -f & f \\ 0 & 0 & f & -f \end{bmatrix} \begin{bmatrix} \theta_{AB} \\ \theta_{BA} \\ \Delta_A \\ \Delta_B \end{bmatrix} , \qquad (6\text{-}15)$$

where, $e_{11} = e_{22} = \frac{\Theta}{\Psi}\frac{1}{\tan\delta L} - \frac{1}{\Psi L}$,

$$e_{12} = e_{21} = \frac{1}{\Psi L} - \frac{\Theta}{\Psi}\frac{1}{\sin\delta L}\ ,\quad \text{and}$$

$$f = k^2 L\ ,$$

where $\Theta = \delta - \dfrac{\xi\delta}{\Pi}$.

The stiffness matrix for the static beam-column becomes

$$\begin{bmatrix} M_{AB} \\ M_{BA} \\ V_{AB} \\ V_{BA} \end{bmatrix} = \frac{1}{e_{11}^2 - e_{12}^2} \begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} & \bar{k}_{14} \\ \bar{k}_{21} & \bar{k}_{22} & \bar{k}_{23} & \bar{k}_{24} \\ \bar{k}_{31} & \bar{k}_{32} & \bar{k}_{33} & \bar{k}_{34} \\ \bar{k}_{41} & \bar{k}_{42} & \bar{k}_{43} & \bar{k}_{44} \end{bmatrix} \begin{bmatrix} \theta_{AB} \\ \theta_{BA} \\ \frac{\Delta_A}{L} \\ \frac{\Delta_B}{L} \end{bmatrix}\ , \qquad (6\text{-}16)$$

where

$$\bar{k}_{11} = \bar{k}_{22} = e_{11}EI\ ,$$

$$\bar{k}_{12} = \bar{k}_{21} = -e_{12}EI\ ,$$

$$\bar{k}_{13} = \bar{k}_{24} = (e_{11} + e_{12})EI\ ,$$

$$\bar{k}_{14} = \bar{k}_{23} = -(e_{11} + e_{12})EI\ ,$$

$$\bar{k}_{31} = \bar{k}_{42} = -(e_{11} + e_{12})\frac{EI}{L}\ ,$$

$$\bar{k}_{32} = \bar{k}_{41} = (e_{11} + e_{12})\frac{EI}{L}\ ,$$

$$\bar{k}_{33} = \bar{k}_{44} = \left[-2(e_{11} + e_{12}) - f(e_{12}^2 - e_{11}^2)\right]\frac{EI}{L}\ ,\quad \text{and}$$

$$\bar{k}_{34} = \bar{k}_{43} = \left[2(e_{11} + e_{12}) - f(e_{12}^2 - e_{11}^2)\right]\frac{EI}{L}\ .$$

CHAPTER VII
DISCUSSIONS

For the free vibration of the beam-column, the addition of the combined effects of shear stress and rotary inertia yields a coupled-set of linear differential equations of motion. The uncoupling procedure for these equations produces a fourth-order differential operator, which yields two independent sets of natural frequencies.

If the effect of shear stress or rotary inertia or both are eliminated, the fourth-order equation reduces to second-order equation, and yields only one set of natural frequencies.

Equating to zero the natural frequencies, the critical buckling loads are obtained. If shear stress, rotary inertia and bending stress are included, the Timoshenko buckling load is produced. If only shear stress is retained, the Pure Shear critical load is produced. The Euler buckling load is obtained if only bending stress is retained.

The Timoshenko buckling load is less than but approximately equal to Euler buckling load for the value L/r greater than 100. The Timoshenko buckling load is identical to Pure Shear buckling load for an L/r ratio <1.0. For a very stubby beam-column, (i.e. L/r is less than 10) the Euler buckling load mathematically increases indefinitely but has no physical meaning. The Pure Shear buckling load is independent of the L/r ratio. However it is physically valid for the L/r ratio less than approximately 1.0.

The lower set of natural frequencies of the Timoshenko theory are less than but approximately equal to the frequencies given by the Bernoulli-Euler theory for an L/r ratio is larger than 100, and are almost identical to the Pure Shear vibration theory for an L/r ratio <1.0. There is no physical meaning for the upper set of natural frequencies in Timoshenko theory. The natural frequencies of free vibration by the Bernoulli-Euler theory varies linearly with the r/L ratio. For an L/r ratio is smaller than 10, the Timoshenko theory yields lower frequency values. The natural frequency given by the Pure Shear theory is independent to L/r ratio and applied in general if L/r ratio is less than 1.0.

The natural frequencies and critical buckling loads, in Timoshenko and Bernoulli-Euler theory, decrease if the L/r ratio increases. For a beam-column, the natural frequencies decrease if the applied axial force increases. The percentage decrease become significant as the applied axial force approaches the critical buckling load.

The formulation of stiffness and flexibility matrices, including the effects of shear and rotary inertia, excluing the effects of shear and rotary inertia and excluding the transverse and rotary inertia, (i.e. static case) produce a four by four matrix with two by two symmetrical sub-matrices. This matrix formulation allows for the solution of dynamically loaded portal frame buckling problem including the effect of sidesway.

CHAPTER VIII

CONCLUSIONS

The following conclusions are determined from the resulting theoritical analysis:

The critical buckling loads of a beam-column given by the Bernoulli-Euler, the Timoshenko and the Pure Shear theory are valid in the following regions:

$r/L < 0.01$	Bernoulli-Euler theory is valid
$0.01 < r/L < 1.0$	Timoshenko theory is valid
$r/L > 1.0$	Pure Shear theory is valid

The natural frequencies of free vibration of the vibrating beam are valid over the following regions:

$r/L < 0.10$	Bernoulli-Euler theory holds
$0.10 < r/L < 1.0$	Timoshenko theory holds
$r/L > 1.0$	Pure Shear theory holds

The natural frequencies of free vibration of the beam-column are valid as follows:

For $1/4\ P_{cr} \leq P \leq 3/4\ P_{cr}$

$r/L < 0.10$	Bernoulli-Euler theory is valid
$0.10 < r/L < 1.0$	Timoshenko theory is valid
$r/L > 1.0$	Pure Shear theory is valid

also for $3/4\ P_{cr} \leq P \leq 0.99\ P_{cr}$

$r/L < 0.10$	Bernoulli-Euler theory holds
$0.10 < r/L < 0.80$	Timoshenko theory holds
$r/L > 0.80$	Pure Shear theory holds

The effects of shear stress and rotary inertia on the natural frequencies and critical buckling loads of a long slender beam-column (i.e. L/r is greater than 100) is small in comparision with the effect of bending stress. For short stubby beam-column, (i.e. $L/r < 60$) the effect of bending stress is small in comparision with shear stress.

As the value of the parameter $\frac{KE}{G}$ increases, the natural frequencies and critical buckling loads are slightly decreased.

The Timoshenko theory yields the best results for all values of the L/r ratio for the beam-column.

APPENDIX I

TABLES

$n\frac{r}{L}$	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
$\frac{P_{nT}}{AE}$	0	.0761	.1807	.2424	.2752	.2937	.3047	.3118	.3166	.3200	.3224

TABLE 1

$n\frac{r}{L}$	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
$\left[\frac{\omega_{-T}L}{n\pi c}\right]$	0	.2681	.4070	.4739	.5093	.5297	.5424	.5507	.5565	.5606	.5636
$\left[\frac{\omega_{+T}L}{n\pi c}\right]$	∞	2.1538	1.4185	1.2182	1.1336	1.0899	1.0644	1.0483	1.0375	1.0300	1.0244

TABLE 2

$$\frac{P}{AE} = .25 \frac{P_{cr}}{AE}$$

$n\frac{r}{L}$	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
$\frac{\Omega_{-T} L}{n \pi c}$	0	.2322	.3528	.4111	.4419	.4595	.4704	.4775	.4823	.4858	.4884
$\frac{\Omega_{+T} L}{n \pi c}$	∞	2.1536	1.4170	1.2162	1.1315	1.0881	1.0629	1.0470	1.0365	1.0291	1.0237

TABLE 3a

$$\frac{P}{AE} = .5 \frac{P_{cr}}{AE}$$

$n\frac{r}{L}$	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
$\frac{\Omega_{-T} L}{n \pi c}$	0	.1895	.2883	.3362	.3614	.3756	.3845	.3903	.3942	.3970	.3990
$\frac{\Omega_{+T} L}{n \pi c}$	∞	2.1533	1.4157	1.2143	1.1297	1.0865	1.0615	1.0459	1.0356	1.0283	1.0231

TABLE 3b

$$\frac{P}{AE} = .9 \frac{P_{cr}}{AE}$$

$n\frac{r}{L}$	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
$\frac{\Omega_{-T} L}{n \pi c}$	0	.0849	.1293	.1504	.1619	.1683	.1724	.1749	.1766	.1776	.1785
$\frac{\Omega_{+T} L}{n \pi c}$	∞	2.1529	1.4136	1.2115	1.1272	1.0844	1.1598	1.0445	1.0344	1.0274	1.0223

TABLE 3d

$$\frac{P}{AE} = .99\ \frac{P_{cr}}{AE}$$

$n\frac{r}{L}$	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
$\frac{\Omega_{-T} L}{n \pi c}$	0	.0282	.0410	.0469	.0517	.0523	.0542	.0553	.0558	.0560	.0563
$\frac{\Omega_{+T} L}{n \pi c}$	∞	2.1528	1.4132	1.2110	1.1267	1.0839	1.0595	1.0443	1.0342	1.0272	1.0221

TABLE 3e

$n\frac{r}{L}$	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
$\left[\frac{\omega_T L}{n\pi c}\right]$	0	.2660	.3912	.4417	.4646	.4764	.4833	.4876	.4904	.4924	.4938

TABLE 4

	$\frac{P}{P_{cr}}$ \ $n\frac{r}{L}$	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
$\frac{\Omega_{T} L}{n \pi C}$	.25	0	.2308	.3424	.3895	.4116	.4234	.4302	.4345	.4374	.4394	.4408
	.5	0	.1888	.2826	.3240	.3442	.3552	.3616	.3657	.3684	.3704	.3717
	.75	0	.1338	.2020	.2336	.2495	.2584	.2637	.2671	.2693	.2710	.2721
	.9	0	.0847	.1286	.1496	.1603	.1664	.1700	.1724	.1740	.1751	.1759
	.99	0	.0268	.0408	.0476	.0512	.0532	.0544	.0552	.0557	.0561	.0564

TABLE 5

$$\frac{P_{nT}}{P_{nE}} = \frac{1}{1 + \frac{KE}{G}\left(\frac{n\pi}{12}\right)^2\left(\frac{h}{L}\right)^2}$$

$$\frac{KE}{G} = 3$$

n \ $\frac{h}{L}$	0	.125	.25	.375	.5	.625	.75	.825	1
1	1	.963	.866	.742	.618	.509	.419	.346	.288
2	1	.866	.617	.419	.288	.206	.153	.117	.092
3	1	.742	.419	.243	.153	.103	.074	.056	.043

TABLE 6

$$\frac{P_{nT}}{P_{nE}} = \frac{1}{1 + \frac{KE}{G}\left(\frac{n\pi}{12}\right)^2\left(\frac{h}{L}\right)^2} \qquad \frac{KE}{G} = 3.5$$

n \ h/L	0	.125	.25	.375	.5	.625	.75	.825	1
1	1	.957	.848	.712	.582	.471	.382	.312	.258
2	1	.848	.582	.382	.258	.182	.134	.102	.080
3	1	.712	.382	.215	.134	.090	.064	.048	.037

TABLE 7

$$\frac{P_{nT}}{P_{nE}} = \frac{1}{1 + \frac{KE}{G}\left(\frac{n\pi}{\frac{L}{r}}\right)^2} \qquad \frac{KE}{G} = 3$$

n \ $\frac{L}{r}$	0	10	20	30	40	50	60	70	80	90	100	110	120
1	0	.772	.931	.968	.982	.988	.992	.994	.995	.996	.997	.998	.998
2	0	.458	.772	.884	.931	.955	.968	.976	.982	.986	.988	.990	.992
3	0	.273	.600	.772	.857	.904	.931	.948	.960	.968	.974	.978	.982

TABLE 8

$$\frac{P_{nT}}{P_{nE}} = \frac{1}{1 + \frac{KE}{G}\left(\frac{n\pi}{\frac{L}{r}}\right)^2}$$

$$\frac{KE}{G} = 3.5$$

n \ L/r	0	10	20	30	40	50	60	70	80	90	100	110	120
1	0	.743	.921	.963	.979	.986	.990	.993	.995	.996	.997	.997	.998
2	0	.420	.743	.867	.921	.948	.963	.973	.979	.983	.986	.989	.990
3	0	.243	.563	.743	.837	.889	.921	.940	.954	.963	.970	.975	.979

TABLE 9

$$\frac{\Omega_{rn}^2}{\Omega_{on}^2} = \frac{1}{1+\left(\frac{n\pi}{\frac{L}{r}}\right)^2}$$

n \ L/r	0	10	20	30	40	50	60	70	80	90	100	110	120
1	0	.910	.976	.989	.994	.996	.997	.998	.998	.999	.999	.999	.999
2	0	.717	.910	.958	.976	.984	.989	.992	.994	.995	.996	.997	.997
3	0	.530	.818	.910	.947	.966	.976	.982	.986	.989	.991	.993	.994

TABLE 10

$$\frac{\Omega_{Gn}^2}{\Omega_{on}^2} = \frac{1-\beta}{1-\frac{KE}{G}\left(\frac{n\pi}{\frac{L}{r}}\right)^2-\beta}$$

$\frac{KE}{G} = 3$ $\quad n = 1$

β \ $\frac{L}{r}$	0	10	20	30	40	50	60	70	80	90	100	110	120
0	0	.772	.931	.968	.982	.988	.992	.994	.995	.996	.997	.998	.998
.25	0	.717	.910	.958	.976	.984	.989	.992	.993	.995	.996	.997	.997
.5	0	.629	.871	.938	.965	.976	.984	.988	.990	.992	.994	.996	.996
.75	0	.458	.771	.883	.932	.954	.969	.976	.980	.984	.988	.992	.992
1	0	0	0	0	0	0	0	0	0	0	0	0	0

TABLE 11

$$\frac{\Omega^2_{Gn}}{\Omega^2_{on}} = \frac{1-\beta}{1-\frac{KE}{G}\left(\frac{n\pi}{\frac{L}{r}}\right)^2-\beta}$$

$\frac{KE}{G} = 3.5$ $n = 1$

β \ $\frac{L}{r}$	0	10	20	30	40	50	60	70	80	90	100	110	120
0	0	.743	.921	.963	.979	.986	.990	.993	.995	.996	.997	.997	.998
.25	0	.684	.897	.951	.972	.981	.987	.990	.993	.995	.996	.996	.997
.5	0	.591	.854	.929	.959	.972	.980	.986	.990	.992	.994	.994	.996
.75	0	.420	.745	.867	.921	.946	.961	.973	.980	.984	.988	.988	.992
1	0	0	0	0	0	0	0	0	0	0	0	0	0

TABLE 12

$$\frac{\Omega^2_{-n}}{\Omega^2_{on}} = \frac{\Omega^2_{Gn}}{\Omega^2_{on}} \left[\frac{1}{1 + \left(\frac{n\pi}{\frac{L}{r}}\right)^2 \alpha \left\{1 - \beta(1-\alpha)\right\}} \right]$$

$\frac{KE}{G} = 3$ $n = 1$

β \ $\frac{L}{r}$	0	10	20	30	40	50	60	70	80	90	100	110	120
0	0	.717	.910	.958	.976	.984	.989	.992	.993	.995	.996	.997	.997
.25	0	.667	.890	.848	.970	.980	.986	.990	.991	.994	.995	.996	.996
.5	0	.589	.852	.928	.959	.972	.981	.986	.988	.990	.993	.995	.995
.75	0	.430	.755	.874	.926	.950	.966	.974	.979	.983	.987	.991	.991
1	0	0	0	0	0	0	0	0	0	0	0	0	0

TABLE 13

$$\frac{\Omega^2_{-n}}{\Omega^2_{on}} = \frac{\Omega^2_{Gn}}{\Omega^2_{on}} \left[\frac{1}{1+\left(\frac{n\pi}{\frac{L}{r}}\right)^2 \alpha \left\{1-\beta\,(1-\alpha)\right\}} \right]$$

$\frac{KE}{G} = 3.5$ $n = 1$

β \ $\frac{L}{r}$	0	10	20	30	40	50	60	70	80	90	100	110	120
0	0	.692	.901	.962	.973	.982	.987	.991	.993	.995	.996	.996	.997
.25	0	.640	.877	.941	.966	.977	.984	.988	.991	.994	.995	.995	.996
.5	0	.555	.836	.919	.953	.968	.977	.984	.988	.990	.993	.993	.995
.75	0	.397	.729	.858	.916	.942	.958	.971	.979	.983	.987	.987	.991
1	0	0	0	0	0	0	0	0	0	0	0	0	0

TABLE 14

$$\frac{\Omega^2_{-n}}{\Omega^2_{Gn}} = \left[\frac{1}{1+\left(\frac{n\pi}{\frac{L}{r}}\right)^2 \alpha\left\{1-\beta(1-\alpha)\right\}} \right]$$

$\frac{KE}{G} = 3$ $n = 1$

β \ $\frac{L}{r}$	0	10	20	30	40	50	60	70	80	90	100	110	120
0	0	.929	.977	.990	.994	.996	.997	.998	.998	.999	.999	.999	.999
.25	0	.930	.978	.990	.994	.996	.997	.998	.998	.999	.999	.999	.999
.5	0	.936	.978	.990	.994	.996	.997	.998	.998	.999	.999	.999	.999
.75	0	.939	.979	.990	.994	.996	.997	.998	.999	.999	.999	.999	.999
1	0	.944	.979	.990	.994	.996	.997	.998	.999	.999	.999	.999	.999

TABLE 15

$$\frac{\Omega^2_{+n}}{\Omega^2_{Gn}} = \frac{G}{KE} \; \frac{\left[\frac{L}{r}\right]^4}{\frac{\Omega^2_{-n}}{\Omega^2_{on}} (n\pi)^4 (1-\beta\alpha)\alpha} - \frac{\Omega^2_{-n}}{\Omega^2_{Gn}} \qquad \frac{KE}{G} = 3 \qquad n = 1$$

β \ $\frac{L}{r}$	0	10	20	30	40	50	60	70	80	90	100	110	120
0	0	61	645	2988	9139	21998	45203	83324	141861	226551	344607	503528	713145
.25	0	81	860	3984	12188	29334	60294	111101	189215	301969	459478	671597	952137
.5	0	122	1290	5979	18275	44013	87993	166662	283741	453576	689229	1007067	1426306
.75	0	244	2580	11956	36561	87979	180781	333455	567057	906391	1378591	2014223	2852738
1	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

TABLE 17

$$\frac{\Omega^2_{+n}}{\Omega^2_{Gn}} = \frac{G}{KE} \frac{\left[\frac{L}{r}\right]^4}{\frac{\Omega^2_{-n}}{\Omega^2_{on}} (n\pi)^4 (1 - \beta a)a} - \frac{\Omega^2_{-n}}{\Omega^2_{Gn}}$$

$\frac{KE}{G} = 3.5$ $\quad n = 1$

β \ $\frac{L}{r}$	0	10	20	30	40	50	60	70	80	90	100	110	120
0	0	56	565	2564	7882	18932	38902	71564	121595	194184	259377	432462	611267
.25	0	75	754	3452	10512	25255	51855	95486	162184	258830	393838	576619	815298
.5	0	112	1129	5177	15764	37883	77823	143146	243206	388779	590767	864943	1222548
.75	0	224	2259	10352	31507	75765	155653	286147	486046	776906	1181649	1730053	2445204
1	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

TABLE 18

$$\frac{\Omega_{+n}^2}{\Omega_{Gn}^2} = \frac{G}{KE} \frac{\left[\frac{L}{r}\right]^4}{\frac{\Omega_n^2}{\Omega_{on}^2} (n\pi)^4 (1-\beta\alpha)\alpha} - \frac{\Omega_{-n}^2}{\Omega_{Gn}^2}$$

$\frac{KE}{G} = 3.5$ $n = 1$

β \ $\frac{L}{r}$	0	10	20	30	40	50	60	70	80	90	100	110	120
0	0	56	565	2564	7882	18932	38902	71564	121595	194184	259377	432462	611267
.25	0	75	754	3452	10512	25255	51855	95486	162184	258830	393838	576619	815298
.5	0	112	1129	5177	15764	37883	77823	143146	243206	388779	590767	864943	1222548
.75	0	224	2259	10352	31507	75765	155653	286147	486046	776906	1181649	1730053	2445204
1	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

TABLE 18

$$\frac{\Omega_{+n}^2}{\Omega_{-n}^2} = \left[\frac{G}{KE} \frac{\left\{1 + \frac{KE}{G}\left(\frac{n\pi}{\frac{L}{r}}\right)^2\right\}^2}{\left(\frac{n\pi}{\frac{L}{r}}\right)^4 \left(1-\beta\right)} \left[1 + \left(\frac{n\pi}{\frac{L}{r}}\right)^2 \alpha \left\{1 - \beta\left(1-\alpha\right)\right\}\right]^2 - 1 \right] \qquad \frac{KE}{G} = 3 \qquad n = 1$$

β \ $\frac{L}{r}$	0	10	20	30	40	50	60	70	80	90	100	110	120
0	0	66	660	3018	9194	22086	45339	83491	142145	226778	344952	504032	494744
.25	0	87	880	4024	12262	29452	60475	111324	189594	302271	459938	672269	953090
.5	0	130	1319	6046	18385	44190	88258	166996	284310	454485	689919	1008075	1427734
.75	0	260	2635	12077	36782	88332	181325	334123	567625	907298	1379971	2016239	2855594
1	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

TABLE 19

$$\frac{\Omega^2_{+n}}{\Omega^2_{-n}} = \left[\frac{G}{KE} \frac{\left\{1 + \frac{KE}{G}\left(\frac{n\pi}{\frac{L}{r}}\right)^2\right\}^2}{\left(\frac{n\pi}{\frac{L}{r}}\right)^4 \left(1-\beta\right)} \left[1 + \left(\frac{n\pi}{\frac{L}{r}}\right)^2 \alpha \left\{1 - \beta\left(1-\alpha\right)\right\}\right]^2 - 1 \right] \qquad \frac{KE}{G} = 3.5 \qquad n = 1$$

β \ $\frac{L}{r}$	0	10	20	30	40	50	60	70	80	90	100	110	120
0	0	60	578	2593	7930	19008	39109	71707	121839	193992	295673	432895	611879
.25	0	80	771	3490	10576	25356	52011	95677	162509	259089	394232	577196	816114
.5	0	119	1153	5235	15859	38035	78057	143433	243693	369558	591358	865809	1223772
.75	0	237	2308	10457	31665	76067	156121	286720	486536	777684	1182832	1731785	2447652
1	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

TABLE 20

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