VIBRATION OF LUMPED-MASS DYNAMICAL SYSTEMS USING THE CHOLESKY TRANSFORMATION

by

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ABSTRACT

VIBRATION OF LUMPED-MASS
DYNAMICAL SYSTEMS USING THE CHOLESKY TRANSFORMATION

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The purpose of the work described in this thesis was to develop the mathematical solution to the forced vibration problem of multi-degree of freedom dynamical systems common to the field of structural dynamics.

Closed-form solution of the linear equation of motion for both the free and forced vibration problems were formulated utilizing the Cholesky theorem of triangular matrices. Both the damped and undamped dynamical systems were investigated with a sample numerical example presented for each case.

A comparison of the above method with the classical solutions was made to determine the overall numerical efficiency of the approach.

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LIST OF SYMBOLS

SYMBOL	DEFINITION	REFERENCE
[A]	Symmetric matrix	
[B]	Symmetric matrix	
[C]	Diagonal cosine matrix	Eq. (14)
[D]	Symmetric matrix	
[G]	Symmetric matrix	Eq. (24)
[I]	Identity matrix	
[K]	Stiffness matrix	Eq. (1)
$[K_1]$	Symmetric matrix	Eq. (4)
[L]	Lower triangular matrix	Eq. (3)
[L*]	Unit lower triangular matrix	Eq. (6)
[M]	Mass matrix	Eq. (1)
$[M_d]^2$	Diagonal matrix	Eq. (6)
[Q]	Symmetric matrix	
[S]	Diagonal sine matrix	Eq. (14)
$[\hat{S}]$	Diagonal $\sin(t-\tau)$ matrix	Eq. (14)
[U]	Partition Eigen-vector matrix of damped vibration problem	Eq. (26a)
[V]	Eigen-vector matrix	Eq. (15)
[KC]	Damped stiffness matrix partitioned	Eq. (20)
[MK]	Damped mass matrix partitioned	Eq. (20)
$[\Lambda]$	Root matrix	Eq. (10a)
$[\Lambda_g]$	Diagonal root, partitioned, damped matrix	Eq. (27c)
$[\Lambda_\omega]$	Natural frequency matrix	Eq. (10b)

LIST OF SYMBOLS

SYMBOLS	DEFINITION	REFERENCE
$\{a\}$	Initial displacement vector	Eq. (14)
$\{b\}$	Initial velocity vector	Eq. (14)
$\{f(t)\}$	Arbitrary forcing function vector	Eq. (8)
$\{g(t)\}$	Arbitrary forcing function vector	Eq. (11)
$\{h(t)\}$	Arbitrary forcing function vector	Eq. (12)
$\{u\}$	Eigen-vector	Eq. (2)
$\{v\}$	Eigen-vector	Eq. (4)
$\{x\}$	Displacement vector	Eq. (1)
$\{\dot{x}\}$	Velocity vector	Eq. (18)
$\{\ddot{x}\}$	Acceleration vector	Eq. (1)
$\{x\}_o$	Initial displacement	Eq. (17a)
$\{\dot{x}\}_o$	Initial velocity vector	Eq. (17b)
$\{y\}$	Associated displacement vector	Eq. (8)
$\{z\}$	Associated displacement vector	Eq. (9)
a_i	Constants	Eq. (13)
b_i	Constants	Eq. (13)
h_i	Scalar forcing function component	Eq. (30)
t	Time	Eq. (13)
w_i	Scalar displacement component	Eq. (31)
z_i	Scalar displacement component	Eq. (12)
λ	Characteristic root	Eq. (2)
ω	Scalar natural frequency	Eq. (13)
τ	Time	Eq. (13)

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CHAPTER I

Introduction

The problem of the forced vibration of lumped-mass systems has been investigated by a number of authors. A vector-type infinite series approach is considered by Crandall[4]*. More recently, concise matrix-type solutions have been formulated for both the damped and the undamped system. A classical matrix solution of a lumped-mass system is presented by Tse[2] where a nonsingular matrix must be determined which simultaneously diagonalizes the mass and stiffness matrices. Cauchey[3] investigates the condition of classical normal modes in the linear damped dynamic systems. Recently, the Cholesky transformation method for the free vibration of undamped systems was presented by Timoshenko[1] et al.

The formation of linear equations of motion of lumped-mass dynamical systems in structural dynamic problems yield equations which are expressed in the matrix form:

$$[[A]+\lambda[B]]\{x\}=\{f(t)\}$$

For the case of undamped systems the matrix [B] (i.e., the mass matrix) is symmetric and positive definite.

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*The superscript refers to the literature cited in the bibliography.

The matrix [A] (i.e.,the stiffness matrix) is symmetric only. The classical approach to the solution of the problem requires the determination of a nonsingular matrix [U] which simultaneously diagonalizes matrices [A] and [B]. This type of problem is termed the generalized eigenvalue-eigenvector problem (GEEP). The determination of the matrix [U] requires a considerable number of numerical operations.

The Cholesky transformation method allows the matrix [B] to be replaced by the product of a nonsingular lower triangular matrix with its transpose,that is:

$$[L][L]^T=[M].$$

This mathematical form allows the former equation to be transformed to the form:

$$[[D]+\lambda[I]]\{y\}=\{g(t)\},$$

where matrix [D] is symmetric. The form of the above equation is termed the classical eigenvalue-eigenvector problem (CEEP). This form of the equation of motion is more easily solved since the number of numerical operations in the solution is greatly reduced.

For the case of a damped system the matrices [A] and [B] are cast in a symmetric partitioned form,however, [B] is no longer positive definite. If the Cholesky transformation is applied for this case,the matrix [L] exists but it may be singular,nonunique,and possess complex(i.e. real and imaginary) components.

CHAPTER 11

2.1 Undamped Vibration Problem

The equations of motion for the vibration of a multi-degree of freedom dynamical system is written,

$$[M]\{\ddot{x}\}+[K]\{x\}=\{f(t)\}. \quad (1)$$

Where the mass matrix [M] is positive definite and symmetric,and the stiffness matrix [K] is symmetric only.

2.2 Free Vibration Problem

Equating to zero the right hand side of equation (1), and noting $\{x(t)\}=e^{\lambda^{k}t}\{u\}$,it follows that,

$$[[K]+\lambda[M]]\{u\}=\{0\} \quad (2)$$

which is uniquely the form (GEEP). The Cholesky transformation yields the condition

$$[L][L]^T=[M], \quad (3)$$

where [L] is lower triangular,real,and nonsingular. Premultiplying equation (2) by $[L]^{-1}$,noting $[L]^{-T}[L]^{-1}=[I]$, $[L]^T\{u\}=\{v\}$,together with equation (3) one obtains the equation (CEEP)

$$[[K_1]+\lambda[I]]\{v\}=\{0\}, \quad (4)$$

where $[K_1]=[L]^{-1}[K][L]^{-T}$ which is symmetric.

The determinant of the coefficient matrix of the vector {v} in equation (4) yields the characteristic equation of the matrix form with its usual classical type invariant coefficients. The roots of this equation (i.e. the values of λ) yield terms containing the natural frequency of free vibration. For stable oscillation the values of λ are always negative. Hence, the values of λ are complex with zero real.

The individual vectors $\{v\}$ corresponding to each value of λ are determined using the form of equation (4). The vectors are combined into a single orthogonal matrix $[V]$ where the following orthogonal property holds:

$$[V]^T[K_1][V]=-[\Lambda]=[\Lambda_\omega]^2. \tag{5}$$

The matrix $[\Lambda]$ is a diagonal matrix with components $\lambda_i, i=1,\ldots,n$, and the matrix $[\Lambda_\omega]$ is a diagonal matrix with components equal to the natural frequencies of free vibration of the system.

2.3 Unit Triangular Matrix Form

In addition to the transformation simplification in part (2.2), the matrix $[M]$ is replaced by the product of three matrices as follows

$$[M]=[L^*][M_d]^2[L^*]^T, \tag{6}$$

where $[L^*]$ is a lower unit triangular matrix and $[M_d]$ is a diagonal matrix, with $[L]=[L^*][M_d]$. Substituting equation (6) and (1) into equation (2) and proceeding in a manner similar to that in section (2.2), it follows that

$$[[K_1^*]+\lambda[I]]\{v\}=\{0\}, \tag{7}$$

where $[K_1^*]=[K_1^*]^T$, that is, $[K_1^*]$ is symmetric.

2.4 Forced Vibration Problem

The closed form solution to equation (1) is obtained in Duhamel integral form in the following manner. Equation (3) is substituted into equation (1) and the result is premultiplied by $[L]^{-1}$ yielding

$$[L]^{-1}[L][L]^T\{\ddot{x}\}+[L]^{-1}[K][L]^{-T}[L]^T\{x\}=[L]^{-1}\{f(t)\}.$$

The transformation $\{y\}=[L]^T\{x\}$ is substituted into the above equation which gives

$$[I]\{\ddot{y}\}+[K_1]\{y\}=[L]^{-1}\{f(t)\}. \tag{8}$$

Substitution of the additional tramsformation

$$\{y\}=[V]\{z\} \tag{9}$$

into equation (8), with premultiplication by the matrix $[V]^T$ gives the diagonal matrix equations,

$$[I]\{\ddot{z}\}-[\Lambda]\{z\}=[V]^T[L]^{-1}\{f(t)\}, \text{ or} \tag{10a}$$

$$[I]\{\ddot{z}\}+[\Lambda_\omega]^2\{z\}=[V]^T[L]^{-1}\{f(t)\} \tag{10b}$$

The component form of the latter matrix equation is written as

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}\begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \\ \vdots \\ \ddot{z}_n \end{Bmatrix}+\begin{bmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_n^2 \end{bmatrix}\begin{Bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{Bmatrix}=\begin{Bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{Bmatrix}. \tag{11}$$

The general i^{th} scalar equation of matrix equation (11) is

$$z_i+\omega_i^2 z_i=h_i(t), \tag{12}$$

which possesses the following integral solution:

$$z_i(t)=a_i\cos\omega_i t+b_i\sin\omega_i t+\frac{1}{\omega_i}\int_{\tau=0}^{\tau=t} h_i(\tau)\sin\omega_i(t-\tau)d\tau. \tag{13}$$

Recasting equation (13) into matrix form gives

$$\{z\}=[C]\{a\}+[S]\{b\}+[\Lambda_\omega]^{-1}\int_{\tau=0}^{\tau=t}[\hat{S}]\{h(\tau)\}d\tau, \qquad (14)$$

where [C], [S], and $[\hat{S}]$ are diagonal matrices with components $\cos\omega_i t$, $\sin\omega_i t$, and $\sin\omega_i(t-\tau)$, respectively. Noting the combined transformation equation

$$\{z\}=[V]^T[L]^T\{x\} \qquad (15)$$

together with equation (14), the general solution of equation (1) becomes

$$\{x(t)\}=[L]^{-T}[V][C]\{a\}+[L]^{-T}[V][S]\{b\}+$$
$$[L]^{-T}[V][\Lambda_\omega]^{-1}\int_{\tau=0}^{\tau=t}[S][V]^T[L]^{-1}\{f(t)\}d\tau. \qquad (16)$$

Applying the initial conditions at t=0,

$$\{x(0)\}=\{x_0\} \text{ and}$$
$$\{\ddot{x}(0)\}=\{\ddot{x}_0\},$$

it follows from equation (16) that

$$\{a\}=[V]^T[L]^T\{x_0\} \text{ and} \qquad (17a)$$
$$\{b\}=[\Lambda_\omega]^{-1}[V]^T[L]^T\{\ddot{x}_0\}. \qquad (17b)$$

2.5 Summary of Results

2.5a Free Vibration Problem

1. $[L][L]^T=[M]$

2. $[L]=[L^*][M_d]$

3. $[K_1]=[L]^{-1}[K][L]^{-T}=[K_1]^T$

4. $[[K_1]+\lambda[I]]\{v\}=\{0\}$

5. $[V][V]^T=[V]^T[V]=[I]$

6. $[V]^T[K_1][V]=-[\Lambda]=[\Lambda_\omega]^2$

2.5b Forced Vibration Problem

1. $$\{x(t)\}=[L]^{-T}[V][C][V]^T[L]^T\{x_0\}+ [L]^{-T}[V][S][\Lambda_\omega]^{-1}[V]^T[L]^T\{x_0\}+ [L]^{-T}[V][\Lambda_\omega]^{-1}\int_{\tau=0}^{\tau=t}[S][V]^T[L]^{-1}\{f(\tau)\}d\tau$$

2.6 Numerical Example

Referring to the problem presented by Crandall, the following numerical matrices are considered:

$$[M]=\begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} \qquad [K]=\begin{vmatrix} 4 & 1 \\ 1 & 1.5 \end{vmatrix}$$

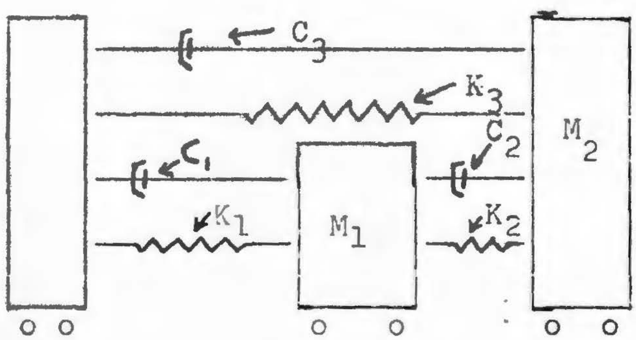

Fig. 1. Modeled system

Using the theory developed in sections (2.2) and (2.3) one obtains the following matrices when the damping matrix, [C], is set equal to zero.

$$[L]=\begin{vmatrix} \sqrt{3} & 0 \\ \frac{2\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \end{vmatrix} \qquad [L^*]=\begin{vmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{vmatrix}$$

$$[M_d]=\begin{vmatrix} \sqrt{3} & 0 \\ 0 & \frac{\sqrt{6}}{3} \end{vmatrix} \qquad [K_1]=\begin{vmatrix} \frac{4}{3} & \frac{-5\sqrt{2}}{6} \\ \frac{-5\sqrt{2}}{6} & \frac{35}{12} \end{vmatrix}$$

$$[V]=\begin{vmatrix} .467 & .883 \\ .876 & -.470 \end{vmatrix} \qquad [\Lambda_\omega]=\begin{vmatrix} 3.348 & 0 \\ 0 & .318 \end{vmatrix}$$

The natural frequencies of free vibration become,

$$\omega_1=3.348 \text{ cycles/sec}$$

$$\omega_2=.318 \text{ cycles/sec}$$

CHAPTER III

3.1 Damped Vibration Problem

The standard matrix equations of motion for the linear damped dynamical systems which are common to structural dynamic problems take the form

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{f(t)\}, \tag{18}$$

where matrices [M], [C], and [K] are symmetric and [M] is positive definite.

Previous work has shown that the solution of equation (18) yields a more compact solution if it is recast in partitioned matrix form. Letting

$$\{x\}=\{y_1\},$$
$$\{\dot{x}\}=\{\dot{y}_1\}=\{y_2\}, \text{ and}$$
$$\{\ddot{x}\}=\{\ddot{y}_1\}=\{\dot{y}_2\},$$

it follows that equation (18) takes the partitioned matrix form

$$\begin{bmatrix}[M] & [0]\\ [0] & -[K]\end{bmatrix}\begin{Bmatrix}\{\dot{y}_2\}\\ \{\dot{y}_1\}\end{Bmatrix}+\begin{bmatrix}[C] & [K]\\ [K] & [0]\end{bmatrix}\begin{Bmatrix}\{y_2\}\\ \{y_1\}\end{Bmatrix}=\begin{Bmatrix}\{f(t)\}\\ \{0\}\end{Bmatrix}. \tag{19}$$

For simplicity, equation (19) is written in the compact form as

$$[MK]\{\dot{y}\}+[KC]\{y\}=\{g(t)\}, \tag{20}$$

Where the matrices [MK] and [KC] are symmetric. The matrix [MK] is not a positive definite matrix.

As stated in the introduction the Cholesky transformation is applicable, however, the matrix [L] may be singular, nonunique, and possess complex number components. Assuming the following condition applies

$$[L][L]^T=[MK],$$

it follows that

$$\begin{bmatrix} [L_{11}] & [0] \\ [L_{21}] & [L_{22}] \end{bmatrix} \begin{bmatrix} [L_{11}]^T & [L_{21}]^T \\ [0] & [L_{22}]^T \end{bmatrix} = \begin{bmatrix} [M] & [0] \\ [0] & -[K] \end{bmatrix}, \quad (21)$$

or

$$[L_{11}][L_{11}]^T=[M], \quad (22a)$$

$$[L_{22}][L_{22}]^T=-[K], \text{ and} \quad (22b)$$

$$[L_{21}][L_{11}]^T=[L_{11}][L_{21}]^T=[0]. \quad (22c)$$

Solution of equation (22a) for the matrix $[L_{11}]$ is unique since the matrix [M] is positive definite. Solution of equation (22b) for the matrix $[L_{22}]$ is more complicated since the matrix -[K] is not positive definite. Assuming there exists a transformation matrix $[L_{22}]$ which satisfies equation (22b), it follows that, for the usual type of stiffness matrices which are defined for linear dynamical systems, the matrix $[L_{22}]$ consists of components which are all complex numbers with zero real parts. In addition matrix $[L_{22}]$ is not unique since the two following equations hold simultaneously.

$$[L_{22}]^T[L_{22}]=-[K] \text{ and}$$

$$[\tilde{L}_{22}]^T[\tilde{L}_{22}]=-[K], \text{ where}$$

the designation $[\tilde{\ }]$ represents the complex conjugate definition. Finally, equation (22c) requires the matrix $[L_{21}]$ to be identically the zero matrix.

Noting equation (21) with appropriate simplifications, equation (20) is rewritten

$$[L][L]^T\{\dot{y}\}+[KC]\{y\}=\{g(t)\}. \tag{23}$$

Substituting $[L]^{-T}[L]^T=[I]$ into the second term of equation (23) premultiplying the equation by $[L]^{-1}$, and noting $\{z\}=[L]^T\{y\}$, one obtains

$$[I]\{\dot{z}\}+[G]\{z\}=[L]^{-1}\{g(t)\}, \tag{24}$$

where $[G]=[L]^{-1}[KC][L]^{-T}$ is a complex symmetric matrix. The form of equation (24) is similar to the (CEEP) form except for the complex components in the matrix [G].

3.2 Free Vibration Problem

For the free vibration problem the right hand side of equation (24) is equated to zero, and the condition $\{z\}=e^{\lambda t}\{u\}$ substituted, yielding

$$[[G]+\lambda[I]]\{u\}=\{0\}. \tag{25}$$

The roots λ of equation (25) must complex conjugate pairs each possessing negative real parts. This is the requirement of any lightly damped vibratory system if stable decay-type oscillation occurs. For each complex value of λ, the associated eigenvector {u} is determined by the solution of equation (25). Defining the matrix [U] whose columns contain the complete set of eigenvectors it follows that

$$[U]^{-1}[U]=[I] \text{ and} \tag{26a}$$

$$[U]^{-1}[G][U]=-[\Lambda_g], \tag{26b}$$

where $-[\Lambda_g]$ is a diagonal matrix with components equal to the individual roots λ obtained by the solution of the determinant form of the left hand side of equation (25). The column vectors of matrix [U] are individually normalized by dividing each vector component by a number equal to the square root of the sum of the products of each component times the associated complex conjugate.

3.3 Forced Vibration Problem

The solution of equation (24) forms the basic solution to the damped vibration problem. Premultiplying equation (24) by $[U]^{-1}$ and substituting $\{z\}=[U]\{w\}$, one obtains

$$[I]\{\dot{w}\}+[U]^{-1}[G][U]\{w\}=[U]^{-1}[L]^{-1}\{g(t)\}\equiv\{h(t)\}. \tag{27a}$$

Noting equation (26b) the above equation becomes

$$[I]\{\dot{w}\}-[\Lambda_g]\{w\}=\{h(t)\}. \tag{27b}$$

Equation (27b) is rewritten in partitioned form as

$$\left[\begin{array}{c|c}[I] & [0]\\ \hline [0] & [I]\end{array}\right]\left\{\begin{array}{c}\{\dot{w}_2\}\\ \hline \{\dot{w}_1\}\end{array}\right\}-\left[\begin{array}{c|c}[\Lambda_2] & [0]\\ \hline [0] & [\Lambda_1]\end{array}\right]\left\{\begin{array}{c}\{w_2\}\\ \hline \{w_1\}\end{array}\right\}=\left\{\begin{array}{c}\{h_2(t)\}\\ \hline \{h_1(t)\}\end{array}\right\}. \tag{28}$$

The general i^{th} equation of the partitioned form is

$$\dot{w}_i-\lambda_i w_i=h_i(t), \tag{29}$$

which possesses the following integral solution form

$$w_i(t)=e^{\lambda_i t}a_i(0)+\int_{\tau=0}^{\tau=t}e^{\lambda_i(t-\tau)}h_i(\tau)d\tau. \tag{30}$$

Recasting equation (30) into partitioned matrix form yields

$$\left\{\begin{array}{c}\{w_2\}\\ \hline \{w_1\}\end{array}\right\}=\left[\begin{array}{c|c}\exp[\Lambda_2] & [0]\\ \hline [0] & \exp[\Lambda_1]\end{array}\right]\left\{\begin{array}{c}\{a_2\}\\ \hline \{a_1\}\end{array}\right\}+ \tag{31}$$

$$\int_{\tau=0}^{\tau=t}\left[\begin{array}{c|c}\exp[\Lambda_2](t-\tau) & [0]\\ \hline [0] & \exp[\Lambda_1](t-\tau)\end{array}\right]\left\{\begin{array}{c}\{h_2(\tau)\}\\ \hline \{h_1(\tau)\}\end{array}\right\}d\tau$$

Noting $\{w\}=[U]^{-1}\{z\}$, $\{z\}=[L]^{T}\{y\}$, and hence $\{w\}=[U]^{-1}[L]^{T}\{y\}$ equation (31) becomes

$$\begin{Bmatrix}\{y_2\}\\ \hline \{y_1\}\end{Bmatrix}=\begin{bmatrix}[L_{11}]^{-T} & [0]\\ \hline [0] & [L_{22}]^{-T}\end{bmatrix}\begin{bmatrix}[U_{11}] & [U_{12}]\\ \hline [U_{21}] & [U_{22}]\end{bmatrix}\begin{bmatrix}\exp[\Lambda_1] & [0]\\ \hline [0] & \exp[\Lambda_2]\end{bmatrix}\begin{Bmatrix}\{a_1\}\\ \hline \{a_2\}\end{Bmatrix}+$$

$$\begin{bmatrix}[L_{11}]^{-T} & [0]\\ \hline [0] & [L_{22}]^{-T}\end{bmatrix}\begin{bmatrix}[U_{11}] & [U_{12}]\\ \hline [U_{21}] & [U_{22}]\end{bmatrix}\int_{\tau=0}^{\tau=t}\begin{bmatrix}\exp[\Lambda_2](t-\tau) & [0]\\ \hline [0] & \exp[\Lambda_1](t-\tau)\end{bmatrix}$$

$$\begin{Bmatrix}\{h_1(\tau)\}\\ \hline \{h_2(\tau)\}\end{Bmatrix}d\tau, \qquad (32)$$

where

$$\begin{Bmatrix}\{h_1(t)\}\\ \hline \{h_2(t)\}\end{Bmatrix}=\begin{bmatrix}[U_{11}] & [U_{12}]\\ \hline [U_{21}] & [U_{22}]\end{bmatrix}^{-1}\begin{bmatrix}[L_{11}]^{-1} & [0]\\ \hline [0] & [L_{22}]^{-1}\end{bmatrix}\begin{Bmatrix}\{f(t)\}\\ \hline \{0\}\end{Bmatrix}. \qquad (32b)$$

3 Numerical Example

Following the numerical example of Crandall the mass, damping, and stiffness matrices (fig. 1) are

$$[M]=\begin{vmatrix}3 & 2\\ 2 & 2\end{vmatrix}, \qquad [C]=\begin{vmatrix}0.14 & 0.04\\ 0.04 & 0.06\end{vmatrix}, \text{and} \qquad [K]=\begin{vmatrix}4 & 1\\ 1 & 1.5\end{vmatrix}.$$

It follows by the previous theory that

$$[L_{11}]=\begin{vmatrix}\sqrt{3} & 0\\ \frac{2\sqrt{3}}{3} & \frac{\sqrt{6}}{3}\end{vmatrix}, \qquad [L_{22}]=\begin{vmatrix}\pm 2i & 0\\ \pm\frac{i}{2} & \pm\frac{i\sqrt{5}}{2}\end{vmatrix},$$

$$[G]=[G]^T=\begin{vmatrix}0.0467 & -0.0377 & -1.155i & 0.0\\ -0.0377 & 0.1033 & 1.021i & -1.369i\\ -1.155i & 1.021i & 0.0 & 0.0\\ 0.0 & -1.369i & 0.0 & 0.0\end{vmatrix}$$

Note that the matrix [G] is complex. The characteristic equation which yields the roots of the determinant equation (i.e. $|[G]+\lambda[I]|=0$) becomes

$$\lambda^4+0.0150\lambda^3+4.225\lambda^2+0.1851\lambda+2.500=0$$

The four roots of λ are determined as

$$\lambda_{1,2}=-0.014\pm 0.8397i \text{ and}$$

$$\lambda_{3,4}=-0.061\pm i.8818i \ .$$

These roots are the same as those given by Crandall. Observation of the characteristic equation shows that there is no sign change in the coefficients. This condition prevents any real roots of the equation from existing. This is expected since the damping in the system must produce roots(i.e. values of λ) which are complex conjugates with negative real parts.

The matrix [U] equals

$$\left[\begin{array}{cc:cc}
\frac{1}{1.6028} & \frac{1}{1.6028} & \frac{1}{3.0057} & \frac{1}{3.0057} \\
\frac{0.533-0.001i}{1.6028} & \frac{0.533+0.001i}{1.6028} & \frac{-1.875-0.0073i}{3.0057} & \frac{-1.875+0.007i}{3.0057} \\
\hdashline
\frac{0.727-0.011i}{1.6028} & \frac{-0.727-0.011i}{1.6028} & \frac{1.630-0.0488i}{3.0057} & \frac{-1.630-0.048i}{3.0057} \\
\frac{0.869-0.016i}{1.6028} & \frac{-0.869-0.016i}{1.6028} & \frac{-1.363+0.0388i}{3.0057} & \frac{1.363+0.038i}{3.0057}
\end{array}\right]$$

The $[\Lambda_g]$ matrix is produced by $[U]^{-1}[G][U]$. The $[\Lambda_g]$ takes the form

$$\left[\begin{array}{cc:cc}
0.014-0.840i & 0.0 & 0.0 & 0.0 \\
0.0 & 0.014+0.840i & 0.0 & 0.0 \\
\hdashline
0.0 & 0.0 & 0.060-1.882i & 0.0 \\
0.0 & 0.0 & 0.0 & 0.060+1.882i
\end{array}\right]$$

CHAPTER IV

Discussion

A closed-form solution of the undamped dynamical system is obtained for the forced vibration problem. The Cholesky transformation is applied to the positive definite mass matrix [M]. Hence, the matrices determined in the solution are unique. This transformation yields a solution in a compact and precise form.

A closed-form solution of the damped vibration problem is obtained assuming the Cholesky transformation applies. Since the partitioned mass matrix is no longer positive definite, a solution is obtained which possesses nonunique parts. Utilization of the transformation is validated by comparing the resulting numerical values with those obtained via classical techniques. In all cases the values of the natural frequencies of free vibration correlate with those obtained by the classical techniques.

CHAPTER V

Conclusions

The principal advantage of the use of the Cholesky transformation for the solutions of linear dynamical systems is that the number of numerical computations which must be performed is reduced by approximately sixty(60) percent. This condition becomes meaningful in systems possessing a large number of degrees of freedom which are most efficiently solved using computer techniques.

Since the transformation increases the efficiency of the mathematical operations, time required for computer usage is noticeably reduced. This directly minimizes the cost for analysing the systems.

The previous analysis justifies the use of the Cholesky transformation for linear damped dynamical systems where no matrix possesses a positive definite form. In this case the [L] matrix used in the formulation is nonunique since it may be replaced by its conjugate without effecting the final solution.

The damped vibrations problem requires the diagonalization of a matrix [G] where $[G] = [G]^T$ and where [G] is complex. This case is not covered in the available mathematical literature. The usual congruent transformations do not apply and a reversion to the inverse technique for diagonalization is a basic requirement.

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APPENDIX A

Classical Eigenvalue-eigenvector Problem

CLASSICAL EIGENVALUE-EIGENVECTOR PROBLEM

The solution of equation (1) using the (CEEP) is formulated here for comparison purposes. The eigenvector matrix [U] which simultaneously diagonalizes matrices [A] and [B] is determined by the solution of the equation

$$[[K]\{x\}+\lambda[M]]\{x\}=\{f(t)\}. \tag{A1}$$

The solution of the determinant equation

$$|[K]+\lambda[M]|=\{0\} \tag{A2}$$

is obtained first. The roots of λ are substituted back into equation (A1) and the associated eigenvectors which compose the matrix [U] are obtained. It follows that

$$[U]^T[M][U]=[\Lambda_m], \tag{A3a}$$

$$[U]^T[K][U]=[\Lambda_k], \text{ and} \tag{A3b}$$

$$[\Lambda_k]=[\Lambda_m][\Lambda_w]^2 , \tag{A3c}$$

where $[\Lambda_m]$ and $[\Lambda_k]$ are both diagonal matrices. Premultplying equation (A1) by $[U]^T$ and noting that $\{y\}=[U]\{z\}$ one obtains

$$[U]^T[M]\{z\}+[U]^T[K][U]\{z\}=[U]^T\{f(t)\}. \tag{A4}$$

Noting equations (A3a) and (A3b),equation (A4) reduces to the diagonal form

$$[\Lambda_m]\{z\}+[\Lambda_k]\{z\}=[U]^T\{f(t)\}. \tag{A5}$$

Premultiplying equation (A5) by $[\Lambda_m]^{-1}$ and noting equation (A3c),one obtains

$$[I]\{z\}+[\Lambda_w]^2\{z\}=[\Lambda_m]^{-1}[U]^T\{f(t)\}. \tag{A6}$$

For the numerical problem given in Chapter 2,the following results are formulated.

$$[\Lambda_k] = \begin{bmatrix} \frac{9}{5} & 0 \\ 0 & \frac{99}{17} \end{bmatrix} \qquad [U] = \begin{bmatrix} \frac{1}{5} & \frac{4}{17} \\ \frac{-2}{5} & \frac{1}{17} \end{bmatrix}$$

$$[\Lambda_m] = \begin{bmatrix} \frac{9}{5} & 0 \\ 0 & \frac{18}{17} \end{bmatrix} \qquad [\Lambda_w] = \begin{bmatrix} 1 & 0 \\ 0 & \frac{2}{11} \end{bmatrix}$$

APPENDIX B

Characteristic Equation and Matrix Invariants

CHARACTERISTIC EQUATION AND MATRIX INVARIANTS

Given

$$[Q]=[Q]^T=\begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix},$$

It follows that the three tensor invariants are:

I_1= Trace of $[Q]=q_{11}+q_{22}=q_{33}$.

I_2= sum of the determinant minors of the principle diagonal= $\begin{vmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{vmatrix} + \begin{vmatrix} q_{11} & q_{13} \\ q_{13} & q_{33} \end{vmatrix} + \begin{vmatrix} q_{22} & q_{23} \\ q_{23} & q_{33} \end{vmatrix}.$

I_3= Major determinant of [Q]

The characteristic equation is written as

$$\lambda^3-I_1\lambda^2+I_2\lambda-I_3=0$$

APPENDIX C

Modeling Example

MODELING EXAMPLE

C.1 Two Story Frame

The modeling of a two story frame into a spring mass system produces equations of vibration that are in the form of equation (1). The girders are assumed to be infinitely ridid as compared to the columns. K_1 (i.e. column stiffness) is expressed as

$$k_1 = \frac{12\ EI}{l^3}$$

for the fix-fix conditions.

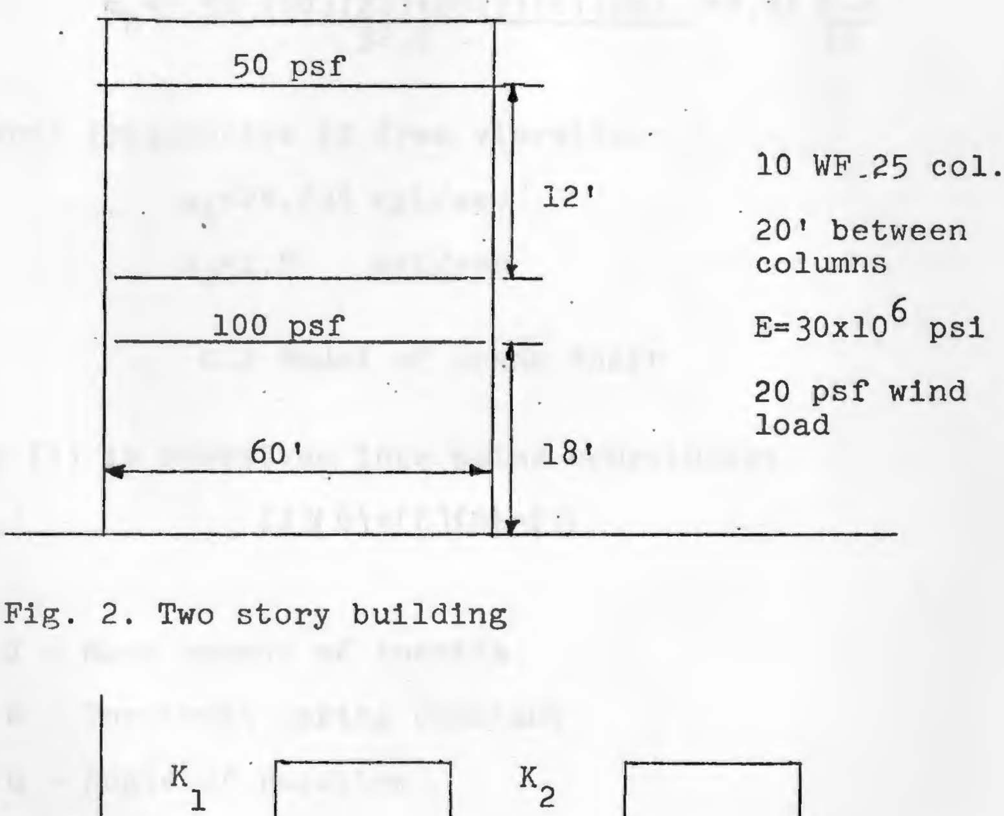


Fig. 2. Two story building

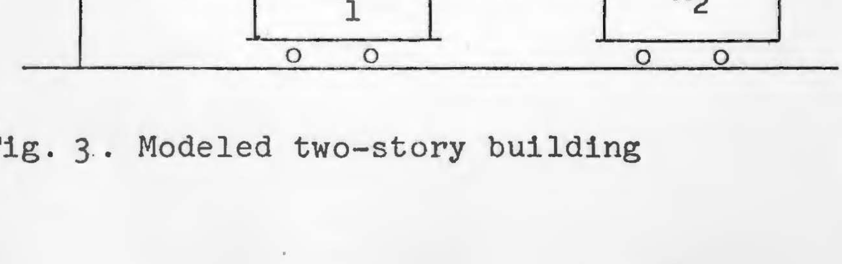


Fig. 3. Modeled two-story building

MODELING EXAMPLE

The equations of equilibrium are written in matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The spring and mass constant are

$$k_1 = \frac{2\,(12)(30\times10^6)(133.2)}{(18)^3\,(12)^3} = 9.52 \text{ kips/in}$$

$$k_2 = \frac{2\,(12)(30\times10^6)(133,2)}{(12)^3\,(12)^3} = 32.12 \text{ kips/in}$$

$$m_1 = \frac{100\,(60)(20)+20(2)(15)(20)}{32.2} = 4.10\ \frac{\text{k-s}}{\text{in}}$$

$$m_2 = \frac{.50\,(60)(20)+20(2)(6)(20)}{32.2} = 2.01\ \frac{\text{k-s}}{\text{in}}$$

The natural frequencies of free vibration

$\omega_1 = 24.634$ cyl/sec

$\omega_2 = 1.5$ cyl/sec

C.2 Model of Crank Shaft

Equation (1) is rewritten into polar ccordinates

$$[J]\{\ddot{\theta}\}+[K]\{\theta\}=\{T\}$$

where

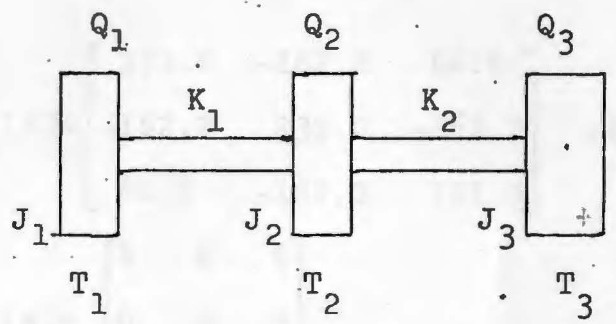
J - Mass moment of inertia

k - Torsional spring constant

Q - Angle of rotation

T - applied troque

MODELING EXAMPLE



The equations of equilibrium are written in matrix form:

$$\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{\theta} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} \theta \\ \theta \\ \theta \end{Bmatrix} = \begin{Bmatrix} T \\ T \\ T \end{Bmatrix}$$

C.3 Modeling of Beam Structures

By lumping the masses and loads at discrete pionts along the beam the dynamic responses can be determined. The motion equation that governs is;

$$m_n\ddot{y}_n = k_{n1}y_{n1} + k_{n2}y_{n2} + \dots + k_{nn}y_n ,$$

in matrix form

$$[M]\{\ddot{y}_n\}+[K]\{y_n\}=\{0\}.$$

The stiffness coefficients can be calculated by the method of moments distribution.

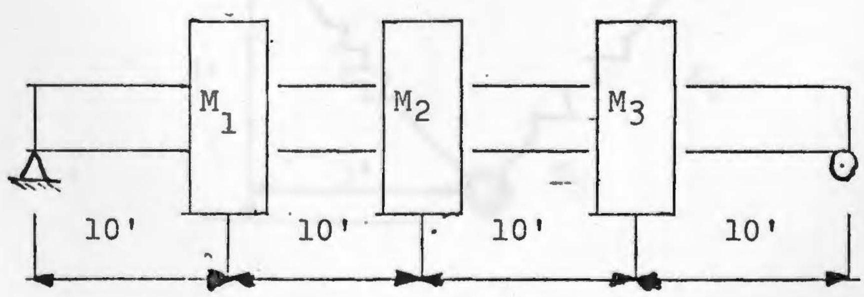


Fig. 4 Modeled beam

MODLEING EXAMPLE

$$[K]=\begin{bmatrix} 171.0 & -162.2 & 66.6 \\ -162.2 & 235.0 & -162.2 \\ 66.6 & -162.2 & 171.0 \end{bmatrix} \text{ kips/in}$$

$$[M]=\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$M_1=4$ kips, $M_2=3$ kips, $M_3=4$ kips ; where [K] is the stiffness matrix and [M] the mass matrix.

C.4 Model Truss

A simple truss is moleled by placing the mass of each member at the nodes and having the members act as springs. The stiffness coefficients is equal to

$$k_i = \frac{AE}{L}$$

where A is the area of the member, E , the modulus of elasticity, L ; length of the member. An example of a model truss is

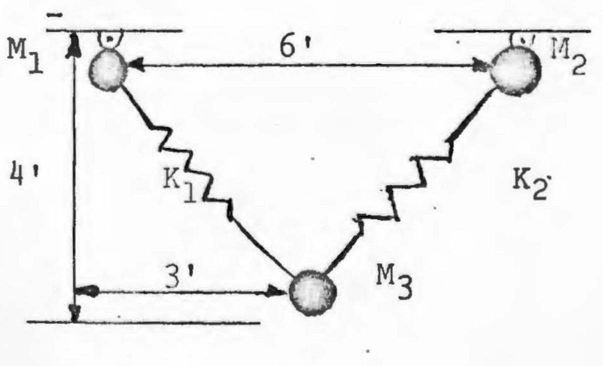


Fig. 5 Model truss

MODELING EXAMPLE

$M_1 = M_2 = M_3 = 2$ kips $\qquad$ $A = 3$ in.^{2}

$E = 30\text{x}10^6$ psi

$$K_1 = K_2 = \frac{3\ (30\text{x}10^6)}{5(12)} = 1500 \text{ kips/in.}$$

The equilibrium equation matrix is

$$[2]\{\ddot{v}\}+[1500]\{v\}=\{0\}\ .$$

The natural frequency of the system is

$$\lambda = 54.78 \text{ cyl./sec.}\ .$$