MATRIX METHODS IN THE NONLINEAR
THEORY OF ELASTICITY

by

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ABSTRACT

MATRIX METHODS IN THE NONLINEAR THEORY OF ELASTICITY

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The purpose of this thesis is to introduce the concepts of matrix algebra and matrix calculus to the field of nonlinear elasticity in order to bridge the gap between a theoritical tensor analysis approach requiring extensive complex mathematics and a basic scalar component approach requiring an extensive memory capacity.

The general nonlinear theory of elasticity including the strain-displacement equations, the equations of equilibrium, and the stress-strain laws are derived in matrix form for the general case.

Three special cases of the general theory are considered:

a) Elongations and shears are small in comparison to unity.

b) Elongations, shears, and angles of rotation are small in comparison to unity.

c) Classical linear elasticity equations.

The nonlinear theory of elasticity, being an essential generalization of the classical theory permits an approach to the solution of a series of important problems which do not arise in the latter theory because of its limitations.

The special case of the large deflection of a thin rod is considered to illustrate the nonlinear theory.

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LIST OF NOTATIONS

SYMBOL	DEFINITION
a_2, a_1, a_0	The three matrix invariants of $[\mathcal{E}]$
b_2, b_1, b_0	The three matrix invariants of $[D]$
b'_2, b'_1, b'_0	The three matrix invariants of $[e]$
b''_2, b''_1, b''_0	The three matrix invariants of $[\omega]$
c_2, c_1, c_0	The three matrix invariants of $[\nabla_0^*]$
$[D]$	$= [e] + [\omega]$
E_1, E_2, E_3	The relative elongations
E	Young's modulus of elasticity
$\{F_x^*\}$	Vector of body forces
$\{f_x^*\}$	Vector of surface forces
H_1, H_2, H_3	Lame coefficients
$[I]$	The identity matrix
$\{\bar{i}\}$	Unit vectors of rectangular Cartesian coordinate system
$\{\tilde{i}\}$	Unit vectors of tangents to the curve line of curvilinear coordinate system
$[J]$	Jacobian matrix
$\{n\}$	External normal unit vector
δR_1	The virtual work due to body forces
δR_2	The virtual work of the surface forces
S_1, S_2, S_3	Surface areas
S	The intensity of tangential stresses
T	The intensity of shearing strains
$\{u\}$	Vector of displacements

SYMBOL	DEFINITION
V	Volume
W	Total work done
x_1, x_2, x_3	Rectangular Cartesiàn coordinates
$\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$	Curvilinear coordinates
$*$	Deformed state
$\sim$	Curvilinear system
COF[ ]	Cofactor of the matrix
$[\alpha]$	$[COF[J]]^T$
Ψ	$= -\frac{\partial \Phi}{\partial a_1}$
Φ	Specific strain energy
$[\varepsilon]$	Strain matrix
$[\sigma]$	Stress matrix
$[\Lambda]$	Orthogonal transformed matrix
μ	Poisson's ratio
$\phi_{12}, \phi_{13}, \phi_{23}$	Shears
ϕ_1, ϕ_2, ϕ_3	Euler angles of rotations
α, β, γ	Euler angles of rotations
$\alpha_1, \alpha_2, \alpha_3$	Orthogonal curvilinear lines
ψ_1, ψ_2, ψ_3	angles of rotation
$\times$	Cross product of vectors
$\|[\ ]\|$	Determinant of matrix

LIST OF FIGURES

CHAPTER I

INTRODUCTION

1.1 Introduction

The basic concepts in the theory of Nonlinear Elasticity of Elastic Solids has by tradition been incorporated in a more broad course entitled Continuum Mechanics which includes the principles of both solid and fluid mechanics. This classical approach requires a through knowledge of Tensor Analysis including tensor algebra and tensor calculus.[(1,2)*] Most approaches utilize curvilinear tensor notation including notations of contravariant and covariant tensors.[(3,4)] Cartesian tensor notation is usually considered as a special case. Some authors[(5,6)] have introduced a combination of both tensor analysis and matrix analysis in nonlinear solid mechanics.

Probably the most well known text in this area is that written by Novozhilov[(7)] which totally eliminates the use of tensor operations. Atmost, a reader requires an elementary course in partial differentiation as prerequisite to reading the text which presents all concepts using "scalar operations." This scalar approach produces an extensive number of equations with no commonality among them.

* Numbers in parenthesis referred to Literature Cited.

The reader is confronted with a requirement of recalling literally hundreds of complex equations in order to understand the principles.

The purpose of this thesis is to introduce the concepts of matrix algebra and matrix calculus to the field of nonlinear elasticity in order to bridge the gap between a theoritical tensor approach requiring extensive mathematics and a basic scalar approach requiring extensive memory capacity. The matrix approach has a prime advantage of forming a common basis for all mathematical operations as well as forming a direct connection for interpretation of mathematical results to real, physical, engineering problems.

1.2 Coordinates

Given the positions of the points of the body in its initial state (i.e., before deformation) and in its terminal state (i.e., after deformation), determine the change in the distance between two arbitary infinitely near points of the body caused by its transition from the first state to the second.

Let the positions of the points of the body in its initial state be described by their projections x_1, x_2, x_3 on the axes of some rectangular system of Cartesian Coordinates X_1, X_2, X_3 .

Furthermore, let the points of the body undergo displacements with components u_1, u_2, u_3 regarded as preassigned functions of x_1, x_2 and x_3 along the same axes. Then the terminal position of an arbitary point of the body is given the Cartesian coordinates

$$\overset{*}{X}_1 = X_1 + u_1(X_1, X_2, X_3)$$
$$\overset{*}{X}_2 = X_2 + u_2(X_1, X_2, X_3) \qquad (1\text{-}1)$$
$$\overset{*}{X}_3 = X_3 + u_3(X_1, X_2, X_3)$$

The functions u_1, u_2, u_3 as well as their partial derivatives with respect to X_1, X_2 and X_3 are assumed continuous. This restriction is called the continuity condition of the deformation.

It follows from equation (1-1) that the terminal position of the points of the body are described in two cases:

Case I -

X_1, X_2, X_3 are rectangular Cartesian coordinates for the initial state and become curvilinear coordinates for the terminal state. When X_1, X_2 and X_3 are considered as curvilinear coordinates of the deformed body, they are marked with tildes ($\sim$) for the curvilinear coordinates and stars (*) for the deformed coordinates (See figure (I-1)).

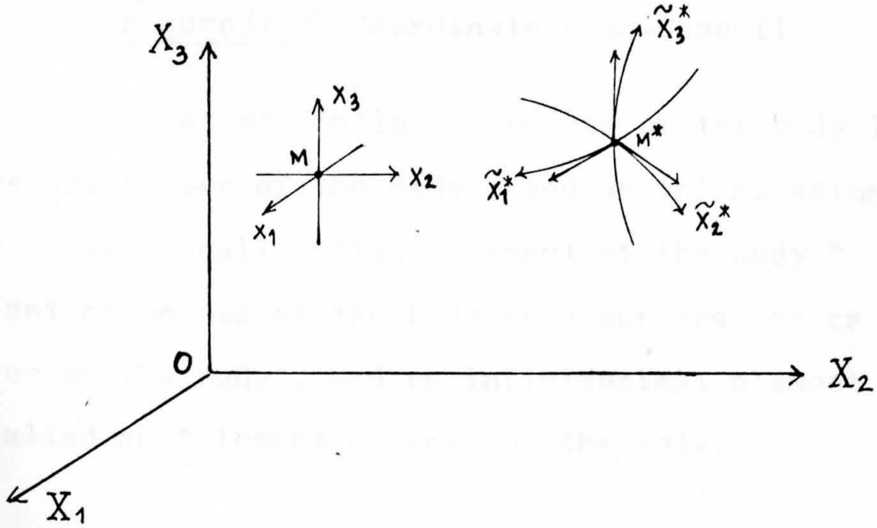


Figure (I-1) Coordinate Axes, Case I

Thus X_1, X_2, X_3 are Cartesian coordinates for the initial state. $\tilde{X}_1^*, \tilde{X}_2^*, \tilde{X}_3^*$ are Curvilinear coordinates for the deformed body.

Case II

X_1^*, X_2^*, X_3^* are rectangular Cartesian coordinates for the deformed body and $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3$ are curvilinear coordinates for the body before deformation (See figure(I-2)).

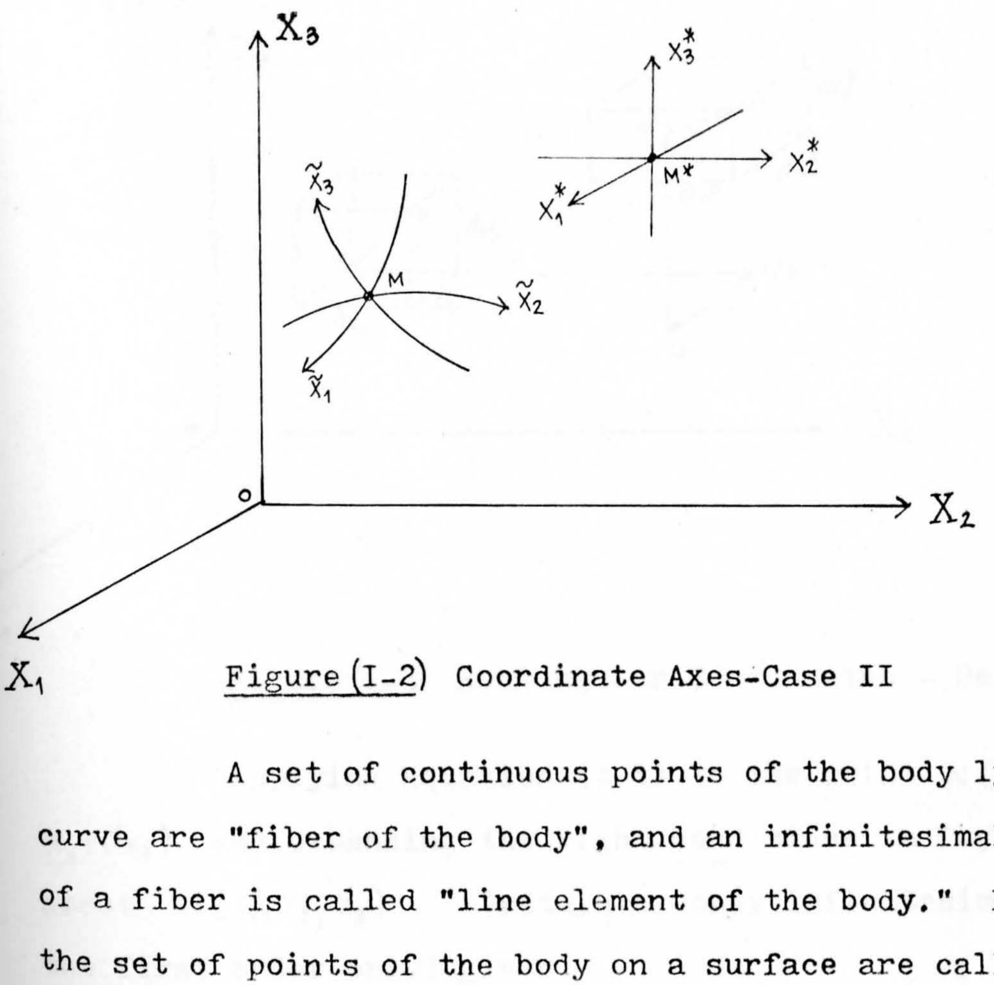


Figure (I-2) Coordinate Axes-Case II

A set of continuous points of the body lying on a curve are "fiber of the body", and an infinitesimal element of a fiber is called "line element of the body." Further the set of points of the body on a surface are called a "layer of the body", and on infinitesimal element of a layer is called an "element of area of the body."

1.3 Angular Directions of the Coordinate Lines

As a result of the deformation the point $M(x_1, x_2, x_3)$ is displaced to the position M* having the Cartesian coordinates $\overset{*}{x}_1, \overset{*}{x}_2, \overset{*}{x}_3$ whereas the point $N(x_1 + dx_1,\ x_2 + dx_2,\ x_3 + dx_3)$ infinitesimaly near M is displaced to the position N* having coordinates $\overset{*}{x}_1 + d\overset{*}{x}_1, \overset{*}{x}_2 + d\overset{*}{x}_2, \overset{*}{x}_3 + d\overset{*}{x}_3$. The vector $\overrightarrow{MN}$ has the projections dx_1, dx_2, dx_3. The vector $\overrightarrow{\overset{*}{M}\overset{*}{N}}$ has the projections $d\overset{*}{x}_1, d\overset{*}{x}_2, d\overset{*}{x}_3$.

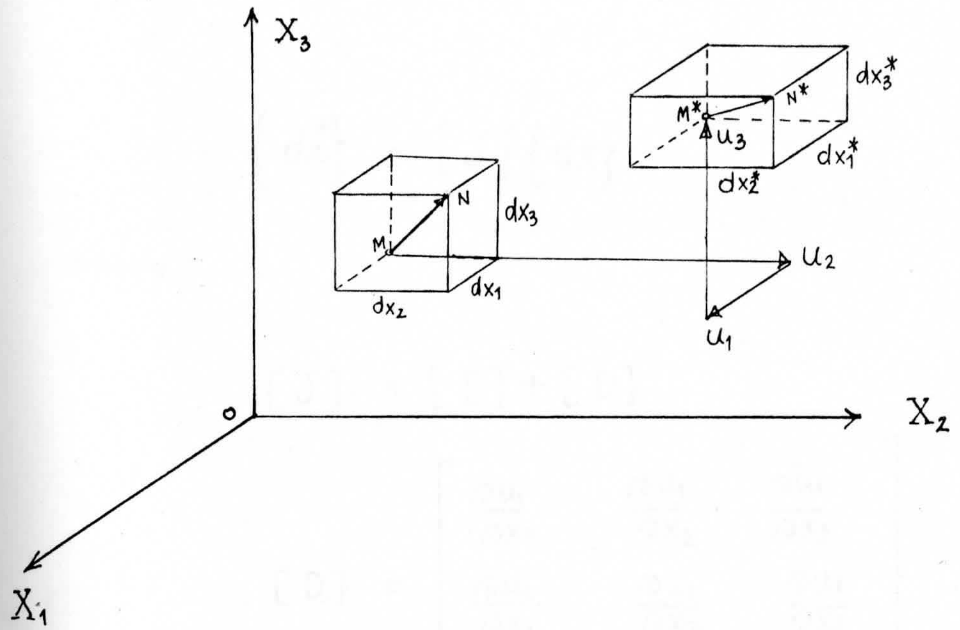


Figure (I-3) Rectangular Coordinates - Deformed Geometry

Applying Equation (1-1) to the point $N(x_1+dx_1,\ x_2+dx_2,\ x_3+dx_3)$ and expanding the right-hand sides in Taylor Series about $M(x_1, x_2, x_3)$ (retaining only infinitesimals of the first order only) gives

$$\begin{Bmatrix} dx_1^* \\ dx_2^* \\ dx_3^* \end{Bmatrix} = \begin{bmatrix} 1+\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & 1+\frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & 1+\frac{\partial u_3}{\partial x_3} \end{bmatrix} \begin{Bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{Bmatrix}$$

$$\{dx^*\} = [J]\{dx\} \qquad (1\text{-}2)$$

where

$$[J] = [I] + [D] \qquad (1\text{-}3)$$

$$[D] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial u} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \qquad (1\text{-}4a)$$

$$[I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (1\text{-}4b)$$

Introducing the notation

$$e_{11} = \frac{\partial u_1}{\partial x_1}, \quad e_{22} = \frac{\partial u_2}{\partial x_2}, \quad e_{33} = \frac{\partial u_3}{\partial x_3}$$

$$e_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}, \quad e_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}, \quad e_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}$$

$$2\omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \quad 2\omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \quad 2\omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

then $[J]$ is expressed in the form

$$[J] = [I] + [D] = [I] + [\omega] + [e] \tag{1-5}$$

where

$$[e] = \begin{bmatrix} e_{11} & \frac{1}{2}e_{12} & \frac{1}{2}e_{13} \\ \frac{1}{2}e_{12} & e_{22} & \frac{1}{2}e_{23} \\ \frac{1}{2}e_{13} & \frac{1}{2}e_{23} & e_{33} \end{bmatrix} ; \qquad [e] = [e]^T \tag{1-6a}$$

$$[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} ; \qquad [\omega] = -[\omega]^T \tag{1-6b}$$

Noting Equations (1-4a), (1-4c), (1-4d), it follows that

$$[e] = \frac{1}{2}([D]^T + [D]) \quad ; \quad [\omega] = \frac{1}{2}([D]^T - [D]) \tag{1-6c}$$

$$[D] = \begin{bmatrix} e_{11} & \frac{1}{2}e_{12}-\omega_3 & \frac{1}{2}e_{13}+\omega_2 \\ \frac{1}{2}e_{12}+\omega_3 & e_{22} & \frac{1}{2}e_{23}-\omega_1 \\ \frac{1}{2}e_{13}-\omega_2 & \frac{1}{2}e_{23}+\omega_1 & e_{33} \end{bmatrix} \tag{1-7a}$$

$$[J] = \begin{bmatrix} 1+e_{11} & \frac{1}{2}e_{12}-\omega_3 & \frac{1}{2}e_{13}+\omega_2 \\ \frac{1}{2}e_{12}+\omega_3 & 1+e_{22} & \frac{1}{2}e_{23}-\omega_1 \\ \frac{1}{2}e_{13}-\omega_2 & \frac{1}{2}e_{23}+\omega_1 & 1+e_{33} \end{bmatrix} \tag{1-7b}$$

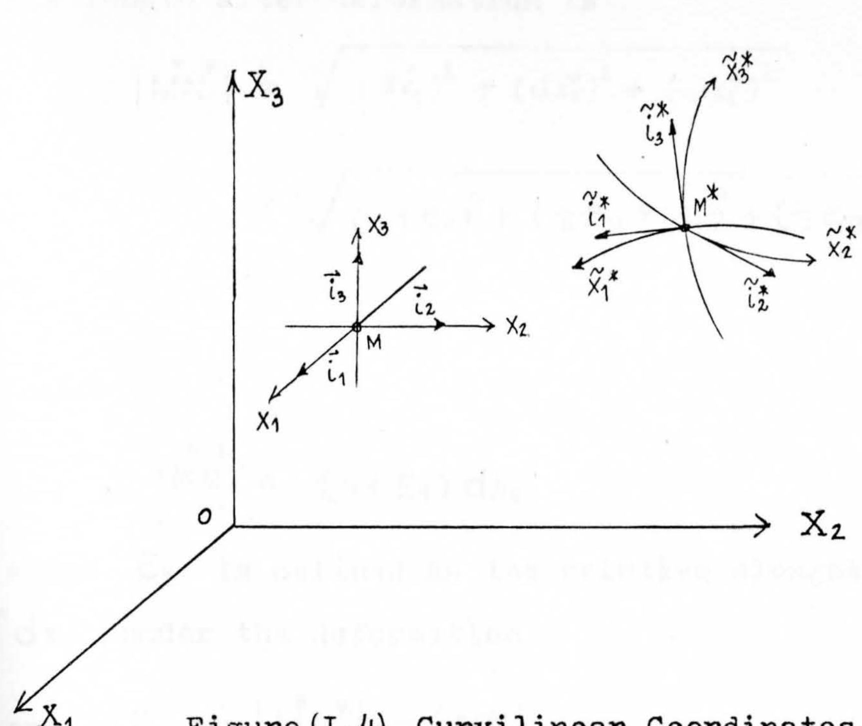
and finally,

$$\{dx^*\} = \big[[I]+[e]+[\omega]\big]\{dx\} \tag{1-7c}$$

The geometry of the coordinates of the deformation divided into two cases (Case I and Case II) from the previous section (1.1) are considered.

Case I

Before deformation the line elements which pass through the point M are parallel to the X_1-, X_2-, X_3- axes (rectangular cartesian coordinate), after the result of deformation, they become elements of arc of the lines $\tilde{X}_1^*$, $\tilde{X}_2^*$, $\tilde{X}_3^*$ in the deformed body as shown in figure (I-4)



<u>Figure (I-4)</u> Curvilinear Coordinates-Case I

where $\tilde{i}_1^*, \tilde{i}_2^*, \tilde{i}_3^*$ are the unit vectors tangent to the curve lines $\tilde{X}_1^*, \tilde{X}_2^*, \tilde{X}_3^*$ at point M*, and i_1, i_2, i_3 are the unit vectors of the X_1, X_2, X_3 axes respectively.

Let the line element MN before deformation be parallel to the X_1 - axis and have the projections

$$(MN)_{x_1} = dx_1, \quad (MN)_{x_2} = 0, \quad (MN)_{x_3} = 0 .$$

Then according to (1-7) its projections after deformation are

$$dx_1^* = (1 + e_{11})\, dx_1 \quad , \quad dx_2^* = (\tfrac{1}{2} e_{12} + \omega_3)\, dx_1 ,$$

$$dx_3^* = (\tfrac{1}{2} e_{13} - \omega_2)\, dx_1 .$$

Its length after deformation is

$$|\overset{*}{M}\overset{*}{N}| = \sqrt{(dx_1^*)^2 + (dx_2^*)^2 + (dx_3^*)^2}$$

$$= \sqrt{(1+e_{11})^2 + (\tfrac{1}{2}e_{12}+\omega_3)^2 + (\tfrac{1}{2}e_{13}-\omega_2)^2},$$

also

$$|\overset{*}{M}\overset{*}{N}| = (1+E_1)\,dx_1 \tag{1-8}$$

where E_1 is defined as the relative elongation of element dx_1 under the deformation

or $$E_1 = \frac{|\overset{*}{M}\overset{*}{N}| - |MN|}{|MN|} \tag{1-9a}$$

or $$|\overset{*}{M}\overset{*}{N}| = (1+E_1)\,|MN|$$

$$= (1+E_1)\,dx_1 \tag{1-9b}$$

$\overrightarrow{\overset{*}{M}\overset{*}{N}}$ (in this case before deformation parallel to X_1 axis) is expressed in terms of the projections on X_1-, X_2-, X_3- axes.

Thus $$\overset{*}{M}\overset{*}{N} = dx_1^*\,\vec{i}_1 + dx_2^*\,\vec{i}_2 + dx_3^*\,\vec{i}_3 \tag{1-10}$$

The unit vector tangent to the arc line $\overset{*}{M}\overset{*}{N}$ is denoted by $\tilde{i}_1^*$.

with $$\tilde{i}^* = \frac{dx_1^*\,\vec{i}_1}{|\overset{*}{M}\overset{*}{N}|} + \frac{dx_2^*\,\vec{i}_2}{|\overset{*}{M}\overset{*}{N}|} + \frac{dx_3^*\,\vec{i}_3}{|\overset{*}{M}\overset{*}{N}|}$$

or $$= \frac{(1+e_{11})\,dx_1\,\vec{i}_1}{(1+E_1)\,dx_1} + \frac{(\tfrac{1}{2}e_{12}+\omega_3)\,dx_1\,\vec{i}_2}{(1+E_1)\,dx_1} + \frac{(\tfrac{1}{2}e_{13}-\omega_2)\,dx_1\,\vec{i}_3}{(1+E_1)\,dx_1}$$

hence,

$$\tilde{i}_1^* = \frac{(1+e_{11})\,\vec{i}_1}{(1+E_1)} + \frac{(\frac{1}{2}e_{12}+\omega_3)\,\vec{i}_2}{(1+E_1)} + \frac{(\frac{1}{2}e_{13}-\omega_2)\,\vec{i}_3}{(1+E_1)} \tag{1-11a}$$

By applying analogous arguments to the line elements dx_2 and dx_3, one obtains

$$\tilde{i}_2^* = \frac{(\frac{1}{2}e_{12}-\omega_3)\,\vec{i}_1}{(1+E_2)} + \frac{(1+e_{22})\,\vec{i}_2}{(1+E_2)} + \frac{(\frac{1}{2}e_{23}+\omega_1)\,\vec{i}_3}{(1+E_2)} \tag{1-11b}$$

$$\tilde{i}_3^* = \frac{(\frac{1}{2}e_{13}+\omega_2)\,\vec{i}_1}{(1+E_3)} + \frac{(\frac{1}{2}e_{23}-\omega_1)\,\vec{i}_2}{(1+E_3)} + \frac{(1+e_{33})\,\vec{i}_3}{(1+E_3)} \tag{1-11c}$$

In matrix form the latter equations become

$$\{\tilde{i}^*\} = [A]^T\{\vec{i}\} \tag{1-12}$$

where

$$\{\tilde{i}^*\} = \begin{Bmatrix} \tilde{i}_1^* \\ \tilde{i}_2^* \\ \tilde{i}_3^* \end{Bmatrix} \quad ; \quad \{\vec{i}\} = \begin{Bmatrix} \vec{i}_1 \\ \vec{i}_2 \\ \vec{i}_3 \end{Bmatrix}$$

where $\tilde{i}_1^*, \tilde{i}_2^*, \tilde{i}_3^*$ are the unit vectors tangent to $\tilde{X}_1^*, \tilde{X}_2^*, \tilde{X}_3^*$ respectively.

and

$$[A] = \begin{bmatrix} \frac{1+e_{11}}{1+E_1} & \frac{\frac{1}{2}e_{12}-\omega_3}{1+E_2} & \frac{\frac{1}{2}e_{13}+\omega_2}{1+E_3} \\ \frac{\frac{1}{2}e_{12}+\omega_3}{1+E_1} & \frac{1+e_{22}}{1+E_2} & \frac{\frac{1}{2}e_{23}-\omega_1}{1+E_3} \\ \frac{\frac{1}{2}e_{13}-\omega_2}{1+E_1} & \frac{\frac{1}{2}e_{23}+\omega_1}{1+E_2} & \frac{1+e_{33}}{1+E_3} \end{bmatrix} \tag{1-13a}$$

where E_1, E_2, E_3 are defined as the relative elongation of element MN, which are parallel to X_1^-, X_2^-, X_3^- axes <u>before deformation</u> respectively, and where the matrix $[A]$ is written in terms of direction cosines as

$$[A] = \begin{bmatrix} \cos(\tilde{i}_1^*, \vec{i}_1) & \cos(\tilde{i}_2^*, \vec{i}_1) & \cos(\tilde{i}_3^*, \vec{i}_1) \\ \cos(\tilde{i}_1^*, \vec{i}_2) & \cos(\tilde{i}_2^*, \vec{i}_2) & \cos(\tilde{i}_3^*, \vec{i}_2) \\ \cos(\tilde{i}_1^*, \vec{i}_3) & \cos(\tilde{i}_2^*, \vec{i}_3) & \cos(\tilde{i}_3^*, \vec{i}_3) \end{bmatrix}. \tag{1-13b}$$

Noting Equations (1-7b) and (1-13a), it follows that,

$$[A]^T = \left[\frac{1}{1+E}\right][J]^T; \quad \text{or} \quad [A] = [J]\left[\frac{1}{1+E}\right] \tag{1-14}$$

where

$$\left[\frac{1}{1+E}\right] = \begin{bmatrix} \frac{1}{1+E_1} & 0 & 0 \\ 0 & \frac{1}{1+E_2} & 0 \\ 0 & 0 & \frac{1}{1+E_3} \end{bmatrix} \tag{1-15}$$

Consideration of Equation (1-2)

$$\{dx\} = [J]^{-1}\{dx^*\} \qquad (1\text{-}16)$$

Taking $\quad [J]^{-1} = \dfrac{[\alpha]}{|[J]|}$

it follows that

$$[\alpha] = [COF[J]]^T \qquad (1\text{-}17)$$

with

$$[\alpha] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} = \left[COF[J]\right]^T$$

$$[\alpha] = \begin{bmatrix} (1+e_{22})(1+e_{33})-(\tfrac{1}{4}e_{23}^2-\omega_1^2); & -(\tfrac{1}{2}e_{12}+\omega_3)(1+e_{33})+(\tfrac{1}{2}e_{13}-\omega_2)(\tfrac{1}{2}e_{23}-\omega_1); & +(\tfrac{1}{2}e_{12}+\omega_3)(\tfrac{1}{2}e_{23}+\omega_1)-(\tfrac{1}{2}e_{13}-\omega_2)(1+e_{22}); \\ -(\tfrac{1}{2}e_{12}-\omega_3)(1+e_{33})+(\tfrac{1}{2}e_{23}+\omega_1)(\tfrac{1}{2}e_{13}+\omega_2); & (1+e_{11})(1+e_{33})(\tfrac{1}{4}e_{13}^2-\omega_2^2); & -(1+e_{11})(\tfrac{1}{2}e_{23}+\omega_1)+(\tfrac{1}{2}e_{13}-\omega_2)(\tfrac{1}{2}e_{12}-\omega_3); \\ (\tfrac{1}{2}e_{12}-\omega_3)(\tfrac{1}{2}e_{23}-\omega_1)-(1+e_{22})(\tfrac{1}{2}e_{13}+\omega_2); & -(1+e_{11})(\tfrac{1}{2}e_{23}-\omega_1)+(\tfrac{1}{2}e_{12}-\omega_3)(\tfrac{1}{2}e_{13}+\omega_2); & (1+e_{11})(1+e_{22})-(\tfrac{1}{4}e_{12}^2-\omega_3^2); \end{bmatrix}$$

Noting Equation (1-14), one obtains

$$COF[J]^T = [COF[J]]^T = COF[[1+E]+[A]^T] = COF[1+E]\,COF[A]^T$$

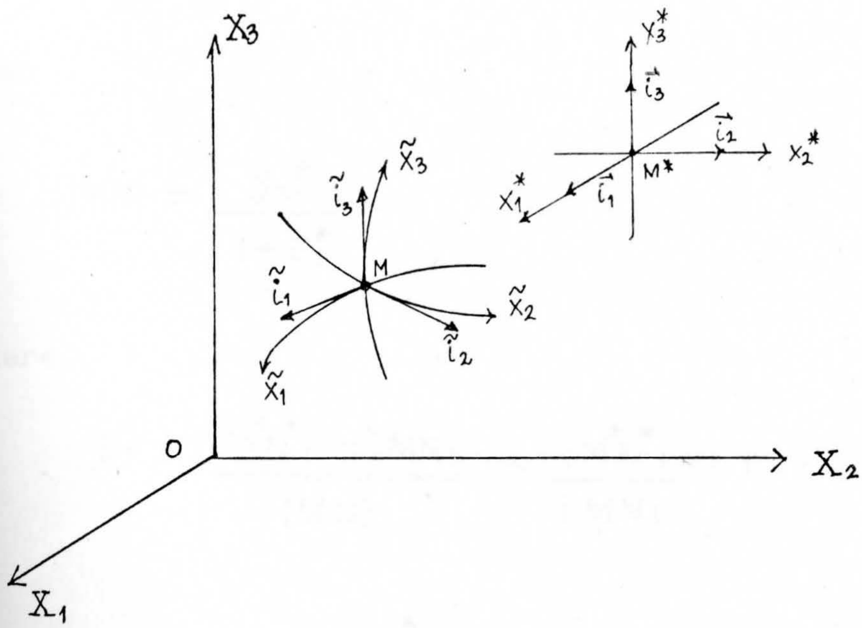
hence $\quad [\alpha] = COF[1+E]\,COF[[A]^T] \qquad (1\text{-}18)$

where

$$[1+E] = \begin{bmatrix} 1+E_1 & 0 & 0 \\ 0 & 1+E_2 & 0 \\ 0 & 0 & 1+E_3 \end{bmatrix}^{-1} = \left[\frac{1}{1+E}\right]^{-1}$$

Case II

Before deformation the line elements of the body which pass through the point M are the elements of the lines $\tilde{X}_1^*$, $\tilde{X}_2^*$, $\tilde{X}_3^*$ and become parallel to X_1, X_2, X_3 axes (Rectangular Cartesian Coordinate) after deformation. (See Figure (I-5))



Figure(I-5) Curvilinear Coordinates-Case II

Let now examine the line element dx_1^* , i.e., the line element parallel to the X_1 - Axis after deformation. According to (1-9), its projections before deformation are

$$dx_1 = \frac{\alpha_{11}}{|[J]|} dx_1^* \quad ; \quad dx_2 = \frac{\alpha_{21}}{|[J]|} dx_1^* \quad ; \quad dx_3 = \frac{\alpha_{31}}{|[J]|} dx_1^*$$

Its length before deformation is

$$|MN| = \sqrt{(dx_1)^2 + (dx_2)^2 + (dx_3)^2}$$

$$= \sqrt{\alpha_{11}^2 + \alpha_{21}^2 + \alpha_{31}^2} \quad \frac{dx_1^*}{|[J]|}$$

also

$$|MN| = \frac{dx_1^*}{1+E_1^*} \qquad (1\text{-}19)$$

where

$$E_1^* = \frac{|M^*N^*| - |MN|}{|MN|} = \frac{|M^*N^*|}{|MN|} - 1$$

$$= \frac{dx_1^*}{\frac{1}{|[J]|}\sqrt{\alpha_{11}^2 + \alpha_{21}^2 + \alpha_{31}^2} \; dx_1^*} - 1$$

$$E_1^* = \frac{|[J]|}{\sqrt{\alpha_{11}^2 + \alpha_{21}^2 + \alpha_{31}^2}} - 1 \qquad (1\text{-}20)$$

E_1^* is defined as the relative elongation of element MN under deformation, which is parallel to X_1 axis after deformation.

$\overrightarrow{MN}$ (in this case before deformation is the curve line $\tilde{X}_1$) is expressed in terms of the projections on $X_{\bar{1}}, X_{\bar{2}}, X_{\bar{3}}$ axes as follows,

$$\overrightarrow{MN} = dx_1 \vec{i}_1 + dx_2 \vec{i}_2 + dx_3 \vec{i}_3$$

The unit vector tangent to the arc line MN is denoted by $\tilde{i}_1$

with
$$\tilde{i}_1 = \frac{dx_1}{|MN|}\vec{i}_1 + \frac{dx_2}{|MN|}\vec{i}_2 + \frac{dx_3}{|MN|}\vec{i}_3$$

or
$$\tilde{i}_1 = \frac{(1+E_1^*)\alpha_{11}}{|[J]|}\vec{i}_1 + \frac{(1+E_1^*)\alpha_{21}}{|[J]|}\vec{i}_2 + \frac{(1+E_1^*)\alpha_{31}}{|[J]|}\vec{i}_3 \tag{1-21a}$$

Analogously

$$\tilde{i}_2 = \frac{(1+E_2^*)\alpha_{12}}{|[J]|}\vec{i}_1 + \frac{(1+E_2^*)\alpha_{22}}{|[J]|}\vec{i}_2 + \frac{(1+E_2^*)\alpha_{23}}{|[J]|}\vec{i}_3 \tag{1-21b}$$

$$\tilde{i}_3 = \frac{(1+E_3^*)\alpha_{13}}{|[J]|}\vec{i}_1 + \frac{(1+E_3^*)\alpha_{23}}{|[J]|}\vec{i}_2 + \frac{(1+E_3^*)\alpha_{33}}{|[J]|}\vec{i}_3 \tag{1-21c}$$

In matrix form the latter equations become

$$\{\tilde{i}\} = [B]^T\{\vec{i}\} \tag{1-22}$$

where

$$\{\tilde{i}\} = \begin{Bmatrix} \tilde{i}_1 \\ \tilde{i}_2 \\ \tilde{i}_3 \end{Bmatrix}$$

$\tilde{i}_1, \tilde{i}_2, \tilde{i}_3$ are denoted the unit vectors tangent to line $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3$ respectively (before deformation).

and where

$$[B] = \frac{1}{|[J]|} \begin{bmatrix} (1+E_1^*)\alpha_{11} & (1+E_2^*)\alpha_{12} & (1+E_3^*)\alpha_{13} \\ (1+E_1^*)\alpha_{21} & (1+E_2^*)\alpha_{22} & (1+E_3^*)\alpha_{23} \\ (1+E_1^*)\alpha_{31} & (1+E_2^*)\alpha_{32} & (1+E_3^*)\alpha_{33} \end{bmatrix} \qquad (1\text{-}23a)$$

E_1^*, E_2^*, E_3^* are the relative elongations of the element MN, which are parallel to X_1, X_2, X_3 axes <u>after deformation</u> respectively.

Matrix $[B]$ is written in terms of direction cosines as

$$[B] = \begin{bmatrix} \cos(\tilde{i}_1, \vec{i}_1) & \cos(\tilde{i}_2, \bar{i}_1) & \cos(\tilde{i}_3, \bar{i}_1) \\ \cos(\tilde{i}_1, \vec{i}_2) & \cos(\tilde{i}_2, \bar{i}_2) & \cos(\tilde{i}_3, \bar{i}_2) \\ \cos(\tilde{i}_1, \vec{i}_3) & \cos(\tilde{i}_2, \bar{i}_3) & \cos(\tilde{i}_3, \bar{i}_3) \end{bmatrix} \qquad (1\text{-}23b)$$

Noting Equations (1-22), (1-23a), it follows that

$$\{\tilde{i}\} = \frac{1}{|[J]|}[1+E^*][\alpha]^T\{\bar{i}\} \tag{1-24}$$

where

$$[1+E^*] = \begin{bmatrix} 1+E_1^* & 0 & 0 \\ 0 & 1+E_2^* & 0 \\ 0 & 0 & 1+E_3^* \end{bmatrix}$$

Noting Equations (1-22), (1-24), it follows that

$$[B]^T = \frac{1}{|[J]|}[1+E^*][\alpha]^T. \tag{1-25}$$

In order to determine the relationship between matrices $[A]$ and $[B]$, consider the Equation (1-14), it follows that

$$[J] = [A][1+E]$$

$$[J]^{-1} = [\tfrac{1}{1+E}][A]^{-1}$$

$$[\alpha] = |[J]|[\tfrac{1}{1+E}][A]^{-1}$$

Consideration of Equation (1-25) gives

$$[B] = \frac{1}{|[J]|}[\alpha][1+E^*]$$

Hence,

$$[B] = \left[\frac{1}{1+E}\right][A]^{-1}[1+E^*] \tag{1-26}$$

$$[B][1+E^*]^{-1} = \left[\frac{1}{1+E}\right][A]^{-1}.$$

CHAPTER II

GEOMETRY OF STRAIN

2.1 Strain Components

The square of the distance between the points M and N (See Figure (I-3)) before deformation is

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 \tag{2-1a}$$

and after deformation is

$$ds^{*2} = dx_1^{*2} + dx_2^{*2} + dx_3^{*2} \tag{2-1b}$$

Thus,

$$\begin{aligned} ds^{*2} - ds^2 &= \{dx^*\}^T\{dx^*\} - \{dx\}^T\{dx\} \\ &= \{dx\}^T[J]^T[J]\{dx\} - \{dx\}^T\{dx\} \\ &= \{dx\}^T\left[[J]^T[J] - [I]\right]\{dx\} \end{aligned} \tag{2-1c}$$

Defining E_{MN} as the relative elongation at the point M in the direction of the point N, then

$$E_{MN} = \frac{ds^* - ds}{ds}$$

or $$E_{MN} + 1 = \frac{ds^*}{ds}$$

it follows that

$$E_{MN}^2 + 2E_{MN} + 1 = \frac{ds^{*2}}{ds^2}$$

with

$$E_{MN}\left(1 + \frac{1}{2}E_{MN}\right) = \frac{1}{2}\left(\frac{ds^{*2} - ds^2}{ds^2}\right) \tag{2-2}$$

In matrix form this is written

$$E_{MN}\left(1+\frac{1}{2}E_{MN}\right) = \left\{\frac{dx}{ds}\right\}^T [\varepsilon] \left\{\frac{dx}{ds}\right\} \tag{2-3a}$$

where

$$\left\{\frac{dx}{ds}\right\} = \left\{\begin{matrix} \frac{dx_1}{ds} \\ \frac{dx_2}{ds} \\ \frac{dx_3}{ds} \end{matrix}\right\} \tag{2-3b}$$

where $[\varepsilon]$ is defined as the strain component matrix in the form

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{11} & \frac{1}{2}\varepsilon_{12} & \frac{1}{2}\varepsilon_{13} \\ \frac{1}{2}\varepsilon_{12} & \varepsilon_{22} & \frac{1}{2}\varepsilon_{23} \\ \frac{1}{2}\varepsilon_{13} & \frac{1}{2}\varepsilon_{23} & \varepsilon_{33} \end{bmatrix} \tag{2-3c}$$

It follows from Equation (2-1c) that

$$\frac{1}{2}\,\frac{ds^{*2}-ds^2}{ds^2} = \frac{1}{2}\,\frac{\{dx\}^T\left[[J]^T[J]-[I]\right]\{dx\}}{ds^2}$$

$$= \frac{1}{2}\left\{\frac{dx}{ds}\right\}^T\left[[J]^T[J]-[I]\right]\left\{\frac{dx}{ds}\right\}$$

$$\left\{\frac{dx}{ds}\right\}^T[\varepsilon]\left\{\frac{dx}{ds}\right\} = \left\{\frac{dx}{ds}\right\}^T\frac{1}{2}\left[[J]^T[J]-[I]\right]\left\{\frac{dx}{ds}\right\} \tag{2-3d}$$

Comparing both sides of the latter equations yields

$$[\varepsilon] = \frac{1}{2}\left[[J]^T[J]-[I]\right] \tag{2-4a}$$

Upon substituting Equations (1-3) and (1-5), one obtains

$$[\varepsilon] = \frac{1}{2}\left[\left[[D]^T+[I]\right]\left[[D]+[I]\right] - [I]\right]$$

$$= \frac{1}{2}\left[[D]^T[D] + [D] + [D]^T + [I] - [I]\right] \tag{2-4b}$$

$$= \frac{1}{2}\left[[D]^T + [D] + [D]^T[D]\right]$$

$$= \frac{1}{2}\left[2[e] + [e]^2 - [\omega][e] + [e][\omega] - [\omega]^2\right]$$

Finally,

$$[\varepsilon] = [e] + \frac{1}{2}\left[[e]^2+[e][\omega]-[\omega][e]-[\omega]^2\right] \qquad (2\text{-}4c)$$

Let

$$\frac{dx_1}{ds} = \lambda_1 \;, \quad \frac{dx_2}{ds} = \lambda_2 \;, \quad \frac{dx_3}{ds} = \lambda_3$$

or

$$\left\{\frac{dx}{ds}\right\} = \{\lambda\} = \begin{Bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{Bmatrix} \qquad (2\text{-}5a)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the direction cosines of Vector MN, it follows from Equation (2-3a) that

$$E_{MN}\left(1+\frac{1}{2}E_{MN}\right) = \{\lambda\}^T[\varepsilon]\{\lambda\} \qquad (2\text{-}5b)$$

If element under consideration is parallel to the X_1- axis before deformation, one obtains

$$E_{MN} = E_1 \qquad ; \qquad \frac{dx_1}{ds} = 1$$

$$E_1\left(1+\frac{1}{2}E_1\right) = \left\{\frac{dx_1}{ds}, 0, 0\right\}[\varepsilon]\begin{Bmatrix}\frac{dx_1}{ds}\\ 0\\ 0\end{Bmatrix}$$

Thus,

$$E_1\left(1+\frac{1}{2}E_1\right) = \varepsilon_{11} \;;$$

or

$$E_{11} = \sqrt{1+2\varepsilon_{11}} - 1 \;.$$

Analogously,

$$E_2\left(1+\frac{1}{2}E_2\right) = \varepsilon_{22} \;; \quad \text{or} \quad E_2 = \sqrt{1+2\varepsilon_{22}} - 1$$

$$E_3\left(1+\frac{1}{2}E_3\right) = \varepsilon_{33} \;; \quad \text{or} \quad E_3 = \sqrt{1+2\varepsilon_{33}} - 1$$

Therefore, the strain components $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$ characterize the elongation of those line elements which, before deformation, are parallel to the co-ordinates axes.

In order to clarify the physical meaning of the strain components $\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}$, a determination of the direction cosines of the angles which the vectors $\tilde{i}_1^*, \tilde{i}_2^*, \tilde{i}_3^*$ form with one another (i.e., the cosines of the angles between the tangents to the lines $\tilde{x}_1^*, \tilde{x}_2^*, \tilde{x}_3^*$ passing through the point M*) From Equation (1-12), one obtains

$$\tilde{i}_1^* = \cos(\tilde{i}_1^* x_1)\,\vec{i}_1 + \cos(\tilde{i}_1^*, x_2)\,\vec{i}_2 + \cos(\tilde{i}_1^*, x_3)\,\vec{i}_3$$

$$\tilde{i}_2^* = \cos(\tilde{i}_2^*, x_1)\,\vec{i}_1 + \cos(\tilde{i}_2^*, x_2)\,\vec{i}_2 + \cos(\tilde{i}_2^*, x_3)\,\vec{i}_3$$

$$(\tilde{i}_1^* \cdot \tilde{i}_2^*) = |\tilde{i}_1^*|\,|\tilde{i}_2^*| \cos(\tilde{i}_1^*, \tilde{i}_2^*) \qquad (2\text{-}7)$$

$$\cos(\tilde{i}_1^*, \tilde{i}_2^*) = \cos(\tilde{i}_1^*, x_1)\cos(\tilde{i}_2^*, x_1) + \cos(\tilde{i}_1^*, x_2)\cos(\tilde{i}_2^*, x_2)$$
$$+ \cos(\tilde{i}_1^*, x_3)\cos(\tilde{i}_2^*, x_3) \qquad (2\text{-}8a)$$

Replacing the direction cosines by their values given in (1-13a) and simplifying yields

$$\cos(\tilde{i}_1^*, \tilde{i}_2^*) = \frac{\varepsilon_{12}}{(1+E_1)(1+E_2)} \qquad (2\text{-}8b)$$

Before deformation, the angle between the line elements dx_1, dx_2 is a right angle.

Let ϕ_{12} denote the angular increment due to the deformation, then

$$\cos(\tilde{i}_1^*, \tilde{i}_2^*) = \cos(\pi/2 - \phi_{12})$$
$$= \sin\phi_{12} = \frac{\varepsilon_{12}}{(1+E_1)(1+E_2)} \qquad (2\text{-}8c)$$

Analogously,

$$\sin\phi_{13} = \frac{\varepsilon_{13}}{(1+E_1)(1+E_3)}$$

$$\sin\phi_{23} = \frac{\varepsilon_{23}}{(1+E_2)(1+E_3)}$$

The angles $\phi_{12}, \phi_{13}, \phi_{23}$ are called "shears."

It follows from the above equations, that the strain components $\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}$ characterize the shears, and that if these three strain components vanish, then the angles between the line elements dx_1, dx_2, dx_3 remain right angles after deformation.

2.2 Transformation of Strain Components under Change of Axes.

A given deformation is considered in two different Cartesian coordinate systems. In all such cases it is characterized completely by the six strain components, whose values, however, depend on the choice of directions of the coordinate axes.

Consider, together with the basic system X_1, X_2, X_3, another system X_1', X_2', X_3' the directions of whose axes relative to the axes of the first system are given in the following equations.

$$\begin{aligned} X_1' &= \lambda_{11} X_1 + \lambda_{21} X_2 + \lambda_{31} X_3 \\ X_2' &= \lambda_{12} X_1 + \lambda_{22} X_2 + \lambda_{32} X_3 \\ X_3' &= \lambda_{13} X_1 + \lambda_{23} X_2 + \lambda_{33} X_3 \end{aligned} \tag{2-9a}$$

In matrix form

$$\{X'\} = [\lambda]^T \{X\} \tag{2-9b}$$

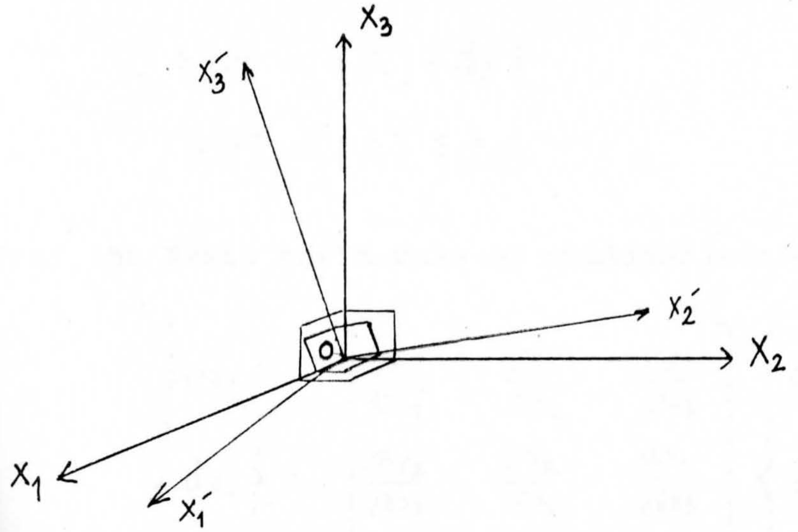


Figure II-1 Change of Rectangular Coordinate Axes

Defining $[\lambda]$ as direction cosines matrix

$$[\lambda] = \lambda_{ij} \quad ; \quad i = 1,2,3 \; ; \; j = 1,2,3 .$$

where i $\Rightarrow$ first system

j $\Rightarrow$ second system

Since both systems are rectangular, $[\lambda]$ is the orthogonal matrix, hence

$$[\lambda]^T[\lambda] = [I]$$
$$[\lambda]^T = [\lambda]^{-1} \qquad (2\text{-}10)$$

where

$$[\lambda] = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} = \begin{bmatrix} \cos(x_1,x_1') & \cos(x_1,x_2') & \cos(x_1,x_3') \\ \cos(x_2,x_1') & \cos(x_2,x_2') & \cos(x_2,x_3') \\ \cos(x_3,x_1') & \cos(x_3,x_2') & \cos(x_3,x_3') \end{bmatrix} \qquad (2\text{-}11)$$

The projections on the axes of the first system of a line element having the components dx_1', dx_2', dx_3' along the axes of the second system, are given by

$$\{dx\} = [\lambda]\{dx'\} \tag{2-12a}$$

or

$$\{dx'\} = [\lambda]^T\{dx\} \tag{2-12b}$$

From the basic chain-rule of multivariate calculus, one obtains

$$\begin{Bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{Bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial x_1'} & \frac{\partial x_1}{\partial x_2'} & \frac{\partial x_1}{\partial x_3'} \\ \frac{\partial x_2}{\partial x_1'} & \frac{\partial x_2}{\partial x_2'} & \frac{\partial x_2}{\partial x_3'} \\ \frac{\partial x_3}{\partial x_1'} & \frac{\partial x_3}{\partial x_2'} & \frac{\partial x_3}{\partial x_3'} \end{bmatrix} \begin{Bmatrix} dx_1' \\ dx_2' \\ dx_3' \end{Bmatrix} \tag{2-12c}$$

Noting Equations (2-12c) and (2-12a), it follows that

$$[\lambda] = \begin{bmatrix} \frac{\partial x_1}{\partial x_1'} & \frac{\partial x_1}{\partial x_2'} & \frac{\partial x_1}{\partial x_3'} \\ \frac{\partial x_2}{\partial x_1'} & \frac{\partial x_2}{\partial x_2'} & \frac{\partial x_2}{\partial x_3'} \\ \frac{\partial x_3}{\partial x_1'} & \frac{\partial x_3}{\partial x_2'} & \frac{\partial x_3}{\partial x_3'} \end{bmatrix} \tag{2-12d}$$

Recalling Equation (2-1), the left-hand side represents the increment of the square of the distance between the points M and N, resulting from the deformation. The choice of these points is independent of the choice of the coordinate system, therefore, the left-hand side of Equation (2-1) is also independent of it, and remains invariant under a change of axes, Noting Equations (2-1c) and (2-2), it follows that

$$ds^{*2} - ds^2 = \{dx\}^T[[J]^T[J] - [I]]\{dx\}$$

$$E_{MN}(1 + \tfrac{1}{2}E_{MN})ds^2 = \{dx\}^T[\varepsilon]\{dx\} \qquad \text{(2-13a)}$$

or $$E_{MN}(1 + \tfrac{1}{2}E_{MN})ds^2 = \{dx'\}^T[\varepsilon']\{dx'\} \qquad \text{(2-13b)}$$

Substitute Equation (2-12), into Equation (2-13a) gives

$$E_{MN}(1 + \tfrac{1}{2}E_{MN})ds^2 = \{dx'\}^T[\Lambda]^T[\varepsilon][\Lambda]\{dx'\} \qquad \text{(2-13c)}$$

Comparing Equations (2-13c) to Equation (2-13b) yields

$$[\varepsilon'] = [\Lambda]^T[\varepsilon][\Lambda] \qquad \text{(2-14)}$$

Hence, it is clear that Equation (2-14) gives the desired law of transformation of the strain matrix in passing from one rectangular coordinate system to another rectangular coordinate system.

2.3 Principal Axes of Strain

According to the classical theory of eigen-value - eigen-vector problem, it follows that

$$[\varepsilon][\Lambda] = [\Lambda][\varepsilon_d] \qquad \text{(2-15)}$$

where

$$[\varepsilon_d] = \begin{bmatrix} \epsilon_1^p & 0 & 0 \\ 0 & \epsilon_2^p & 0 \\ 0 & 0 & \epsilon_3^p \end{bmatrix}$$

and

$\epsilon_1^p, \epsilon_2^p, \epsilon_3^p$ = The extremal values of the strains components $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$ (Principal strains).

Thus Equation (2-14) is rewritten as

$$[\varepsilon'] = [\lambda]^T[\lambda][\varepsilon_d] \quad (2\text{-}16a)$$

$$[\varepsilon'] = [\varepsilon_d] \quad (2\text{-}16b)$$

Therefore Equation (2-16b) exists by the condition of Equation (2-15).

Furthermore, $\varepsilon'_{11} = \epsilon^p_1$, $\varepsilon'_{22} = \epsilon^p_2$, $\varepsilon'_{33} = \epsilon^p_3$ and $\varepsilon'_{12} = \varepsilon'_{13} = \varepsilon'_{23} = 0$.

Also, matrix $[\lambda]$ is the direction cosines of the principal axes of this principal strains.

Note further that the eigen-values - eigen-vectors problem also gives the following equation

$$\left[[\varepsilon] - \epsilon^p[I]\right]\{\lambda\} = \{0\} \quad (2\text{-}17)$$

where ϵ^p's are defined as eigen-values

and $\{\lambda\}^p$ = eigen-vectors

For non-zero value of $\{\lambda\}$, it follows that

$$\left|[\varepsilon] - \epsilon^p[I]\right| = 0 \quad (2\text{-}18a)$$

which yields the characteristic equation (of this matrix $[\varepsilon]$) which is solved directly for the eigen-values. The general form of Equation (2-18a) becomes

$$(\epsilon^p)^3 - a_2(\epsilon^p)^2 + a_1(\epsilon^p)^1 - a_0(\epsilon^p)^0 = 0 \quad (2\text{-}18b)$$

where

$$a_2 = \text{Trace of the matrix}$$
$$= \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \epsilon_1^P + \epsilon_2^P + \epsilon_3^P \qquad (2\text{-}18c)$$

$$a_1 = \text{Sum of the determinant minors of the diagonal components of matrix}$$
$$= \varepsilon_{11}\varepsilon_{22} + \varepsilon_{11}\varepsilon_{33} + \varepsilon_{22}\varepsilon_{33} - \tfrac{1}{4}(\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2)$$
$$= \epsilon_1^P\epsilon_2^P + \epsilon_1^P\epsilon_3^P + \epsilon_2^P\epsilon_3^P \qquad (2\text{-}18d)$$

$$a_0 = \text{The determinant of matrix}$$
$$= \varepsilon_{11}\varepsilon_{22}\varepsilon_{33} - \tfrac{1}{4}(\varepsilon_{11}\varepsilon_{23}^2 + \varepsilon_{22}\varepsilon_{13}^2 + \varepsilon_{33}\varepsilon_{12}^2 - \varepsilon_{12}\varepsilon_{13}\varepsilon_{23})$$
$$= \epsilon_1^P\epsilon_2^P\epsilon_3^P \qquad (2\text{-}18f)$$

The roots of the Equation (2-18b), $\epsilon_1^P, \epsilon_2^P, \epsilon_3^P$ are the eigen-value of matrix $[\varepsilon]$.

The eigen-values of Equation (2-18b) are individually substituted into Equation (2-17) and the corresponding eigen-vectors $\{\lambda\}^P$ are obtained which directly define the direction cosines of the principal axes. These vectors are then combined to form the columns of the matrix $[\lambda]$ which is the same matrix $[\lambda]$ in Equation (2-15).

Thus, it shows that for every point of the body one can choose three mutually perpendicular direction X_1^P, X_2^P, X_3^P for which the strain components $\varepsilon_{11}^P, \varepsilon_{22}^P, \varepsilon_{33}^P$ (and consequently also the relative elongations E_1^P, E_2^P, E_3^P) have extremal values, whereas the strain components $\varepsilon_{12}^P, \varepsilon_{13}^P, \varepsilon_{23}^P$ (and consequently also the shears $\phi_{12}^P, \phi_{13}^P, \phi_{23}^P$) are equal to zero.

These three directions are called "the principal axes of strain" at the point $M(x_1, x_2, x_3)$, and denote the corresponding extremal values of the strain components ε_{11}, ε_{22}, ε_{33} by ϵ_1^p, ϵ_2^p, ϵ_3^p.

As a result of the deformation, the fibers along the directions $\epsilon_1^p, \epsilon_2^p, \epsilon_3^p$ which remain mutually perpendicular may undergo a certain rotations.

The unit vectors of the principal axes after the deformation are denoted as $\epsilon_1^{p*}, \epsilon_2^{p*}, \epsilon_3^{p*}$ (i.e., the directions possessed after the deformation by fibers which, before deformation, had the directions $\epsilon_1^p, \epsilon_2^p, \epsilon_3^p$).

The angles between the mutually perpendicular vectors $\epsilon_1^p, \epsilon_2^p, \epsilon_3^p$ and the mutually perpendicular vectors $\epsilon_1^{p*}, \epsilon_2^{p*}, \epsilon_3^{p*}$ characterize the rotation which an infinitesimal element of the body about the point M undergoes as a result of the deformation.

2.4 Transformation of the Parameters $e_{11}, e_{22}, e_{33}, e_{12}, e_{13}, e_{23}$ and $\omega_1, \omega_2, \omega_3$ under Change of Co-ordinate Axes

The components along the new axes, of the displacement of an arbitary point of the body, are expressed in terms of its components along the old axes by the obvious formulas

$$\{u'\} = [\Lambda]^T \{u\} \tag{2-19a}$$

where

$$\{u'\} = \begin{Bmatrix} u'_1 \\ u'_2 \\ u'_3 \end{Bmatrix} \; ; \; \{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} .$$

From the basic chain-rule of multivariate calculus, one obtains

$$\begin{Bmatrix} \frac{\partial}{\partial x'_1} \\ \frac{\partial}{\partial x'_2} \\ \frac{\partial}{\partial x'_3} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial x'_1} & \frac{\partial x_2}{\partial x'_1} & \frac{\partial x_3}{\partial x'_1} \\ \frac{\partial x_1}{\partial x'_2} & \frac{\partial x_2}{\partial x'_2} & \frac{\partial x_3}{\partial x'_2} \\ \frac{\partial x_1}{\partial x'_3} & \frac{\partial x_2}{\partial x'_3} & \frac{\partial x_3}{\partial x'_3} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \tag{2-19b}$$

Noting Equation (2-12d), it follows that

$$\{\nabla'\} = [\Lambda]^T \{\nabla\} \tag{2-19c}$$

where

$$\{\nabla'\} = \begin{Bmatrix} \frac{\partial}{\partial x'_1} \\ \frac{\partial}{\partial x'_2} \\ \frac{\partial}{\partial x'_3} \end{Bmatrix} \; ; \; \{\nabla\} = \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} .$$

According to Equation (1-4a)

$$\{\nabla'\}\{u'\}^T = [D']^T \tag{2-20a}$$

$$[D']^T = [\Lambda]^T \{\nabla\}\{u\}^T [\Lambda]$$
$$= [\Lambda]^T [D]^T [\Lambda] \qquad (2\text{-}20b)$$
$$[D'] = [\Lambda]^T [D][\Lambda] \qquad (2\text{-}20c)$$

Equation (1-6c) is also written as

$$[e'] = \tfrac{1}{2}\left[[D'] + [D']^T\right] \qquad (2\text{-}21)$$

Substitution of Equations (2-20b) and (2-20c) gives

$$[e'] = \tfrac{1}{2}\left[[\Lambda]^T[D][\Lambda] + [\Lambda]^T[D]^T[\Lambda]\right]$$
$$= [\Lambda]^T \tfrac{1}{2}\left[[D] + [D]^T\right][\Lambda]$$
$$[e'] = [\Lambda]^T[e][\Lambda] \qquad (2\text{-}22)$$

Thus it follows that, under a change of Cartesian co-ordinates axes, the given parameters matrix $[e]$ transform according to the same transformation law as the strain matrix $[\varepsilon]$. Consider the transformation formulas for $\omega_1, \omega_2, \omega_3$ under a change of co-ordinate axes. Since according to Equation (1-6c), the same simplification is used with the parameters matrix $[\omega]$

$$[\omega'] = [\Lambda]^T[\omega][\Lambda] \qquad (2\text{-}23a)$$

where

$$[\omega'] = \begin{bmatrix} 0 & -\omega_3' & \omega_2' \\ \omega_3' & 0 & -\omega_1' \\ -\omega_2' & \omega_1' & 0 \end{bmatrix} \qquad (2\text{-}23b)$$

After matrix multiplication, the components of Equation (2-23a) become

$$\omega_1' = \omega_1(\lambda_{22}\lambda_{33}-\lambda_{32}\lambda_{23})+\omega_2(\lambda_{32}\lambda_{13}-\lambda_{12}\lambda_{33})+\omega_3(\lambda_{12}\lambda_{23}-\lambda_{22}\lambda_{13})$$
$$\omega_2' = \omega_1(\lambda_{31}\lambda_{23}-\lambda_{21}\lambda_{33})+\omega_2(\lambda_{11}\lambda_{33}-\lambda_{13}\lambda_{31})+\omega_3(\lambda_{21}\lambda_{13}-\lambda_{23}\lambda_{11})$$
$$\omega_3' = \omega_1(\lambda_{21}\lambda_{32}-\lambda_{22}\lambda_{31})+\omega_2(\lambda_{31}\lambda_{22}-\lambda_{11}\lambda_{32})+\omega_3(\lambda_{11}\lambda_{22}-\lambda_{21}\lambda_{12}).$$

which are written in the matrix form

$$\{\omega'\} = [\mathrm{COF.}[\lambda]]^T\{\omega\} \qquad (2\text{-}24)$$

where

$$\{\omega'\} = \begin{Bmatrix}\omega_1'\\ \omega_2'\\ \omega_3'\end{Bmatrix} \quad ; \quad \{\omega\} = \begin{Bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{Bmatrix}$$

Because of $[\lambda]$ is an orthogonal matrix, it possessed the following properties

a) $[\lambda]^T[\lambda] = [I]$

b) $[\lambda]^T = [\lambda]^{-1}$

c) $|[\lambda]| = 1.$

$$[\lambda]^{-1} = \frac{[\mathrm{COF.}[\lambda]]^T}{|[\lambda]|}$$
$$= [\mathrm{COF}[\lambda]]$$
$$[\lambda]^T = [\mathrm{COF}[\lambda]]^T \qquad (2\text{-}25)$$

d) $[\lambda] = [\mathrm{COF}[\lambda]]$

Thus, Equation (2-24) reduces to the form

$$\{\omega'\} = [\lambda]^T\{\omega\} \qquad (2\text{-}26)$$

This shows that, under a change of coordinates, the parameters ω_1, ω_2, ω_3 transform as the projection of the axial vector $\vec{\omega}$ whose length is

$$|\vec{\omega}| = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} \tag{2-27a}$$

$$= \tfrac{1}{2}\,\text{Trace}\left[[\omega]^T[\omega]\right] \tag{2-27b}$$

and whose directions are given by the cosines

$$\begin{aligned} \cos(\vec{\omega}, x_1) &= \frac{\omega_1}{|\vec{\omega}|} \\ \cos(\vec{\omega}, x_2) &= \frac{\omega_2}{|\vec{\omega}|} \\ \cos(\vec{\omega}, x_3) &= \frac{\omega_3}{|\vec{\omega}|} \end{aligned} \tag{2-27c}$$

2.5 Geometrical Meaning of the Parameters $\omega_1, \omega_2, \omega_3$

The point M is imagined to coincide with the point M*, and the origin of the coordinate system X_1, X_2, X_3 is transferred to this common point (without changing the directions of the axes), (See Figure (II-2)). MN and M^*N^* have the following projections.

$$\overrightarrow{MN} \sim dx_1, dx_2, dx_3 .$$

$$\overrightarrow{M^*N^*} \sim dx_1^*, dx_2^*, dx_3^* .$$

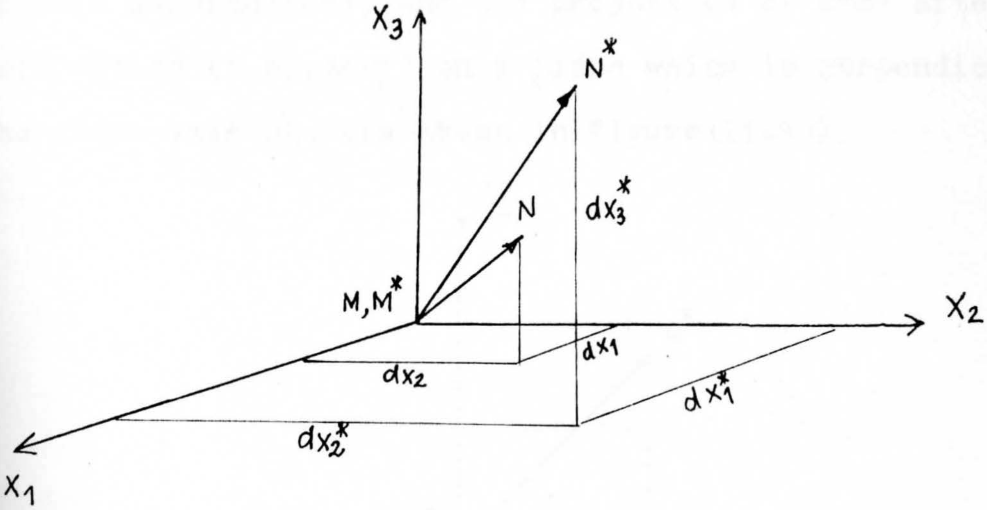


Figure (II-2) Rotation of Line Elements

Under a deformation, however, not only do the relative directions of the fibers change, but also their absolute directions. In view of this, and infinitesimal element of volume of the body in its initial position undergoes a certain rotation, in addition to a deformation, in passing to the terminal position.

The term rotation, as applied to an element of volume which, in the process of displacement, alters not only its position but also its dimensions and form, will be understood to represent the mean value of the rotations experienced by the totality of line elements belonging to the given element of volume.

Let the angle of rotation of a fiber which rotates about an axis $\boxed{H}$, to which it is perpendicular to before deformation, be defined by the angle between this fiber MN (before deformation), and the projection of M*N* after deformation (i.e., MN_1^*) on a plane which is perpendicular to the given axis $\boxed{H}$ (as shown in Figure(II-3)).

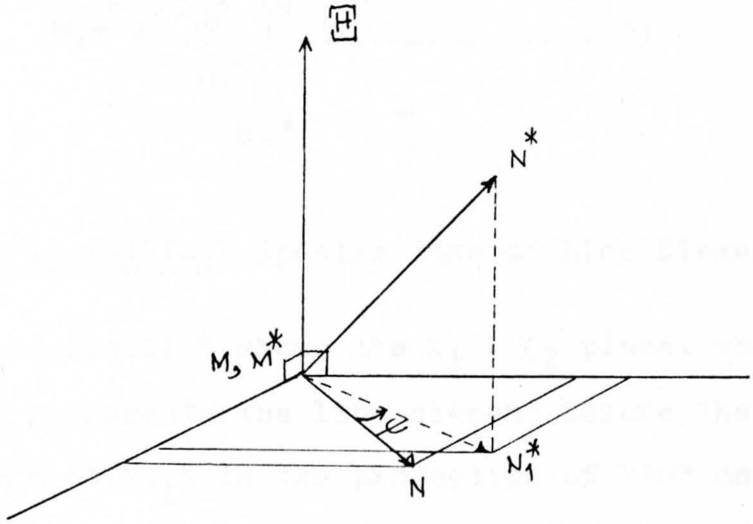
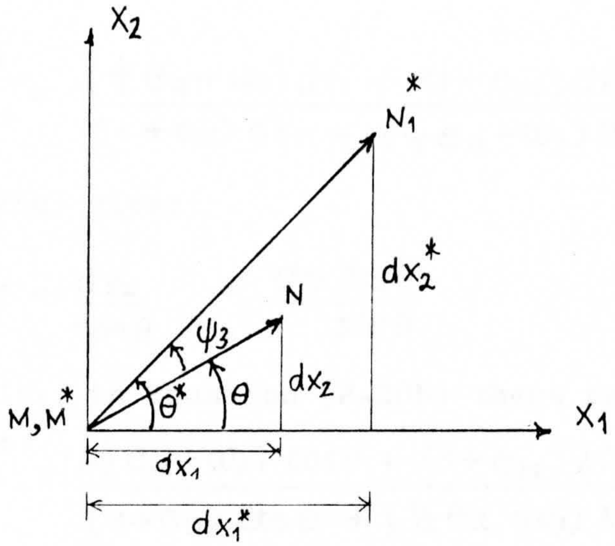


Figure (II-3) Projections of Line Elements and Rotation Angles

To clarify the magnitudes characterizing the rotation which a neighborhood of the point M undergoes as a result of the displacements u_1, u_2, u_3, Equation (1-7c) is applied for the special case where the line element MN is perpendicular to the X_3-axis with $dx_3 = 0$ (See Fugure(II-4)).



Figure(II-4) Special Case of Line Element Rotation

Figure II-4 shows the $X_1 - X_2$ plane, where the segment MN represents the line element before the deformation and the segment MN_1* is the projection of M*N* on the plane under consideration. From the Figure (II-3) it is clear that

$$\tan\theta = \frac{dx_2}{dx_1} \quad ; \quad \tan\theta^* = \frac{dx_2^*}{dx_1^*} \qquad (2\text{-}28a)$$

From Equation (1-7c), it follows that

$$\{dx^*\} = [[I]+[e]+[\omega]]\{dx\}$$

with

$$\{dx\} = \begin{Bmatrix} dx_1 \\ dx_2 \\ 0 \end{Bmatrix}$$

Replacing dx_1^*, dx_2^* by their values in latter equations, one obtains

$$\tan\theta^* = \frac{(\frac{1}{2}e_{12}+\omega_3)\,dx_1 + (1+e_{22})\,dx_2}{(1+e_{11})\,dx_1 + (\frac{1}{2}e_{12}-\omega_3)\,dx_2} \qquad (2\text{-}28b)$$

Equation (2-28a) gives

$$dx_1 = \frac{dx_2}{\tan\theta} = \frac{dx_2\cos\theta}{\sin\theta} \qquad (2\text{-}28c)$$

Eliminating dx_1, in Equation (2-28b) there results

$$\tan\theta^* = \frac{(\frac{1}{2}e_{12}+\omega_3)\cos\theta + (1+e_{22})\sin\theta}{(1+e_{11})\cos\theta + (\frac{1}{2}e_{12}-\omega_3)\sin\theta} \qquad (2\text{-}28d)$$

From Figure (II-3) it follows that

$$\psi_3 = \theta^* - \theta. \qquad \text{or} \qquad \theta^* = \psi_3 + \theta$$

i.e., ψ_3 = angle of rotation of MN about the X_3-axis.

Noting the following identity

$$\tan(\psi_3+\theta) = \frac{\tan\theta + \tan\psi_3}{1-\tan\theta\tan\psi_3} = \frac{(\frac{1}{2}e_{12}+\omega_3)\cos\theta + (1+e_{22})\sin\theta}{(1+e_{11})\cos\theta + (\frac{1}{2}e_{12}-\omega_3)\sin\theta} \qquad (2\text{-}28f)$$

$$= \frac{a}{b}$$

with

$$a - a\tan\theta\tan\psi_3 = b\tan\theta + b\tan\psi_3$$

$$\tan\psi_3(b+a\tan\theta) = a - b\tan\theta$$

or

$$\tan\psi_3 = \frac{a-b\tan\theta}{b+a\tan\theta}$$

It follows from Equation (2-28f) that

$$a - b\tan\theta = (\tfrac{1}{2}e_{12} + \omega_3)\cos\theta + (1+e_{22})\sin\theta - (1+e_{11})\sin\theta - (\tfrac{1}{2} - \omega_3)\frac{\sin^2\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta}\left\{\frac{1}{2}(e_{22} - e_{11})\sin 2\theta + \omega_3 + \frac{1}{2}e_{12}\cos 2\theta\right\}$$

Also,

$$b + a\tan\theta = \frac{1}{\cos\theta}\left(1 + \frac{1}{2}e_{12}\sin 2\theta + e_{11}\cos^2\theta + e_{22}\sin^2\theta\right)$$

Finally,

$$\tan\psi_3 = \frac{\omega_3 + \tfrac{1}{2}e_{12}\cos 2\theta + \tfrac{1}{2}(e_{22} - e_{11})\sin 2\theta}{1 + e_{11}\cos^2\theta + e_{22}\sin^2\theta + \tfrac{1}{2}e_{12}\sin 2\theta} \qquad (2\text{-}28g)$$

The mean value of tan ψ_3 in the interval from $\theta = 0$ to $\theta = 2\pi$ (i.e., its mean value for all the fibers perpendicular to X_3-axis before the deformation) is given by the expression

$$\overline{\tan\psi_3} = \frac{1}{2\pi}\int_0^{2\pi} \tan\psi_3 \, d\theta = I_1 + I_2 \qquad (2\text{-}29)$$

Here

$$I_1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{\omega_3 \, d\theta}{1 + e_{11}\cos^2\theta + e_{22}\sin^2\theta + \tfrac{1}{2}e_{12}\sin 2\theta}, \qquad (2\text{-}29a)$$

$$I_2 = \frac{1}{2\pi}\int_0^{2\pi} \frac{\tfrac{1}{2}e_{12}\cos 2\theta + \tfrac{1}{2}(e_{22} - e_{11})\sin 2\theta}{1 + e_{11}\cos^2\theta + e_{22}\sin^2\theta + \tfrac{1}{2}e_{12}\sin 2\theta}\, d\theta \qquad (2\text{-}29b)$$

The integral I_2 is evaluated by making the substitution

$$f = 1 + e_{11}\cos^2\theta + e_{22}\sin^2\theta + \tfrac{1}{2}e_{12}\sin 2\theta \qquad (2\text{-}30a)$$

$$df = (2\sin\theta\cos\theta(e_{22} - e_{11}) + e_{12}\cos 2\theta)\,d\theta \qquad (2\text{-}30b)$$

which yields

$$I_2 = \frac{1}{4\pi} \int_{\theta=0}^{\theta=2\pi} \frac{df}{f}$$

$$= \frac{1}{4\pi} (\ln f) \Big|_{\theta=0}^{\theta=2\pi} = 0 \tag{2-31a}$$

The integral I_1 is reducible to the form

$$I_1 = \frac{\omega_3}{\pi} \int_0^{2\pi} \frac{d\theta}{2 + 2e_{11}\cos^2\theta + 2e_{22}\sin^2\theta + e_{12}\sin 2\theta}$$

$$= \frac{\omega_3}{\pi} \int_0^{2\pi} \frac{d\theta}{2 + (1+\cos 2\theta)e_{11} + (1-\cos 2\theta)e_{22} + e_{12}\sin 2\theta}$$

$$= \frac{\omega_3}{\pi} \int^{2\pi} \frac{d\theta}{2 + e_{11} + e_{22} + (e_{11} - e_{22})\cos 2\theta + e_{12}\sin 2\theta}$$

$$= \frac{\omega_3}{\pi} \int_0^{2\pi} \frac{d\theta}{2 + e_{11} + e_{22} + [(e_{11} - e_{22})^2 + e_{12}^2]^{1/2} \sin(2\theta + \beta)} \tag{2-31b}$$

where

$$\beta = \sin^{-1}\left(\frac{e_{11} - e_{12}}{\sqrt{(e_{11} - e_{22})^2 + e_{12}^2}}\right)$$

$$\beta = \cos^{-1}\left(\frac{e_{12}}{\sqrt{(e_{11} - e_{22})^2 + e_{12}^2}}\right) \tag{2-32}$$

Now let $\phi = 2\theta + \beta$ or $\theta = \frac{(\phi + \beta)}{2}$ with $d\theta = \frac{1}{2}\,d\phi$

and at $\theta = 0,\ \phi = \beta$

$\theta = 2\pi;\ \phi = 4\pi + \beta$

hence,

$$I_1 = \frac{\omega_3}{2\pi}\int_{\beta}^{4\pi+\beta} \frac{d\phi}{2 + e_{11} + e_{22} + [(e_{11}-e_{22})^2 + e_{12}^2]^{1/2}\sin\phi} \tag{2-33a}$$

$$= \frac{\omega_3}{2\pi}\int_{\beta}^{4\pi+\beta} \frac{d\phi}{A + B\sin\phi}$$

$$= \frac{\omega_3}{2\pi}\left|\frac{2}{\sqrt{A^2-B^2}}\tan^{-1}\left(\frac{A\tan\phi/2 + B}{\sqrt{A^2-B^2}}\right)\right|_{\beta}^{4\pi+\beta}$$

$$= \frac{1}{2\pi}\cdot\frac{2\omega_3}{\sqrt{(2+e_{11}+e_{22})^2-(e_{11}-e_{22})^2-e_{12}^2}}\cdot \left|\tan^{-1}\frac{(2+e_{11}+e_{22})\tan\phi/2 + \sqrt{(e_{11}-e_{22})^2+e_{12}^2}}{\sqrt{(2+e_{11}+e_{22})^2-(e_{11}-e_{22})^2-e_{12}^2}}\right|_{\beta}^{4\pi+\beta} \tag{2-33b}$$

Consider

$$\sqrt{(2+e_{11}+e_{22})^2-(e_{11}-e_{22})^2-e_{12}^2} = 2\sqrt{1+e_{11}+e_{22}+e_{11}e_{22}-\tfrac{1}{4}e_{12}^2},$$

Equation (2-33b) becomes

$$\tan\psi_3 = \frac{1}{2\pi}\cdot\frac{\omega_3}{\sqrt{1+e_{11}+e_{22}+e_{11}e_{22}-\tfrac{1}{4}e_{12}^2}}\cdot\left|\tan^{-1}\frac{(2+e_{11}+e_{22})\tan\phi/2+\sqrt{(e_{11}-e_{21})^2+e_{12}^2}}{2\sqrt{1+e_{11}+e_{22}+e_{11}e_{22}-\tfrac{1}{4}e_{12}^2}}\right|_{\beta}^{4\pi+\beta} \tag{2-33c}$$

Since the last function of the right hand side of Equation (2-33c) is multi-valued, the result obtained is indefinite. This indefiniteness, however, is removed by taking into account the fact that as e_{11}, e_{22}, e_{12} tend to zero, the integral I_1 (and therefore also $\overline{\tan\psi_3}$) must tend to ω_3, as from Equation (2-33a). Consequently in Equation (2-33c) one obtains

$$\left| \tan^{-1}\left[\frac{(2+e_{11}+e_{22})\tan\Phi/2+\sqrt{(e_{11}-e_{22})^2+e_{12}^2}}{2\sqrt{1+e_{11}+e_{22}+e_{11}e_{22}-\tfrac{1}{4}e_{12}^2}}\right]\right|_{\beta}^{4\pi+\beta}$$

$$= \left| \tan^{-1}\left[\frac{2\tan\Phi/2}{2}\right]\right|_{\beta}^{4\pi+\beta}$$

$$= \left|\Phi/2\right|_{\beta}^{4\pi+\beta}$$

$$= 2\pi \tag{2-34}$$

which leads to the following expression for $\tan\psi_3$

$$\overline{\tan\psi_3} = \frac{\omega_3}{\sqrt{(1+e_{11})(1+e_{22})-\tfrac{1}{4}e_{12}^2}} . \tag{2-35a}$$

Analogously

$$\overline{\tan\psi_2} = \frac{\omega_2}{\sqrt{(1+e_{11})(1+e_{33})-\tfrac{1}{4}e_{13}^2}} \tag{2-35b}$$

$$\overline{\tan\psi_1} = \frac{\omega_1}{\sqrt{(1+e_{22})(1+e_{33})-\tfrac{1}{4}e_{23}^2}} \tag{2-35c}$$

which determine the mean values of the tangents of the angles of rotation about the X_1- and X_2-axes, of the line elements of the body perpendicular to these axes before the deformation.

The three parameters $\overline{\tan\psi_1}$, $\overline{\tan\psi_2}$, $\overline{\tan\psi_3}$ characterize the rotation of an infinitesimal volume containing the point M; they are proportional to ω_1, ω_2 and ω_3, and vanish whenever these parameters are equal to zero.

It is clear from (2-26) that if $\omega_1, \omega_2, \omega_3$ are equal to zero in some co-ordinate system X_1, X_2, X_3, then they are equal to zero in any other coordinate system. It follows that if the relations

$$\omega_1 = \omega_2 = \omega_3 = 0 \tag{2-36}$$

holds at some point of the body, then, in the mean, the line elements passing through this point will not undergo a rotation to any axis passing through this point.

2.6 Fibers Preserving Direction Under Deformation

Now consider in the conditions for absence of rotation and establish the fact that at every point in the body, there exists at least one fiber which preserves its direction under a deformation. For such a fiber the vectors MN and M^*N^* (Figure (II-1)) are identical in direction which implies that their projections satisfy the relations

$$\frac{dx_1^*}{dx_1} = \frac{dx_2^*}{dx_2} = \frac{dx_3^*}{dx_3} = Z = \text{Constant}, \tag{2-37a}$$

where (2-37b)

$$Z = \frac{|M^*N^*|}{|MN|} = 1 + E$$

and E is the elongation in the direction MN.

Thus

$$\{dx^*\} = (1+E)\{dx\} \tag{2-37c}$$

According to Equation (1-2), it follows that

$$(1+E)\{dx\} = [J]\{dx\}$$

$$(1+E)[I]\{dx\} = [J]\{dx\}$$

$$[J]\{dx\} - (1+E)[I]\{dx\} = \{0\}$$

$$[[J] - (1+E)[I]]\{dx\} = \{0\} \tag{2-38a}$$

By dividing Equation (2-38a) by $|MN|$, Equation (2-38a) is rewritten in the form

$$\left[[J]-(1+E)[I]\right]\left\{\frac{dx}{|MN|}\right\} = \{0\} \qquad (2\text{-}38b)$$

or

$$\left[[J]-(1+E)[I]\right]\{\lambda\} = \{0\} \qquad (2\text{-}38c)$$

For non-zero value of $\{\lambda\}$, it follows that

$$\left|[J]-(1+E)[I]\right| = 0 \qquad (2\text{-}38d)$$

or

$$\left|[D]-E[I]\right| = 0 \,. \qquad (2\text{-}38e)$$

Since

$$\begin{aligned}[J]-(1+E)[I] &= [D]+[I]-[I]-E[I]\\ &= [D]-E[I]\end{aligned}$$

the Equation (2-38c) yields the characteristic equation of the matrix $[D]$ which is solved directly for the eigenvalues. The general form of Equation (2-38c) becomes

$$(E)^3 - b_2(E)^2 + b_1(E) - b_o = 0 \qquad (2\text{-}39a)$$

where

$$b_2 = e_{11} + e_{22} + e_{33} = \text{Trace of matrix } [D] \qquad (2\text{-}39b)$$

$$b_1 = e_{11}e_{22} + e_{11}e_{33} + e_{22}e_{33} - \tfrac{1}{4}(e_{12}^2 + e_{13}^2 + e_{23}^2) + \omega_1^2 + \omega_2^2 + \omega_3^2 \qquad (2\text{-}39c)$$

$$\begin{aligned}b_o = {} & e_{11}e_{22}e_{33} + \tfrac{1}{4}(e_{12}e_{23}e_{13} - e_{22}e_{13}^2 - e_{33}e_{12}^2 - e_{11}e_{23}^2)\\ & + \omega_1^2 e_{11} + \omega_2^2 e_{22} + \omega_3^2 e_{33} + \omega_1\omega_2 e_{12} + \omega_1\omega_3 e_{13} + \omega_2\omega_3 e_{23}\\ = {} & |[D]|\end{aligned} \qquad (2\text{-}39d)$$

The roots of the Equation (2-39a) $E^{①}$, $E^{②}$, $E^{③}$ are the eigen-values of $[D]$

The quantities b_2, b_1, b_o, remain invariant under a transformation of coordinates. Recalling the three invariants of $[e]$ as

$$b_2' = e_{11} + e_{22} + e_{33} \quad (2\text{-}40a)$$

$$b_1' = e_{11}e_{22} + e_{11}e_{33} + e_{22}e_{33} - 1/4(e_{12}^2 + e_{13}^2 + e_{23}^2) \quad (2\text{-}40b)$$

$$b_o' = e_{11}e_{22}e_{33} + 1/4(e_{12}e_{13}e_{23} - e_{11}e_{23}^2 - e_{22}e_{13}^2 - e_{33}e_{12}^2) \quad (2\text{-}40c)$$

and the three invariants of $[\omega]$ as

$$b_2'' = 0 \quad (2\text{-}40d)$$

$$b_1'' = \omega_1^2 + \omega_2^2 + \omega_3^2 \quad (2\text{-}40e)$$

$$b_o'' = 0 \quad (2\text{-}40f)$$

it follows that

$$b_1 = b_1' + b_1'' \quad (2\text{-}40g)$$

$$b_o = b_o' + \omega_1^2 e_{11} + \omega_2^2 e_{22} + \omega_3^2 e_{33} + \omega_1\omega_2 e_{12} + \omega_1\omega_3 e_{13} + \omega_2\omega_3 e_{23} \quad (2\text{-}40h)$$

must also be invariants.

Since Equation (2-39a) is a cubic equation with real coefficients, at least one root must be real which implies that they exist at least one direction for which the rotation is zero.

2.6a The General Picture of the Deformation in the Neighborhood of an Arbitary Point of the Body

It follows from Equation (1-2) that the projections of the vector $\overrightarrow{M^{*}N^{*}}$ (i.e., the projections of an arbitary line element of the body after deformation) are connected by means of linear relations with the projections of the vector $\overrightarrow{MN}$ (i.e., with the projections of the same element before deformation). Correspondingly, the inverse relations expressible by Equation (1-16) are also linear. The coefficients in Equation (1-2) and Equation (1-16) are to be taken constant and equal to their values at the point M, thus the deformation of an infinitesimal region containing the point M is described by a linear transformation with constant coefficients.

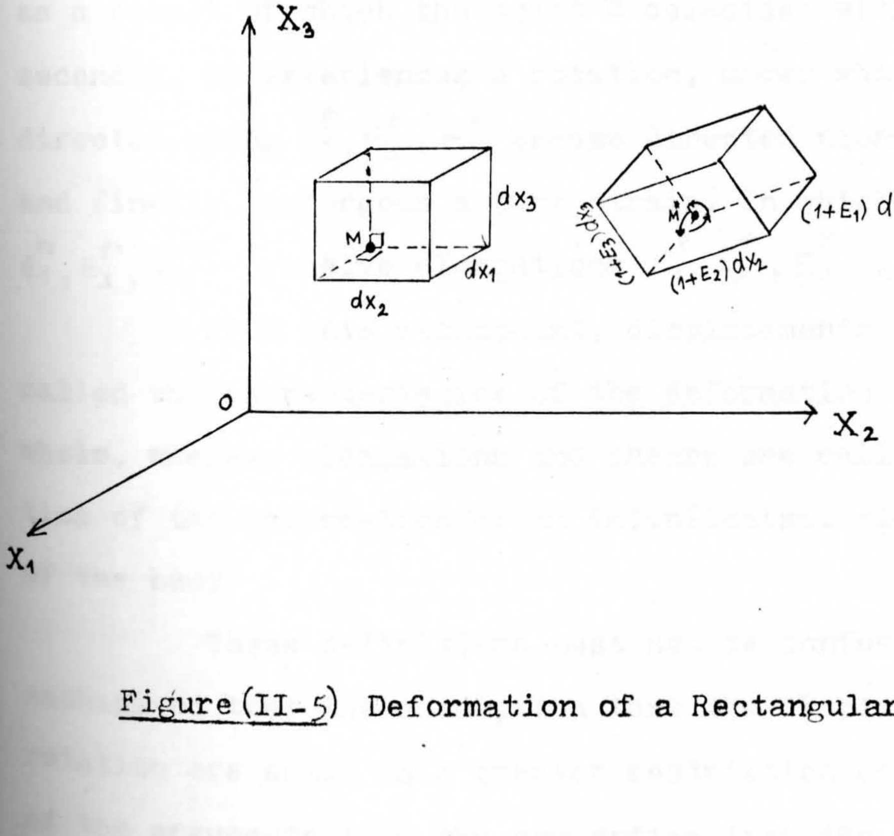


Figure (II-5) Deformation of a Rectangular Parallelopied.

In particular, the rectangular parallelopiped with edges dx_1, dx_2, dx_3 parallel to the coordinate axes is transformed by the deformation into an oblique parallelopiped with edges $(1+E_1)dx_1, (1+E_2)dx_2, (1+E_3)dx_3$ forming angles $(\pi/2-\phi_{12}), (\pi/2-\phi_{13}), (\pi/2-\phi_{23})$ as shown in Figure (II-5).

In case of principal axes, the parallelopiped whose edges before deformation coincide with the principal axes at the point in question is still rectangular after the deformation, and has edges $(1+E_1^p)da_1, (1+E_2^p)da_2, (1+E_3^p)da_3$ where a_1, a_2, a_3 are the lengths of the edges before deformation.

The foregoing gives some idea of the character of the deformation of an infinitesimal region surrounding the point M. Under a deformation, this region first undergoes a translation, as a result of which the point M coincides with the point M*; secondly, it experiences a rotation, under which the fibers directed along $\epsilon_1^p, \epsilon_2^p, \epsilon_3^p$ become directed along $\epsilon_1^{p*}, \epsilon_2^{p*}, \epsilon_3^{p*}$; and finally, undergoes a pure strain, in which the fibers $\epsilon_1^{p*}, \epsilon_2^{p*}, \epsilon_3^{p*}$ receive elongations E_1^p, E_2^p, E_3^p .

From this standpoint, displacements and rotation are called the characteristics of the deformation of a body as a whole, whereas elongations and shears are called the characteristics of the deformation of an infinitesimal element of volume of the body.

These definitions must not be confused. It should be emphasized that the assumption that the displacements and rotation are small is a greater restriction of the generality of the arguments than the assumption that the strain components are small. The first assumption implies the second, but the converse is false. It must also be remarked that, in those cases where the necessity of small displacements is indicated, it is ordinarily not specified what they must be small in comparison with. Such a specification, however, is absolutely necessary, since displacements are dimensional quantities.

Thus in conclusion, the term "small deformation," means the smallness of the elongations and shears compared to unity.

2.7 Change in Volume

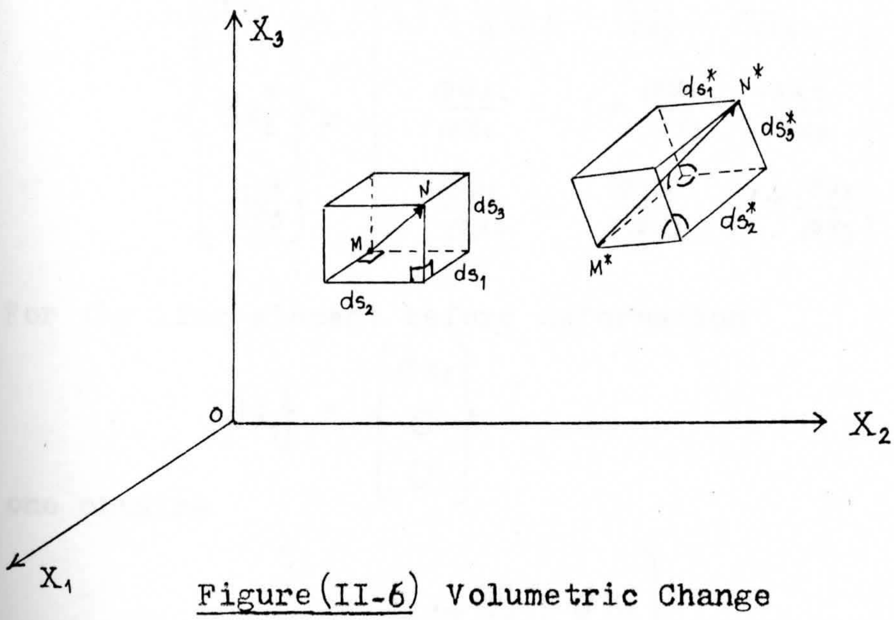


Figure (II-6) Volumetric Change

An infinitesimal rectangular parallelopiped with edges dx_1, dx_2, dx_3 parallel to the coordinate axes is transformed by the deformation into an oblique parallelopiped with edges ds_1^*, ds_2^*, ds_3^*, forming angles $(\pi/2 - \phi_{12}), (\pi/2 - \phi_{13}), (\pi/2 - \phi_{23})$ in Figure (II-6).

Noting

$$V = dx_1\, dx_2\, dx_3 \quad = \text{ The volume of the element before deformation} \qquad (2\text{-}41a)$$

$$V^* = [\,(d\vec{s}_1^{\,*} \times d\vec{s}_2^{\,*}) \cdot d\vec{s}_3^{\,*}\,] \qquad (2\text{-}41b)$$

= The volume of the oblique parallelopiped.

It follows that $\vec{ds}_1^*, \vec{ds}_2^*, \vec{ds}_3^*$ is expressed into the vector form from Equation (1-2) as

$$\{ds_1^*\} = \{dx^*\} = [J]\{ds\} = [J]\{dx\}$$

$$\begin{Bmatrix} dx_1^* \\ dx_2^* \\ dx_3^* \end{Bmatrix} = \begin{bmatrix} \left(1+\frac{\partial u_1}{\partial x_1}\right) & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & 1+\frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & 1+\frac{\partial u_3}{\partial x_3} \end{bmatrix} \begin{Bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{Bmatrix}$$

For the line element before deformation

$$\{ds_1\} = \begin{Bmatrix} dx_1 \\ 0 \\ 0 \end{Bmatrix}$$

one obtains

$$\{ds_1^*\} = \begin{Bmatrix} \left(1+\frac{\partial u_1}{\partial x_1}\right) dx_1 \\ \frac{\partial u_2}{\partial x_1}\, dx_1 \\ \frac{\partial u_3}{\partial x_1}\, dx_1 \end{Bmatrix} \qquad (2\text{-}42a)$$

Analogously,

$$\{ds_2^*\} = \begin{Bmatrix} \frac{\partial u_1}{\partial x_2} dx_2 \\ (1+\frac{\partial u_2}{\partial x_2}) dx_2 \\ \frac{\partial u_3}{\partial x_2} dx_2 \end{Bmatrix} \qquad (2\text{-}42b)$$

$$\{ds_3^*\} = \begin{Bmatrix} \frac{\partial u_1}{\partial x_3} dx_3 \\ \frac{\partial u_2}{\partial x_3} dx_3 \\ (1+\frac{\partial u_3}{\partial x_3}) dx_3 \end{Bmatrix} \qquad (2\text{-}42c)$$

which also before deformation are given as

$$\{ds_2\} = \begin{Bmatrix} 0 \\ dx_2 \\ 0 \end{Bmatrix} \; ; \; \{ds_3\} = \begin{Bmatrix} 0 \\ 0 \\ dx_3 \end{Bmatrix} \quad \text{and}$$

Equation (2-41b) is expressed in matrix form as

$$(V^*) = \left\{[ds_1^*]\{ds_2^*\}\right\}^T\{ds_3^*\}$$

$$= \{ds_2^*\}^T[ds_1^*]^T\{ds_3^*\}$$

The component form becomes

$$(V^*) = \left\{\frac{\partial u_1}{\partial x_2}, (1+\frac{\partial u_2}{\partial x_2}), \frac{\partial u_3}{\partial x_2}\right\} \begin{bmatrix} 0 & \frac{\partial u_3}{\partial x_1} & -\frac{\partial u_2}{\partial x_1} \\ -\frac{\partial u_3}{\partial x_1} & 0 & (1+\frac{\partial u_1}{\partial x_1}) \\ \frac{\partial u_2}{\partial x_1} & -(1+\frac{\partial u_1}{\partial x_1}) & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_3} \\ (1+\frac{\partial u_3}{\partial x_3}) \end{Bmatrix} dx_1 dx_2 dx_3$$

The determinant of $[J]$ is defined as

$$|[J]| = \left|\begin{bmatrix} \left(1+\frac{\partial u_1}{\partial x_1}\right) & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \left(1+\frac{\partial u_2}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \left(1+\frac{\partial u_3}{\partial x_3}\right) \end{bmatrix}\right|$$

$$= \left\{ \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_3} - \left(1+\frac{\partial u_2}{\partial x_2}\right)\frac{\partial u_1}{\partial x_3},\ \left(1+\frac{\partial u_1}{\partial x_1}\right) - \frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_3},\ \left(1+\frac{\partial u_1}{\partial x_1}\right)\left(\frac{\partial u_2}{\partial x_2}+1\right) - \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1} \right\} \begin{Bmatrix} \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_3}{\partial x_2} \\ \left(1+\frac{\partial u_3}{\partial x_3}\right) \end{Bmatrix}$$

$$= \left\{ \frac{\partial u_1}{\partial x_2}, \left(1+\frac{\partial u_2}{\partial x_2}\right), \frac{\partial u_3}{\partial x_2} \right\} \begin{bmatrix} 0 & \frac{\partial u_3}{\partial x_1} & -\frac{\partial u_2}{\partial x_1} \\ -\frac{\partial u_3}{\partial x_1} & 0 & \left(1+\frac{\partial u_1}{\partial x_1}\right) \\ \frac{\partial u_2}{\partial x_1} & -\left(1+\frac{\partial u_1}{\partial x_1}\right) & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_3}{\partial x_2} \\ \left(1+\frac{\partial u_3}{\partial x_3}\right) \end{Bmatrix}$$

Therefore, (V^*) is rewritten the form

$$(V^*) = |[J]|\, dx_1 dx_2 dx_3$$

$$= |[J]|\,(V) \tag{2-43a}$$

or $$\frac{V^*}{V} = |[J]|$$

Defining Δ as the relative change in volume due to deformation.

or $$\Delta = \left(\frac{V^* - V}{V}\right) \tag{2-43b}$$

or $$= \frac{V^*}{V} - 1$$

and $$\frac{V^*}{V} = 1 + \Delta$$

hence

$$|[J]| = 1 + \Delta \tag{2-43c}$$

2.8 The Theory of Small Deformation (Case 2)

The equations derived in the previous sections place no restrictions on the elongations and shears as compared to unity. A restriction in the size of these parameters is now accounted for in this section. Introducing into Equation (2-6) the approximation that $E_{1,2,3} \ll 1$, it follows that

$$E_1 \approx \varepsilon_{11} \quad , \quad E_2 \approx \varepsilon_{22} \quad , \quad E_3 \approx \varepsilon_{33} \tag{2-46a}$$

Further, Equation (2-8c) is reduced by taking into consideration that $E_{1,2,3} \ll 1$, it follows that

$$\phi_{12} \approx \varepsilon_{12} \quad , \quad \phi_{13} \approx \varepsilon_{13} \quad , \quad \phi_{23} \approx \varepsilon_{23} \tag{2-46b}$$

where

$$\sin\phi_{12} \approx \phi_{12} \quad , \quad \sin\phi_{13} \approx \phi_{13} \quad , \quad \sin\phi_{23} \approx \phi_{23} \tag{2-46c}$$

Thus for the small relative deformations, the components $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$ are identified with E_1, E_2, E_3 respectively, and $\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}$ are identified with $\phi_{12}, \phi_{13}, \phi_{23}$ respectively. Therefore, the increment of volume Δ in Equation (2-45) is reduced to the form

$$\begin{aligned} \Delta &\approx E_1 + E_2 + E_3 \\ &\approx \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = a_2 = \epsilon_1^p + \epsilon_2^p + \epsilon_3^p \end{aligned} \tag{2-46d}$$

Analogously, the Equation (1-12) and (1-22) take the form

$$\{\tilde{i}^*\} \approx [J]^T \{i\} \tag{2-46e}$$

$$\{\tilde{i}\} \approx [\alpha]^T \{i\} \tag{2-46f}$$

where

$$[1+E] \approx [I]$$

$$[1+E^*] \approx [I]$$

$$|[J]| \approx (1+E_1)(1+E_2)(1+E_3) \approx 1$$

Squaring both sides yields

$$(1+\Delta)^2 = (|[J]|)^2$$

or

$$= |[J]^T[J]|$$

From Equation (2-4a), it follows that

$$2[\varepsilon]+[I] = [J]^T[J]$$

Thus

$$|[J]^T[J]| = |2[\varepsilon]+[I]|$$

or

$$(1+\Delta)^2 = |2[\varepsilon]+[I]| \qquad (2\text{-}44)$$

In case of principal axes $[\varepsilon]$ changes to the form

$$[\epsilon^p] = \begin{bmatrix} \epsilon_1^p & 0 & 0 \\ 0 & \epsilon_2^p & 0 \\ 0 & 0 & \epsilon_3^p \end{bmatrix}$$

thus

$$(1+\Delta)^2 = |2[\epsilon^p]+[I]|$$

$$= (2\epsilon_1^p+1)(2\epsilon_2^p+1)(2\epsilon_3^p+1)$$

or

$$\Delta = \sqrt{(2\epsilon_1^p+1)(2\epsilon_2^p+1)(2\epsilon_3^p+1)} - 1$$

By using Equation (2-6), one obtains the form

$$\Delta = (1+E_1^p)(1+E_2^p)(1+E_3^p) - 1 \qquad (2\text{-}45)$$

where E_1^p, E_2^p, E_3^p are the principal elongations at the point where the change in volume is calculated.

2.9 The Case of Small Deformation and Small Angles of Rotation (Case 3)

If the angles of rotation as well as the strain components are small compared to unity, then the directions of the vectors $\tilde{i}^*_1, \tilde{i}^*_2, \tilde{i}^*_3$ and $\tilde{i}_1, \tilde{i}_2, \tilde{i}_3$ with obviously deviate. from those of X_1, X_2, X_3 by only a small amount.

As a result, the diagonal members of $[A]$, and $[B]$ (See Equations (1-13a) and (1-23a)) differ from unity only by quantities of the second order which the remaining members of these matrix are quantities of the first order (if the maximum value of an angle of rotation is taken to be a quantity of the first order).

Considering the two dimensional axes of matrix $[A], [B]$

$$\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} 1+\frac{\phi^2}{2!}+\cdots & \phi-\frac{\phi^3}{3!}+\cdots \\ -(\phi-\frac{\phi^3}{3!}+\cdots) & 1+\frac{\phi^2}{2!}+\cdots \end{bmatrix} \qquad (2\text{-}47a)$$

with $\cos\phi \approx 1$, $\sin\phi \approx \phi$ for small angles of rotations ϕ

(2-47b)

Noting Equations (1-18), (1-14) with $E_{1,2,3} \ll 1$, one obtains

$$[\alpha] = \mathrm{COF}\,[A]^T$$

or

$$[\alpha] = \mathrm{COF}\,[J]^T. \qquad (2\text{-}48a)$$

Equation (2-48a) is expressed as follow

$$\alpha_{33} = (1+e_{11})(1+e_{22}) - (\tfrac{1}{4}e_{12}^2 - \omega_3^2)$$

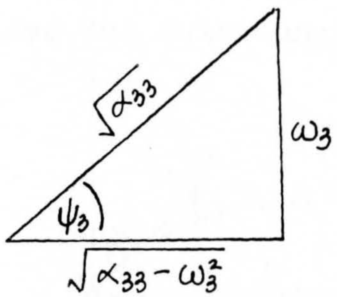
Hence

$$\alpha_{33} - \omega_3^2 = (1+e_{11})(1+e_{22}) - \tfrac{1}{4}e_{12}^2 \qquad (2\text{-}48b)$$

Thus Equation (2-35a) is rewritten in the form

$$\overline{\tan\psi_3} \approx \frac{\omega_3}{\sqrt{\alpha_{33} - \omega_3^2}} \tag{2-48c}$$

Consider the following definitions



$$\overline{\sin\psi_3} = \frac{\omega_3}{\sqrt{\alpha_{33}}}$$

$$\overline{\cos\psi_3} = \frac{\sqrt{\alpha_{33} - \omega_3^2}}{\sqrt{\alpha_{33}}}$$

In accordance with Equations (1-23a) and (1-23b)

$\alpha_{33} \approx \cos(\tilde{i}_3, i_3)$ for a small deformation $(E_3^* \ll 1)$

therefore

$$\overline{\sin\psi_3} = \frac{\omega_3}{\sqrt{\cos(\tilde{i}_3, i_3)}} \tag{2-48d}$$

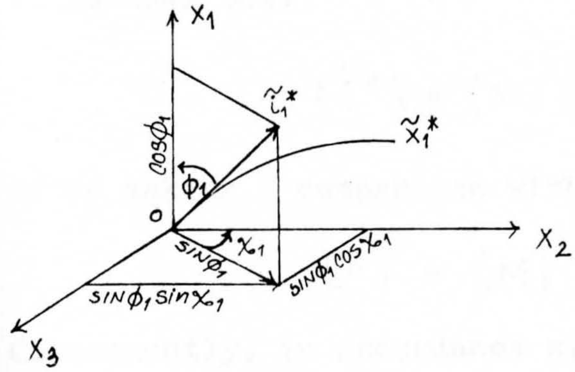
It is noted above that in the present case the cosine of the angle between the X_3-axis and vector $\tilde{i}_3$ differs from unity only by a quantity of the second order. Moreover, since the rotations is small, $\overline{\psi}_3$ differs from $\overline{\sin\psi_3}$ only by quantities of the third order. Hence, neglecting the squares of the angle of rotation compared to unity, Equation (2-48d) is rewritten as follows:

$$\overline{\psi}_3 \approx \omega_3 \tag{2-49a}$$

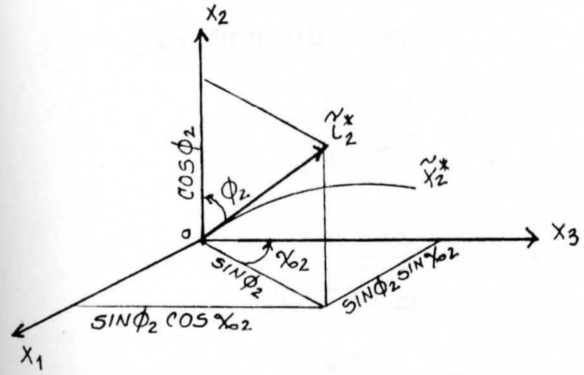
Analogously

$$\bar{\psi}_2 \quad \omega_2 \qquad ; \qquad \bar{\psi}_1 \approx \omega_1 \qquad (2\text{-}49b,c)$$

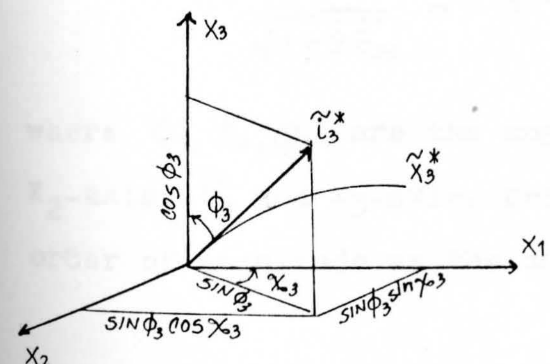
Furthermore, the formulas for the strain components $[\varepsilon]$ are simplified under the assumption that the angles of rotation and the strain components are small comparedto unity as follows: Consider the Euler angles:



$$\{\tilde{i}_1^*\} = \begin{Bmatrix} \cos\phi_1 \\ \sin\phi_1 \cos\chi_{o1} \\ \sin\phi_1 \sin\chi_{o1} \end{Bmatrix} \qquad (2\text{-}50a)$$



$$\{\tilde{i}_2^*\} = \begin{Bmatrix} \sin\phi_2 \sin\chi_{o2} \\ \cos\phi_2 \\ \cos\chi_{o2} \sin\phi_2 \end{Bmatrix} \qquad (2\text{-}50b)$$



$$\{\tilde{i}_3^*\} = \begin{Bmatrix} \sin\phi_3 \cos\chi_{o3} \\ \sin\phi_3 \sin\chi_{o3} \\ \cos\phi_3 \end{Bmatrix} \qquad (2\text{-}50c)$$

Figure (II-7) Euler Angles for Rotations

Let these vectors be combined to form the column of the matrix $[M]$ with

$$[M] = \begin{bmatrix} \cos\phi_1 & \sin\phi_2 \sin\chi_2 & \sin\phi_3 \cos\chi_3 \\ \sin\phi_1 \cos\chi_1 & \cos\phi_2 & \sin\phi_3 \sin\chi_3 \\ \sin\phi_1 \sin\chi_1 & \sin\phi_2 \cos\chi_2 & \cos\phi_3 \end{bmatrix} \tag{2-50d}$$

It follows that

$$\{\tilde{i}^*\} = [M]^T \{i\} \quad . \tag{2-50e}$$

After making a comparison with Equation (1-12), one obtains

$$[A]^T = [M]^T \tag{2-51}$$

Consequently, in accordance with Equations (2-51), and (2-6), (2-41a), one obtains

$$\frac{1+e_{11}}{\sqrt{1+2\varepsilon_{11}}} \approx 1 - \frac{\phi_1^2}{2} \tag{2-52a}$$

$$\frac{1+e_{22}}{\sqrt{1+2\varepsilon_{22}}} \approx 1 - \frac{\phi_2^2}{2} \tag{2-52b}$$

$$\frac{1+e_{33}}{\sqrt{1+2\varepsilon_{33}}} \approx 1 - \frac{\phi_3^2}{2} \tag{2-52c}$$

where ϕ_1, ϕ_2, ϕ_3 are the angles between $\tilde{i}_1^*$ and X_1-axis, $\tilde{i}_2^*$ and X_2-axis, $\tilde{i}_3^*$ and X_3-axis, respectively, and also are the same order of magnitude as the angles of rotation.

Consider the binomial equation

$$(a+b)^n = a^n + n a^{(n-1)} b + (n-1) a^{(n-2)} b^2 + \cdots \quad (2\text{-}53a)$$

$$\begin{aligned}(1+e_{11})(1+2\varepsilon_{11})^{1/2} &= (1+e_{11})(1^{-1/2} + (-\tfrac{1}{2})(1^{-3/2})(2\varepsilon_{11}) + \cdots \\ &= (1+e_{11})(1-\varepsilon_{11}) \\ &= 1+e_{11} - \varepsilon_{11} - \varepsilon_{11}e_{11} \quad .\end{aligned} \quad (2\text{-}53b)$$

Since the strain components are assumed to be small incomparison to unity, the product of "ε" and "e" are neglected. Thus, Equation (2-53b) becomes

$$1+e_{11}-\varepsilon_{11} = 1-\phi_1^2/2$$

or

$$\varepsilon_{11}-e_{11} = \frac{\phi_1^2}{2} \quad . \quad (2\text{-}54a)$$

Analogously

$$\varepsilon_{22}-e_{22} = \frac{\phi_2^2}{2} \quad (2\text{-}54b)$$

$$\varepsilon_{33}-e_{33} = \frac{\phi_3^2}{2} \quad . \quad (2\text{-}54c)$$

Thus in this case, the quantities e_{11}, e_{22}, e_{33} differ from the corresponding strain components $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$ only by magnitudes of the same order as the squares of the angles of rotation. Furthermore, in accordance with Equations (2-51), and by using the definition of Equation (2-47b) then the off diagonal terms of Equation (1-13a) are expressed as:

$$\begin{aligned}&\tfrac{1}{2}e_{12}+\omega_3 \approx \phi_1\cos\chi_1 \; ; \; \tfrac{1}{2}e_{12}-\omega_3 \approx \phi_2\sin\chi_2 \\ &\tfrac{1}{2}e_{13}+\omega_2 \approx \phi_3\cos\chi_3 \; ; \; \tfrac{1}{2}e_{13}-\omega_2 \approx \phi_1\sin\chi_1 \\ &\tfrac{1}{2}e_{23}+\omega_1 \approx \phi_2\cos\chi_2 \; ; \; \tfrac{1}{2}e_{23}-\omega_1 \approx \phi_3\sin\chi_3\end{aligned} \quad (2\text{-}55)$$

Now consider the shear strain components $\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}$ obtained from the dot product between Equations (2-50a), (2-50b) and (2-50c), it follows that

$$(\tilde{i}_1^* \cdot \tilde{i}_2^*) = \{\tilde{i}_1^*\}^T \{\tilde{i}_2^*\}$$

$$\cos(\tilde{i}_1^*, \tilde{i}_2^*) = \sin\phi_{12} = \cos\phi_1 \sin\phi_2 \sin\gamma_2 + \cos\phi_2 \sin\phi_1 \cos\gamma_1 + \sin\phi_1 \sin\phi_2 \sin\gamma_1 \cos\gamma_2 \quad (2\text{-}56a)$$

$$\cos(\tilde{i}_1^*, \tilde{i}_3^*) = \sin\phi_{13} = \cos\phi_1 \sin\phi_3 \cos\gamma_3 + \cos\phi_3 \sin\phi_1 \sin\gamma_1 + \sin\phi_1 \sin\phi_3 \cos\gamma_1 \sin\gamma_3 \quad (2\text{-}56b)$$

$$\cos(\tilde{i}_2^*, \tilde{i}_3^*) = \sin\phi_{23} = \cos\phi_2 \sin\phi_3 \sin\gamma_3 + \cos\phi_3 \sin\phi_2 \cos\gamma_2 + \sin\phi_2 \sin\phi_3 \sin\gamma_2 \cos\gamma_3 \quad (2\text{-}56c)$$

By comparing Equations (2-3c) and (2-37b) with the above equations, noting the condition that $E_{1,2,3} \ll 1$, and omitting all terms containing ϕ to higher than the second power, one obtains

$$\begin{aligned} \varepsilon_{12} &\approx \phi_2 \sin\gamma_2 + \phi_1 \cos\gamma_1 + \phi_1\phi_2 \sin\gamma_1 \cos\gamma_2 \\ \varepsilon_{13} &\approx \phi_3 \cos\gamma_3 + \phi_1 \sin\gamma_1 + \phi_1\phi_3 \cos\gamma_1 \sin\gamma_3 \\ \varepsilon_{23} &\approx \phi_3 \sin\gamma_3 + \phi_2 \cos\gamma_2 + \phi_2\phi_3 \sin\gamma_2 \cos\gamma_3 \end{aligned} \quad (2\text{-}57)$$

Combination of Equation (2-55) with Equation (2-57), yields

$$\begin{aligned} \varepsilon_{12} - e_{12} &\approx \phi_1\phi_2 \sin\gamma_1 \cos\gamma_2 \\ \varepsilon_{13} - e_{13} &\approx \phi_1\phi_3 \cos\gamma_1 \sin\gamma_3 \\ \varepsilon_{23} - e_{23} &\approx \phi_2\phi_3 \sin\gamma_2 \cos\gamma_3 \end{aligned} \quad (2\text{-}58)$$

which implies that the parameters e_{12}, e_{13}, e_{23} differ from the corresponding strain components only by quantities of the same order as the products of the angles of rotation.

Consider Equation (2-4c)

$$[\varepsilon] = [e] + \frac{1}{2}\left[[e]^2 + [e][\omega] - [\omega][e] - [\omega]^2\right]$$

It is seen that the squares of the parameters matrix $[e]$ may neglected, because they are the same order as the fourth powers of the angles of rotation, thus Equation (2-4c) is reduced into

$$[\varepsilon] \approx [e] + \tfrac{1}{2}\left[[e][\omega] - [\omega][e] - [\omega]^2\right] , \quad (2\text{-}59a)$$

and also $[e][\omega]$, $[\omega][e]$ have the same power as the cubes of angles of rotation, so they may neglected in comparison with $[\omega]^2$, thus

$$[\varepsilon] \approx [e] - \tfrac{1}{2}[\omega]^2 . \quad (2\text{-}59b)$$

These equations are correct to within the accuracy obtainable by neglecting the angles of rotation and the strains in comparison to unity.

2.10 The Transition to the Equations of the Classical Theory (Case 4)

Assuming that the squares and products of the angles of rotation may be neglected in comparison with $[e]$, Equation (2-49c) reduces to

$$[\varepsilon] \approx [e] = \frac{1}{2}\left[[D]^T + [D]\right] \qquad (2\text{-}60)$$

These are the equations of the classical theory of elasticity.

It is seen from the two proceeding sections that the expressions for the strain components become linear only under the two following conditions:

a) The elongation, shears, and angles of rotation must be small compared to unity.

b) The terms of the second degree in the angles of rotation appearing in Equation (2-59b) must be small compared to the corresponding strain components.

The last requirement can be formulated, roughly speaking, as the condition that the squares of the angles of rotation be negligibly small compared to the elongations and shears. If the body is MASSIVE, i.e., is of the same order of magnitude in all three of its dimensions, then condition (a) implies condition (b).

This is not true if the body is flexible, i.e., if its dimensions in one or two directions is essentially small compared to its remaining dimensions (rod, plate, shell).

In this case the angles of rotation may considerably exceed the elongations and shears, so that Equation (2-60) are in general not applicable to such bodies. This implies that the linear Equation (2-60) is to be used primarily in analyzing the deformation of massive bodies, while the non linear Equation (2-4c) and (2-59b) are applicable to deformation of flexible bodies.

2.11 On the Transition to Curvilinear Coordinates

It has been assumed up to now that the positions of the points of a body are expressed in terms of Cartesian coordinates X_1, X_2, X_3 . In the solution of some engineering problems, it is more convenient to use orthogonal curvilinear coordinates.

Let the curvilinear co-ordinates be related to the Cartesian coordinates in accordance with the equations

$$X_1 = f_1(\alpha_1, \alpha_2, \alpha_3)\ , \quad X_2 = f_2(\alpha_1, \alpha_2, \alpha_3)\ , \quad X_3 = f_3(\alpha_1, \alpha_2, \alpha_3)\ .$$

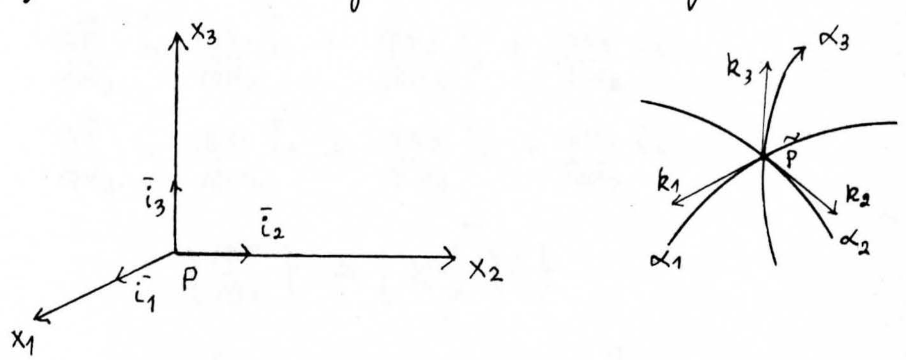


Figure (II-8) Curvilinear Coordinate Axes

These equations determine three families of curves, the coordinate lines $\alpha_1, \alpha_2, \alpha_3$. Denote the unit vectors tangent to the coordinate lines by k_1, k_2, k_3 respectively, as shown in Figure (II-8).

Since the curvilinear co-ordinates are assumed to be orthogonal, k_1, k_2, k_3 form at every point a mutually perpendicular trihedral of local coordinate axes (reference is made to local axes because, unlike a Cartesian system, the directions of these axes change from one point to another).

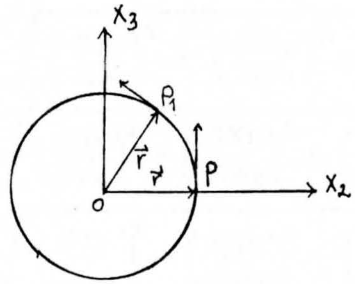


Figure (II-9) Example of Curvilinear Coordinates

Considering vector $\vec{r}$ in Figure (II-9)

$$\vec{r} = x_1 \bar{i}_1 + x_2 \bar{i}_2 + x_3 \bar{i}_3 \tag{2-62a}$$

$$\frac{\partial \vec{r}}{\partial \alpha_1} = \frac{\partial x_1}{\partial \alpha_1} \bar{i}_1 + \frac{\partial x_2}{\partial \alpha_1} \bar{i}_1 + \frac{\partial x_3}{\partial \alpha_1} \bar{i}_3$$

$$\frac{\partial \vec{r}}{\partial \alpha_2} = \frac{\partial x_1}{\partial \alpha_2} \bar{i}_1 + \frac{\partial x_2}{\partial \alpha_2} \bar{i}_2 + \frac{\partial x_3}{\partial \alpha_2} \bar{i}_3 \tag{2-62b}$$

$$\frac{\partial \vec{r}}{\partial \alpha_3} = \frac{\partial x_1}{\partial \alpha_3} \bar{i}_1 + \frac{\partial x_2}{\partial \alpha_3} \bar{i}_2 + \frac{\partial x_3}{\partial \alpha_3} \bar{i}_3$$

$$\left\{\frac{\partial r}{\partial \alpha}\right\} = [K]^T \{i\} \tag{2-62c}$$

where

$$[K] = \begin{bmatrix} \frac{\partial x_1}{\partial \alpha_1} & \frac{\partial x_1}{\partial \alpha_2} & \frac{\partial x_1}{\partial \alpha_3} \\ \frac{\partial x_2}{\partial \alpha_1} & \frac{\partial x_2}{\partial \alpha_2} & \frac{\partial x_2}{\partial \alpha_3} \\ \frac{\partial x_3}{\partial \alpha_1} & \frac{\partial x_3}{\partial \alpha_2} & \frac{\partial x_3}{\partial \alpha_3} \end{bmatrix} \tag{2-62d}$$

and

$$\left\{\frac{\partial \vec{r}}{\partial \alpha}\right\} = \begin{Bmatrix} \frac{\partial \vec{r}}{\partial \alpha_1} \\ \frac{\partial \vec{r}}{\partial \alpha_2} \\ \frac{\partial \vec{r}}{\partial \alpha_3} \end{Bmatrix} . \tag{2-62e}$$

The unit vectors in $\alpha_1, \alpha_2, \alpha_3$ directions are expressed as follow

$$\bar{k}_1 = \frac{\frac{\partial \vec{r}}{\partial \alpha_1}}{H_1} \quad , \quad \bar{k}_2 = \frac{\frac{\partial \vec{r}}{\partial \alpha_2}}{H_2} \quad , \quad \bar{k}_3 = \frac{\frac{\partial \vec{r}}{\partial \alpha_2}}{H_3} \quad . \tag{2-63a}$$

where

$$H_1 = \frac{\partial \vec{r}}{\partial \alpha_1} = \sqrt{\left(\frac{\partial x_1}{\partial \alpha_1}\right)^2 + \left(\frac{\partial x_2}{\partial \alpha_1}\right)^2 + \left(\frac{\partial x_3}{\partial \alpha_1}\right)^2}$$

$$H_2 = \frac{\partial \vec{r}}{\partial \alpha_2} = \sqrt{\left(\frac{\partial x_1}{\partial \alpha_2}\right)^2 + \left(\frac{\partial x_2}{\partial \alpha_2}\right)^2 + \left(\frac{\partial x_3}{\partial \alpha_2}\right)^2} \tag{2-63b}$$

$$H_3 = \frac{\partial \vec{r}}{\partial \alpha_3} = \sqrt{\left(\frac{\partial x_1}{\partial \alpha_3}\right)^2 + \left(\frac{\partial x_2}{\partial \alpha_3}\right)^2 + \left(\frac{\partial x_3}{\partial \alpha_3}\right)^2} \; .$$

Writing the latter equations in the matrix form gives

$$\{\bar{k}\} = [R]^T \{\bar{i}\} = \left[\tfrac{1}{H}\right][K]^T \{\bar{i}\} \tag{2-63c}$$

where

$$[R] = [K]\left[\tfrac{1}{H}\right] \tag{2-63d}$$

$$\left[\tfrac{1}{H}\right] = \begin{bmatrix} \frac{1}{H_1} & 0 & 0 \\ 0 & \frac{1}{H_2} & 0 \\ 0 & 0 & \frac{1}{H_3} \end{bmatrix} ; \; [R] = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} . \tag{2-63e}$$

According to Equation (2-63c), matrix $[R]$ is the transformation matrix from $\{\bar{i}\}$ to $\{\bar{k}\}$.

Since both sets of coordinates axes are orthogonal then

$$[R]^T[R] = [R][R]^T = [I] \, . \tag{2-63f}$$

Assuming that at each point in the field, the $\alpha_1, \alpha_2, \alpha_3$ axes are rotated so that they coincide with the x_1, x_2, x_3 axes at point P. (See Figure (II-10)), it follows that the vector $\begin{Bmatrix} d\alpha_1 \\ d\alpha_2 \\ d\alpha_3 \end{Bmatrix}$ has the same direction as the vector $\begin{Bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{Bmatrix}$.

Hence, the matrix $[K]$ becomes diagonal with

$$\frac{\partial x_1}{\partial \alpha_2} = \frac{\partial x_1}{\partial \alpha_3} = \frac{\partial x_2}{\partial \alpha_1} = \frac{\partial x_2}{\partial \alpha_3} = \frac{\partial x_3}{\partial \alpha_1} = \frac{\partial x_3}{\partial \alpha_2} = 0 \qquad (2\text{-}64a)$$

and the components of diagonal matrix $\left[\frac{1}{H}\right]$ reduce to the form

$$H_1 = \frac{\partial x_1}{\partial \alpha_1}, \quad H_2 = \frac{\partial x_2}{\partial \alpha_2}, \quad H_3 = \frac{\partial x_3}{\partial \alpha_3}. \qquad (2\text{-}64b)$$

As a result, at point P the units vectors coincide

or
$$\{\bar{k}\} = \{\bar{i}\} \qquad (2\text{-}64c)$$

and the $[R]$ matrix becomes

$$[R] = [I]. \qquad (2\text{-}64d)$$

According to the well-known chain rule of multivariate calculus

$$\{\nabla_\alpha\} = [K]^T\{\nabla_k\} \qquad (2\text{-}65a)$$

$$\{\nabla_x\} = [\hat{K}]^T\{\nabla_\alpha\} \qquad (2\text{-}65b)$$

with
$$[\hat{K}][K] = [I].$$

From Equations (2-63d) and (2-64d)

$$[R] = [K]\left[\frac{1}{H}\right] = [I]$$

$$[\hat{K}][K]\left[\frac{1}{H}\right] = [\hat{K}][I]$$

or
$$\left[\frac{1}{H}\right] = [\hat{K}]. \qquad (2\text{-}65c)$$

Therefore, Equation (2-65b) is rewritten as

$$\{\nabla_x\} = \left[\frac{1}{H}\right]\{\nabla_\alpha\} \tag{2-65d}$$

where

$$\{\nabla_x\} = \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \; ; \; \{\nabla_\alpha\} = \begin{Bmatrix} \frac{\partial}{\partial \alpha_1} \\ \frac{\partial}{\partial \alpha_2} \\ \frac{\partial}{\partial \alpha_3} \end{Bmatrix} . \tag{2-65e}$$

Thus,

$$\begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{H_1}\frac{\partial}{\partial \alpha_1} \\ \frac{1}{H_2}\frac{\partial}{\partial \alpha_2} \\ \frac{1}{H_3}\frac{\partial}{\partial \alpha_3} \end{Bmatrix} \tag{2-65f}$$

Consider the point P in Figure (II-10). Upon differentiating Equation (2-63f) with respect to x_1, one obtains

$$\left[\frac{\partial R}{\partial x_1}\right][R]^T + [R]\left[\frac{\partial R}{\partial x_1}\right]^T = 0. \tag{2-66a}$$

which when evaluated at point P (i.e. $[R] = [I]$), it follows that

$$\left[\frac{\partial R}{\partial x_1}\right] + \left[\frac{\partial R}{\partial x_1}\right]^T = 0. \tag{2-66b}$$

Equation (2-66b) defines matrix $\left[\frac{\partial R}{\partial x_1}\right]$ as a skew-symmetric matrix, or

$$\frac{\partial R_{ii}}{\partial x_1} = 0 \quad ; \quad \frac{\partial R_{jk}}{\partial x_1} = -\frac{\partial R_{kj}}{\partial x_1} \quad ; \quad k \neq j = 1,2,3 . \tag{2-66c}$$

Similary at point P these relations remain valid if x_1 is replaced by $x_2, x_3, \alpha_1, \alpha_2, \alpha_3$.

Consider the component $\frac{\partial R_{32}}{\partial X_1}$. By using Equation (2-65e)

$$\frac{\partial R_{32}}{\partial X_1} = \frac{1}{H_1}\frac{\partial R_{32}}{\partial \alpha_1} \qquad = \frac{1}{H_1}\frac{\partial}{\partial \alpha_1}\left(\frac{1}{H_2}\frac{\partial X_3}{\partial \alpha_2}\right)$$

$$= \frac{1}{H_1}\left(\frac{1}{H_2}\frac{\partial^2 X_3}{\partial \alpha_1 \partial \alpha_2} + \frac{\partial X_3}{\partial \alpha_2}\frac{\partial}{\partial \alpha_1}\left(\frac{1}{H}\right)\right).$$

The function $\frac{\partial X_3}{\partial \alpha_2} = o$ at point P, thus

$$\frac{\partial R_{32}}{\partial X_1} = \frac{1}{H_1}\frac{1}{H_2}\frac{\partial^2 X_3}{\partial \alpha_1 \partial \alpha_2}. \tag{2-67a}$$

Noting the following differentiation

$$\frac{\partial R_{31}}{\partial X_2} = \frac{1}{H_2}\frac{\partial}{\partial \alpha_2}\left(\frac{1}{H_1}\frac{\partial X_3}{\partial \alpha_1}\right)$$

$$= \frac{1}{H_2}\left[\frac{1}{H_1}\frac{\partial^2 X_3}{\partial \alpha_1 \partial \alpha_2} + \frac{\partial X_3}{\partial \alpha_1}\frac{\partial}{\partial \alpha_2}\left(\frac{1}{H_1}\right)\right]$$

$$= \frac{1}{H_1}\frac{1}{H_2}\frac{\partial^2 X_3}{\partial \alpha_1 \partial \alpha_2} \quad . \tag{2-67b}$$

The following equality holds:

$$\frac{\partial R_{31}}{\partial X_2} = \frac{\partial R_{32}}{\partial X_1}. \tag{2-67c}$$

Analogously,

$$\frac{\partial R_{21}}{\partial X_3} = \frac{\partial R_{23}}{\partial X_1} \quad ; \quad \text{and} \quad \frac{\partial R_{13}}{\partial X_2} = \frac{\partial R_{12}}{\partial X_3}. \tag{2-67d}$$

According to Equation (2-66c)

$$\frac{\partial R_{32}}{\partial X_1} = -\frac{\partial R_{23}}{\partial X_1} = -\frac{\partial R_{21}}{\partial X_3} = \frac{\partial R_{12}}{\partial X_3} = \frac{\partial R_{13}}{\partial X_2} = -\frac{\partial R_{31}}{\partial X_2}. \tag{2-67e}$$

Comparison of Equations (2-67c) with (2-66c) gives

$$\frac{\partial R_{32}}{\partial X_1} = \frac{\partial R_{31}}{\partial X_2} = 0$$

$$\frac{\partial R_{23}}{\partial X_1} = \frac{\partial R_{21}}{\partial X_3} = \frac{\partial R_{12}}{\partial X_3} = \frac{\partial R_{13}}{\partial X_2} = 0 \,. \qquad (2\text{-}67f)$$

And also $\frac{\partial R_{32}}{\partial \alpha_1} = \frac{\partial R_{31}}{\partial \alpha_2} = \frac{\partial R_{23}}{\partial \alpha_1} = \frac{\partial R_{21}}{\partial \alpha_3} = \frac{\partial R_{12}}{\partial \alpha_3} = \frac{\partial R_{13}}{\partial \alpha_2} = 0 \,.$

Now consider the component $\frac{\partial R_{32}}{\partial X_3}$ at point P. Equation (2-63d) gives

$$R_{32} = \frac{1}{H_2} \frac{\partial X_3}{\partial \alpha_2}$$

then

$$\frac{\partial R_{32}}{\partial \alpha_3} = \frac{1}{H_2} \frac{\partial^2 X_3}{\partial \alpha_2 \partial \alpha_3} + \frac{\partial X_3}{\partial \alpha_2} \frac{\partial}{\partial \alpha_3}\left(\frac{1}{H_2}\right)$$

or

$$H_2 \cdot \frac{\partial R_{32}}{\partial \alpha_3} = \frac{\partial^2 X_3}{\partial \alpha_2 \partial \alpha_3} \,. \qquad (2\text{-}68a)$$

In accordance with Equation (2-63b)

$$H_3^2 = \left(\frac{\partial X_1}{\partial \alpha_3}\right)^2 + \left(\frac{\partial X_2}{\partial \alpha_3}\right)^2 + \left(\frac{\partial X_3}{\partial \alpha_3}\right)^2 \,. \qquad (2\text{-}68b)$$

Differentiating Equation (2-63b) with respect to α_2 yields when evaluated at point P

$$2 H_3 \frac{\partial H_3}{\partial \alpha_2} = 2 \frac{\partial X_1}{\partial \alpha_3} \frac{\partial^2 X_1}{\partial \alpha_2 \partial \alpha_3} + 2 \frac{\partial X_2}{\partial \alpha_3} \frac{\partial^2 X_2}{\partial \alpha_2 \partial \alpha_3} + 2 \frac{\partial X_3}{\partial \alpha_3} \frac{\partial^2 X_3}{\partial \alpha_2 \partial \alpha_3}$$

or

$$\frac{\partial H_3}{\partial \alpha_2} = \frac{\partial^2 X_3}{\partial \alpha_2 \partial \alpha_3} \qquad (2\text{-}68c)$$

thus,

$$\frac{\partial H_3}{\partial \alpha_2} = H_2 \frac{\partial R_{32}}{\partial \alpha_3} = \frac{\partial^2 X_3}{\partial \alpha_2 \partial \alpha_3} \,.$$

Because $\left[\frac{\partial R}{\partial \alpha}\right]$ is a skew-symmetrix, it follows that

$$\frac{\partial R_{32}}{\partial \alpha_3} = \frac{1}{H_2} \frac{\partial H_3}{\partial \alpha_2} = - \frac{\partial R_{23}}{\partial \alpha_3} \,. \qquad (2\text{-}68d)$$

Analogously

$$
\begin{aligned}
\frac{\partial R_{31}}{\partial \alpha_3} &= \frac{1}{H_1}\frac{\partial H_3}{\partial \alpha_1} = -\frac{\partial R_{13}}{\partial \alpha_3} \\
\frac{\partial R_{21}}{\partial \alpha_2} &= \frac{1}{H_1}\frac{\partial H_2}{\partial \alpha_1} = -\frac{\partial R_{12}}{\partial \alpha_2} \\
\frac{\partial R_{23}}{\partial \alpha_2} &= \frac{1}{H_3}\frac{\partial H_2}{\partial \alpha_3} = -\frac{\partial R_{32}}{\partial \alpha_2} \\
\frac{\partial R_{12}}{\partial \alpha_1} &= \frac{1}{H_2}\frac{\partial H_1}{\partial \alpha_2} = -\frac{\partial R_{21}}{\partial \alpha_1} \\
\frac{\partial R_{13}}{\partial \alpha_1} &= \frac{1}{H_3}\frac{\partial H_1}{\partial \alpha_3} = -\frac{\partial R_{31}}{\partial \alpha_1} .
\end{aligned}
\qquad (2\text{-}68d)
$$

Combining the values from Equations (2-67f) and (2-68d) yields

$$
\left[\frac{\partial R}{\partial X_1}\right] = \frac{1}{H_1}\left[\frac{\partial R}{\partial \alpha_1}\right] = \frac{1}{H_1}\begin{bmatrix} 0 & \frac{1}{H_2}\frac{\partial H_1}{\partial \alpha_2} & \frac{1}{H_3}\frac{\partial H_1}{\partial \alpha_3} \\ -\frac{1}{H_2}\frac{\partial H_1}{\partial \alpha_2} & 0 & 0 \\ -\frac{1}{H_3}\frac{\partial H_1}{\partial \alpha_3} & 0 & 0 \end{bmatrix} \qquad (2\text{-}69a)
$$

$$
\left[\frac{\partial R}{\partial X_2}\right] = \frac{1}{H_2}\left[\frac{\partial R}{\partial \alpha_2}\right] = \frac{1}{H_2}\begin{bmatrix} 0 & -\frac{1}{H_1}\frac{\partial H_2}{\partial \alpha_1} & 0 \\ \frac{1}{H_1}\frac{\partial H_2}{\partial \alpha_1} & 0 & \frac{1}{H_3}\frac{\partial H_2}{\partial \alpha_3} \\ 0 & -\frac{1}{H_3}\frac{\partial H_2}{\partial \alpha_3} & 0 \end{bmatrix} \qquad (2\text{-}69b)
$$

$$\left[\frac{\partial R}{\partial X_3}\right] = \frac{1}{H_3}\left[\frac{\partial R}{\partial \alpha_3}\right] = \frac{1}{H_3}\begin{bmatrix} 0 & 0 & -\frac{1}{H_1}\frac{\partial H_3}{\partial \alpha_1} \\ 0 & 0 & -\frac{1}{H_2}\frac{\partial H_3}{\partial \alpha_2} \\ \frac{1}{H_1}\frac{\partial H_3}{\partial \alpha_1} & \frac{1}{H_2}\frac{\partial H_3}{\partial \alpha_2} & 0 \end{bmatrix} \quad (2\text{-}69c)$$

Consider the displacement vector $\vec{u}$

$$\{u_x\} = \begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{Bmatrix}$$ with respect to rectangular Cartesian axes

and also $\{u_\alpha\} = \begin{Bmatrix} u_{\alpha 1} \\ u_{\alpha 2} \\ u_{\alpha 3} \end{Bmatrix}$ W.R.T. curvilinear coordinate axes.

It follows that

$$\{u_\alpha\} = [R]^T \{u_x\} \quad (2\text{-}70a)$$

$$\{u_x\} = [R]\{u_\alpha\}$$

$$\frac{\partial}{\partial X_1}\{u_x\} = \frac{\partial}{\partial X_1}\left\{[R]\{u_\alpha\}\right\}$$

$$= \left[\frac{\partial}{\partial X_1}[R]\right]\{u_\alpha\} + [R]\left\{\frac{\partial}{\partial X_1}\{u_\alpha\}\right\}. \quad (2\text{-}70b)$$

Substituting the values of $\left[\frac{\partial}{\partial X_1}[R]\right]$ from Equation (2-69a) and Equating $[R] = [I]$ for the condition at point P., one obtains

$$\frac{\partial\{u_x\}}{\partial X_1} = \frac{1}{H_1}\begin{bmatrix} 0 & \frac{1}{H_2}\frac{\partial H_1}{\partial \alpha_2} & \frac{1}{H_3}\frac{\partial H_1}{\partial \alpha_3} \\ -\frac{1}{H_2}\frac{\partial H_1}{\partial \alpha_2} & 0 & 0 \\ -\frac{1}{H_3}\frac{\partial H_1}{\partial \alpha_3} & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_{\alpha 1} \\ u_{\alpha 2} \\ u_{\alpha 3} \end{Bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \frac{\partial u_{\alpha 1}}{\partial X_1} \\ \frac{\partial u_{\alpha 2}}{\partial X_1} \\ \frac{\partial u_{\alpha 3}}{\partial X_1} \end{Bmatrix},$$

or finally the three terms

$$\begin{Bmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} \\ \frac{\partial u_3}{\partial x_1} \end{Bmatrix} = \begin{bmatrix} \frac{1}{H_1}\frac{1}{H_2}\frac{\partial H_1}{\partial \alpha_2} u_{\alpha 2} + \frac{1}{H_1}\frac{1}{H_3}\frac{\partial H_1}{\partial \alpha_3} u_{\alpha 3} + \frac{1}{H_1}\frac{\partial u_{\alpha 1}}{\partial \alpha_1} \\ \frac{1}{H_1}\frac{\partial u_{\alpha 2}}{\partial \alpha_1} - \frac{1}{H_1}\frac{1}{H_2}\frac{\partial H_1}{\partial \alpha_2} u_{\alpha 1} \\ \frac{1}{H_1}\frac{\partial u_{\alpha 3}}{\partial \alpha_1} - \frac{1}{H_1}\frac{1}{H_3}\frac{\partial H_1}{\partial \alpha_3} u_{\alpha 1} \end{bmatrix} . \qquad (2\text{-}70c)$$

Analogously

$$\begin{Bmatrix} \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_2} \\ \frac{\partial u_3}{\partial x_2} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{H_2}\frac{\partial u_{\alpha 1}}{\partial \alpha_2} - \frac{1}{H_1}\frac{1}{H_2}\frac{\partial H_2}{\partial \alpha_1} u_{\alpha 2} \\ \frac{1}{H_2}\frac{\partial u_{\alpha 2}}{\partial \alpha_2} + \frac{1}{H_1}\frac{1}{H_2}\frac{\partial H_2}{\partial \alpha_1} u_{\alpha 1} + \frac{1}{H_2}\frac{1}{H_3}\frac{\partial H_2}{\partial \alpha_3} u_{\alpha 3} \\ \frac{1}{H_2}\frac{\partial u_{\alpha 3}}{\partial \alpha_2} - \frac{1}{H_2}\frac{1}{H_3}\frac{\partial H_2}{\partial \alpha_3} u_{\alpha 2} \end{Bmatrix} \qquad (2\text{-}70d)$$

$$\begin{Bmatrix} \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_3} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{H_3}\frac{\partial u_{\alpha 1}}{\partial \alpha_3} - \frac{1}{H_1}\frac{1}{H_3}\frac{\partial H_3}{\partial \alpha_1} u_{\alpha 3} \\ \frac{1}{H_3}\frac{\partial u_{\alpha 2}}{\partial \alpha_3} - \frac{1}{H_2}\frac{1}{H_3}\frac{\partial H_3}{\partial \alpha_2} u_{\alpha 3} \\ \frac{1}{H_3}\frac{\partial u_{\alpha 2}}{\partial \alpha_3} + \frac{1}{H_1}\frac{1}{H_3}\frac{\partial H_3}{\partial \alpha_1} u_{\alpha 1} + \frac{1}{H_2}\frac{1}{H_3}\frac{\partial H_3}{\partial \alpha_2} u_{\alpha 2} \end{Bmatrix} \qquad (2\text{-}70e)$$

Thus,

$$\tilde{e}_{11} = \frac{\partial u_1}{\partial x_1} = \frac{1}{H_1}\frac{1}{H_2}\frac{\partial H_1}{\partial \alpha_2} u_{\alpha 2} + \frac{1}{H_1}\frac{1}{H_3}\frac{\partial H_1}{\partial \alpha_3} u_{\alpha 3} + \frac{1}{H_1}\frac{\partial u_{\alpha 1}}{\partial \alpha_1}$$

$$\tilde{e}_{22} = \frac{\partial u_2}{\partial x_2} = \frac{1}{H_2}\frac{\partial u_{\alpha 2}}{\partial \alpha_2} + \frac{1}{H_1}\frac{1}{H_2}\frac{\partial H_2}{\partial \alpha_1} u_{\alpha 1} + \frac{1}{H_2}\frac{1}{H_3}\frac{\partial H_2}{\partial \alpha_3} u_{\alpha 3}$$

$$\tilde{e}_{33} = \frac{\partial u_3}{\partial x_3} = \frac{1}{H_3}\frac{\partial u_{\alpha 2}}{\partial \alpha_3} + \frac{1}{H_1}\frac{1}{H_3}\frac{\partial H_3}{\partial \alpha_1} u_{\alpha 1} + \frac{1}{H_2}\frac{1}{H_3}\frac{\partial H_3}{\partial \alpha_2} u_{\alpha 2} \qquad (2\text{-}71)$$

$$\tilde{e}_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = \frac{1}{H_2}\frac{\partial u_{\alpha 1}}{\partial \alpha_2} - \frac{1}{H_1}\frac{1}{H_2}\frac{\partial H_2}{\partial \alpha_1} u_{\alpha 2} + \frac{1}{H_1}\frac{\partial u_{\alpha 2}}{\partial \alpha_1} - \frac{1}{H_1}\frac{1}{H_2}\frac{\partial H_1}{\partial \alpha_2} u_{\alpha 1}$$

$$= \frac{1}{H_1}\frac{1}{H_2}\left(\frac{H_1 \frac{\partial u_{\alpha 1}}{\partial \alpha_1} - \frac{\partial H_1}{\partial \alpha_1} u_{\alpha 1}}{H_1^2}\right) H_1^2 + \frac{1}{H_1}\frac{1}{H_2}\left(\frac{H_2 \frac{\partial u_{\alpha 2}}{\partial \alpha_1} - u_{\alpha 2}\frac{\partial H_2}{\partial \alpha_1}}{H_2^2}\right) H_2^2$$

$$= \frac{H_1}{H_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u_{\alpha 1}}{H_1}\right) + \frac{H_2}{H_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_{\alpha 2}}{H_2}\right)$$

$$\tilde{e}_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = \frac{H_1}{H_3}\frac{\partial}{\partial \alpha_3}\left(\frac{u_{\alpha 1}}{H_1}\right) + \frac{H_3}{H_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_{\alpha 3}}{H_3}\right)$$

$$\tilde{e}_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = \frac{H_3}{H_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u_{\alpha 3}}{H_3}\right) + \frac{H_2}{H_1}\frac{\partial}{\partial \alpha_3}\left(\frac{u_{\alpha 2}}{H_2}\right)$$

$$2\tilde{\omega} = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} = \frac{1}{H_2}\frac{\partial u_{\alpha 3}}{\partial \alpha_2} - \frac{1}{H_2}\frac{1}{H_3}\frac{\partial H_2}{\partial \alpha_3}u_{\alpha 2} - \frac{1}{H_3}\frac{\partial u_{\alpha 2}}{\partial \alpha_3} + \frac{1}{H_2}\frac{1}{H_3}\frac{\partial H_3}{\partial \alpha_2}u_{\alpha 3}$$

$$= \frac{1}{H_2}\frac{1}{H_3}\left[\left(H_3\frac{\partial u_{\alpha 3}}{\partial \alpha_2} + u_{\alpha 3}\frac{\partial H_3}{\partial \alpha_2}\right) - \left(H_2\frac{\partial u_{\alpha 2}}{\partial \alpha_3} + u_{\alpha 2}\frac{\partial H_2}{\partial \alpha_3}\right)\right]$$

$$\tilde{\omega}_1 = \frac{1}{2H_2H_3}\left[\frac{\partial}{\partial \alpha_2}\left(H_3 u_{\alpha 3}\right) - \frac{\partial}{\partial \alpha_3}\left(H_2 u_{\alpha 2}\right)\right]$$

$$\tilde{\omega}_2 = \frac{1}{2H_1H_3}\left[\frac{\partial}{\partial \alpha_3}\left(H_1 u_{\alpha 1}\right) - \frac{\partial}{\partial \alpha_1}\left(H_3 u_{\alpha 3}\right)\right]$$

$$\tilde{\omega}_3 = \frac{1}{2H_1H_2}\left[\frac{\partial}{\partial \alpha_1}\left(H_2 u_{\alpha 2}\right) - \frac{\partial}{\partial \alpha_2}\left(H_1 u_{\alpha 1}\right)\right]$$

Finally matrix $[\tilde{\varepsilon}]$ in the orthogonal curvilinear co-ordinate is written as follow

$$[\tilde{\varepsilon}] = [\tilde{e}] + \frac{1}{2}\left[[\tilde{e}]^2 - [\tilde{\omega}][\tilde{e}] + [\tilde{e}][\tilde{\omega}] - [\tilde{\omega}]^2\right]. \quad (2\text{-}73)$$

2.12 Summary

Case 1 General Nonlinear Equation.

Elongations:

$$\{\tilde{i}^*\} = \left[\frac{1}{1+E}\right][J]^T\{i\} \; ; \quad \{\tilde{i}\} = [1+E^*]\frac{[\alpha]^T}{|[J]|}\{i\}.$$

$$[\alpha] = \mathrm{COF}[J]^T$$

$$E_1(1+\tfrac{1}{2}E_1) = \varepsilon_{11}$$

$$E_2(1+\tfrac{1}{2}E_2) = \varepsilon_{22}$$

$$E_3(1+\tfrac{1}{2}E_3) = \varepsilon_{33}.$$

Shears:

$$\sin\phi_{12} = \frac{\varepsilon_{12}}{(1+E_1)(1+E_2)},$$

$$\sin\phi_{13} = \frac{\varepsilon_{13}}{(1+E_1)(1+E_3)},$$

$$\sin\phi_{23} = \frac{\varepsilon_{23}}{(1+E_2)(1+E_3)}.$$

Angle of rotation: (mean values)

$$\overline{\tan\psi_1} = \frac{\omega_1}{\sqrt{(1+e_{22})(1+e_{33})-\tfrac{1}{4}e_{23}^2}},$$

$$\overline{\tan\psi_2} = \frac{\omega_2}{\sqrt{(1+e_{11})(1+e_{33})-\tfrac{1}{4}e_{13}^2}},$$

$$\overline{\tan\psi_3} = \frac{\omega_3}{\sqrt{(1+e_{11})(1+e_{22})-\tfrac{1}{4}e_{12}^2}}.$$

Change in Volume:

$$\Delta = (1+E_1^p)(1+E_2^p)(1+E_3^p) - 1.$$

General nonlinear strain equation:

$$[\varepsilon] = [e] + \frac{1}{2}\left[[e]^2 + [e][\omega] - [\omega][e] - [\omega]^2\right].$$

Case 2 Small Deformation

The elongations and shear parameters are small in comparison to unity. Thus,

$$E_1, E_2, E_3 \ll 1$$

or

$$\left[\frac{1}{1+E}\right] \approx [I]$$

$$[1+E^*] \approx [I]$$

$$|[J]| \approx 1,$$

and

$$\sin\phi_{12} \approx \phi_{12}$$
$$\sin\phi_{13} \approx \phi_{13}$$
$$\sin\phi_{23} \approx \phi_{23}.$$

Elongation:

$$E_1 \approx \varepsilon_{11}, \quad E_2 \approx \varepsilon_{22}, \quad E_3 \approx \varepsilon_{33}.$$

Shears:

$$\phi_{12} \approx \varepsilon_{12}, \quad \phi_{13} \approx \varepsilon_{13}, \quad \phi_{23} \approx \varepsilon_{23}.$$

Angles of Rotations:

$$\overline{\tan}\,\psi_1 = \frac{\omega_1}{\sqrt{(1+e_{22})(1+e_{33}) - \frac{1}{4}e_{23}^2}}$$

$$\overline{\tan}\,\psi_2 = \frac{\omega_2}{\sqrt{(1+e_{11})(1+e_{33}) - \frac{1}{4}e_{13}^2}}$$

$$\overline{\tan}\,\psi_3 = \frac{\omega_3}{\sqrt{(1+e_{11})(1+e_{22}) - \frac{1}{4}e_{12}^2}}.$$

Change in Volume:

$$\Delta \approx E_1 + E_2 + E_3 \approx \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

$$\{\tilde{i}^*\} = [J]^T\{i\}$$

$$\{\tilde{i}\} \simeq [\alpha]^T\{i\}$$

General nonlinear strain equation:

$$[\varepsilon] = [e] + \tfrac{1}{2}\left[[e]^2 + [e][\omega] - [\omega][e] - [\omega]^2\right]$$

<u>Case 3</u> Small Deformation and Small Angle of Rotation

In addition to the elongations and shear parameters, the rotation angles are small in comparison to unity.

Thus,

$$E_{1,2,3} \ll 1.$$

and

$$\cos\phi \approx 1 + \phi^2/2! \quad , \quad \sin\phi \simeq \phi - \phi^3/3!$$

$$\cos\phi \approx 1 \quad , \quad \sin\phi \approx \phi .$$

Elongation:

$$E_1 \approx \varepsilon_{11} \quad ; \quad E_2 \approx \varepsilon_{22} \; ; \quad \varepsilon_{33} \approx E_3 .$$

Shear:

$$\phi_{12} \approx \varepsilon_{12} \quad ; \quad \phi_{13} \approx \varepsilon_{13} \; ; \quad \phi_{23} \approx \varepsilon_{23} .$$

Angle of rotation:

$$\overline{\psi}_1 \approx \omega_1 \quad , \quad \overline{\psi}_2 \approx \omega_2 \; , \; \overline{\psi}_3 \approx \omega_3 ,$$

$$\varepsilon_{11} - e_{11} \approx \phi_1^2/2$$

$$\varepsilon_{22} - e_{22} \approx \phi_2^2/2$$

$$\varepsilon_{33} - e_{33} \approx \phi_3^2/2$$

$$\varepsilon_{12} - e_{12} \approx \phi_1\phi_2 \sin\gamma_1 \cos\gamma_2$$

$$\varepsilon_{13} - e_{13} \approx \phi_1\phi_3 \cos\gamma_1 \sin\gamma_3$$

$$\varepsilon_{23} - e_{13} \approx \phi_2\phi_3 \sin\gamma_2 \cos\gamma_3$$

where ϕ_i = Euler angle of rotation.

The nonlinear strain equation reduces into

$$[\varepsilon] \simeq [e] - \tfrac{1}{2}[\omega]^2.$$

Case 4 Equation of Classical Theory

Neglecting the square of the angles of rotation compare to $[e]$;

Elongation:

$$E_1 \approx \varepsilon_{11}\ ,\ E_2 \approx \varepsilon_{22}\ ,\ E_3 \approx \varepsilon_{33}\ .$$

Shear:

$$\phi_{12} \approx \varepsilon_{12}\ ,\ \phi_{13} \approx \varepsilon_{13}\ ,\ \phi_{23} \approx \varepsilon_{23}.$$

Angle of rotation:

$$\bar{\psi}_1 \approx \omega_1\ ,\ \bar{\psi}_2 \approx \omega_2\ ,\ \bar{\psi}_3 \approx \omega_3\ .$$

The nonlinear strain equation reduces into:

$$[\varepsilon] \simeq [e].$$

CHAPTER III

THE EQUILIBRIUM OF AN ELEMENT OF VOLUME OF A BODY

3.1 Stresses

In this chapter the investigation of the conditions for the equilibrium of an arbitary infinitesimal element of volume of the deformed body is considered.

It is necessary to apply to this isolated element forces distributed over its surface which represent the effect of the surrounding medium on this element. Consider an element of area dA^* on the given surface. Its orientation is described by a unit vector $\vec{n}^*$ along the normal, which is regarded as positive if directed toward the exterior of the element of volume in question; denoting

∇dA^* as the force acting on the element of area

$\vec{\nabla}$ as the vector representing the intensity of the surface loading on the area .

The magnitude and direction of $\vec{\nabla}$ depend on the position of the area (which is specified by the coordinates x^*_1, x^*_2, x^*_3 of its centroid) as well as on the orientation of the area (i.e., on $\vec{n}^*$). The triple x^*_1, x^*_2, x^*_3 however, determines a radius vector $\vec{r}$ extending from the origin of coordinates to the centroid of the area, so that

$$\vec{\nabla} = \vec{\nabla}(\vec{r}, \vec{n}^*) \tag{3-1a}$$

Thus, ∇ is a function of two vectors, and is odd with respect to $\vec{n}^*$

$$\nabla(\vec{r},\vec{n}^*) = -\nabla(\vec{r},-\vec{n}^*) \qquad (3\text{-}1b)$$

The vector $\vec{\nabla}$ is called the stress. In sequel it is marked with a subscript indicating the direction of the normal to the area on which it acts, "$*$" indicating the strained state, and "$\sim$" indicating the curvilinear coordinate system.

Consider an element of volume which is a tetrahedron, three of whose edges are parallel to the coordinate axes X_1, X_2, X_3 and equal to dx_1^*, dx_2^*, dx_3^* respectively as shown in Figure (III-1a).

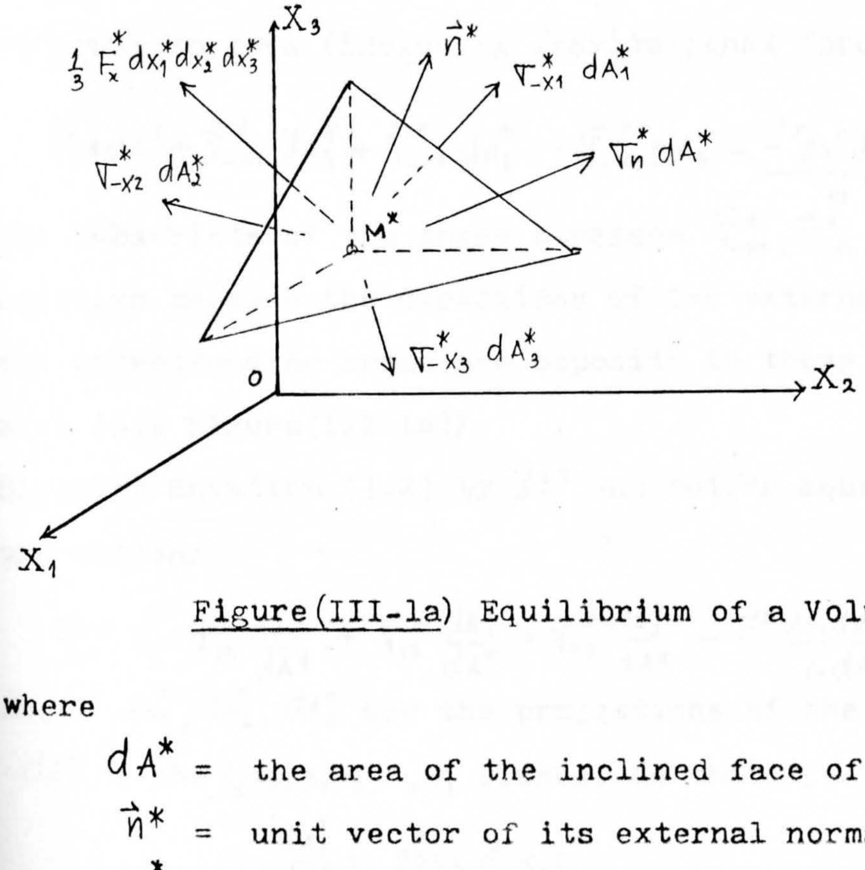


Figure (III-1a) Equilibrium of a Volume Element

where

dA^* = the area of the inclined face of the tetrahedron;

$\vec{n}^*$ = unit vector of its external normal

F_x^* = is the mean value of the specific body force acting on the tetrahedron.

∇^*_{-x1} = the stress on the area, perpendicular to X_1-axis

∇^*_{-x2} = the stress on the area, perpendicular to X_2-axis

∇^*_{-x3} = the stress on the area, perpendicular to X_3-axis

dA_1^* = the area of the face of the tetrahedron which is normal to X_1-axis = $\frac{1}{2} dx_2^* dx_3^*$

dA_2^* = the area of the face of the tetrahedron which is normal to X_2-axis = $\frac{1}{2} dx_1^* dx_3^*$

dA_3^* = $\dfrac{dx_1^* dx_2^*}{2}$.

For the given element to be in equilibrium, it is necessary, first of all, that the sum of all the forces acting on it be equal to zero (including gravitational forces). Thus,

$$\vec{\nabla}_{n^*} dA^* + \vec{\nabla}^*_{-x1} dA_1^* + \vec{\nabla}^*_{-x2} dA_2^* + \vec{\nabla}^*_{-x3} dA_3^* + \frac{\vec{F}^* dx_1^* dx_2^* dx_3^*}{6} = 0. \quad (3\text{-}2)$$

The subscripts of the three stresses $\vec{\nabla}^*_{-x1}, \vec{\nabla}^*_{-x2}, \vec{\nabla}^*_{-x3}$ are negative because the directions of the external normals to the corresponding areas are opposite to those of the coordinate axes (See Figure(III-1a)).

Dividing Equation (3-2) by dA^* and noting Equation (3-1b), one obtains

$$\vec{\nabla}_n^* = \vec{\nabla}^*_{x1} \frac{dA_1^*}{dA^*} + \vec{\nabla}^*_{x2} \frac{dA_2^*}{dA^*} + \vec{\nabla}^*_{x3} \frac{dA_3^*}{dA^*} - \vec{F}^* \frac{dx_1^* dx_2^* dx_3^*}{6\, dA^*} \quad (3\text{-}3)$$

where dA_1^*, dA_2^*, dA_3^* are the projections of the inclined face dA^* on the x_2-x_3, x_1-x_3, x_1-x_2 planes, so that

$$\begin{aligned} \frac{dA_1^*}{dA^*} &= \cos(n^*, x_1) \\ \frac{dA_2^*}{dA^*} &= \cos(n^*, x_2) \\ \frac{dA_3^*}{dA^*} &= \cos(n^*, x_3). \end{aligned} \quad (3\text{-}4)$$

Furthermore, the fraction $\frac{dx_1^* \, dx_2^* \, dx_3^*}{6\, dA^*}$ represents the ratio of the volume of the tetrahedron to the area of its inclined face, and is therefore a magnitude of the order of the linear dimension of the tetrahedron (i.e., and infinitesimal quantity). Hence, the last term in Equation (3-3) is also an infinitesimal, and is neglected.

Combining Equations (3-3) and (3-4) yields the auchy's equations as

$$\vec{\nabla}_n^* = \vec{\nabla}_{X1}^* \cos(n^*, x_1) + \vec{\nabla}_{X2}^* \cos(n^*, x_2) + \vec{\nabla}_{X3}^* \cos(n^*, x_3) \qquad (3\text{-}5)$$

Considering the following definitions:

$\nabla_{n11}^*, \nabla_{n12}^*, \nabla_{n13}^*$ - the projections of $\vec{\nabla}_{n1}^*$ on X_1, X_2, X_3 axes.

$\nabla_{11}^*, \nabla_{12}^*, \nabla_{13}^*$ - the projections of ∇_{X1}^* on X_1, X_2, X_3 axes.

$\nabla_{21}^*, \nabla_{22}^*, \nabla_{23}^*$ - the projections of ∇_{X2}^* on X_1, X_2, X_3 axes.

$\nabla_{31}^*, \nabla_{32}^*, \nabla_{33}^*$ - the projections of ∇_{X3}^* on X_1, X_2, X_3 axes.

Thus, Equation (3-5) is rewritten as

$$\begin{Bmatrix} \nabla_{n11}^* \\ \nabla_{n12}^* \\ \nabla_{n13}^* \end{Bmatrix} = \begin{Bmatrix} \nabla_{11}^* \\ \nabla_{12}^* \\ \nabla_{13}^* \end{Bmatrix} \cos(n_1^*, x_1) + \begin{Bmatrix} \nabla_{21}^* \\ \nabla_{22}^* \\ \nabla_{23}^* \end{Bmatrix} \cos(n_1^*, x_2) + \begin{Bmatrix} \nabla_{31}^* \\ \nabla_{32}^* \\ \nabla_{33}^* \end{Bmatrix} \cos(n_1^*, x_3) \qquad (3\text{-}6a)$$

By mean of the expression of vector $\vec{\nabla}_{n1}^*$ of an inclined face which is indicated by unit vector $\vec{n}_1^*$ normal to this face, Equation (3-6a) is

$$\begin{Bmatrix} \nabla_{n11}^* \\ \nabla_{n12}^* \\ \nabla_{n13}^* \end{Bmatrix} = \begin{bmatrix} \nabla_{11}^* & \nabla_{21}^* & \nabla_{31}^* \\ \nabla_{12}^* & \nabla_{22}^* & \nabla_{32}^* \\ \nabla_{13}^* & \nabla_{23}^* & \nabla_{33}^* \end{bmatrix} \begin{Bmatrix} \cos(n_1^*, x_1) \\ \cos(n_1^*, x_2) \\ \cos(n_1^*, x_3) \end{Bmatrix}, \qquad (3\text{-}6b)$$

$$\{\vec{\nabla}_{n1}^{*}\} = [\nabla_o^{*}]^T \{n_1^{*}\} \qquad (3\text{-}6c)$$

where

$$\{n_1^{*}\} = \begin{Bmatrix} n_{11}^{*} \\ n_{12}^{*} \\ n_{13}^{*} \end{Bmatrix} = \begin{Bmatrix} \cos(n_1^{*}, x_1) \\ \cos(n_2^{*}, x_2) \\ \cos(n_3^{*}, x_3) \end{Bmatrix} . \qquad (3\text{-}6d)$$

Analogously for the other two vectors $\vec{\nabla}_{n2}^{*}, \vec{\nabla}_{n3}^{*}$ (See Figure(III-1b)) are also be expressed as

$$\{\nabla_{n2}^{*}\} = [\nabla_o^{*}]^T \{n_2^{*}\} \qquad (3\text{-}6e)$$

$$\{\nabla_{n3}^{*}\} = [\nabla_o^{*}]^T \{n_3^{*}\} \qquad (3\text{-}6f)$$

where

$$\{\nabla_{n2}^{*}\} = \begin{Bmatrix} \nabla_{n21}^{*} \\ \nabla_{n22}^{*} \\ \nabla_{n23}^{*} \end{Bmatrix} ; \{n_2^{*}\} = \begin{Bmatrix} n_{21}^{*} \\ n_{22}^{*} \\ n_{23}^{*} \end{Bmatrix} = \begin{Bmatrix} \cos(n_2^{*}, x_1) \\ \cos(n_2^{*}, x_2) \\ \cos(n_2^{*}, x_3) \end{Bmatrix}$$

$$\{\nabla_{n3}^{*}\} = \begin{Bmatrix} \nabla_{n31}^{*} \\ \nabla_{n32}^{*} \\ \nabla_{n33}^{*} \end{Bmatrix} ; \{n_3^{*}\} = \begin{Bmatrix} n_{31}^{*} \\ n_{32}^{*} \\ n_{33}^{*} \end{Bmatrix} = \begin{Bmatrix} \cos(n_3^{*}, x_1) \\ \cos(n_3^{*}, x_2) \\ \cos(n_3^{*}, x_3) \end{Bmatrix} .$$

These vectors are then combined to form the columns of the matrix $[\nabla_N^{*}]^T$ and matrix $[C]$ as follow

$$[\{\nabla_{n1}^{*}\}\{\nabla_{n2}^{*}\}\{\nabla_{n3}^{*}\}] = [\nabla_o^{*}][\{n_1^{*}\}\{n_2^{*}\}\{n_3^{*}\}]$$

$$[\nabla_n^{*}]^T = [\nabla_o^{*}][C] \qquad (3\text{-}7)$$

where

$$[\nabla_n^*] = \begin{bmatrix} \nabla_{n11}^* & \nabla_{n12}^* & \nabla_{n13}^* \\ \nabla_{n21}^* & \nabla_{n22}^* & \nabla_{n23}^* \\ \nabla_{n31}^* & \nabla_{n23}^* & \nabla_{n33}^* \end{bmatrix} \qquad (3\text{-}8a)$$

$$[C]^T = \begin{bmatrix} \cos(n_1^*, x_1) & \cos(n_1^*, x_2) & \cos(n_1^*, x_3) \\ \cos(n_2^*, x_1) & \cos(n_2^*, x_2) & \cos(n_2^*, x_3) \\ \cos(n_3^*, x_1) & \cos(n_3^*, x_2) & \cos(n_3^*, x_3) \end{bmatrix} \qquad (3\text{-}8b)$$

Thus, Equation (3-7) becomes

$$[\nabla_n^*] = [C]^T [\nabla_o^*] \qquad (3\text{-}9a)$$

$$[\nabla_o^*] = [C]^{-T} [\nabla_n^*] \qquad (3\text{-}9b)$$

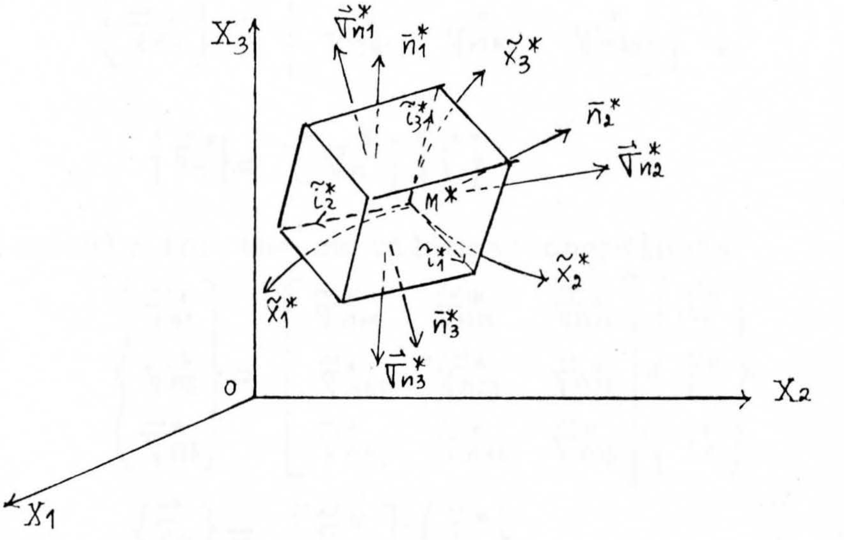


Figure (III-1b) Curvilinear Equilibrium Element

where $[\nabla_n^*]$ is the projections of the vectors $\vec{\nabla}_{n1}^*$, $\vec{\nabla}_{n2}^*$ and $\vec{\nabla}_{n3}^*$ on the Cartesian coordinate system (X_1, X_2, X_3 axes) (See Figure (III-1b)).

$$[\tilde{\nabla}_n^*] = \begin{bmatrix} \tilde{\nabla}_{n11}^* & \tilde{\nabla}_{n12}^* & \tilde{\nabla}_{n13}^* \\ \tilde{\nabla}_{n21}^* & \tilde{\nabla}_{n22}^* & \tilde{\nabla}_{n23}^* \\ \tilde{\nabla}_{n31}^* & \tilde{\nabla}_{n32}^* & \tilde{\nabla}_{n33}^* \end{bmatrix} \tag{3-9c}$$

In determining the relation between matrix $[\tilde{\nabla}_n^*]$ and $[\nabla_n^*]$ one obtains

$$\begin{aligned} \vec{\nabla}_{n1}^* &= \nabla_{n11}^* \bar{i}_1 + \nabla_{n12}^* \bar{i}_2 + \nabla_{n13}^* \bar{i}_3 \\ \vec{\nabla}_{n2}^* &= \nabla_{n21}^* \bar{i}_1 + \nabla_{n22}^* \bar{i}_2 + \nabla_{n23}^* \bar{i}_3 \\ \vec{\nabla}_{n3}^* &= \nabla_{n31}^* \bar{i}_1 + \nabla_{n32}^* \bar{i}_2 + \nabla_{n33}^* \bar{i}_3 \end{aligned} \tag{3-10a}$$

In matrix forms, the latter equation becomes

$$\begin{Bmatrix} \vec{\nabla}_{n1}^* \\ \vec{\nabla}_{n2}^* \\ \vec{\nabla}_{n3}^* \end{Bmatrix} = \begin{bmatrix} \nabla_{n11}^* & \nabla_{n12}^* & \nabla_{n13}^* \\ \nabla_{n21}^* & \nabla_{n22}^* & \nabla_{n23}^* \\ \nabla_{n31}^* & \nabla_{n32}^* & \nabla_{n33}^* \end{bmatrix} \begin{Bmatrix} \bar{i}_1 \\ \bar{i}_2 \\ \bar{i}_3 \end{Bmatrix}$$

$$\{\vec{\nabla}_n^*\} = [\nabla_n^*]\{\vec{i}\} \tag{3-10b}$$

Analogously for the curvilinear coordinate

$$\begin{Bmatrix} \vec{\nabla}_{n1}^* \\ \vec{\nabla}_{n2}^* \\ \vec{\nabla}_{n3}^* \end{Bmatrix} = \begin{bmatrix} \tilde{\nabla}_{n11}^* & \tilde{\nabla}_{n12}^* & \tilde{\nabla}_{n13}^* \\ \tilde{\nabla}_{n21}^* & \tilde{\nabla}_{n22}^* & \tilde{\nabla}_{n23}^* \\ \tilde{\nabla}_{n31}^* & \tilde{\nabla}_{n32}^* & \tilde{\nabla}_{n33}^* \end{bmatrix} \begin{Bmatrix} \tilde{i}_1^* \\ \tilde{i}_3^* \\ \tilde{i}_/^* \end{Bmatrix}$$

$$\{\vec{\nabla}_n\} = [\tilde{\nabla}_n^*]\{\tilde{i}^*\} \tag{3-10c}$$

Then,

$$[\nabla_n^*]\{\bar{i}\} = [\tilde{\nabla}_n^*]\{\tilde{i}^*\} \tag{3-10d}$$

According to Equation (1-12), it follows that

$$[\nabla_n^*]\{\bar{i}\} = [\tilde{\nabla}_n^*][A]^T\{\bar{i}\} \tag{3-10e}$$

or

$$[\nabla_n^*] = [\tilde{\nabla}_n^*][A]^T \tag{3-10f}$$

3.2 Transformation of Stress Components Under Change of Coordinate System

Consider another rectangular system x_1', x_2', x_3', the directions of whose axes relative to the axes of the first system x_1, x_2, x_3 are given by $[\Lambda]$.

Therefore, it follows from Equation (2-12b)

$$\{\sigma_{n1}^{*'}\} = [\Lambda]^T\{\sigma_{n1}^{*}\} \qquad (3\text{-}11a)$$

$$\{n_1^{*'}\} = [\Lambda]^T\{n_1^{*}\} \qquad (3\text{-}11b)$$

According from Equation (3-6b)

$$\{\sigma_{n1}^{*'}\} = [\sigma_o^{*'}]^T\{n_1^{*'}\} \qquad (3\text{-}11c)$$

$$[\Lambda]^T\{\sigma_{n1}^{*}\} = [\sigma_o^{*'}]^T[\Lambda]^T\{n_1^{*}\}$$

$$\{\sigma_{n1}^{*}\} = [\Lambda][\sigma_o^{*'}]^T[\Lambda]^T\{n_1^{*}\} \qquad (3\text{-}11d)$$

Comparing Equations (3-11d) and (3-6b), one obtains

$$[\sigma_o^{*}]^T = [\Lambda][\sigma_o^{*'}]^T[\Lambda]^T$$

or

$$[\sigma_o^{*'}]^T = [\Lambda]^T[\sigma_o^{*}]^T[\Lambda] \qquad (3\text{-}11e)$$

It will be shown later in this section that

$$[\sigma_o^{*}] = [\sigma_o^{*}]^T \qquad (3\text{-}12)$$

Thus Equation (3-11) is rewritten as follows:

$$[\sigma_o^{*'}] = [\Lambda]^T[\sigma_o^{*}][\Lambda] \qquad (3\text{-}13)$$

Comparing (3-13) with (2-14) one can see that the transformation of the stress components under a change of axes is similar to that of the strain $[\varepsilon]$.

For this reason, the series of results proved in the preceding chapter for the strain components are immediately asserted also for the stress components. Thus the principal normal stresses $\sigma_1^p, \sigma_2^p, \sigma_3^p$ (the extremal values of the normal stresses at the point M^*) and the principal axes of the state of stress (the directions of the normals to the axes on which those $\sigma_1^p, \sigma_2^p, \sigma_3^p$ act) are determined as follows: according to Equation (3-6c)

$$\{\sigma_n^*\} = [\sigma_o^*]\{n^*\} = \sigma^p[I]\{n^*\} \qquad (3\text{-}14a)$$

or

$$\big[[\sigma_o^*] - \sigma^p[I]\big]\{n^*\} = \{0\} \qquad (3\text{-}14b)$$

For the non-zero value of $\{n^*\}$

$$\big|[\sigma^*] - \sigma^p[I]\big| = 0 \qquad (3\text{-}14c)$$

which yields the characteristic equation of matrix which is solved directly for the eigen-values. The general form of Equation (3-14c)

$$(\sigma^p)^3 - C_2(\sigma^p)^2 + C_1(\sigma^p) - C_o = 0 \qquad (3\text{-}14d)$$

where

$$C_2 = \sigma_{11}^* + \sigma_{22}^* + \sigma_{33}^* = \sigma_1^p + \sigma_2^p + \sigma_8^p \qquad (3\text{-}15a)$$

$$\begin{aligned} C_1 &= \sigma_{11}^*\sigma_{22}^* + \sigma_{11}^*\sigma_{33}^* - \sigma_{12}^{*2} - \sigma_{13}^{*2} - \sigma_{23}^{*2} \\ &= \sigma_1^p\sigma_2^p + \sigma_1^p\sigma_3^p + \sigma_2^p\sigma_3^p \end{aligned} \qquad (3\text{-}15b)$$

$$\begin{aligned} C_o &= \sigma_{11}^*\sigma_{22}^*\sigma_{33}^* + 2\sigma_{12}^*\sigma_{13}^*\sigma_{23}^* - \sigma_{11}^*\sigma_{23}^{*2} - \sigma_{22}^*\sigma_{13}^{*2} - \sigma_{33}^*\sigma_{12}^{*2} \\ &= \sigma_1^p\sigma_2^p\sigma_3^p . \end{aligned} \qquad (3\text{-}15c)$$

3.3 Conditions for Equilibrium of an Elementary Volume Isolated from a Deformed Body

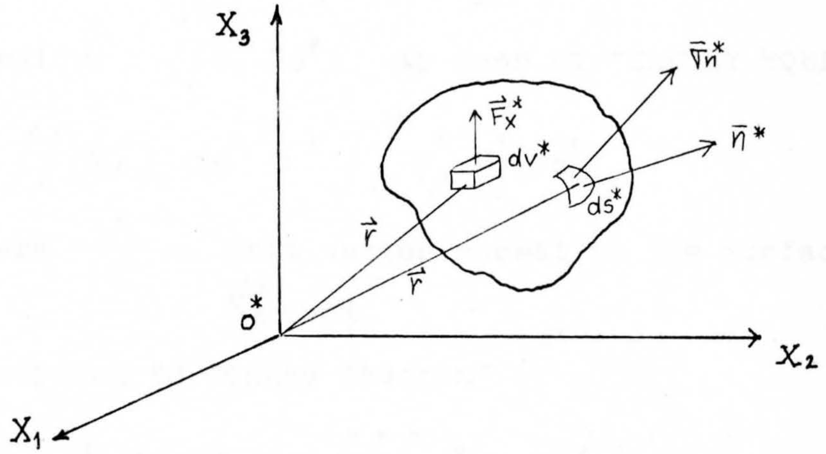


Figure (III-2) Equilibrium of an Elementary Volume

From a deformed body, the static equilibrium equation of an elementary volume is written in the form

$$\iiint \vec{F}_x^* \, dV^* + \iint \vec{\nabla}_n^* \, ds^* = 0 \qquad (3\text{-}16)$$

where $\vec{F}_X^*$ = the mean value of the specific body force (Body force pen unit volume)

$\vec{\nabla}_N^*$ = the stress on the surface area

$\vec{n}^*$ = the unit vector normal to the surface area (as shown in Figure (III-2)).

From the definition of "Guass Theorem"

$$\iiint \nabla_{ji}^* n_j^* \, ds^* = \iiint \nabla_{ji,j}^* \, dv^* \qquad (3\text{-}17)$$

where $\sigma_{ji}^* = [\sigma_o^*]$

$n_j^* = \vec{n}^* = \{n^*\}$

$\sigma_{ji,j}^* = \{\nabla\}^T[\sigma_o^*]$

Consider $\iint \vec{\sigma}_n^* \, ds^*$ by mean of "CAUCHY EQUATION"

$$\iint (\sigma_n^*)_i \, ds^* = \iint \sigma_{ji}^* n_j^* \, ds^* \tag{3-18a}$$

where n_j^* = unit vector normal to the surface area which $\vec{\sigma}_n^*$ act.

According to "Gauss Theorem"

$$\iint \sigma_{ji}^* n_j^* \, ds^* = \iiint \sigma_{ji,j}^* \, dv^* \tag{3-18b}$$

Thus Equation (3-16) is rewritten as

$$\iiint (F_x^*)_i \, dv^* + \iiint \sigma_{ji,j}^* \, dv^* = 0. \tag{3-19a}$$

$$\iiint \left((F_x^*)_i + \sigma_{ji,j}^*\right) dv^* = 0. \tag{3-19b}$$

$$\sigma_{ji,j}^* + (F_x^*)_i = 0. \tag{3-19c}$$

or

$$\begin{aligned} \frac{\partial \sigma_{11}^*}{\partial x_1^*} + \frac{\partial \sigma_{21}^*}{\partial x_2^*} + \frac{\partial \sigma_{31}^*}{\partial x_3^*} + F_{x1}^* &= 0 \\ \frac{\partial \sigma_{12}^*}{\partial x_1^*} + \frac{\partial \sigma_{22}^*}{\partial x_2^*} + \frac{\partial \sigma_{32}^*}{\partial x_3^*} + F_{x2}^* &= 0 \\ \frac{\partial \sigma_{13}^*}{\partial x_1^*} + \frac{\partial \sigma_{23}^*}{\partial x_2^*} + \frac{\partial \sigma_{33}^*}{\partial x_3^*} + F_{x3}^* &= 0 \end{aligned} \tag{3-19d}$$

Matrix form of Equation (3-19d) becomes

$$\{\nabla^*\}^T[\sigma_o^*] + \{F_x^*\}^T = \{0\}^T \tag{3-20}$$

where

$$\{\nabla^*\} = \begin{Bmatrix} \frac{\partial}{\partial x_1^*} \\ \frac{\partial}{\partial x_2^*} \\ \frac{\partial}{\partial x_3^*} \end{Bmatrix} \quad ; \quad \{F_x^*\} = \begin{Bmatrix} F_{x1}^* \\ F_{x2}^* \\ F_{x3}^* \end{Bmatrix} .$$

Equation (3-20) is the equation of equilibrium for every point in the deformed body. In accordance with the Figure III-2 and by using the indicial tensor notations and also the permutation symbol (3rd order tensor), it is shown that

$$\iiint (\vec{r} \times \vec{F}_x^*)\, dv^* = \iiint \epsilon_{ijk}\, X_j (F_x^*)_k\, dv^* \qquad (3\text{-}21a)$$

$$\iint (\vec{r} \times \vec{\nabla}_n^*)\, ds^* = \iint \epsilon_{ijk}\, X_j (\nabla_n^*)_k\, ds^* \qquad (3\text{-}21b)$$

where $\epsilon_{ijk} = 1$ if $i \neq j \neq k$

$\epsilon_{ijk} = 0$ if $i = j$, or $j = k$, or $k = i$

$\epsilon_{ijk} = -1$ if i, j, k are not in order, i.e., ikj etc.

According to the "CAUCHY EQUATION", Equation (3-21b) is rewritten in this form

$$\iint (\vec{r} \times \vec{\nabla}_n^*)\, ds^* = \iint \epsilon_{ijk}\, x_j^*\, \nabla_{\ell k}^*\, n_\ell^*\, ds^* \qquad (3\text{-}21c)$$

By using "Guass Theorem," one obtains

$$\iint (\vec{r} \times \vec{\nabla}_n^*)\, ds^* = \iiint (\epsilon_{ijk}\, x_j^*\, \nabla_{\ell k}^*)_{,\ell}\, dV^*$$

$$= \iiint (\epsilon_{ijk}\, X_{j,\ell}^*\, \nabla_{\ell k}^* + \epsilon_{ijk}\, X_j^*\, \nabla_{\ell k,\ell}^*)\, dV^* . \qquad (3\text{-}21d)$$

Now consider the following term,

$$X_{j,\ell}^* = \{\nabla^*\}\{X^*\}^T \qquad (3\text{-}21e)$$

$$X^*_{j,l} = \begin{bmatrix} \frac{\partial x_1^*}{\partial x_1^*} & \frac{\partial x_1^*}{\partial x_2^*} & \frac{\partial x_1^*}{\partial x_3^*} \\ \frac{\partial x_2^*}{\partial x_1^*} & \frac{\partial x_2^*}{\partial x_2^*} & \frac{\partial x_2^*}{\partial x_3^*} \\ \frac{\partial x_3^*}{\partial x_1^*} & \frac{\partial x_3^*}{\partial x_2^*} & \frac{\partial x_3^*}{\partial x_3^*} \end{bmatrix}$$

The off diagonal terms of $X^*_{j,l}$ are equal to zero. Thus, Equation (3-21e) is rewritten as follow

$$X^*_{j,l} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [I] = \delta_{jl} \qquad (3\text{-}21f)$$

where δ_{jl} = KRONECKER DELTA

if $j = l \quad \delta jl = 1$

$j \neq l \quad \delta jl = 0$

Equation (3-21d) is rewritten as follow

$$\iint (\vec{r} \times \vec{\sigma}_n^*) ds^* = \iiint (\epsilon_{ijk} \delta_{jl} \sigma^*_{lk} + \epsilon_{ijk} X_j^* \sigma^*_{lk,l}) dv^*$$

$$= \iiint (\epsilon_{ijk} \sigma^*_{jk} + \epsilon_{ijk} X_j^* \sigma^*_{lk,l}) dv^*. \qquad (3\text{-}21g)$$

In accordance with the Figure (III-2), the static equilibrium Equation (3-16) is written in the form

$$\iint (\vec{r} \times \vec{\sigma}_n^*) ds^* + \iiint (\vec{r} \times \vec{F}_x^*) dV^* = 0 . \qquad (3\text{-}22a)$$

Substuting Equations (3-21a), (3-21g) into Equation (3-22a) the equilibrium equation takes the form

$$\iiint (\epsilon_{ijk} \sigma^*_{jk} + \epsilon_{ijk} x_i^* \sigma^*_{lk,l} + \epsilon_{ijk} x_j^* (F_x^*)_k) dV^* = 0 , \qquad (3\text{-}22b)$$

$$\iiint \{ \epsilon_{ijk} \sigma^*_{jk} + \epsilon_{ijk} x^*_j (\sigma^*_{\ell k,\ell} + (F_x)^*_k) \} dv^* = 0 . \qquad (3\text{-}22c)$$

In accordance with Equation (3-19c), the second term of the left-hand side of Equation (3-22c) is equal to zero, thus, Equation (3-22c) is rewritten as follow

$$\iiint \epsilon_{ijk} \sigma^*_{jk} \, dv^* = 0 \qquad (3\text{-}22d)$$

$$\epsilon_{ijk} \sigma^*_{jk} = 0 . \qquad (3\text{-}22e)$$

For the value of $i = 1$.

$$\begin{aligned} \epsilon_{1jk} \sigma^*_{jk} &= \epsilon_{11k} \sigma^*_{1k} + \epsilon_{12k} \sigma^*_{2k} + \epsilon_{13k} \sigma^*_{3k} \\ &= \epsilon_{121} \sigma^*_{21} + \epsilon_{122} \sigma^*_{22} + \epsilon_{123} \sigma^*_{23} + \epsilon_{131} \sigma^*_{31} \\ &\quad + \epsilon_{132} \sigma^*_{32} + \epsilon_{133} \sigma^*_{33} . \end{aligned} \qquad (3\text{-}22f)$$

By using the properties of "Permutation symbol", Equation (3-22f) becomes

$$\epsilon_{ijk} \sigma^*_{jk} = \sigma^*_{23} - \sigma^*_{32} = 0 \qquad (3\text{-}22g)$$

$$\sigma^*_{23} = \sigma^*_{32} \qquad (3\text{-}22h)$$

Analogously

$$\sigma^*_{12} = \sigma^*_{21} \qquad (3\text{-}22h)$$

$$\sigma^*_{13} = \sigma^*_{31} . \qquad (3\text{-}22h)$$

Thus, it is concluded that

$$\sigma^*_{ij} = \sigma^*_{ji} \qquad (3\text{-}22i)$$

$$[\sigma^*_\circ] = [\sigma^*_\circ]^T \qquad (3\text{-}22j)$$

3.4 Transformation of the Equation of Equilibrium of an Element of Volume to the Cartesian Coordinates of the Points of the Body Before its Deformation

In passing from differentiation with respect to X_1^*, X_2^*, X_3^* to differentiation with respect to X_1, X_2, X_3 it follows, by using the definition of chain-rule, that

$$\{\nabla^*\} = [R_c]^T \{\nabla\} \tag{3-23a}$$

where

$$[R_c] = \begin{bmatrix} \frac{\partial X_1}{\partial X_1^*} & \frac{\partial X_1}{\partial X_2^*} & \frac{\partial X_1}{\partial X_3^*} \\ \frac{\partial X_2}{\partial X_1^*} & \frac{\partial X_2}{\partial X_2^*} & \frac{\partial X_3}{\partial X_3^*} \\ \frac{\partial X_3}{\partial X_1^*} & \frac{\partial X_3}{\partial X_2^*} & \frac{\partial X_3}{\partial X_3^*} \end{bmatrix} \tag{3-23b}$$

Also by use of the chain-rule, one obtains

$$\{dx\} = [R_c]\{dx^*\} \tag{3-23c}$$

In accordance with Equation (1-2), it follows that

$$[R_c] = [J]^{-1} \tag{3-23d}$$

$$= \frac{[\alpha]}{|[J]|} = \frac{[COF[J]]^T}{|[J]|} \tag{3-23e}$$

$$[R_c]^T = \frac{[\alpha]^T}{|[J]|} \tag{3-23f}$$

Then Equation (3-23a) is written in the form

$$\{\nabla^*\} = \frac{[\alpha]^T}{|[J]|}\{\nabla\} \tag{3-24a}$$

$$\{\nabla^*\}^T = \{\nabla\}^T \frac{[\alpha]}{|[J]|} \tag{3-24b}$$

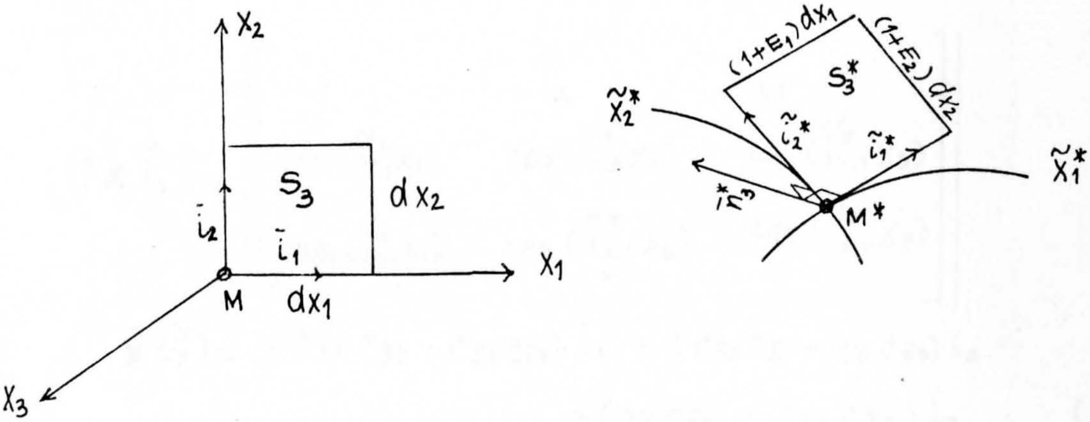
Note $\{\nabla\}^T$ does not operate on $\frac{[\alpha]}{|[J]|}$

Therefore, Equation (3-20) is rewritten as follow

$$\{\nabla\}^T \frac{[\alpha]}{|[J]|}[\nabla_0^*] + \{F_x^*\}^T = \{0\}^T$$

$$\{\nabla\}^T[\alpha][\nabla_0^*] + |[J]|\{F_x^*\}^T = \{0\}^T \qquad (3\text{-}25)$$

The combination of matrix $[\alpha]$ and $[\nabla_0^*]$ has a definite physical meaning which is interpretable by the following considerations :



Before Deformation After Deformation

<u>Figure(III-3)</u> Geometry of Undeformed and Deformed Elements

Supposing that a rectangular area perpendicular to X_3 - axis and with sides dx_1, dx_2 is isolated from the body before the deformation. As a result of the deformation, this area becomes a parallelogram, the directions of whose sides are given by $\tilde{i}_1^*$, $\tilde{i}_2^*$ (Equation(1-13b)). Consequently, the unit vector in the direction of the normal to the given area is found from the equation

$$(\tilde{i}_1^* \times \tilde{i}_2^*) = \tilde{n}_3^* \sin(\tilde{i}_1^*, \tilde{i}_2^*) \qquad (3\text{-}26a)$$

where

$$\tilde{i}_1^* = \cos(\tilde{i}_1^*, X_1)\,\bar{i}_1 + \cos(\tilde{i}_1^*, X_2)\,\bar{i}_2 + \cos(\tilde{i}_1^*, X_3)\,\bar{i}_3$$

$$\tilde{i}_2^* = \cos(\tilde{i}_2^*, X_1)\,\bar{i}_1 + \cos(\tilde{i}_2^*, X_2)\,\bar{i}_2 + \cos(\tilde{i}_2^*, X_3)\,\bar{i}_3$$

$\tilde{n}_3^*$ = the unit vector in the direction of the normal to the plane of $\tilde{i}_1^*$ and $\tilde{i}_2^*$.

$$(\tilde{i}_1^* \times \tilde{i}_2^*) = \left\|\begin{bmatrix} \bar{i}_1 & \bar{i}_2 & \bar{i}_3 \\ \cos(\tilde{i}_1^*, X_1) & \cos(\tilde{i}_1^*, X_2) & \cos(\tilde{i}_1^*, X_3) \\ \cos(\tilde{i}_2^*, X_1) & \cos(\tilde{i}_2^*, X_2) & \cos(\tilde{i}_2^*, X_3) \end{bmatrix}\right\|$$

$$(\tilde{i}_1^* \times \tilde{i}_2^*) = (C_{\tilde{1}2} C_{\tilde{2}3} - C_{\tilde{2}2} C_{\tilde{1}3})\,\bar{i}_1 + (C_{\tilde{1}3} C_{\tilde{2}1} - C_{\tilde{1}1} C_{\tilde{2}3})\,\bar{i}_2$$
$$+ (C_{\tilde{1}1} C_{\tilde{2}2} - C_{\tilde{2}1} C_{\tilde{1}2})\,\bar{i}_3 \qquad (3\text{-}26b)$$

where

$$C_{\tilde{1}2} \Rightarrow \cos(\tilde{i}_1^*, X_2), \quad C_{\tilde{1}1} \Rightarrow \cos(\tilde{i}_1^*, X_1), \quad C_{\tilde{2}3} \Rightarrow \cos(\tilde{i}_2^*, X_3), \text{ etc.}$$

Analogously

$$(\tilde{i}_3^* \times \tilde{i}_1^*) = (C_{\tilde{3}2} C_{\tilde{1}3} - C_{\tilde{1}2} C_{\tilde{3}3})\,\bar{i}_1 + (C_{\tilde{1}1} C_{\tilde{3}3} - C_{\tilde{3}1} C_{\tilde{1}3})\,\bar{i}_2$$
$$+ (C_{\tilde{3}1} C_{\tilde{1}2} - C_{\tilde{1}1} C_{\tilde{3}2})\,\bar{i}_3 \qquad (3\text{-}26c)$$

$$(\tilde{i}_2^* \times \tilde{i}_3^*) = (C_{\tilde{2}2} C_{\tilde{3}3} - C_{\tilde{2}3} C_{\tilde{3}2})\,\bar{i}_1 + (C_{\tilde{3}1} C_{\tilde{2}3} - C_{\tilde{2}1} C_{\tilde{3}3})\,\bar{i}_2$$
$$+ (C_{\tilde{2}1} C_{\tilde{3}2} - C_{\tilde{3}1} C_{\tilde{2}2})\,\bar{i}_3 \qquad (3\text{-}26d)$$

In matrix forms, these equations become

$$\{\tilde{i}_\alpha^* \times \tilde{i}_\beta^*\} = [\mathrm{COF}[A]]^T \{\bar{i}\} \qquad (3\text{-}26e)$$

where

$$\{\tilde{i}_\alpha^* \times \tilde{i}_\beta^*\} = \begin{Bmatrix} \tilde{i}_2^* \times \tilde{i}_3^* \\ \tilde{i}_3^* \times \tilde{i}_1 \\ \tilde{i}_1^* \times \tilde{i}_2^* \end{Bmatrix}$$

In accordance with Equation (3-26a), it follows that

$$\{\tilde{i}^*_\alpha \times \tilde{i}^*_\beta\} = [\sin]\{\tilde{n}^*\} \tag{3-27a}$$

where

$$[\sin] = \begin{bmatrix} \sin(\tilde{i}^*_2, \tilde{i}^*_3) & 0 & 0 \\ 0 & \sin(\tilde{i}^*_3, \tilde{i}^*_1) & 0 \\ 0 & 0 & \sin(\tilde{i}^*_1, \tilde{i}^*_2) \end{bmatrix} \tag{3-27b}$$

$\{\tilde{n}^*\} = \begin{Bmatrix} \tilde{n}^*_1 \\ \tilde{n}^*_2 \\ \tilde{n}^*_3 \end{Bmatrix}$ = unit vectors normal to the planes $\tilde{i}^*_2 - \tilde{i}^*_3$, $\tilde{i}^*_3 - \tilde{i}^*_1$, $\tilde{i}^*_1 - \tilde{i}^*_2$ respectively. (3-27c)

By the comparison of the Equations (3-27a) and (3-26e), one obtains

$$[\sin]\{\tilde{n}^*\} = [\mathrm{COF}[A]]^T\{\bar{i}\}\,. \tag{3-28}$$

Referring to Figure (III-3), the following equations hold:

$S_3 = dx_1\,dx_2$ = area of the rectangular before deformation

$$S^*_3 = (1+E_1)(1+E_2)\,dx_1\,dx_2\,\sin(\tilde{i}^*_1, \tilde{i}^*_2)$$

where S^*_3 = area of the parallelogram after deformation.

Thus,

$$\frac{S^*_3}{S_3} = (1+E_1)(1+E_2)\sin(\tilde{i}^*_1, \tilde{i}^*_2) \tag{3-29a}$$

Analogously

$$\frac{S^*_2}{S_2} = (1+E_1)(1+E_3)\sin(\tilde{i}^*_3, \tilde{i}^*_1) \tag{3-29b}$$

$$\frac{S^*_1}{S_2} = (1+E_2)(1+E_3)\sin(\tilde{i}^*_2, \tilde{i}^*_3) \tag{3-29c}$$

In matrix form, the latter equations become

$$[S^*/S] = [COF[1+E]][\sin] \tag{3-29d}$$

where

$$[S^*/S] = \begin{bmatrix} S_1^*/S_1 & 0 & 0 \\ 0 & S_2^*/S_2 & 0 \\ 0 & 0 & S_3^*/S_3 \end{bmatrix} \tag{3-29e}$$

Consider, the unit vectors normal to $\tilde{i}_1^* - \tilde{i}_2^*$, $\tilde{i}_2^* - \tilde{i}_3^*$, $\tilde{i}_1^* - \tilde{i}_3^*$ planes, they are expressed as follows:

$$\begin{aligned} \tilde{n}_1^* &= \cos(\tilde{n}_1^*, x_1)\,\bar{i}_1 + \cos(\tilde{n}_1^*, x_2)\,\bar{i}_2 + \cos(\tilde{n}_1^*, x_3)\,\bar{i}_3 \\ \tilde{n}_2^* &= \cos(\tilde{n}_2^*, x_2)\,\bar{i}_1 + \cos(\tilde{n}_2^*, x_2)\,\bar{i}_2 + \cos(\tilde{n}_2^*, x_3)\,\bar{i}_3 \\ \tilde{n}_3^* &= \cos(\tilde{n}_3^*, x_1)\,\bar{i}_1 + \cos(\tilde{n}_3^*, x_2)\,\bar{i}_2 + \cos(\tilde{n}_3^*, x_3)\,\bar{i}_3 . \end{aligned} \tag{3-30a}$$

Thus,

$$\{\tilde{n}^*\} = [C]^T \{\bar{i}\} \tag{3-30b}$$

where

$$\{\tilde{n}^*\} = \begin{Bmatrix} \tilde{n}_1^* \\ \tilde{n}_2^* \\ \tilde{n}_3^* \end{Bmatrix} ; \quad [C] = \begin{bmatrix} \cos(\tilde{n}_1^*, x_1) & \cos(\tilde{n}_2^*, x_1) & \cos(\tilde{n}_3^*, x_1) \\ \cos(\tilde{n}_1^*, x_2) & \cos(\tilde{n}_2^*, x_2) & \cos(\tilde{n}_3^*, x_2) \\ \cos(\tilde{n}_1^*, x_3) & \cos(\tilde{n}_2^*, x_3) & \cos(\tilde{n}_3^*, x_3) \end{bmatrix}$$

Substituting Equation (3-30b) into Equation (3-28), and combining gives

$$[\sin][C]^T\{\bar{i}\} = [COF[A]]^T\{\bar{i}\} \tag{3-31a}$$

or

$$[\sin][C]^T = [COF[A]]^T \tag{3-31b}$$

Multiplying both sides by $[COF[1+E]]$ gives

$$[COF[1+E]][\sin][C]^T = [COF[1+E]][COF[A]]^T$$

According to Equation (1-18), the above equation is rewritten in the form

$$[COF[1+E]][\sin][C]^T = [\alpha] \tag{3-31c}$$

Again according to Equation (3-29d), it is rewritten as

$$[s^*/s][C]^T = [\alpha] \tag{3-31d}$$

Substituting Equation (3-31d) into the Equation (3-25), one obtains

$$\{\nabla\}^T[s^*/s][C]^T[\nabla_o^*] + |[J]|\{F_x^*\}^T = \{0\}^T \tag{3-32a}$$

By substituting $[\nabla_o^*]$ from Equation (3-9b) gives

$$\{\nabla\}^T[s^*/s][C]^T[C]^{-T}[\nabla_n^*] + |[J]|\{F_x^*\}^T = \{0\}^T \tag{3-32b}$$

or

$$\{\nabla\}^T[s^*/s][\nabla_n^*] + |[J]|\{F_x^*\}^T = \{0\}^T \tag{3-32c}$$

Equation (3-32c) is assumed by Equation (3-20) if the positions of the points of the deformed body are determined not by the Cartesian coordinates x_1^*, x_2^*, x_3^* but by the curvilinear coordinates $\tilde{x}_1^*, \tilde{x}_2^*, \tilde{x}_3^*$ (which are the Cartesian co-ordinates for the body in its initial state). Thus, in changing matrix $[\nabla_n^*]$ to the matrix $[\tilde{\nabla}_n^*]$ (in the direction $\tilde{i}_1^*, \tilde{i}_2^*, \tilde{i}_3^*$) by using Equation (3-10f) gives

$$\{\nabla\}^T[s^*/s][\tilde{\nabla}_n^*][A]^T + |[J]|\{F_x^*\}^T = \{0\}^T \tag{3-32d}$$

By setting up matrix $[\nabla_R]$

$$[\nabla_R] = [S^*/S][\tilde{\nabla}_n^*][\frac{1}{1+E}] \qquad (3\text{-}33)$$

where

$[\nabla_R]$ defined by Equation (3-33) are not, strictly speaking, stresses. They can be called stresses referred to the dimensions of an element of volume before, not after the deformation.

Thus Equation (3-32d) is rewritten as

$$\{\nabla\}^T[S^*/S][\tilde{\nabla}_n^*][\frac{1}{1+E_1}][J]^T + |[J]|\{F_x^*\}^T = \{0\}^T \qquad (3\text{-}34a)$$

$$\{\nabla\}^T[\nabla_R][J]^T + |[J]|\{F_x^*\}^T = \{0\}^T \qquad (3\text{-}34b)$$

Equation (3-34b) comprises the equations of equilibrium of the nonlinear theory (Case 1).

3.5 Simplification of the Equations of Equilibrium in the Case of Small Elongations and Shears (Case 2)

The ratios $\frac{S_1^*}{S_1}, \frac{S_2^*}{S_2}, \frac{S_3^*}{S_3}, \frac{V^*}{V}$ differ from unity only by magnitudes of the same order as the elongations and shears. Hence, they are set equal to unity for small deformations and also the conditions $E_{1,2,3} \ll 1$, the Equation (3-32c) assumes the form

$$\{\nabla\}^T[\nabla_n^*] + \{F_x^*\}^T \simeq \{0\}^T \qquad (3\text{-}35a)$$

and Equation (3-32d) changes to

$$\{\nabla\}^T[\tilde{\nabla}_n^*][A]^T + \{F_x^*\}^T \simeq \{0\}^T \qquad (3\text{-}35b)$$

In addition, neglecting the relative elongations E_1, E_2, E_3 in comparison with unity, Equation (3-33) is rewritten as

$$[\nabla_R] = [\tilde{\nabla}_n^*] \qquad (3\text{-}36)$$

Equation (3-34b) is rewritten as

$$\{\nabla\}^T[\tilde{\nabla}_n^*][J]^T + \{F_x^*\}^T \simeq \{0\}^T \tag{3-37a}$$

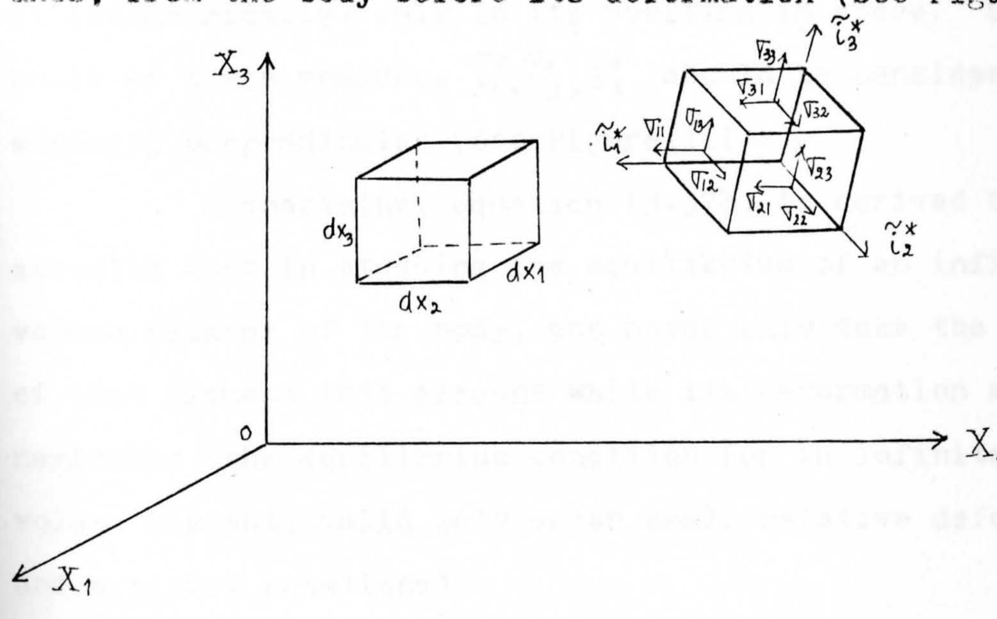
For convenience, matrix $[\tilde{\nabla}_n^*]$ is replaced by $[\nabla]$ where

$$[\nabla] = \begin{bmatrix} \nabla_{11} & \nabla_{12} & \nabla_{13} \\ \nabla_{21} & \nabla_{22} & \nabla_{23} \\ \nabla_{31} & \nabla_{32} & \nabla_{33} \end{bmatrix}.$$

Thus, Equation (3-37a) is rewritten in the form

$$\{\nabla\}^T[\nabla][J]^T + \{F_x^*\}^T \simeq \{0\}^T \tag{3-37b}$$

A diagram is used to clarify the geometrical nature of the simplifications of these equations. Isolate a rectangular parallelopiped, with edges dx_1, dx_2, dx_3 parallel to the X_1, X_2, X_3 axes, from the body before its deformation (See Figure(III-4)).



Figure(III-4) Rectangular Parallelopiped
Before and After Deformation

As a result of the deformation, this rectangular parallelopiped becomes an oblique one, with edges $(1+E_1)dx_1$, $(1+E_2)dx_2$, $(1+E_3)dx_3$ forming the angles $(\pi/2-\phi_{12})$, $(\pi/2-\phi_{13})$, $(\pi/2-\phi_{23})$.

However, the angles of rotation are large relative to the shears ϕ_{12}, ϕ_{13}, ϕ_{23} then ϕ_{12}, ϕ_{13}, ϕ_{23} may be neglected in comparison with the former in projecting the forces. This means that the examined parallelopiped can also be represented by a rectangular one after deformation (Figure (III-4)). Moreover, the smallness of the elongations and shears allows one to ignore distinctions between its dimensions before and after deformation. It is thus permissible to represent the parallelopiped after the deformation, as equal to the paparallelopiped before the deformation, but differing from it (geometrically) only in its position in space. On the basis of these remarks, $\tilde{i}_1^*$, $\tilde{i}_2^*$, $\tilde{i}_3^*$ should be considered as mutually perpendicular (See Figure (III-4)).

Summarizing, Equation (3-37b) is derived by assuming that in studying the equilibrium of an infinitesimal volume element of the body, one needs only take the rotation of that element into account while its deformation may be neglected (the equilibrium condition for an infinitesimal volume element, valid only under small relative deformation and arbitary rotations).

3.6 Case 3, Simplification of the Equilibrium Equations for small Rotations

If the angles of rotation are small compared to unity, then, by (section 2.9), the parameters $[e]$ differs from the strain components $[\varepsilon]$ only by quantities of the same order as the squares of the angles of rotation. Thus, Equations (3-37b) are simplified by neglecting the strains and the squares of the angles of rotation as compared to the first powers of the angle of rotation.

Consider matrix $[J]$

$$[J] = [I] + [e] + [\omega] \tag{3-38a}$$

By the above remarks, it reduces to

$$[J] \approx [I] + [\omega] \tag{3-38b}$$

Thus, Equation (3-37b) is rewritten as

$$\{\nabla\}^T [\nabla] [[I] - [\omega]] + \{F_x^*\}^T = \{0\}^T \tag{3-39a}$$

$$\{\nabla\}^T [[\nabla] - [\nabla][\omega]] + \{F_x^*\}^T = \{0\}^T. \tag{3-39b}$$

3.7 Case 4, Transition to the Classical Equations of Equilibrium

The next step in the simplifying process is to assume that the angles of rotation are so small that the terms in Equation (3-39b) which contain them as factors are neglected in comparison with the terms which do not.

Equation (3-39b) then reduces to

$$\{\nabla\}^T [\nabla] + \{F_x\}^T = \{0\}^T. \tag{3-40a}$$

Equation (3-40a) is derived by neglecting the rotations of volume element when all the forces acting on it are projected, i.e., by identifying the direction $\tilde{i}_1^*$, $\tilde{i}_2^*$, $\tilde{i}_3^*$ with X_1, X_2, X_3.

In this case, the stress components $[\nabla]$ in the directions of the local trihedral of the curvilinear co-ordinate system $\tilde{i}_1^*, \tilde{i}_2^*, \tilde{i}_3^*$ are identical with $[\nabla_o^*]$ the stress components along the X_1-, X_2-, X_3- axes. Hence Equation (3-40a) are also be written in the form

$$\{\nabla\}^T[\nabla_o^*] + \{F_x\}^T = \{0\}^T, \qquad (3\text{-}40b)$$

which combined with the Equation (3-22j) of the form

$$[\nabla_o^*]^T = [\nabla_o^*]$$

are the conditions of equilibrium for a volume element in the classical theory of elasticity.

3.8 Transition to Curvilinear Coordinates

In the preceding discussion the points of the body are referred to a Cartesian coordinate system. Such a coordinate system is convenient for bodies which are bounded by mutually perpendicular planes, but is much less convenient if the body is bounded by curved surfaces. Hence the curvilinear coordinates should always be selected in such a way that the bounding surfaces of the body should at the same time be also coordinate surfaces. This will result in an especially simple formulation of the boundary conditions. In this connection a discussion of the conditions of equilibrium for a body whose points are referred to an arbitary orthogonal curvilinear coordinate system $\alpha_1, \alpha_2, \alpha_3$ follows.

To shorten the calculations involved in this transformation it has been already noted, that the equations of equilibrium of a volume element in the nonlinear theory are similar in appearance to the corresponding equations of the classical theory.

In the nonlinear theory the conditions of equilibrium for an element referred to Cartesian coordinates reduce to the Equation (3-32d)

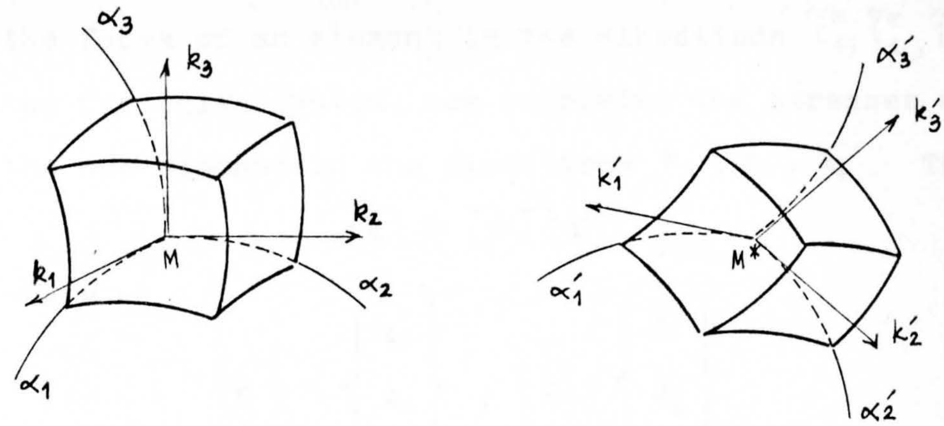
$$\{\nabla\}^T [S^*/S][\tilde{\nabla}_n^*][\tfrac{1}{1+E}][J]^T + |[J]|\{F_x^*\}^T = \{0\}^T \tag{3-32d}$$

thus, in the linear theory, assumes the form

$$\{\nabla\}^T [\nabla] + \{F_x\}^T = \{0\}^T \tag{3-40a}$$

thus

$$[\nabla] \Leftarrow [S^*/S][\tilde{\nabla}_n^*][\tfrac{1}{1+E}][J]^T. \tag{3-41}$$



Before Deformation After Deformation

Figure(III-5) Curvilinear Coordinate System

If the points of the body are referred to curvilinear coordinates coordinates (Chapter II, Section 2), an infinitesimal volume element is isolated which is bounded by the six coordinate surfaces of the curvilinear system chosen. As a result of the deformation, this element changes its position in space (due to displacement and rotation) and, moreover, changes its dimensions and form. Its edges, initially equal to $k_1 H_1 d\alpha_1$, $k_2 H_2 d\alpha_2$, $k_3 H_3 d\alpha_3$, now become

$$k_1' H_1 (1+E_{\alpha 1}) d\alpha_1 , k_2' H_2 (1+E_{\alpha 2}) d\alpha_2 , k_3' H_3 (1+E_{\alpha 3}) d\alpha_3$$

where k_1', k_2', k_3' are the unit vectors in the directions of the linear elements which, in the unstrained state, coincided with the vectors k_1, k_2, k_3.

The cosines of the angles between the trihedrals k_1, k_2, k_3 and k_1', k_2', k_3' are given by Equation (1-12) in which the values of the parameters matrix $[\tilde{e}]$ and $[\tilde{\omega}]$ are determined from Equations (2-71) and (2-72).

In analogy with the resolution of the stresses acting on the faces of an element in the directions $\tilde{i}_1^*, \tilde{i}_2^*, \tilde{i}_3^*$ in the Cartesian system, now resolving the stresses acting on the new element in the directions k_1', k_2', k_3'. Thus,

$$\{k'\} = [A]^T \{k\} \tag{3-42}$$

where

$$\{k'\} = \begin{Bmatrix} k_1' \\ k_2' \\ k_3' \end{Bmatrix} \; ; \; \{k\} = \begin{Bmatrix} k_1 \\ k_2 \\ k_3 \end{Bmatrix}$$

In books on the classical theory of elasticity (See, e.g., Love's Mathematical Theory of Elasticity, P. 90) it is proved that, in an orthogonal curvilinear coordinate system, Equation (3-40a) is replaced by the following three scalar equations:

$$\frac{1}{H_1H_2H_3}\left\{\frac{\partial}{\partial\alpha_1}(H_2H_3\sigma_{\alpha 11}) + \frac{\partial}{\partial\alpha_2}(H_3H_1\sigma_{\alpha 21}) + \frac{\partial}{\partial\alpha_3}(H_1H_2\sigma_{\alpha 31})\right\} + \frac{1}{H_1H_2}\frac{\partial H_1}{\partial\alpha_2}\sigma_{\alpha 12}$$
$$+ \frac{1}{H_1H_3}\frac{\partial H_1}{\partial\alpha_3}\sigma_{\alpha 13} - \frac{1}{H_1H_2}\frac{\partial H_2}{\partial\alpha_1}\sigma_{\alpha 22} - \frac{1}{H_1H_3}\frac{\partial H_3}{\partial\alpha_1}\sigma_{\alpha 33} + F_{\alpha 1} = 0$$

$$\frac{1}{H_1H_2H_3}\left\{\frac{\partial}{\partial\alpha_1}(H_2H_3\sigma_{\alpha 12}) + \frac{\partial}{\partial\alpha_2}(H_3H_1\sigma_{\alpha 22}) + \frac{\partial}{\partial\alpha_3}(H_1H_2\sigma_{\alpha 32})\right\} + \frac{1}{H_2H_3}\frac{\partial H_2}{\partial\alpha_3}\sigma_{\alpha 23}$$
$$+ \frac{1}{H_2H_1}\frac{\partial H_2}{\partial\alpha_1}\sigma_{\alpha 21} - \frac{1}{H_2H_3}\frac{\partial H_3}{\partial\alpha_2}\sigma_{\alpha 33} - \frac{1}{H_2H_1}\frac{\partial H_1}{\partial\alpha_2}\sigma_{\alpha 11} + F_{\alpha 2} = 0 \tag{3-43}$$

$$\frac{1}{H_1H_2H_3}\left\{\frac{\partial}{\partial\alpha_1}(H_2H_3\sigma_{\alpha 13}) + \frac{\partial}{\partial\alpha_2}(H_3H_1\sigma_{\alpha 23}) + \frac{\partial}{\partial\alpha_3}(H_1H_2\sigma_{\alpha 33})\right\} + \frac{1}{H_3H_1}\frac{\partial H_3}{\partial\alpha_1}\sigma_{\alpha 31}$$
$$+ \frac{1}{H_3H_2}\frac{\partial H_3}{\partial\alpha_2}\sigma_{\alpha 32} - \frac{1}{H_3H_1}\frac{\partial H_1}{\partial\alpha_3}\sigma_{\alpha 11} - \frac{1}{H_3H_2}\frac{\partial H_2}{\partial\alpha_3}\sigma_{\alpha 22} + F_{\alpha 3} = 0$$

Here

H_1, H_2, H_3 are the Lame coefficients Equation (2-63b)

$F_{\alpha 1}, F_{\alpha 2}, F_{\alpha 3}$ are the projections of the specific body force on the directions $\alpha_1, \alpha_2, \alpha_3$

$\vec{\nabla}_{\alpha 1}, \vec{\nabla}_{\alpha 2}, \vec{\nabla}_{\alpha 3}$ are the stresses on the areas perpendicular to the dihedrals $[k_2', k_3']$, $[k_3', k_1']$, $[k_1', k_2']$

$\nabla_{\alpha 11}, \nabla_{\alpha 12}, \nabla_{\alpha 13}$ are the components of the stress $\vec{\nabla}_{\alpha 1}$ along k_1, k_2, k_3 ;

$\nabla_{\alpha 21}, \nabla_{\alpha 22}, \nabla_{\alpha 23}$ are the components of the stress $\vec{\nabla}_{\alpha 2}$ along k_1, k_2, k_3 ;

$\nabla_{\alpha 31}, \nabla_{\alpha 32}, \nabla_{\alpha 33}$ are the components of the stress $\vec{\nabla}_{\alpha 3}$ along k_1, k_2, k_3 .

In the linear theory no distinction is made between k_1, k_2, k_3 and k_1', k_2', k_3'. Equation (3-43) are the equations of equilibrium of the linear theory referred to the orthogonal curvilinear coordinates $\alpha_1, \alpha_2, \alpha_3$.

Hence,

$$\{\nabla_\alpha\} = [\nabla_\alpha]\{k\} \tag{3-44a}$$

where

$$[\nabla_\alpha] = \begin{bmatrix} \nabla_{\alpha 11} & \nabla_{\alpha 12} & \nabla_{\alpha 13} \\ \nabla_{\alpha 21} & \nabla_{\alpha 22} & \nabla_{\alpha 23} \\ \nabla_{\alpha 31} & \nabla_{\alpha 32} & \nabla_{\alpha 33} \end{bmatrix} \tag{3-44b}$$

$$\{\nabla_\alpha\} = \begin{Bmatrix} \vec{\nabla}_{\alpha 1} \\ \vec{\nabla}_{\alpha 2} \\ \vec{\nabla}_{\alpha 3} \end{Bmatrix} \tag{3-44c}$$

Turning now to the nonlinear theory and taking into account the similarity of Equation (3-41), it may be concluded that

$$[\sigma_\alpha] = [s^*/s][\tilde{\sigma}_\alpha^*][A] \tag{3-45}$$

where

$$[\tilde{\sigma}_\alpha^*] = \begin{bmatrix} \tilde{\sigma}_{\alpha 11}^* & \tilde{\sigma}_{\alpha 12}^* & \tilde{\sigma}_{\alpha 13}^* \\ \tilde{\sigma}_{\alpha 21}^* & \tilde{\sigma}_{\alpha 22}^* & \tilde{\sigma}_{\alpha 23}^* \\ \tilde{\sigma}_{\alpha 31}^* & \tilde{\sigma}_{\alpha 32}^* & \tilde{\sigma}_{\alpha 33}^* \end{bmatrix}$$

$[\tilde{\sigma}_\alpha^*]$ = the projections of the stresses $\vec{\sigma}_{n1}^*$, $\vec{\sigma}_{n2}^*$, $\vec{\sigma}_{n3}^*$ on k_1', k_2', k_3' (after deformation).

In accordance with Equation (1-14), Equation (3-45) becomes

$$[\sigma_\alpha] = [s^*/s][\tilde{\sigma}_\alpha^*]\left[\frac{1}{1+E}\right][J]^T \tag{3-46}$$

Denoting

$$[\sigma_\alpha^R] = [s^*/s][\tilde{\sigma}_\alpha^*]\left[\frac{1}{1+E}\right] \tag{3-47}$$

where

$$[\sigma_\alpha^R] = \begin{bmatrix} \sigma_{\alpha 11}^R & \sigma_{\alpha 12}^R & \sigma_{\alpha 13}^R \\ \sigma_{\alpha 21}^R & \sigma_{\alpha 22}^R & \sigma_{\alpha 23}^R \\ \sigma_{\alpha 31}^R & \sigma_{\alpha 32}^R & \sigma_{\alpha 33}^R \end{bmatrix}$$

Then Equation (3-46) becomes

$$[\sigma_\alpha] = [\sigma_\alpha^R][J]^T \tag{3-48a}$$

$$[\sigma_\alpha] = [\sigma_\alpha^R][[I]+[e]-[\omega]] \tag{3-48b}$$

In order to transform Equation (3-43) into the equations of the nonlinear theory, besides replacing the Equation (3-48a), it is also necessary to replace $F_{\alpha 1}, F_{\alpha 2}, F_{\alpha 3}$ respectively, by

$$\frac{V^*}{V}F_{\alpha 1}^*\ ,\quad \frac{V^*}{V}F_{\alpha 2}^*\ ,\quad \frac{V^*}{V}F_{\alpha 3}^*$$

where $F^*_{\alpha 1}, F^*_{\alpha 2}, F^*_{\alpha 3}$ are the projections on k_1, k_2, k_3 of the specific body forces relative to the strained body, while

$$V^*/V = |[J]| = (1+\Delta)$$

where Δ = the volume increment.

The above rules for transforming the system (Equation (3-34b)) to the orthogonal curvilinear coordinates are established without neglecting any terms (Case 1). Hence substitution of Equation (3-48a) into Equation(3-43) will make the latter correspond precisely to Equation (3-34b). For Case 2, if the elongations and shears are negligibly small compared to unity, Equation (3-48a) is simplified by identifying the matrix $[\nabla^R_\alpha]$ with the matrix $[\tilde{\nabla}^*_\alpha]$ (section 3.5). For Case 3, in addition, the angles of rotation are small compared to unity, Equation (3-48) becomes

$$[\nabla_\alpha] = [\tilde{\nabla}^*_\alpha][[I] - [\omega]] \tag{3-49}$$

Finally Case 4, the angles of rotation are small quantities of the same order of magnitudes as the strain components, the products of the stresses by the angles of rotation are neglected in Equation (3-49). The result is

$$[\nabla_\alpha] = [\tilde{\nabla}^*_\alpha] \tag{3-50}$$

In this case, Equation (3-43) become identical with the equations of equilibrium of the linear theory referred to the orthogonal curvilinear coordinates $\alpha_1, \alpha_2, \alpha_3$.

3.9 Summary

Case 1 General Nonlinear Equilibrium Equations

$$\{\nabla\}^T[s^*/s][\tilde{\nabla}_n^*][\tfrac{1}{1+E}][J]^T + |[J]|\{F_x^*\}^T = \{0\}^T$$

or

$$\{\nabla\}^T[\nabla][J]^T + |[J]|\{F_x^*\}^T = \{0\}^T$$

where

$$[\nabla] = [s^*/s][\tilde{\nabla}_n^*][J]^T$$

Case 2 Small Deformation

The elongations and shear parameters are small in comparison to unity:

Thus,

$$E_{1,2,3} \ll 1 \quad , [s^*/s] \approx [I] \quad , \quad |[J]| \approx 1.$$

Then,

$$[\nabla] \approx [\tilde{\nabla}_n^*]$$

and

$$\{\nabla\}^T[\nabla][J]^T + \{F_x^*\}^T \simeq \{0\}^T$$

Case 3 Small Deformations and Small Angles of Rotation

In addition to the elongations and shear parameters the rotation angles are small in comparison to unity:

Thus,

$$[J] \approx [I] + [\omega]$$

or

$$[J]^T \approx [I] - [\omega]$$

Noting the equation of Case 2

$$\{\nabla\}^T[\nabla][J]^T + \{F_x^*\}^T = \{0\}^T$$

The following reduction occurs

$$\{\nabla\}^T[\nabla][[I]-[\omega]] + \{F_x^*\}^T = \{0\}^T$$

or

$$\{\nabla\}^T[[\nabla]-[\nabla][\omega]] + \{F_x^*\}^T = \{0\}^T.$$

Case 4 Transition to the Classical Equations of Equilibrium

Neglecting $[\nabla][\omega]$ compared to $[\nabla]$, in Case 3 it follows that

$$\{\nabla\}^T[\nabla] + \{F_x^*\}^T \simeq \{0\}^T$$

and noting Equations (3-40a) and (3-20)

$$\{\nabla\}^T \approx \{\nabla^*\}^T$$

$$[\nabla] \approx [\nabla_o^*] .$$

CHAPTER IV

STRAIN ENERGY, BOUNDARY CONDITIONS, STRESS-STRAIN LAW

4.1 Strain Energy

The system of differential equations derived in the last chapter, which expresses the conditions of body in a state of strain, contains more unknowns than equations. Indeed, it consists of six Equations (3-34b); (3-22j) containing twelve unknowns (nine stresses and three displacement components).

Hence, the problem of the equilibrium of a deformed solid body remains indeterminate until six supplementary equations are established. These relate the stress components to the displacement components and express the law according to which the material of the given body resists various forms of deformation. But at the present time, the relation between stresses and strains, which differs for different materials, is established mainly by experiment. Some general properties inherent in this relation can, however, be explained theoretically.

It is assumed that the process of deformation is isothermal and that the work expended on changing the volume and form of an arbitary infinitesimal rectangular parallelopiped isolated from the body is independent of the manner in which the transition from the initial state of this element to the strained state is realized.

In other words, the role of the dissipative (non conservative) forces in the process of interaction of the particles of the body undergoing deformation is negligible compared to the role of the conservative force.

A body which satisfies this assumption must return to its initial dimensions and form after the load on it is removed (ideally elastic).

The work required to deform an infinitesimal parallelopiped of an elastic body is expressed in theform

$$dW = Q(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23})\, dx_1\, dx_2\, dx_3 \qquad (4\text{-}1)$$

The form of this function depends on the physical properties of the given material, but it independent of the dimensions and shape of the body. On the other hand, the strain components always expressable in terms of the three principal strain components $\epsilon_1^p, \epsilon_2^p, \epsilon_3^p$ and the direction cosines of the principal axes of strain $\epsilon_1^p, \epsilon_2^p, \epsilon_3^p$ with respect to the x_1-, x_2-, x_3- axes.

Here, the direction cosines are regarded as functions of three independent quantities, e.g., the Euler angles θ, ϕ and ψ which determine the orientation of the trihedral $\epsilon_1^p, \epsilon_2^p, \epsilon_3^p$ relative to the trihedral x_1, x_2, x_3. Hence, Equation (4-1) is also rewritten as

$$dw = Q(\epsilon_1^p, \epsilon_2^p, \epsilon_3^p, \theta, \phi, \psi)\, dx_1 dx_2 dx_3 \qquad (4\text{-}2)$$

Equation (4-2) as well as Equation (4-1) assumes that the body reacts to deformations differently in different directions, i.e., it assumes that the material of the body is an isotropic. If the physical properties of the body are the same in all directions, the work expended in deforming a volume element would not depend on quantities which vary with a rotation of the coordinate axes, but would be a function only the invariant quantities. It follows that for an isotropic body

$$dw = Q(\epsilon_1^p, \epsilon_2^p, \epsilon_3^p)\, dx_1, dx_2, dx_3 \tag{4-3}$$

The three independent invariants, $\epsilon_1^p, \epsilon_2^p, \epsilon_3^p$ are of value because they have a simple physical meaning, especially for small deformations. Mathematically, however, they are inconvenient because, in order to express them in terms of the strain components, the cubic Equation (2-18b) would have to be solved.

In view of this, it is more expendient to express the work of deformation on an element of an isotropic body as a function of the three coefficients of Equation (2-18b) (a_2, a_1, a_0) rather than in terms of the roots by this equation. Then the work done in deforming an elementary parallelopiped of an isotropic body is most conveniently written in the form

$$dw = \Phi(a_2, a_1, a_0)\; dx_1\, dx_2\, dx_3 \tag{4-4}$$

It follows that the work done in deforming the whole body is

$$W = \iiint \Phi(a_2, a_1, a_0)\, dx_1\, dx_2\, dx_3 \tag{4-5}$$

where the integration must be extended over the whole volume of the body in its UNSTRAINED STATE.

$\Phi(a_2, a_1, a_0)$ = The work of deformation or the strain energy referred to a unit volume of the body in it unstrained (specific strain energy).

W = Total work done in deforming the whole body

$dx_1 dx_2 dx_3 = dv$ = The volume of an infinitesimal element of the body before deformation.

4.2 The Principle of Virtual Displacements

Assigning to the displacements $u_1(x_1,x_2,x_3), u_2(x_1,x_2,x_3)$ $u_3(x_1,x_2,x_3)$ virtual increments $\delta u_1, \delta u_2, \delta u_3$ respectively, which are regarded as arbitary continuous functions of x_1, x_2, x_3 equal to zero at those points where the values of the displacements are given, then the strain energy changes by the amount δW and this must be equal to the work done by all the exterior forces applied to the body in effecting the above virtual displacement.

Hence, it follows that

$$\delta W = \delta R_1 + \delta R_2 \tag{4-6a}$$

where

δR_1 = The virtual work due to body forces

Referring to Equation (3-16)

$$\delta R_1 = \iiint [F^*_{x1} \delta u_1 + F^*_{x2} \delta u_2 + F^*_{x3} \delta u_3] \frac{V^*}{V} dx_1 dx_2 dx_3 \tag{4-6b}$$

where V^* = Volume element of the strained body.

Note, the integration in Equation (4-6b) must be extended over the body in its initial state.

δR_2 = The virtual work of the surface forces.

$$= \iint [f^*_{x1} \delta u_1 + f^*_{x2} \delta u_2 + f^*_{x3} \delta u_3] \frac{S^*_n}{S_n} dA \tag{4-6c}$$

where $f^*_{x1}, f^*_{x2}, f^*_{x3}$ are the components along the X_1-, X_2-, x_3-axes of the force acting on a unit area of the surface of the deformed body.

$\frac{S_n^*}{S_n}$ = The ratio of the elements of area in the terminal and initial states.

dA = Area of a surface element in the initial state.

All the volume and surface integrals appearing in Equations (4-6b) and (4-6c) are now to be extended over the limits of the body in the unstrained state (and not in the strained state) is a great convenience, since the limits of integration are now independent of any unknown quantities For convenience, Equation (4-6b), (4-6c) are rewritten by using the definition of TRACE (Apprendix II) as follow:

$$\delta R_1 = \iiint \text{Trace} [\{\delta u\}\{F_x^*\}]^T |[J]| \, dx_1 dx_2 dx_3 \qquad (4\text{-}7a)$$

$$\delta R_2 = \iint \text{Trace} [\{\delta u\}\{f_x^*\}^T] \frac{S_n^*}{S_n} \, dA \qquad (4\text{-}7b)$$

where

$$\{\delta u\} = \begin{Bmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \end{Bmatrix} \; ; \; \{F_x^*\} = \begin{Bmatrix} F_{x1}^* \\ F_{x2}^* \\ F_{x3}^* \end{Bmatrix} \; ; \; \{f_x^*\} = \begin{Bmatrix} f_{x1}^* \\ f_{x2}^* \\ f_{x3}^* \end{Bmatrix} .$$

4.3 Derivation of the Differential Equations of Equilibrium of a Deformed Isotropic Body from the Principle of Virtual Displacements.

On the assumption that the body is homogeneous and isotropic and that the dissipative forces play a neglible role in the deformation, then

$$\begin{aligned} \delta W &= \delta \iiint \Phi(a_2, a_1, a_0) \, dx_1 dx_2 dx_3 \\ &= \iiint \delta \Phi(a_2, a_1, a_0) \, dx_1 dx_2 dx_3 . \end{aligned} \qquad (4\text{-}8)$$

On the other hand, by using Chain-Rule and definition of TRACE

$$\delta[\Phi(a_2,a_1,a_0)] = \frac{\partial\Phi}{\partial\varepsilon_{11}}\delta\varepsilon_{11} + \frac{\partial\Phi}{\partial\varepsilon_{22}}\delta\varepsilon_{22} + \frac{\partial\Phi}{\partial\varepsilon_{33}}\delta\varepsilon_{33} + \frac{\partial\Phi}{\partial\varepsilon_{12}}\delta\varepsilon_{12} + \frac{\partial\Phi}{\partial\varepsilon_{13}}\delta\varepsilon_{13} + \frac{\partial\Phi}{\partial\varepsilon_{23}}\delta\varepsilon_{23}$$

$$= \mathrm{Trace}\left[\left[\frac{\partial\Phi}{\partial\varepsilon}\right][\delta\varepsilon]\right] \qquad (4\text{-}9a)$$

where

$$\left[\frac{\partial\Phi}{\partial\varepsilon}\right] = \begin{bmatrix} \frac{\partial\Phi}{\partial\varepsilon_{11}} & \frac{\partial\Phi}{\partial\varepsilon_{12}} & \frac{\partial\Phi}{\partial\varepsilon_{13}} \\ \frac{\partial\Phi}{\partial\varepsilon_{12}} & \frac{\partial\Phi}{\partial\varepsilon_{22}} & \frac{\partial\Phi}{\partial\varepsilon_{23}} \\ \frac{\partial\Phi}{\partial\varepsilon_{13}} & \frac{\partial\Phi}{\partial\varepsilon_{23}} & \frac{\partial\Phi}{\partial\varepsilon_{33}} \end{bmatrix} \qquad (4\text{-}9b)$$

and

$$[\delta\varepsilon] = \begin{bmatrix} \delta\varepsilon_{11} & \frac{1}{2}\delta\varepsilon_{12} & \frac{1}{2}\delta\varepsilon_{13} \\ \frac{1}{2}\delta\varepsilon_{12} & \delta\varepsilon_{22} & \frac{1}{2}\delta\varepsilon_{23} \\ \frac{1}{2}\delta\varepsilon_{13} & \frac{1}{2}\delta\varepsilon_{23} & \delta\varepsilon_{33} \end{bmatrix} \qquad (4\text{-}9c)$$

According to Equation (2-4b)

$$2[\varepsilon] = [D] + [D]^T + [D]^T[D]$$

$$2[\delta\varepsilon] = [\delta D] + [\delta D] + [\delta D]^T[D] + [D]^T[\delta D]$$

$$= [\delta D]^T[[I]+[D]] + [[I]+[D]^T][\delta D]$$

$$= [\delta D]^T[J] + [J]^T[\delta D]$$

thus,

$$[\delta\varepsilon] = \frac{1}{2}[\delta D]^T[J] + \frac{1}{2}[J]^T[\delta D] \qquad (4\text{-}10)$$

Accordance with Equation (1-4a)

$$[D] = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \qquad (4\text{-}11a)$$

Introducing

$$\left[\frac{\partial\Phi}{\partial D}\right] = \begin{bmatrix} \frac{\partial\Phi}{\partial d_{11}} & \frac{\partial\Phi}{\partial d_{12}} & \frac{\partial\Phi}{\partial d_{13}} \\ \frac{\partial\Phi}{\partial d_{21}} & \frac{\partial\Phi}{\partial d_{22}} & \frac{\partial\Phi}{\partial d_{23}} \\ \frac{\partial\Phi}{\partial d_{31}} & \frac{\partial\Phi}{\partial d_{32}} & \frac{\partial\Phi}{\partial d_{33}} \end{bmatrix} , \tag{4-11b}$$

it follows from Equation (2-4b) that

$$2[\varepsilon] = [D] + [D]^T + [D]^T[D]$$

or

$$\begin{bmatrix} 2\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & 2\varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & 2\varepsilon_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 2d_{11}+d_{11}^2+d_{21}^2+d_{31}^2 , & d_{12}+d_{21}+d_{11}d_{12}+d_{21}d_{22}+d_{31}d_{32} , & d_{13}d_{31}+d_{11}d_{13}+d_{21}d_{23}+d_{31}d_{33} \\ & 2d_{22}+d_{12}^2+d_{22}^2+d_{32}^2 & d_{23}+d_{32}+d_{12}d_{13}+d_{22}d_{23}+d_{32}d_{33} \\ \text{SYSMMETRIC} & & 2d_{33}+d_{13}^2+d_{23}^2+d_{33}^2 \end{bmatrix} \tag{4-11c}$$

Differentiating both sides of Equation (4-11c) with respect to each component of matrix (D) gives

$$\begin{aligned} &\frac{\partial\varepsilon_{11}}{\partial d_{11}} = 1 + d_{11} \quad ; \quad \frac{\partial\varepsilon_{22}}{\partial d_{11}} = 0 \quad ; \quad \frac{\partial\varepsilon_{33}}{\partial d_{11}} = 0 \\ &\frac{\partial\varepsilon_{12}}{\partial d_{11}} = d_{12} \quad ; \quad \frac{\partial\varepsilon_{13}}{\partial d_{11}} = d_{13} \quad ; \quad \frac{\partial\varepsilon_{23}}{\partial d_{11}} = 0 . \end{aligned} \tag{4-11d}$$

By the well-known chain rule

$$\frac{\partial\Phi}{\partial d_{11}} = \frac{\partial\varepsilon_{11}}{\partial d_{11}}\frac{\partial\Phi}{\partial\varepsilon_{11}} + \frac{\partial\varepsilon_{22}}{\partial d_{11}}\frac{\partial\Phi}{\partial\varepsilon_{22}} + \frac{\partial\varepsilon_{12}}{\partial d_{11}}\frac{\partial\Phi}{\partial\varepsilon_{12}} + \frac{\partial\varepsilon_{33}}{\partial d_{11}}\frac{\partial\Phi}{\partial\varepsilon_{33}} + \frac{\partial\varepsilon_{13}}{\partial d_{11}}\frac{\partial\Phi}{\partial\varepsilon_{13}} + \frac{\partial\varepsilon_{23}}{\partial d_{11}}\frac{\partial\Phi}{\partial\varepsilon_{23}} \tag{4-11e}$$

$$= (1+d_{11})\frac{\partial\Phi}{\partial\varepsilon_{11}} + d_{12}\frac{\partial\Phi}{\partial\varepsilon_{12}} + d_{13}\frac{\partial\Phi}{\partial\varepsilon_{13}} \tag{4-11f}$$

Analogously

$$\frac{\partial \Phi}{\partial d_{22}} = d_{21}\frac{\partial \Phi}{\partial \varepsilon_{12}} + (1+d_{22})\frac{\partial \Phi}{\partial \varepsilon_{22}} + d_{23}\frac{\partial \Phi}{\partial \varepsilon_{23}}$$

$$\frac{\partial \Phi}{\partial d_{33}} = d_{31}\frac{\partial \Phi}{\partial \varepsilon_{13}} + d_{32}\frac{\partial \Phi}{\partial \varepsilon_{23}} + (1+d_{33})\frac{\partial \Phi}{\partial \varepsilon_{33}}$$

$$\frac{\partial \Phi}{\partial d_{12}} = (1+d_{11})\frac{\partial \Phi}{\partial \varepsilon_{12}} + d_{12}\frac{\partial \Phi}{\partial \varepsilon_{22}} + d_{13}\frac{\partial \Phi}{\partial \varepsilon_{23}}$$

$$\frac{\partial \Phi}{\partial d_{13}} = (1+d_{11})\frac{\partial \Phi}{\partial \varepsilon_{13}} + d_{12}\frac{\partial \Phi}{\partial \varepsilon_{23}} + d_{13}\frac{\partial \Phi}{\partial \varepsilon_{33}} \qquad (4\text{-}11f)$$

$$\frac{\partial \Phi}{\partial \varepsilon_{31}} = d_{31}\frac{\partial \Phi}{\partial \varepsilon_{11}} + d_{32}\frac{\partial \Phi}{\partial \varepsilon_{12}} + (1+d_{33})\frac{\partial \Phi}{\partial \varepsilon_{13}}$$

$$\frac{\partial \Phi}{\partial d_{21}} = d_{21}\frac{\partial \Phi}{\partial \varepsilon_{11}} + (1+d_{22})\frac{\partial \Phi}{\partial \varepsilon_{12}} + d_{23}\frac{\partial \Phi}{\partial \varepsilon_{13}}$$

$$\frac{\partial \Phi}{\partial d_{23}} = d_{21}\frac{\partial \Phi}{\partial \varepsilon_{13}} + (1+d_{22})\frac{\partial \Phi}{\partial \varepsilon_{23}} + d_{23}\frac{\partial \Phi}{\partial \varepsilon_{33}}$$

$$\frac{\partial \Phi}{\partial d_{32}} = d_{31}\frac{\partial \Phi}{\partial \varepsilon_{12}} + d_{32}\frac{\partial \Phi}{\partial \varepsilon_{22}} + (1+d_{33})\frac{\partial \Phi}{\partial \varepsilon_{23}}.$$

Equations (4-11f) becomes in matrix form

$$\begin{bmatrix} \frac{\partial \Phi}{\partial D_{11}} & \frac{\partial \Phi}{\partial D_{12}} & \frac{\partial \Phi}{\partial D_{13}} \\ \frac{\partial \Phi}{\partial D_{21}} & \frac{\partial \Phi}{\partial D_{22}} & \frac{\partial \Phi}{\partial D_{23}} \\ \frac{\partial \Phi}{\partial D_{31}} & \frac{\partial \Phi}{\partial D_{32}} & \frac{\partial \Phi}{\partial D_{33}} \end{bmatrix} = \begin{bmatrix} 1+d_{11} & d_{12} & d_{13} \\ d_{21} & 1+d_{22} & d_{23} \\ d_{31} & d_{32} & 1+d_{33} \end{bmatrix} \begin{bmatrix} \frac{\partial \Phi}{\partial \varepsilon_{11}} & \frac{\partial \Phi}{\partial \varepsilon_{12}} & \frac{\partial \Phi}{\partial \varepsilon_{13}} \\ \frac{\partial \Phi}{\partial \varepsilon_{12}} & \frac{\partial \Phi}{\partial \varepsilon_{22}} & \frac{\partial \Phi}{\partial \varepsilon_{23}} \\ \frac{\partial \Phi}{\partial \varepsilon_{13}} & \frac{\partial \Phi}{\partial \varepsilon_{23}} & \frac{\partial \Phi}{\partial \varepsilon_{33}} \end{bmatrix} \qquad (4\text{-}12a)$$

$$\left[\frac{\partial \Phi}{\partial D}\right] = \left[[I] + [D]\right]\left[\frac{\partial \Phi}{\partial \varepsilon}\right] \qquad (4\text{-}12b)$$

or

$$\left[\frac{\partial \Phi}{\partial D}\right] = [J]\left[\frac{\partial \Phi}{\partial \varepsilon}\right] \qquad (4\text{-}12c)$$

Substituting the value from Equation (4-10) into Equation (4-9a), one obtains

$$\delta[\Phi(a_2,a_1,a_0)] = \text{Trace}\left[\left[\frac{\partial \Phi}{\partial \varepsilon}\right][\delta \varepsilon]\right]$$

$$\delta[\Phi(a_2,a_1,a_0)] = \text{Trace}\left[\left[\frac{\partial\Phi}{\partial\varepsilon}\right]\left[\frac{1}{2}[\delta D]^T[J] + \frac{1}{2}[J]^T[\delta D]\right]\right]$$

$$= \text{Trace}\left[\frac{1}{2}\left[\frac{\partial\Phi}{\partial\varepsilon}\right][\delta D]^T[J]\right] + \text{Trace}\left[\frac{1}{2}\left[\frac{\partial\Phi}{\partial\varepsilon}\right][J]^T[\delta D]\right]$$

$$= \text{Trace}\left[\frac{1}{2}[\delta D]^T[J]\left[\frac{\partial\Phi}{\partial\varepsilon}\right]\right]$$

$$+ \text{Trace}\left[\frac{1}{2}[\delta D]^T[J]\left[\frac{\partial\Phi}{\partial\varepsilon}\right]\right]$$

$$= \text{Trace}\left[[\delta D]^T[J]\left[\frac{\partial\Phi}{\partial\varepsilon}\right]\right] \qquad (4\text{-}13a)$$

Noting Equation (4-12c) gives

$$\delta[\Phi(a_2,a_1,a_0)] = \text{Trace}\left[[\delta D]^T\left[\frac{\partial\Phi}{\partial D}\right]\right] \qquad (4\text{-}13b)$$

Since $[\delta D] = [\{\nabla\}\{\delta u\}^T]^T$, (4-13c)

it follows that*

$$\delta[\Phi(a_2,a_1,a_0)] = \text{Trace}\left[[\{\nabla\}\{\delta u\}^T]\left[\frac{\partial\Phi}{\partial D}\right]\right]$$

which after tedious computation is shown as equal to

$$= \text{Trace}\left[\{\nabla\}\left\{\{\delta u\}^T\left[\frac{\partial\Phi}{\partial D}\right]\right\}\right] - \text{Trace}\left[\{\delta u\}\left\{\{\nabla\}^T\left[\frac{\partial\Phi}{\partial D}\right]\right\}\right] \qquad (4\text{-}13d)$$

Consider Guass's Theorem*

$$\iiint \text{Trace}[\{\nabla\}\{b\}^T]\,dx_1dx_2dx_3 = \iint \text{Trace}[\{b\}\{n\}^T]\,dA \qquad (4\text{-}13e)$$

where

$\{b\}$ = any vector in the x_1, x_2, x_3 coordinate

$\{n\}$ = Unit Vector normal to surface area

$\vec{n} = \cos(n,x_1)\,\bar{i}_1 + \cos(n,x_2)\,\bar{i}_2 + \cos(n,x_3)\,\bar{i}_3$ (4-13f)

(*) See appendix II for all the definition of TRACE

Substitute all the values into Equation (4-8)

$$\delta W = \iiint Trace\left[\{\nabla\}\{\{\delta u\}^T[\tfrac{\partial \Phi}{\partial D}]\}\right] dx_1dx_2dx_3 - \iiint Trace\left[\{\delta u\}\{\{\nabla\}^T[\tfrac{\partial \Phi}{\partial D}]^T\}\right] dx_1dx_2dx_3$$

$$= \iint Trace\left[[\tfrac{\partial \Phi}{\partial D}]^T\{\delta u\}\{n\}^T\right] dA - \iiint Trace\left[\{\delta u\}\{\{\nabla\}^T[\tfrac{\partial \Phi}{\partial D}]^T\}\right] dx_1dx_2dx_3$$

$$= \iint Trace\left[\{\delta u\}\{n\}^T[\tfrac{\partial \Phi}{\partial D}]\right] dA - \iiint Trace\left[\{\delta u\}\{\{\nabla\}^T[\tfrac{\partial \Phi}{\partial D}]^T\}\right] dx_1dx_2dx_3 \qquad (4\text{-}14a)$$

In accordance with Equation (4-6a)

$$\delta R_1 + \delta R_2 - \delta W = 0 \quad , \qquad (4\text{-}14b)$$

substituting the values of δR_1, δR_2 and δW, then Equation (4-14b) is rewritten as

$$\iint Trace\left[\{\delta u\}\{\tfrac{S_n^*}{S_n}\{f_x^*\}^T - \{n\}^T[\tfrac{\partial \Phi}{\partial D}]^T\}\right] dA$$

$$+ \iiint Trace\left[\{\delta u\}\{|[J]|\{F_x^*\}^T + \{\nabla\}^T[\tfrac{\partial \Phi}{\partial D}]^T\}\right] dV = 0 \qquad (4\text{-}14c)$$

Since, the principle of virtual displacements, Equation (4-14c) must be satisfied for arbitary values of $\delta u_1, \delta u_2, \delta u_3$, the following equation must hold at all interior points of the body

$$\{\nabla\}^T[\tfrac{\partial \Phi}{\partial D}]^T + |[J]|\{F_x^*\}^T = \{0\}^T \qquad (4\text{-}15a)$$

together with the equation on all surface points of the body.

$$\frac{S_n^*}{S_n}\{f_x^*\}^T - \{n\}^T[\tfrac{\partial \Phi}{\partial D}]^T = \{0\}^T \qquad (4\text{-}15b)$$

According to Equation (4-12c), then Equation (4-15a) is written in the form

$$\{\nabla\}^T[\tfrac{\partial \Phi}{\partial \varepsilon}][J]^T + |[J]|\{F_x^*\}^T = \{0\}^T \qquad (4\text{-}15c)$$

4.4 The Relation between Stress and Strain Components.

Comparing Equation (4-15c) with Equations (3-34b). both express the conditions of equilibrium of a volume element of the deformed body which initially is a rectangular parallelopiped with edges dx_1, dx_2, dx_3 parallel to the x_1-, x_2-, x_3- axes. It is seen that one of these systems are transformed into the other by setting

$$[\nabla_R] = \left[\frac{\partial\Phi}{\partial\varepsilon}\right] \tag{4-16}$$

It follows immediately from the above equation that

$$[\nabla_R] = [\nabla_R]^T \tag{4-17a}$$

since

$$\left[\frac{\partial\Phi}{\partial\varepsilon}\right] = \left[\frac{\partial\Phi}{\partial\varepsilon}\right]^T \tag{4-17b}$$

Consider the well-known chain rule of mutivariate calculus

$$\left[\frac{\partial\Phi}{\partial\varepsilon}\right] = \frac{\partial\Phi}{\partial a_2}\left[\frac{\partial a_2}{\partial\varepsilon}\right] + \frac{\partial\Phi}{\partial a_1}\left[\frac{\partial a_1}{\partial\varepsilon}\right] + \frac{\partial\Phi}{\partial a_0}\left[\frac{\partial a_0}{\partial\varepsilon}\right] \tag{4-18a}$$

$$= \frac{\partial\Phi}{\partial a_2}[I] + \frac{\partial\Phi}{\partial a_1}\left[[I]a_2 - [\varepsilon]\right] + \frac{\partial\Phi}{\partial a_0}\left[COF[\varepsilon]\right] \tag{4-18b}$$

where

$$a_2 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

$$\left[\frac{\partial a_2}{\partial\varepsilon}\right] = [I] \tag{4-18c}$$

$$a_1 = \varepsilon_{11}\varepsilon_{22} + \varepsilon_{11}\varepsilon_{33} + \varepsilon_{22}\varepsilon_{33} - \tfrac{1}{4}(\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2)$$

$$\left[\frac{\partial a_1}{\partial\varepsilon}\right] = \begin{bmatrix} \varepsilon_{22}+\varepsilon_{33} & -\tfrac{1}{2}\varepsilon_{12} & -\tfrac{1}{2}\varepsilon_{13} \\ -\tfrac{1}{2}\varepsilon_{12} & \varepsilon_{11}+\varepsilon_{33} & -\tfrac{1}{2}\varepsilon_{23} \\ -\tfrac{1}{2}\varepsilon_{13} & -\tfrac{1}{2}\varepsilon_{23} & \varepsilon_{11}+\varepsilon_{22} \end{bmatrix}$$

$$= \left[a_2[I] - [\varepsilon]\right] \tag{4-18d}$$

Also

$$a_o = \varepsilon_{11}\varepsilon_{22}\varepsilon_{33} - \tfrac{1}{4}(\varepsilon_{11}\varepsilon_{23}^2 + \varepsilon_{22}\varepsilon_{13}^2 + \varepsilon_{33}\varepsilon_{12}^2 - \varepsilon_{12}\varepsilon_{13}\varepsilon_{23})$$

$$\left[\frac{\partial a_o}{\partial \varepsilon}\right] = \begin{bmatrix} (\varepsilon_{22}\varepsilon_{33} - \tfrac{1}{4}\varepsilon_{23}^2) & (\tfrac{1}{4}\varepsilon_{13}\varepsilon_{23} - \tfrac{1}{2}\varepsilon_{33}\varepsilon_{12}) & (\tfrac{1}{4}\varepsilon_{12}\varepsilon_{23} - \tfrac{1}{2}\varepsilon_{22}\varepsilon_{13}) \\ & (\varepsilon_{11}\varepsilon_{33} - \tfrac{1}{4}\varepsilon_{13}^2) & (\tfrac{1}{4}\varepsilon_{12}\varepsilon_{13} - \tfrac{1}{2}\varepsilon_{11}\varepsilon_{23}) \\ \text{SYSMMETRIC.} & & (\varepsilon_{11}\varepsilon_{22} - \tfrac{1}{4}\varepsilon_{12}^2) \end{bmatrix}$$

$$= \left[\mathrm{COF}\ [\varepsilon]\right] \tag{4-18e}$$

Hence

$$[\nabla_R] = \frac{\partial \Phi}{\partial a_2}[I] + \frac{\partial \Phi}{\partial a_1}\left[[I]a_2 - [\varepsilon]\right] + \frac{\partial \Phi}{\partial a_o}\left[\mathrm{COF}[\varepsilon]\right] \tag{4-19}$$

Equation (4-19) is the general statement of relation which must exist between the stress and strain components. In deriving this equation two assumptions have been used

1. The body is isotropic
2. The dissipative forces due to the interaction of the particles of the body are small enough to be neglected in comparison with the conservative forces.

In conclusion, it should be noted that Equations (4-9a) and (4-16) imply that Equation (4-8) may be rewritten as

$$\delta W = \iiint \mathrm{Trace}\left[[\nabla_R][\delta\varepsilon]\right] dx_1 dx_2 dx_3 \tag{4-20}$$

This equation is a generalization of the analogous expression of the classical theory of elasticity to the case of deformations of arbitary magnitude.

4.5 Boundary Conditions

Equation (4-15b) expresses the conditions which must be satisfied at those points of the bounding surface where the surface loading is prescribed but the displacements are not.

Now consider Equation (3-32c) compared with Equation (4-15a), it follows that

$$\left[{}^{s^*}\!/_{s} \right] \left[\overset{*}{\nabla_n} \right] = \left[\frac{\partial \Phi}{\partial D} \right]^T \quad (4\text{-}21a)$$

$$\left[\overset{*}{\nabla_n} \right]^T \left[{}^{s^*}\!/_{s} \right] = \left[\frac{\partial \Phi}{\partial D} \right] \quad (4\text{-}21b)$$

Substituting Equation (4-21b) above into Equation (4-15b), it follows that

$$\left[\overset{*}{\nabla_n} \right]^T \left[{}^{s^*}\!/_{s} \right] \left\{ \overset{*}{n} \right\} = \frac{\overset{*}{S_n}}{S_n} \left\{ f_x^* \right\} \quad (4\text{-}21c)$$

where $\left[\overset{*}{\nabla_n} \right]$ may be expressed in terms of the strain components. The left-hand side of Equation (4-21c) may be regarded as functions of the displacements.

After these substitutions have been made, the given expressions become the mathematical formulation of the conditions which must be imposed on the displacements at those points of the bounding surface of the body at which u_1, u_2, u_3 are not given directly.

4.6 The Simplification of The Derived Equations in the Case of a Small Deformation.

All the equations that have derived from the beginning of this chapter are all for Case 1 (general nonlinear case) which may be simplified for the case of small deformation as follow:

Case 2. The Case of Small Deformation.

If the deformation is small, its components are neglected in those equations where they appear together with terms of order unity with

$$\left[S^*/S \right] = \left[I \right] \qquad (4\text{-}22a)$$

$$V^*/V = |[J]| = 1. \qquad (4\text{-}22b)$$

$$\frac{\overset{*}{Sn}}{Sn} = 1. \qquad (4\text{-}22c)$$

Thus, Equations (4-15b), (4-15d) are rewritten as

$$\{\nabla\}^T\left[\frac{\partial\Phi}{\partial\varepsilon}\right]\left[[I]+[e]-[\omega]\right] + \{F_x^*\}^T = \{0\}^T \qquad (4\text{-}23a)$$

$$[\nabla_N^*]^T\{n\} = \{f_x^*\} \qquad (4\text{-}23b)$$

Case 3. The Case of Small Deformation and Small Angles of Rotation.

If the angles of rotation, as well as the strain components, are small compare to unity, then Equation (4-23a) are simplified by neglecting $[e]$ in comparison with matrices $[I]$ and $[\omega]$.

Hence, Equation (4-23a) reduces to

$$\{\nabla\}^T\left[\frac{\partial\Phi}{\partial\varepsilon}\right]\left[[I]-[\omega]\right] + \{F_x^*\}^T = \{0\}^T \qquad (4\text{-}24a)$$

$$\{\nabla\}^T\left[\left[\frac{\partial\Phi}{\partial\varepsilon}\right] - \left[\frac{\partial\Phi}{\partial\varepsilon}\right][\omega]\right] + \{F_x^*\}^T = \{0\}^T. \qquad (4\text{-}24b)$$

Case 4 The Transition to the Equation of the Classical Theory

With this degree of accuracy, the only other simplification possible consists of neglecting the product of $\left[\frac{\partial\Phi}{\partial\varepsilon}\right]$ and $[\omega]$ in comparison with only the matrix $\left[\frac{\partial\Phi}{\partial\varepsilon}\right]$. So the Equation (4-24b) reduces into

$$\{\nabla\}^T\left[\frac{\partial\Phi}{\partial\varepsilon}\right] + \{F_x\}^T = \{0\}^T. \qquad (4\text{-}25)$$

Now representing the function $\Phi(a_2, a_1, a_0)$ as a power series in the three parameters a_2, a_1, a_0. No negative powers can appear in the series, for otherwise the specific strain energy would tend to infinity for infinitesimal displacements of the points of the body from their initial position, which is unacceptable.

Furthermore, if the strain enegy of the body is to be zero in the initial state (the body to be free of all stresses), then the series must begin with terms which contain the strain components to the second power. Under these conditions, it is written as

$$\begin{aligned}\Phi(a_2, a_1, a_0) = {} & A_1 a_2^2 + A_2 a_1 \\ & + B_1 a_2^3 + B_2 a_2 a_1 + B_3 a_0 \\ & + C_1 a_2^4 + C_2 a_2^2 a_1 + C_3 a_2 a_0 + C_4 a_1^2 \\ & + D_1 a_2^5 + D_2 a_2^3 a_1 + D_3 a_2^2 a_0 + D_4 a_2 a_1^2 + D_5 a_1 a_0 \\ & + \cdots\cdots\end{aligned} \qquad (4\text{-}26)$$

where

A_j's = the coefficients of those terms which contain the strain components to the second power

B_j's = correspond to the terms containing the strain components to the third power

C_j's = correspond to the terms containing the strain components to the fourth power.

The series (4-2b) is regarded as the general expression for the strain energy of an isotropic body which, in its initial state, is free from any internal forces.

4.7 Hooke's law.

Assuming that the strain components are infinitely small, then, whatever the relative magnitudes of the physical constants A_j, B_j, C_j...., their influence is nullified by the infinitesimal smallness of the strains. Therefore, only those terms in the series (4-26) which contain the strain components to the smallest (i.e., second) power need be retained.

Thus Equation (4-2b) reduces to

$$\Phi(a_2, a_1, a_0) = A_1 a_2^2 + A_2 a_1 \qquad (4\text{-}27a)$$

$$\frac{\partial \Phi}{\partial a_2} = 2A_1 a_2$$

$$\frac{\partial \Phi}{\partial a_1} = A_2 \quad ; \quad \frac{\partial \Phi}{\partial a_0} = 0.$$

Equation (4-19) is rewritten as follow

$$[\nabla] = 2A_1 a_2 [I] + A_2 [[I] a_2 - [\varepsilon]] \qquad (4\text{-}27b)$$

or

$$\nabla_{11} = 2A_1(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + A_2(\varepsilon_{22} + \varepsilon_{33})$$

$$\nabla_{22} = 2A_1(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + A_2(\varepsilon_{11} + \varepsilon_{33})$$

$$\nabla_{33} = 2A_1(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + A_2(\varepsilon_{11} + \varepsilon_{22}) \qquad (4\text{-}27c)$$

$$\nabla_{12} = -\frac{1}{2} A_2 \varepsilon_{12}$$

$$\nabla_{13} = -\frac{1}{2} A_2 \varepsilon_{13}$$

$$\nabla_{23} = -\frac{1}{2} A_2 \varepsilon_{23}$$

Put into matrix form, it is written as

$$\begin{Bmatrix} \nabla_{11} \\ \nabla_{22} \\ \nabla_{33} \\ \nabla_{12} \\ \nabla_{13} \\ \nabla_{23} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} 2A_1 & 2A_1+A_2 & 2A_1+A_2 & 0 & 0 & 0 \\ (2A_1+A_2) & 2A_1 & 2A_1+A_2 & 0 & 0 & 0 \\ (2A_1+A_2) & (2A_1+A_2) & 2A_1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\frac{A_2}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{A_2}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{A_2}{2} \end{array}\right] \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{Bmatrix} \qquad (4\text{-}27d)$$

A_1 and A_2 are replaced by two new constants E and μ where

$$A_2 = -\frac{E}{1+\mu} \quad ; \quad 2A_1 + A_2 = \frac{\mu E}{(1+\mu)(1-2\mu)}$$

$$2A_1 = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)} \qquad (4\text{-}27e)$$

E = Young's modulus, μ = Poisson's ratio

It follows that

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hdashline \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = \frac{E}{(1+\mu)} \left[\begin{array}{ccc:ccc} \frac{(1-\mu)}{(1-2\mu)} & \frac{\mu}{(1-2\mu)} & \frac{\mu}{(1-2\mu)} & 0 & 0 & 0 \\ & \frac{(1-\mu)}{(1-2\mu)} & \frac{\mu}{(1-2\mu)} & 0 & 0 & 0 \\ \text{SYSMMETRY} & & \frac{(1-\mu)}{(1-2\mu)} & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{array} \right] \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \hdashline \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{Bmatrix}$$

(4-27f)

or

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \hdashline \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{Bmatrix} = \left[\begin{array}{ccc:ccc} \frac{1}{E} & -\frac{\mu}{E} & -\frac{\mu}{E} & 0 & 0 & 0 \\ -\frac{\mu}{E} & \frac{1}{E} & -\frac{\mu}{E} & 0 & 0 & 0 \\ -\frac{\mu}{E} & -\frac{\mu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \frac{2(1+\mu)}{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2(1+\mu)}{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(1+\mu)}{E} \end{array} \right] \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hdashline \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix}$$

(4-27g)

Equation (4-27f) expresses the well-known law of James Hooke. It follows from the above that for every material a range of small deformations can be established for which Hooke's law is approximately valid.

Hence, as soon as Hooke's law looses its validity the problem of ascertaining the stress-strain relation is complicated drastically. A further complication arises from the fact that the part of the dissipative forces increases substantially after the limit of proportionality ispassed.

4.8 On the Applicability of Equation (4-19) to Elastic-Plastic Deformations.

The basic relations of Hencky's theory of plasticity are derived from Equation (4-19) by introducing suitable assumptions regarding the nature of the dependence of the derivatives $\frac{\partial \Phi}{\partial a_2}, \frac{\partial \Phi}{\partial a_1}, \frac{\partial \Phi}{\partial a_0}$ on the strain components. In order to show this, it is necessary to use Equation (4-19) to establish a relation between the two invariants of the stress tensor.

$$C_2^* = \sigma_{R11}^2 + \sigma_{R22}^2 + \sigma_{R33}^2 \qquad (4\text{-}28a)$$

$$\begin{aligned} C_2^{*2} - 3C_1^* = {} & \sigma_{R11}^2 + \sigma_{R22}^2 + \sigma_{R33}^2 + 2\left(\sigma_{R11}\sigma_{R22} + \sigma_{R22}\sigma_{R33} + \sigma_{R11}\sigma_{R33}\right) \\ & - 3\left(\sigma_{R11}\sigma_{R22} + \sigma_{R11}\sigma_{R33} + \sigma_{R22}\sigma_{R33}\right) + 3\left(\sigma_{R12}^2 + \sigma_{R13}^2 + \sigma_{R23}^2\right) \end{aligned}$$

$$C_2^{*2} - 3C_1^* = \sigma_{R11}^2 + \sigma_{R22}^2 + \sigma_{R33}^2 - (\sigma_{R11}\sigma_{R22} + \sigma_{R11}\sigma_{R33} + \sigma_{R22}\sigma_{R33}) + 3(\sigma_{R12}^2 + \sigma_{R13}^2 + \sigma_{R23}^2).$$

$$= \tfrac{1}{2}\sigma_{R11}^2 - \sigma_{R11}\sigma_{R22} + \tfrac{1}{2}\sigma_{R22}^2 + \tfrac{1}{2}\sigma_{R11}^2 - \sigma_{R11}\sigma_{R33} + \tfrac{1}{2}\sigma_{R33}^2 + \tfrac{1}{2}\sigma_{R22}^2 - \sigma_{R22}\sigma_{R33} + \tfrac{1}{2}\sigma_{R33}^2 + 3(\sigma_{R12}^2 + \sigma_{R13}^2 + \sigma_{R23}^2).$$

$$= \tfrac{1}{2}(\sigma_{R11} - \sigma_{R22})^2 + \tfrac{1}{2}(\sigma_{R11} - \sigma_{R33})^2 + \tfrac{1}{2}(\sigma_{R22} - \sigma_{R33})^2 + 3(\sigma_{R12}^2 + \sigma_{R13}^2 + \sigma_{R23}^2)$$

$$= \tfrac{1}{2}\left\{(\sigma_{R11} - \sigma_{R22})^2 + (\sigma_{R11} - \sigma_{R33})^2 + (\sigma_{R22} - \sigma_{R33})^2 + 6(\sigma_{R12}^2 + \sigma_{R13}^2 + \sigma_{R23}^2)\right\} \quad (4\text{-}28b)$$

According to the left-hand side of Equation (4-19) C_2^* and C_1^* are calculated in the easier way by referring to the principal strains

Thus

$$[\varepsilon] = \begin{bmatrix} \epsilon_1^p & 0 & 0 \\ 0 & \epsilon_2^p & 0 \\ 0 & 0 & \epsilon_3^p \end{bmatrix} \quad (4\text{-}28c)$$

Equation (4-19) is expressed as

$$[\sigma_R] = \begin{bmatrix} \frac{\partial\Phi}{\partial a_2} & 0 & 0 \\ 0 & \frac{\partial\Phi}{\partial a_2} & 0 \\ 0 & 0 & \frac{\partial\Phi}{\partial a_2} \end{bmatrix} + \frac{\partial\Phi}{\partial a_1}\begin{bmatrix} \varepsilon_2+\varepsilon_3 & 0 & 0 \\ 0 & \varepsilon_1+\varepsilon_3 & 0 \\ 0 & 0 & \varepsilon_1+\varepsilon_2 \end{bmatrix} + \frac{\partial\Phi}{\partial a_0}\begin{bmatrix} \varepsilon_2\varepsilon_3 & 0 & 0 \\ 0 & \varepsilon_1\varepsilon_3 & 0 \\ 0 & 0 & \varepsilon_1\varepsilon_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial\Phi}{\partial a_2} + \frac{\partial\Phi}{\partial a_1}(\epsilon_2^p+\epsilon_3^p) + \frac{\partial\Phi}{\partial a_0}(\epsilon_2^p\epsilon_3^p) & 0 & 0 \\ 0 & \frac{\partial\Phi}{\partial a_2} + \frac{\partial\Phi}{\partial a_1}(\epsilon_1^p+\epsilon_3^p) + \frac{\partial\Phi}{\partial a_0}(\epsilon_1^p\epsilon_3^p) & 0 \\ 0 & 0 & \frac{\partial\Phi}{\partial a_2} + \frac{\partial\Phi}{\partial a_1}(\epsilon_1^p+\epsilon_2^p) + \frac{\partial\Phi}{\partial a_0}\epsilon_1^p\epsilon_2^p \end{bmatrix} \quad (4\text{-}28d)$$

The two invariants are calculated as follow

$$C_2^* = 3\frac{\partial\Phi}{\partial a_2} + 2\frac{\partial\Phi}{\partial a_1}(\epsilon_1^p+\epsilon_2^p+\epsilon_3^p) + \frac{\partial\Phi}{\partial a_0}(\epsilon_1^p\epsilon_2^p+\epsilon_1^p\epsilon_3^p+\epsilon_2^p\epsilon_3^p)$$

$$= 3\frac{\partial\Phi}{\partial a_2} + 2\frac{\partial\Phi}{\partial a_1}a_2 + \frac{\partial\Phi}{\partial a_0}a_1 \qquad (4\text{-}29a)$$

$$C_1^* = \left(\frac{\partial\Phi}{\partial a_2} + \frac{\partial\Phi}{\partial a_1}(\epsilon_1^p+\epsilon_3^p) + \frac{\partial\Phi}{\partial a_0}\epsilon_1^p\epsilon_3^p\right)\left(\frac{\partial\Phi}{\partial a_2} + \frac{\partial\Phi}{\partial a_1}(\epsilon_2^p+\epsilon_1^p) + \frac{\partial\Phi}{\partial a_0}\epsilon_1^p\epsilon_2^p\right)$$

$$+\left(\frac{\partial\Phi}{\partial a_2} + \frac{\partial\Phi}{\partial a_1}(\epsilon_1^p+\epsilon_3^p) + \frac{\partial\Phi}{\partial a_0}\epsilon_1^p\epsilon_3^p\right)\left(\frac{\partial\Phi}{\partial a_2} + \frac{\partial\Phi}{\partial a_1}(\epsilon_2^p+\epsilon_3^p) + \frac{\partial\Phi}{\partial a_0}\epsilon_2^p\epsilon_3^p\right)$$

$$+\left(\frac{\partial\Phi}{\partial a_2} + \frac{\partial\Phi}{\partial a_1}(\epsilon_1^p+\epsilon_2^p) + \frac{\partial\Phi}{\partial a_0}\epsilon_1^p\epsilon_2^p\right)\left(\frac{\partial\Phi}{\partial a_2} + \frac{\partial\Phi}{\partial a_1}(\epsilon_2^p+\epsilon_3^p) + \frac{\partial\Phi}{\partial a_0}\epsilon_2^p\epsilon_3^p\right)$$

$$= \left(\frac{\partial\Phi}{\partial a_2} + \frac{\partial\Phi}{\partial a_1}(\epsilon_1^p+\epsilon_3^p) + \frac{\partial\Phi}{\partial a_0}\epsilon_1^p\epsilon_3^p\right)\Big(2\frac{\partial\Phi}{\partial a_2} + \frac{\partial\Phi}{\partial a_1}(\epsilon_1^p+2\epsilon_2^p+\epsilon_3^p)$$

$$+\frac{\partial\Phi}{\partial a_0}(\epsilon_1^p\epsilon_2^p+\epsilon_2^p\epsilon_3^p)\Big) + \left(\frac{\partial\Phi}{\partial a_2} + \frac{\partial\Phi}{\partial a_1}(\epsilon_1^p+\epsilon_2^p) + \frac{\partial\Phi}{\partial a_0}\epsilon_1^p\epsilon_2^p\right)\Big(\frac{\partial\Phi}{\partial a_2} +$$

$$+\frac{\partial\Phi}{\partial a_1}(\epsilon_2^p+\epsilon_3^p) + \frac{\partial\Phi}{\partial a_0}\epsilon_2^p\epsilon_3^p\Big). \qquad (4\text{-}29b)$$

$$= 3\left(\frac{\partial\Phi}{\partial a_2}\right)^2 + \frac{\partial\Phi}{\partial a_2}\frac{\partial\Phi}{\partial a_1}(4a_2) + \frac{\partial\Phi}{\partial a_0}\frac{\partial\Phi}{\partial a_2}(2a_1) + \left(\frac{\partial\Phi}{\partial a_1}\right)^2 a_2^2$$

$$+\frac{\partial\Phi}{\partial a_1}\frac{\partial\Phi}{\partial a_0}(3a_0 + a_1a_2) + \left(\frac{\partial\Phi}{\partial a_0}\right)^2 a_0a_2$$

$$C_2^{*2} - 3C_1^* = \left(\frac{\partial\Phi}{\partial a_1}\right)^2(a_2^2-3a_1) + \frac{\partial\Phi}{\partial a_1}\frac{\partial\Phi}{\partial a_0}(a_2a_1-9a_0) + \left(\frac{\partial\Phi}{\partial a_0}\right)^2(a_1^2-3a_2a_0) \qquad (4\text{-}29c)$$

Experiments show that the character of only Equation (4-29c) is drastically changed by the transition from elastic to plastic deformations, while Equation (4-29a) (which gives the connection between the average value of the three principal stresses and the strain invariants) changes so little that it can be extended intact, with no serious error, to the plastic range. However, according to Equation (4-21f), in the elastic range

$$\frac{1}{3}\left(\sigma_{R11}+\sigma_{R22}+\sigma_{R33}\right) = \frac{1}{3}\frac{E}{(1-2\mu)}a_2 \qquad (4\text{-}30a)$$

Hence, extending this relation to the plastic range as well, yields

$$\frac{\partial\Phi}{\partial a_2}+\frac{2}{3}a_2\frac{\partial\Phi}{\partial a_1}+\frac{1}{3}a_1\frac{\partial\Phi}{\partial a_0} = \frac{1}{3}\frac{E}{(1-2\mu)}a_2 \qquad (4\text{-}30b)$$

Further more, according to experiment, the stress invariant $(C_2^{*2}-3C_1)$ can be taken to depend only on the combination of the strain invariants a_2, a_1 i.e., on the quantity $a_2^2-3a_1$. In order to bring the Equation (4-29c) into agreement with this fact it suffices to set

$$\frac{\partial\Phi}{\partial a_0} = 0. \qquad (4\text{-}30c)$$

and to regard $\frac{\partial\Phi}{\partial a_1}$ as a function of $a_2^2-3a_1$ alone. Taking into account these assumptions as well as Equation (4-30b), Equation (4-29) assumes the form

$$\frac{\partial\Phi}{\partial a_2}-\frac{2}{3}a_2\Psi(T) = \frac{E}{3(1-2\mu)}a_2 \qquad (4\text{-}31a)$$

$$S = T\cdot\Psi(T) \qquad (4\text{-}31b)$$

where $$S = \frac{2}{\sqrt{3}} \sqrt{(C_2^{2*} - 3C_1^*)} \tag{4-31c}$$

$$T = \frac{2}{\sqrt{3}} \sqrt{(a_2^2 - 3a_1)} \tag{4-31d}$$

$$\Psi(T) = -\frac{\partial \Phi}{\partial a_1} \tag{4-31e}$$

In the theory of Plasticity, it is denoted that

S = the intensity of tangential stresses

T = the intensity of shearing strain

Noting Equations (4-31a), (4-28a) and (4-30a), it follows that

$$\frac{\partial \Phi}{\partial a_2} = \frac{E}{3(1-2\mu)} a_2 + \frac{2}{3} a_2 \Psi(T).$$

$$= \frac{1}{3} C_2^* + \frac{2}{3} a_2 \Psi(T). \tag{4-32a}$$

Returning now to Equation (4-19) and substituting in it the values in the Equations (4-31e) and (4-32a), one obtains

$$[\nabla] = \left(\frac{1}{3} C_2^* + \frac{2}{3} a_2 \Psi(T)\right)[I] - \Psi(T)\left[[I]a_2 - [\varepsilon]\right]. \tag{4-32b}$$

Equation (4-32b) is precisely the stress-strain relation proposed by Hencky for elastic-plastic bodies.

Thus, Equation (4-32b) for the theory of plasticity is a special case of Equation (4-19). In other words, in spite of the irreversibility of a plastic deformation, it can be described by means of equations derived on the explicit assumption that the deformation is reversible. It should, however, be noted that the use of Equation (4-19) in the theory of plasticity is admissible only if the process of deformation is an active one, i.e., only if the deformation, during all its intermediate stages, is monotonic in the direction of increasing intensity of shearing strain. If unloading takes place during deformaton, Equations (4-32b) are no longer valid.

4.9 On The Simplest Variants of Nonlinear Stress-Strain Relations.

Suppose that the deformations are so large as to render Hooke's law inexact. Then as a second approximation, one can retain in Equation (4-26) those terms which contain the strain components to the third degree in addition to those containing them to the second degree. It is clear that the description of the elastic properties of the materail in this case requires a knowledge of five physical constants.

$$\Phi(a_2, a_1, a_0) = A_1 a_2^2 + A_2 a_1 + B_1 a_2^3 + B_2 a_2 a_1 + B_3 a_0 \qquad (4\text{-}33a)$$

Differentiating Φ with respect to a_2, a_1, a_0, one obtains

$$\begin{aligned} \frac{\partial \Phi}{\partial a_2} &= 2a_2 A_1 + 3a_2^2 B_1 + B_2 a_1 \\ \frac{\partial \Phi}{\partial a_1} &= A_2 + B_2 a_2 \\ \frac{\partial \Phi}{\partial a_0} &= B_3 \end{aligned} \qquad (4\text{-}33b)$$

Substituting these values into Equation (4-19), gives

$$\begin{aligned} [\nabla_R] &= (2a_2 A_1 + 3a_2^2 B_1 + a_1 B_2)[I] + (A_2 + B_2 a_2)([I]a_2 - [E]) + B_3[COF[E]] \\ &= a_2[I](2A_1 + A_2) - A_2[E] + 3a_2^2 B_1[I] + B_2 a_1[I] + B_2 a_2^2[I] \\ &\qquad - B_2 a_2[E] + B_3[COF[E]] \end{aligned}$$

Replacing A_1, A_2, B_1, B_2 and B_3 by the new constants E, $\mu, \beta_1, \beta_2, \beta_3$, yields

$$[\sigma_R] = \frac{E}{(1+\mu)} \left\{ \left(\frac{\mu}{(1-2\mu)} a_2 + \beta_2 a_2^2 - (\beta_1+\beta_3) a_1 \right) [I] + \left(1+(\beta_1+\beta_3)a_2\right)[\varepsilon] + \beta_3 [COF[\varepsilon]] \right\} \quad (4\text{-}34)$$

where

$$2A_1 + A_2 = \frac{\mu E}{(1+\mu)(1-2\mu)} \quad ; \quad A_2 = -\frac{E}{(1+\mu)}$$

$$3B_1 + B_2 = \frac{E}{(1+\mu)} \beta_2 \quad ; \quad B_3 = \frac{E}{(1+\mu)} \beta_3 \quad ; \quad B_2 = \frac{-E(\beta_1+\beta_3)}{(1+\mu)}. \quad (4\text{-}35)$$

It is essential to note that the second approximation differs from the first only in terms which are "even" functions of the strain components, i.e., terms which remain invariant if the signs of all strain components appearing in them are changed.

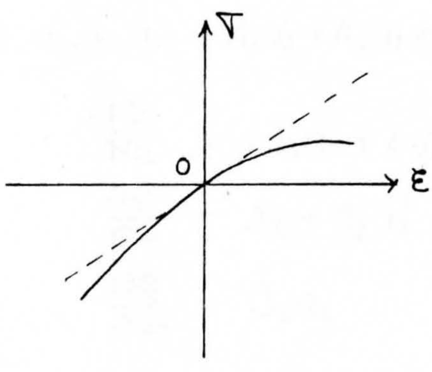


Figure (IV-1a)

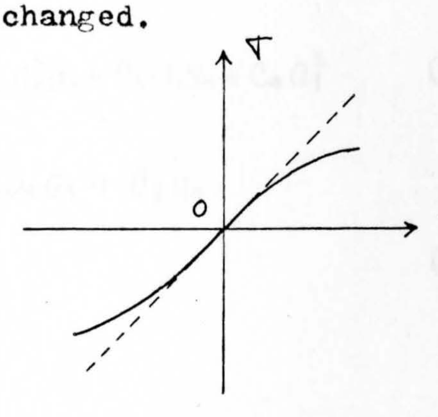


Figure (IV-1b)

The Extension-Compression Curve

The extension-compression curve for such a material must lie wholly on one side of its tangent at the origin O (Figure (IV-1a)). However, the majority of materials have extension-compression curves of the form shown in Figure IV-1b. It follows that deviations from Hooke's law are ordinarily conditioned not so much by terms containing the strains to even powers as by terms containing them to odd powers. In view of this, Equation (4-34) by no means yields all possible variants of extension-compression curves.

In the light of the above remarks, it is interesting to investigate the forms of the nonlinear stress-strain relation in which the stresses are odd functions of the strains. Thus, assuming the specific strain energy to be of the form

$$\Phi(a_2, a_1, a_0) = A_1 a_2^2 + A_2 a_1 + C_1 a_2^4 + C_2 a_2^2 a_1 + C_3 a_2 a_0 + C_4 a_1^2 \qquad (4\text{-}36a)$$

$$\begin{aligned} \frac{\partial \Phi}{\partial a_2} &= 2A_1 a_2 + 4a_2^3 C_1 + 2C_2 a_2 a_1 + C_3 a_0 \\ \frac{\partial \Phi}{\partial a_1} &= A_2 + C_2 a_2^2 + 2C_4 a_1 \\ \frac{\partial \Phi}{\partial a_0} &= C_3 a_2 \end{aligned} \qquad (4\text{-}36b)$$

Substituting these values into Equation (4-36a) gives

$$\begin{aligned} [\nabla_R] &= (2A_1 a_2 + 4a_2^3 C_1 + 2C_2 a_2 a_1 + C_3 a_0)[I] + (A_2 + C_2 a_2^2 + 2C_4 a_1) \\ &\qquad [[I]a_2 - [\varepsilon]] + C_3 a_2 [COF[\varepsilon]]. \\ &= a_2[I](2A_1 + A_2) + a_2^3[I](4C_1 + C_2) + 2a_2 a_1[I](C_2 + C_4) \\ &\qquad + C_3 a_0[I] - (A_2 + C_2 a_2^2 + 2C_4 a_1)[\varepsilon] + C_3 a_2[COF[\varepsilon]]. \end{aligned}$$

$$[\nabla_R] = \left\{ a_2(2A_1+A_2) + a_2^3(4C_1+C_2) + 2a_1a_2(C_2+C_4) + a_0C_3 \right\}[I]$$

$$- (A_2 + C_2a_2^2 + 2C_4a_1)[\varepsilon] + C_3a_2[\mathrm{COF}[\varepsilon]] . \qquad (4\text{-}37)$$

or

$$[\nabla_R] = \frac{E}{(1+\mu)}\left[\left\{\frac{\mu}{(1-2\mu)}a_2 + a_2^3(\gamma_3) - (2\gamma_1+\gamma_2+2\gamma_4)a_1a_2 + \gamma_4a_0\right\}[I]\right.$$

$$\left. + (1+(\gamma_1+\gamma_4)a_2^2 + \gamma_2a_1)[\varepsilon] + \gamma_4a_2[\mathrm{COF}[\varepsilon]]\right] \qquad (4\text{-}38)$$

where

$$2A_1 + A_2 = \frac{E\mu}{(1+\mu)(1-2\mu)}$$

$$A_2 = -\frac{E}{(1+\mu)} \qquad (4\text{-}39)$$

$$4C_1 = \frac{E}{(1+\mu)}(\gamma_1+\gamma_3+\gamma_4)$$

$$C_2 = -\frac{E}{(1+\mu)}(\gamma_1+\gamma_4)$$

$$C_3 = \frac{E\gamma_4}{(1+\mu)}$$

$$2C_4 = -\frac{E\gamma_2}{(1+\mu)}$$

Here $E, \mu, \gamma_1, \gamma_2, \gamma_3, \gamma_4$ are six physical constants, of which the last five are dimension-less and the first has the dimension of a stress.

Hence Equation (4-38) is the nonlinear relation between stresses and strains with the sic physical constants.

4.10 Summary

Case 1 Minimization of the total work

Taking into account the characteristic of the material referred to as the limits of proportionality of an isotropic material, it follows that the minimization of the total work expression yields

$$\{\nabla\}^T\left[\frac{\partial\Phi}{\partial\varepsilon}\right][J]^T + |[J]|\{F_x^*\}^T = \{0\}^T$$

or

$$\{\nabla\}^T\left[\frac{\partial\Phi}{\partial\varepsilon}\right]\left[[I]+[e]-[\omega]\right] + |[J]|\{F_x^*\}^T = \{0\}^T.$$

Comparison of the latter two equations with Equations (3-34b) yields

$$[\nabla] = \left[\frac{\partial\Phi}{\partial\varepsilon}\right].$$

The relationship between stress and strain for the four cases is summarized below

$$[\nabla] = \frac{\partial\Phi}{\partial a_2}[I] + \frac{\partial\Phi}{\partial a_1}\left[[I]a_2-[\varepsilon]\right] + \frac{\partial\Phi}{\partial a_0}\left[\mathrm{COF}[\varepsilon]\right],$$

for Case 1 and 2

$$[\varepsilon] = [e] + \frac{1}{2}\left[[e]^2+[e][\omega]-[\omega][e]-[\omega]^2\right],$$

for Case 3

$$[\varepsilon] \simeq [e] - \frac{1}{2}[\omega]^2,$$

for Case 4

$$[\varepsilon] \approx [e].$$

CHAPTER V

PROBLEMS ON THE DEFORMATION OF FLEXIBLE BODIES

5.1 Deformation of Rods (First Approximation)

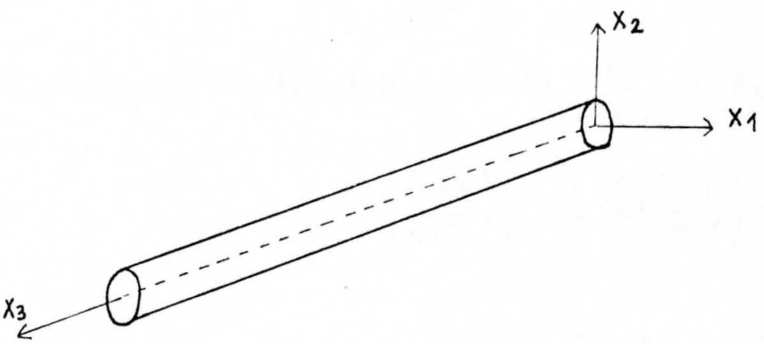


Figure (V-1) Thin Prismatic Rod

Consider a thin prismatic rod of arbitary cross section as shown (See Figure(V-1)) The origin of the coordinate system X_1, X_2, X_3 is placed at the center of gravity of the area of one of the ends of the rod, and the X_3-axis is directed along the rod. The X_1 - and X_2-axis lie along the principal axes of initia of the cross-section. The parameters $U_1(x_1, x_2, x_3)$ $U_2(x_1, x_2, x_3)$, $u_3(x_1, x_2, x_3)$ denote the displacements of an arbitary point of the rod due to a deformation.
Since the variations of the x_1 and x_2 coordinates in this problem are substantially smaller than the variation of the x_3 coordinate, it is assumed that the power series expansions of

the displacements in x_1 and x_2 coverge rapidly enough within limits which are of interest. Accordingly, the displacements of an arbitary point of the bar are expressed in the form

$$u_1(x_1, x_2, x_3) = u_1(0,0,x_3) + x_1\left(\frac{\partial u_1}{\partial x_1}\right)_0 + x_2\left(\frac{\partial u_1}{\partial x_2}\right)_0 + \frac{1}{2}x_1^2\left(\frac{\partial^2 u_1}{\partial x_1^2}\right) + \frac{1}{2}x_2^2\left(\frac{\partial^2 u_1}{\partial x_2^2}\right)_0 + x_1x_2\left(\frac{\partial^2 u_1}{\partial x_1 \partial x_2}\right) + \cdots$$

$$u_2(x_1, x_2, x_3) = u_2(0,0,x_3) + x_1\left(\frac{\partial u_2}{\partial x_1}\right)_0 + x_2\left(\frac{\partial u_2}{\partial x_2}\right)_0 + \frac{1}{2}x_1^2\left(\frac{\partial^2 u_2}{\partial x_1^2}\right)_0 + \frac{1}{2}x_2^2\left(\frac{\partial^2 u_2}{\partial x_2^2}\right) + x_1x_2\left(\frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right) + \cdots \tag{5-1}$$

$$u_3(x_1, x_2, x_3) = u_3(0,0,x_3) + x_1\left(\frac{\partial u_3}{\partial x_1}\right)_0 + x_2\left(\frac{\partial u_3}{\partial x_2}\right)_0 + \frac{1}{2}x_1^2\left(\frac{\partial^2 u_3}{\partial x_1^2}\right)_0 + \frac{1}{2}x_2^2\left(\frac{\partial^2 u_3}{\partial x_2^2}\right)_0 + x_1x_2\left(\frac{\partial^2 u_3}{\partial x_1 \partial x_2}\right)_0 + \cdots$$

where the operation $(\quad)_0$ implies evaluation at the point $x_1 = x_2 = 0$

Denoting

$$\hat{u}_1 = u_1(0,0,x_3)\ ;\ \hat{u}_2 = u_2(0,0,x_3)\ ;\ \hat{u}_3 = u_3(0,0,x_3) \tag{5-2a}$$

$$\vartheta_1 = \left(\frac{\partial u_1}{\partial x_1}\right)_0\ ;\ \psi_1 = \left(\frac{\partial u_2}{\partial x_1}\right)_0\ ;\ \chi_{01} = \left(\frac{\partial u_3}{\partial x_1}\right)$$

$$\vartheta_2 = \left(\frac{\partial u_1}{\partial x_2}\right)_0\ ;\ \psi_2 = \left(\frac{\partial u_2}{\partial x_2}\right)_0\ ;\ \chi_{02} = \left(\frac{\partial u_3}{\partial x_2}\right) \tag{5-2b}$$

$$\bar{u}_1 = \frac{1}{2}x_1^2\left(\frac{\partial^2 u_1}{\partial x_1^2}\right)_0 + \frac{1}{2}x_2^2\left(\frac{\partial^2 u_1}{\partial x_2^2}\right)_0 + x_1x_2\left(\frac{\partial^2 u_1}{\partial x_1 \partial x_2}\right)_0 + \cdots$$

$$\bar{u}_2 = \frac{1}{2}x_2^2\left(\frac{\partial^2 u_2}{\partial x_1^2}\right)_0 + \frac{1}{2}x_2^2\left(\frac{\partial^2 u_2}{\partial x_2^2}\right)_0 + x_1x_2\left(\frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right)_0 + \cdots \tag{5-2c}$$

$$\bar{u}_3 = \frac{1}{2}x_1^2\left(\frac{\partial^2 u_3}{\partial x_1^2}\right)_0 + \frac{1}{2}x_2^2\left(\frac{\partial^2 u_3}{\partial x_2^2}\right)_0 + x_1x_2\left(\frac{\partial^2 u_3}{\partial x_1 \partial x_2}\right)_0 + \cdots$$

where

(a) $\hat{u}_1, \hat{u}_2, \hat{u}_3$ are the displacements of the points on the axis of the rod and consequently, are functions of X_3 alone

(b) $1+\vartheta_1, \vartheta_2, \psi_1, 1+\psi_2, \chi_1, \chi_2$ are of the same order of magnitude as the direction cosines of those fibers in the strained state which were initially parallel to the to the X_1- and X_2 -axes (Equation 1-13b). In addition, it is assumed that these parameters (some or all of them) may substantially exceed the elongations and shears, also are functions of X_3 alone.

(c) $\bar{u}_1, \bar{u}_2, \bar{u}_3$ contain all remaining terms, (beginning with the fourth), of the power series for the displacements. It is clear from this that for $X_1 = 0$, $X_2 = 0$

$$\bar{u}_1 = \bar{u}_2 = \bar{u}_3 = \frac{\partial \bar{u}_1}{\partial x_1} = \frac{\partial \bar{u}_2}{\partial x_1} = \frac{\partial \bar{u}_1}{\partial x_2} = \frac{\partial \bar{u}_2}{\partial x_2} = \frac{\partial \bar{u}_3}{\partial x_1} = \frac{\partial \bar{u}_3}{\partial x_2} = 0$$

In addition, $\bar{u}_1, \bar{u}_2, \bar{u}_3$ are regarded as the correction terms in Equation (5-1), which are very small in comparison with the remaining terms.

Thus, Equations (5-1) becomes

$$\begin{aligned} u_1(x_1,x_2,x_3) &= \hat{u}_1(x_3) + x_1\vartheta_1(x_3) + x_2\vartheta_2(x_3) + \bar{u}_1(x_1,x_2,x_3) \\ u_2(x_1,x_2,x_3) &= \hat{u}_2(x_3) + x_1\psi_1(x_3) + x_2\psi_2(x_3) + \bar{u}_2(x_1,x_2,x_3) \quad (5\text{-}3) \\ u_3(x_1,x_2,x_3) &= \hat{u}_3(x_3) + x_1\chi_1(x_3) + x_2\chi_2(x_3) + \bar{u}_3(x_1,x_2,x_3) \end{aligned}$$

Since the first two rows of the first matrix on the right hand side of Equation (5-4c) is zero, one obtains

$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \vartheta_1 & \psi_1 & \chi_1 \\ \vartheta_2 & \psi_2 & \chi_2 \\ \frac{\partial \hat{u}_1}{\partial x_3} & \frac{\partial \hat{u}_2}{\partial x_3} & \frac{\partial \hat{u}_3}{\partial x_3} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_1\frac{\partial \vartheta_1}{\partial x_3} + x_2\frac{\partial \vartheta_2}{\partial x_3} ; & x_1\frac{\partial \psi_1}{\partial x_3} + x_2\frac{\partial \psi_2}{\partial x_3} ; & x_1\frac{\partial \chi_1}{\partial x_3} + x_2\frac{\partial \chi_2}{\partial x_3} \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{\partial \bar{u}_1}{\partial x_1} & \frac{\partial \bar{u}_2}{\partial x_1} & \frac{\partial \bar{u}_3}{\partial x_1} \\ \frac{\partial \bar{u}_1}{\partial x_2} & \frac{\partial \bar{u}_2}{\partial x_2} & \frac{\partial \bar{u}_3}{\partial x_2} \\ \frac{\partial \bar{u}_1}{\partial x_3} & \frac{\partial \bar{u}_2}{\partial x_3} & \frac{\partial \bar{u}_3}{\partial x_3} \end{bmatrix} \tag{5-4d}$$

In symbolic form the latter equation becomes

$$[D]^T = [\hat{D}]^T + [K]^T + [\bar{D}]^T \tag{5-4e}$$

or

$$[D] = [\hat{D}] + [K] + [\bar{D}]$$

Substituting the values of matrix $[D]$ and $[D]^T$ into Equation (2-4b) gives

$$[\varepsilon] = \frac{1}{2}\left[[D] + [D]^T + [D]^T[D]\right]$$

$$= \frac{1}{2}\Big[[\hat{D}] + [K] + [\bar{D}] + [\hat{D}]^T + [K] + [\bar{D}] + [\hat{D}]^T[\hat{D}]$$

$$+ [\hat{D}]^T[K] + [\hat{D}]^T[\bar{D}] + [K]^T[\hat{D}] + [K]^T[K] + [K]^T[\bar{D}]$$

$$+ [\bar{D}]^T[\hat{D}] + [\bar{D}]^T[K] + [\bar{D}]^T[\bar{D}]\Big]$$

$$[\varepsilon] = [\hat{\varepsilon}] + \frac{1}{2}\left[[\hat{J}][K] + [K]^T[\hat{J}] + [K]^T[K]\right] + \frac{1}{2}\Big[[\bar{D}] + [\bar{D}]^T$$

$$+ [\hat{D}][\bar{D}] + [K]^T[\bar{D}] + [\bar{D}]^T[\hat{D}] + [\bar{D}][K] + [\bar{D}]^T[\bar{D}]\Big] \tag{5-5a}$$

where

$$[\hat{\varepsilon}] = \frac{1}{2}\left[[\hat{D}] + [\hat{D}]^T + [\hat{D}]^T[\hat{D}]\right] \tag{5-5b}$$

Also Equation (5-3) can be rewritten into the matrix form as follows:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{Bmatrix} + \begin{bmatrix} \vartheta_1 & \vartheta_2 & 0 \\ \psi_1 & \psi_2 & 0 \\ \chi_{o1} & \chi_{o2} & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{Bmatrix} \qquad (5\text{-}4a)$$

with

$$\begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \{u_1, u_2, u_3\} = \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \{\hat{u}_1, \hat{u}_2, \hat{u}_3\} + \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \{x_1\vartheta_1 + x_2\vartheta_2, \psi_1 x_1 + \psi_2 x_2, \chi_{o1} x_1 + \chi_{o2} x_2\}$$

$$+ \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \{\bar{u}_1, \bar{u}_2, \bar{u}_3\} \qquad (5\text{-}4b)$$

or

$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial \hat{u}_1}{\partial x_1} & \frac{\partial \hat{u}_2}{\partial x_1} & \frac{\partial \hat{u}_3}{\partial x_1} \\ \frac{\partial \hat{u}_1}{\partial x_2} & \frac{\partial \hat{u}_2}{\partial x_2} & \frac{\partial \hat{u}_3}{\partial x_2} \\ \frac{\partial \hat{u}_1}{\partial x_3} & \frac{\partial \hat{u}_2}{\partial x_3} & \frac{\partial \hat{u}_3}{\partial x_3} \end{bmatrix} + \begin{bmatrix} \vartheta_1 & \psi_1 & \chi_{o1} \\ \vartheta_2 & \psi_2 & \chi_{o2} \\ x_1\frac{\partial \vartheta_1}{\partial x_3} + x_2\frac{\partial \vartheta_2}{\partial x_3}; & x_1\frac{\partial \psi_1}{\partial x_3} + x_2\frac{\partial \psi_2}{\partial x_3}; & x_1\frac{\partial \chi_{o1}}{\partial x_3} + x_2\frac{\partial \chi_{o2}}{\partial x_3} \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{\partial \bar{u}_1}{\partial x_1} & \frac{\partial \bar{u}_2}{\partial x_1} & \frac{\partial \bar{u}_3}{\partial x_1} \\ \frac{\partial \bar{u}_1}{\partial x_2} & \frac{\partial \bar{u}_2}{\partial x_2} & \frac{\partial \bar{u}_3}{\partial x_3} \\ \frac{\partial \bar{u}_1}{\partial x_3} & \frac{\partial \bar{u}_2}{\partial x_3} & \frac{\partial \bar{u}_3}{\partial x_3} \end{bmatrix} \qquad (5\text{-}4c)$$

represents the strain matrix for points on the X_3 axis of the rod and consequently are functions of X_3 alone; also for these points

$$[\hat{J}] = [I] + [\hat{D}]$$

$$[\hat{\varepsilon}] = \begin{bmatrix} \hat{\varepsilon}_{11} & \frac{1}{2}\hat{\varepsilon}_{12} & \frac{1}{2}\hat{\varepsilon}_{13} \\ & \hat{\varepsilon}_{22} & \frac{1}{2}\hat{\varepsilon}_{23} \\ \text{symmetric} & & \varepsilon_{33} \end{bmatrix} \qquad (5\text{-}5c)$$

In the first approximation of the deformation of rods are neglected in comparison with the remaining terms in Equation (5-3). Then Equations (5-5a) become $([\bar{D}] \approx [0])$

$$[\varepsilon] = [\hat{\varepsilon}] + \frac{1}{2}\left[[\hat{J}]^T[K] + [K]^T[\hat{J}] + [K]^T[K]\right] \qquad (5\text{-}6)$$

This accuracy is not adequate for the deformation of rods, since a solution in this form cannot be subjected to boundary conditions which arise in practice. Hence, in studying the deformation of rods, it becomes necessary to take the displacements in the form given in the more complicated Equation (5-5a) rather than in the Equation (5-6). However, for greater clarity, it is convenient to assume that Equation (5-6) is adequate. After completing the computations, it will be found that enough terms have not retained to give a full solution of the problem. At this point the necessary corrections will be introduced. This will cause no special difficulties, since by this time the reader will have a complete picture of the method.

In the first approximation, Equation (5-6) is expressed in the strain components as follow:

$$\varepsilon_{11} = \hat{\varepsilon}_{11} \quad ; \quad \varepsilon_{22} = \hat{\varepsilon}_{22} \quad , \quad \varepsilon_{12} = \hat{\varepsilon}_{12}$$

$$\begin{aligned}
\varepsilon_{13} = \hat{\varepsilon}_{13} &+ x_1\left(\frac{\partial \vartheta_1}{\partial x_3} + \vartheta_1 \frac{\partial \vartheta_1}{\partial x_3} + \psi_1 \frac{\partial \psi_1}{\partial x_3} + \chi_1 \frac{\partial \chi_1}{\partial x_3}\right) \\
&+ x_2\left((1+\vartheta_1)\frac{\partial \vartheta_2}{\partial x_3} + \psi_1 \frac{\partial \psi_2}{\partial x_3} + \chi_1 \frac{\partial \chi_2}{\partial x_3}\right)
\end{aligned}$$

$$\begin{aligned}
\varepsilon_{23} = \hat{\varepsilon}_{23} &+ x_1\left(\vartheta_2 \frac{\partial \vartheta_1}{\partial x_3} + (1+\psi_2)\frac{\partial \psi_1}{\partial x_3} + \chi_2 \frac{\partial \chi_1}{\partial x_3}\right) \qquad (5\text{-}7) \\
&+ x_2\left(\frac{\partial \psi_2}{\partial x_3} + \vartheta_2 \frac{\partial \vartheta_2}{\partial x_3} + \psi_2 \frac{\partial \psi_2}{\partial x_3} + \chi_2 \frac{\partial \chi_2}{\partial x_3}\right)
\end{aligned}$$

$$\begin{aligned}
\varepsilon_{33} = \hat{\varepsilon}_{33} &+ x_1\left(\frac{\partial \hat{u}_1}{\partial x_3}\frac{\partial \vartheta_1}{\partial x_3} + \frac{\partial \hat{u}_2}{\partial x_3}\frac{\partial \psi_1}{\partial x_3} + \left(1+\frac{\partial \hat{u}_3}{\partial x_3}\right)\frac{\partial \chi_1}{\partial x_3}\right) \\
&+ x_2\left(\frac{\partial \hat{u}_1}{\partial x_3}\frac{\partial \vartheta_2}{\partial x_3} + \frac{\partial \hat{u}_2}{\partial x_3}\frac{\partial \psi_2}{\partial x_3} + \left(1+\frac{\partial \hat{u}_3}{\partial u_3}\right)\frac{\partial \chi_2}{\partial x_3}\right) \\
&+ \frac{x_1^2}{2}\left[\left(\frac{\partial \vartheta_1}{\partial x_3}\right)^2 + \left(\frac{\partial \psi_1}{\partial x_3}\right)^2 + \left(\frac{\partial \chi_1}{\partial x_3}\right)^2\right] + \frac{x_2^2}{2}\left[\left(\frac{\partial \vartheta_2}{\partial x_3}\right)^2 + \left(\frac{\partial \psi_2}{\partial x_3}\right)^2 + \left(\frac{\partial \chi_2}{\partial x_3}\right)^2\right] \\
&+ x_1 x_2\left[\frac{\partial \vartheta_1}{\partial x_3}\frac{\partial \vartheta_2}{\partial x_3} + \frac{\partial \psi_1}{\partial x_3}\frac{\partial \psi_2}{\partial x_3} + \frac{\partial \chi_1}{\partial x_3}\frac{\partial \chi_2}{\partial x_3}\right]
\end{aligned}$$

Also Equation (5-5b) expressed into

$$\begin{aligned}
\hat{\varepsilon}_{11} &= \vartheta_1 + \tfrac{1}{2}\left(\vartheta_1^2 + \psi_1^2 + \chi_1^2\right) \\
\hat{\varepsilon}_{22} &= \psi_2 + \tfrac{1}{2}\left(\vartheta_2^2 + \psi_2^2 + \chi_2^2\right) \\
\hat{\varepsilon}_{12} &= \vartheta_2(1+\vartheta_1) + \psi_1(1+\psi_2) + \chi_1\chi_2 \qquad (5\text{-}8)
\end{aligned}$$

$$\hat{\varepsilon}_{13} = (1+\vartheta_1)\frac{\partial \hat{u}_1}{\partial x_3} + \psi_1 \frac{\partial \hat{u}_2}{\partial x_3} + \chi_1 \left(1+\frac{\partial \hat{u}_3}{\partial x_3}\right)$$

$$\hat{\varepsilon}_{23} = \vartheta_2 \frac{\partial \hat{u}_1}{\partial x_3} + (1+\psi_2)\frac{\partial \hat{u}_2}{\partial x_3} + \chi_2 \left(1+\frac{\partial \hat{u}_3}{\partial x_3}\right)$$

$$\hat{\varepsilon}_{33} = \frac{\partial \hat{u}_3}{\partial x_3} + \frac{1}{2}\left[\left(\frac{\partial \hat{u}_1}{\partial x_3}\right)^2 + \left(\frac{\partial \hat{u}_2}{\partial x_3}\right)^2 + \left(\frac{\partial \hat{u}_3}{\partial x_3}\right)^2\right]$$

Denoting

$$k_{11} = \frac{\partial \hat{u}_1}{\partial x_3}\frac{\partial \vartheta_1}{\partial x_3} + \frac{\partial \hat{u}_2}{\partial x_3}\frac{\partial \psi_1}{\partial x_3} + \left(1+\frac{\partial \hat{u}_3}{\partial x_3}\right)\frac{d\chi_1}{dx_3}$$

$$k_{22} = \frac{d\hat{u}_1}{dx_3}\frac{\partial \vartheta_2}{\partial x_3} + \frac{\partial \hat{u}_2}{\partial x_3}\frac{d\psi_2}{dx_3} + \left(1+\frac{\partial \hat{u}_3}{\partial x_3}\right)\frac{\partial \chi_2}{\partial x_3} \quad (5\text{-}9)$$

$$k_{12} = (1+\vartheta_1)\frac{\partial \vartheta_2}{\partial x_3} + \psi_1 \frac{\partial \psi_2}{\partial x_3} + \chi_1 \frac{\partial \chi_2}{\partial x_3}$$

and

$$v_{11} = \frac{1}{2}\left[\left(\frac{\partial \vartheta_1}{\partial x_3}\right)^2 + \left(\frac{\partial \psi_1}{\partial x_3}\right)^2 + \left(\frac{\partial \chi_1}{\partial x_3}\right)^2\right]$$

$$v_{22} = \frac{1}{2}\left[\left(\frac{\partial \vartheta_2}{\partial x_3}\right)^2 + \left(\frac{\partial \psi_2}{\partial x_3}\right)^2 + \left(\frac{\partial \chi_2}{\partial x_3}\right)^2\right] \quad (5\text{-}10)$$

$$v_{12} = \frac{\partial \vartheta_1}{\partial x_3}\frac{\partial \vartheta_2}{\partial x_3} + \frac{\partial \psi_1}{\partial x_3}\frac{\partial \psi_2}{\partial x_3} + \frac{\partial \chi_1}{\partial x_3}\frac{\partial \chi_2}{\partial x_3}$$

then Equation (5-7) is rewritten in the form

$$\varepsilon_{11} = \hat{\varepsilon}_{11}, \quad \varepsilon_{22} = \hat{\varepsilon}_{22}, \quad \varepsilon_{12} = \hat{\varepsilon}_{12}$$

$$\varepsilon_{13} = \hat{\varepsilon}_{13} + x_1 \frac{d\hat{\varepsilon}_{11}}{dx_3} + x_2 k_{12}$$

$$\varepsilon_{23} = \hat{\varepsilon}_{23} + x_1\left(\frac{d\hat{\varepsilon}_{12}}{dx_3} - k_{12}\right) + x_2 \frac{d\hat{\varepsilon}_{22}}{dx_3} \quad (5\text{-}11)$$

$$\varepsilon_{33} = \hat{\varepsilon}_{33} + x_1 k_{11} + x_2 k_{22} + x_1^2 v_{11} + x_2^2 v_{22} + x_1 x_2 v_{12}$$

where the quantities k_{11}, k_{22}, k_3 are functions of x_3 alone v_{11}, v_{22}, v_{12} are the coefficients.

As was pointed out before, all the parameters $\vartheta_1, \vartheta_2, \psi_1, \psi_2, \chi_1, \chi_2$ or at least some of the, must be regarded as substantially exceeding the strain components, since in the bending of a thin rod some, or all, of the angles of rotation are large in comparison with the elongations and shears. For the same reason, the derivatives $\frac{d\hat{u}_1}{dx_3}, \frac{d\hat{u}_2}{dx_3}, (1+\frac{d\hat{u}_3}{dx_3})$ (Equation I-13a) posses the same property. Hence it follows that the right-hand sides of Equation (5-8) must represent small differences of large terms, thus

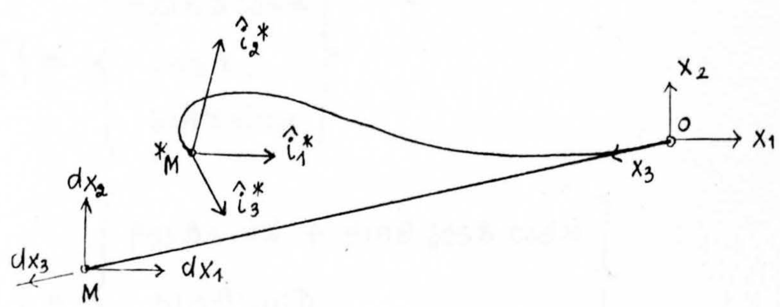
$$[\hat{\varepsilon}] = 0 \tag{5-12}$$

Here, naturally, the equation should not be interpreted as meaning that all the strain components of the rod along its axis are negligible. Equation (5-8) can be rewritten as follow

$$
\begin{aligned}
[\hat{\varepsilon}] &= \tfrac{1}{2}\left[[\hat{D}]+[\hat{D}]^T+[\hat{D}]^T[\hat{D}]\right] \\
0 &= [\hat{D}]+[\hat{D}]^T+[\hat{D}]^T[\hat{D}] \\
[I] &= [\hat{D}]+[\hat{D}]^T[\hat{J}]+[I] \\
[I] &= [\hat{J}]+[\hat{D}]^T[\hat{J}] \\
[I] &= [\hat{J}][\hat{J}]^T .
\end{aligned}
\tag{5-13a}
$$

Then

$$
\begin{aligned}
[\hat{J}] &= \text{ORTHOGONAL MATRIX} \\
[I] &= [\hat{J}]^T[\hat{J}]
\end{aligned}
\tag{5-13b}
$$



Figure(V-2) Geometric Deformation of Point on the Axis of the Thin Rod

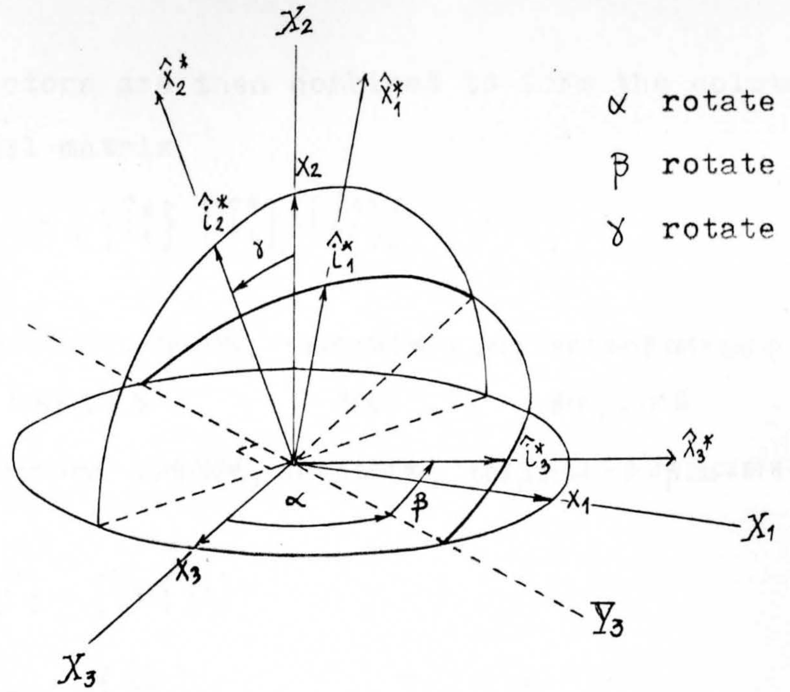


Figure (V-3) Euler Angles of Rotations

In accordance with Figure (V-3), a system of three mutually perpendicular directions are defined with respect to one another by means of the three Euler angles. It may be easily shown that

$$\{\hat{i}_1^*\} = \begin{Bmatrix} \cos\beta\cos\gamma\cos\alpha - \sin\beta\sin\alpha \\ \cos\beta\sin\gamma \\ -\cos\beta\cos\gamma\sin\alpha - \sin\beta\cos\alpha \end{Bmatrix}$$

$$\{\hat{i}_2^*\} = \begin{Bmatrix} -\sin\gamma\cos\alpha \\ \cos\gamma \\ \sin\gamma\sin\alpha \end{Bmatrix} \qquad (5\text{-}14)$$

$$\{\hat{i}_3^*\} = \begin{Bmatrix} \cos\beta\sin\gamma + \sin\beta\cos\gamma\cos\alpha \\ \sin\beta\sin\gamma \\ \cos\beta\cos\alpha - \sin\beta\cos\gamma\sin\alpha \end{Bmatrix}$$

These vectors are then combined to form the columns of the orthogonal matrix

$$[EU] = [\{\hat{i}_1^*\}\ \{\hat{i}_2^*\}\ \{\hat{i}_3^*\}]$$

$$= \begin{bmatrix} \cos\beta\cos\gamma\cos\alpha - \sin\beta\sin\alpha; & -\sin\gamma\cos\alpha; & \cos\beta\sin\gamma + \sin\beta\cos\gamma\cos\alpha \\ \cos\beta\sin\gamma & \cos\gamma & \sin\beta\sin\gamma \\ -\cos\beta\cos\gamma\sin\alpha - \sin\beta\cos\alpha; & \sin\gamma\sin\alpha; & \cos\beta\cos\alpha - \sin\beta\cos\gamma\sin\alpha \end{bmatrix} \quad (5\text{-}15)$$

Thus,

$$\{\hat{i}^*\} = [EU]^T\{\bar{i}\} \quad (5\text{-}16)$$

where

$$\{\hat{i}^*\} = \begin{Bmatrix} \hat{i}_1^* \\ \hat{i}_2^* \\ \hat{i}_3^* \end{Bmatrix} \text{ and } [EU]^T = [EU]^T$$

According to Figure(V-2), as a result of the deformation, the point M is displaced by the amounts $\hat{u}_1, \hat{u}_2, \hat{u}_3$ and assumes the position M^* while the line elements are directed along $\hat{i}_1^*, \hat{i}_2^*, \hat{i}_3^*$. If the angles of rotation of the elements of the rod are large in comparison with the shears, the latter may be neglected in determining the directions $\hat{i}_1^*, \hat{i}_2^*, \hat{i}_3^*$. With this approximation, $\hat{i}_1^*, \hat{i}_2^*, \hat{i}_3^*$ are taken as orthogonal and the parameters

$$1+\vartheta_1\ ,\ \vartheta_2\ ,\ \psi_1\ ,\ 1+\psi_2\ ,\ \chi_1\ ,\ \chi_2\ ,\ \frac{d\hat{u}_1}{dx_3}\ ,\ \frac{d\hat{u}_2}{dx_3}\ ,\ 1+\frac{d\hat{u}_3}{dx_3}$$

become equal to the direction cosines of $\hat{i}_1^*, \hat{i}_2^*, \hat{i}_3^*$ if the elongations are neglected in comparison with unity.

Recalling $[A] = [J][\frac{1}{1+E}]$

it follows that $[A] \approx [J]$

or $\{\hat{i}^*\} \approx [\hat{J}]^T \{\bar{i}\}$ (5-17)

Comparing Equation (5-17) with Equation (5-16), one obtains

$$[\hat{J}]^T = [EU]^T \qquad (5\text{-}18a)$$

since both are orthogonal matrices, thus, it follows from Equations (5-13a) and (5-13b) that

$$[\hat{J}] = [EU]^T \qquad (5\text{-}18b)$$

since the orthogonal condition $[EU]^T[EU] = [EU][EU]^T$.

Thus

$$\begin{bmatrix} 1+\vartheta_1 & \vartheta_2 & \frac{\partial \hat{u}_1}{\partial X_3} \\ \psi_1 & 1+\psi_2 & \frac{\partial \hat{u}_2}{\partial X_3} \\ \chi_1 & \chi_2 & 1+\frac{\partial \hat{u}_3}{\partial X_3} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\beta\cos\gamma\cos\alpha - \sin\beta\sin\alpha; & \cos\beta\sin\gamma; & -\cos\beta\cos\gamma\sin - \sin\beta\cos\alpha \\ -\sin\gamma\cos\gamma & \cos\gamma & \sin\gamma\sin\alpha \\ \cos\beta\sin\alpha + \sin\beta\cos\gamma\cos\alpha; & \sin\beta\sin\gamma; & \cos\beta\cos\alpha - \sin\beta\cos\gamma\sin\gamma \end{bmatrix} \qquad (5\text{-}18c)$$

Differentiating the matrix $[EU]$ with respect to X_3 and expanding yields

$$\frac{\partial \vartheta_1}{\partial X_3} = \cos\beta\cos\gamma(-\sin\alpha)\frac{d\alpha}{dX_3} + \cos\alpha\left(\cos\beta(-\sin\gamma)\frac{d\gamma}{dX_3} + \cos\gamma(-\sin\beta)\frac{d\beta}{dX_3}\right) - \left[\sin\beta\cos\alpha\frac{d\alpha}{dX_3} + \sin\alpha\cos\beta\frac{d\beta}{dX_3}\right]$$

$$= -(\cos\beta\cos\gamma\sin\alpha + \sin\beta\cos\alpha)\frac{d\alpha}{dX_3} - (\cos\alpha\cos\gamma\sin\beta + \sin\alpha\cos\beta)\frac{d\beta}{dX_3} - \cos\alpha\cos\beta\sin\gamma\frac{d\gamma}{dX_3}$$

$$= \frac{d\hat{u}_1}{dX_3}\frac{d\alpha}{dX_3} - \chi_1\frac{d\beta}{dX_3} - \cos\alpha\,\vartheta_2\frac{d\gamma}{dX_3} \qquad (5\text{-}19)$$

$$\frac{\partial \vartheta_2}{\partial x_3} = \cos\beta \cos\gamma \frac{d\gamma}{dx_3} - \sin\gamma \sin\beta \frac{d\beta}{dx_3}$$

$$= -\chi_2 \frac{d\beta}{dx_3} + \cos\beta (1+\psi_2) \frac{d\gamma}{dx_3}$$

$$\frac{d\chi_1}{dx_3} = (\cos\beta \cos\alpha - \cos\gamma \sin\beta \sin\alpha) \frac{d\alpha}{dx_3} + (\cos\gamma \cos\alpha \cos\beta$$

$$- \sin\alpha \sin\beta) \frac{d\beta}{dx_3} - \cos\alpha \sin\beta \sin\gamma \frac{d\gamma}{dx_3}$$

$$= \left(\frac{d\hat{u}_3}{dx_3} + 1\right) \frac{d\alpha}{dx_3} + (\vartheta_1 + 1) \frac{d\beta}{dx_3} - \psi_1 \sin\beta \frac{d\gamma}{dx_3}$$

$$\frac{d\psi_1}{dx_3} = \sin\gamma \sin\alpha \frac{d\alpha}{dx_3} - \cos\alpha \cos\gamma \frac{d\gamma}{dx_3}$$

$$= \frac{d\hat{u}_2}{dx_3} \frac{d\alpha}{dx_3} - \cos\alpha (1+\psi_2) \frac{d\gamma}{dx_3}$$

$$\frac{d\psi_2}{dx_3} = - \sin\gamma \frac{d\gamma}{dx_3}$$

$$\frac{d\chi_1}{dx_3} = \sin\beta \cos\gamma \frac{d\gamma}{dx_3} + \sin\gamma \cos\beta \frac{d\beta}{dx_3}$$

$$= \sin\beta (1+\psi_2) \frac{d\gamma}{dx_3} + \vartheta_2 \frac{d\beta}{dx_3}$$

Substituting Equation (5-19) into Equation (5-9), one obtains

$$k_{11} = \frac{d\alpha}{dx_3} + \frac{d\beta}{dx_3} \cos\gamma$$

$$k_{22} = \cos\alpha \sin\gamma \frac{d\beta}{dx_3} - \sin\gamma \frac{d\gamma}{dx_3} \qquad (5\text{-}20a)$$

$$k_{12} = \cos\alpha \frac{d\gamma}{dx_3} + \sin\alpha \sin\gamma \frac{d\beta}{dx_3}$$

or in matrix form as

$$\begin{Bmatrix} k_{11} \\ k_{22} \\ k_{12} \end{Bmatrix} = \begin{bmatrix} 1 & \cos\gamma & 0 \\ 0 & \cos\alpha \sin\gamma & -\sin\alpha \\ 0 & \sin\alpha \sin\gamma & \cos\alpha \end{bmatrix} \begin{Bmatrix} \frac{d\alpha}{dx_3} \\ \frac{d\beta}{dx_3} \\ \frac{d\gamma}{dx_3} \end{Bmatrix} \qquad (5\text{-}20b)$$

thus,

$$\begin{Bmatrix} \frac{d\alpha}{dx_3} \\ \frac{d\beta}{dx_3} \\ \frac{d\gamma}{dx_3} \end{Bmatrix} = \begin{bmatrix} 1 & \cos\gamma & 0 \\ 0 & \cos\alpha\sin\gamma & -\sin\alpha \\ 0 & \sin\alpha\sin\gamma & \cos\alpha \end{bmatrix}^{-1} \begin{Bmatrix} k_{11} \\ k_{22} \\ k_{33} \end{Bmatrix} . \qquad (5\text{-}20c)$$

By using the inverse operation, Equation (5-20c) becomes

$$\begin{Bmatrix} \frac{d\alpha}{dx_3} \\ \frac{d\beta}{dx_3} \\ \frac{d\gamma}{dx_3} \end{Bmatrix} = \begin{vmatrix} 1 & -\frac{\cos\gamma\cos\alpha}{\sin\gamma} & -\frac{\sin\alpha\cos\gamma}{\sin\gamma} \\ 0 & \frac{\cos\alpha}{\sin\gamma} & \frac{\sin\alpha}{\sin\gamma} \\ 0 & -\sin\alpha & \cos\alpha \end{vmatrix} \begin{Bmatrix} k_{11} \\ k_{22} \\ k_{12} \end{Bmatrix} \qquad (5\text{-}20d)$$

Substituting the values from Equations (5-19) into Equation (5-10) and using Equation (5-20a), the relations between k_{11}, k_{22}, k_{12} and v_{11}, v_{22}, v_{12} are

$$\begin{aligned} v_{11} &= \tfrac{1}{2}(k_{11}^2 + k_{12}^2) \\ v_{22} &= \tfrac{1}{2}(k_{22}^2 + k_{12}^2) \qquad (5\text{-}20e) \\ v_{12} &= k_{11}k_{22}. \end{aligned}$$

If these values of the coefficients v_{11}, v_{22}, v_{12} are substituted into the last of Equations (5-11) for (ε_{33}), the result becomes

$$\begin{aligned} \varepsilon_{33} = \hat{\varepsilon}_{33} + X_1 k_{11} + X_2 k_{22} + \tfrac{1}{2}(k_{11}^2 + k_{12}^2)X_1^2 \\ + \tfrac{1}{2}(k_{22}^2 + k_{12}^2)X_2^2 + k_{11}k_{22}X_1X_2 . \qquad (5\text{-}21) \end{aligned}$$

It is seen that the terms corresponding to these coefficients may be neglected, being quantities of the same order as the squares of the elongations and shears.

With this approximation, Equation (5-11) becomes

$$\begin{aligned}
&\varepsilon_{11} = \hat{\varepsilon}_{11}\,, \quad \varepsilon_{22} = \hat{\varepsilon}_{22}\,, \quad \varepsilon_{12} = \hat{\varepsilon}_{12} \\
&\varepsilon_{13} = \hat{\varepsilon}_{13} + x_1 \frac{d\hat{\varepsilon}_{11}}{dx_3} + x_2 k_{12} \\
&\varepsilon_{23} = \hat{\varepsilon}_{23} + x_2 \frac{d\hat{\varepsilon}_{22}}{dx_3} + x_1 \left(\frac{d\hat{\varepsilon}_{12}}{dx_3} - k_{12} \right) \\
&\varepsilon_{33} = \hat{\varepsilon}_{33} + x_1 k_{11} + x_2 k_{22}
\end{aligned} \qquad (5\text{-}22)$$

where k_{11}, k_{22}, k_{12} are determined by Equation (5-20a). Since the derivatives $\frac{d\hat{\varepsilon}_{11}}{dx_3}, \frac{d\hat{\varepsilon}_{22}}{dx_3}, \frac{d\hat{\varepsilon}_{12}}{dx_3}$ are ordinarily small compared to k_{11}, k_{12}, k_{22} which characterize the curvature of the axis of the rod in the strained state, the terms in Equation (5-22) containing these derivatives may be neglected. Hence, it follows that

$$\begin{aligned}
&\varepsilon_{11} = \hat{\varepsilon}_{11}\,, \quad \varepsilon_{22} = \hat{\varepsilon}_{22}\,, \quad \varepsilon_{12} = \hat{\varepsilon}_{12} \\
&\varepsilon_{13} = \hat{\varepsilon}_{13} + x_2 k_{12} \\
&\varepsilon_{23} = \hat{\varepsilon}_{23} - x_1 k_{12} \\
&\varepsilon_{33} = \hat{\varepsilon}_{33} + x_1 k_{11} + x_2 k_{22}
\end{aligned} \qquad (5\text{-}23)$$

since in Equation (5-23), the terms $x_2 k_{12}, x_1 k_{12}, x_1 k_{11}, x_2 k_{22}$ are of the same order of magnitude as the strain components.

These equations are based on the assumption that the elongations and shears are negligibly small in comparison with unity and the angles of rotation of the elements of the rod. However, in deriving Equation (5-23), it was postulated that only the first three terms of the Taylor series for the displacements need be retained. This assumption is not correct, as is seen by applying Equations (5-23) to the special case in which the rod is not bent but only twisted uniformly along its whole length.

For this case

$$\alpha = \beta = 0 \quad , \quad \gamma = \tau x_3 \tag{5-24a}$$

where τ is a constant coefficient. Substituting these values of the Euler angles into Equation (5-18c), the results are

$$\begin{aligned} \vartheta_1 &= -(1-\cos\tau x_3) \\ \vartheta_2 &= \sin\tau x_3 \\ \psi_1 &= -\sin\tau x_3 \\ \psi_2 &= -(1-\cos\tau x_3) \end{aligned} \tag{5-24b}$$

$$\chi_{o1} = \chi_{o2} = \frac{d\hat{u}_1}{dx_3} = \frac{d\hat{u}_2}{dx_3} = \frac{d\hat{u}_3}{dx_3} = 0.$$

Substituting the values above into Equation (5-3), the results of the displacements are

$$\begin{aligned} u_1 &= -x_1(1-\cos\tau x_3) + x_2\sin\tau x_3 \\ u_2 &= -x_1\sin\tau x_3 - x_2(1-\cos\tau x_3) \\ u_3 &= 0 \end{aligned} \tag{5-24c}$$

(by neglecting $\bar{u}_1, \bar{u}_2, \bar{u}_3$ in this first approximation). For the strain components, substitution into Equation (5-23), gives

$$\begin{aligned} \varepsilon_{11} &= \varepsilon_{22} = \varepsilon_{12} = \varepsilon_{33} = 0 \\ \varepsilon_{13} &= x_2\tau \quad , \quad \varepsilon_{23} = -x_1\tau \quad , \end{aligned} \tag{5-24d}$$

These expressions coincide with the "old" theory of torsion, rather than with the Saint-Venant Theory. The former, as is well-known, is inadequate since it does not permit the freeing of the lateral surface of the rod from stresses, which is essential in this problem. Hence it is clear that the general Equations (5-23) are also inadequate and must be corrected so as to yield Sain-Venant's Theory of torsion as a special case. In order to correct the results of this section the second approximation has to be derived by adding the remaining terms $\bar{u}_1, \bar{u}_2, \bar{u}_3$.

5.2 Deformation of Rods (Second Approximation)

In this second approximation, the whole Equation (5-5a) is used. Then it follows that

$$[\varepsilon] = [\hat{\varepsilon}] + \tfrac{1}{2}\big[[\hat{J}]^T[K] + [K]^T[\hat{J}] + [K]^T[K]\big] + \tfrac{1}{2}\big[[\bar{D}] + [\bar{D}]^T + [\hat{D}]^T[\bar{D}] + [K]^T[\bar{D}] + [\bar{D}]^T[\hat{D}] + [\bar{D}]^T[K] + [\bar{D}]^T[\bar{D}]\big].$$

As in the preceding section, Equation (5-5a) is written in the form

$$[\varepsilon] = [\text{First Approximation}] + \tfrac{1}{2}\big[[\bar{D}]^T + [\bar{D}] + [\hat{D}]^T[\bar{D}] + [K]^T[\bar{D}] + [\bar{D}]^T[\hat{D}] + [\bar{D}]^T[K]\big] \qquad (5\text{-}25a)$$

where [First Approximation] is the same as Equation (5-23). In accordance with Equation (5-5c), Equation (5-25a) is rewritten in the form

$$[\varepsilon] = [\text{First Approximation}] + \tfrac{1}{2}\big[[\hat{J}]^T[\bar{D}] + [\bar{D}]^T[\hat{J}] \; [K]^T[\bar{D}] + [\bar{D}]^T[K]\big] \qquad (5\text{-}25b)$$

Denoting

$$[G]^T = [\hat{J}]^T + [K]^T \qquad (5\text{-}25c)$$

$$[G] = [\hat{J}] + [K]$$

Equation (5-25b) becomes

$$[\varepsilon] = [\text{First Approximation}] + \tfrac{1}{2}\left[[G]^T[\bar{D}] + [\bar{D}]^T[G]\right] \qquad (5\text{-}25d)$$

Denoting

$$\{U\} = [G]^T\{\bar{u}\} \qquad (5\text{-}26a)$$

where

$$\{U\} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \text{supplementary displacements}$$

$$\{\bar{u}\} = \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ u_3 \end{Bmatrix} \qquad (5\text{-}26b)$$

$$[G] = \begin{bmatrix} 1+\vartheta_1 & \vartheta_2 & \frac{d\hat{u}_1}{dx_3} + x_1\frac{d\vartheta_1}{dx_3} + x_2\frac{d\vartheta_2}{dx_3} \\ \psi_1 & 1+\psi_2 & \frac{d\hat{u}_2}{dx_3} + x_1\frac{d\psi_1}{dx_3} + x_3\frac{d\psi_2}{dx_3} \\ \chi_1 & \chi_2 & \left(1+\frac{d\hat{u}_3}{dx_3}\right) + x_1\frac{d\chi_1}{dx_3} + x_3\frac{d\chi_2}{dx_3} \end{bmatrix}$$

U_1, U_2, U_3 are the functions of all three co-ordinates x_1, x_2, x_3 with the following properties:

(a) They are small in comparison with the lateral dimensions of the rod and their derivatives are of the same order of magnitude as the strain components. Thus, the products of pairs of derivatives and product of a derivative by a quantity of the order of magnitude of the strain components are neglected.

(b) For $X_1 = 0$, $X_2 = 0$

$$U_1 = U_2 = U_3 = \frac{\partial U_1}{\partial X_1} = \frac{\partial U_2}{\partial X_1} = \frac{\partial U_1}{\partial X_2} = \frac{\partial U_2}{\partial X_2} = \frac{\partial U_3}{\partial X_1} = \frac{\partial U_3}{\partial X_2} = 0.$$

Differentiating Equation (5-26a) with respect to x_1, x_2, x_3, one obtains

$$\frac{d}{dx_1}\{U\} = \left[\frac{d}{dx_1}[G]^T\right]\{\bar{u}\} + [G]^T\frac{d}{dx_1}\{\bar{u}\}$$

$$\frac{d}{dx_1}\{U\} - \left[\frac{d}{dx_1}[G]^T\right]\{\bar{u}\} = [G]^T\frac{d}{dx_1}\{\bar{u}\} \qquad (5\text{-}27a)$$

Analogously

$$\frac{d}{dx_2}\{U\} - \left[\frac{d}{dx_2}[G]^T\right]\{\bar{u}\} = [G]^T\frac{d}{dx_2}\{\bar{u}\}$$

$$\frac{d}{dx_3}\{U\} - \left[\frac{d}{dx_3}[G]^T\right]\{\bar{u}\} = [G]^T\frac{d}{dx_3}\{\bar{u}\}$$

These vectors are then combined to form the columns of the matrices $[P]$ and $[\bar{D}]$ as follow:

$$\left[\left\{\frac{d}{dx_1}\{U\}-\left[\frac{d}{dx_1}[G]^T\right]\{\bar{u}\}\right\}\left\{\frac{d}{dx_2}\{U\}-\left[\frac{d}{dx_2}[G]^T\right]\{\bar{u}\}\right\}\left\{\frac{d}{dx_3}\{U\}-\left[\frac{d}{dx_3}[G]^T\right]\{\bar{u}\}\right\}\right]$$

$$= [G]^T\left[\left\{\frac{d}{dx_1}\{\bar{u}\}\right\}\left\{\frac{d}{dx_2}\{\bar{u}\}\right\}\left\{\frac{d}{dx_3}\{\bar{u}\}\right\}\right]$$

or

$$[P] = [G]^T[\bar{D}] \qquad (5\text{-}27b)$$

$$[P]^T = [\bar{D}]^T[G]$$

Equation (5-25d) becomes

$$[\varepsilon] = [\text{First Approximation}] + \tfrac{1}{2}\left[[P]+[P]^{T}\right] \tag{5-28a}$$

where

$$[P]=\begin{bmatrix} \dfrac{dU_1}{dx_1} & \dfrac{dU_1}{dx_2} & \left(\dfrac{dU_1}{dx_3}-\bar{u}_1\dfrac{d\vartheta_1}{dx_3}-\bar{u}_2\dfrac{d\psi_1}{dx_3}-\bar{u}_3\dfrac{d\chi}{dx_3}\right) \\ \dfrac{dU_2}{dx_1} & \dfrac{dU_2}{dx_2} & \left(\dfrac{dU_2}{dx_3}-\bar{u}_1\dfrac{d\vartheta_2}{dx_3}-\bar{u}_2\dfrac{d\psi_2}{dx_3}-\bar{u}_3\dfrac{d\chi_2}{dx_3}\right) \\ \left(\dfrac{dU_3}{dx_1}-\bar{u}_1\dfrac{d\vartheta_1}{dx_3}-\bar{u}_2\dfrac{d\psi_1}{dx_3}-\bar{u}_3\dfrac{d\chi_1}{dx_3}\right); & \left(\dfrac{dU_3}{dx_2}-\bar{u}_1\dfrac{d\vartheta_2}{dx_3}-\bar{u}_2\dfrac{d\psi_2}{dx_3}-\bar{u}_3\dfrac{d\chi_2}{dx_3}\right); & P_{33} \end{bmatrix} \tag{5-28b}$$

$$P_{33} = \frac{dU_3}{dx_3} - \bar{u}_1\left(\frac{d^2\hat{u}_1}{dx_3^2} + x_1\frac{d^2\vartheta_1}{dx_3^2} + x_2\frac{d^2\vartheta_2}{dx_3^2}\right) - \bar{u}_2\left(\frac{d^2\hat{u}_2}{dx_3^2} + x_1\frac{d^2\psi_1}{dx_3^2} + x_2\frac{d^2\psi_2}{dx_3^2}\right) - \bar{u}_3\left(\frac{d^2\hat{u}_3}{dx_3} + x_1\frac{d^2\chi_1}{dx_3} + x_2\frac{d^2\chi_2}{dx_3}\right)$$

The Equation (5-28a) is expressed into the terms of strain components as follow

$$\varepsilon_{11} = \hat{\varepsilon}_{11} + \frac{dU_1}{dx_1}\ ; \quad \varepsilon_{22} = \hat{\varepsilon}_{22} + \frac{dU_2}{dx_2}\ ; \quad \varepsilon_{12} = \hat{\varepsilon}_{12} + \frac{dU_1}{dx_2} + \frac{dU_2}{dx_1}$$

$$\varepsilon_{13} = \hat{\varepsilon}_{13} + x_2 k_{12} + \frac{\partial U_1}{\partial x_3} + \frac{\partial U_3}{\partial x_1} - 2\underline{\left(\bar{u}_1\frac{d\vartheta_1}{dx_3} + \bar{u}_2\frac{d\psi_1}{dx_3} + \bar{u}_3\frac{d\chi_1}{dx_3}\right)}$$

$$\varepsilon_{23} = \hat{\varepsilon}_{23} - x_1 k_{12} + \frac{\partial U_2}{\partial x_3} + \frac{\partial U_3}{\partial x_2} - 2\underline{\left(\bar{u}_1\frac{d\vartheta_2}{dx_3} + \bar{u}_2\frac{d\psi_2}{dx_3} + \bar{u}_3\frac{d\chi_2}{dx_3}\right)}$$

$$\begin{aligned}\varepsilon_{33} = \hat{\varepsilon}_{33} + x_1 k_{11} + x_2 k_{22} + \frac{dU_3}{dx_3} &- \bar{u}_1\underline{\left(\frac{d^2\hat{u}_1}{dx_3^2} + x_1\frac{d^2\vartheta_1}{dx_3^2} + x_2\frac{d^2\vartheta_2}{dx_3^2}\right)} \\ &- \underline{\bar{u}_2\left(\frac{d^2\hat{u}_2}{dx_3^2} + x_1\frac{d^2\psi_1}{dx_3^2} + x_2\frac{d^2\psi_2}{dx_3^2}\right)} \\ &- \underline{\bar{u}_3\left(\frac{d^2\hat{u}_3}{dx_3^2} + x_1\frac{d^2\chi_1}{dx_3^2} + x_2\frac{d^2\chi_2}{dx_3^2}\right)}\ .\end{aligned} \tag{5-29}$$

In addition to the second approximation, the possibility of which was established in the first approximation, the underlined terms in Equation (5-29) may also be neglected.

Denoting

$$\mathfrak{X}_1 = \bar{u}_1 \frac{d\vartheta_1}{dx_3} + \bar{u}_2 \frac{d\psi_1}{dx_3} + \bar{u}_3 \frac{d\chi_1}{dx_3}$$

$$\mathfrak{X}_2 = \bar{u}_1 \frac{d\vartheta_2}{dx_3} + \bar{u}_2 \frac{d\psi_2}{dx_3} + \bar{u}_3 \frac{d\chi_2}{dx_3} \qquad (5\text{-}30a)$$

Substituting Equation (5-19) into the Equation (5-30), yields

$$\mathfrak{X}_1 = \bar{u}_1\left(\frac{d\hat{u}_1}{dx_3}\frac{d\alpha}{dx_3} - \chi_1\frac{d\beta}{dx_3} - \cos\alpha\,\vartheta_2\frac{d\gamma}{dx_3}\right) + \bar{u}_2\left(\frac{d\hat{u}_2}{dx_3}\frac{d\alpha}{dx_3} - \cos\alpha(1+\psi_2)\frac{d\gamma}{dx_3}\right)$$

$$+ \bar{u}_3\left(-\psi_1\sin\beta\frac{d\gamma}{dx_3} + (\vartheta_1+1)\frac{d\beta}{dx_3} + \left(\frac{d\hat{u}_3}{dx_3}+1\right)\frac{d\alpha}{dx_3}\right)$$

$$= \frac{d\alpha}{dx_3}\left(\bar{u}_1\frac{d\hat{u}_1}{dx_3} + \bar{u}_2\frac{d\hat{u}_2}{dx_3} + \bar{u}_3\left(\frac{d\hat{u}_3}{dx_3}+1\right)\right) + \left(-\chi_1\bar{u}_1 + (\vartheta_1+1)\bar{u}_3\right)\frac{d\beta}{dx_3}$$

$$- \left(\bar{u}_1\cos\alpha\,\vartheta_2 + \bar{u}_2\cos\alpha(1+\psi_2) + \bar{u}_3\psi_1\sin\beta\right)\frac{d\gamma}{dx_3}$$

$$(5\text{-}30b)$$

$$\mathfrak{X}_1 = J_{11}\frac{d\alpha}{dx_3} + J_{12}\frac{d\beta}{dx_3} + J_{13}\frac{d\gamma}{dx_3}$$

$$\mathfrak{X}_2 = \frac{d\beta}{dx_3}\left(-\chi_2\bar{u}_1 + \vartheta_2\bar{u}_3\right) + \frac{d\gamma}{dx_3}\left(\bar{u}_1\cos\beta(1+\psi_2) - \bar{u}_2\sin\gamma + \bar{u}_3\sin\beta(1+\psi_2)\right)$$

$$\mathfrak{X}_2 = J_{21}\frac{d\alpha}{dx_3} + J_{22}\frac{d\beta}{dx_3} + J_{23}\frac{d\gamma}{dx_3}$$

where $J_{21} = 0$

Writing Equation (5-30b) into the matrix form gives

$$\begin{Bmatrix}\mathfrak{X}_1\\ \mathfrak{X}_2\end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13}\\ J_{21} & J_{22} & J_{23}\end{bmatrix}\begin{Bmatrix}\frac{d\alpha}{dx_3}\\ \frac{d\beta}{dx_3}\\ \frac{d\gamma}{dx_3}\end{Bmatrix} \qquad (5\text{-}30c)$$

Substituting Equation (5-20d) into the Equation (5-30c) above yields

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \end{bmatrix} \begin{bmatrix} 1 & -\frac{\cos\gamma\cos\alpha}{\sin\gamma} & -\frac{\sin\alpha\cos\gamma}{\sin\gamma} \\ 0 & \frac{\cos\alpha}{\cos\gamma} & \frac{\sin\alpha}{\sin\gamma} \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} k_{11} \\ k_{22} \\ k_{12} \end{Bmatrix} \quad (5\text{-}30d)$$

(5-30d)

Equation (5-30d) is expressed in terms of X_1 and X_2 as follow

$$X_1 = J_{11} k_{11} + \left(-J_{11} \frac{\cos\gamma\cos\alpha}{\sin\gamma} + J_{12} \frac{\cos\alpha}{\sin\gamma} - J_{13}\sin\alpha\right) k_{22}$$

$$+\left(-J_{11} \frac{\sin\alpha\cos\gamma}{\sin\gamma} + J_{12} \frac{\sin\alpha}{\sin\gamma} + J_{13} \frac{\sin\alpha}{\sin\gamma}\right) k_{12} \quad (5\text{-}31a)$$

$$X_2 = J_{21} k_{11} + \left(J_{22} \frac{\cos\alpha}{\sin\gamma} - J_{23}\sin\alpha\right) k_{22}$$

$$+\left(J_{22} \frac{\sin\alpha}{\sin\gamma} + J_{23}\cos\alpha\right) k_{12} \quad (5\text{-}31b)$$

By considering the individual terms in Equations (5-31a) and (5-31b), it follows that

$$J_{22} \frac{\cos\alpha}{\sin\gamma} - J_{23}\sin\alpha$$

$$= -\chi_2 \bar{u}_1 \frac{\cos\alpha}{\sin\gamma} + \bar{u}_3 \vartheta_2 \frac{\cos\alpha}{\sin\gamma} - \left(\bar{u}_1 \cos\beta(1+\psi_2) - \bar{u}_2 \sin\gamma + \sin\beta(1+\psi_2)\bar{u}_3\right)\sin\alpha$$

$$= \bar{u}_1\left(-\chi_2 \frac{\cos\alpha}{\sin\gamma} - \cos\beta\sin\alpha(1+\psi_2)\right) + \bar{u}_2 \sin\gamma\sin\alpha + \bar{u}_3\left(\vartheta_3 \frac{\cos\alpha}{\sin\gamma} - \sin\alpha\sin\beta(1+\psi_3)\right)$$

$$= \bar{u}_1\left(\frac{d\hat{u}_1}{dx_3}\right) + \bar{u}_2\left(\frac{d\hat{u}_2}{dx_3}\right) + \bar{u}_3\left(1 + \frac{d\hat{u}_3}{dx_3}\right)$$

$$= J_{11} \quad (5\text{-}32a)$$

Analogously

$$J_{22} \frac{\sin\alpha}{\sin\gamma} + J_{23}\cos\alpha = (1+\vartheta_1)\bar{u}_1 + \psi_1 \bar{u}_2 + \chi_1 \bar{u}_3$$

$$= U_1 \quad (5\text{-}32b)$$

$$-J_{11}\frac{\cos\gamma\cos\alpha}{\sin\gamma} + J_{12}\frac{\cos\alpha}{\sin\gamma} - J_{13}\sin\alpha = 0 \tag{5-32c}$$

$$-J_{11}\frac{\sin\alpha\cos\gamma}{\sin\gamma} + J_{12}\frac{\sin\alpha}{\sin\gamma} + J_{13}\cos\alpha = -U_2 \tag{5-32d}$$

In accordance with Equation (5-26a), U_3 is expressed into the following form

$$U_3 = \frac{d\hat{u}_1}{dx_3}\bar{u}_1 + \frac{d\hat{u}_2}{dx_3}\bar{u}_2 + \left(1+\frac{d\hat{u}_3}{dx_3}\right)\bar{u}_3 + x_1\left(\frac{d\vartheta_1}{dx_3}\bar{u}_1 + \frac{d\psi_1}{dx_3}\bar{u}_2 + \frac{d\chi_1}{dx_3}\bar{u}_3\right)$$
$$+ x_2\left(\frac{d\vartheta_2}{dx_3}\bar{u}_1 + \frac{d\psi_2}{dx_3}\bar{u}_2 + \frac{d\chi_2}{dx_3}\bar{u}_3\right)$$

or

$$U_3 = J_{11} + x_1\mathbb{X}_1 + x_2\mathbb{X}_2$$
$$J_{11} = U_3 - x_1\mathbb{X}_1 - x_2\mathbb{X}_2 \tag{5-32e}$$

Taking all the above into account, Equation (5-31a), (5-31b) are rewritten as follow:

$$\mathbb{X}_1 = (U_3 - x_1\mathbb{X}_1 - x_2\mathbb{X}_2)k_{11} - U_2 k_{12}$$
$$\mathbb{X}_2 = (U_3 - x_1\mathbb{X}_1 - x_2\mathbb{X}_2)k_{22} - U_1 k_{12} \tag{5-33a}$$

or

$$\mathbb{X}_1 k_{22} = (U_3 - x_1\mathbb{X}_1 - x_2\mathbb{X}_2)k_{11}k_{22} - U_2 k_{12}k_{22}$$
$$\mathbb{X}_2 k_{11} = (U_3 - x_1\mathbb{X}_1 - x_2\mathbb{X}_2)k_{22}k_{11} - U_1 k_{12}k_{11} \tag{5-33b}$$

By subtracting Equations (5-33b), it follows that

$$\mathbb{X}_2 k_{11} - \mathbb{X}_1 k_{22} = U_1 k_{12}k_{11} + U_2 k_{12}k_{22}$$

or $$\mathbb{X}_2 = \mathbb{X}_1\frac{k_{22}}{k_{11}} + U_1 k_{12} + U_3\frac{k_{12}k_{22}}{k_{11}}$$

and

$$\mathbb{X}_1 = \frac{U_3 k_{11} - U_2(1 + x_2 k_{22})k_{12} - U_1 x_2 k_{11}k_{12}}{(1 + x_1 k_{11} + x_2 k_{22})} \tag{5-34a}$$

Analogously,

$$\mathrm{X}_2 = \frac{U_3 k_{22} - U_1 (1 + x_2 k_{11}) k_{12} - U_2 x_1 k_{22} k_{12}}{(1 + x_1 k_{11} + x_2 k_{22})} \qquad (5\text{-}34a)$$

In accordance with Equations (5-23), $x_1 k_{11}, x_2 k_{22}$ are of the same order of magnitude as the strain components, which are small compared to the unity, and also $x_2 k_{11} k_{12}, x_1 k_{22} k_{12}$ may be omitted. Thus, Equation (5-34a) is rewritten as follow

$$\begin{aligned} \mathrm{X}_1 &\approx U_3 k_{11} - U_2 k_{12} \\ \mathrm{X}_2 &\approx U_3 k_{22} + U_1 k_{12} \end{aligned} \qquad (5\text{-}34b)$$

Thus, the functions $\mathrm{X}_1, \mathrm{X}_2$ in Equation (5-34b) are of the same order of magnitude as the product of U_1, U_2, U_3 by the curvature parameters of the axis of the rod in the strained state. But the supplymentary displacements U_1, U_2, U_3 are always very small compared to the lateral dimensions of rods. Hence, since the products of the curvature parameters k_{11}, k_{22}, k_{12} of axis of the rod by the lateral dimensions are of the same order of magnitude as the strain components, one may conclude that X_1 and X_2 are always small in comparison with the elongations and shears. Similarly, it may be shown that the three last terms in the last of Equation (5-29) may be omitted.

Hence, with these approximation, the following expressions for the strain components of a thin initially prismatic bar are obtained:

$$\begin{aligned} &\varepsilon_{11} = \hat{\varepsilon}_{11} + \frac{\partial U_1}{\partial x_1}, \quad \varepsilon_{22} = \hat{\varepsilon}_{22} + \frac{\partial U_2}{\partial x_2}, \quad \varepsilon_{12} = \hat{\varepsilon}_{12} + \frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1} \\ &\varepsilon_{13} = \hat{\varepsilon}_{13} + x_2 k_{12} + \frac{\partial U_1}{\partial x_3} + \frac{\partial U_3}{\partial x_1} \\ &\varepsilon_{23} = \hat{\varepsilon}_{23} - x_1 k_{12} + \frac{\partial U_2}{\partial x_3} + \frac{\partial U_3}{\partial x_2} \\ &\varepsilon_{33} = \hat{\varepsilon}_{33} + x_1 k_{11} + x_2 k_{22} + \frac{\partial U_3}{\partial x_3} \end{aligned} \qquad (5\text{-}35)$$

Adjusting the supplementary displacements U_1, U_2, U_3, one may bring Equations (5-35) into agreement with boundary conditions on the lateral surface of rod.

5.3 Pure Torsion

By subjecting a rod to a uniform torsion along its whole length, it follows that

$$\alpha = \beta = 0$$

$$\gamma = \tau X_3 \qquad \text{where } \tau = \text{CONSTANT.}$$

Substituting α, β, γ into Equations (5-20a), gives

$$k_{11} = k_{22} = 0 \quad , \quad k_{12} = \tau \tag{5-36a}$$

Furthermore, by neglecting the strains which are uniformly distributed along the cross-section of the rod, yields

$$[\hat{\varepsilon}] = 0 \tag{5-36b}$$

with

$$\{\hat{u}\} = 0 \tag{5-36c}$$

Noting the above, Equation (5-4a) is written in the form

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} -(1-\cos\tau X_3) & \sin\tau X_3 & 0 \\ -\sin\tau X_3 & -(1-\cos\tau X_3) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} + \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{Bmatrix} \tag{5-36d}$$

Also Equation (5-35) is rewritten in the form

$$\begin{aligned} \varepsilon_{11} &= \frac{\partial U_1}{\partial X_1} \quad , \quad \varepsilon_{22} = \frac{\partial U_2}{\partial X_2} \quad , \quad \varepsilon_{12} = \frac{\partial U_1}{\partial X_2} + \frac{\partial U_2}{\partial X_1} \\ \varepsilon_{13} &= X_2 \tau + \frac{\partial U_1}{\partial X_3} + \frac{\partial U_3}{\partial X_1} \\ \varepsilon_{23} &= -X_1 \tau + \frac{\partial U_2}{\partial X_3} + \frac{\partial U_3}{\partial X_2} \\ \varepsilon_{33} &= \frac{\partial U_3}{\partial X_3} \end{aligned} \tag{5-36e}$$

Setting

$$U_1 = U_2 = 0 \quad , \quad U_3 = k_{12}(X_3)\,\phi(X_1, X_2) = \tau\,\phi(X_1 X_2) \tag{5-37a}$$

then, Equations (5-36e) becomes

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{12} = \varepsilon_{33} = 0$$
$$\varepsilon_{13} = \tau\left(\frac{\partial\phi}{\partial x_1} + x_2\right) \qquad (5\text{-}37b)$$
$$\varepsilon_{23} = \tau\left(\frac{\partial\phi}{\partial x_2} - x_1\right)$$

and thus yields the equations of Saint-Venant's Theory of torsion. It is noted that the displacements of points of the twisted rod are determined by Equation (5-36d) not by the expressions of the classical theory, (i.e., the first approximation). Thus, in accordance with Equation (5-26a) it is shown that

$$\{\bar{u}\} = [G]^{-T}\{U\} \qquad (5\text{-}37c)$$

Substituting Equations (5-24b) into Equation (5-37c) gives

$$[G] = \begin{bmatrix} \cos\tau x_3 & \sin\tau x_3 & (-x_1\tau\sin\tau x_3 + x_2\tau\cos\tau x_3) \\ -\sin\tau x_3 & \cos\tau x_3 & (-x_1\tau\cos\tau x_3 - x_2\tau\sin\tau x_3) \\ 0 & 0 & 1 \end{bmatrix}$$

Supposing that $\tau x_3 \ll 1$, i.e., assuming that the angles of rotation under torsion are negligibly small compared to unity, one obtains

$$\sin\tau x_3 \approx \tau x_3 \qquad (5\text{-}38a)$$
$$\cos\tau x_3 \approx 1.$$

Then, the matrix $[G]$ becomes

$$[G] = \begin{bmatrix} 1 & \tau x_3 & (-x_1\tau^2 x_3 + x_2\tau) \\ -\tau x_3 & 1 & (-x_1\tau - x_2\tau^2 x_3) \\ 0 & 0 & 1 \end{bmatrix}$$

with

$$|[G]| = 1 + \tau^2 x_3^2$$

and

$$[G]^{-T} = \begin{bmatrix} \frac{1}{(1+\tau^2 x_3^2)} & \frac{\tau x_3}{(1+\tau^2 x_3^2)} & 0 \\ \frac{-\tau x_3}{1+\tau^2 x_3^2} & \frac{1}{1+\tau^2 x_3^2} & 0 \\ -\tau x_2 & \tau x_1 & 1 \end{bmatrix} \quad (5\text{-}38b)$$

Thus, Equation (5-37c) becomes

$$\begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{Bmatrix} = \begin{bmatrix} \frac{1}{(1+\tau^2 x_3^2)} & \frac{\tau x_3}{(1+\tau^2 x_3^2)} & 0 \\ -\frac{\tau x_3}{1+\tau^2 x_3^2} & \frac{1}{1+\tau^2 x_3^2} & 0 \\ -\tau x_2 & \tau x_1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \tau \phi(x_1, x_2) \end{Bmatrix} \quad (5\text{-}38c)$$

It follows that,

$$\bar{u}_1 = 0 \; ; \quad \bar{u}_2 = 0 \quad ; \quad \bar{u}_3 = \tau \phi(x_1, x_2) . \quad (5\text{-}38d)$$

Substituting Equations (5-38d) into Equation (5-36d), yields

$$\begin{aligned} u_1 &= -x_2 x_3 \tau \\ u_2 &= x_1 x_3 \tau \\ u_3 &= \tau \phi(x_1, x_2) \end{aligned} \quad (5\text{-}38e)$$

These are the classical displacement components for a slender rod in pure torsion subject to negligibly small rotation restrictions in comparison to unity.

5.4 The Final Expressions for the Strain Components of a Thin Rod

It can be seen that in the general case of the deformation of a rod (when it is subjected not only to twisting but also to bending), Equations (5-23) are inadequate. It may furthermore be seen that the necessary corrections which must be introduced into these equations have the same character in the general case as they do in the case of pure torsion. More specifically, these corrections must be allowed to remove the stresses which twist the rod and act on its lateral surface, which arise unavoidably in using Equations (5-23) (for rods of non-circular cross-sections). Hence an attempt is made to construct a general theory of deformation of thin rod by setting, as in the preceding section,

$$U_1 = U_2 = 0 \quad , \quad U_3 = k_{12}(x_3) \cdot \phi(x_1, x_2) \tag{5-39}$$

Equations (5-36e) then assume the forms

$$\begin{aligned} \varepsilon_{11} &= \hat{\varepsilon}_{11} \;, \quad \varepsilon_{22} = \hat{\varepsilon}_{22} \;, \quad \varepsilon_{12} = \hat{\varepsilon}_{12} \\ \varepsilon_{13} &= \hat{\varepsilon}_{13} + \left(\frac{\partial \phi}{\partial x_1} + x_2\right) k_{12} \\ \varepsilon_{23} &= \hat{\varepsilon}_{23} + \left(\frac{\partial \phi}{\partial x_2} - x_1\right) k_{12} \\ \varepsilon_{33} &= \hat{\varepsilon}_{33} + x_1 k_{11} + k_{22} x_2 + \phi(x_1, x_2) \frac{d k_{12}}{d x_3} \end{aligned} \tag{5-40}$$

Equation (5-40) above are actually adequate for the problem at hand. With them as a basis, a consistent theory of deformation of flexible rods may be constructed, restricted only by the assumption that the elongations and shears are negligible when compared to unity. The error in this theory is estimated by comparing the elongations and shears with the angles of rotation, since the former are neglected in comparison with the latter in Equation (5-36e).

CHAPTER VI

DISCUSSION AND CONCLUSIONS

6.1 Discussion

The tradition, established in the majority of books on the theory of Elasticity, refer to the equation

$$[\varepsilon] = [e] + \tfrac{1}{2}\left[[e]^2 + [e][\omega] - [\omega][e] - [\omega]^2\right]$$

as the "components of a finite deformation." This evitably implies that the equation

$$[\varepsilon] \approx [e]$$

of the classical theory are the "components of an infinitesimal deformation." Chapter II makes it completely clear, however, that the degree of smallness of the elongations and shears compared to unity is not at all a sufficient criterion for passing from former equation to the latter equation. The magnitude of the angles of rotation play an essential role transforming the general case to the special case (i.e., the classical linear case).

In some problems the use of the linear equations of elasticity is inadmissible even for very small elongations and shears (compression of a thin rod, bending of a thin plate). In other problems the linear equations are applicable even though the elongations and shear are much larger (extension of rod, bending a thick plate).

Thus, both the nonlinear theory (case 1) and the classical theory of Elasticity (case 4) deal with finite deformations, and, moreover, as a rule, with deformations of the same order of smallness. Otherwise, the classical theory would have no practical significance. The difference in approach of these two theories in dealing with the determination of strain consists only in that the linear theory neglects the influence of rotations on elongations and shears, while the nonlinear theory takes it into account.

As a result, the nonlinear theory embraces all problem dealing with the elastic deformation of bodies, while the linear theory applies only to a particular group of problems.

It has been shown that nonlinearity is introduced into the theory of elasticity in three ways.

1. The formulas for the strain components (Equation (2-4c))

$$[\varepsilon] = [e] + \tfrac{1}{2}\left[[e]^2 + [e][\omega] - [\omega][e] - [\omega]^2\right]$$

2. The equations of equilibrium of a volume element of the body (Equation (3-34b))

$$\{\nabla\}^T[\nabla_R][J]^T + |[J]|\{F_x^*\}^T = \{0\}^T$$

3. The stress-strain equations (Equation (4-19))

$$[\nabla_R] = \frac{\partial \Phi}{\partial a_2}[I] + \frac{\partial \Phi}{\partial a_1}\left[[I]a_2 - [\varepsilon]\right] + \frac{\partial \Phi}{\partial a_0}\left[\mathrm{COF}[\varepsilon]\right]$$

For the first two sets mentioned, the retention of the nonlinear terms, is conditioned by geometric considerations, i.e., the necessity of taking into account the angles of rotation in determining changes of dimension in the line elements and in formulating the conditions of equilibrium of a volume element. On the other hand, nonlinear terms appear in the third set if the strain exceeds in magnitude certain physical constants characteristic of the material examined, that is, the limits of proportinality. It follows that there are four types of problems in the theory of elasticity.

1. Those having both materially and geometrical linearity;
2. Those which are materially nonlinear but geometrically linear;
3. Those linear materially but nonlinear geometrically;
4. Those nonlinear both materially and geometrically

In problems of the first type, the angles of rotation are of the same order of magnitude as the elongations and shears, while the elongations do not exceed the limit of proportionality of the given material. The simplest example of this type of problem is the extension of a straight rod by forces which keep the stresses within the limit of proportionality.

In this problems of the second type, the angles of rotation may be neglected in projecting the forces which act on a volume element and in determining strains. However, the elongations exceed the limit of proportionality and this requires a nonlinear stress-strain relation. The example given above becomes a problem of this type if it is complicated by the assumption that the stresses in the rod exceed the limit of proportionality.

In problems of the third type, the angles of rotation are essentially large (with strains not exceeding the limit of proportionality). An example of this type of problem is illustrated by the bending of a thin (steel) strip. It is well known that strips of high strength material can straighten out without traces of residual deformation after having their ends brought together. Tnis condition reinforces the fact that in these strips, even for large displacements and angles of rotation, the stresses do not exceed the yield point (which, for steel, is close to the limit of proportionality).

Finally, in problems of the fourth type, the strains exceed the limit of proportionality and the angles of rotation are so large that it is necessary to retain nonlinear terms both in the stress-strain equations, the equations of equilibrium of an element, as well as in the formulas for the strain components. The preceding example becomes one of this type if it is complicated by assuming that the stresses in the bent strip exceed the limit of proportionality.

6.2 Conclusions

The complete theory of Nonlinear Elasticity has been formulated in this thesis utilizing the basic concepts of matrix algebra, matrix transformations and matrix calculus. The nonlinear equations of the strain components, the equations of equilibrium, and the stress-strain relationships are formulated efficiently and completely in the total component form using matrix techniques. This gives the reader a broad over view of the total problems without reliance upon the mathematical complexity of tensor calculus operation, or the extensive memory capacity of a strict scalar components approached.

Matrix techniques although initially apply only to the classical theory of Elascity have been shown in this thesis to be even more efficient in their operations in formulating and understanding the general nonlinear Elasticity theory. Infact, the reduction from the general nonlinear theory to the intermidate theories and finally to the classical theory is most easily understood using matrix these technique, since the required reduction in mathematical equations are performed by a systematically neglecting higher order terms in equations consisting of matrix series terms.

It has been shown consistently throughout this thesis that basic matrix definitions play a fundamental role in the formulation of the nonlinear theory. These operations include the eigenvalue eigenvector problem, the concept of the three matrix invariants, the concept of spectral decomposition, the the definition of the trace of the matrix, together with the more basic definition of a matrix transpose, a matrix inverse, the cofactor matrix, as well as the notion of a nonsymmetric matrix, a sysmmetric matrix, a skew sysmatic matrix, orthogonal matrix and a diagonal matrix.

APPENDIX I

The comparison between the notations that are used in this thesis and in the book "Foundations of the Nonlinear Theory of Elasticity" by Novozhilov are given as follows :

CHAPTER I

Thesis	Novozhilov
X_1 , X_2 , X_3	X , Y , Z
x_1 , x_2 , x_3	x , y , z
x_1^* , x_2^* , x_3^*	ξ , η , ζ
u_1 , u_2 , u_3	u , v , w
$\lvert[J]\rvert$	D
E_1 , E_2 , E_3	E_x , E_y , E_z
$e_{11}, e_{22}, e_{33}, e_{12}, e_{13}, e_{23}$	$e_{xx}, e_{yy}, e_{zz}, e_{xy}, e_{xz}, e_{yz}$
$\tilde{i}_1^*$, $\tilde{i}_2^*$, $\tilde{i}_3^*$	i_1 , i_2 , i_3
ω_1 , ω_2 , ω_3	ω_x , ω_y , ω_z

CHAPTER II

Thesis	Novozhilov
$\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}$	$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{yz}$
λ_1 , λ_2 , λ_3	λ , η , ν
ϕ_{12} , ϕ_{13} , ϕ_{23}	φ_{xy} , φ_{xz} , φ_{yz}
ϵ_1^p , ϵ_2^p , ϵ_3^p	ε_1 , ε_2 , ε_3
ψ_1 , ψ_2 , ψ_3	ψ_x , ψ_y , ψ_z
$\tilde{e}_{11}, \tilde{e}_{22}, \tilde{e}_{33}, \tilde{e}_{12}, \tilde{e}_{13}, \tilde{e}_{23}$	$e_{11}, e_{22}, e_{33}, e_{12}, e_{13}, e_{23}$
$\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{\omega}_3$	ω_1 , ω_2 , ω_3
$\tilde{\varepsilon}_{11}, \tilde{\varepsilon}_{22}, \tilde{\varepsilon}_{33}, \tilde{\varepsilon}_{12}, \tilde{\varepsilon}_{13}, \tilde{\varepsilon}_{23}$	$\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}$

CHAPTER III

Thesis	Novozhilov
dA^*	$d\Omega$
$\vec{\sigma}^*_{x1}, \vec{\sigma}^*_{x2}, \vec{\sigma}^*_{x3}$	$\sigma_\xi, \sigma_\eta, \sigma_\zeta$
S^*_1, S^*_2, S^*_3	S^*_x, S^*_y, S^*_z
$F^*_{x1}\ F^*_{x2}\ F^*_{x3}$	$F^*_\xi\ F^*_\eta\ F^*_\zeta$
$\vec{\sigma}^*_{n1}, \vec{\sigma}^*_{n2}, \vec{\sigma}^*_{n3}$	$\sigma_{n1}, \sigma_{n2}, \sigma_{n3}$
$\begin{bmatrix} \sigma^*_{11}, & \sigma^*_{12}, & \sigma^*_{13} \\ \sigma^*_{21} & \sigma^*_{22} & \sigma^*_{23} \\ \sigma^*_{31} & \sigma^*_{32} & \sigma^*_{33} \end{bmatrix}$	$\begin{bmatrix} \sigma_{\xi\xi}, & \sigma_{\xi\eta} & \sigma_{\xi\zeta} \\ \sigma_{\eta\xi} & \sigma_{\eta\eta} & \sigma_{\eta\zeta} \\ \sigma_{\zeta\xi} & \sigma_{\zeta\eta} & \sigma_{\zeta\zeta} \end{bmatrix}$
$\begin{bmatrix} \tilde{\sigma}^*_{11} & \tilde{\sigma}^*_{12} & \tilde{\sigma}^*_{13} \\ \tilde{\sigma}^*_{21} & \tilde{\sigma}^*_{22} & \tilde{\sigma}^*_{23} \\ \tilde{\sigma}^*_{31} & \tilde{\sigma}^*_{32} & \tilde{\sigma}^*_{33} \end{bmatrix}$	$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$
$\begin{bmatrix} \sigma_{R11} & \sigma_{R12} & \sigma_{R13} \\ \sigma_{R21} & \sigma_{R22} & \sigma_{R23} \\ \sigma_{R31} & \sigma_{R32} & \sigma_{R33} \end{bmatrix}$	$\begin{bmatrix} \sigma^*_{xx} & \sigma^*_{xy} & \sigma^*_{xz} \\ \sigma^*_{yx} & \sigma^*_{yy} & \sigma^*_{yz} \\ \sigma^*_{zx} & \sigma^*_{zy} & \sigma^*_{zz} \end{bmatrix}$
$\begin{bmatrix} \sigma_{\alpha 11} & \sigma_{\alpha 12} & \sigma_{\alpha 13} \\ \sigma_{\alpha 21} & \sigma_{\alpha 22} & \sigma_{\alpha 23} \\ \sigma_{\alpha 31} & \sigma_{\alpha 32} & \sigma_{\alpha 33} \end{bmatrix}$	$\begin{bmatrix} \widehat{\alpha_1\alpha_1} & \widehat{\alpha_1\alpha_2} & \widehat{\alpha_1\alpha_3} \\ \widehat{\alpha_2\alpha_1} & \widehat{\alpha_2\alpha_2} & \widehat{\alpha_2\alpha_3} \\ \widehat{\alpha_3\alpha_1} & \widehat{\alpha_3\alpha_2} & \widehat{\alpha_3\alpha_3} \end{bmatrix}$

CHAPTER IV

Thesis	Novozhilov
dW	dA
Q	$\mathfrak{Z}$
f^*_{x1}, f^*_{x2}, f^*_{x3}	f^*_ξ f^*_η f^*_ζ

CHAPTER V

Thesis	Novozhilov
k_{11}, k_{22}, k_{12}	k_{xx}, k_{yy}, k_{xy}
ϑ_{11}, ϑ_{22}, ϑ_{12}	ϑ_{xx}, ϑ_{yy}, ϑ_{xy}
$\mathbf{U}_1$, $\mathbf{U}_2$, $\mathbf{U}_3$	$\mathfrak{U}$, $\mathfrak{V}$, $\mathfrak{W}$
$\underline{X}_1$, $\underline{X}_2$	X, Y

APPENDIX II

The definition and the relation of the trace of the matrices is expressed as follow

$$\text{Trace}\,[A] = a_{11} + a_{22} + a_{33} + a_{44} + \cdots\cdots a_{nn}$$

where

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn} \end{bmatrix}$$

= square matrix

$$\text{Trace}\,[A] = \text{Trace}\,[A]^T$$

$$\text{Trace}\,\big[[A] \pm [B]\big] = \text{Trace}\,[A] \pm \text{Trace}\,[B]$$

$$\text{Trace}\,\big[[A][B]\big] = \text{Trace}\,\big[[B][A]\big]$$

$$\text{Trace}\,\big[\{a\}\{b\}^T\big] = \{a\}^T\{b\}$$

$$\text{Trace}\,\big[\{\nabla\}\{\{a\}^T[A]\}\big]$$

$$= \text{Trace}\,\big[[\{\nabla\}\{a\}^T][A]\big] + \text{Trace}\,\big[\{a\}\{\{\nabla\}^T[A]\}^T\big]$$

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