QUOTIENTS OF TOPOLOGICAL SPACES

by

Barbara A. Bilas

Submitted in Partial Fulfillment of the Requirements

for the Degree of

Master of Science

in the

Mathematics

Program

Albert J. Klein   March 9, 1977

Adviser   Date

Leon Rand   3/14/77

Dean of the Graduate School   Date

YOUNGSTOWN STATE UNIVERSITY

March, 1977

ABSTRACT

QUOTIENTS OF TOPOLOGICAL SPACES

Barbara S. Bilas

Master of Science

Youngstown State University, 1977

The purpose of this thesis is to examine the concept of quotient spaces by means of the identification map and the identification topology. The transference of basic topological properties from the domain of an identification function to its range (or vice versa) is explored. In addition a search for maximal and minimal topologies on the domain space of a function that insure an identification mapping provides some original results and thoughts on the topic of quotients of topological spaces.

ACKNOWLEDGEMENTS

I would like to bestow my sincere thanks to Dr. Albert J. Klein, whose time, thoughtful advice and encouragement made this thesis possible. Also a special thanks to my mother for her assistance in the typing of this paper.

TABLE OF CONTENTS

		PAGE
ABSTRACT		ii
ACKNOWLEDGEMENTS		iii
TABLE OF CONTENTS		iv
LIST OF SYMBOLS		v
CHAPTER		
I.	INTRODUCTION	1
II.	GENERAL FACTS CONCERNING THE IDENTIFICATION TOPOLOGY AND QUOTIENT SPACES	3
	Equivalence Relations and Partitions	3
	The Identification Map and Topology	7
	Quotient and Decomposition Space Approaches to the Identification Topology	14
III.	TRANSFERENCE OF TOPOLOGICAL PROPERTIES BY THE IDENTIFICATION MAPPING	23
	Transference by the Identification Mapping with no Additional Hypotheses	23
	Transference of Topological Properties Utilizing Added Hypotheses	30
IV.	MINIMAL AND MAXIMAL TOPOLOGIES ON THE DOMAIN SPACE OF AN IDENTIFICATION FUNCTION	43
	The Minimal Topology	44
	The Maximal Topology	46
BIBLIOGRAPHY		56
REFERENCES		57

LIST OF SYMBOLS

SYMBOL	DEFINITION
$f^{-1}(U)$	The inverse image of the set U under the function, f.
R	The set of real numbers (unless otherwise distinguished).
Z^{+}	The set of positive integers.
$\in$	Greek letter designating the phrase, "is an element of".
ϕ	The empty set.
$\tilde{\mathcal{J}}_B$	The topology for or depending on B.

Note: All other symbols used are standard mathematical notation or will be defined within the context of their usage.

Chapter I

INTRODUCTION

The discussion of the identification function presented in this thesis is not a new area of mathematical investigation. It apparently first began with a theorem of R. L. Moore[1] in the 1920's, although he was probably not the originator of this idea. His approach was that of decomposition spaces which sought to decompose the plane into curves. Next in the 1930's came George T. Whyburn whose basic interest was in the area of Complex Analysis, where all non-constant analytic functions are open maps. With this added emphasis, Whyburn developed the quasi-compact map which, as will be seen, is merely a special case of the identification.[2] Bringing these two ideas together in the 1940's was the Bourbaki Committee with the concept of the quotient set. Of underlying interest throughout all these investigations is the transference of topological properties. E. A. Michaels deals with this question for more advanced topological properties in his publication, "Quintuple Quotient Questions" published in 1972[3].

---

[1] R. L. Moore, "Foundations of Point Set Theory," A. M. S. Colloquium Publication, XIII, (1932).

[2] G.T. Whyburn, "Open and Closed Mappings," Duke Math Journal, 17 (1950) 69-74.

[3] E. A. Michaels, "Quintuple Quotient Questions," General Topology and its Applications, 2 (1972) 91-138.

But, it may be asked, what precisely is the significance of the identification function? Initially it may be observed that all open and closed surjections are identification functions. And, as has already been mentioned, Complex Variables holds that all analytic functions aside from constant functions are open maps. Also any continuous function from a compact space to a Hausdorf space is an identification. Secondly, it will be seen that the property of a function being an identification is slightly stronger than continuity. Thus the identification function is a generalization which includes several significant and commonly encountered types of maps.

In the next three chapters, it will be endeavored to present a clear and concise discussion of the identification function. In Chapter II, an initial discussion of the identification function will be presented. Definitions and theorems concerning the identification function as well as decomposition spaces, quasi-compact functions and the quotient set will be presented and related. The intriguing question of the transference of basic topological properties will be discussed in Chapter III. Finally in Chapter IV the questions about minimal and maximal topologies on the domain space of the identification function are investigated with some interesting results.

Chapter II

GENERAL FACTS CONCERNING THE IDENTIFICATION TOPOLOGY AND QUOTIENT SPACES

Equivalence Relations and Partitions

This section of definitions clears the way for the exploration of the identification mapping and identification topology. The discussion begins with a definition.

Definition 1

Let X be any set. If R is a relation from X to X (i.e. a subset of X x X) then R is an equivalence relation if and only if the following conditions hold;

(1) $(x,x) \in R$ for all $x \in X$ (Reflexive Property)

(2) If $(x,y) \in R$ then $(y,x) \in R$ (Symmetric Property)

(3) If $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$ (Transitive Property).

Of special importance with reference to an equivalence relation is the concept of an equivalence class.

Definition 2

Let X be a set. If R is an equivalence relation on X and $x \in X$, then $R(x) = \{y \in X \mid (x,y) \in R\}$ is called the equivalence class in X determined by x with respect to R.

As a matter of notation R(A) will be used to represent the set consisting of all $y \in X$ such that $(x,y) \in R$ for all x in A.

Now the first result can be stated.

Theorem 1

Let R be an equivalence relation on a set X. Suppose also that $\{A_\alpha : \alpha \in \Delta\}$ is an indexed family of subsets of X. Then the following two results hold:

(1) $R\left(\bigcup_{\alpha\in\Delta} A_\alpha\right) = \bigcup_{\alpha\in\Delta} R(A_\alpha)$

(2) $R(A_\delta \cap A_\beta) \subseteq R(A_\delta) \cap R(A_\beta)$, $\delta, \beta \in \Delta$

Proof

Proof for (1).

First it must be shown that $R\left(\bigcup_{\alpha\in\Delta} A_\alpha\right) \subseteq \bigcup_{\alpha\in\Delta} R(A_\alpha)$. Let $y \in R\left(\bigcup_{\alpha\in\Delta} A_\alpha\right)$. Then there exists $x \in X$ such that $(x,y) \in R$ and $x \in \bigcup_{\alpha\in\Delta} A_\alpha$. x must be in at least one A_γ $\gamma\in\Delta$. Therefore, $y \in R(a_\gamma)$.

Now it must be shown that $\bigcup_{\alpha\in\Delta} R(A_\alpha) \subseteq R\left(\bigcup_{\alpha\in\Delta} A_\alpha\right)$. If $y \in \bigcup_{\alpha\in\Delta} R(A_\alpha)$ then there exists an x such that $(x,y) \in R$ and $x \in A_\delta$ for some $\delta \in \Delta$. $y \in R(A_\delta)$, $\delta\in\Delta$, $x \in \bigcup_{\alpha\in\Delta} A_\alpha$ implies $y \in R\left(\bigcup_{\alpha\in\Delta} A_\alpha\right)$.

Proof for (2).

If $y \in R(A_\delta \cap A_\beta)$, then there exists an x such that $(x,y) \in R$ and $x \in A_\delta \cap A_\beta$. So for $x \in A_\delta$, $y \in R(A_\delta)$; and similarly if $x \in A_\beta$, $y \in R(A_\beta)$. Hence $y \in R(A_\delta) \cap R(A_\beta)$.

It should be noted that the proof of theorem 1 did not actually make use of the first hypothesis - that R is an equivalence relation. The result holds, in fact, even if R is just a relation (i.e. a subset of X x X). The proof of this fact follows identically to that of theorem 1.

It is now appropriate to turn to the second of the two previously mentioned introductory topics, the partition.

Definition 3

If X is a nonempty set, then $\{A_\alpha : \alpha \in \Delta\} \subseteq X$ forms a partition of X if and only if the following three conditions hold:

(1) $A_\alpha \neq \phi$ for each $\alpha \in \Delta$

(2) $\bigcup_{\alpha \in \Delta} A_\alpha = X$

(3) For $\alpha, \beta \in \Delta$ either $A_\alpha = A_\beta$ or $A_\alpha \cap A_\beta = \phi$.

The precise relationship between the partition and equivalence relations presents itself in the next result which is stated without proof.

Theorem 2

If R is an equivalence relation on X with x, y $\in$ X then either $R(x) = R(y)$ or $R(x) \cap R(y) = \phi$.

A corollary to this result spells out the aforementioned relationship.

Corollary 1

If R is an equivalence relation on the set X, then the equivalence classes of R form a partition on X.

Proof

What must be done is to show that $\{R(x): x \in X\}$ forms a partition of X.

Since $x \in R(x)$ by the reflexive property of an equivalence relation, $R(x) \neq \phi$ for all $x \in X$ and $\bigcup_{x \in X} R(x) = X$.

Finally the fact that either $R(x) = R(y)$ or $R(x) \cap R(y) = \phi$ for any two partition elements follows immediately from theorem 2.

It is also possible to reverse the process and show that any partition of a set determines an equivalence relation. However, first it is necessary to define the relation determined by a partition.

Definition 4

If P is a partition of the set X, the relation determined by the partition is defined as follows: For $x,y \in X$, $(x,y) \in R$ if and only if x and y are in the same partition element.

Utilizing this definition the following can now be proved.

Theorem 3

If P is a partition of a set X then the relation determined by P is an equivalence relation.

Proof

Let $P = \{A_\alpha : \alpha \in \Delta\}$. Since each $x \in X$ belongs to A_α for some $\alpha \in \Delta$ by partition specification (2), it follows that $(x,x) \in R$. Hence R is reflexive.

Also if $(x,y) \in R$ then x and y belong to the same partition element. Therefore $(y,x) \in R$. R is symmetric.

Finally, if x and y are in the same partition element, A_α, and y and z are in the same partition element, A_β, then x and z are in the same partition element. This result follows from partition specification (3): If $y \in A_\alpha$ and $y \in A_\beta$ then $A_\alpha = A_\beta$.

It is clear that an equivalence relation determines a partition which in turn determines an equivalence relation identical to the original. Analogously, a partition determines an equivalence relation which determines a partition identical to the original.

With all the results amassed thus far, it is now possible to consider not only our original set, X, but also the set of equivalence classes on X with respect to an equivalence relation, R. This set will be referred to as the quotient set.

Definition 5

Given a set, X, and an equivalence relation R on X, X/R denotes the quotient set of X relative to R. Its elements are the equivalence classes of X under R.

The Identification Map and Topology

Lemma 1

If $f: X \to Y$, where X and Y are sets, and X has the topology $\mathcal{T}_X$, then $\mathcal{T}_Y = \{V: f^{-1}(V) \in \mathcal{T}_X\} \subseteq Y$ forms a topology on Y.

Proof

Clearly, Y and ϕ are elements of $\mathcal{T}_f$. Since $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$ where $f^{-1}(U)$ and $f^{-1}(V) \in \mathcal{T}_X$. Hence $f^{-1}(U \cap V) \in \mathcal{T}_X$ which implies $U \cap V \in \mathcal{T}_f$. Finally, the case for arbitrary unions of open sets must be considered. For $U_\gamma \in \mathcal{T}_f$, $f^{-1}(\bigcup_{\gamma \in \Gamma} U_\gamma) = \bigcup_{\gamma \in \Gamma} f^{-1}(U_\gamma) \in \mathcal{T}_X$. Therefore, $\bigcup_{\gamma \in \Gamma} U_\gamma \in \mathcal{T}_f$.

Definition 6

Let $(X, \mathcal{T}_X)$ and Y be a topological space and a set, respectively. If f is a mapping from X onto Y, then the topology $\mathcal{T}_f$, as described in the immediately preceding lemma, forms the identification topology on Y with respect to f and $\mathcal{T}_X$.

It should be mentioned at this time that requiring the function, f, to be onto does not weaken the definition of the identification topology. If $f:X \to Y$ is not onto, then $f(X)$ is both open and closed in $\mathcal{T}_f$ and for all $y \notin f(X)$, $\{y\} \in \mathcal{T}_f$. It is clear that the omission of the onto requirement is not advantageous.

In a manner analogous to what has already appeared, the definition of an identification function can now be presented.

Definition 7

If the sets X and Y together with their respective topologies form topological spaces and if $f:X \to Y$ is a surjection then f is an identification function if and only if the topology on Y is the identification topology.

An immediate consequence, therefore, of this definition is that whenever Y has the identification topology, $f:X \to Y$ is continuous.

The following points out two important classes of identification functions: the open continuous surjections and the closed continuous surjections.

Theorem 4

If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $f:X \to Y$ is a continuous open surjection, then f is an identification function.

Proof

It must be shown that the topology on Y is identical to the desired identification topology.

For $U \in \mathcal{T}_f$, $f^{-1}(U) \in \mathcal{T}_X$. Also since f is a surjection $f(f^{-1}(U)) = U$. Furthermore since f is open, U is open in Y. Hence $U \in \mathcal{T}_Y$. Now for $U \in \mathcal{T}_Y$, $f^{-1}(U)$ is open in X since f is continuous. Therefore by the definition of the identification topology, $U \in \mathcal{T}_f$. Hence the conclusion $\mathcal{T}_Y = \mathcal{T}_f$ holds.

Specifications for an open set in the identification topology are given by the definition. The following lemma delineates exactly what can be classified as a closed set in the topology.

Lemma 2

If f is an identification function from X to Y and if $F \subseteq Y$, then F is closed relative to $\mathcal{T}_f$ if and only if $f^{-1}(F)$ is closed relative to $\mathcal{T}_X$.

Proof

If F is closed relative to $\mathcal{T}_f$, then $f^{-1}(F)$ is closed relative to $\mathcal{T}_X$ by the continuity of f.

Conversely, if $f^{-1}(F)$ is closed relative to $\mathcal{T}_X$, then $X \setminus f^{-1}(F) \in \mathcal{T}_X$. However $X \setminus f^{-1}(F) = f^{-1}(Y \setminus F)$. Hence $Y \setminus F$ is open relative to $\mathcal{T}_f$ and, therefore, F is closed relative to $\mathcal{T}_f$.

A parallel result to that of Theorem 4 can be achieved through the use of Lemma 2. It is stated without proof as the following theorem.

Theorem 5

If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $f:X \to Y$ is a continuous closed surjection, then f is an identification function.

Corollary 2

If $f:X \to Y$ is a continuous surjection from a compact space, X, to a Hausdorff space, Y, it is an identification function.

Proof

If F is a closed subset of X then F is compact. f(F) is also compact. But a compact subset of a T_2 space must be closed. Hence f(F) is closed which implies that the mapping, f, is closed. The conclusion follows immediately through an application of theorem 5.

In the next result, a special case of function composition aids in isolating the identification function.

Theorem 6

If $f:X \to Y$ is a continuous function and there exists a continuous function, g, such that g maps Y into X with fg the identity map then f is an identification function.

Proof

In proving f to be an identification function it is necessary to show two things: First, that f is a surjection and second that the image space of f has the identification topology.

If $y \in Y$, then $g(y) = x$, $x \in X$. By hypothesis, $f(x) = y$. Therefore f is a surjection.

Now it remains to show that Y has the identification topology. If $U \in \mathcal{T}_Y$, $f^{-1}(U)$ is open by the continuity of f. This implies that $U \in \mathcal{T}_f$. If $U \in \mathcal{T}_f$ then $f^{-1}(U) \in \mathcal{T}_X$. Since g is continuous, $g^{-1}(f^{-1}(U)) \in \mathcal{T}_Y$. It should be noted that by hypothesis fg is a surjection. Therefore, the following holds:

$$g^{-1}(f^{-1}(U)) = fg(g^{-1}(f^{-1}(U))) = fg(fg^{-1}(U)) = U \in \mathcal{T}_Y.$$

In theorem 7 function composition again comes into play.

Theorem 7

If $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$ are topological spaces with f mapping X to Y and g mapping Y to Z identification functions, then the composition $gf:X \to Z$ is an identification function.

Proof

Clearly gf is a continuous surjection. Now it remains to show that Z holds the identification topology. If $U \in \mathcal{T}_Z$, $(gf)^{-1}(U) \in \mathcal{T}_X$ by the continuity of gf. Therefore, $U \in \mathcal{T}_{gf}$. If $U \in \mathcal{T}_{gf}$ then $(gf)^{-1}(U) = f^{-1}(g^{-1}(U)) \in \mathcal{T}_X$. Since f is an identification function $g^{-1}(U) \in \mathcal{T}_Y$. Also because g is an identification function, $U \in \mathcal{T}_Z$.

In the last three theorems, it was necessary to prove that a particular topology was contained in the identification topology, and in each case the proof was identical. This fact leads to a result formalized in the next theorem.

Theorem 8

If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces with $f:X \to Y$ an identification function, then $\mathcal{T}_Y$ is the largest topology on Y which makes f continuous.

Proof

If U is an open set in any topology that makes f continuous, $f^{-1}(U) \in \mathcal{T}_X$. Hence U is an element of the identification topology.

Continuity is again of major concern in the next result.

Theorem 9

If $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$ are topological spaces with $f:X \to Y$ an identification function and g any function mapping Y to Z, then g is continuous if and only if gf is continuous.

Proof

If it is assumed first that g is continuous, then the result follows directly since g and f are both continuous.

Conversely, if it is assumed that gf is continuous, then for $U \in \mathcal{T}_Z$, $gf^{-1}(U) = f^{-1}(g^{-1}(U)) \in \mathcal{T}_X$. Since f is an identification function, $g^{-1}(U) \in \mathcal{T}_Y$ which yields the result.

Now the discussion is turned to the identification topology for subspaces. If f is an identification mapping from X to Y and $S \subseteq Y$ then S can have two topologies:

(1) $\mathcal{T}_S$, the subspace topology.

(2) $\mathcal{T}_{fS}$, the topology determined by the surjection $f:f^{-1}(S) \to S$.

Since $f:f^{-1}(S) \to S$ is continuous when S carries $\mathcal{T}_S$, it follows that $\mathcal{T}_S \subseteq \mathcal{T}_{fS}$. However, the reverse set containment does not always hold as is illustrated by the following example.

Let S be the set of irrationals in $[0, 1]$, $Y = \{1\} \cup S$, and Y has the identification topology determined by $f:I \to Y$ where $f(x) = x$ if $x \in S$ and $f(x) = 1$ otherwise; the only nonempty open sets in Y containing 1 are those of the form $f(W)$ where $W \subseteq I$ is open and contains $I \setminus S$. Thus the set $S \cap (0,1/2) \notin \mathcal{T}_S$; however, $S \cap (0, 1/2) \in \mathcal{T}_{fS}$.

The next theorem gives several sufficient conditions for equality.

Theorem 10

If $f:X \to Y$ is an identification function and $S \subseteq Y$ and if either

(1) S is open (or closed) in Y or

(2) f is an open (or closed) map,

then $\mathcal{T}_S = \mathcal{T}_{fS}$.

Proof

By previous comments, $\mathcal{T}_S \subseteq \mathcal{T}_{fS}$.

Utilizing assumption (1) (where S is open), if $U \in \mathcal{T}_{fS}$ then $f^{-1}(U)$ is open in the open $f^{-1}(S)$. Therefore $U \in \mathcal{T}_S$. When S is closed the proof follows similarly.

If f is an open map, then $U \in \mathcal{T}_{fS}$ implies that $f^{-1}(U)$ is open in $f^{-1}(S)$ so that $f^{-1}(U) = f^{-1}(S) \cap V$ where V is open in X. Therefore, $U = S \cap f(V)$ and since f is an open map, U is open in $\mathcal{T}_S$. The proof for f, a closed map, again follows similarly.

Quotient and Decomposition Space Approaches to the Identification Topology

If $(X, \mathcal{T}_X)$ is a topological space and R an equivalence relation on X, there is a surjective projection function $p:X \to X/R$ given by $p(x) = R(x)$ for each $x \in X$. This projection function is the device by which X/R is topologized.

Definition 8

If $(X, \mathcal{T}_X)$ is a space, R an equivalence relation on X, and $p:X \to X/R$, the projection function, then the quotient topology on X/R is the identification topology induced by p and $\mathcal{T}_X$.

The set X/R, together with the quotient topology is called a quotient space.

Therefore, with the quotient topology on X/R, p is an identification function so that the results of the previous section are now applicable.

Theorem 11

If $(X, \mathcal{T})$ is a space and R is an equivalence relation on X with X/R the respective quotient space, then $p: X \to X/R$ is open (closed) if and only if for each open (closed) $U \subseteq X$, $R(U) = \bigcup_{x \in U} R(x)$ is open (closed) in X.

Proof

If it is assumed that U is open in X then $p^{-1}(p(U)) \in \mathcal{T}$. However, the inverse image of elements in X/R yields not only the set U, but R (U), i.e. $\bigcup_{x \in U} R(x)$. Hence, $R(U) \in \mathcal{T}$.

Conversely, since $p: X \to X/R$ is an identification function and $p^{-1}(p(U)) = R(U)$ which is open in X, then p(U) is open in X/R.

If $f: X \to Y$ is a surjective function, a relation can be defined on X as follows: For any two points $x_1, x_2 \in X$, $(x_1, x_2) \in R$ if and only if $f(x_1) = f(x_2)$. It can easily be shown that R is an equivalence relation and since it is induced by the function, f, it is denoted by R (f). Now the quotient set can be formed; and since f is a surjection, a bijective function $h: Y \to X/R(f)$ may be defined in the following manner. For each $y \in Y$, $h(y) = R(a)$ if and only if $f(a) = y$. This leads to the following result.

Theorem 12

If $f: X \to Y$ is a continuous surjection, then $h: Y \to X/R(f)$ is a homeomorphism if and only if f is an identification function.

Proof

From the preceding remarks, it is clear that $p = hf$.

Assuming initially that h is a homeomorphism, it should be noted that $f = h^{-1} p$ and that both p and h^{-1} are identification functions. Theorem 7 now yields the result.

Conversely, if f is an identification function, then by Theorem 9 the map h is continuous since p is continuous and f is an identification function. For similar reasons, h^{-1} is continuous. Therefore, h is a homeomorphism.

Thus far in the discussion, two approaches have been taken to quotient spaces. These two approaches are those of the identification function and the equivalence relation. Given an identification function, $f:X \to Y$, a quotient space can be defined on the image space of f through $\mathcal{J}_f$ and $R(f)$. However, it is also possible to begin with an equivalence relation, R, yielding the corresponding quotient set, X/R. This quotient set may now be given an identification topology corresponding to the projection function, $p:X \to X/R$. As can be seen the relation between the two approaches is very close and is delineated in the last theorem.

In contrast to the equivalence relation and the identification function, the partition has been seen mainly as a result of the given equivalence relation. The next definitions and results cast a new light on the usefulness of the partition and its relationship to previously mentioned results concerning open and closed mappings. The following definition and theorem present generalities about partitions and the topology that can be induced on them.

Definition 9

Let $(X, \mathcal{T}_X)$ be a topological space and P a partition of X. R is the equivalence relation defined by P. P together with the quotient topology for $X/R = P$ is called the decomposition space of X determined by P, or simply, the decomposition space P.

It should be noted that the natural projection $p: X \to P$ (X/R) is an identification.

Theorem 13

If $(X, \mathcal{T})$ is a topological space and P is a partition of X with $S \subseteq P$, then S is open (closed) in the decomposition space P if and only if $\bigcup\{A \mid A \in S\}$ is open (closed) in X.

Proof

Let $p: X \to P$ be the natural projection. Clearly, $p^{-1}(S) = \bigcup\{A \mid A \in S\}$. The result now follows from lemma 1 and definition 6 (open) and from lemma 2 (closed).

The following definitions single out important special types of decomposition spaces.

Definition 10

P is called an upper semicontinuous decomposition of $(X, \mathcal{T})$ provided that for each closed subset, F, in X the union of the collection of all elements of P that intersect F is closed in X, i.e. for all F closed $\bigcup\{A \in P \mid A \cap F \neq \phi\}$ is closed in X.

Example 1

Let $(X, \mathcal{T})$ be a topological space with F_0 a closed subset of X. Let $P = \{F_0\} \cup \{\{x\} \mid x \in X \setminus F_0\}$. Then clearly P is an upper semicontinuous decomposition space. P is sometimes called the decomposition space obtained by identifying the closed set F_0 to a point.

Definition 11

P is called a lower semicontinuous decomposition of $(X, \mathcal{T})$ provided that for each open set U in X the union of the collection of all elements in P that intersect U is open in X, i.e. for all U open $\bigcup\{A \in P \mid A \cap U \neq \phi\}$ is open in X.

Example 2

If $(X, \mathcal{T})$ is a topological space and $U_0 \subseteq X$ is open, then $P = \{U_0\} \cup \{\{x\} \mid x \in X \setminus U_0\}$ is a lower semicontinuous decomposition.

Theorem 14

If $(X, \mathcal{T})$ is a topological space and P is a decomposition space of X then P is an upper semicontinuous decomposition space if and only if the natural projection is closed.

Proof

Let $p: X \rightarrow P$ be the natural projection for $S \subseteq X$, $p(S) = \{A \in P \mid A \cap S \neq \phi\}$. From this the result is clear.

The next theorem, which will be presented without proof, offers a result parallel to that of the last theorem concerning the lower semicontinuous decomposition.

Theorem 15

If X is a topological space and P is a decomposition space of X then P is a lower semicontinuous decomposition space if and only if the natural projection is open.

The next two results link upper semicontinuous decomposition and lower semicontinuous decomposition to open and closed sets respectively.

Theorem 16

If $(X, \mathcal{T})$ is a topological space and P is a decomposition of X then P is an upper semicontinuous decomposition if and only if for all $U \in \mathcal{T}$, $\bigcup \{A \mid A \in P \text{ and } A \subset U\}$ is open.

Proof

Clearly for $U \in \mathcal{T}$, $A \cap (X \setminus U) \neq \emptyset$ or $A \subset U$ for all $A \in P$. Therefore, $(\bigcup \{A \mid A \in P \text{ and } A \cap (X \setminus U) \neq \emptyset\}) \cup (\bigcup \{A \mid A \in P \text{ and } A \subset U\}) = X$. From this both results follow immediately.

Theorem 17

If $(X, \mathcal{T})$ is a topological space and P is a decomposition of X then P is a lower semicontinuous decomposition if and only if for all F closed in $(X, \mathcal{T})$, $\bigcup \{A \mid A \in P \text{ and } A \subset F\}$ is closed.

The proof of this last theorem follows in very much the same way as did its counterpart. For this reason, no proof will be presented here.

A very important means for connecting the three approaches to quotient spaces is the idea of point inverses.

Definition 12

If $f:X \to Y$ is a surjective mapping then $P = \{ f^{-1}(y) \mid y \in Y \}$ is a decomposition of X where $f^{-1}(y)$ are called point inverses.

Theorem 18

$f:X \to Y$ is an open continuous surjection with $P = \{f^{-1}(y) \mid y \in Y\}$, if and only if P is a lower semicontinuous decomposition of X.

Proof

If f is an open continuous mapping, with U an open subset of X, then $f^{-1} f[U]$ is equal to the union of the collection of all elements of P that intersect U. Since U is open, $f[U]$ is open because f is an open mapping. Also since f is continuous $f^{-1}f[U]$ is open in X. Hence P is a lower semicontinuous decomposition of X.

Conversely if P forms a lower semicontinuous decomposition then for U open in X, $\bigcup\{ f^{-1}(y) \mid y \in f(U)\} = \{x \mid x \in f^{-1}f(U)\}$ is open in X. $f(U)$ is, therefore, open in Y by the results of definition 6.

Once again a parallel result can be stated with reference to an upper semicontinuous decomposition of X.

Theorem 19

$f:X \rightarrow Y$ is closed continuous surjection with $P = \{f^{-1}(y) \mid y \in Y\}$ if and only if P is an upper semicontinuous decomposition of X.

The proof of this theorem follows in much the same way as does its counterpart and, hence, will not be presented here.

The last definition and the last theorem lead to the following definition.

Definition 13

If $f:X \rightarrow Y$, then $A \subseteq X$ such that $A = f^{-1}(C)$ where $C \subseteq f[X]$ is an inverse set.

Corresponding to the concept of an inverse set is the mapping which uses this idea, the quasi-compact mapping.

Definition 14

If $f:X \rightarrow Y$ is a surjection such that for F, a closed inverse set, $f(F)$ is closed; and for U, an open inverse set, $f(U)$ is open, then f is said to be a quasi-compact mapping.

The following theorem links the idea of a quasi-compact mapping and the identification function.

Theorem 20

If $f:X \rightarrow Y$ is a surjection with the respective topologies $\mathcal{J}_X$ and $\mathcal{J}_Y$ on X and Y, then $\mathcal{J}_Y$ is the identification topology for Y determined by f and $\mathcal{J}_X$ if and only if f is continuous and quasi-compact.

Proof

First it will be assumed that f is an identification function. If U is an open inverse set then $f^{-1}(f(U)) = U \in \mathcal{T}_X$. So since $\mathcal{T}_Y$ is the identification topology $f(U) \in \mathcal{T}_Y$, the same construction will work for F, a closed inverse set in X. Hence, f is quasi-compact.

Conversely, if it is assumed that f is continuous and quasi-compact, then it must be shown that $\mathcal{T}_Y$ is actually the identification topology, $\mathcal{T}_f$. By theorem 8, $\mathcal{T}_Y \subset \mathcal{T}_f$ the reverse set containment will now be shown. If $W \in \mathcal{T}_f$, then $f^{-1}(W) \in \mathcal{T}_X$ and $f^{-1}(W)$ is an inverse set for which $ff^{-1}[W] = W$. Since f is quasi-compact $W \in \mathcal{T}_Y$. Therefore the two topologies are equal.

CHAPTER III

TRANSFERENCE OF TOPOLOGICAL PROPERTIES BY THE IDENTIFICATION MAPPING

In this chapter various topological properties are tested on identification maps to check whether or not the image space of the function has a certain property when the domain space has the property. The case of checking the reverse, that is whether or not the domain space possesses a certain topological property when the image space does, turned out much the same in every instance - it was proved false by a simple counterexample.

Then in the second section added hypotheses are investigated to aid in the transference of the topological properties when the identification mapping alone proved insufficient.

The topological properties to be considered are the following: first countable, second countable, T_0, T_1, T_2, T_3, T_4, regular, normal, completely regular, connected, locally connected, compact, locally compact, separable, and metrizable.

Transference by the Identification mapping with no Additional Hypotheses

First Countable

If the domain space of an identification function is first countable, the image space need not be. This is illustrated by the following example.

Example 1[4]

Let X be the space of real numbers with the usual topology, Y the set consisting of 0 and all $x \in R$ that are not integers, and define $f: X \to Y$ as follows: $f(X) = x$ for all $x \in R$ that are not integers; $f(x) = 0$ for all x such that x is an integer. From this information the identification topology can easily be deduced. What must be shown is that the space Y is not first countable for each $y \in Y$. The element in Y that is the exception will be shown to be 0. The proof is as follows.

Let $\{U_i\}$ be a countable basis of Y at 0. It can be shown that an open set can be constructed so that no basis element is a subset of that open set. Let $\theta = \bigcup_{i=0}^{\infty} \theta_i$ be an open set in X such that $\theta_0 = (-\infty, \frac{1}{2})$, $\theta_1 = (1 - \varepsilon_1, 1 + \varepsilon_1) \subsetneq f^{-1}(U_1)$, $\theta_2 = (2 - \varepsilon_2, 2 + \varepsilon_2) \subsetneq f^{-1}(U_2)$, ..., $\theta_n = (n - \varepsilon_n, n + \varepsilon_K) \subsetneq f^{-1}(U_n)$ where $0 < \varepsilon_i < \frac{1}{2}$ for all i. Because θ is an open inverse set $f[\theta]$ is an open set in Y and, for all i, $U_i \nsubseteq f[\theta]$ by the choice of the ε_i's. Therefore the conclusion on the transference of the first countable property is substantiated. Also note that one can obtain the point inverse decomposition space by identifying the closed set of integers to a point. Therefore, f is closed.

Second Countable

Again using example 1 it is clear that for $f: X \to Y$, in identification mapping, the property of second countability is not necessarily transferred from the domain space to the image space.

---

[4] John L. Kelley, General Topology (Princeton, New Jersey: D. Van Nostrand Company, Inc., 1955), p.104.

T_0

It is clear from a simple example that the fact that the domain of an identification function is T_0 does not necessarily imply that the image has the same property. The following provides the desired example.

Example 2

Let X be R, with the usual topology and $Y = \{0, 1\}$ then the topology on Y induçed by the map $f: X \to Y$ defined by $f(x) = 0$, $x \in R \setminus Q$ (Q, the set of rational numbers), $f(x) = 1$, $x \in Q$ is not T_0. The identification topology would be simply the indiscrete topology on $\{0,1\}$.

T_1

Utilizing the results of example 2, it can be concluded that the transference of the T_1 property cannot always be guaranteed by an identification mapping.

T_2

Again with the results of example 2, the non-transference of the T_2 property by an identification mapping is shown.

Regular

Upon the examination of the next example it will be shown that if the domain space of an identification function is regular then the image space need not be.

Example 3

Let X be $[0,1]$ with the usual topology and $Y=\{0,1\}$, then the function $f:X\to Y$ mapping $f(x)=1$ for $x\in[0,\frac{1}{2})$ and $f(x)=0$ for $x\in[\frac{1}{2},1]$ gives rise to an identification topology on Y that is not regular despite the fact that X is. This topology is $\{\{1\},\{0,1\},\phi\}$.

T_3

The topological property, T_3, is not necessarily transferred by an identification function. This fact is proved by example 3.

Normal

Once again it is found that another topological property is not transferred by an identification function. Normality is definitely not preserved in the following example.

Example 4

Let X be $[0,1]$ with the usual topology and $Y=\{a,b,c\}$ then the function $f:X\to Y$ mapping $f(x)=a$, for $x\in[0,\frac{1}{4}]$; $f(x)=b$, for $x\in(\frac{1}{4},\frac{1}{2})$; and $f(x)=c$, for $x\in[\frac{1}{2},1]$ induces an identification topology such that Y with this topology is not normal. The identification topology on Y is $\{\{a,b\}$, $\{b\}$, $\{b,c\}$, $\{a,b,c\}$, $\phi\}$. Hence the closed sets $\{a\}$ and $\{c\}$ cannot be separated.

T_4

Since the domain space of the identification function described in example 4 is also T_1 it follows, by example 4, that the T_4 property is not preserved by an identification mapping.

Completely Regular

In reviewing the example cited to show that the property of being regular is not transferred by an identification function (example 3), it becomes clear that the example is also sufficient for illustrating the non-transference of the completely regular property. It follows again by example 3 that if the domain space of an identification mapping is $T_{3\frac{1}{2}}$ the image space need not be.

Separable

In contrast to all cases considered thus far, an identification function transfers separability. This does not come as any surprise, however, since it is known that the transference holds for any continuous surjection. Properties unique to the identification mapping were not made use of here. A short proof illustrates this fact.

Let $f:X\to Y$ be a continuous surjection. If D is a countable dense set in X, then $\bar{D} = X$ which implies $f(\bar{D}) = Y$. However, $\overline{f(D)} \subset Y$ but $f(\bar{D}) \subset \overline{f(D)}$ by continuity. Hence $Y = \overline{f(D)}$ and Y is separable.

Connected

It is clear, once again, that the transference of this topological property holds, not due to the fact that an identification function is used but that the function in question is a continuous surjection. The proof of this fact is standard and will be omitted here.

Locally Connected

In the case for local connectivity transference can be shown. The proof is as follows.

Theorem 1

If $(X, \mathcal{J}_X)$, and $(Y, \mathcal{J}_Y)$ are topological spaces, then the identification function $f: X \to Y$ with X locally connected insures the fact that Y is locally connected.

Proof

In this proof it must be shown that components of an arbitrary open set in the image space of an identification function are themselves open.

Let $f: X \to Y$ be an identification function with X locally connected. If Q is a component of any set $U \subseteq Y$ then $f^{-1}(Q)$ is the union of a collection of components of $f^{-1}(U)$; for if R is a component of $f^{-1}(U)$ that intersects $f^{-1}(Q)$ then $f(R) \subseteq Q$ since $f(R)$ is connected, lies in U, and intersects Q. Hence $R \subseteq f^{-1} f(R) \subseteq f^{-1}(Q)$.

If Q is a component of any open set, U, in Y, then $f^{-1}(Q)$ is open in X since $f^{-1}(Q)$ is the union of a collection of components of the open set $f^{-1}(U)$ where each component is open by the local connectedness of the space X. Therefore the fact that $f^{-1}(Q)$ is open in X implies that Q is open in Y relative to the identification topology. Hence Y is locally connected.

Compact

Using only the criteria specified by a continuous surjection, compactness of the domain space of an identification function is transferred to the image space. The proof is, again, standard and will not be presented here.

Locally Compact

The topological property, locally compact, is not transferred by the identification mapping. An instance of this non-transference can be found in example 1. Clearly the domain space of the cited function is locally compact. However the image space Y is not. A short justification of this claim is now appropriate.

Suppose Y is locally compact, then for any $\theta \in \mathcal{J}_Y$ there exists a compact neighborhood of 0 contained in θ. Call it θ_1. By the structure of the mapping there exists an ε_i for all i in the set of integers such that $[i-\varepsilon_i, i+\varepsilon_i] \subseteq f^{-1}[\theta_1]$ where $0 < \varepsilon_i < \frac{1}{2}$ for all i. For n, an integer, $\{f(n+\varepsilon_n)\} = \{n+\varepsilon_n\}$ form a sequence in θ_1. By the compactness of θ_1 there exists a cluster point, x, in θ_1. Clearly $x \neq 0$ since the image of $U - \{(i - \varepsilon_i/2,\ i + \varepsilon_i/2) \mid$ i is an integer$\}$ would not contain any points of the sequence. Now it remains to pick a $\delta > 0$ such that $(x-\delta, x+\delta)$ in X contains no integers. This is an open inverse set which contains at most finitely many terms of the sequence. Hence Y is not locally compact.

Metrizable

Example 3 serves to illustrate the fact that a metrizable domain space in an identification function does not necessarily insure a metrizable image space.

Transference of Topological Properties Utilizing Added Hypotheses

In showing the transference of topological properties from the domain to the image space of an identification function one finds that the added hypotheses of upper semicontinuous and/or lower semicontinuous decompositions on the domain spaces induced by point inverses to be of great assistance. It is interesting to note that for an identification function, f, to say that f induces an upper semicontinuous decomposition on its domain space is equivalent to saying f is a closed map. In a parallel result, for f again an identification mapping, to say that f induces lower semicontinuous decomposition on its domain space is equivalent to saying f is an open map. These facts follow directly from the results in Chapter II.

In the first theorem presented in this section an upper semicontinuous decomposition of the domain space of an identification function is introduced as an added hypothesis and yields some desirous results with reference to normality.

Theorem 2

If $f: X \to Y$ is an identification function with X normal and if point inverses with respect to f induce an upper semicontinuous decomposition of X, then Y is normal.[5]

Proof

If F_1 and F_2 are closed sets in Y then $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are closed in X. By normality of X, there exists θ_1 and θ_2 open in X such that $f^{-1}(F_1) \subseteq \theta_1$, and $f^{-1}(F_2) \subseteq \theta_2$ where $\theta_1 \cap \theta_2 = \phi$. Also by theorem 15 of Chapter II, $\bigcup\{f^{-1}(y) \mid f^{-1}(y) \subseteq \theta_1\} = \hat{\theta}_1$, and $\bigcup\{f^{-1}(y) \mid f^{-1}(y) \subseteq \theta_2\} = \hat{\theta}_2$ are open in X and are inverse sets. Hence $f[\hat{\theta}_1]$, $f[\hat{\theta}_2]$ are open in Y with $F_1 \subseteq f[\hat{\theta}_1]$, $F_2 \subseteq f[\hat{\theta}_2]$ and $f[\hat{\theta}_1] \cap f[\hat{\theta}_2] = \phi$. Therefore Y is normal.

Next assuming a lower semicontinuous decomposition on the domain space of an identification function, the property of first countability is transferred.

Theorem 3

If $f: X \to Y$ is an identification function with X first countable and if the point inverses with respect to f induce a lower semicontinuous decomposition on X, then Y is first countable.[6]

Proof

Consider $y \in Y$, $x \in f^{-1}(y)$, and $\theta \in \mathcal{T}_Y$ such that $y \in \theta$, then

---

[5] Robert H. Kasriel, Undergraduate Topology (Philadelphia: W. B. Saunders Company, 1971), p. 237.

[6] Kasriel, p. 237.

$f^{-1}(\mathcal{O}) \in \mathcal{T}_X$. Since X is first countable there exists a countable local basis $\{U_n\}$ at x then $U_i \subseteq f^{-1}(\mathcal{O})$ for some i. Also since the point inverses induce a lower semicontinuous decomposition on X, $C_n = \bigcup\{f^{-1}(z) \mid z \in Y$ and $f^{-1}(z) \cap U_n \neq \phi\}$ is an open inverse set in X which implies $f(C_n)$ is open in Y for all n. The only thing that remains to be proved is that $f(C_i) \subseteq \mathcal{O}$. Since $U_i \subseteq f^{-1}(\mathcal{O})$, $f(U_i) \subseteq f\,f^{-1}(\mathcal{O}) = \mathcal{O}$. However $f(U_i) = f(C_i)$ by construction. Hence $f(C_i) \subseteq \mathcal{O}$ and the $\{f(C_n)\}$'s form a countable local basis for y. Y is first countable.

A lower semicontinuous decomposition again comes into play as an added hypothesis to aid in the transference of second countability from the domain space to the image space of identification function.

<u>Theorem 4</u>

If $f{:}X \to Y$ is an identification function with X second countable and if the point inverses with respect to f induce a lower semicontinuous decomposition on X, then Y is second countable.[7]

<u>Proof</u>

If $\{U_i\}$ forms a countable basis of X then $C_i = \bigcup\{f^{-1}(z) \mid z \in Y$ and $f^{-1}(z) \cap U_i \neq \phi\}$ is an open inverse set in X. Therefore $f[C_i]$ is an open set in Y. It remains to be shown that the $\{f[C_i]\}$'s form a basis for Y.

---

[7]Kasriel, p. 237.

Consider $\mathcal{O} \in \mathcal{J}_Y$ with $y \in \mathcal{O}$ and $x \in f^{-1}(y)$ then $x \in f^{-1}(\mathcal{O}) \in \mathcal{J}_X$. By the second countability of X there exists $U_i \subseteq f^{-1}(\mathcal{O})$ such that $x \in U_i$. Hence by construction, as in theorem 3 of this chapter, $y \in f[C_i] \subseteq \mathcal{O}$. This gives the desired result Y is second countable.

The added hypothesis of a lower semicontinuous decomposition of the domain space of an identification mapping aids in the transference of the locally compact property.

<u>Theorem 5</u>

If $f\colon X \to Y$ is an identification mapping with X locally compact and if f induces a lower semicontinuous decomposition on X, then Y is locally compact.[8]

<u>Proof</u>

Let $\mathcal{O}$ be an open set in Y and let $y \in \mathcal{O}$. It must be shown that $\mathcal{O}$ contains a compact neighborhood of y. Since $x \in f^{-1}(\mathcal{O})$ is open in X and X is locally compact, $f^{-1}(\mathcal{O})$ contains a K and U such that $x \in U \in \mathcal{J}_X$ and K is a compact subset of X with $U \subseteq K$. By lower semicontinuity, $f(U) \in \mathcal{J}_Y$ $y = f(x) \in f(U) \in \mathcal{J}_Y$ and $f(K)$ is compact and $f(U) \subseteq f(K) \subseteq \mathcal{O}$. Hence Y is locally compact.

Further results concerning the separation properties may be obtained with upper and lower semicontinuous decomposition induced on the

---

[8]Kasriel, p. 237.

domain space by the identification function. However, before the separation properties are considered one result must be presented to facilitate the discussion.

Theorem 6

If f: $X \to Y$ is an identification function then Y is T_1 if and only if $f^{-1}(y)$ is closed for all y Y.

Proof

If it is first assumed that Y is T_1 then it follows that $\{y\}$ is closed for all $y \in Y$. Since Y holds the identification topology, $\{y\}$ is closed in Y if and only if $f^{-1}(y)$ is closed for all $y \in Y$.

Next if it is assumed that $f^{-1}(y)$ is closed for all $y \in Y$ then it follows immediately that $\{y\}$ is closed in Y since Y has the identification topology. Hence Y is T_1.

This leads to the next result for T_1.

Theorem 7

If f: $X \to Y$ is an identification function with X, T_1, and if f induces an upper semicontinuous decomposition on X then Y is T_1.

Proof

It should be noted that it is sufficient to show that $f^{-1}(y)$ is closed for all $y \in Y$.

For x, $y \in Y$ and for $\hat{x} \in f^{-1}(x)$ and $\hat{y} \in f^{-1}(y)$, $\{\hat{x}\}$ and $\{\hat{y}\}$ are closed sets in X. Hence by the upper semicontinuity of X, $\bigcup\{f^{-1}(z) \mid z \in Y$ and $f^{-1}(z) \cap \{\hat{x}\} \neq \phi\} = f^{-1}(x)$ and $\bigcup\{f^{-1}(z) \mid z \in Y$ and $f^{-1}(z) \cap \{\hat{y}\} \neq \phi\} = f^{-1}(y)$ are closed. Hence $\{x\}$ and $\{y\}$ are closed in Y. Y is, therefore, T_1.

With the results now available, it is possible to present a result of theorems 1 and 6.

Corollary 1

If f: $X \rightarrow Y$ is an identification mapping and if f induces an upper semicontinuous decomposition on X with X, T_4, then Y is T_2.

Proof

If $x, y \in Y$ where $x \neq y$, then $f^{-1}(x)$ and $f^{-1}(y)$ are closed in X by theorem 6, Hence $\{x\}$ and $\{y\}$ are closed in Y and by the normality gives as a result in Theorem 1, there exist open sets θ_x and θ_y in Y such that $x \in \theta_x$ and $y \in \theta_y$ with $\theta_x \cap \theta_y = \phi$.

A result for T_4 can be found with relation to upper semicontinuity of a decomposition on X induced by an identification mapping f .

Corollary 2

If f: $X \rightarrow Y$ is an identification mapping and if f induces an upper semicontinuous decomposition on X with X, T_4, then Y is T_4.

Proof

The proof is a direct result of theorems 1 and 6.

In proceeding with the discussion of the transference of these topological properties, it now becomes apparent that the singular added hypothesis of an upper or a lower semicontinuous decomposition is no longer sufficient in obtaining further results. Hence another concept is introduced - that of the perfect map.

Definition 1

If f: $X \to Y$ is a closed continuous surjection such that $f^{-1}(y)$ is compact for all $y \in Y$ then f is called a perfect map.

As is obvious, this possible hypothesis is stronger than upper semicontinuity when an identification map is considered. If f is an identification map, f, perfect, implies f induces an upper semicontinuous decomposition on X.

However, before results are introduced on topological properties not yet mentioned, another result using the hypothesis of a perfect map and a topological property already considered will be presented.

Theorem 8

If f: $X \to Y$ is an identification mapping with X, T_2, and f perfect, then Y is T_2.

Proof

For x, $y \in Y$ with $x \neq Y$, $f^{-1}(x)$ and $f^{-1}(y)$ are compact, disjoint closed sets in X. Hence there exists $\mathcal{O}_1$ and $\mathcal{O}_2$ open sets in X such that $f^{-1}(x) \subseteq \mathcal{O}_1$ and $f^{-1}(y) \subseteq \mathcal{O}_2$. Further by the upper semicontinuous decomposition induced by f, the sets C_1 and C_2 can be defined such that $C_1 = \cup\{ f^{-1}(z) \mid z \in Y$ and $f^{-1}(z) \subseteq \mathcal{O}_1\}$ and $C_2 = \cup\{f^{-1}(z) \mid z \in Y$ and $f^{-1}(z) \subseteq \mathcal{O}_1\}$ with C_1 and C_2 open inverse sets. Therefore $x \in f[C_1]$ and $y \in f[C_2]$ with $f[C_1] \cap f[C_2] = \phi$. Hence Y is T_2.

Again making use of definition 1 of this chapter the following result can be obtained for the topological property of regularity.

Theorem 9

If $f: X \rightarrow Y$ is a perfect mapping with X regular then Y is regular.

Proof

If $y \in Y$ and $F \subseteq Y$ where F is closed in Y, then $f^{-1}(F)$ is closed in X. Let $x \in f^{-1}(y)$ such that $f(x) = y$. Since X is regular, there exists an $\mathcal{O}_1$ and U_1 such that $x \in \mathcal{O}_1$, $f^{-1}(F) \subseteq U_1$ with $\mathcal{O}_1 \cap U_1 = \phi$. This works in a similar way for all $x \in f^{-1}(y)$. Hence the $\{\mathcal{O}_\alpha\}$'s form an open cover for $f^{-1}(y)$ and the corresponding $\{U_\alpha\}$'s form an open cover for $f^{-1}(F)$. Now since $f^{-1}(y)$ is compact there exists a finite subcover composed of $\mathcal{O}_i$'s for $f^{-1}(y)$ with $f^{-1}(y) \subseteq \bigcup_{i=1}^{n} \mathcal{O}_i = \overline{\mathcal{O}}$ open in X. Correspondingly, $\bigcap_{i=1}^{n} U_i = \overline{U}$ is open in X and contains $f^{-1}(F)$. Clearly, $\overline{\mathcal{O}} \cap \overline{U} = \phi$. By the upper semicontinuity of the function, sets C_1 and C_2 may be defined for $\overline{\mathcal{O}}$ and $\overline{U}$ respectively as in theorem 7. The result is of course disjoint open inverse sets containing $f^{-1}(y)$ and $f^{-1}(F)$. Hence Y is regular.

From this result and that for T_1 it is now possible to state a corollary that will yield a result for T_3.

Corollary 3

If $f: X \rightarrow Y$ is a perfect map with X, T_3, then Y is also T_3.

The next property to be investigated is that of complete regularity. Here the added hypothesis of an open perfect map yields the results.

Theorem 10

If $f:X\to Y$ is an open perfect mapping with X completely regular, then Y is also completely regular.

Proof

If F is a closed set in Y and $y \in Y$ such that $y \notin F$, then $f^{-1}(F)$ is closed in X and $f^{-1}(y)$ is a compact set in Y. Because $f^{-1}(y)$ is compact and X is completely regular, it is possible to construct a family of open sets $\{\mathcal{O}_d \mid d \in D\}$ such that $f^{-1}(y) \subseteq \mathcal{O}_0, \overline{\mathcal{O}}_s \subseteq \mathcal{O}_t \subseteq X \setminus f^{-1}[F]$ whenever $s < t$ and D is dense in $[0, 1]$.[9] Next the following open inverse sets may be formed through properties of an upper semicontinuous decomposition $U_d = \bigcup \{ f^{-1}(z) \mid f^{-1}(z) \subseteq \mathcal{O}_d \}$. Hence $f[U_d] = V_d$ which is open in Y. Clearly $y \in V_d \subseteq Y \setminus F$ for all $d \in D$. Now it must be shown that for $t < s$, $\overline{V_d} \subseteq V_s$. Since $U_d \subseteq \mathcal{O}_d \subseteq \overline{\mathcal{O}}_d \subseteq \mathcal{O}_s$ and since f induces a lower semicontinuous decomposition, $\bigcup\{f^{-1}(z) \mid f^{-1}(z) \subseteq \overline{\mathcal{O}}_d\}$ is a closed set containing U_d and contained in U_s. This implies that $\overline{U}_d \subseteq U_s$ which yields $\overline{V_d} = \overline{f(U_d)} = f(\overline{U}_d) \subseteq f(U_s) = V_s$ and therefore the result.

The topic now to be considered in this section is perhaps the most complex. It is the topic of metrizability. For convenience the discussion will be divided into three sections each representing differing added hypotheses that aid in the transference of metrizability from the domain to the image space of an identification function.

---

9 Kelly, pp. 114,142.

Theorem 11

If $f:X\to Y$ is perfect with X a separable metric space and f inducing a lower semicontinuous decomposition on X then Y is metrizable.

Proof

The proof of this theorem is as follows. First realize that a separable metric space is also second countable. Second, since f is perfect, Y is T_3 (corollary 3). Finally the result is obtained through an application of Urysohn's Metrization Theorem.

In the next variation, the hypotheses are strengthened by ommitting the separability of X but weakened by requiring that f be perfect. The following definitions are also needed for the next theorem.

Definition 2

If ρ is a metric on X, $S\subseteq X$, and $\varepsilon>0$, $V_\varepsilon(S)=\{x \mid \rho(x,s)<\varepsilon \text{ for } s\in S\}$.

Definition 3

If ρ is a metric on X and K_1, K_2 are closed bounded subsets of X, $H(K_1,K_2)=\inf\{\varepsilon \mid K_1\subseteq V_\varepsilon(K_2) \text{ and } K_2\subseteq V_\varepsilon(K_1)\}$. H is the Hausdorff metric determined by ρ.[10]

Theorem 12

If $f:X\to Y$ is open and perfect with X metrizable by ρ_X then $\rho(y_1,y_2) = H(f^{-1}(y_1),f^{-1}(y_2))$ is a metric for Y and $\mathcal{T}_\rho=\mathcal{T}_Y$ where H is the Hausdorff metric.

---

[10] Felix Hausdorff, Set Theory, trans. John R. Auman, et al. (Second edition, New York, Chelsea Publishing Company, 1962), pp.166-172.

Proof

It is true that φ is a metric on Y since $f^{-1}(y)$ is compact and so closed and bounded in X.

Secondly it must be shown that $\mathcal{T}_Y = \mathcal{T}_\varphi$. Consider $y_0 \in U \in \mathcal{T}_Y$ such that $f^{-1}(y_0) \subseteq f^{-1}(U) \in \mathcal{T}_X$. By the Lesbesque Covering Lemma there exists an $\varepsilon > 0$ such that $V_\varepsilon\ (f^{-1}(y_0)) = \{z \mid z \in X$ and $\varphi_X\ (x_0, z) < \varepsilon$ where φ_X is the metric on $X^*\} \subseteq f^{-1}(U)$. Now consider $y \in B_\varphi[\,y_0, \varepsilon]$. With this specification it is clear that $\varphi(y, y_0) < \varepsilon$. Therefore $H(f^{-1}(y), f^{-1}(y_0)) < \varepsilon$ and so $f^{-1}(y) \subseteq V_\varepsilon\ (f^{-1}(y_0)) \subseteq f^{-1}(U)$. Hence $y \in U$, thus giving $U \in \mathcal{T}_\varphi$.

Next the reverse set inclusion must be shown. If $\varepsilon > 0$ and $y_0 \in Y$ then $U_0 = \bigcup\{f^{-1}(z) \mid f^{-1}(z) \subseteq V_{\varepsilon/2}\ (f^{-1}(y_0))\}$ with $f^{-1}(y_0) \subseteq U_0$ is open in X since f induces an upper semicontinuous decomposition on X. Also, since $f^{-1}(y_0)$ is compact, there exist $x_1, \ldots, x \in f^{-1}(y_0)$ such that $f^{-1}(y_0) \subseteq \bigcup_{i=1}^{k} B_{\varphi_X}(x_i, \varepsilon/4)$. Let $U_i = \bigcup\{f^{-1}(y) \mid f^{-1}(y) \cap B_{\varphi_X}(x_i, \varepsilon/4) \neq \Phi\}$. Note that since $x_i \in f^{-1}(y_0) \subseteq U_i$ and that because f is open (i.e. f induces a lower semicontinuous decomposition of X), U_i is an open inverse set in X. Hence $\bigcap_{i=0}^{k} U$ is also an open inverse set which implies $f[\bigcap_{i=0}^{k} U_i] \in \mathcal{T}_Y$. Now it must be shown that $y_0 \in f[\bigcap_{i=0}^{k} U_i] \subseteq B_\varphi(y_0; \varepsilon)$. It is known that $f^{-1}(y_0) \subseteq \bigcap_{i=0}^{k} U_i$ which implies $y_0 \in f[\bigcap_{i=0}^{k} U_i]$. If $y \in f[\bigcap_{i=0}^{k} U_i]$ then $f^{-1}(y) \subseteq U_0$ and so $f^{-1}(y) \subseteq V_{\varepsilon/2}\ (f^{-1}(y_0))$. If $x \in f^{-1}(y_c)$ there exists an x_i such that $\varphi_X(x_i, x) < \varepsilon/4$. $f^{-1}(y) \subseteq U_i$, therefore, there exists an $x^1 \in f^{-1}(y)$ such that $\varphi_X(x_i, x^1) < \varepsilon/4$. This results in the fact that $\varphi_X(x, x^1) < \varepsilon/2$ and that $x \in V_{\varepsilon/2}(f^{-1}(y))$. Hence $f^{-1}(y_0) \subseteq V_{\varepsilon/2}(f^{-1}(y)$ or $\varphi(y, y_0) \leq \varepsilon/2 < \varepsilon$ which yields $y \in B_\varphi[\,y_0; \varepsilon]$. Therefore $B_\varphi[y_0; \varepsilon] \in \mathcal{T}_Y$.

*and $x_0 \in f^{-1}(y_0)$

The final consideration and reference to metrizability is the strongest result whose proof can be found in Dugundji, page 236.

Theorem 13

If $f:X \to Y$ is a perfect mapping with X metrizable then Y is also.

Thus far in the discussion the only results being investigated are those that transfer a topological property from the domain space of an identification function to the range space. This is the case due to the fact that very strong hypotheses are needed to transfer the aforementioned properties in the opposite direction. This discussion will be pursued in more depth in Chapter IV. As for the hypotheses available at present, there is little that can be accomplished. Two results are immediately available; however, the first concerning compactness is presented in the next theorem.

Theorem 14

If $f:X \to Y$ is a perfect identification mapping where Y is compact then X is also compact.

Proof

If $\{\theta_i\}$ forms an open cover of X, for all $y \in Y$ only a finite number of θ_x's cover $f^{-1}(y)$. Let V_y be the union of the θ_x's covering $f^{-1}(y)$ for a particular $y \in Y$. Then U_{y_j} can be defined as $\cup\{f^{-1}(y_j) \mid f^{-1}(y_j) \subset V_{y_j}\}$ an open inverse set. Hence $\{f(U_{y_i})\}$ forms an open cover of Y. By the compactness of Y, there exists a finite subcover composed of the sets $\{f(U_{y1}), f(U_{y2}), \ldots, f(U_{yn})\}$. Hence the corresponding $V_{y1}, \ldots, V_{yn}$ form a finite subcover of θ_x's for X. X is compact.

A second result concerning transference of a topological property from the image space to the domain space of an identification function is now presented.

Theorem 15

If $f:X \to Y$ is an identification function where Y is connected and $f^{-1}(y)$ is connected for all $y \in Y$ then X is connected.

Proof

Suppose that X is not connected. Then there exists $C \subseteq X$ such that C is both open and closed in X and $C \neq X$ or $C \neq \phi$. Consider, now, $x \in C$. Then $f^{-1}f(x) \cap C \neq \phi$. Furthermore $f^{-1}f(x) \cap C \subseteq f^{-1}f(x)$. If C is an inverse set with $f^{-1}(y) \subset C$ for all $f^{-1}(y) \cap C \neq \phi$ $(y \in Y)$, then the connectedness of Y would be contradicted. If C is not an inverse set there exists an $x^1 \in X$ such that $f^{-1}(f(x^1) \cap C \neq \phi$ and $f^{-1}f(x^1) \not\subseteq C$. This implies that $f^{-1}f(x^1) \cap C$ is a subset of $f^{-1}f(x^1)$ that is both open and closed in the subspace topology.

Chapter IV

MINIMAL AND MAXIMAL TOPOLOGIES ON THE DOMAIN SPACE OF AN IDENTIFICATION FUNCTION

Thus far in the discussion the only elements of the identification function that have been in question were the identification function itself and the topology on the image space of a function. Now in this chapter a new concern comes into play - that of the topology on the domain space of an identification function.

In order to motivate the discussion of maximal and minimal topologies it is first necessary to illustrate the fact that given a continuous surjection and a topology on the image space of that surjection that there exists at least one topology on the domain space of the function that makes it an identification. There may be several as the next example illustrates.

Example 1

If $f:X \to Y$ is a mapping from X onto Y where X is the set of the reals and $Y=\{0, 1\}$ then the cofinite as well as the usual topology make f an identification function where the topology on Y is the indiscrete topology with $f(x)=0$, $x \in Q$, and $f(x)=1$, $x \in R \setminus Q$.

The set consisting of all topologies that make $f:X \to Y$ an identification function where $\mathcal{J}_Y$ is the topology on Y will be referred to as $Q(f, \mathcal{J}_Y)$.

The question now asked is, "Is there a smallest member of $Q(f,\mathcal{T}_Y)$?" The answer to this question is yes. This element will be referred to as the minimal topology.

The Minimal Topology

Lemma 1

If $f:X\to Y$ is a surjection with $\mathcal{T}_Y$ the topology on Y and $f^{-1}\mathcal{T}_Y = \{f^{-1}(U) \mid U \in \mathcal{T}_Y\}$ then $f^{-1}\mathcal{T}_Y$ is the smallest element of $Q(f,\mathcal{T}_Y)$.

The easy proof is omitted.

Definition 1

If $f:X\to Y$ is a surjection with $\mathcal{T}_Y$ the topology on Y then $f^{-1}\mathcal{T}_Y$ is called the minimal topology.

It might be asked whether or not the above is the only element of a particular $Q(f,\mathcal{T}_Y)$. The next theorem yields results in this area.

Theorem 1

If $f:X\to Y$ is a surjective mapping with $\mathcal{T}_Y$ the topology on Y, then $f^{-1}\mathcal{T}_Y$ is the only topology in $Q(f,\mathcal{T}_Y)$ if and only if f is one to one.

Proof

If it is assumed that f is one to one and that f is an identification function then it must also be true that f is an open surjection since $f^{-1}f(U) = U$ for all $U \in \mathcal{T}_X$. Hence the minimal topology is the only possible topology on X in $Q(f,\mathcal{T}_Y)$.

Conversely, if $f(x_\alpha) = f(x_\beta)$ with $x_\alpha \neq x_\beta$, then $\{\phi, \{x_\alpha\}, X\} \vee f^{-1}\mathcal{T}_Y$ is in $Q(f, \mathcal{T}_Y)$. However, this is in contradiction of the fact that f is the only element in $Q(f, \mathcal{T}_Y)$. Therefore f is one to one.

The definition and clarification of the concept of the minimal topology now gives rise to a new added hypothesis that is especially use-useful in transferring topological properties in an identification mapping from the image space of the function to the domain space. It is indeed readily apparent that under the hypothesis $\mathcal{T}_X = f^{-1}\mathcal{T}_Y$ the properties of first countability, second countability, regularity, normality, separability, local compactness, compactness, connectivity, and local connectivity are preserved. This is not the case, however, for the separation properties. In order to transfer any one of the separation properties from the image space of an identification function to its domain space, a homeomorphism is necessary. The case for T_0 is illustrated.

Theorem 2

If $f: X \to Y$ is an identification function with the respective topologies $\mathcal{T}_X = f^{-1}\mathcal{T}_Y$ and $\mathcal{T}_Y$ on X and Y with Y, T_0, then X is T_0 if and only if f is one to one.

Proof

Assuming first that X with the minimal topology is T_0, suppose f is not one to one. Then there exists $x, y \in X$ such that $f(x) = f(y)$ with $x \neq y$. However there can be no open set containing x but not y (or vice versa), since all the open sets in X are of the form

$f^{-1}(V)$, $V \in \mathcal{J}_Y$, and cannot separate elements of X that map to the same element in Y. Hence, if $\mathcal{J}_X = f^{-1}\mathcal{J}_Y$ and X is T_0 then f must be one to one.

Conversely if f is one to one and Y is T_0, it is clear that $x, y \in X$ can be separated by T_0 specifications by merely separating f(x) and f(y). Hence X is T_0.

Metrizability because of its link with the T_2 property, also requires a homeomorphism in transference from the image space of an identification function to the domain space.

The Maximal Topology

Having discussed the idea of a minimal topology the more complex analysis of a maximal topology on the domain space of an identification topology remains. As can be ascertained by referring to example one of this chapter, what must be sought is a maximal topology, not a largest topology. This can be seen through the realization that $\{\phi, \{x\}, R\} \in Q(f, \mathcal{J}_Y)$ for all $x \in R$. Therefore a largest topology would have to be discrete. The discrete topology is not an element of $Q(f, \mathcal{J}_Y)$. The quest is then to come up with some formulation that will describe a maximal topology on the domain space of an identification function. Nevertheless some identification functions lend themselves to having a largest topology. The next definitions and theorem give results in this area.

Definition 2

If $f:X \to Y$ is an identification function then $K_f = \{x \mid f^{-1}f(x) = \{x\}, x \in X\}$ is the kernel of f. $L_f = f[K_f]$.

In addition to this definition this next concept will prove useful in the study of the maximal topology.

Definition 3

Suppose $\mathcal{J}_\alpha$ and $\mathcal{J}_\beta$ are two topologies on a set X. Let $B = \mathcal{J}_\alpha \cup \mathcal{J}_\beta$. Thus B is the collection of all sets either open in $\mathcal{J}_\alpha$ or $\mathcal{J}_\beta$. The topology generated by B is called the suprememum of $\mathcal{J}_\alpha$ and $\mathcal{J}_\beta$, written $\mathcal{J}_\alpha \vee \mathcal{J}_\beta$.

Theorem 3

If $f:X \to Y$ is an identification function, then $Q(f, \mathcal{J}_Y)$ contains a largest element if and only if for all $y \in Y \setminus L_f$ $\{y\} \in \mathcal{J}_Y$ where $\tilde{\mathcal{J}}_Y$ is the topology on Y and $L_f \in \mathcal{J}_Y$.

Proof

If it is first assumed that there exists a largest element in $Q(f, \mathcal{J}_Y)$, namely $\mathcal{J}_L$, then for $y \in Y \setminus L_f$ and $x \in f^{-1}(y)$, $f^{-1}\mathcal{J}_Y \vee \{\phi, \{x\}, X\} \in Q(f, \mathcal{J}_Y)$. Therefore for all $x \in f^{-1}(y)$, $\{x\} \in \mathcal{J}_L$ which implies $f^{-1}(y) \in \mathcal{J}_L$. Hence $\{y\} \in \mathcal{J}_Y$.

Continuing with the assumption, one is able to pick $x_1, x_2 \in f^{-1}(y)$ with $x_1(y) \neq x_2(y)$ and define $S_1 = \{x_1(y) \mid y \in Y \setminus L_f\}$ and $S_2 = \{x_2(y) \mid y \in Y \setminus L_f\}$. It must now be shown that $f^{-1}\mathcal{T}_Y \vee \{\phi, K_f \cup S_i, X\} \in Q(f, \mathcal{T}_Y)$ for $i = 1,2$. First it should be noted that the basic open sets in the described topology are of two types: $U \cap X = U$ where $U \in f^{-1}\mathcal{T}_Y$ or $U \cap (K_f \cup S_i)$ where $U \in f^{-1}\mathcal{T}_Y$. Both types of open sets are closed under unions. Therefore a typical open set is of the form $U_1 \cup (U_2 \cap (K_f \cup S_i))$ where $U_1, U_2 \in f^{-1}\mathcal{T}_Y$. If $U_1 \cup (U_2 \cap (K_f \cup S_i))$ is an inverse set then $U_1 \cup (U_2 \cap (K_f \cup S_i)) = U_1 \cup U_2$. The set inclusions $U_1 \cup (U_2 \cap (K_f \cup S_i)) \subseteq U_1 \cup U_2$ is indeed obvious. The reverse inclusion merits investigation. If $x \in U_1 \cup U_2$ then $x \in U_1$ which yields the result or $x \in U_2$ and $y = f(x)$; then $x \in K_f$ implies $x \in U_2 \cap (K_f \cup S_i)$. Otherwise $x_i(y) \in U_2 \cap S_i \subseteq U_1 \cup (U_2 \cap (K_f \cup S_i))$. Since this is an inverse set $x \in f^{-1}(y) \subseteq U_1 \cup (U_2 \cap (K_f \cup S_i))$. Thus any open inverse set is already open in $f^{-1}\mathcal{T}_Y$ and $f^{-1}\mathcal{T}_Y \vee \{\phi, K_f \cup S_i, X\} \in Q(f, \mathcal{T}_Y)$. Therefore, $K_f \cup S_2$ and $K_f \cup S_1$ are both in the largest topology and so is $(K_f \cup S_1) \cap (K_f \cup S_2) = K_f$. Since K_f is an open inverse set by definition, $f[K_f] = L_f \in \mathcal{T}_Y$.

Conversely if $L_f \in \mathcal{T}_Y$ and $\{y\} \in \mathcal{T}_Y$ for $y \in Y \setminus L_f$ a largest topology exists. Let $\mathcal{T}_L$ be defined by $\mathcal{T}_L = f^{-1}\mathcal{T}_Y \vee \{U \mid U = X$ or $U \subsetneq X \setminus K_f\}$. Two things must now be shown: first, that $\mathcal{T}_L \in Q(f, \mathcal{T}_Y)$ and second that if $\mathcal{T} \in Q(f, \mathcal{T}_Y)$ then $\mathcal{T} \subseteq \mathcal{T}_L$. Returning to the first item to be proved, it is clear that $\mathcal{T}_L$ on X allows f to preserve its property of being a continuous surjection. Now suppose $f^{-1}(S) \in \mathcal{T}_L$ where $S \subseteq Y$. It now remains to show that $S \in \mathcal{T}_Y$. One may now consider $x \in K_f \cap f^{-1}(S)$. There exists a $U_x \in f^{-1}\mathcal{T}_Y$ and $U_2 \in \{U \mid U = X$ or $U \subseteq X \setminus K_f\}$ such that $x \in U_x \cap U_2 \subseteq f^{-1}(S)$. By definition U_2 must be X. Hence

$x \in U_x \subseteq f^{-1}(S)$ and $\bigcup\{U_x \mid x \in f^{-1}(S) \cap K_f\} = V \in f^{-1}\mathcal{T}_Y$. Therefore $S = f(V) \cup (\bigcup_{\alpha \in \Gamma} y_\alpha)$ where $y_\alpha \in Y \setminus L_f$ and $\mathcal{T}_L$ preserves the properties of an identification function.

Returning now to the second consideration mentioned one must consider any $\mathcal{T} \in Q(f, \mathcal{T}_Y)$ and show $\mathcal{T} \subseteq \mathcal{T}_L$. Suppose first that $U \in \mathcal{T}$ and consider $x \in U$. If $x \notin K_f$ then by hypothesis $\{x\} \in \mathcal{T}_L$, $\{x\} \subseteq U$ and U is a $\mathcal{T}_L$ neighborhood of x. If $x \in K_f$ then $x \in K_f \cap U$ which is an open inverse set. Hence $f(K_f \cap U) \in \mathcal{T}_Y$. But $K_f \cap U = f^{-1}f(K_f \cap U) \in \mathcal{T}_L$. So $x \in K_f \cap U \subseteq U$ and U is a $\mathcal{T}_L$ neighborhood of x. However since U is a neighborhood for $x \in K_f$ and $x \notin K_f$ it can be concluded that U is an open set in $\mathcal{T}_L$. Hence $\mathcal{T} \subset \mathcal{T}_L$ and the proof is complete.

With these results it is now possible to embark on a search for maximal topologies.

First before the actual pinpointing of a maximal topology can be made, an analysis concerning Zorn's lemma now seems applicable. The question is, do all chains of identification topologies have a maximal element? In order for this question to be answered, a lemma must now be presented.

Lemma 1

If $(X, \mathcal{T})$ is a topological space with $x_0 \in X$ then $\mathcal{T} \vee \{\phi, \{x_0\}, X\} = \mathcal{T} \cup \{U \cup \{x_0\} \mid U \in \mathcal{T}\}$.

Proof

Clearly by the definition of the supremum of two topologies $\mathcal{T} \cup \{U \cup \{x_0\} \mid U \in \mathcal{T}\}$ is contained in $\mathcal{T} \vee \{\phi, \{x_0\}, X\}$. To show the reverse one must consider $V \in \mathcal{T} \vee \{\phi, \{x_0\}, X\}$. It must be shown that $V \setminus \{x_0\} \in \mathcal{T}$.

If $x_0 \notin V$, $V\setminus\{x_0\} = V$; hence $V\setminus\{x_0\} \in \mathcal{T}$. If $x_0 \in V$, $V = (V\setminus\{x_0\}) \cup \{x_0\}$ which says that $V\setminus\{x_0\}$ is indeed an element of $\mathcal{T}$.

Example 2

If $X = [0, 1]$ and $Y = [0, 1]\setminus\{\frac{1}{n} \mid n\in Z^+\}$ and f maps X into Y by $f(x) = 0$ if $x = \frac{1}{n}$ for some $n\in Z^+$, and $f(x) = x$ if $x \neq \frac{1}{n}$ for $n \in Z^+$; and the topology on Y is co-finite with $\mathcal{T}_0 = f^{-1}\mathcal{T}_Y$ and $\mathcal{T}_{n+1} = \mathcal{T}_n \vee \{\phi, \{\frac{1}{n+1}\}, X\}$ the following conclusions can be made.

First, it can be shown that for all n $f:(X, \mathcal{T}_n) \to (Y, \mathcal{T}_Y)$ is an identification function. This can be shown by induction. The initial case $n = 0$ is true by previous work. Now assuming the conclusion for $\mathcal{T}_n$, suppose $f^{-1}[S] \in \mathcal{T}_{n+1}$. It must be shown that $S \in \mathcal{T}_Y$. If $\frac{1}{n+1} \notin f^{-1}[S]$, $f^{-1}[S] \in \mathcal{T}_n$ which implies that $S \in \mathcal{T}_Y$. If $\frac{1}{n+1} \in f^{-1}[S]$, $f^{-1}[S] = U \cup \{\frac{1}{n+1}\}$ and $0 \in S$. This implies that $f^{-1}\{0\} \subseteq S$. So if $S = f[V]$, $U \in \mathcal{T}_n$ implies U equals the union of V and some finite subset of $\{1, 1/2, 1/3, \ldots, 1/n\}$ where $V \in f^{-1}\mathcal{T}_Y$.

Let $\mathcal{T}_\infty$ be the topology with basis $\bigcup_{n=0}^{\infty} \mathcal{T}_n$. $\mathcal{T}_\infty$ is the smallest topology which contains $\mathcal{T}_n$ for all n.

Finally, it can be demonstrated that $f:(X, \mathcal{T}_\infty) \to (Y, \mathcal{T}_Y)$ is not an identification. This is true since $\{\frac{1}{n} \mid n\in Z^+\} \in \mathcal{T}_\infty$ which would make $f^{-1}(0)$ an open set when $\{0\}$ is not open in Y.

Therefore, from the above example it is clear that there exists a chain of identification topologies based on a certain f and $\mathcal{T}$ with no upper bound. Hence the Zorn's Lemma argument fails to produce even at the very least any evidence of a maximal element.

There is at least one instance for which a maximal topology for a function $f:X\to Y$ and $\mathcal{J}_Y$ can be constructed. The construction begins with the following definition.

Definition 4

If $f:X\to Y$ with $\mathcal{J}_Y$ the topology on Y with no topology on X and $A\subseteq X$ such that $f|_A$ is 1-1 and onto then $\mathcal{J}_{DIS\setminus A}$ is the topology for X with base $\{X, A\}\cup\{\{x\}\mid x\notin A\}$ and $\mathcal{J}_M = \mathcal{J}_{DIS\setminus A} \vee f^{-1}\mathcal{J}_Y$.

The next two theorems yield results that confirm $\mathcal{J}_M$ is a maximal topology.

Theorem 4

Let $f:X\to Y$ be a surjection with $\mathcal{J}_Y$ the topology on Y. If X has the topology $\mathcal{J}_M$, then f is an identification function.

Proof

Suppose there exists a set $V\in\mathcal{J}_M$ that destroys the identification map, i.e. V an inverse set such that $f(V)\notin\mathcal{J}_Y$. Then for all $x\in V$ either $x\in A$ or $x\notin A$. If $x\notin A$ then $\{x\}\in\mathcal{J}_M$. Let $V_1=\{x\mid x\in V \text{ and } x\in A\}$. Then $V=V_1 \bigcup_{\substack{x\in V\\ x\notin A}} \{x\}$. Also by the construction of A, $f(V_1) = f(V)$. $f(V_1)$, however, is not an open set which implies that $V_1\notin\mathcal{J}_M$. This in turn yields the result that $V\notin\mathcal{J}_M$. Also f is continuous since $f^{-1}\mathcal{J}_Y\subset\mathcal{J}_M$.

Next there remains the question of whether or not the topology is actually maximal.

Theorem 5

Let $f: X \to Y$ be an identification function with $\mathcal{T}_Y$, the topology on Y, and $\mathcal{T}_M$ the topology on X. Then if there exists $\mathcal{T}$ a topology for the domain such that f with this topology on X is an identification and $\mathcal{T}_M \subseteq \mathcal{T}$ then $\mathcal{T} = \mathcal{T}_M$.

Proof

Suppose there exists a $U \in \mathcal{T}$ and $U \notin \mathcal{T}_M$. Then for all $x \in U$, either $x \in A$ (as described in the definition of $\mathcal{T}_M$) or $x \notin A$. Again, since $\mathcal{T}_M \subseteq \mathcal{T}$, if $x \notin A$, $\{x\} \in \mathcal{T}$. Let $U_1 = \{ x \mid x \in U \text{ and } x \in A \}$. Then $U = U_1 \bigcup_{x \notin A,\, x \in U} \{x\}$. Now since A is open in $\mathcal{T}$ $A \cap U = U_1$ which is also open in $\mathcal{T}$. Since $U \notin \mathcal{T}_M$ it will be assumed that $U_1 \neq \phi$. However, because $U \notin \mathcal{T}_M$ there is no member $V \in f^{-1}\mathcal{T}_Y$ such that $V \cap A = U_1$. So $f(U_1)$ is not open in Y. But $f^{-1} f(U_1)$ is open in $\mathcal{T}$. The identification function is then destroyed. Hence U cannot be an open set. $\mathcal{T}_M = \mathcal{T}$.

Thus far the main concern in this discussion is to start with the minimal topology and construct a topology which has been found to be maximal. There has been no guarantee, however, that any $\mathcal{T}_X \in Q(f, \mathcal{T}_Y)$ can be contained in such a maximal topology. The next theorem yields results in this area.

Theorem 6

Let $f: X \to Y$ be an identification function with $\mathcal{T}_X, \mathcal{T}_Y$ the topologies on X and Y respectively. There exists an $A \subseteq X$ such that $\mathcal{T}_X \subseteq \mathcal{T}_M$ if and only if $f|_A$ is a homeomorphism from A to Y. The topology on A is the relative topology based on $\mathcal{T}_X$.

Proof

If there exists an $A \subseteq X$ such that $\mathcal{T}_X \subseteq \mathcal{T}_M$, then in addition to being 1-1 and onto $f|_A$ is an identification function since for all $U \subseteq Y$, $f^{-1}(U) \cap A$ is open in the subspace topology if and only if $U \in \mathcal{T}_Y$. Now it remains to show that $f|_A$ is an open map. Suppose there exists a set $U \in \mathcal{T}_A$ such that $f|_A(U) \notin \mathcal{T}_Y$. Then since $f|_A^{-1} \; f|_A(U) = U$, due to the property of $f|_A$ being 1-1, the identification map would be destroyed. Hence, there exists no such U and $f|_A$ is an open map and, therefore, a homeomorphism from A to Y.

Conversely, suppose $f|_A$ is a homeomorphism from A to Y and A has the subspace topology. If $U \in \mathcal{T}_X$ then $U = U_1 \cup U_2$ where $U_1 = \{x \mid x \in U \text{ and } x \notin A\}$ and $U_2 = \{x \mid x \in U \text{ and } x \in A\}$. Clearly $U_1 \in \mathcal{T}_M$. In addition $U_2 = U \cap A$ which is open in $\mathcal{T}_M$. Hence $U \in \mathcal{T}_M$.

Despite the results given in the last theorem there are instances for which $\mathcal{T}_X \in Q(f, \mathcal{T}_Y)$ is not contained in a $\mathcal{T}_M$ as described. In other words there will not always exist a set $A \subseteq X$ such that $f|_A$ with the relative topology on the domain is a homeomorphism. The following example provides a case in point.

Example 3

Consider $f: X \to Y$ where $X = \{-3, -2, 0, 1, 2, 3\}$ and $Y = \{a, b, c\}$; $f(-3) = a$, $f(-2) = a$, $f(0) = b$, $f(1) = b$, $f(2) = c$, and $f(3) = c$. A base for $\mathcal{T}_X$ is $\{\{-3, -2\}, \{3, 2\}, \{2, 3, 0\}, \{-2, -3, 1\}\}$. $\mathcal{T}_Y =$ $\{\{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \phi\}$.

Clearly f is an identification function. It is also easily seen that for all possible choices for a set $A \subseteq X$ that satisfies requirements for the construction of $\mathcal{T}_M$, $f|_A$ is not a homeomorphism. Hence by the previous theorem $\mathcal{T}_X \not\leq \mathcal{T}_M$. Even more importantly, this example illustrates that the described maximal topology is not the only maximal topology possible for an identification function. This statement can be made due to the fact that the identification function of the previous example has a finite domain and range, and therefore, only a finite number of topologies in $Q(f, \mathcal{T}_Y)$.

Returning for the moment to the described maximal topology it seems now appropriate to discuss the transference of topological properties from the domain space to the range space of an identification function and vice-versa when a maximal topology, $\mathcal{T}_M$, is the topology of the domain space.

When transference from the domain space to the range space is considered, the answers are quite clear. Since the specified set A is both open, closed, and homeomorphic to Y; and since all the properties discussed in Chapter III are either hereditary, F-hereditary, or G-hereditary, the transference is automatic.

Transference from the range space back to the domain space is not as all-encompassing. Nevertheless the facts are easily seen and are presented here without proof. Separation properties and, hence, metrizability are transferred. Connectivity is not transferred unless A, as described with reference to this maximal topology, is equal to X. compactness is not transferred unless $X \setminus A$ is finite. Local connectivity as well as local compactness are both transferred. Second countability and separability are transferred on the condition that $X\setminus A$ is countable. First countability is also transferred.

What now remains in this discussion of the maximal topology are some unanswered questions. The first of these is quite apparent from the discussion following the last example: What if any is the characterization of the other maximal topology or topologies? Also it is presently unknown whether or not all topologies on the domains of identification functions are contained in a maximal topology. Therefore it can be said that what has been found is only one of a possible many in the class of maximal topologies on the domain space of an identification function. An infinite number of circumstances concerning identification functions remain to be investigated.

BIBLIOGRAPHY

Books

Dugundji, James. Topology. Boston: Allyn and Bacon, Inc., 1966.

Hausdorff, Felix. Set Theory. Translated by John R. Auman et alia. Second Edition. New York: Chelsea Publishing Company.

Kasriel, Robert H. Undergraduate Topology. Philadelphia: N. B. Saunders Company, 1971.

Kelley, John L. General Topology. Princeton, New Jersey: D. Van Nostrand Company, Inc., 1955.

Articles

Michaels, E. A. "Quintuple Quotient Questions." General Topology and its Applications, 2 (1972), 91-138.

Moore, R.L. "Foundations of Point Set Theory." American Mathematical Society Colloquium Publication, XIII, (1932).

Whyburn, George T. "Open and Closed Mappings." Duke Math Journal, 17, (1950), 69-74.

REFERENCES

Long, Paul E. An Introduction to General Topology. Columbus, Ohio: Charles E. Merrill Publishing Company, 1971.

Munkres, James R. Topology: A First Course. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1975.