FORCED VIBRATION OF STRUCTURAL MODELS WITH EQUAL FREQUENCIES

by

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ABSTRACT

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The purpose of this thesis is to present methods for analyzing the displacements and forces developed in structures that are subjected to an arbitrary dynamic loadings.

Single-Degree of freedom systems are discussed and concept of dynamic load factor is introduced.

The response equations of multidegree dynamic systems are formulated using matrix methods. A computer program utilizing the Jacobi method is presented which solves for the natural frequencies and mode shapes of the dynamic system.

A Finite Difference technique is introduced which efficiently solves for maximum response and maximum force conditions. A computer program is utilized to augment the procedure.

Finally, the response of a four degree of freedom system is considered in which two of the natural frequencies are numerically close.

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LIST OF NOTATION

SYMBOL	DEFINITION
A	Cross-sectional area of menber
DLF	Dynamic load factor
E	Young's modulus of elasticity
F	Force
I	Moment of inertia
$[k]$	Stiffness matrix
k	Spring constant
L	Length of member
m	Mass per length
$[M]$	Mass matrix
n	Degrees of freedom
t_i	Rise time
t_d	Time duration
T	Natural period
$\{u\}$	Mode shape vector
$\{v\}$	Displacement vector
W	Weight
$\{x\}$	Eigenvector
y	Displacement
ω	Natural circular frequency
λ	Eigenvalue

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CHAPTER I

INTRODUCTION

1.1 Background

The primary purpose of this thesis is to present methods, with primary emphasis on matrix methods, for analyzing the displacement and force developed in any given type of structure when it is subjected to an arbitrary dynamic loading. The interest in structural design for dynamic loading has been increasing steadily over the years. This is in part due to advancing technology, which has made possible more accurate design. It is also due to the fact that more daring structures (larger, lighter, etc.) are being attempted, and these are more susceptive to dynamic effects because they are generally more flexible and have longer periods.

1.2 Definition

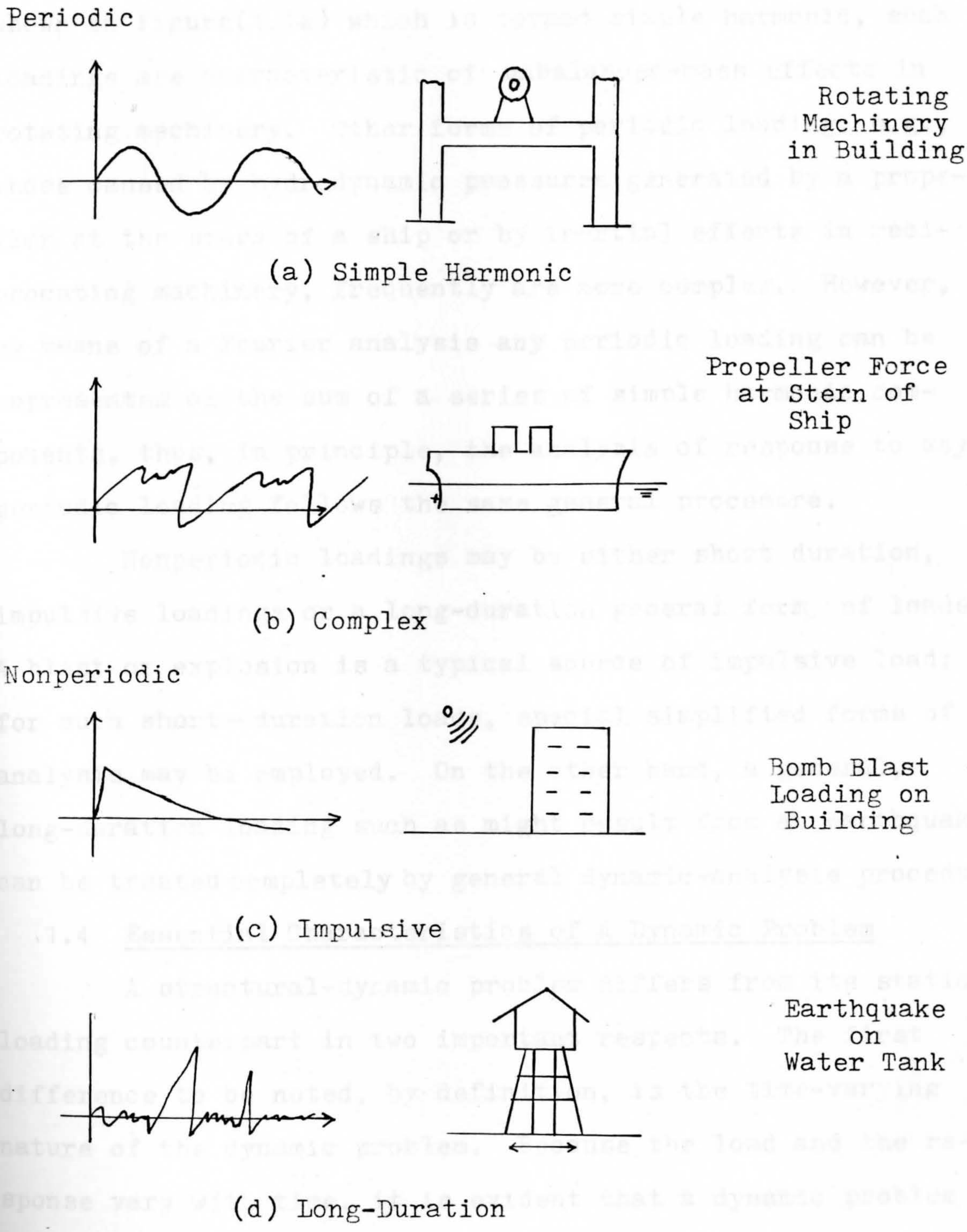
The term dynamic may be defined simply as time-varying, thus, a dynamic load is any load of which the magnitude, direction or position varies with time. Similarly, the structural response due to a dynamic load (i.e., the resulting displacement and stresses) is also time-varying or dynamic. In fact, no structural loads (with the possible exception of dead load) are really static, since they must be applied to the structure in some manner, and this involves a time variation of force. It is obvious, however, that if the magnitude of force varies slowly enough, it will not have dynamic effect and can be treated as static. "Slowly enough" is not definite, and apparently the question of whether or not a load is dynamic is a relative matter. It turns out that the natural period of the structure is the significant parameter, and if the load varies slowly relative to this period, it may be considered static. The natural period, loosely defined, is the time required for the structure to go through one cycle of free vibration, i.e., vibration after the force causing the motion has been removed or has ceased to vary.

1.3 Types of Prescribed Loading

Almost any type of structural system may be subjected to one form or another of dynamic loading during its lifetime. If the time varition of loading is fully known, even though it may be highly oscillatory or irregular in character, it will be referred to herein as a prescribed dynamic loading. From an analytical standpoint, it is convenient to divide the loadings into two basic categories, periodic and nonperiodic[(1)*]. Some typical forms of prescribed loadings and examples of situation in which such loadings might be developed are shown in Figure(1.1).

As indicated in Figure(1.1a) and (1.1b), periodic loadings are repetitive loads which exhibit the same time varition successively for a large number of cycles. The simplest periodic loading is the sinusoidal variation

*Number in parenthesis refers to literature cited in the Bibliography.

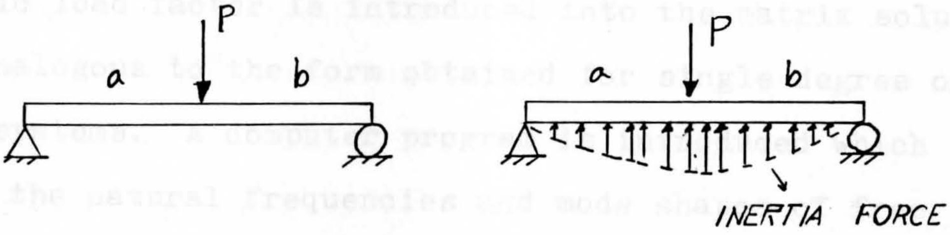


Figure(1.1) Characteristics and Sources of Typical Dynamic Loadings shown in Figure(1.1a) which is termed simple harmonic, such loadings are characteristic of unbalanced-mass effects in rotating mechinery. Other forms of periodic loading, e.g., those caused by hydrodynamic pressures generated by a propeller at the stern of a ship or by inertial effects in reciprocating machinery, frequently are more complex. However, by means of a Fourier analysis any periodic loading can be represented as the sum of a series of simple harmonic components, thus, in principle, the analysis of response to any periodic loading follows the same general procedure.

Nonperiodic loadings may be either short duration, impulsive loadings or a long-duration general form of loads. A blast or explosion is a typical source of impulsive load; for such short—duration loads, special simplified forms of analysis may be employed. On the other hand, a general, long-duration loading such as might result from an earthquake can be treated completely by general dynamic-analysis procedures.

1.4 Essential Characteristics of A Dynamic Problem

A structural-dynamic problem differs from its static loading counterpart in two important respects. The first difference to be noted, by definition, is the time-varying nature of the dynamic problem. Because the load and the response vary with time, it is evident that a dynamic problem does not have a simple solution, as a static problem does; instead the analyst must establish a succession of solutions corresponding to all times of interest in the response history. Thus a dynamic analysis is clearly more complex and time-consuming than a static problem analysis. However, a more fundamental distinction between static and dynamic problems is illustrated in Figure(1.2). If a simple beam is subjected to a static load p, as shown in Figure(1.2a), its internal moments and shears and deflected shape depend upon the given load and can be computed from p by established principles of force equilbrium. On the other hand, If the load p(t) is applied dynamically, as shown in Figure(1.2b), the resulting displacement of the beam are associated with acceleration which produce inertia forces resisting the acceleration. Thus the internal moments and shears in the beam in Figure(1.2b) must equilibrate not only the externally applied force but also the inertia forces resulting from the acceleration of the beam.



(a) Static Loading (b) Dynamic Loading

Figure(1.2) Basic Difference Between Static and Dynamic Loads

Inertia forces which resist accelerations of the structure in this way are the most important distinguishing characteristic of a structural-dynamic problem. In general, if the inertia forces represent a significant portion of the total load equilibrated by the internal elastic forces of the structure, then the dynamic character of the problem must be accounted for in its solution. On other hand, if the motions are so slow that the inertia forces are negligibly small, the analysis for any desired instant of time may be made by the static structural-analysis procedures even though the load and response may be time-varying.

1.5 Thesis Procedure

The format of the thesis includes the following four major areas of investigation:

1. The dynamic analysis of a single degree of freedom vibratory system is investigated using the concept of dynamic load factor.

2. The dynamic analysis of multidegree of freedom vibratory systems is formulated utilizing matrix methods. The concept of dynamic load factor is introduced into the matrix solutions, analogous to the form obtained for single degree of freedom systems. A computer program is introduced which computes the natural frequencies and mode shapes of free vibration. The mathematical difficulties involved in the "direct" solution of the family of coupled differential equations is observed.

3. The "indirect" technique of Finite-Difference analysis is utilized in a matrix formulation process to obtain the response solutions of multidegree of freedom systems. This procedure eliminates the need of consideration of the concept

of DLF. As will be shown, the method is effective and efficient for the determination of response provided a digital computer is available.

4. An investigation of a multidegree of freedom system with two numerically close frequencies is carried out. Using the finite difference analysis. The purpose of this particular problem is to determine the variation of induced structure forces, illustrating the fact that these forces take on maximum values as any two of the frequencies approach one another.

The importance of investigating a structural system with close vibrational frequencies was brought out dramatically by a recent highrise building failure. In 1977, because of inproper design, the John Hancock Building in Boston had a serious wind force problem. This was primarily due to the fact that the bending natural frequency of the building was close to the torsional frequency. As a result of excessive vibratory deformations the entire window surface of the building had to be replaced. A vibration tuner had to be placed in the building to eliminate torsional frequencies. The cost of these construction changes was in the order of twenty five million dollars. Currently pending in court is a suit against the designer in excess of forty five million dollars.

CHAPTER II

DYNAMIC RESPONSE OF SINGLE-DEGREE SYSTEMS

2.1 Introduction

The dynamic response of spring-mass systems with a single-degree of freedom are discussed in this chapter. These problems have found many practical applications in structural dynamics, since many structural systems in engineering are idealized conveniently into a spring-mass system with a single-degree of freedom. For example, the frame structure shown in Figure(2.1a) is often represented by the simple mass-spring system shown in Figure(2.1b)

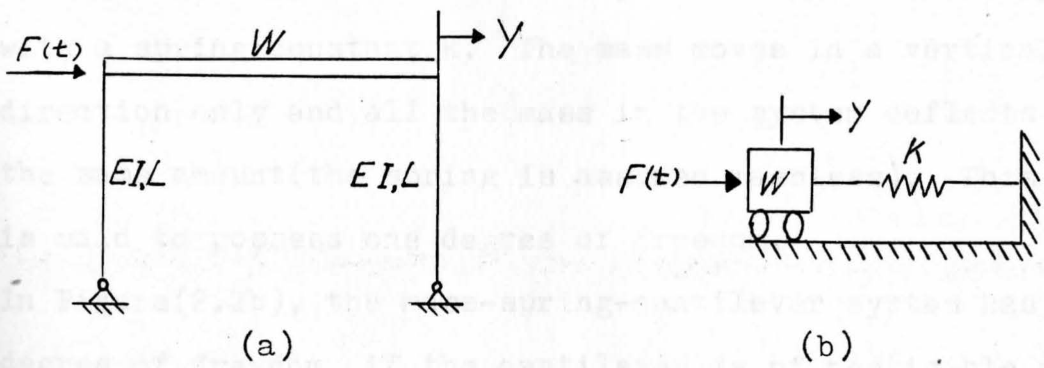
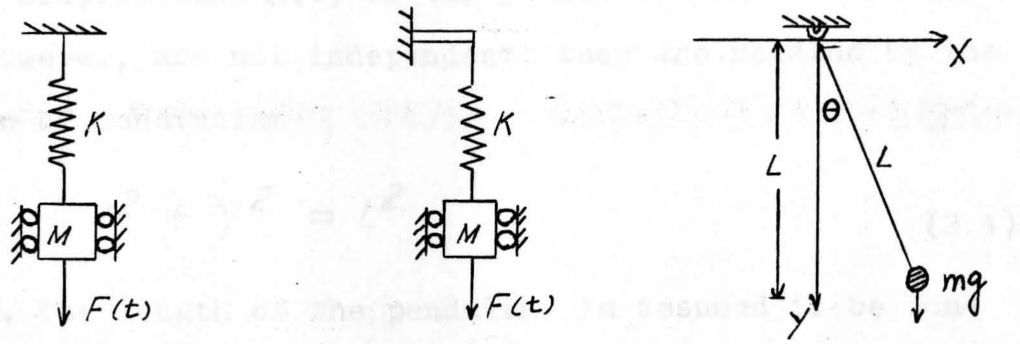


Fig.(2.1) Structure Idealized As Spring-Mass System

A system generally consists of many, or infinitely many, mass particles. If the interrelationship of the masses is such that only one spatial coordinate is required to define the configuration of the system, or in other words, the position of the system at any instant can be defined in terms of a single coordinate, it is said to possess one degree of freedom. A configuration is defined as the geometric location of all the masses of a system in space. Several single-degree of freedom systems are shown in Figure(2.2).



(a)Spring-Mass System (b)Spring in Series (c)Simple Pendulum

Fig.(2.2) Systems with One Degree of Freedom

In Figure(2.2a), the mass is suspended from a coiled spring with a spring constant k. The mass moves in a vertical direction only and all the mass in the system deflects by the same amount(the spring is assumed massless). This system is said to possess one degree of freedom.

In Figure(2.2b), the mass-spring-cantilever system has one degree of freedom, if the cantilever is of negligible mass and mass M is constrained to move vertically. By neglecting the inertia effect of the cantilever and considering only its elasticity, the cantilever is assumed to be a spring which is placed in series with the other spring k of the system. A spring, with equivalent stiffness to the two springs in series, is defined, and the system reduces to the case of the spring-mass system of Figure(2.1a)

In Figure(2.2c), a simple pendulum is constrained to move in the X-Y plane. The configuration is defined either by the rectangular cartesian coordinates X(t) and Y(t) or by the angular displacement $\theta(t)$ of the pendulum. The X-Y coordinates, however, are not independent; they are related by the equation of constraint

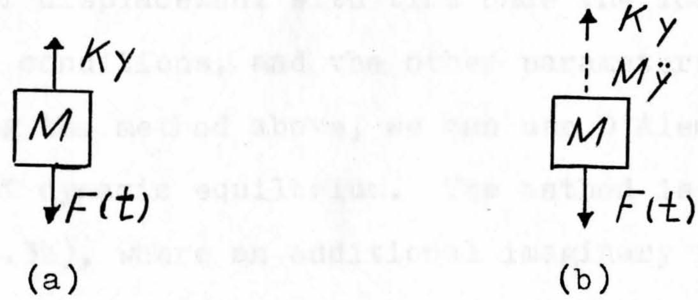
$$X^2 + Y^2 = L^2 \tag{2.1}$$

where L, the length of the pendulum, is assumed to be constant. Thus, if X(t) is chosen arbitrarily, Y(t) is determined from Equation(2.1). It is more convenient to choose a single coordinate $\theta(t)$ to define the configuration of this system.

2.2 Response of An Undamped System

2.2-1 Equation of Motion

First, consider the system shown in Figure(2.1a). One isolates the mass as shown in Figure(2.2a)



Fig(2.3) Dynamic Equilbrium —— One Degree System

The external forces, which are the applied force F(t) and the spring force Ky, are applied to the mass. It is assumed here that the spring is linear (i.e., the force in the spring is always equal to the spring constant times the displacement). Note that the weight, or gravity force, does not appear in the figure. This implies that the displacement y is measured from the neutral position, that is, the static position which the mass would take if only the force of gravity were acting. After isolating the mass, using NEWTON'S SECOND LAW F=Ma, the equation of motion is determined, wherein F is the net or algebraic sum of the forces acting on the mass, and the positive direction of force is the same as that for displacement or acceleration. Thus, the equation of motion for this system is

$$F(t) - Ky = M\ddot{y} \tag{2.2}$$

This differential equation may be solved to determine the variation of displacement with time once the loading function, the initial conditions, and the other parameters are known. Besides using the method above, we can use D'Alembert's Principle of dynamic equilbrium. The method is illustrated in Figure(2.3b), where an additional imaginary force is applied to the mass. This is the inertia force, and is equal to the product of the mass and the acceleration, with direction opposite to positive displacement. Having added this force, the situation shown in Figure(2.2b) is exactly similar to a problem in static equilbrium. The equilbrium equation becomes

$$F(t) - Ky - M\ddot{y} = 0 \qquad (2.3)$$

It is seen that this approach results is exactly the same equation as that previously obtained. In general, the second approach given is more convenient, especially when distributed masses are involved.

2.2-2 Free Vibration

After obtaining the equation of motion, one considers a special case where F equals zero. Motion will occur only if the system is given an initial disturbance, which may take the form of an initial displacement y (i.e., the mass is displaced and then released at t=0), or an initial velocity (i.e.,The velocity is produced by an impluse or impact or a combination of the two). The resulting motion, unaffected by an external force, is called free vibration. The equation of motion for this case is simply

$$\ddot{y} + \frac{K}{M} y = 0 \qquad (2.4)$$

and solution of the above differential equation is

$$y = C_1 \sin\sqrt{\frac{K}{M}}\, t + C_2 \cos\sqrt{\frac{K}{M}}\, t \qquad (2.5)$$

by letting $\omega^2 = K/M$, we have

$$Y = C_1 \sin \omega t + C_2 \cos \omega t \qquad (2.6)$$

in which the constants C_1 and C_2 may be expressed in terms of the initial condition(i.e., the displacement y_o and velocity $\dot{y}_o$ at time t = 0), which initiated the free vibration of the system.

First, at t = 0, $y = y_o$, Equation(2.6) may be written as

$$Y_o = C_1 \sin \omega(0) + C_2 \cos \omega(0)$$

therefore

$$C_2 = y_o$$

Differentiating Equation(2.6) and substituting at t = 0; $\dot{y} = \dot{y}_o$, we obtain

$$\dot{Y}_o = C_1 \omega \cos \omega(0) + C_2 \omega \sin \omega(0)$$

therefore

$$C_1 = \dot{y}_o/\omega$$

Substituting these expressions for the constants into Equation(2.6), yields the solution for zero external load as

$$Y = \frac{\dot{Y}_o}{\omega} \sin \omega t + Y_o \cos \omega t \qquad (2.7)$$

2.2-3 Natural Period and Frequency

The free vibration discussed above is said to be harmonic; that is, y varies sinusoidally with t. The motion is completely repetitive if there is no damping in the system. Harmonic motion is defined by a maximum amplitude and a natural period, which is the time required for the motion to complete one cycle. The initial conditions affect only the amplitude of the vibration. The parameter ω in Equation(2.7) is called the natural circular frequency, or, Natural circular frequency

$$\omega = \sqrt{\frac{K}{M}} \quad \text{rad/sec} \tag{2.8}$$

Since one complete cycle occurs for each angular increment $\omega t = 2\pi$, the natural period of the system is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{M}{K}} \quad \text{sec} \tag{2.9}$$

Note that the natural period and frequency are characteristics of the system and depend only upon the mass and spring constant. The natural frequency is defined as the inverse of the natural period, or the number of cycles per unit of time, or,

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{K}{M}} \quad \text{cps} \tag{2.10}$$

2.2-4 Forced Vibration

Consider the case in which the motion is the result of an applied force F(t). It will be assumed that the system begins at rest; i.e., both the velocity and displacement are zero at t = 0. Obviously, this is not a necessary condition and a solution could be obtained for the combination of the two effects.

To begin with a simple case, assumed that F(t) has a constant magnitude F_1 which is suddenly applied and remains constant indefinitely. For this situation Equation(2.1) becomes

$$\ddot{y} + \frac{K}{M} y = \frac{F_1}{K} \qquad (2.11)$$

The solution of this equation is

$$y = C_1 \sin \omega t + C_2 \cos \omega t + \frac{F_1}{K} \qquad (2.12)$$

where the constants C_1 and C_2 are determined by the initial conditions. Substituting into Equation(2.12), $y_0 = 0$; $t = 0$ one obtains

$$0 = C_1 \sin(0) + C_2 \cos(0)$$

therefore

$$C_2 = -\frac{F_1}{K}$$

C_1 is obtained by substituting $\dot{y}_0 = 0$ at $t = 0$ into the differentiated form of Equation(2.12) giving

$$\dot{y} = C_1 \omega \cos(0) - C_2 \omega \sin(0) = 0$$

therefore $C_1 = 0$

If these values of C_1 and C_2 are substitued into Equation(2.12) the final solution is obtained as

$$y = \frac{F_1}{K}(1 - \cos\omega t) \qquad (2.13)$$

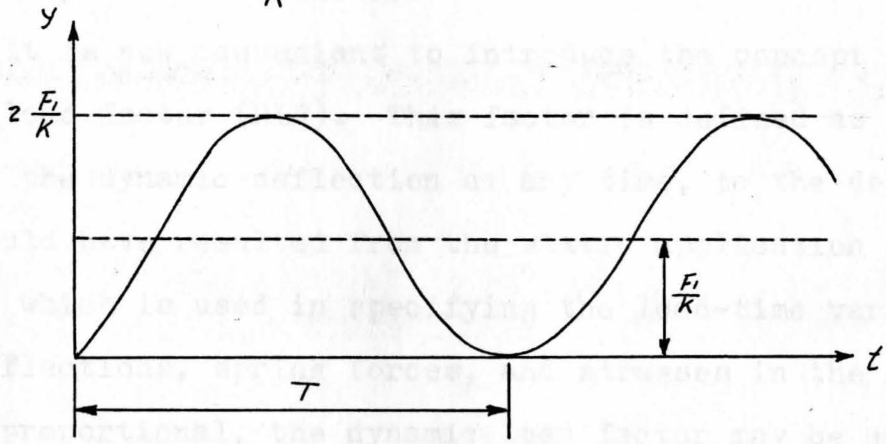


Fig.(2.4) Response of Undamped One-Degree System to Suddenly Applied Constant Force.

This solution for a suddenly applied constant load is plotted in Figure(2. 4).

It will be observed that the solution just obtained is very similar to the previous solution for free vibration(see Figure(2.2)). The only difference is that the axis of the vibration has been shifted by the amount equal to F_1/k. It should also be noted that the maximum displacement F_1/k is exactly twice the displacement which would occur if the load F_1 were applied statically. Thus, we reach an elementary but very important conclusion : If a constant force is suddenly applied to a linear elastic system, the resulting displacement is exactly twice that for the same force applied statically. The same observation is true regarding the dynamic force in the spring, which is proportional to the displacement. Furthermore, since the spring-mass system represents an actual structure, the same statement may be made regarding both dynamic deflections and stresses in that structure.

2.2-5 Dynamic Load Factor

It is now convenient to introduce the concept of the dynamic load factor (DLF). This factor is defined as the ratio of the dynamic deflection at any time, to the deflection which would have resulted from the static application of the load F_1, which is used in specifying the load-time variation[2]. Since deflections, spring forces, and stresses in the structure are all proportional, the dynamic load factor may be applied to any of these in order to obtain the ratio of dynamic to static effects.

In the preceding example, which involved a suddenly applied constant load, the static deflection is F_1/k. Thus the dynamic load factor is given by

$$DLF = \frac{y}{y_{st}} = \frac{y}{F_1/k} = \frac{ky}{F_1} \qquad (2.14)$$

Substituting Equation(2.13) for y gives

$$DLF = 1 - \cos \omega t \qquad (2.15)$$

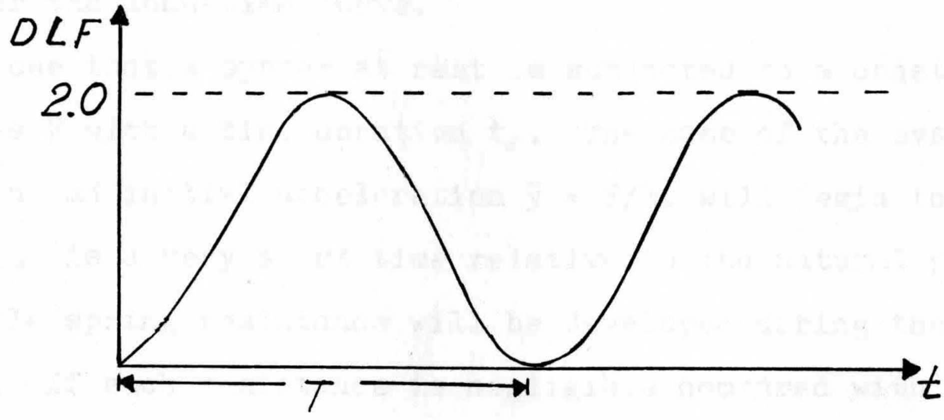


Fig.(2.5) Dynamic Load Factor (DLF) for an Undamped One-Degree System Subject to a Suddenly Applied Constant Force.

Thus, the dynamic load factor for this case is as shown in Figure(2.5). It is apparent that the dynamic load factor is nondimensional and independent of the magnitude of load. It is because of this fact that it is convenient to use.

In many structural problems only the maximum value of the DLF is of interest. In the case just considered, this maximum is 2, which immediately indicates that all maximum displacements, forces, and stresses due to the dynamic load are twice the values that would be obtained from a static analysis for the load F_1.

2.3 Various Forcing Functions (Undamped Systems)

2.3-1 Generalized Linear-Systems Theory

Before discussing responses for various load-time functions, it is convenient to obtain a general solution applicable to any such a function. First, however, let us recall the concept of impulse, which is defined as the area under tha load-time curve.

Suppose that a system at rest is subjected to a constant force F with a time duration t_d. The mass of the system, having an initial acceleration $\ddot{y} = F/M$, will begin to move. If t_d is a very short time relative to the natural period, little spring resistance will be developed during the time t_d. If such resistance is negligible compared with F, the acceleration can be considered constant and the net effect will be a velocity imparted to the mass. The value of this velocity at time t_d will be

$$y = \ddot{y}\, t_d = \frac{F}{M}\, t_d = \frac{I}{M} \tag{2.16}$$

where I is the applied impulse equal to the area under the load-time curve. If the assumption stated above and implied by Equation(2.16) is valid, I is said to be a pure impulse. To give a quantitative feeling for this concept, it may be said that the error in Equation(2.16) is negligible if t_d is smaller than about one-tenth of the natural period. Obviously, in such cases, the actual shape of the load-time function during the time t_d is of small importance.

Turning now to a general load function such as shown in Figure(2.6), consider the area in the element of time $d\tau$ to be a pure impulse.

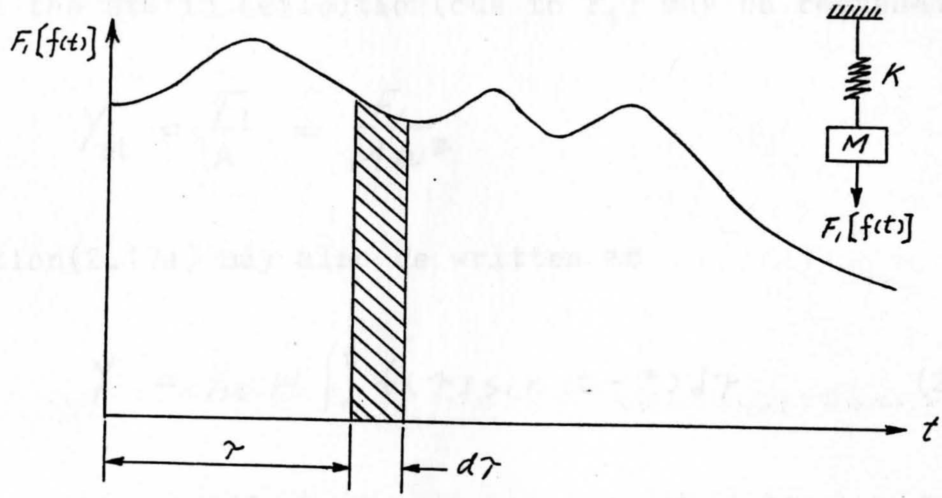


Fig.(2.6) Linear-System Theory —— Impulse Element

This causes an increment of velocity at τ equal to $F_1 f(\tau)\frac{d\tau}{M}$ which may be considered as an initial velocity imparted to a system at rest. The displacement at a later time due to this single element of impulse is given by Equation(2.4). If $\dot{Y}_0$ is the initial velocity just defined and if Y_0 is taken as zero (since there is no initial displacement corresponding to the effect of this impulse), thus, one obtains

$$\frac{F_1 f(\tau) d\tau}{M\omega} \sin\omega(t-\tau)$$

which is the displacement at time t due to the load applied during $d\tau$. Since the system is linear, superposition may be employed and the total displacement at t is the sum of the effects of all elements of impulse between zero and t, thus

$$Y = \int_0^t \frac{F_1 f(\tau) d\tau}{M\omega} \sin\omega(t-\tau) \qquad (2.17a)$$

Since the static deflection(due to F_1) may be represented by

$$Y_{st} = \frac{F_1}{k} = \frac{F_1}{M\omega^2}$$

Equation(2.17a) may also be written as

$$Y = Y_{st}\,\omega \int_0^t f(\tau)\sin(t-\tau)\,d\tau \qquad (2.17b)$$

To mak the equation even more general, the effects of initial displacement and velocity may be included by superimposing Equations(2.7) and (2.17b) as

$$Y = Y_0 \cos\omega t + \frac{\dot{Y}_0}{\omega}\sin\omega t + Y_{st}\int_0^t f(\tau)\sin\omega(t-\tau)\,d\tau \qquad (2.18)$$

where y_0 and $\dot{y}_0$ are the displacement and velocity(if any) at t = 0. Equation(2.18) is a perfectly general expression for the response of an undamped, linearly elastic one-degree system subjected to any load function and/or initial conditions. A closed solution is of course possible only if the integral can be evaluated. Applications of Equation(2.18) are illustrated below.

2.3-2 Constant Force With a Finite Rise Time and Limited Duration

After formulating the general equation for the response of an undamped, linearly elastic one-degree system, one may conveniently investigate a loading with finite rise time and limited duration as shown in Figure(2.7a), where t_i is the rising time and t_d is the limited duration.

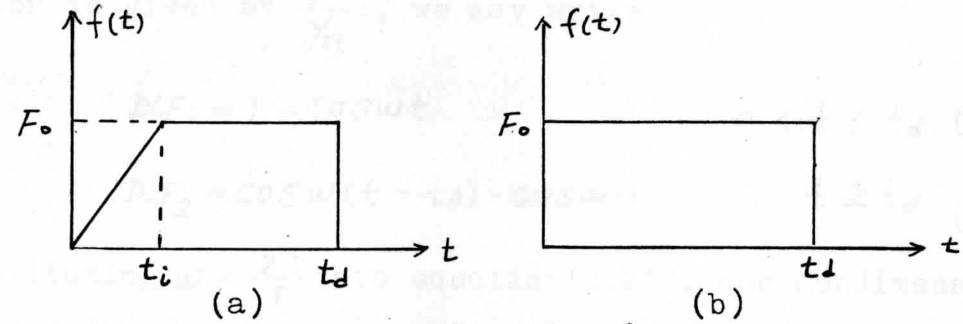


Fig.(2.7) (a) Controlled Pulses (b) Rectangular-Pulse Load

First consider the special case t = 0, the case of a suddenly applied constant load shown in Figure(2.7b). The system starts at rest,(i.e., $y_o = 0$, $\dot{y}_o = 0$), with no damping force present. One computes the response in two stages.

For the first stage

$$0 < t \leq t_d \qquad f(t) = 1$$

Substituting f(t) in Equation(2.17b) and integrating the function from 0 to t, yields

$$y_1(t) = \frac{F_1}{K}(1-\cos\omega t) \tag{2.19}$$

which defines the response $t < t_d$.

For the second stage, that is, the response beyond t_d,

$$t \geq t_d \qquad f(t) = 0$$

$$y_2(t) = \frac{F_1}{K}\left[\int_0^{t_d} \sin\omega(t-\tau)d\tau\right] \tag{2.20}$$

$$= \frac{F_1}{K}\left[\cos\omega(t-t_d) - \cos\omega t\right]$$

Since $\frac{F_1}{K}$ is the static deflection and the dynamic load factor is given by $\frac{y}{y_{st}}$, we may write

$$DLF_1 = 1 - \cos\omega t \qquad 0 < t \leq t_d \quad (2.21a)$$

$$DLF_2 = \cos\omega(t - t_d) - \cos\omega t \qquad t \geq t_d \quad (2.21b)$$

Substituting $\omega = \frac{2\pi}{T}$ into equation(2.21), one nondimensionalizes the time parameter as follows

$$DLF_1 = 1 - \cos 2\pi \frac{t}{t_d} \frac{t_d}{T} \quad (2.22a)$$

$$DLF_2 = \cos 2\pi\left(\frac{t}{t_d} \frac{t_d}{T} - \frac{t_d}{T}\right) - \cos 2\pi \frac{t}{t_d} \frac{t_d}{T} \quad (2.22b)$$

where T is the natural period. This latter form serves to emphasize the fact that the ratio of the time duration of the load function to natural period, rather than the actual value of either quantity, is the important parameter. The maximum values of Dynamic Load Factor are computed by maximizing the respective time functions shown in Equations (2.22a) and (2.22b).

The results of the maximization process are listed as

$$DLF_1 = 1 - \cos\pi = 2 \qquad \frac{t_d}{T} \geq \frac{1}{2} \quad (2.23a)$$

$$DLF_2 = -2 \sin\frac{t_d}{T}\pi \qquad 0 < \frac{t_d}{T} < \frac{1}{2} \quad (2.23b)$$

Equation(2.23) which is plotted in Figure(2.9), shows that as the time duration of the load approachs zero, the maximum deflection, or stress, also diminishes to zero. Also for the case where $t_d/T > 0.5$, the maximum response of the system is the same as if the time duration of the load had been infinite. For this case the dynamic load factor has a constant value equal to two.

Graphs such as shown in Figure(2.9) are extremely useful for design purpose. For a given load function one need know only the natural period in order to read from the chart the maximum DLF and hence the ratio of maximum dynamic to static stress. In the derivation of the chart no damping has been inclued because it would have no significant effect. The maximum dynamic load factor usually corresponds to the first peak of response, and the amount of damping normally encountered in structures is not sufficient to decrease appreciably this value.

Consider the case shown in Figure(2.7a). The response is computed by using Equation(2.17b) in three stages. For the first stage

$$0 < t \leq t_d$$

$$f(\tau) = \frac{\tau}{t_i}$$

$$y_1(t) = \frac{F_o}{m\omega}\int_{\tau=0}^{\tau=t} \frac{\tau}{t_i} \sin\omega(t-\tau)\,d\tau \qquad (2.24)$$

After integrating we obtain

$$y_1(t) = \frac{F_1}{m\omega^2}\frac{1}{\omega t_i}(\omega t - \sin\omega t) \quad (2.25a)$$

and $$DLF_1 = \frac{y}{y_{st}} = \frac{1}{\omega t_i}(\omega t - \sin\omega t) \quad (2.25b)$$

which defines the response for $t < t_i$.

For the second stage $t_i < t \le t_d$

$$f(\tau) = 1$$

$$y_2(t) = \frac{F_1}{m\omega}\left[\int_{\tau=0}^{\tau=t_i}\frac{\tau}{t_i}\sin\omega(t-\tau)d\tau + \int_{\tau=t_i}^{\tau=t}\sin\omega(t-\tau)d\tau\right]$$

$$= \frac{F_1}{m\omega^2}\left[1 + \frac{1}{\omega t_i}(\sin\omega(t-t_i) - \sin\omega t)\right] \quad (2.26a)$$

and $$DLF_2 = 1 + \left[\frac{1}{\omega t_i}(\sin(t-t_i) - \sin\omega t)\right] \quad (2.26b)$$

which gives the response between t_i and t_d.

For the last stage $t > t_d$

$$f(\tau) = 0$$

$$y_3(t) = \frac{F_1}{m\omega}\left[\int_{\tau=0}^{\tau=t_i}\frac{\tau}{t_i}\sin\omega(t-\tau)d\tau + \int_{\tau=t_i}^{\tau=t_d}\sin\omega(t-\tau)d\tau\right]$$

$$= \frac{F_1}{m\omega^2}\left[\frac{1}{\omega t_i}\sin\omega(t-t_i) - \sin\omega t\right] + \cos\omega(t-t_d) \quad (2.27a)$$

and $$DLF_3 = \frac{1}{\omega t_i}\left[\sin\omega(t-t_i) - \sin\omega t\right] + \cos\omega(t-t_d) \quad (2.27b)$$

which is the response beyond t_d.

Substituting $\omega = \frac{2\pi}{T}$ into Equations(2.25b) (2.26b) (2.27b) one obtains

$$DLF_1 = 2\pi \frac{t_d}{t_i}\frac{t}{t_d} - \frac{\sin 2\pi \frac{t_d}{T}\frac{t}{t_d}}{\frac{t_i}{t_d}\frac{t_d}{T}}$$

$$DLF_2 = 1 + \frac{1}{2\pi \frac{t_i}{t_d}\frac{t_d}{T}}\left[\sin 2\pi\left(\frac{t}{t_d}\frac{t_d}{T} - \frac{t_i}{t_d}\frac{t_d}{T}\right) - \sin 2\pi \frac{t}{t_d}\frac{t_d}{T}\right]$$

$$DLF_3 = \frac{1}{2\pi \frac{t_i}{t_d}\frac{t_d}{T}}\left[\sin 2\pi\left(\frac{t}{t_d}\frac{t_d}{T} - \frac{t_i}{t_d}\frac{t_d}{T}\right) - \sin 2\pi \frac{t}{t_d}\frac{t_d}{T}\right] + \cos 2\pi\left(\frac{t}{t_d}\frac{t_d}{T} - \frac{t_d}{T}\right)$$

To determine the maximum dynamic load factor, one differentiates D_1, D_2, and D_3 with respect to time parameter $\frac{t}{t_d}$, and after considerable algebraic manipulation obtains

$$DLF_{1\,max} = \frac{t_d}{t_i}\frac{n}{t_d/T} \tag{2.28a}$$

$$DLF_{2\,max} = 1 - \frac{1}{\pi \frac{t_i}{t_d}\frac{t_d}{T}} \cos n\pi \sin n\pi \frac{t_i}{t_d}\frac{t_d}{T} \tag{2.28b}$$

$$DLF_{3\,max} = \frac{h}{\beta} \tag{2.28c}$$

with

$$h = \left[\beta^2 + \sin^2\beta - 2\beta \sin\beta \cos(\alpha - \beta)\right]^{1/2}$$

$$\beta = \frac{\omega t_i}{2} \qquad \alpha = \omega t_d$$

Figure(2.10), shows the plot of the latter factors (DLF)max for different values of time parameter ($\frac{t_i}{t_d}$). Here the effect of rise time is apparent. It is clear from the Figure(2.10) that when the time parameter $\frac{t_i}{t_d}$ is decreasing, then the value of DLFmax is increasing. Also, if is very small compared to the whole time period t_d, the affect is essentially the same as for a suddenly applied load. This observation is of significance in practical design since it indicates that smaller rise times may be ignored. A peculiarity of this type of load pulse is the fact that, if $\frac{t_d}{T}$ is reciprocal of $\frac{t_i}{t_d}$, the response is the same as if F_1 had been applied statically. (i.e., DLF=1.0) Consider now the final case where $\frac{t_i}{t_d}$ =1. The load function starts at zero and reaches a maximum value at t_d , after t_d , f(t)=0. This case is shown in Figure(2.8).

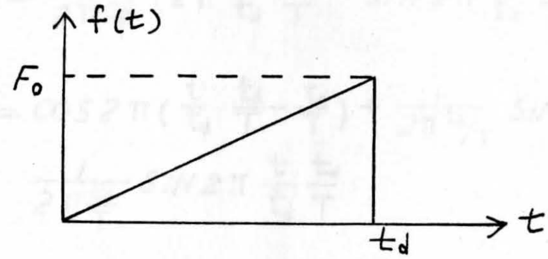


Fig.(2.8) Triangular Load Pulses

For the first stage, $0 < t \leq t_d$

load time function $f(\tau)=\frac{\tau}{t_d}$, then

$$y_1(t)=\frac{F_0}{m\omega}\int_{\tau=0}^{\tau=t}\frac{\tau}{t_d}\sin\omega(t-\tau)\,d\tau$$

$$=\frac{F_0}{m\omega}\frac{1}{\omega t_d}(\omega t-\sin\omega t) \qquad (2.29a)$$

and $DLF_1 = \frac{1}{\omega t_d}(\omega t-\sin\omega t)$ (2.29b)

For the second stage, $t \geq t_d$; $f(\tau)=0$

$$Y_2(t) = \frac{F_o}{m\omega}\int_{\tau=0}^{\tau=t_d} \frac{\tau}{t_d} \sin\omega(t-\tau)\,d\tau$$

$$= \frac{F_o}{m\omega^2}\left[\cos\omega(t-t_d) + \frac{1}{\omega t_d}\sin\omega(t-t_d) - \frac{1}{\omega t_d}\sin\omega t\right] \quad (2.30a)$$

and $DLF_2 = \cos\omega(t-t_d) + \frac{1}{\omega t_d}\sin\omega(t-t_d) - \frac{1}{\omega t_d}\sin\omega t$ (2.30b)

Substituting $\omega = \frac{2\pi}{T}$ in Equation(2.29b) (2.30b) gives

$$DLF_1 = \frac{1}{2\pi\frac{t_d}{T}}\left(2\pi\frac{t}{t_d}\frac{t_d}{T} - \sin 2\pi\frac{t}{t_d}\frac{t_d}{T}\right)$$

$$DLF_2 = \cos 2\pi\left(\frac{t}{t_d}\frac{t_d}{T} - \frac{t_d}{T}\right) + \frac{1}{2\pi\, t_d/T}\sin 2\pi\left(\frac{t}{t_d}\frac{t_d}{T} - \frac{t_d}{T}\right) - \frac{1}{2\pi\frac{t_d}{T}}\sin 2\pi\frac{t}{t_d}\frac{t_d}{T}$$

Differentiating D_1 D_2 respect to time parameter $\frac{t}{t_d}$, after some simplification one obtains

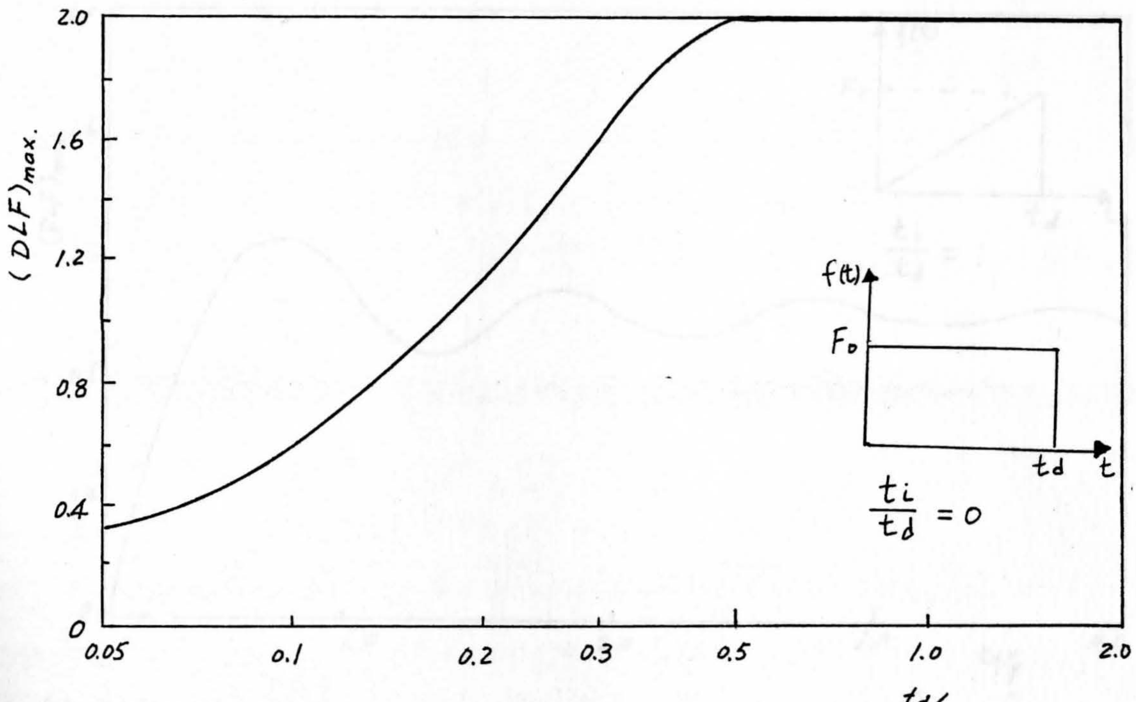
$$DLF_{1\,max} = \frac{1}{t_d/T} \quad (2.31a)$$

$$DLF_{2\,max} = \frac{h}{\beta} \quad (2.31b)$$

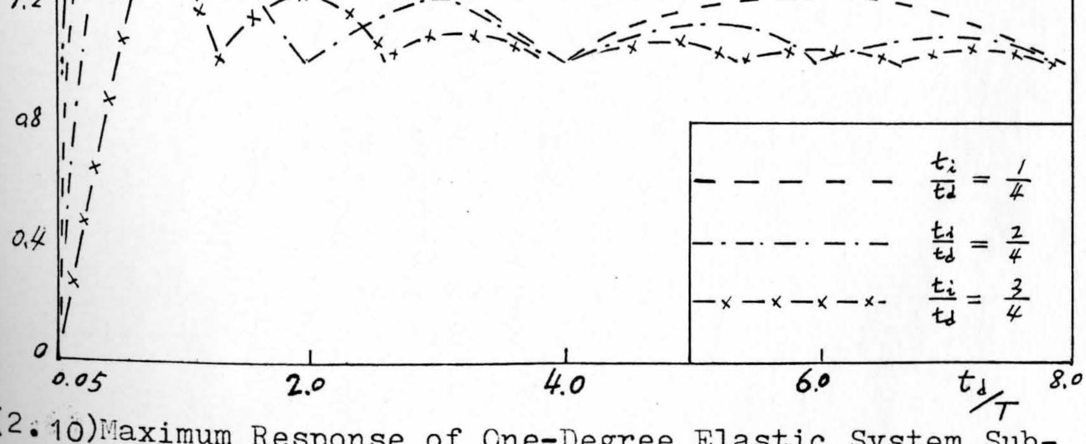
with $h = (\sin^2\beta + \beta^2 - \beta\sin 2\beta)^{1/2}$

$$\beta = \frac{\omega t_d}{2}$$

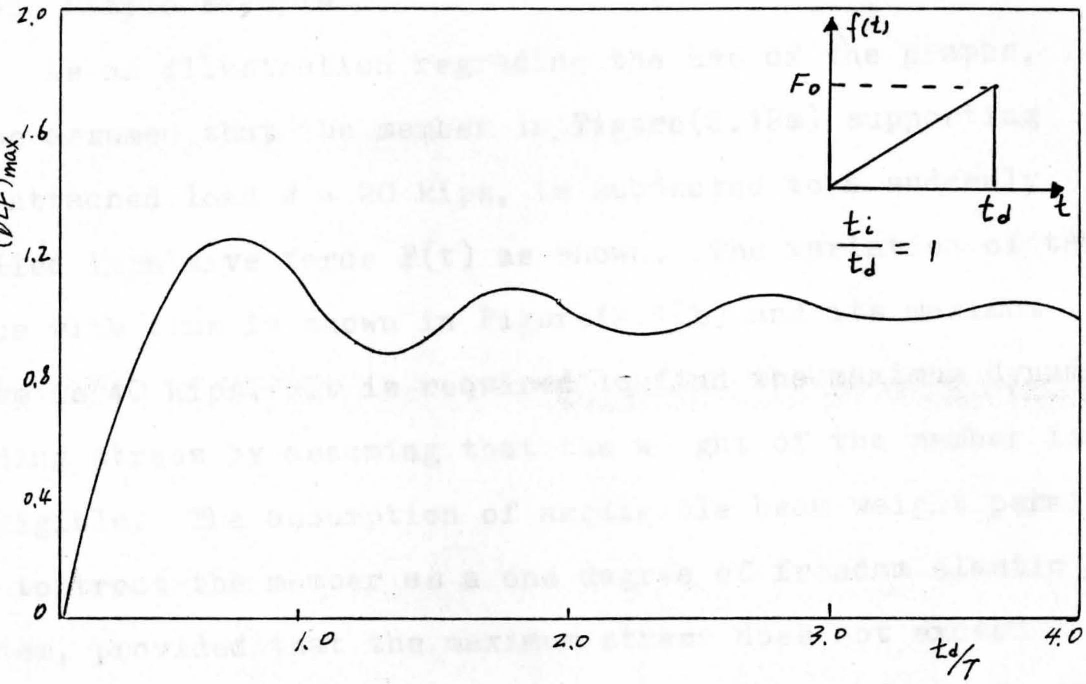
A plot of the maximum Dynamic Load Factor as a function of $\frac{t_d}{T}$ is shown in Figure(2.11).



Fig(2.9) Maximum Response of One-Degree Elastic Systems Subjected to Rectangular load pulses.



Fig(2.10)Maximum Response of One-Degree Elastic System Subjected to Controlled Load.



Fig(2.11)Maximum Response of One-Degree Elastic System Subjected to Triangular Load with Finite Rise Time.

2.3-3 Sample Example

As an illustration regrading the use of the graphs, let it be assumed that the member in Figure(2.12a) supporting the attached load W = 20 kips, is subjected to a suddenly applied impulsive force F(t) as shown. The variation of this force with time is shown in Figure(2.12b) and its maximum value is 40 kips. It is required to find the maximum dynamic bending stress by assuming that the weight of the member is negligible. The assumption of negligible beam weight permits one to treat the member as a one degree of freedom elastic system, provided that the maximum stress does not exceed the elastic limit of the material. The idealized one degree of freedom system is shown in Figure(2.12c). The spring constant k is determined by applying to the center of the beam a force p that is capable of producing a unit vertical displacement at the point, thus

$$K = \frac{48EI}{\ell^3}$$

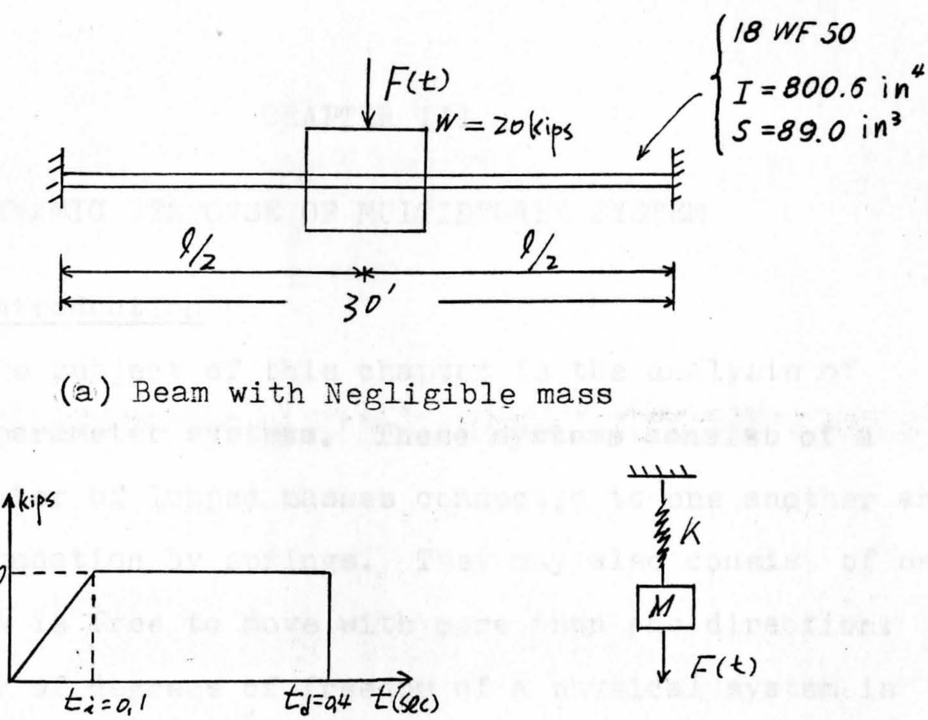
$$= \frac{48 \times 30 \times 10^3 \times 800.6}{(30 \times 12)^3} = 24.71 \text{ kips/in}$$

The natural period of the system is

$$T = 2\pi \sqrt{\frac{m}{K}}$$

$$= 2\pi \sqrt{\frac{20}{386 \times 24.71}}$$

$$= 0.288 \text{ sec}$$



(a) Beam with Negligible mass

(b) Load Function (c) Idealized System

Fig.(2.12) Sample Example of Dynamic Load Factor

$$\frac{t_d}{T} = 1.39 \qquad \frac{t_i}{t_d} = \frac{1}{4}$$

Entering $\frac{t_d}{T}$ and $\frac{t_i}{t_d}$ into Figure(2.10), one obtain

$$(DLF)_{max} = 1.81$$

The maximum dynamic stress σ_{max} is equal to static σ_{st} caused by 40 kips force, multiplied by the $(DLF)_{max}$, that is

$$\sigma_{MAX} = \frac{M_{st}}{S} (DLF)_{MAX}$$

$$= \frac{40/4 \times 360 \times 1.81}{89.0}$$

$$= 73.21 \text{ Ksi}$$

which may be greater than the material yield stress.

CHAPTER III

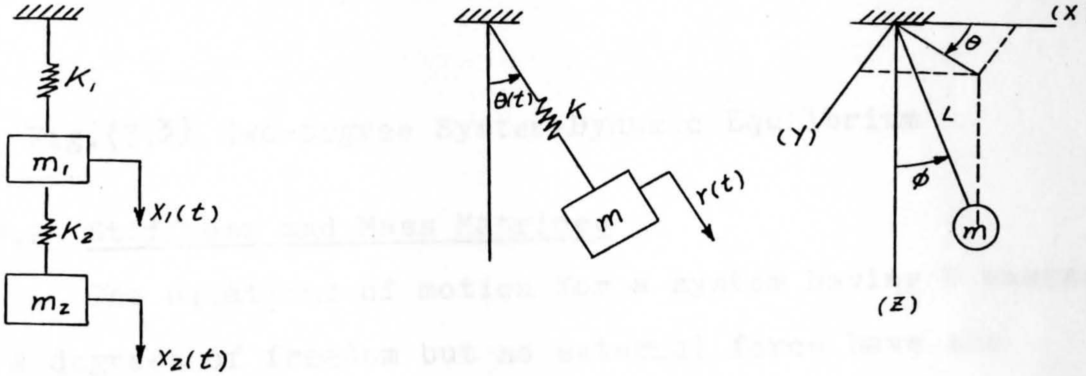
DYNAMIC RESPONSE OF MULTIDEGREE SYSTEM

3.1 Introduction

The subject of this chapter is the analysis of discrete-parameter systems. These systems consist of a finite number of lumped masses connected to one another and to the foundation by springs. They may also consist of one mass which is free to move with more than one direction. The number of degrees of freedom of a physical system is equal to the number of independent spatial coordinates necessary to define the configuration of the system. Several two-degree of freedom systems shown in Figure(3.1), are briefly described as follows:

1. The two-spring-two-mass system of Figure(3.1a) possesses two degrees of freedom if the masses are constrained to move in the vertical direction. The two spatial coordinates defining the configuration are $X_1(t)$ and $X_2(t)$.
2. The spring-mass system shown in Figure(3.1b) was described previously as a one-degree-of-freedom system. If the mass m, however, is allowed to oscillate along the axis of the spring as well as to swing from side to side, the system possesses two degrees of freedom, $r(t)$ and $\theta(t)$.
3. The pendulum in space shown in Figure(3.1c) can be described by the $\theta(t)$ and $\phi(t)$ coordinates as well as by the $X(t)$, $Y(t)$, and $Z(t)$ coordinates. The latter are related

by the equation of constraint $X^2 + Y^2 + Z^2 = L^2$. Thus, this pendulum has only two degrees of freedom.



(a) 2-mass-2 Spring System (b) Spring-mass System (c) Spherical Pendulum

Fig.(3.1) Multidegree Systems

It may be stated that, for each degree of freedom, there is an independent differential equation of motion. For example, the equations for the two degrees system shown in Figure(3.2), obtained by considering the dynamic equilbrium of the two masses, are

$$M_1\ddot{Y}_1 + K_1Y_1 - K_2(Y_2 - Y_1) = F_1(t) \tag{3.1}$$

$$M_2\ddot{Y}_2 + K_2(Y_2 - Y_1) = F_2(t) \tag{3.2}$$

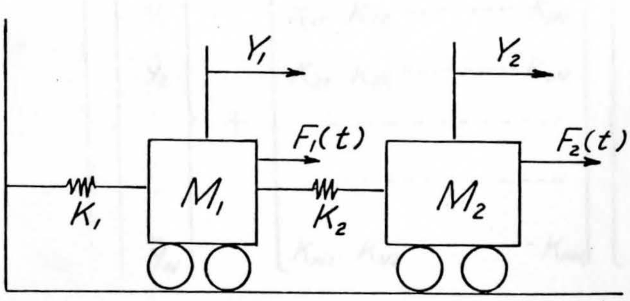


Fig.(3.2) Two Degree of Freedom Dynamic System

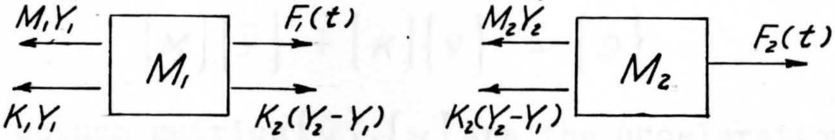


Fig.(3.3) Two-degree System-Dynamic Equilbrium

3.2 Stiffness and Mass Matrices

The equations of motion for a system having N masses and N degrees of freedom but no external force have the following form:

$$\begin{array}{l} M_1\ddot{Y}_1 + K_{11}Y_1 + K_{12}Y_2 + \text{--------} + K_{1N}Y_N = 0 \\ M_2\ddot{Y}_2 + K_{21}Y_1 + K_{22}Y_2 \text{---------} + K_{2N}Y_N = 0 \\ \text{-----------------------} = 0 \\ M_N\ddot{Y}_N + K_{N1}Y_1 + K_{N2}Y_2 + \text{--------} + K_{NN}Y_N = 0 \end{array} \tag{3.3}$$

where the K's are stiffness coefficients, which are spring constants or combinations thereof, and y's are the displacements of the lumped masses, and M's are masses of the particles. When dealing with simultaneous equations of this type, it is convenient to use matrix notation. Equation (3.3) is rewritten in matrix notation as

$$\begin{bmatrix} m_1 & & & 0 \\ & m_2 & & \\ & & \ddots & \\ 0 & & & m_N \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \vdots \\ \ddot{y}_N \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1N} \\ K_{21} & K_{22} & \cdots & K_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K_{N1} & K_{N2} & \cdots & K_{NN} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ - \\ - \\ y_N \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ - \\ - \\ 0 \end{Bmatrix}$$

or in matrix symbolic form as

$$[M]\{\ddot{y}\} + [K]\{y\} = \{0\} \qquad (3.4)$$

where the column matrics $\{\ddot{y}\}$, $\{y\}$ are the acceleration vector and the displacement vector, respectively, at time t. The square matrix

$$[m] = \begin{bmatrix} m_1 & & & 0 \\ & m_2 & & \\ & & \ddots & \\ 0 & & & m_N \end{bmatrix}$$

is known as the mass matrix.

The square matrix

$$[K] = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1N} \\ K_{21} & K_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ K_{N1} & K_{N2} & \cdots & K_{NN} \end{bmatrix}$$

is known as the stiffness matrix.

It is very important to note that, for the linear structural models, $[K]$ is symmetric (i.e., $[K]^T = [K]$), and the mass matrix is a diagonal matrix, that is, $m_{ij} = 0$, when $i \neq j$.

3.3 Natural Frequencies and Mode Shapes

If the system is vibrating in a normal mode, one makes the substitutions

$$\{y\} = \{a_n\} \sin \omega_n t \qquad \{\ddot{y}\} = -\omega_n^2 \{a_n\} \sin \omega_n t$$

into Equation(3.4) and obtains

$$\left([K] - \omega_n^2[M]\right)\{a_n\} = 0 \tag{3.5}$$

where $\{a_n\}$ is a column matrix of amplitudes. It's components define the mode shape function of the model. Noting that $\{a_n\}$ can not be zero, then it follows according to Cramer's rule that

$$\left|[K] - \omega_n^2[M]\right| = 0$$

or

$$\begin{vmatrix} K_{11}-M_1\omega^2 & K_{12} & \cdots & K_{1N} \\ K_{12} & K_{22}-M_2\omega^2 & \cdots & \\ \vdots & & & \vdots \\ K_{1N} & \cdots & \cdots & -K_{NN}-M_N\omega^2 \end{vmatrix} = 0 \tag{3.6}$$

From the expansion of this determinant, one obtains a frequency equation which can be solved for ω. There is one real root for each normal mode, thus, N natural frequencies are obtained. Since there is no basic difference in concept between a two-degree system and multidegree system, a procedure for the determination of the natural frequency is illustrated below by involving a two degree system. Considering the undamped two-degree system shown in Figure (3.2), assume there are no external forces applied to the masses, i.e., $F_1(t)=0$, $F_2(t)=0$.

Newton's equations of motion require that

$$-k_1 y_1 + k_2(y_2 - y_1) = m_1 \ddot{y}_1$$
$$-k_2(y_2 - y_1) = m_2 \ddot{y}_2 \qquad (3.7)$$

Rearranging Equation(3.7) one obtains

$$m_1 \ddot{y}_1 + k_1 y_1 - k_2(y_2 - y_1) = 0$$
$$m_2 \ddot{y}_2 + k_2(y_2 - y_1) = 0$$

which when arranged in matrix form become

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (3.8a)$$

let $\{y\} = \{a_n\} \sin \omega_n t$ and $\{\ddot{y}\} = \{a_n\} - \omega^2 \sin \omega_n t$, Equation(3.8) is written as

$$\begin{bmatrix} (k_1 + k_2) - m_1 \omega_n^2 & -k_2 \\ -k_2 & k_2 - m_1 \omega_n^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (3.8b)$$

Since $\{a_n\}$ is not equal to $\{0\}$, we have

$$\begin{vmatrix} (k_1 + k_2) - m_1 \omega_n^2 & -k_2 \\ -k_2 & k_2 - m_1 \omega_n^2 \end{vmatrix} = 0$$

$$[(k_1 + k_2) - m_1 \omega_n^2](k_2 - m_1 \omega_n^2) - k_2^2 = 0$$

$$\omega_1^2 = \frac{m_1 k_2 + m_2(k_2 + k_1) - \sqrt{[m_1 k_2 + m_2(k_1 + k_2)]^2 - 4 k_1 k_2 m_1 m_2}}{2 m_1 m_2}$$

$$\omega_2^2 = \frac{m_1 k_2 + m_2(k_2 + k_1) - \sqrt{[m_1 k_2 + m_2(k_1 + k_2)]^2 - 4 k_1 k_2 m_1 m_2}}{2 m_1 m_2}$$

For a special case suppose that all spring constants are equal to k, and both masses are equal to M then we have

$$\omega_1^2 = \frac{3-\sqrt{5}}{2}\frac{k}{m} = 0.382\frac{k}{m} \qquad \omega_1 = 0.618\sqrt{\frac{k}{m}}$$

$$\omega_2^2 = \frac{3+\sqrt{5}}{2}\frac{k}{m} = 2.618\frac{k}{m} \qquad \omega_2 = 1.618\sqrt{\frac{k}{m}}$$

These are the natural frequencies or eigenvalues of the two normal modes. The smaller frequency ω_1, corresponds to the fundamental, or first, mode, while ω_2 is the frequency of the second mode.

Having the natural frequencies of the multidegree system represented by Equations(3.3), the characteristic shapes of the modes are obtained by the use of Equations(3.5). If the value of ω^2 for a particular mode is substituted into these N equations, there are then exactly N unknowns, namely, the characteristic amplitudes $a_1 \ . \ . \ . \ a_n$ of that mode. Since the right side of Equations(3.5) is zero, unique values of the a's are not obtained. However, it is possible to obtain the relative values of all amplitudes, or in other words, the ratio of any two. If an arbitrary value is given one amplitude, all others are then fixed in magnitude. A set of such arbitrary amplitudes defines the characteristic shape, since the latter is not dependent upon absolute values of amplitude. In mathematical terms, a set of modal amplitudes is known as a characteristic vector.

It is not surprising that unique values of the characteristic amplitudes are unobtainable. We are here dealing with free vibration, the cause of which has not been defined by either initial conditions or forcing functions. The important point is that the amplitudes of a normal mode are always in the same proportion; i.e., the shape is maintained, regardless of the cause of the vibration.

To illustrate the above, consider again the two-degree system shown in Figure(3.3) for which the natural frequencies or eigenvalues were found to be $0.618\sqrt{\frac{k}{m}}$ and $1.618\sqrt{\frac{k}{m}}$.

Since the k's were taken to be equals, as were the M's, Equations(3.8b) becomes

$$\begin{bmatrix} 2k - m_1\omega_n^2 & -k \\ -k & k - m_2\omega_n^2 \end{bmatrix} \begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Substituting ω_1, the frequency of the first mode, into the first equation yields

$$(1.618k)\,a_{11} - k\,a_{21} = 0 \longrightarrow a_{11} = 0.618\,a_{21}$$
$$(-k)\,a_{11} + (0.618k)\,a_{21} = 0$$

which defines the characteristic shape or eigenvector of the first mode. The same result would have been obtained by substitution into the second equation. The notation adopted is that the first subscript on the a indicates the mass, or point on the structure at which the amplitude occurs, and the second subscript designates the mode. Substituting ω_2 into either equation yields

$$(-0.618k)a_{12} + (-k)\,a_{22} = 0$$

$$(-k)a_{12} - (1.618k)a_{22} = 0 \longrightarrow a_{12} = -1.618\,a_{22}$$

which defines the characteristic shape of the second mode. If it is desired to assign arbitrary values to the amplitudes, the two modal shapes could be indicated by

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & -1.618 \\ 1.618 & 1 \end{bmatrix}$$

The two characteristic shapes, i.e., the motions associated with the normal modes, are indicated in Figure(3.4). In the first mode the two masses move in the same direction and when m_1 moves one unit then m_2 moves 1.618 units. In the second mode when m_2 moves one unit then m_1 moves 1.618 units in the opposite direction. In both cases the motions of the two masses are in phase; i.e., the maximum displacements are attained simultaneously. The neutral point of the vibration is the static dead-load position, and the a's are in reality amplitudes of the total motion. It should be intuitively obvious that the type of distortion associated with the first mode should as we have shown, have a lower natural frequency than that associated with the second mode. We can normalize the vector columns of latter matrix to obtain

$$[U] = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} 0.8507 & 0.525 \\ -0.5257 & 0.8507 \end{bmatrix}$$

It should be noted that the latter matrix $[U]$ is an orthogonal matrix, that is, $[U]^T[U] = [I]$

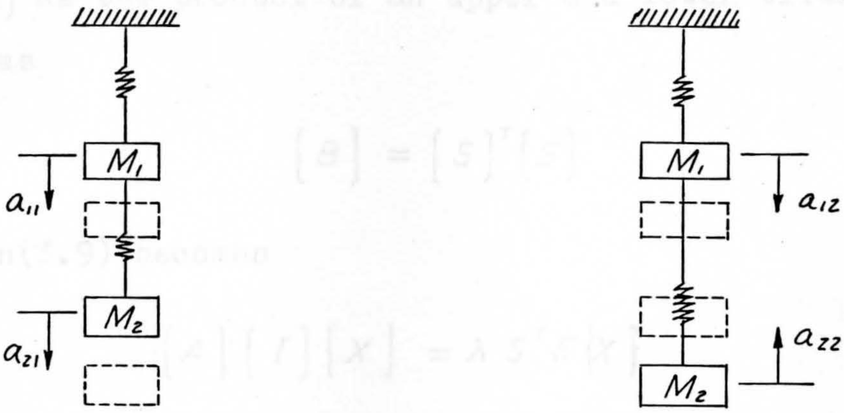


Fig.(3.4) Characteristic Shapes of Normal Modes

3.4 Mathematical Formulation of The Natural Frequency And Mode Shape Equations

3.4-1 The Cholesky Transformation

It will become apparent that solutions of natural frequency and mode shape function become extremely cumbersome as the number of modes increases. For this reason other procedures have been devised. A method utilizing the digital computer is discussed in this section.
First consider Equation(3.5), it is a general form of eigenvalue and eigenvetor problem. The expression of the equation is

$$[A]\{X\} = \lambda [B]\{X\} \qquad (3.9)$$

where matrix $[A]$ is the symmetric stiffness matrix, $[B]$ is the symmetric mass matrix, λ's are eigenvalues, and $\{X\}$ is eigenvector. From previous work, we know mass matrix $[M]$ is a diagonal and positive definite matrix. One may decompose $[B]$ as the product of an upper and lower triangular matrix as

$$[B] = [S]^T [S] \tag{3.10}$$

Equation(3.9) becomes

$$[A] [I] \{X\} = \lambda S^T S \{X\} \tag{3.11}$$

where upon substituting for unit matrix it follows that,

$$A S^{-1} S X = \lambda S^T S X \tag{3.12}$$

Noting that

$$(S^{-1})^T = (S^T)^{-1}$$

and premultiply Equation(3.12) by $(S^T)^{-1}$, one obtains

$$(S^{-1})^T A S^{-1} S X = \lambda (S^T)^{-1} S^T S X = \lambda S X \tag{3.13}$$

Defining a new vector

$$\{X'\} = S\{X\} \tag{3.14}$$

Equation(3.13) is written as

$$[[S]^{-1}]^T A S^{-1} \{X\} = \lambda \{X'\} \tag{3.15}$$

Defining

$$H = (S^{-1})^T A\, S^{-1} \tag{3.16}$$

Equation(3.15) becomes

$$[H]\{X'\} = \lambda\,\{X'\} \tag{3.17}$$

where $[H] = [H]^T$, that is, $[H]$ is symmetric.
It should be noted that the eigenvalues of Equation(3.17) are identical to those of Equation(3.9). Also, the eigenvectors of Equation(3.17) are related to those of Equation (3.9) through Equation(3.14)
Equation(3.17) is a basic form of $[A]\{X\} = \lambda\,\{X\}$. Let us consider the n matrix equations corresponding to each eigenvalue, in terms of the normalized eigenvectors

$$\begin{aligned} [A]\,\{X_1'\} &= \lambda_1\,\{X_1'\} \\ [A]\,\{X_2'\} &= \lambda_2\,\{X_2'\} \\ &- - - - \\ &- - - - \\ [A]\,\{X_n'\} &= \lambda_n\,\{X_n'\} \end{aligned} \tag{3.18}$$

or in compact form

$$[A]\,(X_1'\ X_2' \text{-----} X_n') = (\lambda_1 X_1' \text{---------} \lambda_n X_n') \tag{3.19}$$

letting

$$\{Q\} = (X_1'\ X_2' \text{---------} X_n')$$

so

$$(\lambda_1 X_1',\ \lambda_2 X_2' \text{-------------} \lambda_n X_n') = Q\,\lambda \tag{3.20}$$

where λ is the diagonal matrix of eigenvalues

$$\lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \tag{3.21}$$

Equation(3.19) can then be written as

$$[A][Q] = [Q][\lambda] \tag{3.22}$$

Next we examine the matrix product

$$[B] = [Q]^T [Q] \tag{3.23}$$

The coefficient of $[B]$ located on the ith row and jth column are given by

$$b_{ij} = X_i'^T X_j' \tag{3.24}$$

corresponding to the scalar product of the ith and jth eigenvectors, associated with the two different eigenvalues λ_i and λ_j by the expressions

$$A X_i' = \lambda_i X_i' \tag{3.25a}$$

$$A X_j' = \lambda_j X_j' \tag{3.25b}$$

Postmultiplying the transpose of (3.25a) by X'_j, and premultiplying (3.25b) by X_i', we obtain

$$X_i'^T A^T X_j' = \lambda_i X_i'^T X_j' \tag{3.26a}$$

$$X_i'^T A X_j' = \lambda_j X_i^{T'} X_j \tag{3.26b}$$

Subtracting Equation(3.26a) from Equation(3.26b), and considering that $[A] = [A^T]$ for a symmetric matrix A, the result is

$$(\lambda_j - \lambda_i) X_i'^T X_j' = 0 \tag{3.27}$$

which shows that

$$X_i'^T X_j = 0 \tag{3.28}$$

Thus, the two eigenvectors X'_i, X'_j for two different eigenvalues λ_i and λ_j are orthogonal. In the case where i = j we have,

$$X_i'^T X_i' \neq 0 \tag{3.29a}$$

and in particular, if the eigenvectors are normalized

$$X_i'^T X_i' = 1 \tag{3.29b}$$

From this discussion we conclude that

$$b_{ij} = \begin{cases} 0 & for \quad i \neq j \\ 1 & for \quad i = j \end{cases} \tag{3.30a}$$

so that $[B]$ is a unit matrix

$$B = I = Q^T Q$$

and

$$Q^T = Q^{-1} \tag{3.30b}$$

or the eigenvector matrix is orthogonal. Premultiplying both side of equation(3.22) by Q^T, we obtain

$$\left[Q\right]^T\left[A\right]\left[Q\right] = Q^T Q \lambda = \left[\lambda\right] \tag{3.31}$$

which shows that applying to A the orthogonal transformation $Q^T[\ ]Q$, produces a diagonal matrix. The diagonal coefficient of the matrix are then the eigenvalues of system $AX = \lambda X$. The columns of matrix Q are the corresponding eigenvectors. Therefore, our problem reduces to the need to diagonalize the matrix $[A]$.

3.4-2 Jacobi's Method for Eigenvalue Eigenvector Computation

The Jacobi method[3] **is an efficient and effective** process of computing eigenvalues and eigenvectors of a symmetric matrix. The procedure is to zero selected off diagonal terms of the given matrix by preforming a sequence of elementary orthogonal transformations. Consider the symmetric matrix of order 4

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix} \tag{3.32}$$

and assumed the term a_{24} is to be eliminated. Working with the orthogonal transformation matrix

$$R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C & 0 & -S \\ 0 & 0 & 1 & 0 \\ 0 & S & 0 & C \end{bmatrix} \tag{3.33}$$

where $C \equiv \cos\theta$ and $s \equiv \sin\theta$, with θ being a rotation angle to be determined, the result of a matrix operation of type given in Equation(3.31) is

$$R_1^T A R_1 = \left[\begin{array}{c|c|c|c} a_{11} & Ca_{12}+Sa_{14} & a_{13} & -Sa_{12}+Ca_{44} \\ Ca_{12}+Sa_{14} & C^2a_{22}+S^2a_{44}+2Sa_{24} & Ca_{23}+Sa_{34} & -CS(a_{22}-a_{44})+a_{24}(C^2-S^2) \\ a_{13} & Ca_{32}+Sa_{34} & a_{33} & -Sa_{32}+Ca_{34} \\ -Sa_{12}+Ca_{14} & -CS(a_{22}-a_{44})+a_{24}(C^2-S^2) & -Sa_{23}+Ca_{34} & S^2a_{22}+C^2a_{44}-2SCa_{24} \end{array}\right] \tag{3.34}$$

To eliminate the term a_{24} it follows that

$$-\cos\theta\sin\theta(a_{22}-a_{44}+a_{24}(\cos^2\theta-\sin^2\theta)=0 \tag{3.35a}$$

which is transformed into

$$a_{24}\tan^2\theta+\tan\theta(a_{22}-a_{44})-a_{24}=0 \tag{3.35b}$$

and

$$\tan\theta=\frac{-(a_{22}-a_{44})\pm\sqrt{(a_{22}-a_{44})^2+4a_{24}^2}}{2a_{24}} \tag{3.35c}$$

Let us restrict oursolves to one of the roots, for instance

$$\tan\theta=\frac{-(a_{22}-a_{44})+\sqrt{(a_{22}-a_{44})^2+4a_{24}^2}}{2a_{24}} \tag{3.35d}$$

Notice that the other root will be 180° out of phase and would not affect the results. Working with root of Equation (3.35d) is equivalent to considering only the $(-\frac{\pi}{2}<\theta<\frac{\pi}{2})$ interval. Having the $\tan\theta$ one computes

$$\cos\theta=(1+\tan^2\theta)^{-\frac{1}{2}} \tag{3.36a}$$

$$\sin\theta=\cos\theta\tan\theta \tag{3.36b}$$

Jacobi's method consists in applying the above transformation to all the off-diagonal terms until all of them are, to a small error, equal to zero. Normally one starts with the off-diagonal term with the largest absolute value. Assuming that it occupies the location (I,J) the Equation (3.35d) becomes

$$\tan\theta = \frac{-(a_{ii}-a_{jj}) + \sqrt{(a_{ii}-a_{jj})^2 + 4a_{ij}^2}}{2a_{ij}} \tag{3.37}$$

from which one evaluates $\cos\theta$ and $\sin\theta$ using Equations (3.36a) and (3.36b). Next, one builds $[R_1]$ taking a unit matrix and placing $\cos\theta$ in location (I,I) and (J,J), $\sin\theta$ in location (I,J) and $-\sin\theta$ in location (J,I). Performing the orthogonal transformation is equivalent to modifying the ith and jth rows and columns of $[A]$ according to the following scheme:

Row i

$$a_{ii} = \cos^2\theta\, a_{ii} + \sin^2\theta\, a_{jj} - 2\sin\theta\cos\theta\, a_{ij}$$

$$a_{ij} = -\cos\theta\sin\theta(a_{ii}-a_{jj}) + a_{ij}(\cos^2\theta - \sin^2\theta)$$

$$a_{ik} = \cos\theta\, a_{ik} + \sin\theta\, a_{jk} \tag{3.38a}$$

Row j

$$a_{jj} = \sin^2\theta\, a_{ii} + \cos^2\theta\, a_{jj} - 2\cos\theta\sin\theta\, a_{ij}$$

$$a_{ji} = -\cos\theta\sin\theta\,(a_{ii}-a_{jj}) + a_{ji}(\cos^2\theta - \sin^2\theta)$$

$$a_{jk} = -\sin\theta\, a_{ik} + \cos\theta\, a_{jk} \tag{3.38b}$$

$$k = 1, n \quad \text{but} \quad k \neq i \quad k \neq j$$

column i

$$a_{ki} = \cos\theta\, a_{ki} + \sin\theta\, a_{kj} \tag{3.38c}$$
$$k = 1, n \quad \text{WITH} \quad k \neq i,\ k \neq j$$

column j

$$a_{kj} = -\sin\theta\, a_{ki} + \cos\theta\, a_{kj} \tag{3.38d}$$
$$k = 1, n \quad \text{WITH} \quad k \neq i,\ k \neq j$$

Notice that an orthogonal transformation preserves symmetry which allows us to reduce the number of operation required by Equations(3.38a) to (3.38d). One again selects the largest absolute value off-diagonal term, from those that remain different than zero and repeats the transformations outlined above. These transformations are repeatedly applied until no other than zero off-diagonal terms remain. After all transformations have been applied, we obtain

$$R_n^T \text{------} R_3^T R_2^T R_1^T A R_1 R_2 R_3 \text{-----} R_n = Q^T A Q = \lambda \tag{3.39}$$

where the eigenvector matrix $[Q]$ is given by

$$Q = R_1 R_2 R_3 \text{------} R_n \tag{3.40}$$

3.4-3 Computer Programs for Natural Frequencies and Mode Shapes

3.4-3.1 Jacobi Method

A computer program which performs the Jacobi Method on a square symmetric matrix is given in Appendix I. The alogrithm follows the procedure developed in Section 3.4-2.

The accompaning solution of the eigenvalues and eigenvectors is given for a sample problem defined as

$$[A] = \begin{bmatrix} 12 & 6 & -6 & -12 \\ 6 & 4 & -2 & -6 \\ -6 & -2 & 4 & 6 \\ -12 & -6 & 6 & 12 \end{bmatrix}$$

where $[A]$ is symmetric.

The resulting elgenvalue matrix becomes

$$[\lambda] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 30 \end{bmatrix}$$

and the eigenvector matrix is

$$[Q] = \begin{bmatrix} \sqrt{2}/2 & \sqrt{10}/10 & 0 & \sqrt{10}/5 \\ 0 & -\sqrt{10}/5 & \sqrt{2}/2 & \sqrt{10}/10 \\ 0 & \sqrt{10}/5 & \sqrt{2}/2 & -\sqrt{10}/10 \\ \sqrt{2}/2 & \sqrt{10}/10 & 0 & -\sqrt{10}/5 \end{bmatrix}$$

It should be noted that matrix $[Q]$ is orthogonal which satisfies Equations(3.30a), (3.30b) and (3.30c).

3.4-3.2 Generalized Eigenvalue, Eigenvector Problem

A general computerprogram which solves the general form of Equation(3.9) is given in Appendix II. The algorithm utilizes the Cholesky Transformation technique (see Equation (3.10)), and includes the Jacobi Method process as a subroutine[3]. The accompaning solution is given for a sample problem shown in Figure(3.2), where

$$[A] \equiv [K] = k\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$[B] \equiv [M] = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The eigenvalue matrix becomes,

$$|\Lambda_\omega| = \sqrt{\frac{k}{m}}\begin{bmatrix} 0.3820 & 0 \\ 0 & 2.618 \end{bmatrix}$$

The eigenvector matrix is computed as

$$[U] = \begin{bmatrix} 0.526 & 0.850 \\ 0.850 & -0.526 \end{bmatrix}$$

It should be noted for this special problem that $[K][M] = [M][K]$ and, thus, $[U][U]^T = [I]$ or $[U]$ is an orthogonal matrix.

3.5 Modal Analysis of Multidegree System

3.5-1 Matrix Method Formulation

Before one uses matrix notation to calculate the expressions of forced motion of a multidegree system, there are important relationships among mass, stiffness and natural frequency matrices which must be known. free vibration of multidegree systems , it follows from Newton's Law that

$$\left[M \right] \left\{ \ddot{V} \right\} + \left[K \right] \left\{ V \right\} = \left\{ 0 \right\} \tag{3.41}$$

Assuming resulting motions are harmonic, one obtains

$$\left[K \right] \left[U \right] = \left[M \right] \left[U \right] \left[\Lambda_\omega \right]^2 \tag{3.42}$$

where $\left[U \right]$ is the matrix of eigenvectors and $\left[\Lambda_\omega \right]$ is a diagonal matrix of natural frequenices in the form

$$\left[\Lambda_\omega \right] = \begin{bmatrix} \omega_1 & & & & 0 \\ & \omega_2 & & & \\ & & \omega_3 & & \\ & & & \ddots & \\ 0 & & & & \omega_n \end{bmatrix}$$

Letting

$$\left[U\right]^T\left[M\right]\left[U\right]=\left[\Lambda_M\right]$$
$$\left[U\right]^T\left[K\right]\left[U\right]=\left[\Lambda_k\right] \tag{3.43}$$

both $\left[\Lambda_M\right]$ and $\left[\Lambda_K\right]$ are also diagonal matrics.
Premultiplying $\left[U\right]^T$ on both side of equation(3.42) yields

$$\left[U\right]^T\left[K\right]\left[U\right]=\left[U\right]^T\left[M\right]\left[U\right]\left[\Lambda_\omega\right]^2$$

which reduces to $\left[\Lambda_k\right]=\left[\Lambda_M\right]\left[\Lambda_\omega\right]^2$ (3.44)

The latter equation has a direct analogy with the single-degree of freedom system where $k = m\omega^2$.

The matrix equation of motion for forced vibration takes the form

$$\left[M\right]\left\{\ddot{V}\right\}+\left[K\right]\left\{V\right\}=\left\{f(t)\right\} \tag{3.45}$$

Letting $\left\{V\right\}=\left[U\right]\left\{Y\right\}$; $\left\{\ddot{V}\right\}=\left[U\right]\left\{\ddot{Y}\right\}$

Equation(3.45) becomes

$$\left[M\right]\left[U\right]\left\{\ddot{Y}\right\}+\left[K\right]\left[U\right]\left\{Y\right\}=\left\{f(t)\right\} \tag{3.46}$$

Premultiplying $\left[U \right]^T$ on both side of equation(3.46) yields

$$\left[U \right]^T \left[M \right] \left[U \right] \left\{ \ddot{y} \right\} + \left[U \right]^T \left[K \right] \left[U \right] \left\{ y \right\} = \left[U \right]^T \left\{ f(t) \right\}$$

or

$$\left[\Lambda_M \right] \left\{ \ddot{y} \right\} + \left[\Lambda_K \right] \left\{ y \right\} = \left[U \right]^T \left\{ f(t) \right\} \qquad (3.47)$$

Premultiplying $\left[\Lambda_M \right]^{-1}$ on both side of Equation(3.47) gives

$$\left[\Lambda_M \right]^{-1} \left[\Lambda_M \right] \left\{ \ddot{y} \right\} + \left[\Lambda_M \right]^{-1} \left[\Lambda_K \right] \left\{ y \right\} = \left[\Lambda_M \right]^{-1} \left[U \right]^T \left\{ f(t) \right\} \qquad (3.48)$$

Recalling $\qquad \left[\Lambda_K \right] = \left[\Lambda_M \right] \left[\Lambda_\omega \right]^2$

Equation(3.48) reduces to the form

$$\left[I \right] \left\{ \ddot{y} \right\} + \left[\Lambda_\omega \right]^2 \left\{ y \right\} = \left[\Lambda_M \right]^{-1} \left[U \right]^T \left\{ f(t) \right\} \qquad (3.49)$$

Setting $\qquad \left[\Lambda_M \right]^{-1} \left[U \right]^T \left\{ f(t) \right\} = \left\{ g(t) \right\}$

Equation(3.49) becomes

$$\left[I \right] \left\{ \ddot{y} \right\} + \left[\Lambda_\omega \right]^2 \left\{ y \right\} = \left\{ g(t) \right\} \qquad (3.50)$$

or in matrix component form

$$\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \begin{Bmatrix} \ddot{Y}_1 \\ \ddot{Y}_2 \\ \vdots \\ \ddot{Y}_n \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_n^2 \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{Bmatrix} = \begin{Bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{Bmatrix} \tag{3.51}$$

The arbitrary differential equation for the " i^{th} " displacement becomes

$$\ddot{Y}_i + \omega_i^2 Y_i = g_i(t) \qquad i = 1.2.3 - - - - - n \tag{3.52}$$

The solution of Equation(3.52) is determined in Duhamel Integral form as

$$Y_i(t) = a_i \cos \omega_i t + b_i \sin \omega_i t + \frac{1}{\omega_i}\int_{\tau=0}^{\tau=t} g_i(\tau) \sin \omega_i (t-\tau)\, d\tau \tag{3.53}$$

in which a_i and b_i are constants determined from initial displacement y_0 and initial velocity $\dot{y}_0$ as

$$a_i = Y_i^{\circ} \qquad\qquad i = 1.2.3 - - - n$$

$$b_i = \frac{\dot{Y}_i^{\circ}}{\omega_i}$$

Equation(3.53) takes the final matrix form

$$\{y(t)\} = [C]\{a\} + [S]\{b\} + [\Lambda_\omega]^{-1}\int_{\tau=0}^{\tau=t}[\hat{S}][\Lambda_M]^{-1}[U]^T\{f(t)\}\,d\tau$$
$$= [C]\{a\} + [S]\{b\} + [\Lambda_\omega]^{-1}[\Lambda_M]^{-1}\int_{\tau=0}^{\tau=t}[\hat{S}][U]^T\{f(t)\}\,d\tau \qquad (3.54)$$

Recalling $\{V(t)\} = [U]\{y(t)\}$, the final form of the response function becomes

$$\{V(t)\} = [U][C]\{a\} + [U][S]\{b\} + [U][\Lambda_\omega]^{-1}[\Lambda_m]^{-1}\int_{\tau=0}^{\tau=t}[\hat{S}][U]^T\{P(t)\}\,d\tau \qquad (3.55)$$

where

$$[C] = \begin{bmatrix} \cos\omega_1 t & & & 0 \\ & \cos\omega_2 t & & \\ & & \ddots & \\ 0 & & & \cos\omega_n t \end{bmatrix}$$

$$[S] = \begin{bmatrix} \sin\omega_1 t & & & 0 \\ & \sin\omega_2 t & & \\ & & \ddots & \\ 0 & & & \sin\omega_n t \end{bmatrix}$$

$$[\hat{S}] = \begin{bmatrix} \sin\omega_1(t-\tau) & & & 0 \\ & \sin\omega_2(t-\tau) & & \\ & & \ddots & \\ 0 & & & \sin\omega_n(t-\tau) \end{bmatrix}$$

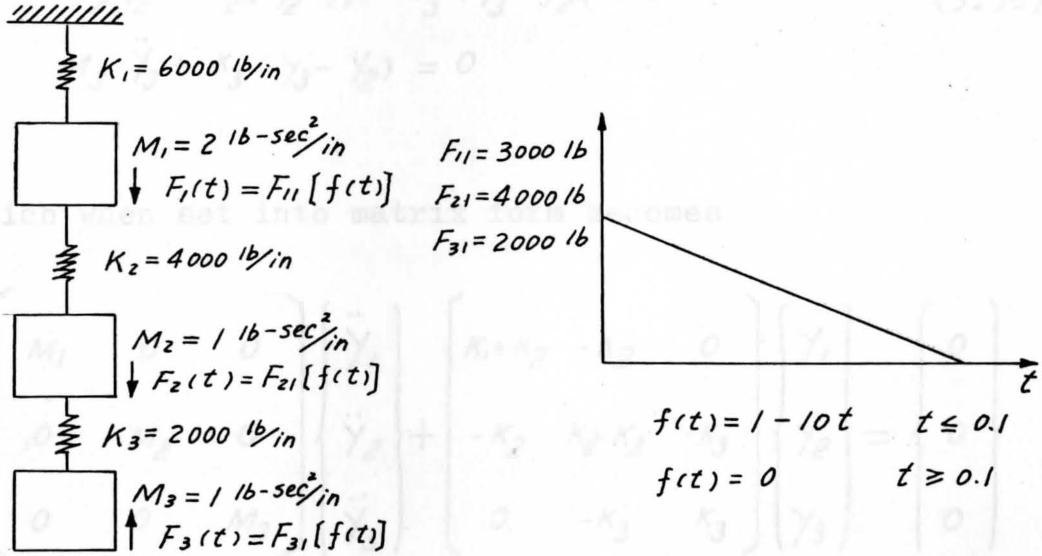
and

$$\{a\} = \{y_1^o \; y_2^o \text{-------} y_n^o\}^T$$

$$\{b\} = \{\dot{y}_1^o/\omega_1 \; \dot{y}_2^o/\omega_2 \text{-------} \dot{y}_n^o/\omega_n\}^T$$

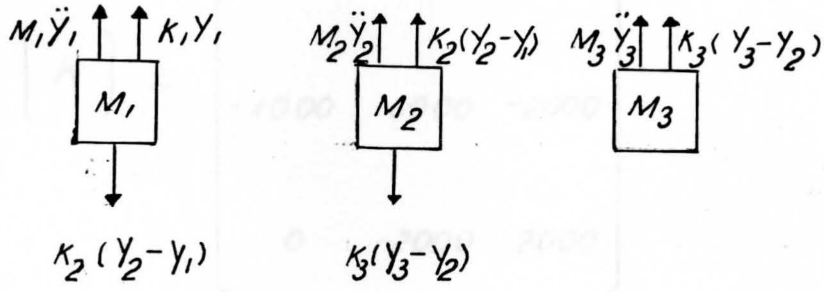
3.5-2 Sample Example

Using the matrix response Equation(3.55) for the multidegree system, a three-degree system shown in Figure(3.5) is investigated to determine maximum response amplitudes.



Figure(3.5) Three-Degree of Freedom System — Forced Vibration

First for free vibration assume all the external forces are equal to zero. Using the diagrams of dynamic equilibrium shown in Figure(3.6) we obtain the equations of free motion as



Fig(3.6) Dynamic Equilbrium — Three-Degree of Freedom

$$
\begin{aligned}
&M_1\ddot{Y}_1 + K_1 Y_1 - K_2(Y_2-Y_1) = 0\\
&M_2\ddot{Y}_2 + K_2(Y_2-Y_1) - K_3(Y_3-Y_2) = 0\\
&M_3\ddot{Y}_3 + K_3(Y_3-Y_2) = 0
\end{aligned}
\tag{3.56}
$$

which when set into matrix form becomes

$$
\begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix}
\begin{Bmatrix} \ddot{Y}_1 \\ \ddot{Y}_2 \\ \ddot{Y}_3 \end{Bmatrix}
+
\begin{bmatrix} K_1+K_2 & -K_2 & 0 \\ -K_2 & K_2+K_3 & -K_3 \\ 0 & -K_3 & K_3 \end{bmatrix}
\begin{Bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}
\tag{3.57}
$$

Taking $m_1 = m_2 = m_3 = 2$ and $k_1 = 6000$; $k_2 = 2k_3 = 4000$, it follows that,

$$
[M] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$\left[K \right] = \begin{bmatrix} 10000 & -4000 & 0 \\ -4000 & 6000 & -2000 \\ 0 & -2000 & 2000 \end{bmatrix}$$

Utilizing the general computer program in Appendix II, one obtains

$$\left[\Lambda_{\omega} \right] = \begin{bmatrix} 790.95 & 0 & 0 \\ 0 & 3473.56 & 0 \\ 0 & 0 & 8735.49 \end{bmatrix}$$

and after normalization the eigenvector matrix becomes

$$\left[U \right] = \begin{bmatrix} 0.2387 & 0.6316 & 0.4566 \\ 0.5024 & 0.4684 & -0.8528 \\ 0.8310 & -0.6357 & 0.2533 \end{bmatrix}$$

Recalling Equation(3.55), with initial conditions as zero

$$\left\{ V(t) \right\} = \left[U \right] \left[\Lambda_{\omega} \right]^{-1} \left[\Lambda_{m} \right]^{-1} \int_{\tau=0}^{\tau=t} \left[S \right] \left[U \right]^{T} \left\{ f(t) \right\} d\tau$$

The forcing function vector $\{f(t)\}$ is assumed in the form shown in Figure(3.5). Noting for the first time interval

$$\{f(t)\} = (1-10t)\begin{Bmatrix} F_{11} \\ F_{21} \\ F_{31} \end{Bmatrix}$$

$$\{V(t)\} = [U][\Lambda_\omega]^{-1}[\Lambda_m]\int_{\tau=0}^{\tau=t}(1-10\tau)[\grave{}\hat{S}\lrcorner][U]^T\{F_o\}d\tau$$

$$= [U][\Lambda_\omega]^{-1}[\Lambda_m]^{-1}\int_{\tau=0}^{\tau=t}(1-10\tau)[\grave{}\hat{S}\lrcorner]d\tau\,[U]^T\{F_o\}$$

$$= [U][\Lambda_\omega]^{-1}[\Lambda_m]^{-1}[\Lambda_\omega]^{-1}\left[(1-\cos\omega t + 10\frac{\sin\omega t}{\omega} - 10t)\right][U]^T\{F_o\} \qquad (3.58)$$

Recalling from Chapter II, the dynamic load factor for the single-degree system with forcing function $f(t) = 1 - \frac{t}{t_d}$ is

$$DLF = 1 - \cos\omega_i t + \frac{\sin\omega_i t}{\omega t_d} - \frac{t}{t_d}$$

$$\{V(t)\} = [U][\Lambda_\omega]^{-1}[\Lambda_m]^{-1}[\Lambda_\omega]^{-1}[DLF][U]^T\{F_o\}$$

$$= [U][\Lambda_k][DLF][U]^T\{F_o\} \qquad (3.59)$$

where matrix $\lceil DLF \rfloor$ is a diagonal matrix.

It is very important to note that each normal mode may be treated as an independent single-degree system. And we can find the dynamic load factor from Biggs.

In Equation(3.59)

$$[\Lambda_K]^{-1} = \left[[U]^T[K][U]\right]^{-1}$$

$$[\Lambda_K]^{-1} = \left[\begin{bmatrix} 0.2387 & 0.5024 & 0.8310 \\ 0.6136 & 0.4684 & -0.6357 \\ 0.4566 & -0.8528 & 0.2523 \end{bmatrix}\begin{bmatrix} 10000 & -4000 & 0 \\ -4000 & 6000 & -2000 \\ 0 & -2000 & 2000 \end{bmatrix}\begin{bmatrix} 0.2387 & 0.6136 & 0.4566 \\ 0.5024 & 0.4684 & -0.8528 \\ 0.8310 & -0.6357 & 0.2533 \end{bmatrix}\right]^{-1}$$

$$= \begin{bmatrix} 835.97 & 0 & 0 \\ 0 & 4782 & 0 \\ 0 & 0 & 10556 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \frac{1}{835.97} & 0 & 0 \\ 0 & \frac{1}{4782} & 0 \\ 0 & 0 & \frac{1}{10556} \end{bmatrix}$$

$$V(t) = [U][\Lambda_K]^{-1}[DLF][U]^T\{F_o\}$$

$$= \begin{bmatrix} 0.2387 & 0.6136 & 0.4566 \\ 0.5024 & 0.4684 & -0.8528 \\ 0.8310 & -0.6357 & 0.2533 \end{bmatrix}\begin{bmatrix} \frac{1}{835.97} & 0 & 0 \\ 0 & \frac{1}{4782} & 0 \\ 0 & 0 & \frac{1}{10556} \end{bmatrix}\begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{bmatrix}\begin{bmatrix} 0.2387 & 0.5024 & 0.8310 \\ 0.6136 & 0.4684 & -0.6357 \\ 0.4566 & -0.8528 & 0.2533 \end{bmatrix}\begin{Bmatrix} 3000 \\ 4000 \\ -2000 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.3036\,D_1 + 0.6398\,D_2 - 0.1102\,D_3 \\ 0.6390\,D_1 + 0.4881\,D_2 + 0.2059\,D_3 \\ 1.0574\,D_1 - 0.6627\,D_2 - 0.0611\,D_3 \end{Bmatrix} \qquad (3.60)$$

The determination of maximum deflection at any point of the system would involve differentiation of Equation(3.60) with respect to time in order to find the time of maximum response. This is obviously a very difficult process. In many cases the practical solution is to proceed graphically and from a plot approximately deduce the time of maximum responses. An upper limit for the maximum response may be obtained by adding numerically the maximum of the modes taken seperately. For example, the upper bound of V_2 can be obtained as follows

From Biggs, $(DLF_1)_{max} = 1.12$

$(DLF_2)_{max} = 1.53$

$(DLF_3)_{max} = 1.68$

so the upper bound of V_2 is

$$V_{2,MAX} < 0.639(1.12) + 0.4881(1.53) + 0.2059(1.68) = 1.8084 \text{ in}$$

For the example, the value just computed is a rather conservative estimate of the maximum displacement. In fact, the true value of the maximum displacement at point 2, as one will see in next chapter, occuring at about 0.044 sec, is 1.3096 in.

CHAPTER IV

FINITE DEFFERENCE ANALYSIS OF MULTIDEGREE SYSTEM

4.1 Introduction

For many dynamic problems of considerable practical interest, analytic solutions to the governing differential equations are very complicated as shown in Chapter III. Fortunately, numerical treatment of these kinds of differential equations can be yield approximate results, acceptable for most practical purposes. Among the numerical techniques presently available, the Finite Difference method is one of the most general. The Finite Difference procedure produces a direct solution to the differential equations for a given set of forcing functions. A time step incrementation process is implemented yielding a direct calculation of the system response parameters. The method involves, at most, simple algebraic processes eliminating the need of integration procedures or the use of the concept of Dynamic Load Factor. The determination of maximum response, the time of maximum response, and the resulting maximum structure forces are simple, direct outputs of the mathematical processes.

In this chapter a FORTRAN program of the Finite Difference method originally formulated by Phimphilia[(4)] is used to determine the response of a multidegree of freedom structural modles. The program is augmented to include as output the actual internal structural forces so that the maximum values of structure forces are determined.

4.2 Mathematical Developement Of The Finite Difference Method

Recalling from Chapter 3, the governing differential equation of a multidegree of freedom structure for forced vibration is

$$\left[M\right]\left\{\ddot{V}\right\}+\left[K\right]\left\{V\right\}=\left\{f(t)\right\} \tag{4.1}$$

where $[M]$ is mass matrix, $[K]$ is stiffness matrix, $\{V\}$ and $\{\ddot{V}\}$ are displacement and acceleration vector at time t, and $\{f(t)\}$ is the external force applied on the system which is vary with time. A vector iteration method is utilized to determine the response vector $\{V\}$ of the dynamic system. Since of mass matrix is a positive definite matrix, its determinant is nonzero, and the inverse of the mass matrix exists. Rearranging Equation(4.1) as follows

$$\left\{\ddot{V}\right\}=\left[M\right]^{-1}\left\{f(t)\right\}-\left[M\right]^{-1}\left[K\right]\left\{V\right\} \tag{4.2}$$

recalling the Taylor Series expansion of a function with one variable, it follows that

$$\begin{aligned} Y_{m+1} &= f(X_m+\Delta X) \\ &= Y(X)_m+\frac{\Delta X}{1!}Y'(X)_m+\frac{(\Delta X)^2}{2!}Y''(X)_m+\frac{(\Delta X)^3}{3!}Y'''(X)_m+\text{------} \end{aligned} \tag{4.3}$$

By using direct analogy to the Taylor expansion for the time varying vector $\{V(t)\}$, by analogy to Equation(4.2), the displacement vector is expanded as follows,

$$\{V(t)\}_{i+1} = \{V(t)\}_i + \frac{\Delta t}{1!}\{\dot{V}(t)\}_i + \frac{(\Delta t)^2}{2!}\{\ddot{V}(t)\}_i + \frac{(\Delta t)^3}{3!}\{\dddot{V}(t)\}_i + \text{------} \tag{4.4}$$

where Δt is time difference between any two continous stations and

$$\Delta t = t_{n+1} - t_n$$

Differentiating Equation(4.4) with respect to time t one obtains

$$\{\dot{V}(t)\}_{i+1} = \{\dot{V}(t)\}_i + \frac{\Delta t}{1!}\{\ddot{V}(t)\}_i + \frac{(\Delta t)^2}{2!}\{\dddot{V}(t)\}_i + \frac{(\Delta t)^3}{3!}\{\overset{iv}{V}(t)\}_i + \text{------} \tag{4.5}$$

$$\{\ddot{V}(t)\}_{i+1} = \{\ddot{V}(t)\}_i + \frac{\Delta t}{1!}\{\dddot{V}(t)\}_i + \frac{(\Delta t)^2}{2!}\{\overset{iv}{V}(t)\}_i + \frac{(\Delta t)^3}{3!}\{\overset{v}{V}(t)\}_i + \text{-----} \tag{4.6}$$

The number of terms in the expansion may be arbitrarily chosen. However, the accuracy of this method is dependent on the terms in the expansion one chooses; the more terms that are chosen the more accurate the results. The simplest solution may be found by neglecting terms on the right hand side of the expansion which contain derivatives higher than the second order. After eliminating these higher order dervatives from Equations(4.4)(4.5)(4.6), we obtain

$$\{V(t)\}_{i+1} = \{V(t)\}_i + \frac{\Delta t}{1!}\{\dot{V}(t)\}_i + \frac{(\Delta t)^2}{2!}\{\ddot{V}(t)\}_i \tag{4.7a}$$

$$\left\{\dot{V}(t)\right\}_{i+1} = \left\{\dot{V}(t)\right\}_{i} + \Delta t \left\{\ddot{V}(t)\right\}_{i} \tag{4.7b}$$

$$\left\{\ddot{V}(t)\right\}_{i+1} = \left\{\ddot{V}(t)\right\}_{i} \tag{4.7c}$$

For a given value of $\{V(t)\}_i$; $\{\dot{V}(t)\}_i$; $\{\ddot{V}(t)\}_i$, one obtains $\{V(t)\}_{i+1}$; $\{\dot{V}(t)\}_{i+1}$; $\{\ddot{V}(t)\}_{i+1}$ directly from Equations(4.7a)(4.7b)(4.7c) respectively. Note from Equation(4.7c) that the acceleration at the end of the time interval is exactly the same as the acceleration at the beginning of the time interval, (i.e., no change of acceleration through whole time period). One defines this procedure as the "Constant Acceleration Method" of iteration. Now permutting the value of n to (n-1) and (n-1) to n in Equations(4.7a)(4.7b)(4.7c), one obtains

$$\left\{V(t)\right\}_{i} = \left\{V(t)\right\}_{i-1} + \frac{\Delta t}{1!}\left\{\dot{V}(t)\right\}_{i-1} - \frac{(\Delta t)^2}{2!}\left\{\ddot{V}(t)\right\}_{i-1} \tag{4.8a}$$

$$\left\{\dot{V}(t)\right\}_{i} = \left\{V(t)\right\}_{i-1} + \frac{\Delta t}{1!}\left\{\dot{V}(t)\right\}_{i-1} \tag{4.8b}$$

$$\left\{\ddot{V}(t)\right\}_{i} = \left\{\ddot{V}(t)\right\}_{i-1} \tag{4.8c}$$

Rearranging the terms in Equation(4.8b)

$$\left\{\ddot{V}(t)\right\}_{i-1} = \frac{1}{\Delta t}\left\{\dot{V}(t)\right\}_{i} - \frac{1}{\Delta t}\left\{\dot{V}(t)\right\}_{i-1} \tag{4.9}$$

Substituting the latter equation into Equation(4.8a) gives

$$\left\{V(t)\right\}_i = \left\{V(t)\right\}_{i-1} + \frac{\Delta t}{2!}\left\{\dot{V}(t)\right\}_{i-1} + \frac{\Delta t}{2!}\left\{\dot{V}(t)\right\}_i$$

or
$$\left\{V(t)\right\}_i - \left\{V(t)\right\}_{i-1} - \frac{\Delta t}{2!}\left\{\dot{V}(t)\right\}_{i-1} + \frac{\Delta t}{2!}\left\{\dot{V}(t)\right\}_i = 0 \tag{4.10}$$

Now, permutting n to (n+1) in Equation(4.10), one obtains

$$\left\{V(t)\right\}_{i+1} - \left\{V(t)\right\}_i - \frac{\Delta t}{2!}\left[\left\{\dot{V}(t)\right\}_i + \left\{\dot{V}(t)\right\}_{i+1}\right] = 0 \tag{4.11}$$

Subtracting Equation(4.10) from Equation(4.11) yields

$$\left\{V(t)\right\}_{i+1} - 2\left\{V(t)\right\}_i + \left\{V(t)\right\}_{i-1} = \frac{\Delta t}{2}\left[\left\{\dot{V}(t)\right\}_n - \left\{\dot{V}(t)\right\}_{n-1}\right] \tag{4.12}$$

Combining Equation(4.12) with Equation(4.9), one obtains

$$\left\{\ddot{V}(t)\right\}_{i-1} = \frac{1}{(\Delta t)^2}\left[\left\{V(t)\right\}_{i+1} - 2\left\{V(t)\right\}_i + \left\{V(t)\right\}_{i-1}\right] \tag{4.13}$$

Then, permutting the form of Equation(4.7c) yields

$$\left\{\ddot{V}(t)\right\}_i = \frac{1}{(\Delta t)^2}\left[\left\{V(t)\right\}_{i+1} - 2\left\{V(t)\right\}_i + \left\{V(t)\right\}_{i-1}\right] \tag{4.14}$$

The convenient form of Equation(4.13) for calculation purposes is written as

$$\left\{V(t)\right\}_{i+1} = 2\left\{V(t)\right\}_i - \left\{V(t)\right\}_{i-1} + (\Delta t)^2\left\{\ddot{V}(t)\right\} \tag{4.15}$$

Finally, at any time t, it follows from Equation(4.2) that

$$\left\{\ddot{V}(t)\right\} = \left[M\right]^{-1}\left\{f(t)\right\} - \left[M\right]^{-1}\left[K\right]\left\{V(t)\right\} \tag{4.2}$$

From the above equations, the following steps are taken in the iteration procedure :

<u>step(1)</u> at t=0, n=0

$$\left\{V(0)\right\} = 0 \qquad \left\{\ddot{V}(0)\right\} = \left[M\right]^{-1}\left\{f(0)\right\}$$

and by choice

$$\left\{V(1)\right\} = \frac{(\Delta t)^2}{2}\left\{f(0)\right\}$$

<u>step(2)</u> at $t=t_1=\Delta t$, n=1

$$\left\{\ddot{V}(\Delta t)\right\} = -\left[M\right]^{-1}\left[K\right]\left\{V(\Delta t)\right\} + \left[M\right]^{-1}\left\{f(\Delta t)\right\}$$

and

$$\left\{V(2\Delta t)\right\} = 2\left\{V(\Delta t)\right\} - \left\{V(0)\right\} + (\Delta t)^2\left\{\ddot{V}(\Delta t)\right\}$$

<u>step(3)</u> at $t=t_2=2\Delta t$, n=2

$$\left\{\ddot{V}(2\Delta t)\right\} = -\left[M\right]^{-1}\left[K\right]\left\{V(2\Delta t)\right\} + \left[M\right]^{-1}\left\{f(2\Delta t)\right\}$$

and

$$\left\{V(3\Delta t)\right\} = 2\left\{V(2\Delta t)\right\} - \left\{V(\Delta t)\right\} + (\Delta t)^2\left\{\ddot{V}(2\Delta t)\right\}$$

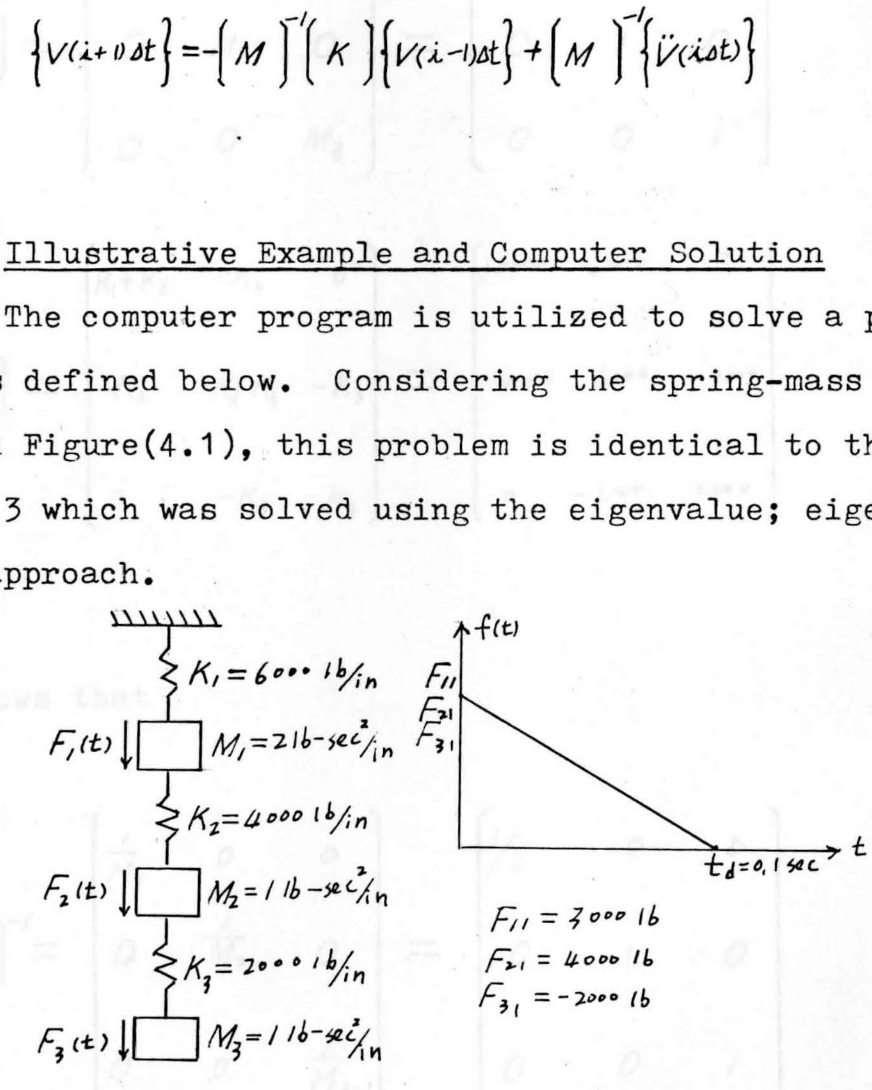
Additional steps similar to the latter steps are taken up to the value of i, and in general

$$\left\{\ddot{V}(i\Delta t)\right\} = -\left[M\right]^{-1}\left[K\right]\left\{V(i\Delta t)\right\} + \left[M\right]^{-1}\left\{f(i\Delta t)\right\}$$

$$\left\{V((i+1)\Delta t)\right\} = -\left[M\right]^{-1}\left[K\right]\left\{V((i-1)\Delta t)\right\} + \left[M\right]^{-1}\left\{\ddot{V}(i\Delta t)\right\}$$

4.3 Illustrative Example and Computer Solution

The computer program is utilized to solve a problem which is defined below. Considering the spring-mass system shown in Figure(4.1), this problem is identical to that of Chapter 3 which was solved using the eigenvalue; eigenvector; matrix approach.



Fig(4.1) Sample Problem Identification

The governing differential equation of response of the system is

$$[M]\{\ddot{V}\}+[K]\{V\}=\{f(t)\}$$

From the dynamic equilibrium conditions one obtains

$$[M]=\begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix}=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[K]=\begin{bmatrix} K_1+K_2 & -K_2 & 0 \\ -K_2 & K_2+K_3 & -K_3 \\ 0 & -K_3 & K_3 \end{bmatrix}=\begin{bmatrix} 10000 & -4000 & 0 \\ -4000 & 6000 & -2000 \\ 0 & -2000 & 2000 \end{bmatrix}$$

It follows that

$$[M]^{-1}=\begin{bmatrix} \frac{1}{M_1} & 0 & 0 \\ 0 & \frac{1}{M_2} & 0 \\ 0 & 0 & \frac{1}{M_3} \end{bmatrix}=\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and from giving data we have $t_d = 0.1$ sec and

$$\{f(t)\} = \begin{Bmatrix} -30000 \times T(m) + 3000 \\ -40000 \times T(m) + 4000 \\ 20000 \times T(m) - 2000 \end{Bmatrix}$$

A requirement of this method is a need to define as input data the time interval Δt. The general recommendation(5) is that it be at least less than one tenth of the lowest value of the structure natural period. This requires a precalculation of the natural frequencies of the structure. For the problem under consideration the lowest value of natural frequency was determined(see chapter 3) to be $\omega_1 = 28.12$ rad/sec, and lowest natural period $T_1 = 0.2234$ sec.

Assuming $\Delta t = 0.0005$ sec, n = 3 and putting $[M]^{-1}.[K]$, $\{f(t)\}$ and t_d as data into the program, the complete program solution is given in Appendix III. From the computer solution we can easily find out that the time of maximum response is 0.044sec and the maximum response is 1.3096 in.

4.4 A System with Close Natural Frequencies

In some structural systems a possibility exists that two of the natural frequencies of free vibration may be numerically close to one another. Although most well engineered and designed structures are not subject to this condition, many improperly designed structures currently exist. It is, therefore, important to be able to determine the response characteristic of these structures when subjected to typical dynamic force conditions.

Consider a four-degree of freedom dynamic system in Figure(4.2)

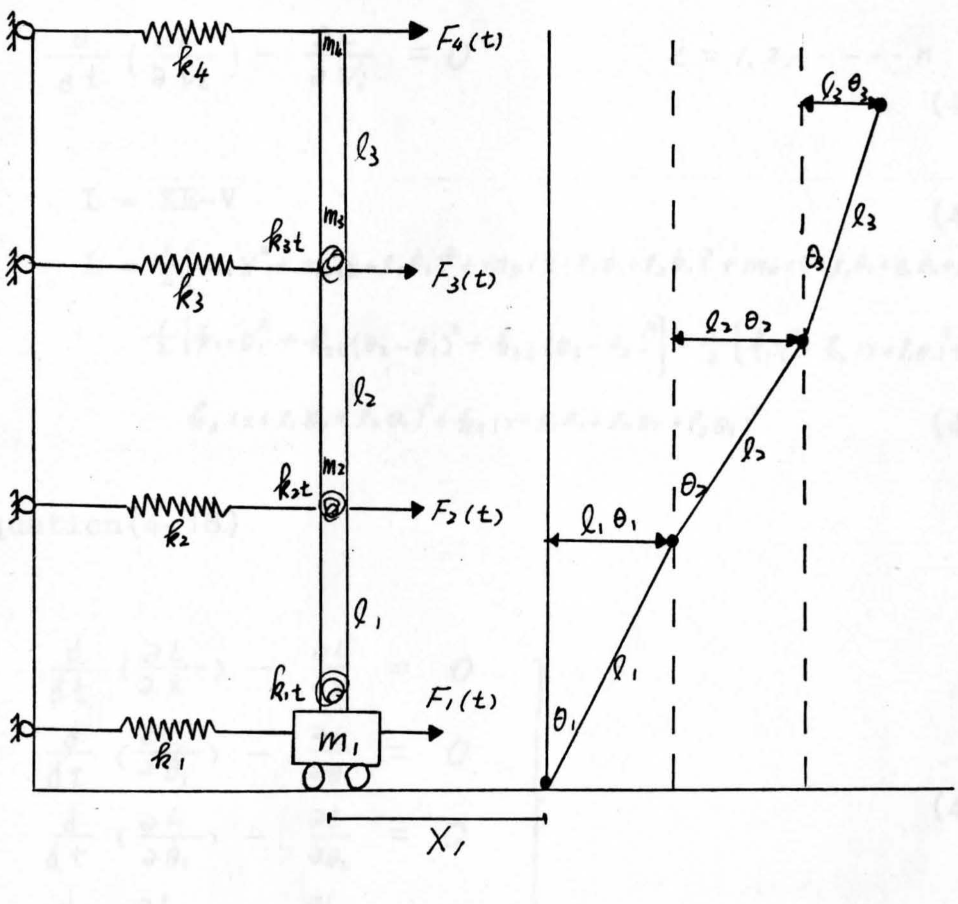


Fig.(4.2) Four-Degree of Freedom Dynamic System

The Kinetic Energy of the system is

$$KE = \frac{1}{2} m_1\dot{X}^2 + \frac{1}{2} m_2(\dot{x}+l_1\dot{\theta}_1)^2 + \frac{1}{2} m_3(\dot{X}+l_1\dot{\theta}_1+l_2\dot{\theta}_2)^2 + \frac{1}{2} m_4(\dot{X}+l_1\dot{\theta}_1+l_2\dot{\theta}_2+l_3\dot{\theta}_3)^2 \tag{4.16}$$

The Poential Energy of the system is

$$V = \frac{1}{2} k_{1t}\,\theta_1^2 + \frac{1}{2} k_{2t}(\theta_2-\theta_1)^2 + \frac{1}{2} k_{3t}(\theta_3-\theta_2)^2 + \frac{1}{2} k_1 X^2 + \frac{1}{2} k_2 (X+l_1\theta_1)^2 + \frac{1}{2} k_3 (X+l_1\theta_1+l_2\theta_2) + \frac{1}{2} k_4 (X+l_1\theta_1+l_2\theta_2+l_3\theta_3) \tag{4.17}$$

Lagrange equation of conservative elastic systems takes the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{V}_i}\right) - \frac{\partial L}{\partial V_i} = 0 \qquad i = 1, 2, - - - - - n \tag{4.18}$$

where $\quad L = KE-V \qquad (4.19)$

$$L = \frac{1}{2}\left[m_1\dot{X}^2 + m_2(\dot{x}+l_1\dot{\theta}_1)^2 + m_3(\dot{x}+l_1\dot{\theta}_1+l_2\dot{\theta}_2)^2 + m_4(\dot{x}+l_1\dot{\theta}_1+l_2\dot{\theta}_2+l_3\dot{\theta}_3)^2\right] - \frac{1}{2}\left[k_{1t}\theta_1^2 + k_{2t}(\theta_2-\theta_1)^2 + k_{3t}(\theta_3-\theta_2)^2\right] - \frac{1}{2}\left[k_1 X^2 + k_2(X+l_1\theta_1)^2 + k_3(X+l_1\theta_1+l_2\theta_2)^2 + k_4(X+l_1\theta_1+l_2\theta_2+l_3\theta_3)^2\right] \tag{4.20}$$

From Equation (4.18)

$$\left.\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) - \frac{\partial L}{\partial X} &= 0\\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} &= 0\\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} &= 0\\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_3}\right) - \frac{\partial L}{\partial \theta_3} &= 0
\end{aligned}\right\} \tag{4.21}$$

After differentiating, one obtains

$$m_1\ddot{x} + m_2(\ddot{x}+l_1\ddot{\theta}_1) + m_3(\ddot{x}+l_1\ddot{\theta}_1+l_2\ddot{\theta}_2) + m_4(\ddot{x}+l_1\ddot{\theta}_1+l_2\ddot{\theta}_2+l_3\ddot{\theta}_3) + k_1x + k_2(x+l_1\theta_1) + k_3(x+l_1\theta_1+l_2\theta_2) + k_4(x+l_1\theta_1+l_2\theta_2+l_3\theta_3) = 0$$

$$m_2l_1(\ddot{x}+l_1\ddot{\theta}_1) + m_3l_1(\ddot{x}+l_1\ddot{\theta}_1+l_2\ddot{\theta}_2) + m_4l_1(\ddot{x}+l_1\ddot{\theta}_1+l_2\ddot{\theta}_2+l_3\ddot{\theta}_3) - k_{1t}\theta_1 + k_{2t}(\theta_2-\theta_1) + k_2(x+l_1\theta_1) + k_3(x+l_1\theta_1+l_2\theta_2) + k_4(x+l_1\theta_1+l_2\theta_2+l_3\theta_3) = 0$$

$$m_3l_2(\ddot{x}+l_1\ddot{\theta}_1+l_2\ddot{\theta}_2) + m_4l_2(\ddot{x}+l_1\ddot{\theta}_1+l_2\ddot{\theta}_2+l_3\ddot{\theta}_3) - k_{2t}(\theta_2-\theta_1) + k_{3t}(\theta_3-\theta_2) + k_3(x+l_1\theta_1+l_2\theta_2) + k_4(x+l_1\theta_1+l_2\theta_2+l_3\theta_3) = 0$$

$$m_4l_3(\ddot{x}+l_1\ddot{\theta}_1+l_2\ddot{\theta}_2+l_3\ddot{\theta}_3) - k_{3t}(\theta_3-\theta_2) + k_4(x+l_1\theta_1+l_2\theta_2+l_3\theta_3) = 0 \qquad (4.22)$$

which possesess the matrix form

$$\begin{bmatrix} M \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \begin{bmatrix} K \end{bmatrix} \begin{Bmatrix} x \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (4.23)$$

where

$$\begin{bmatrix} M \end{bmatrix} = \begin{bmatrix} (m_1+m_2+m_3+m_4) & (m_2+m_3+m_4)l_1 & (m_3+m_4)l_2 & m_4l_3 \\ (m_2+m_3+m_4)l_1 & (m_2+m_3+m_4)l_1^2 & (m_3+m_4)l_1l_2 & m_4l_1l_3 \\ (m_3+m_4)l_2 & (m_3+m_4)l_1l_2 & (m_3+m_4)l_2^2 & m_4l_2l_3 \\ m_4l_3 & m_4l_1l_3 & m_4l_2l_3 & m_4l_3^2 \end{bmatrix}$$

$$\begin{bmatrix} K \end{bmatrix} = \begin{bmatrix} (k_1+k_2+k_3+k_4) & (k_2+k_3+k_4)l_1 & (k_3+k_4)l_2 & k_4l_3 \\ (k_2+k_3+k_4)l_1 & (k_2+k_3+k_4)l_1^2+k_{1t}+k_{2t} & (k_3+k_4)l_1l_2-k_{2t} & k_4l_1l_3 \\ (k_3+k_4)l_2 & (k_3+k_4)l_1l_2-k_{2t} & (k_3+k_4)l_2^2+k_{2t}+k_{3t} & k_4l_2l_3-k_{3t} \\ k_4l_3 & k_4l_1l_3 & k_4l_2l_3-k_{3t} & k_4l_3^2+k_{3t} \end{bmatrix}$$

For convenience, let

$$\begin{aligned} m_1 &= m_2 = m_3 = m_4 = m \\ k_1 &= k_2 = k_3 = k_4 = k \\ \ell_1 &= \ell_2 = \ell_3 = \ell \\ k_{1t} &= k_{2t} = k_{3t} = k_t \end{aligned}$$

Substituting these conditions into $\left[M\right]\left[K\right]$ of Equation(4.23) and defining $k_t/k\ell^2 = \hat{k}$ one obtains

$$\begin{bmatrix} 4m & 3m\ell & 2m\ell & m\ell \\ 3m\ell & 3m\ell^2 & 2m\ell^2 & m\ell^2 \\ 2m\ell & 2m\ell^2 & 2m\ell^2 & m\ell^2 \\ m\ell & m\ell^2 & m\ell^2 & m\ell^2 \end{bmatrix} \begin{Bmatrix} \ddot{x}/\ell \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \begin{bmatrix} 4k & 3k\ell & 2k\ell & k\ell \\ 3k\ell & k\ell^2(3+2\hat{k}) & k\ell^2(2-\hat{k}) & k\ell^2 \\ 2k\ell & k\ell^2(2-\hat{k}) & k\ell^2(2+2\hat{k}) & k\ell^2(1-\hat{k}) \\ k\ell & k\ell^2 & k\ell^2(1-\hat{k}) & k\ell^2(1+\hat{k}) \end{bmatrix} \begin{Bmatrix} x/\ell \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

After nondimensionalizing, one obtains

$$m\ell^2 \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}/\ell \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + k\ell^2 \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & (3+2\hat{k}) & (2-\hat{k}) & 1 \\ 2 & (2-\hat{k}) & (2+2\hat{k}) & (1-\hat{k}) \\ 1 & 1 & (1-\hat{k}) & (1+\hat{k}) \end{bmatrix} \begin{Bmatrix} x/\ell \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Using the latter equation the natural frequenices and mode shapes of the system are obtained. Setting

$$\left[A\right] \equiv \left[K\right] = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & (3+2\hat{k}) & (2-\hat{k}) & 1 \\ 2 & (2-\hat{k}) & 2(1+\hat{k}) & (1-\hat{k}) \\ 1 & 1 & (1-\hat{k}) & (1+\hat{k}) \end{bmatrix}$$

$$[B] \equiv [M] = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

the computer program of Chapter 3 is used to obtain a plot for ω^2 vs $\hat{k}$, which is shown in Figure(4.3). From the figure, it is determined that as $\hat{k}$ increases, the values of ω^2 increases. In the process as $\hat{k}$ decreases toward zero the values of ω_1^2 and ω_2^2 approach the same value, that is unity.

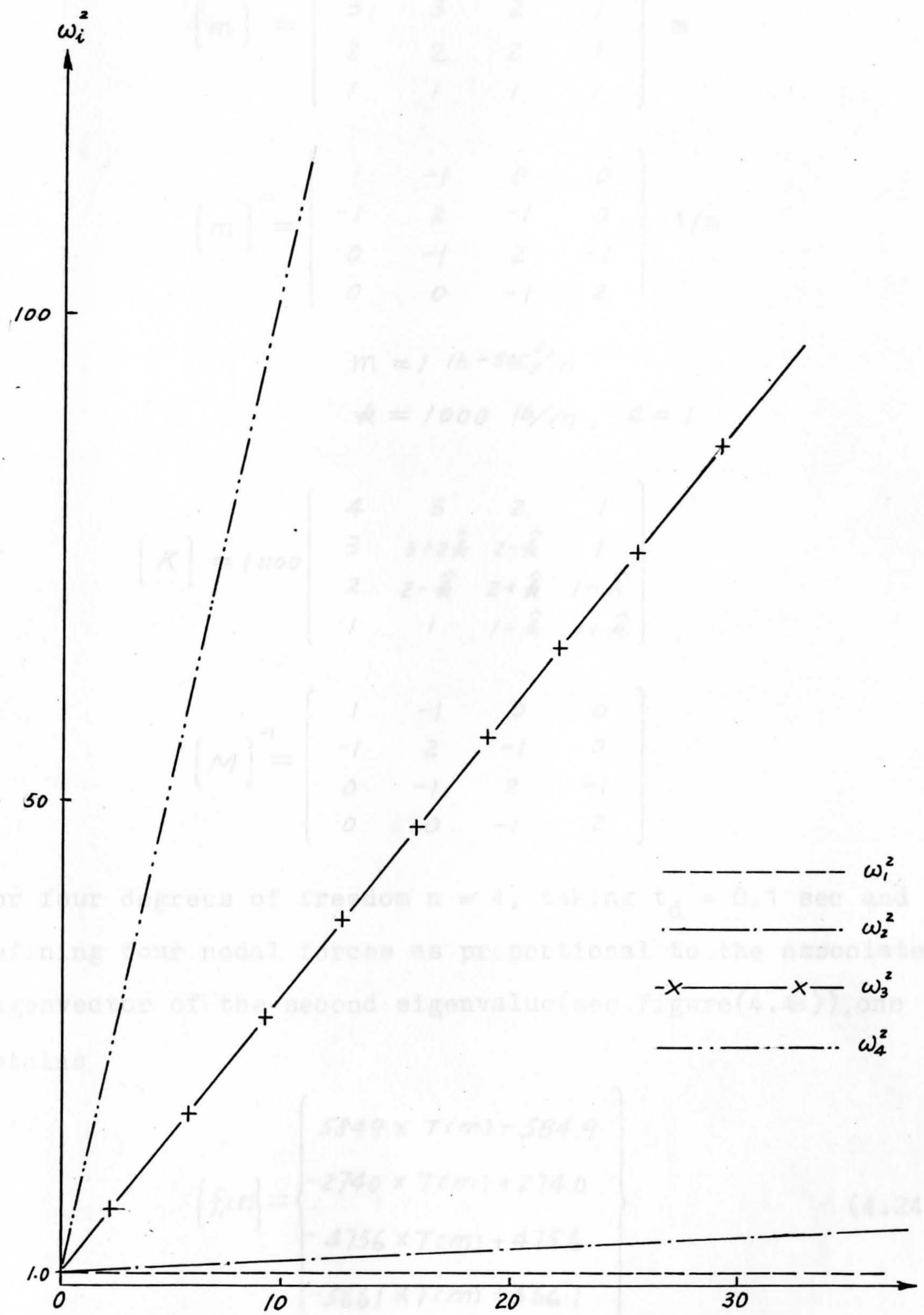
In general, the natural frequenices of this system are

$$\left(\frac{m}{k}\right) \omega_1^2(\hat{k}) = 1$$
$$\left(\frac{m}{k}\right) \omega_2^2(\hat{k}) = 1 + 0.1239\,\hat{k}$$
$$\left(\frac{m}{k}\right) \omega_3^2(\hat{k}) = 1 + 2.9561\,\hat{k}$$
$$\left(\frac{m}{k}\right) \omega_4^2(\hat{k}) = 1 + 1092\,\hat{k}$$

and mode shape function matrix of the system is

$$[U] = \begin{bmatrix} 1 & 0.5957 & 0.3694 & -0.1645 \\ 0 & -0.2791 & 0.6197 & 0.5467 \\ 0 & -0.4844 & 0.1475 & -0.7030 \\ 0 & -0.5767 & -0.6766 & 0.4241 \end{bmatrix}$$

The dynamic response and spring forces of the system are now determined , using



Fig(4.3) Frequency Stiffness Variation

$$\left[m\right] = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \mathrm{m}$$

$$\left[m\right]^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} 1/\mathrm{m}$$

$$m = 1 \ \mathrm{lb\text{-}sec^2/in}$$

$$k = 1000 \ \mathrm{lb/in}, \quad e = 1$$

$$\left[K\right] = 1000 \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3+2\hat{k} & 2-\hat{k} & 1 \\ 2 & 2-\hat{k} & 2+\hat{k} & 1-\hat{k} \\ 1 & 1 & 1-\hat{k} & 1+\hat{k} \end{bmatrix}$$

$$\left[M\right]^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

For four degrees of freedom n = 4, taking t_d = 0.1 sec and defining four nodal forces as proportional to the associated eigenvector of the second eigenvalue(see Figure(4.4a)),one obtains

$$\left\{f_1(t)\right\} = \begin{Bmatrix} 5849 \times T(m) - 584.9 \\ -2740 \times T(m) + 274.0 \\ -4756 \times T(m) + 475.6 \\ -5661 \times T(m) + 566.1 \end{Bmatrix} \qquad (4.24)$$

Substituting the numerical values of $[M]^{-1}$, $[K]$, n, t_d into the program, and obtaining the numerical results one may plot the parameter $\frac{X_{11}^{max}}{\ell}$ vs. $\hat{k}$, θ_{11}^{max} vs. $\hat{k}$, θ_{12}^{max} vs. $\hat{k}$, θ_{13}^{max} vs. $\hat{k}$. The results are shown in Figure(4.5a).

For the purposes of comparison, the forcing functions are changed such as to be proportional to the reciprocal of the second eigenvector components (see Figure(4.4b)). In addition the net force for these forcing function is chosen as to be similar to the net force of the previous set of forcing functions, that is

$$\{f_2(t)\} = \begin{Bmatrix} 2151 \times T(m) - 215.1 \\ -4592 \times T(m) + 459.2 \\ -2643 \times T(m) + 264.3 \\ -2224 \times T(m) + 222.4 \end{Bmatrix} \tag{4.25}$$

From the computer output one may plot $\frac{X_{21}^{max}}{\ell}$ vs. $\hat{k}$, θ_{21}^{max} vs. $\hat{k}$, θ_{22}^{max} vs. $\hat{k}$, θ_{23}^{max} vs. $\hat{k}$. The results are shown in Figure(4.5b).

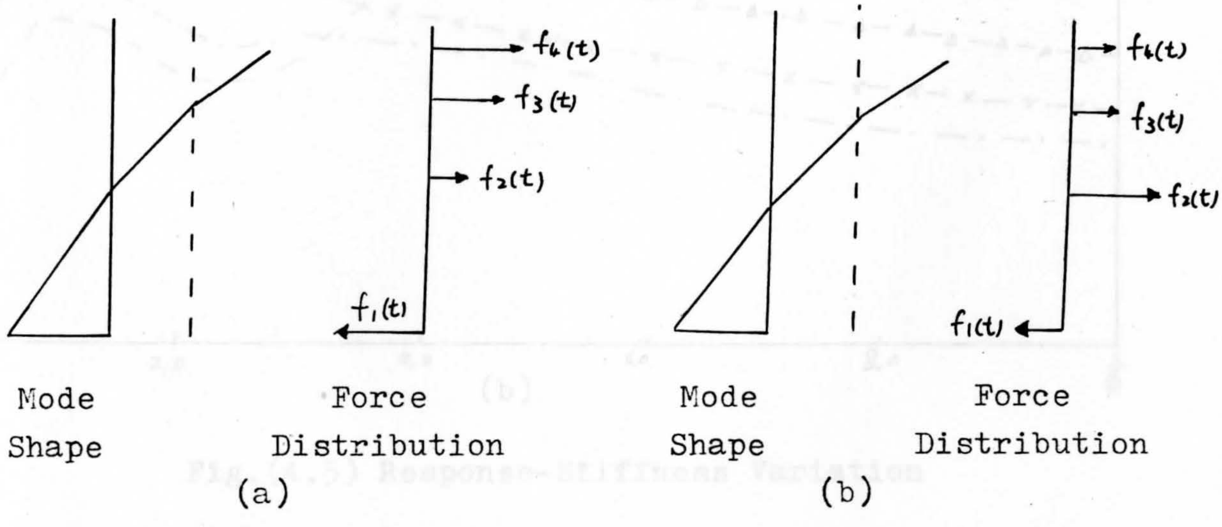


Fig.(4.4) Forcing Function Associated with Second Eigenvector

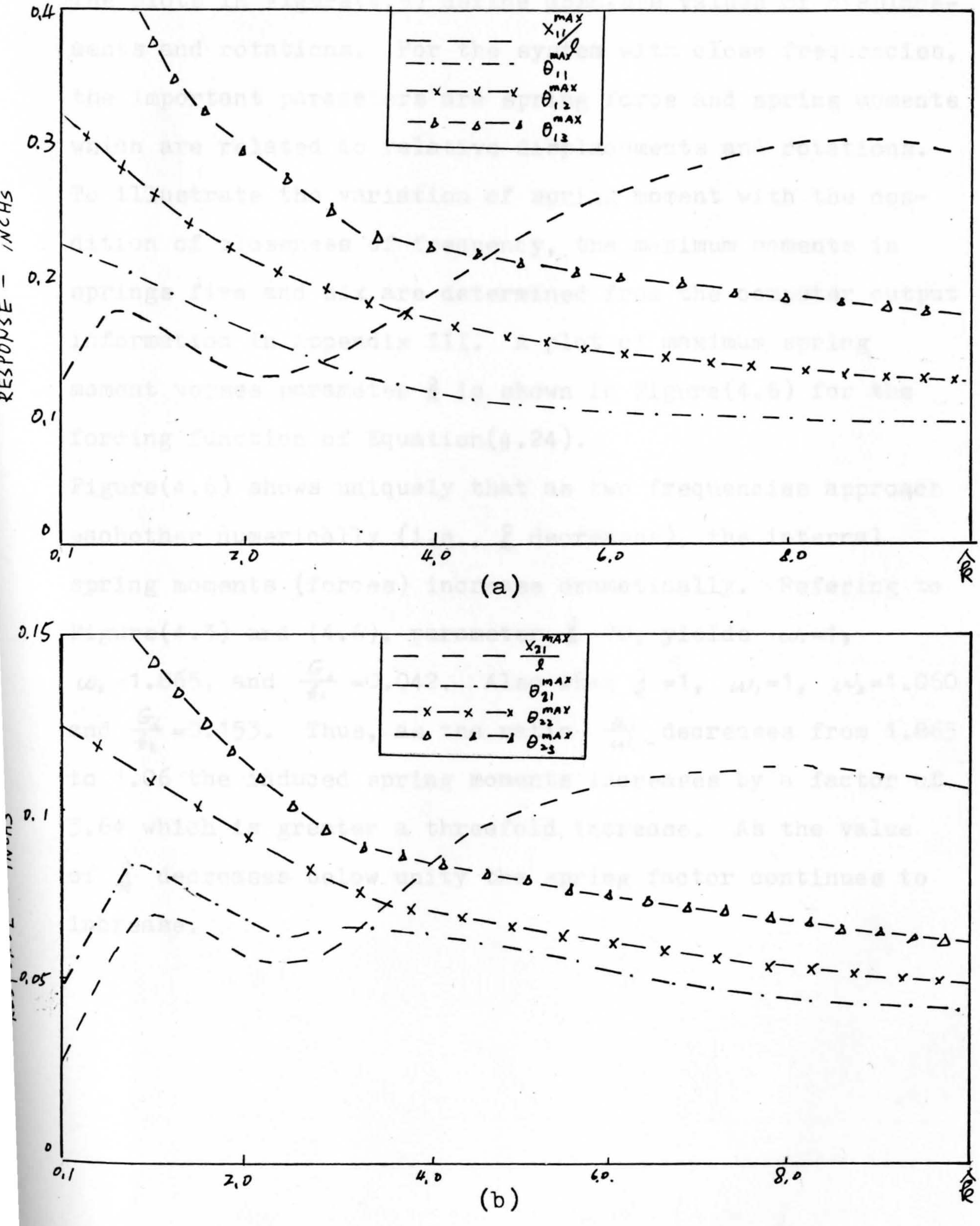


Fig.(4.5) Response-Stiffness Variation

The plots in Figure(4.5) define absolute values of displacements and rotations. For the system with close frequencies, the important parameters are spring force and spring moments which are related to relative displacements and rotations. To illustrate the variation of spring moment with the condition of closeness of frequency, the maximum moments in springs five and six are determined from the computer output information in Appendix III. A plot of maximum spring moment verses parameter $\hat{k}$ is shown in Figure(4.6) for the forcing function of Equation(4.24).

Figure(4.6) shows uniquely that as two frequencies approach eachother numerically (i.e., $\hat{k}$ decreases), the internal spring moments (forces) increase dramatically. Refering to Figure(4.3) and (4.6), parameter $\hat{k}$=20, yields ω_1=1, ω_2=1.865, and $\frac{G_6}{k_t}$=0.042. Also when $\hat{k}$=1, ω_1=1, ω_2=1.060 and $\frac{G_6}{k_t}$=0.153. Thus, as the ratio $\frac{\omega_2}{\omega_1}$ decreases from 1.865 to 1.06 the induced spring moments increases by a factor of 3.64 which is greater a threefold increase. As the value of $\hat{k}$ decreases below unity the spring factor continues to increase.

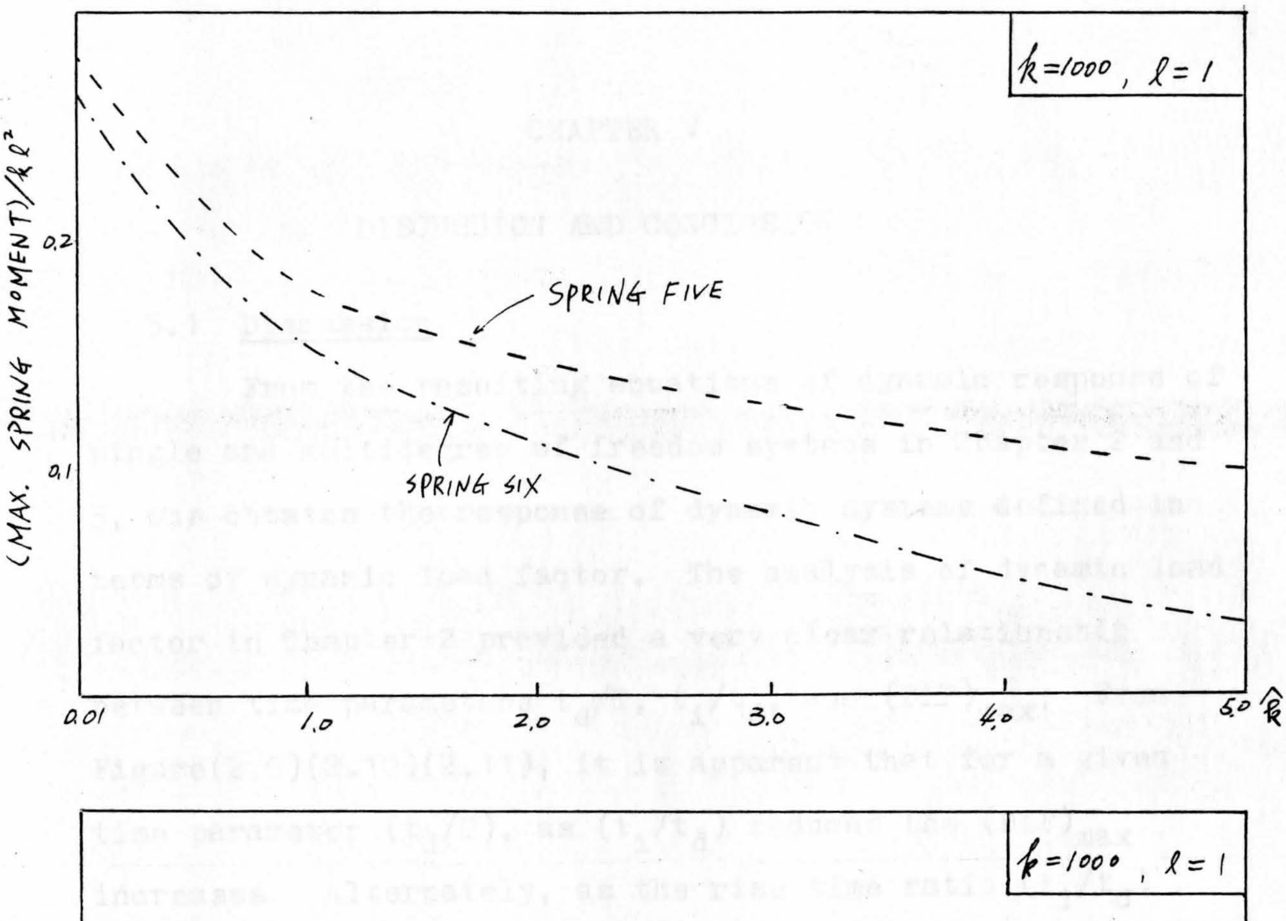


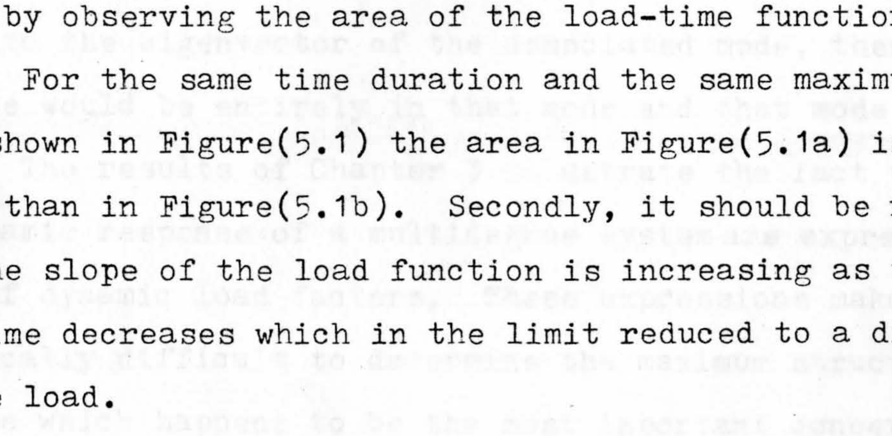
Fig.(4.6) Maximum Spring Moment vs. $\hat{k}$

CHAPTER V

DISCUSSION AND CONCLUSION

5.1 Discussion

From the resulting equations of dynamic response of single and multidegree of freedom systems in Chapter 2 and 3, one obtains the response of dynamic systems defined in terms of dynamic load factor. The analysis of dynamic load factor in Chapter 2 provided a very clear relationship between time parameters t_d/T, t_i/t_d, and $(DLF)_{max}$. From Figure(2.9)(2.10)(2.11), it is apparent that for a given time parameter (t_d/T), as (t_i/t_d) reduces the $(DLF)_{max}$ increases. Alternately, as the rise time ratio (t_i/t_d) increases the response of the system decreases. This is proven by observing the area of the load-time function curve. For the same time duration and the same maximum force shown in Figure(5.1) the area in Figure(5.1a) is larger than in Figure(5.1b). Secondly, it should be noted that the slope of the load function is increasing as the rise time decreases which in the limit reduced to a direct impulse load.



(a) (b)

Fig(5.1) Load Function with Different Rising Time

A n-degree of freedom system possess n dynamic load factors. If one applies the same load functions to all the masses, the n dynamic load factors can be represent by an (n x n) diagonal matrix(see Equation 3.59). From Equation (3.55) it follows that the equation of motion for a multi-degree of freedom system has a similar form as a single degree system. The response for each mass point is the product of the associated normalized eigenvector, times the dynamic load factor, times the modal static deflection given by the product of $[\Lambda_k]^{-1}[U]^T\{f(t)\}$. From the modal static deflection equation one determines that the vibrational is response greatly affected by the distribrtion of applied load. The fact is reflected in the matrix product $[U]^T\{f(t)\}$. The more nearly the load distribution is in proportion to the corresponding eigenvector, the greater the mode participation. In fact, if the load at all points were proportional to the eigenvector of the associated mode, then the response would be entirely in that mode and that mode alone.

The results of Chapter 3 illustrate the fact that the dynamic response of a multidegree system are expressed in terms of dynamic load factors. These expressions make it analytically difficult to determine the maximum structure response which happens to be the most important concern. Fortunately, since early 1950, the introduction of digtal computer has made the use of finite difference methods a practical analytical tool. As a result as shown in Chapter 4 the maximum displacement and time of maximum displacement may be effectively and efficiently determined. This conclusion is illustrated by the solution of the close frequency system discussed in Chapter 4. The finite-difference solution precisely showed that as any two frequencies approach oneanother, the induced internal spring moments/forces increase by a significant factor which in certain cases may be in the order of three to fivefold.

5.2 Conclusions

It is concluded from this study that the dynamic load factor is the most important factor in the analysis of structures subjected to dynamic load.

For multidegree of freedom systems, the dynamic load factor matrix is a diagonal matrix only in the special case where the components of the load vector are identical in the time variable. Otherwise, the matrix of dynamic load factors is an (n x n) matrix with a full complement of terms.

The dynamic response of a multidegree dynamic system possesses a very complicated numerical formulation and computation for the determination of maximum displacements. It has been shown that reliance on the computer techniques is an absolute necessity.

APPENDIX I

Computer Program for the Jacobi Method

```
      SUBROUTINE JACOB(A,V,ERR,N)
C
C ERR:ERROR ALLOWED
C A:SYSTEM MATRIX. AFTER THE COMPUTATIONS ARE COMPLETED ITS DIAGONAL TERMS
C WILL BE THE EIGENVALUES
C V:EACH COLUMN OF THIS ARRREY WILL CONTAIN A SET OF EIGENVECTORS
C
      DOUBLE PRECISION A(10,10),V(10,10)
      DOUBLE PRECISION T,T1,PS,TA,S,C,P
      ERR=0.00000001
      ITM=200
      IT=0
C
C PUT A UNIT MATRIX IN ARRAY V
C
      DO 10 I=1,N
      DO 10 J=1,N
      IF(I-J)3,1,3
    3 V(I,J)=0.
      GO TO 10
    1 V(I,J)=1.
   10 CONTINUE
C
C FIND LARGEST OFF DIAGONAL COEFFICIENT
C
   13 T=0
      M=N-1
      DO 20 I=1,M
      J1=I+1
      DO 20 J=J1,N
      IF(DABS(A(I,J))-T)20,20,2
    2 T=DABS(A(I,J))
      IR=I
      IC=J
   20 CONTINUE
      IF(IT)5,4,5
C
C TAKE FIRST LARGEST OFF DIAGONAL COEFFICIENT
C TIMES ERR AS COMPARISON VALUE FOR ZERO
C
    4 T1=T*ERR
    5 IF(T-T1)999,999,6
C
C COMPUTE TAN(TA),SIN(S),AND COSINE(C) OF ROTATION ANGLE
C
    6 PS=A(IR,IR)-A(IC,IC)
      TA=(-PS+DSQRT(PS*PS+4*T*T))/(2*A(IR,IC))
      C=1./DSQRT(1+TA*TA)
      S=C*TA
C
C MULTIPLY ROTATION MATRIX TIMES V AND STORE IN V
C
      DO 50 I=1,N
      P=V(I,IR)
      V(I,IR)=C*P+S*V(I,IC)
```

```
   50 V(I,IC)=C*V(I,IC)-S*P
      I=1
  100 IF(I-IR)7,200,7
C
C APPLY ORTHOGONAL TRANSFORMATION TO A AND STORE IN A
C
    7 P=A(I,IR)
      A(I,IR)=C*P+S*A(I,IC)
      A(I,IC)=C*A(I,IC)-S*P
      I=I+1
      GO TO 100
  200 I=IR+1
  300 IF(I-IC)8,400,8
    8 P=A(IR,I)
      A(IR,I)=C*P+S*A(I,IC)
      A(I,IC)=C*A(I,IC)-S*P
      I=I+1
      GO TO 300
  400 I=IC+1
  500 IF(I-N)9,9,600
    9 P=A(IR,I)
      A(IR,I)=C*P+S*A(IC,I)
      A(IC,I)=C*A(IC,I)-S*P
      I=I+1
      GO TO 500
  600 P=A(IR,IR)
      A(IR,IR)=C*C*P+2.*C*S*A(IR,IC)+S*S*A(IC,IC)
      A(IC,IC)=C*C*A(IC,IC)+S*S*P-2.*C*S*A(IR,IC)
      A(IR,IC)=0.
      A(IC,IR)=0.
      IT=IT+1
      IF(IT-ITM)13,13,999
  999 RETURN
      STOP
      END
/*
```

APPENDIX II

Computer Program for the Eigenvalue Problem

```
$JOB
C
C THIS PROGRAM COMPUTES THE EIGENVALUES AND EIGENVECTORS
C OF AN EQUATION OF TYPE A *X = LAMBDA * B * X
C N : ACTUAL ORDER OF A AND B
C ERR:ERROR LIMIT USED IN SUBROUTINE JACOB
C V : AUXILIARY ARRAY
C
      DOUBLE PRECISION A(10,10),B(10,10),H(10,10),V(10)
      READ(5,1)N
    1 FORMAT(I2)
      WRITE(6,22)
   22 FORMAT('1'////T20,' ERROR ALLOWED=0.00000001 ')
      DO 10 I=1,N
      READ(5,2)(A(I,J),J=1,N)
      WRITE(6,4)(A(I,J),J=1,N)
    2 FORMAT(8F10.5)
   10 CONTINUE
      DO 20 I=1,N
      READ(5,2)(B(I,J),J=1,N)
      WRITE(6,4)(B(I,J),J=1,N)
    4 FORMAT(21X,8(F10.4,2X))
   20 CONTINUE
      CALL EIGG(A,B,H,V,N,ERR)
      WRITE(6,5)
    5 FORMAT(/T20,' EIGENVALUE MATRIX ')
      DO 30 I=1,N
      WRITE(6,4)(A(I,J),J=1,N)
   30 CONTINUE
      WRITE(6,7)
    7 FORMAT(/T20,' EIGENVECTOR MATRIX ')
      DO 40 I=1,N
      WRITE(6,4)(B(I,J),J=1,N)
   40 CONTINUE
      STOP
      END

      SUBROUTINE EIGG(A,B,H,V,N,ERR)
      DOUBLE PRECISION A(10,10),B(10,10),H(10,10),V(10)
C
C DECOMPOSE MATRIX B USING CHOLESKI'S METHOD
C
      CALL DECOG(B,N)
C
C INVERT MATRIX B
C
      CALL INVCH(B,H,N)
C
C MULTIPLY TRANSPOSE(H) *A * H
C
      CALL BTAB3(A,H,V,N)
C
C COMPUTE THE EIGENVALUES
C
      CALL JACOB(A,B,ERR,N)
C
C COMPUTE THE EIGENVECTORS
C
      CALL MATMB(H,B,V,N)
      RETURN
```

```
      END
      SUBROUTINE DECOG(A,N)
C
C THIS PROGRAM PERFORMS THE DECOMPOSITION OF A SYMMETRIC MATRIX,
C INTO AN UPPER TRIANGULAR MATRIX,FOR POSITIVE DEFINITE MATRICES.
C A : ARRAY ORIGINALLY CONTAINING THE MATRIX TO BE DECOMPOSED.
C AT THE END IT CONTAINS THE UPPER TRIANGULAR MATRIX
C
      DOUBLE PRECISION A(10,10)
      DOUBLE PRECISION D
      IF(A(1,1))1,1,3
    1 WRITE(6,2)
    2 FORMAT('ZERO OR NEGATIVE RADICAND')
      GO TO 200
    3 A(1,1)=DSQRT(A(1,1))
      DO 10 J=2,N
   10 A(1,J)=A(1,J)/A(1,1)
      DO 40 I=2,N
      I1=I-1
      D=A(I,I)
      DO 20 L=1,I1
   20 D=D-A(L,I)*A(L,I)
      IF(A(I,I))1,1,21
   21 A(I,I)=DSQRT(D)
      I2=I+1
      IF(I2.GT.N)GO TO 41
      DO 40 J=I2,N
      D=A(I,J)
      DO 30 L=1,I1
   30 D=D-A(L,I)*A(L,J)
   40 A(I,J)=D/A(I,I)
   41 DO 50 I=2,N
      I1=I-1
      DO 50 J=1,I1
   50 A(I,J)=0.
      DO 31 I=1,N
   31 CONTINUE
  200 RETURN
      END
      SUBROUTINE INVCH(S,A,N)
C
C THIS PROGRAM COMPUTES THE INVERSE OF AN UPPER TRIANGULAR MATRIX,
C STORE IN "S",PLACING THE RESULTS IN "A".
C
      DOUBLE PRECISION A(10,10),S(10,10)
C
C COMPUTE DIAGONAL TERMS OF A
C
      DO 10 I=1,N
   10 A(I,I)=1./S(I,I)
C
C COMPUTE THE TERMS OF KTH DIAGONAL OF A
C
      N1=N-1
      DO 100 K=1,N1
      NK=N-K
      DO 100 I=1,NK
      J=I+K
```

```
      D=0.
      I1=I+1
      IK=I+K
      DO 20 L=I1,IK
   20 D=D+S(I,L)*A(L,J)
  100 A(I,J)=-D/S(I,I)
      DO 42 I=2,N
      I1=I-1
      DO 42 J=1,I1
   42 A(I,J)=0.
      DO 21 I=1,N
   21 CONTINUE
      RETURN
      END

      SUBROUTINE BTAB3(A,B,V,N)
C
C THIS PROGRAM COMPUTES THE MATRIX OPERATION A=TRANSPOSE(B)*A*B,
C WHERE A AND B ARE SQUARE MATRICES
C COMPUTE A*B AND STORE IN A
C
      DOUBLE PRECISION A(10,10),B(10,10),V(10)
      DO 10 I=1,N
      DO 5 J=1,N
      V(J)=0.
      DO 5 K=1,N
    5 V(J)=V(J)+A(I,K)*B(K,J)
      DO 10 J=1,N
   10 A(I,J)=V(J)
C
C COMPUTE TRANSPOSE(B)*A AND STORE IN A
C
      DO 20 J=1,N
      DO 15 I=1,N
      V(I)=0
      V(I)=0
      DO 15 K=1,N
   15 V(I)=V(I)+B(K,I)*A(K,J)
      DO 20 I=1,N
   20 A(I,J)=V(I)
      DO 25 I=1,N
   25 CONTINUE
      RETURN
      END

      SUBROUTINE JACOB(A,V,ERR,N)
C
C ERR:ERROR ALLOWED
C A:SYSTEM MATRIX. AFTER THE COMPUTATIONS ARE COMPLETED ITS DIAGONAL TERM
C WILL BE THE EIGENVALUES
C V:EACH COLUMN OF THIS ARRREY WILL CONTAIN A SET OF EIGENVECTORS
C
      DOUBLE PRECISION A(10,10),V(10,10)
      DOUBLE PRECISION T,T1,PS,TA,S,C,P
      ERR=0.00000001
      ITM=200
      IT=0
C
C PUT A UNIT MATRIX IN ARRAY V
C
```

```
      DO 10 I=1,N
      DO 10 J=1,N
      IF(I-J)3,1,3
    3 V(I,J)=0.
      GO TO 10
    1 V(I,J)=1.
   10 CONTINUE
C
C FIND LARGEST OFF DIAGONAL COEFFICIENT
C
   13 T=0
      M=N-1
      DO 20 I=1,M
      J1=I+1
      DO 20 J=J1,N
      IF(DABS(A(I,J))-T)20,20,2
    2 T=DABS(A(I,J))
      IR=I
      IC=J
   20 CONTINUE
      IF(IT)5,4,5
C
C TAKE FIRST LARGEST OFF DIAGONAL COEFFICIENT
C TIMES ERR AS COMPARISON VALUE FOR ZERO
C
    4 T1=T*ERR
    5 IF(T-T1)999,999,6
C
C COMPUTE TAN(TA),SIN(S),AND COSINE(C) OF ROTATION ANGLE
C
    6 PS=A(IR,IR)-A(IC,IC)
      TA=(-PS+DSQRT(PS*PS+4*T*T))/(2*A(IR,IC))
      C=1./DSQRT(1+TA*TA)
      S=C*TA
C
C MULTIPLY ROTATION MATRIX TIMES V AND STORE IN V
C
      DO 50 I=1,N
      P=V(I,IR)
      V(I,IR)=C*P+S*V(I,IC)
   50 V(I,IC)=C*V(I,IC)-S*P
      I=1
  100 IF(I-IR)7,200,7
C
C APPLY ORTHOGONAL TRANSFORMATION TO A AND STORE IN A
C
    7 P=A(I,IR)
      A(I,IR)=C*P+S*A(I,IC)
      A(I,IC)=C*A(I,IC)-S*P
      I=I+1
      GO TO 100
  200 I=IR+1
  300 IF(I-IC)8,400,8
    8 P=A(IR,I)
      A(IR,I)=C*P+S*A(I,IC)
      A(I,IC)=C*A(I,IC)-S*P
      I=I+1
      GO TO 300
  400 I=IC+1
  500 IF(I-N)9,9,600
```

```
    9 P=A(IR,I)
      A(IR,I)=C*P+S*A(IC,I)
      A(IC,I)=C*A(IC,I)-S*P
      I=I+1
      GO TO 500
  600 P=A(IR,IR)
      A(IR,IR)=C*C*P+2.*C*S*A(IR,IC)+S*S*A(IC,IC)
      A(IC,IC)=C*C*A(IC,IC)+S*S*P-2.*C*S*A(IR,IC)
      A(IR,IC)=0.
      A(IC,IR)=0.
      IT=IT+1
      IF(IT-ITM)13,13,999
  999 RETURN
      END

      SUBROUTINE MATMB(A,B,V,N)
C
C THIS PROGRAM PERFORMS THE MATRIX OPERATION A = A * B
C
      DOUBLE PRECISION A(10,10),B(10,10),V(10)
      DO 20 J=1,N
      DO 16 I=1,N
      V(I)=0.
      DO 16 K=1,N
   16 V(I)=V(I)+A(I,K)*B(K,J)
      DO 20 I=1,N
   20 B(I,J)=V(I)
      RETURN
      END

$ENTRY
```

```
ERROR ALLOWED=0.00000001
 10000.0000  -4000.0000      0.0000
 -4000.0000   6000.0000  -2000.0000
     0.0000  -2000.0000   2000.0000
     2.0000      0.0000      0.0000
     0.0000      1.0000      0.0000
     0.0000      0.0000      1.0000

EIGENVALUE MATRIX
  8735.4895     -0.0000      0.0000
     0.0000   3473.5603     -0.0000
     0.0000      0.0000    790.9502

EIGENVECTOR MATRIX
     0.4154      0.5230      0.2322
    -0.7758      0.3992      0.4887
     0.2304     -0.5418      0.8083

STATEMENTS EXECUTED=      806
```

APPENDIX III

Computer Program for the Finite Difference Method

```
$JOB
C THIS IS A COMPUTER PROGRAM USING FINITE DIFFERENCE METHOD TO DETERMINE
C THE RESPONSE OF A MULTIDEGREE SYSTEM.
C S(I,J) IS THE INVERSE OF MASS MATRIX.
C XK(I,J) IS THE STIFFNESS MATRIX.
      DIMENSION XK(5,5),S(5,5),XXM(5,5),FM(500)
      DIMENSION F0(500),XDD0(500),T(500),X1(500),F(500),PXDD1(500)
      DIMENSION PXD1(500),FF(500,5),XP(500)
      DIMENSION PX(500,5),PDDX(500,5),XX(500,5),PXDD(500,5),PXD(500,5)
      DIMENSION XDD(500,5)
      READ(5,2)N
      READ(5,99)TD
      READ(5,99)DT
      WRITE(6,99)DT
      READ(5,100)((S(I,J),J=1,N),I=1,N)
      WRITE(6,100)((S(I,J),J=1,N),I=1,N)
      READ(5,100)((XK(I,J),J=1,N),I=1,N)
      WRITE(6,100)((XK(I,J),J=1,N),I=1,N)
    2 FORMAT(I2)
  100 FORMAT(30X,3F10.4)
   99 FORMAT(30X,F10.5)
      DO 110 J=1,N
      DO 110 I=1,N
  110 XXM(I,J)=0
      DO 120 M=1,N
      DO 130 I=1,N
      DO 140 K=1,N
  140 XXM(M,I)=S(M,K)*XK(K,I)+XXM(M,I)
  130 CONTINUE
  120 CONTINUE
      WRITE(6,200)
  200 FORMAT(15H PRODUCT MATRIX)
      WRITE(6,100)((XXM(I,J),J=1,N),I=1,N)
      T0=0.0
      F0(1)=-30000.*T0+3000.
      F0(2)=-40000.*T0+4000.
      F0(3)=20000.*T0-2000.
      DO 180 I=1,N
  180 XDD0(I)=0.0
      DO 190 I=1,N
      DO 220 K=1,N
  220 XDD0(I)=S(I,K)*F0(K)+XDD0(I)
  190 CONTINUE
      DO 240 I=1,N
  240 XP(I)=0.0
      DO 260 I=1,N
  260 X1(I)=((DT**2.)/2.)*XDD0(I)
      F(1)=-30000.*DT+3000.
      F(2)=-40000.*DT+4000.
      F(3)=20000.*DT-2000.
      DO 270 M=1,N
      PXDD1(M)=0.0
  270 PXD1(M)=0.0
      DO 280 I=1,N
      DO 290 K=1,N
      PXDD1(I)=XXM(I,K)*X1(K)+PXDD1(I)
      PXD1(I)=S(I,K)*F(K)+PXD1(I)
  290 CONTINUE
  280 CONTINUE
      DO 301 I=1,N
```

```
      L=1
  301 XDD(L,I)=-PXDD1(I)+PXD1(I)
      M=2
      T(M)=2*DT
      DO 310 I=1,N
      PX(L,I)=2.*X1(I)
      PDDX(L,I)=(DT**2.)*XDD(L,I)
      XX(M,I)=PX(L,I)+PDDX(L,I)
  310 CONTINUE
      L=2
  360 CONTINUE
      XX(1,1)=X1(1)
      XX(1,2)=X1(2)
      XX(1,3)=X1(3)
      FF(M,1)=-30000.*T(M)+3000.
      FF(M,2)=-40000.*T(M)+4000
      FF(M,3)=20000.*T(M)-2000.
      DO 315 I=1,N
      PXDD(L,I)=0.0
  315 PXD(L,I)=0.0
      DO 320 I=1,N
      DO 330 K=1,N
      PXDD(L,I)=-XXM(I,K)*XX(M,K)+PXDD(L,I)
      PXD(L,I)=S(I,K)*FF(M,I)+PXD(L,I)
  330 CONTINUE
  320 CONTINUE
      DO 340 I=1,N
      XDD(L,I)=PXDD(L,I)+PXD(L,I)
  340 CONTINUE
      LX=(TD/DT)+1
      LY=2*LX
      DO 350 I=1,N
      J=L+1
      JJ=L-1
      XX(J,I)=2.*XX(L,I)-XX(JJ,I)+DT**2.*XDD(L,I)
  350 CONTINUE
      M=M+1
      T(M)=M*DT
      L=L+1
      IF(L-LX)360,360,370
  370 CONTINUE
      L=LX
  801 CONTINUE
      DO 803 I=1,N
  803 PXDD(L,I)=0.0
      DO 800 I=1,N
      DO 810 K=1,N
      PXDD(L,I)=-XXM(I,K)*XX(L,K)+PXDD(L,I)
  810 CONTINUE
  800 CONTINUE
      DO 820 I=1,N
      FF(L,I)=0.0
      J=L+1
      JJ=L-1
      XX(J,I)=2.*XX(L,I)-XX(JJ,I)+DT**2.*PXDD(L,I)
  820 CONTINUE
      L=L+1
      IF(L-LY)801,802,802
  802 CONTINUE
      DO 807 I=1,N
```

```
      807 FF(L,I)=0.
          MXO=0
          Y=0.0
          WRITE(6,500)
      500 FORMAT('1',1X,7H NUMBER,2X,5H TIME,7X,3H F1,7X,3H X1,8X,
         *3H F2,8X,3H X2,8X,3H F3,8X,3H X3)
          WRITE(6,600)MXO,Y,FO(1),XP(1),FO(2),XP(2),FO(3),XP(3)
          MXO=I
          Y=DT
          WRITE(6,600)MXO,Y,F(1),X1(1),F(2),X1(2),F(3),X1(3)
      600 FORMAT(4X,I3,3X,F6.4,2X,3(F10.4,2X,F8.5,2X))
          DO 700 I=2,LY
          Y=I*DT
          I1=1
          I2=2
          I3=3
      700 WRITE(6,600)I,Y,FF(I,I1),XX(I,I1),FF(I,I2),XX(I,I2),FF(I,I3),XX(I,
         *I3)
          STOP
          END
```

```
$ENTRY
                              0.00050
                              0.5000     0.0000     0.0000
                              0.0000     1.0000     0.0000
                              0.0000     0.0000     1.0000
                          10000.0000-4000.0000     0.0000
                          -4000.0000 6000.0000-2000.0000
                              0.0000-2000.0000 2000.0000
RODUCT MATRIX
                           5000.0000-2000.0000     0.0000
                          -4000.0000 6000.0000-2000.0000
                              0.0000-2000.0000 2000.0000
```

NUMBER	TIME	F1	X1	F2	X2	F3	X3
0	0.0000	3000.0000	0.00000	4000.0000	0.00000	-2000.0000	0.00000
1	0.0005	2985.0000	0.00019	3980.0000	0.00050	-1990.0000	-0.00025
2	0.0010	2970.0000	0.00075	3960.0000	0.00199	-1980.0000	-0.00100
3	0.0015	2955.0000	0.00168	3940.0000	0.00448	-1970.0000	-0.00224
4	0.0020	2940.0000	0.00298	3920.0000	0.00794	-1960.0000	-0.00397
5	0.0025	2925.0000	0.00465	3899.9990	0.01237	-1950.0000	-0.00618
6	0.0030	2910.0000	0.00669	3879.9990	0.01775	-1940.0000	-0.00887
7	0.0035	2894.9990	0.00909	3859.9990	0.02409	-1930.0000	-0.01204
8	0.0040	2879.9990	0.01185	3839.9990	0.03135	-1920.0000	-0.01567
9	0.0045	2865.0000	0.01497	3820.0000	0.03953	-1910.0000	-0.01975
10	0.0050	2850.0000	0.01845	3800.0000	0.04862	-1900.0000	-0.02428
11	0.0055	2835.0000	0.02229	3780.0000	0.05858	-1890.0000	-0.02925
12	0.0060	2819.9990	0.02649	3759.9990	0.06941	-1879.9990	-0.03465
13	0.0065	2805.0000	0.03104	3740.0000	0.08109	-1870.0000	-0.04047
14	0.0070	2790.0000	0.03594	3720.0000	0.09359	-1860.0000	-0.04669
15	0.0075	2775.0000	0.04119	3700.0000	0.10689	-1850.0000	-0.05331
16	0.0080	2759.9990	0.04679	3680.0000	0.12097	-1839.9990	-0.06032
17	0.0085	2745.0000	0.05274	3660.0000	0.13581	-1830.0000	-0.06769
18	0.0090	2730.0000	0.05903	3640.0000	0.15138	-1820.0000	-0.07541
19	0.0095	2715.0000	0.06567	3620.0000	0.16765	-1810.0000	-0.08348
20	0.0100	2700.0000	0.07265	3600.0000	0.18460	-1799.9990	-0.09188
21	0.0105	2685.0000	0.07996	3580.0000	0.20219	-1790.0000	-0.10058
22	0.0110	2670.0000	0.08762	3560.0000	0.22041	-1780.0000	-0.10958
23	0.0115	2655.0000	0.09560	3540.0000	0.23922	-1769.9990	-0.11887
24	0.0120	2640.0000	0.10392	3520.0000	0.25860	-1759.9990	-0.12841
25	0.0125	2625.0000	0.11257	3500.0000	0.27850	-1750.0000	-0.13821
26	0.0130	2610.0000	0.12155	3480.0000	0.29891	-1740.0000	-0.14823
27	0.0135	2595.0000	0.13085	3460.0000	0.31979	-1730.0000	-0.15846
28	0.0140	2580.0000	0.14047	3440.0000	0.34110	-1720.0000	-0.16889
29	0.0145	2565.0000	0.15040	3420.0000	0.36282	-1710.0000	-0.17949
30	0.0150	2550.0000	0.16066	3400.0000	0.38490	-1700.0000	-0.19025
31	0.0155	2535.0000	0.17122	3380.0000	0.40733	-1690.0000	-0.20114
32	0.0160	2520.0000	0.18209	3360.0000	0.43006	-1680.0000	-0.21216
33	0.0165	2505.0000	0.19326	3340.0000	0.45306	-1670.0000	-0.22327
34	0.0170	2490.0000	0.20473	3320.0000	0.47630	-1660.0000	-0.23446
35	0.0175	2475.0000	0.21649	3300.0000	0.49974	-1650.0000	-0.24571
36	0.0180	2460.0000	0.22854	3280.0000	0.52335	-1640.0000	-0.25700
37	0.0185	2445.0000	0.24087	3260.0000	0.54710	-1630.0000	-0.26831
38	0.0190	2430.0000	0.25348	3240.0000	0.57094	-1620.0000	-0.27962
39	0.0195	2415.0000	0.26637	3220.0000	0.59486	-1610.0000	-0.29091
40	0.0200	2400.0000	0.27952	3200.0000	0.61880	-1600.0000	-0.30216
41	0.0205	2385.0000	0.29293	3180.0000	0.64275	-1590.0000	-0.31335
42	0.0210	2370.0000	0.30660	3160.0000	0.66666	-1580.0000	-0.32446
43	0.0215	2355.0000	0.32051	3140.0000	0.69051	-1570.0000	-0.33547
44	0.0220	2340.0000	0.33466	3120.0000	0.71426	-1560.0000	-0.34636
45	0.0225	2325.0000	0.34904	3100.0000	0.73787	-1550.0000	-0.35710
46	0.0230	2310.0000	0.36364	3080.0000	0.76133	-1540.0000	-0.36769
47	0.0235	2295.0000	0.37846	3060.0000	0.78459	-1530.0000	-0.37810
48	0.0240	2280.0000	0.39348	3040.0000	0.80764	-1520.0000	-0.38831
49	0.0245	2265.0000	0.40871	3020.0000	0.83042	-1510.0000	-0.39830
50	0.0250	2250.0000	0.42411	3000.0000	0.85293	-1500.0000	-0.40805
51	0.0255	2235.0000	0.43970	2980.0000	0.87513	-1490.0000	-0.41755
52	0.0260	2220.0000	0.45546	2960.0000	0.89698	-1480.0000	-0.42678
53	0.0265	2205.0000	0.47137	2940.0000	0.91848	-1470.0000	-0.43571
54	0.0270	2190.0000	0.48742	2920.0000	0.93959	-1460.0000	-0.44433
55	0.0275	2175.0000	0.50361	2900.0000	0.96028	-1450.0000	-0.45263
56	0.0280	2160.0000	0.51993	2880.0000	0.98053	-1440.0000	-0.46058
57	0.0285	2145.0000	0.53635	2860.0000	1.00032	-1430.0000	-0.46817
58	0.0290	2130.0000	0.55287	2840.0000	1.01963	-1420.0000	-0.47539

59	0.0295	2115.0000	0.56947	2820.0000	1.03843	-1410.0000	-0.48221
60	0.0300	2100.0000	0.58615	2800.0000	1.05671	-1400.0000	-0.48802
61	0.0305	2085.0000	0.60288	2780.0000	1.07444	-1390.0000	-0.49461
62	0.0310	2070.0000	0.61966	2760.0000	1.09161	-1380.0000	-0.50017
63	0.0315	2055.0000	0.63647	2740.0000	1.10820	-1370.0000	-0.50527
64	0.0320	2040.0000	0.65329	2720.0000	1.12420	-1360.0000	-0.50991
65	0.0325	2025.0000	0.67011	2700.0000	1.13960	-1350.0000	-0.51407
66	0.0330	2010.0000	0.68692	2680.0000	1.15437	-1340.0000	-0.51774
67	0.0335	1995.0000	0.70370	2660.0000	1.16850	-1330.0000	-0.52091
68	0.0340	1980.0000	0.72043	2640.0000	1.18199	-1320.0000	-0.52357
69	0.0345	1965.0000	0.73710	2620.0000	1.19483	-1310.0000	-0.52570
70	0.0350	1950.0000	0.75368	2600.0000	1.20700	-1300.0000	-0.52730
71	0.0355	1935.0000	0.77018	2580.0000	1.21850	-1290.0000	-0.52836
72	0.0360	1920.0000	0.78656	2560.0000	1.22932	-1280.0000	-0.52887
73	0.0365	1905.0000	0.80281	2540.0000	1.23947	-1270.0000	-0.52882
74	0.0370	1890.0000	0.81892	2520.0000	1.24892	-1260.0000	-0.52820
75	0.0375	1875.0000	0.83486	2500.0000	1.25769	-1250.0000	-0.52701
76	0.0380	1860.0000	0.85062	2479.9990	1.26577	-1240.0000	-0.52523
77	0.0385	1845.0000	0.86619	2460.0000	1.27315	-1230.0000	-0.52288
78	0.0390	1830.0000	0.88153	2440.0000	1.27985	-1220.0000	-0.51993
79	0.0395	1815.0000	0.89665	2420.0000	1.28585	-1210.0000	-0.51638
80	0.0400	1800.0000	0.91151	2399.9990	1.29117	-1200.0000	-0.51223
81	0.0405	1785.0000	0.92610	2380.0000	1.29561	-1190.0000	-0.50749
82	0.0410	1770.0000	0.94041	2360.0000	1.29977	-1180.0000	-0.50213
83	0.0415	1755.0000	0.95441	2340.0000	1.30306	-1170.0000	-0.49618
84	0.0420	1740.0000	0.96809	2319.9990	1.30569	-1160.0000	-0.48961
85	0.0425	1725.0000	0.98142	2300.0000	1.30766	-1150.0000	-0.48244
86	0.0430	1710.0000	0.99440	2280.0000	1.30899	-1140.0000	-0.47466
87	0.0435	1695.0000	1.00701	2259.9990	1.30968	-1130.0000	-0.46627
88	0.0440	1680.0000	1.01922	2239.9990	1.30974	-1120.0000	-0.45728
89	0.0445	1665.0000	1.03102	2220.0000	1.30919	-1110.0000	-0.44768
90	0.0450	1650.0000	1.04240	2200.0000	1.30803	-1100.0000	-0.43748
91	0.0455	1635.0000	1.05333	2179.9990	1.30629	-1090.0000	-0.42669
92	0.0460	1620.0000	1.06380	2159.9990	1.30397	-1080.0000	-0.41530
93	0.0465	1605.0000	1.07380	2140.0000	1.30109	-1070.0000	-0.40332
94	0.0470	1590.0000	1.08330	2120.0000	1.29767	-1060.0000	-0.39075
95	0.0475	1575.0000	1.09230	2099.9990	1.29372	-1050.0000	-0.37761
96	0.0480	1560.0000	1.10076	2079.9990	1.28926	-1040.0000	-0.36389
97	0.0485	1545.0000	1.10872	2060.0000	1.28430	-1030.0000	-0.34961
98	0.0490	1530.0000	1.11610	2039.9990	1.27886	-1020.0000	-0.33477
99	0.0495	1515.0000	1.12293	2019.9990	1.27297	-1010.0000	-0.31933
100	0.0500	1499.9990	1.12917	1999.9990	1.26663	-1000.0000	-0.30344
101	0.0505	1485.0000	1.13483	1980.0000	1.25987	-990.0000	-0.28696
102	0.0510	1470.0000	1.13988	1959.9990	1.25270	-980.0000	-0.26997
103	0.0515	1455.0000	1.14431	1939.9990	1.24515	-970.0000	-0.25245
104	0.0520	1439.9990	1.14812	1919.9990	1.23724	-960.0000	-0.23443
105	0.0525	1425.0000	1.15129	1900.0000	1.22898	-950.0000	-0.21591
106	0.0530	1410.0000	1.15382	1879.9990	1.22040	-940.0000	-0.19691
107	0.0535	1395.0000	1.15569	1859.9990	1.21151	-930.0000	-0.17743
108	0.0540	1379.9990	1.15689	1839.9990	1.20233	-920.0000	-0.15749
109	0.0545	1365.0000	1.15742	1819.9990	1.19289	-910.0000	-0.13711
110	0.0550	1350.0000	1.15728	1799.9990	1.18320	-900.0000	-0.11628
111	0.0555	1334.9990	1.15644	1779.9990	1.17329	-890.0000	-0.09503
112	0.0560	1319.9990	1.15491	1759.9990	1.16317	-880.0000	-0.07337
113	0.0565	1305.0000	1.15269	1739.9990	1.15287	-870.0000	-0.05131
114	0.0570	1290.0000	1.14976	1719.9990	1.14239	-860.0000	-0.02886
115	0.0575	1274.9990	1.14613	1699.9990	1.13177	-850.0000	-0.00605
116	0.0580	1259.9990	1.14179	1679.9990	1.12102	-840.0000	0.01712
117	0.0585	1245.0000	1.13674	1659.9990	1.11015	-830.0000	0.04064
118	0.0590	1230.0000	1.13098	1639.9990	1.09919	-820.0000	0.06448

119	0.0595	1214.9990	1.12451	1619.9990	1.08816	-810.0000	0.08663
120	0.0600	1199.9990	1.11733	1599.9990	1.07707	-800.0000	0.11308
121	0.0605	1185.0000	1.10944	1579.9990	1.06593	-790.0000	0.13782
122	0.0610	1170.0000	1.10085	1559.9990	1.05477	-780.0000	0.162[illegible]
123	0.0615	1154.9990	1.09155	1539.9990	1.04360	-770.0000	0.18606
124	0.0620	1139.9990	1.08156	1519.9990	1.03243	-760.0000	0.21355
125	0.0625	1125.0000	1.07087	1500.0000	1.02129	-750.0000	0.23926
126	0.0630	1110.0010	1.05949	1480.0010	1.01017	-740.0007	0.26515
127	0.0635	1095.0000	1.04743	1460.0000	0.99910	-730.0005	0.29126
128	0.0640	1079.9990	1.03470	1439.9990	0.98810	-720.0000	0.31753
129	0.0645	1065.0000	1.02130	1420.0010	0.97716	-710.0007	0.34395
130	0.0650	1050.0000	1.00725	1400.0000	0.96630	-700.0002	0.37051
131	0.0655	1035.0010	0.99255	1380.0010	0.95554	-690.0010	0.39719
132	0.0660	1020.0000	0.97722	1360.0000	0.94488	-680.0005	0.42395
133	0.0665	1005.0000	0.96127	1339.9990	0.93433	-670.0000	0.45067
134	0.0670	990.0010	0.94471	1320.0010	0.92390	-660.0007	0.47752
135	0.0675	975.0002	0.92756	1300.0000	0.91360	-650.0002	0.50404
136	0.0680	960.0015	0.90982	1280.0010	0.90343	-640.0010	0.53169
137	0.0685	945.0007	0.89152	1260.0000	0.89340	-630.0005	0.556[illegible]7
138	0.0690	930.0000	0.87266	1240.0000	0.88352	-620.0000	0.58[illegible]
139	0.0695	915.0010	0.85326	1220.0010	0.87379	-610.0007	0.61314
140	0.0700	900.0002	0.83338	1200.0000	0.86421	-600.0002	0.64020
141	0.0705	885.0015	0.81298	1180.0010	0.85479	-590.0010	0.66722
142	0.0710	870.0007	0.79210	1160.0000	0.84552	-580.0005	0.69419
143	0.0715	855.0000	0.77076	1140.0000	0.83642	-570.0000	0.72109
144	0.0720	840.0012	0.74899	1120.0010	0.82746	-560.0007	0.74790
145	0.0725	825.0005	0.72679	1100.0000	0.81870	-550.0002	0.77461
146	0.0730	810.0015	0.70420	1080.0010	0.81008	-540.0010	0.80121
147	0.0735	795.0007	0.68124	1060.0000	0.80162	-530.0005	0.82767
148	0.0740	780.0000	0.65792	1040.0000	0.79331	-520.0000	0.85399
149	0.0745	765.0012	0.63426	1020.0010	0.78517	-510.0007	0.88015
150	0.0750	750.0005	0.61033	1000.0000	0.77717	-500.0002	0.90619
151	0.0755	734.9998	0.58610	979.9998	0.76932	-490.0000	0.93193
152	0.0760	720.0010	0.56161	960.0012	0.76161	-480.0007	0.95752
153	0.0765	705.0002	0.53689	940.0002	0.75404	-470.0002	0.98239
154	0.0770	690.0012	0.51197	920.0017	0.74600	-460.0010	1.00683
155	0.0775	675.0005	0.48680	900.0007	0.73929	-450.0005	1.03292
156	0.0780	659.9998	0.46160	879.9998	0.73209	-440.0000	1.05755
157	0.0785	645.0010	0.43621	860.0012	0.72501	-430.0007	1.08191
158	0.0790	630.0002	0.41072	840.0002	0.71804	-420.0002	1.10596
159	0.0795	615.0012	0.38515	820.0017	0.71116	-410.0010	1.12975
160	0.0800	600.0007	0.35954	800.0007	0.70436	-400.0005	1.15321
161	0.0805	585.0000	0.33390	779.9998	0.69765	-390.0005	1.17635
162	0.0810	570.0010	0.30827	760.0012	0.69101	-380.0007	1.19915
163	0.0815	555.0002	0.28266	740.0005	0.68442	-370.0002	1.22160
164	0.0820	540.0015	0.25712	720.0017	0.67789	-360.0010	1.24369
165	0.0825	525.0007	0.23166	700.0010	0.67141	-350.0005	1.26540
166	0.0830	510.0000	0.20631	680.0000	0.66495	-340.0000	1.28675
167	0.0835	495.0010	0.18111	660.0015	0.65852	-330.0007	1.30766
168	0.0840	480.0005	0.15606	640.0005	0.65209	-320.0002	1.32819
169	0.0845	465.0015	0.13121	620.0020	0.64567	-310.0010	1.34830
170	0.0850	450.0007	0.10657	600.0010	0.63924	-300.0005	1.36795
171	0.0855	435.0000	0.08213	580.0000	0.63279	-290.0000	1.38721
172	0.0860	420.0012	0.05805	560.0015	0.62632	-280.0007	1.40600
173	0.0865	405.0005	0.03422	540.0005	0.61980	-270.0002	1.42433
174	0.0870	389.9998	0.01071	519.9995	0.61323	-259.9998	1.44219
175	0.0875	375.0007	-0.01247	500.0010	0.60661	-250.0005	1.45957
176	0.0880	360.0002	-0.03527	480.0002	0.59992	-240.0002	1.47643
177	0.0885	345.0012	-0.05769	460.0017	0.59315	-230.0010	1.49254
178	0.0890	330.0005	-0.07970	440.0007	0.58629	-220.0005	1.50673

179	0.0895	314.9995	-0.10127	419.9998	0.57934	-210.0000	1.52409
180	0.0900	300.0010	-0.12239	400.0012	0.57229	-200.0007	1.53893
181	0.0905	285.0002	-0.14303	380.0002	0.56512	-190.0002	1.55324
182	0.0910	270.0012	-0.16317	360.0017	0.55783	-180.0010	1.56700
183	0.0915	255.0005	-0.18280	340.0007	0.55042	-170.0005	1.58022
184	0.0920	240.0000	-0.20189	319.9998	0.54287	-160.0000	1.59287
185	0.0925	225.0010	-0.22042	300.0012	0.53519	-150.0007	1.60496
186	0.0930	210.0002	-0.23839	280.0002	0.52735	-140.0002	1.61648
187	0.0935	195.0015	-0.25576	260.0017	0.51937	-130.0010	1.62742
188	0.0940	180.0007	-0.27254	240.0007	0.51123	-120.0005	1.63777
189	0.0945	165.0000	-0.28869	220.0000	0.50292	-110.0000	1.64753
190	0.0950	150.0010	-0.30421	200.0015	0.49446	-100.0007	1.65663
191	0.0955	135.0002	-0.31909	180.0005	0.48582	-90.0002	1.66524
192	0.0960	119.9978	-0.33331	159.9995	0.47701	-79.9998	1.67317
193	0.0965	105.0007	-0.34685	140.0010	0.46804	-70.0005	1.68049
194	0.0970	90.0000	-0.35972	120.0000	0.45888	-60.0000	1.68719
195	0.0975	75.0012	-0.37190	100.0015	0.44956	-50.0007	1.69326
196	0.0980	60.0005	-0.38337	80.0005	0.44006	-40.0002	1.69869
197	0.0985	44.9978	-0.39414	59.9995	0.43038	-29.9998	1.70348
198	0.0990	30.0007	-0.40420	40.0010	0.42053	-20.0005	1.70762
199	0.0995	15.0000	-0.41353	20.0000	0.41051	-10.0000	1.71112
200	0.1000	0.0012	-0.42215	0.0015	0.40033	-0.0007	1.71397
201	0.1005	0.0000	-0.43003	0.0000	0.38997	0.0000	1.71616
202	0.1010	0.0000	-0.43718	0.0000	0.37946	0.0000	1.71765
203	0.1015	0.0000	-0.44360	0.0000	0.36880	0.0000	1.71853
204	0.1020	0.0000	-0.44928	0.0000	0.35801	0.0000	1.71871
205	0.1025	0.0000	-0.45421	0.0000	0.34708	0.0000	1.71821
206	0.1030	0.0000	-0.45841	0.0000	0.33605	0.0000	1.71702
207	0.1035	0.0000	-0.46186	0.0000	0.32490	0.0000	1.71515
208	0.1040	0.0000	-0.46456	0.0000	0.31367	0.0000	1.71257
209	0.1045	0.0000	-0.46655	0.0000	0.30236	0.0000	1.70930
210	0.1050	0.0000	-0.46780	0.0000	0.29098	0.0000	1.70532
211	0.1055	0.0000	-0.46831	0.0000	0.27955	0.0000	1.70064
212	0.1060	0.0000	-0.46810	0.0000	0.26808	0.0000	1.69524
213	0.1065	0.0000	-0.46716	0.0000	0.25659	0.0000	1.68913
214	0.1070	0.0000	-0.46552	0.0000	0.24509	0.0000	1.68231
215	0.1075	0.0000	-0.46317	0.0000	0.23360	0.0000	1.67476
216	0.1080	0.0000	-0.46012	0.0000	0.22214	0.0000	1.66650
217	0.1085	0.0000	-0.45637	0.0000	0.21071	0.0000	1.65751
218	0.1090	0.0000	-0.45199	0.0000	0.19934	0.0000	1.64779
219	0.1095	0.0000	-0.44691	0.0000	0.18805	0.0000	1.63736
220	0.1100	0.0000	-0.44119	0.0000	0.17684	0.0000	1.62619
221	0.1105	0.0000	-0.43483	0.0000	0.16574	0.0000	1.61431
222	0.1110	0.0000	-0.42734	0.0000	0.15476	0.0000	1.60169
223	0.1115	0.0000	-0.42023	0.0000	0.14393	0.0000	1.58836
224	0.1120	0.0000	-0.41203	0.0000	0.13325	0.0000	1.57430
225	0.1125	0.0000	-0.40325	0.0000	0.12274	0.0000	1.55952
226	0.1130	0.0000	-0.39390	0.0000	0.11243	0.0000	1.54402
227	0.1135	0.0000	-0.38401	0.0000	0.10233	0.0000	1.52780
228	0.1140	0.0000	-0.37358	0.0000	0.09246	0.0000	1.51087
229	0.1145	0.0000	-0.36264	0.0000	0.08283	0.0000	1.49324
230	0.1150	0.0000	-0.35120	0.0000	0.07346	0.0000	1.47489
231	0.1155	0.0000	-0.33929	0.0000	0.06436	0.0000	1.45585
232	0.1160	0.0000	-0.32595	0.0000	0.05556	0.0000	1.43611
233	0.1165	0.0000	-0.31412	0.0000	0.04706	0.0000	1.41568
234	0.1170	0.0000	-0.30095	0.0000	0.03889	0.0000	1.39456
235	0.1175	0.0000	-0.28729	0.0000	0.03105	0.0000	1.37276
236	0.1180	0.0000	-0.27330	0.0000	0.02357	0.0000	1.35036
237	0.1185	0.0000	-0.25895	0.0000	0.01646	0.0000	1.32717
238	0.1190	0.0000	-0.24425	0.0000	0.00972	0.0000	1.30338

239	0.1195	0.0000	-0.22929	0.0000	0.00338	0.0000	1.27895
240	0.1200	0.0000	-0.21402	0.0000	-0.00256	0.0000	1.25388
241	0.1205	0.0000	-0.19848	0.0000	-0.00808	0.0000	1.22813
242	0.1210	0.0000	-0.18269	0.0000	-0.01317	0.0000	1.20186
243	0.1215	0.0000	-0.16669	0.0000	-0.01783	0.0000	1.17494
244	0.1220	0.0000	-0.15048	0.0000	-0.02204	0.0000	1.14741
245	0.1225	0.0000	-0.13410	0.0000	-0.02579	0.0000	1.11930
246	0.1230	0.0000	-0.11756	0.0000	-0.02908	0.0000	1.09062
247	0.1235	0.0000	-0.10089	0.0000	-0.03189	0.0000	1.06138
248	0.1240	0.0000	-0.08411	0.0000	-0.03423	0.0000	1.03159
249	0.1245	0.0000	-0.06724	0.0000	-0.03609	0.0000	1.00127
250	0.1250	0.0000	-0.05031	0.0000	-0.03745	0.0000	0.97043
251	0.1255	0.0000	-0.03333	0.0000	-0.03833	0.0000	0.93908
252	0.1260	0.0000	-0.01635	0.0000	-0.03871	0.0000	0.90725
253	0.1265	0.0000	0.00067	0.0000	-0.03860	0.0000	0.87494
254	0.1270	0.0000	0.01765	0.0000	-0.03799	0.0000	0.84218
255	0.1275	0.0000	0.03460	0.0000	-0.03689	0.0000	0.80897
256	0.1280	0.0000	0.05147	0.0000	-0.03529	0.0000	0.77534
257	0.1285	0.0000	0.06827	0.0000	-0.03320	0.0000	0.74131
258	0.1290	0.0000	0.08497	0.0000	-0.03062	0.0000	0.70689
259	0.1295	0.0000	0.10154	0.0000	-0.02755	0.0000	0.67209
260	0.1300	0.0000	0.11797	0.0000	-0.02401	0.0000	0.63695
261	0.1305	0.0000	0.13425	0.0000	-0.01999	0.0000	0.60148
262	0.1310	0.0000	0.15034	0.0000	-0.01551	0.0000	0.56569
263	0.1315	0.0000	0.16624	0.0000	-0.01058	0.0000	0.52962
264	0.1320	0.0000	0.18193	0.0000	-0.00519	0.0000	0.49328
265	0.1325	0.0000	0.19738	0.0000	0.00063	0.0000	0.45663
266	0.1330	0.0000	0.21259	0.0000	0.00687	0.0000	0.41986
267	0.1335	0.0000	0.22754	0.0000	0.01353	0.0000	0.38283
268	0.1340	0.0000	0.24221	0.0000	0.02059	0.0000	0.34562
269	0.1345	0.0000	0.25659	0.0000	0.02803	0.0000	0.30824
270	0.1350	0.0000	0.27066	0.0000	0.03583	0.0000	0.27072
271	0.1355	0.0000	0.28441	0.0000	0.04400	0.0000	0.23309
272	0.1360	0.0000	0.29783	0.0000	0.05249	0.0000	0.19536
273	0.1365	0.0000	0.31090	0.0000	0.06130	0.0000	0.15756
274	0.1370	0.0000	0.32361	0.0000	0.07041	0.0000	0.11972
275	0.1375	0.0000	0.33596	0.0000	0.07980	0.0000	0.08184
276	0.1380	0.0000	0.34792	0.0000	0.08945	0.0000	0.04397
277	0.1385	0.0000	0.35949	0.0000	0.09933	0.0000	0.00612
278	0.1390	0.0000	0.37067	0.0000	0.10942	0.0000	-0.03169
279	0.1395	0.0000	0.38143	0.0000	0.11971	0.0000	-0.06942
280	0.1400	0.0000	0.39175	0.0000	0.13016	0.0000	-0.10705
281	0.1405	0.0000	0.40170	0.0000	0.14075	0.0000	-0.14455
282	0.1410	0.0000	0.41119	0.0000	0.15147	0.0000	-0.18196
283	0.1415	0.0000	0.42025	0.0000	0.16227	0.0000	-0.21917
284	0.1420	0.0000	0.42886	0.0000	0.17315	0.0000	-0.25619
285	0.1425	0.0000	0.43702	0.0000	0.18406	0.0000	-0.29299
286	0.1430	0.0000	0.44472	0.0000	0.19499	0.0000	-0.32956
287	0.1435	0.0000	0.45197	0.0000	0.20591	0.0000	-0.36586
288	0.1440	0.0000	0.45875	0.0000	0.21678	0.0000	-0.40188
289	0.1445	0.0000	0.46507	0.0000	0.22759	0.0000	-0.43759
290	0.1450	0.0000	0.47092	0.0000	0.23831	0.0000	-0.47297
291	0.1455	0.0000	0.47631	0.0000	0.24890	0.0000	-0.50799
292	0.1460	0.0000	0.48122	0.0000	0.25934	0.0000	-0.54263
293	0.1465	0.0000	0.48566	0.0000	0.26960	0.0000	-0.57687
294	0.1470	0.0000	0.48963	0.0000	0.27965	0.0000	-0.61069
295	0.1475	0.0000	0.49312	0.0000	0.28947	0.0000	-0.64406
296	0.1480	0.0000	0.49614	0.0000	0.29903	0.0000	-0.67697
297	0.1485	0.0000	0.49870	0.0000	0.30829	0.0000	-0.70938
298	0.1490	0.0000	0.50078	0.0000	0.31724	0.0000	-0.74129

299	0.1495	0.0000	0.50240	0.0000	0.32584	0.0000	-0.77267
300	0.1500	0.0000	0.50355	0.0000	0.33406	0.0000	-0.80349
301	0.1505	0.0000	0.50423	0.0000	0.34189	0.0000	-0.83375
302	0.1510	0.0000	0.50445	0.0000	0.34929	0.0000	-0.86342
303	0.1515	0.0000	0.50423	0.0000	0.35625	0.0000	-0.89248
304	0.1520	0.0000	0.50355	0.0000	0.36272	0.0000	-0.92092
305	0.1525	0.0000	0.50242	0.0000	0.36869	0.0000	-0.94871
306	0.1530	0.0000	0.50084	0.0000	0.37414	0.0000	-0.97585
307	0.1535	0.0000	0.49883	0.0000	0.37904	0.0000	-1.00231
308	0.1540	0.0000	0.49638	0.0000	0.38337	0.0000	-1.02805
309	0.1545	0.0000	0.49350	0.0000	0.38711	0.0000	-1.05314
310	0.1550	0.0000	0.49020	0.0000	0.39023	0.0000	-1.07748
311	0.1555	0.0000	0.48649	0.0000	0.39272	0.0000	-1.10108
312	0.1560	0.0000	0.48238	0.0000	0.39456	0.0000	-1.12394
313	0.1565	0.0000	0.47782	0.0000	0.39572	0.0000	-1.14604
314	0.1570	0.0000	0.47289	0.0000	0.39620	0.0000	-1.16737
315	0.1575	0.0000	0.46758	0.0000	0.39597	0.0000	-1.18791
316	0.1580	0.0000	0.46185	0.0000	0.39502	0.0000	-1.20767
317	0.1585	0.0000	0.45575	0.0000	0.39334	0.0000	-1.22662
318	0.1590	0.0000	0.44929	0.0000	0.39090	0.0000	-1.24476
319	0.1595	0.0000	0.44245	0.0000	0.38771	0.0000	-1.26203
320	0.1600	0.0000	0.43525	0.0000	0.38375	0.0000	-1.27857
321	0.1605	0.0000	0.42772	0.0000	0.37901	0.0000	-1.29424
322	0.1610	0.0000	0.41983	0.0000	0.37348	0.0000	-1.30907
323	0.1615	0.0000	0.41160	0.0000	0.36716	0.0000	-1.32305
324	0.1620	0.0000	0.40304	0.0000	0.36003	0.0000	-1.33619
325	0.1625	0.0000	0.39416	0.0000	0.35210	0.0000	-1.34848
326	0.1630	0.0000	0.38496	0.0000	0.34336	0.0000	-1.35992
327	0.1635	0.0000	0.37545	0.0000	0.33382	0.0000	-1.37051
328	0.1640	0.0000	0.36564	0.0000	0.32346	0.0000	-1.38025
329	0.1645	0.0000	0.35553	0.0000	0.31229	0.0000	-1.38913
330	0.1650	0.0000	0.34514	0.0000	0.30031	0.0000	-1.39717
331	0.1655	0.0000	0.33446	0.0000	0.28753	0.0000	-1.40435
332	0.1660	0.0000	0.32351	0.0000	0.27395	0.0000	-1.41069
333	0.1665	0.0000	0.31229	0.0000	0.25958	0.0000	-1.41618
334	0.1670	0.0000	0.30081	0.0000	0.24443	0.0000	-1.42084
335	0.1675	0.0000	0.28908	0.0000	0.22849	0.0000	-1.42465
336	0.1680	0.0000	0.27710	0.0000	0.21179	0.0000	-1.42758
337	0.1685	0.0000	0.26488	0.0000	0.19434	0.0000	-1.42964
338	0.1690	0.0000	0.25243	0.0000	0.17614	0.0000	-1.43120
339	0.1695	0.0000	0.23975	0.0000	0.15722	0.0000	-1.43176
340	0.1700	0.0000	0.22685	0.0000	0.13759	0.0000	-1.43152
341	0.1705	0.0000	0.21375	0.0000	0.11726	0.0000	-1.43050
342	0.1710	0.0000	0.20040	0.0000	0.09625	0.0000	-1.42871
343	0.1715	0.0000	0.18687	0.0000	0.07458	0.0000	-1.42615
344	0.1720	0.0000	0.17315	0.0000	0.05228	0.0000	-1.42284
345	0.1725	0.0000	0.15923	0.0000	0.02936	0.0000	-1.41880
346	0.1730	0.0000	0.14514	0.0000	0.00584	0.0000	-1.41402
347	0.1735	0.0000	0.13086	0.0000	-0.01824	0.0000	-1.40854
348	0.1740	0.0000	0.11641	0.0000	-0.04287	0.0000	-1.40237
349	0.1745	0.0000	0.10179	0.0000	-0.06802	0.0000	-1.39551
350	0.1750	0.0000	0.08701	0.0000	-0.09367	0.0000	-1.38799
351	0.1755	0.0000	0.07208	0.0000	-0.11978	0.0000	-1.37982
352	0.1760	0.0000	0.05699	0.0000	-0.14633	0.0000	-1.37102
353	0.1765	0.0000	0.04175	0.0000	-0.17329	0.0000	-1.36161
354	0.1770	0.0000	0.02640	0.0000	-0.20063	0.0000	-1.35160
355	0.1775	0.0000	0.01089	0.0000	-0.22832	0.0000	-1.34102
356	0.1780	0.0000	-0.00473	0.0000	-0.25632	0.0000	-1.32988
357	0.1785	0.0000	-0.02047	0.0000	-0.28461	0.0000	-1.31821
358	0.1790	0.0000	-0.03635	0.0000	-0.31315	0.0000	-1.30601

359	0.1795	0.0000	-0.05233	0.0000	-0.34191	0.0000	-1.29332
360	0.1800	0.0000	-0.06842	0.0000	-0.37086	0.0000	-1.28016
361	0.1805	0.0000	-0.08460	0.0000	-0.39996	0.0000	-1.26654
362	0.1810	0.0000	-0.10088	0.0000	-0.42918	0.0000	-1.25248
363	0.1815	0.0000	-0.11725	0.0000	-0.45848	0.0000	-1.23802
364	0.1820	0.0000	-0.13370	0.0000	-0.48783	0.0000	-1.22316
365	0.1825	0.0000	-0.15022	0.0000	-0.51719	0.0000	-1.20794
366	0.1830	0.0000	-0.16682	0.0000	-0.54653	0.0000	-1.19237
367	0.1835	0.0000	-0.18348	0.0000	-0.57581	0.0000	-1.17647
368	0.1840	0.0000	-0.20020	0.0000	-0.60501	0.0000	-1.16028
369	0.1845	0.0000	-0.21697	0.0000	-0.63407	0.0000	-1.14381
370	0.1850	0.0000	-0.23379	0.0000	-0.66297	0.0000	-1.12708
371	0.1855	0.0000	-0.25065	0.0000	-0.69167	0.0000	-1.11012
372	0.1860	0.0000	-0.26754	0.0000	-0.72014	0.0000	-1.09295
373	0.1865	0.0000	-0.28445	0.0000	-0.74835	0.0000	-1.07559
374	0.1870	0.0000	-0.30139	0.0000	-0.77625	0.0000	-1.05807
375	0.1875	0.0000	-0.31835	0.0000	-0.80382	0.0000	-1.04040
376	0.1880	0.0000	-0.33528	0.0000	-0.83102	0.0000	-1.02262
377	0.1885	0.0000	-0.35223	0.0000	-0.85782	0.0000	-1.00474
378	0.1890	0.0000	-0.36916	0.0000	-0.88418	0.0000	-0.98679
379	0.1895	0.0000	-0.38608	0.0000	-0.91009	0.0000	-0.96879
380	0.1900	0.0000	-0.40297	0.0000	-0.93550	0.0000	-0.95076
381	0.1905	0.0000	-0.41982	0.0000	-0.96038	0.0000	-0.93272
382	0.1910	0.0000	-0.43663	0.0000	-0.98471	0.0000	-0.91469
383	0.1915	0.0000	-0.45338	0.0000	-1.00845	0.0000	-0.89669
384	0.1920	0.0000	-0.47007	0.0000	-1.03158	0.0000	-0.87876
385	0.1925	0.0000	-0.48669	0.0000	-1.05408	0.0000	-0.86089
386	0.1930	0.0000	-0.50323	0.0000	-1.07591	0.0000	-0.84313
387	0.1935	0.0000	-0.51967	0.0000	-1.09705	0.0000	-0.82548
388	0.1940	0.0000	-0.53602	0.0000	-1.11747	0.0000	-0.80796
389	0.1945	0.0000	-0.55225	0.0000	-1.13717	0.0000	-0.79060
390	0.1950	0.0000	-0.56838	0.0000	-1.15610	0.0000	-0.77341
391	0.1955	0.0000	-0.58434	0.0000	-1.17425	0.0000	-0.75641
392	0.1960	0.0000	-0.60017	0.0000	-1.19160	0.0000	-0.73962
393	0.1965	0.0000	-0.61585	0.0000	-1.20814	0.0000	-0.72306
394	0.1970	0.0000	-0.63137	0.0000	-1.22384	0.0000	-0.70674
395	0.1975	0.0000	-0.64671	0.0000	-1.23869	0.0000	-0.69066
396	0.1980	0.0000	-0.66165	0.0000	-1.25267	0.0000	-0.67489
397	0.1985	0.0000	-0.67680	0.0000	-1.26577	0.0000	-0.65939
398	0.1990	0.0000	-0.69153	0.0000	-1.27798	0.0000	-0.64419
399	0.1995	0.0000	-0.70604	0.0000	-1.28929	0.0000	-0.62931
400	0.2000	0.0000	-0.72031	0.0000	-1.29968	0.0000	-0.61475
401	0.2005	0.0000	-0.73433	0.0000	-1.30915	0.0000	-0.60054
402	0.2010	0.0000	-0.74805	0.0000	-1.31769	0.0000	-0.58669

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