The Compass and Straightedge in the Teaching of Euclidean Geometry with Applications in Gothic Architecture

by

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ABSTRACT

Constructions with compass and straightedge have been widely used in art and design across many cultures. In this way, Greek geometry (primarily through Euclid's *Elements*) has influenced art and architecture across the world. As translations of Euclid's *Elements* arose in various cultures, it inspired artists and gave them a theory of the idealized shapes that were their basic tools; artists repaid their debt to mathematics by making creative advances in the techniques that begged mathematicians to examine the underlying geometry.

Using a compass and straightedge to recreate geometric designs found in artistic masterpieces are wonderful problems of Euclidean geometry for high school students; they are visually interesting and complex, but based on relatively simple underlying principles. The designs found in Gothic cathedrals are especially well suited for this purpose.

Constructions with straightedge and compass provide limitless opportunities for students to apply the results of Euclidean geometry, they allow students to participate in the mathematical process of discovery and justification, and they are a link between geometry and art, architecture, and a variety of trades.

A survey of constructions with compass and straightedge and an examination of how they can be used to enrich the teaching of high school geometry are provided within the framework of applications to Gothic architecture.

The Compass and Straightedge in the Teaching of Euclidean Geometry with Applications in Gothic Architecture

As a first year teacher, I came across a few basic constructions with straightedge and compass scattered through the text that I was using. They each consisted of step by step instructions on how to complete the desired construction and were accompanied by graphics of a cartoon compass completing the described movements. I faintly recalled performing a few clumsy constructions as a high school student myself, and as an undergraduate I had some exposure to the idea that such constructions had been a focus of the ancient Greek mathematicians who provided us with the foundations of Euclidean geometry. I also possessed a vague understanding that some of the constructions they attempted were not possible, but that this was not proven until relatively modern times. So out of respect to the history of geometry, as we came across these constructions in the text, I showed my students how to manipulate a compass and straight-edge to perform the desired figure. We started with constructing a perpendicular bisector of a segment; they learned how to place an arc above and below a line segment using one endpoint of the segment as a pivot and then the other. In fact, after a few attempts, they became rather adept at estimating where to place the mark so that they could use extremely short arcs to find the points needed to construct the bisector. After these obligatory excursions for some historic flavor, we went back to learning the definitions of geometric figures, their properties, and how these properties could be proven in our adopted axiomatic system.

I didn't give the constructions much more thought until about a year later when I began researching the relationship between art and geometry hoping to find some applications of geometry that would lend themselves to good cumulative projects for high school students. It was then that I began to realize the potential of constructions in teaching students Euclidean geometry. Constructions with straightedge and compass provide limitless opportunities for students to apply the results of Euclidean geometry, they allow students to participate in the mathematical process of discovery and justification, and they are a link between geometry and art, architecture, and a variety of trades. What follows is a survey of constructions with compass and straightedge and an examination of how they can be used to enrich the teaching of high school geometry, including an application to gothic architecture.

I. Constructions in the Curriculum

The rules of the game are simple,

1) Given two points, you can draw a line that passes through the two points. Though the theoretical line extends infinitely, for our physical model this means that we can extend it in either direction as far as is needed.
2) Given two points, you can draw a circle with one of the given points as its center and passing through the other point.

Our physical tools for accomplishing these tasks are the collapsible compass and the unmarked straightedge. By collapsible compass we mean that when the compass is picked up, it collapses, and thus it does not "keep track" of the radius of the circle that it has just drawn. Since the straightedge that we are using is also meant to be unmarked, neither of our tools individually provides the function of carrying a distance from one place in our figure to another, but we often use the ability of our compass to carry distances in our constructions. Are we changing the rules? We don't have to wait long to find the answer when examining Euclid's *Elements*; by the third proposition the equivalence of the collapsible compass and straightedge to our modern compass and straightedge has been established. Let us begin by examining these first propositions of Euclid.

<u>Note</u>: The three constructions that follow are based on the Sir Thomas Heath translation of Euclid's *Elements*. In Heath's translation, some terms are used differently than our modern interpretation; for example, the term "line" is used to describe what we would call a line segment and "equal" is used to mean congruent. I have used modern language where appropriate. I have also chosen a list format for the proofs in keeping with the style in many high school texts. In constructions we utilize points of intersection of lines, segments, rays, circles, and arcs to construct new lines and circles. Though Euclid used this method without justification, in a more complete development of geometry (such as [5]), reasoning should be provided to ensure that these intersections exist. For practical purposes of high school level geometry instruction, this is not typically addressed in constructions and will not be provided in this paper.

E.I.1 *To construct an equilateral triangle with a given line segment as one of its sides.*

Let AB be the given line segment.
Construct circle A_B, the circle centered at A and passing through B.
Construct circle B_A.
Label one of the points of intersection C.
Claim: Triangle ABC is the desired triangle.

Proof: AC is congruent to AB as they are both radii of circle A_B.
BC is congruent to AB as they are both radii of circle B_A.
AC is congruent to BC by the transitive property of congruent segments.
Thus ABC is an equilateral triangle. ■

Notice that this construction didn't require a length/distance to be copied at any step, so this construction could just as easily be accomplish with a collapsible compass (if a physical model existed of such a tool). The same is true of the next two propositions.

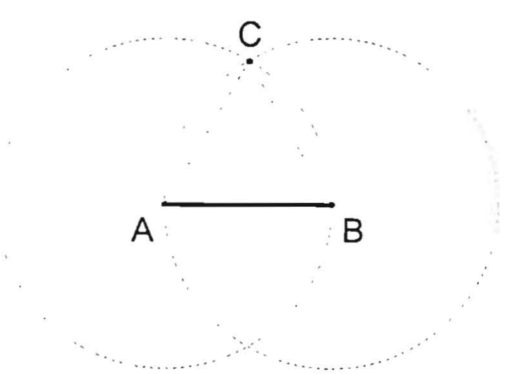


Figure 1.1. First proposition of Euclid. (E.I.1)

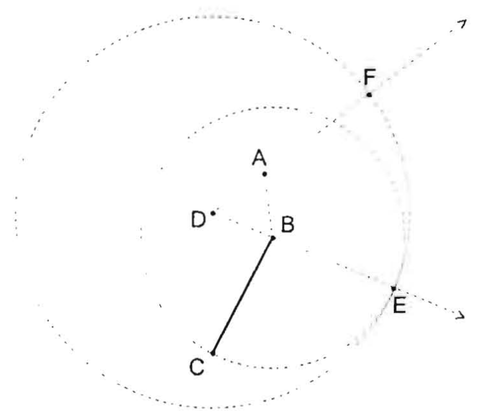


Figure 1.2. Second proposition of Euclid. (E.I.2)

E.I.2 *To place at a given point [as an extremity] a line segment congruent to a given line segment.*

Let A be the given point and BC the given line segment.
Construct line segment AB.
On segment AB, construct equilateral triangle ABD (E.I.1).
Construct circle B_C.
Let E denote the point of intersection of ray DB with circle B_C.
Construct circle with center D and passing through E.
Let F denote the point of intersection of ray DA with circle D_E.
Claim: AF the desired line segment.

Proof: BC is congruent to BE as they are both radii of circle B_C.
DF is congruent to DE as they are both radii of circle D_E.
DA is congruent to DB as they are sides of an equilateral triangle.
The remainder AF is congruent to remainder BE.
AF is congruent to BC by the transitive property of segment congruence. ■

E.I.3 *Given two unequal line segments, to cut off from the greater a segment congruent to the less.*

E.I.3 follows quickly from E.I.2 (for a given larger line segment AX) by simply constructing circle A_F.
The point of intersection of this circle with a ray AX will produce the required cut.

Again notice that Euclid's second and third propositions do not utilize any copying of a distance or length from one part of the figure to another; however, if you look closely at Euclid's third proposition, you will see that the purpose of the construction is to provide a method to "carry" distance with the straightedge and collapsible compass. Once this has been established, we could use this method to copy any length in our construction onto another ray. Unfortunately, the method is a little cumbersome to work through each time we want to carry distance, but since we know that it is possible, we can happily use this ability of our modern compass without having changed the original rules of the game.

Notice that this traditional development allows us to justify our use of a modern compass in performing constructions immediately and without any other geometric results. However, it may not be a logical place to start a high school geometry class. If geometric results are desired before constructions are to be introduced, there is another sequence of constructions that nicely establish the ability of the collapsible compass and straightedge to carry length. This alternative development was presented by Edwin Moise in his text *Elementary Geometry from an Advanced Standpoint*. His presentation, though unlike Euclid's, requires some results of Euclidean geometry to already have been developed. Moise's constructions, M1-M4 below, can be used in place of E.I.1 through E.I.3 to establish the ability to carry distances.

M.1: *To construct the perpendicular bisector of a given segment.*

Let AB be the given line segment.
Construct circle A_B.
Construct circle B_A.
Let C and D denote the points of intersection of A_B with B_A.
Line CD is the desired line.
Let E denote the point of intersection of line CD with line segment AB.
Point E is the bisector of the given segment.

Proof: AC, AD, and AB are congruent as they are radii of circle A_B.
BC, BD, and AB are congruent as they are radii of circle B_A.
Thus AC is congruent to BC and AD is congruent to BD by the transitive property of segment congruence.
So ΔACD is congruent to ΔBCD by SSS.
Angle ACD is congruent to angle BCD as they are corresponding angles in congruent triangles.
ΔACE is congruent to ΔBCE by SAS.
Thus AE congruent to BE by as they are corresponding sides of congruent triangles.
And so E is the desired bisector of AB.
Also, angle AEC is congruent to angle BEC as they are corresponding angles in congruent triangles.
And since they are a linear pair, line CD is perpendicular to line AB. ∎

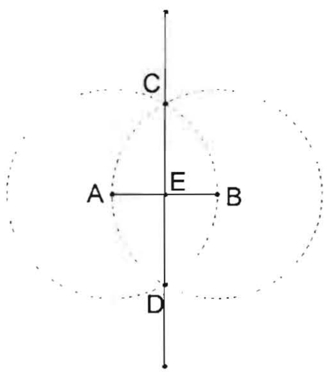


Figure 1.3. First construction of Moise (M.1)

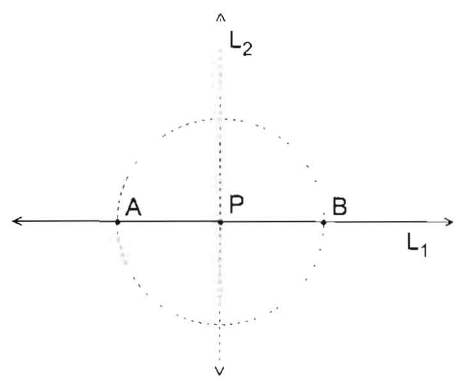


Figure 1.4. Second construction of Moise (M.2)

M.2: *To construct a line perpendicular to a given line, and passing through a given point on the line.*

Let L_1 be the given line and P the given point on this line.
Choose another point A on L.
Construct circle P_A.
The circle P_A will intersect L_1 at A and one other point. Label this other point B.
Construct the perpendicular bisector L_2 of AB as in the previous construction.
Claim: L_2 is the desired line.

M.3: *Given three points A, B, and C to construct rectangle ABED on AB such that AD congruent to AC.*

Let points A, B, and C be the given points.
Construct line AB.
Construct a perpendicular line to AB at A. Label this line l_1.
Construct circle A_C.
Choose one of the points of intersection of l_1 with circle A_C and label it D.
Construct a perpendicular line to l_1 that passes through D. Label it l_2.
Construct a perpendicular line to line AB that passes through B. Label it l_3.
Let E denote the point of intersection of l_2 and l_3.
Claim: ABDE is the desired rectangle.

Proof: Angle BAD is a right angle as l_1 was constructed perpendicular to AB.
Angle ABE is a right angle as l_3 was constructed perpendicular to AB.
Angle ADE is a right angle as l_2 was constructed perpendicular to l_1.
l_1 is parallel to l_3 since they are both perpendicular to AB.
Angle DEB is a right angle since same-side interior angles of parallel lines are supplementary.
Thus ABED is a rectangle.
And AD is congruent to AC as they are radii of circle A_C. ■

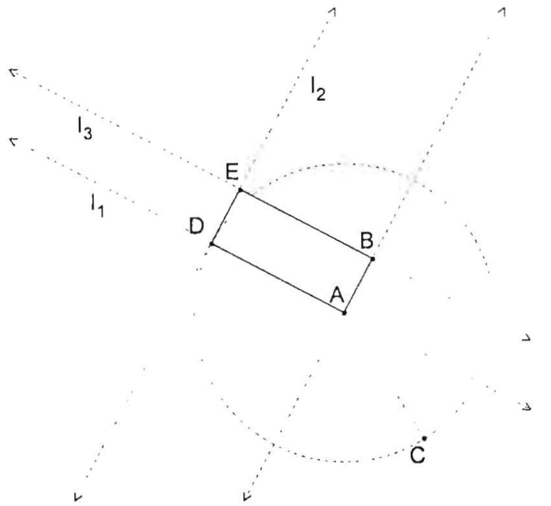


Figure 1.5. Construction M.3

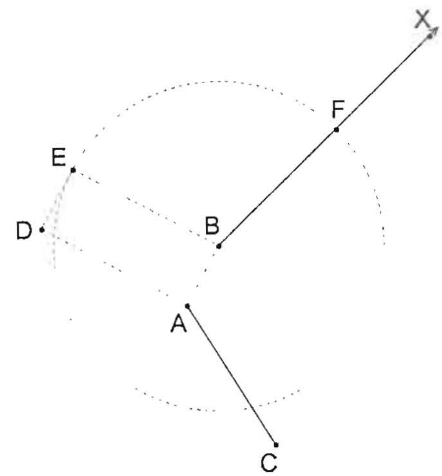


Figure 1.6. Construction M.4

M.4: *Given a line segment and a ray, to construct a segment on the ray congruent to the given segment.*

Let AC be the given segment and BX the given ray.
Construct segment AB.
Construct a rectangle ABED on AB such that AD is congruent to AC.
Construct circle B_E.
Let F denote the intersection of circle B_E with ray BX.
Claim: BF is the desired segment.

Proof: AC congruent to AD by construction.
AD congruent to BE as they are opposite sides of a rectangle.
BE congruent to BF as they are radii of circle B_E.
Thus Ac is congruent to BF by the transitive property of segment congruence. ■

Again, the ability of the collapsible compass and straightedge to transfer lengths has been established. This development requires a few more constructions, but these constructions are time well spent as M.1 and M.2 are fundamental constructions that the majority of later constructions will utilize. These four constructions also provide applications of properties of parallel lines and rectangles as well as triangle congruence postulates.

Whichever development you choose, we have certainly established the validity of using a modern compass to complete our constructions. Let us now look at how geometric constructions can be used to teach students mathematics.

Basic constructions and the instructional method

Students receive many benefits from learning compass and straightedge constructions. Constructions provide an opportunity for students to experience genuine mathematical methods while applying geometric properties. Students can look for construction methods to accomplish the desired outcome (discovery/problem solving), and then write proofs to explain how they know the construction produces the desired result (formal justification based on an axiomatic system). Meanwhile, they are applying the properties of, and relationships between, geometric figures. In other words, constructions can help students to mature mathematically as well as master the content objectives.

Before beginning constructions, students will need to be introduced to the compass and straightedge that they will be using, and the rules must be explained. These rules are repeated here for convenience:

1) Given two points, you can draw a line that passes through the two points.
2) Given two points, you can draw a circle with one of the given points as the center and passing through the other point.

The discussion of the collapsible compass may be left until later when they are more familiar with the concept of constructions. However, after the definition of a circle is discussed and they have had some time to practice using the tools, the ability of the compass to mark off congruent distances with congruent circles should be noted. They are now ready to begin exploring constructions.

The first step in any construction is to pose the problem to the students. The students must then clarify what elements they are starting with and what they will need to construct. After they have explored the problem for a time, the students may get stuck and need a hint as to how the construction may be accomplished. Finding an appropriate technique will not be obvious to most students, and many of the techniques that we employ were discovered and improved upon through the years, so the students may not discover them independently in a short amount of time. Having the students work in small groups may also be helpful. The hints are meant to illuminate a key idea in a common method of constructing the figure, but the students will have to use geometric properties to complete the construction and to feel confident that it produces the desired result. There are often multiple ways that the constructions can be accomplished, and students should be encouraged to explore alternate methods, but they must be able to justify that these alternate methods produce the required figure. After a method has been found, the students should write up a thorough description of the method (including a labeled figure) and then complete a formal justification of why the method produces the desired result. This justification could take a variety of forms depending on the role of proof in their course; here the standard two-column proof format will be used.

Though Euclid began his development of geometry with three constructions (E.I.1–E.I.3), he then went on to prove a variety of results before returning to constructions, which he interspersed throughout *Elements*. The basic constructions rely on triangle congruence postulates or other results, so it seems natural to introduce constructions to our students shortly after they have studied triangle congruence and to explore more complex constructions after the required geometric results are developed. Let us examine the basic constructions that would provide a good introduction to students. We begin with a construction E.I.1 that we have already examined, but with a look at how we could lead students through its development.

C.1. *To construct an equilateral triangle with a given segment as one side.*

Task:

Let AB be the given segment. We wish to construct an equilateral triangle ABC.
What conditions does a figure have to meet to be an equilateral triangle?

(We need to find where to place the third vertex so that it is equidistant to the given endpoints.)

Hints:
Consider all points that are the same distance from A as B is from A. Where are these points?
Consider all points that are the same distance from B as A is from B. Where are these points?
Are there any points that satisfy both requirements?

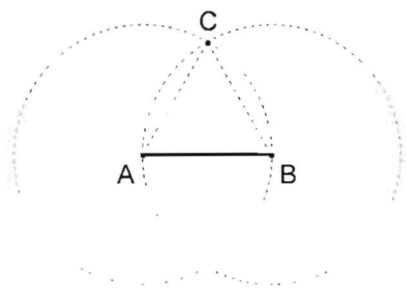


Figure 1.7

The construction and justification are provided in E.I.1 earlier. Notice that the construction and justification reinforce the idea of a circle as a locus of points and the definition of an equilateral triangle. This is a good construction to begin with as it can be introduced early in the development of Euclidean geometry and is very intuitive. The next construction we will examine is not the second of Euclid, but is extremely useful as it is useful in many other constructions.

C.2. *To construct an angle on a given ray that is congruent to a given angle.*

Task:
Let angle A be the given angle and PQ the given ray.
We want to find a point R so that the angle QPR is congruent to angle A.

Hints:
Consider the given angle as one of the angles of a triangle.
What could we use to form a congruent triangle on the given ray? (SAS, ASA, AAS, SSS)
Why wouldn't the SAS postulate, the ASA postulate, or the AAS theorem be practical?
If we want to use the SSS postulate, how can we form congruent sides?

Construction:
Let angle A be the given angle and PQ the given ray.
Construct a circle with center P and passing through Q.
Construct a circle with center A and radius of length PQ.
Let D and E denote the points of intersection of circle A with the rays of angle A.
Construct a circle with center Q and radius of length DE.
Let R denote the point of intersection of this circle and circle P_Q.
Construct ray PR.
Claim: Angle QPR is the desired angle.

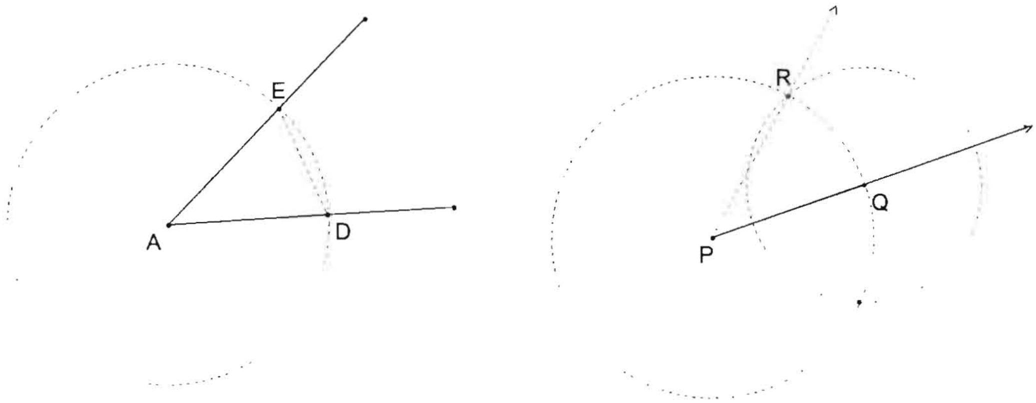


Figure 1.8

Justification:

Given: Angle QPR constructed as described.
Show: Angle QPR is congruent to angle A.

Statement	Reason
1. $\angle QPR$ constructed as described	1. Given
2. Circles A and P_Q are congruent.	2. Def. of congruent circles. (Circle A constructed with same radius as P_Q.)
3. $\overline{PQ} \cong \overline{AD}$, $\overline{PR} \cong \overline{AE}$	3. Radii of congruent circles are congruent.
4. $QR = DE$	4. Circle Q_R was constructed with radius of length DE
5. $\overline{QR} \cong \overline{DE}$	5. Def. of segment congruence.
6. $\Delta DAE \cong \Delta QPR$	6. SSS Triangle Congruence Postulate
7. $\angle DAE \cong \angle QPR$	7. Corresponding parts of congruent triangles are congruent.

This construction may have been more difficult for the students to discover independently, but with some hints, they should be able to see the logic used and begin to get a feel for the thought process in trying to develop constructions. After working through this example, they can apply similar ideas to accomplish the next construction.

C.3. *To bisect a given angle.*

Task:
Let BAC be the given angle.
What needs to be true to have an angle bisector?
(We must construct a ray in the interior of angle BAC so that it cuts the angle into two congruent parts.)

Hints:
Imagine the desired bisector.
How can we make the two angles congruent?
Consider the two angles as angles in congruent triangles.
How could we form these congruent triangles?
Which triangle congruence postulate/theorem would be helpful?
How could we meet the requirements of this postulate/theorem?

Construction:
Let angle BAC be the given angle.
Construct circle A_B.
Let D denote the point of intersection of A_B and ray AC.
Construct an equilateral triangle on DB. Label the third vertex E.
Claim: Ray AE is the desired bisector.

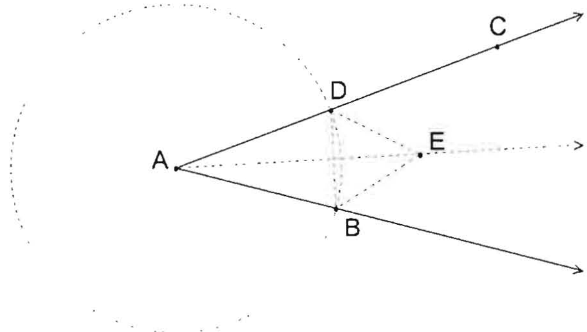


Figure 1.9

Justification:

Given: Ray AE constructed as above.
Show: Ray AE bisects angle BAC.

Statement	Reason
1. Ray AE constructed as above.	1. Given
2. $\overline{AD} \cong \overline{AB}$	2. All radii of a circle are congruent.
3. ΔDEB is equilateral.	3. By construction.
4. $\overline{DE} \cong \overline{BE}$	4. Def of equilateral triangle.
5. $\overline{AE} \cong \overline{AE}$	5. Reflexive property of segment congruence.
6. $\Delta DAE \cong \Delta BAE$	6. SSS Triangle Congruence Postulate
7. $\angle DAE \cong \angle BAE$	7. Corresponding angles of congruent triangles are congruent

The use of the equilateral triangle in the previous construction may not be the first suggested method (congruent circles D_B and B_D could also be used to get the required point E), but it is a nice example of how constructions can call upon earlier constructions in a similar fashion to theorems building upon one another to provide a more concise representation. The next construction makes use of C.2.

C.4 *To construct a line parallel to a given line and through a given point.*

Task:
Let L_1 be the given line and P the given point not on L_1.
We want to construct a line though P that is parallel to L_1.

Hints:
Imagine the desired line.
What theorems do we have that conclude that a pair of lines is parallel?
What are the requirements of these theorems?
Where might we want to construct a transversal?

Construction:
Let L_1 be the given line and P the given point not on L_1.
Choose two points on L_1. Let A and B denote these two points.
Construct line AP.
Construct a point C on line AP so that A-P-C.
Construct a point D so that the angle CPD is congruent to angle PAB (C.2) and they are corresponding angles with respect to lines PD and L_1 and transversal AP.
Construct line PD.
Claim: PD is the desired line.

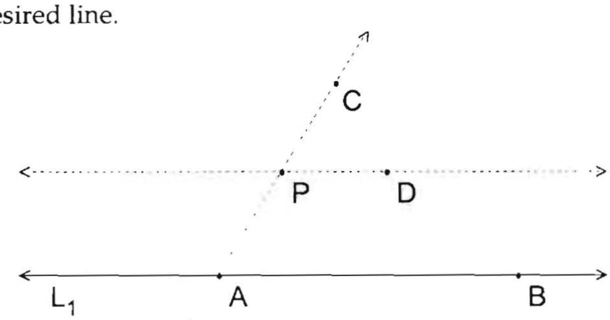


Figure 1.10

Justification:

Given: Line PD constructed as above.
Show: PD passes through P and is parallel to L_1.

Statement	Reason
1. Line PD constructed as above.	1. Given
2. $\angle PAB$ and $\angle CPD$ are corresponding angles with respect to lines L_1 and PD and transversal AP.	2. By construction.
3. $\angle PAB \cong \angle CPD$	3. By construction. (C.2)
4. Lines PD and L_1 are parallel.	4. If two lines are crossed by a transversal and corresponding angles are congruent, then the lines are parallel.
5. Line PD passes through P.	5. By construction.

Parallel lines could be formed in a similar manner by constructing the congruent angle in a different position so that are alternate interior or alternate exterior angles are congruent. Our ability to copy angles (along with our capacity to transfer lengths) also provides us with a variety of methods to construct congruent triangles.

C.5 *To construct a triangle congruent to a given triangle. (With a given point as a vertex and a side on a given ray.)*

Task:
Let ABC be the given triangle and PQ the given ray.
We need to find a point R on ray PQ and a point S not on line PQ so that triangle PRS is congruent to triangle ABC.

Hints:
How do we know when two triangles are congruent?
How can we construct a triangle that meets these requirements?

Method 1: Using the side-side-side (SSS) triangle congruence postulate

Hints:
Mark the point R on PQ so that PR is the same length as AB.
What points are at a distance CA from P?
What points are at a distance BC from R?
What point(s) satisfy both requirements?

Construction:
Let ABC be the given triangle and ray PQ the given ray.
Construct a point R on ray PQ so that PR $\cong$ AB.
Construct a circle with center P and radius equal to AC.
Construct a circle with center R and radius equal to BC.
Let S denote one of the points of intersection of circle P and circle R.
Claim: Triangle PRS is the desired triangle.

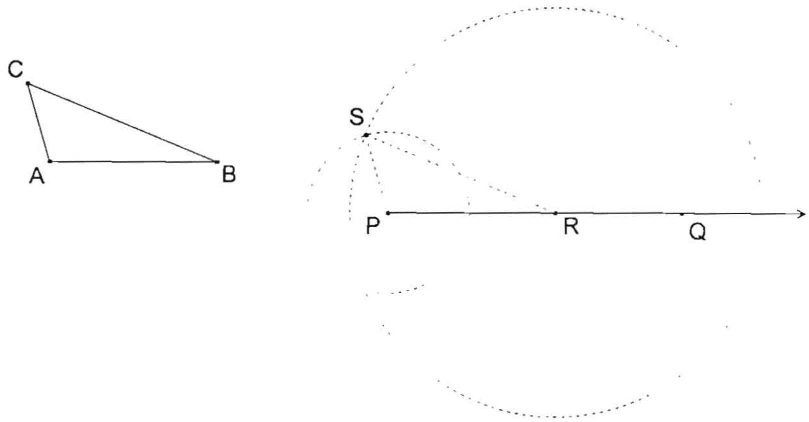


Figure 1.11

Justification:

Given: Triangle *PRS* constructed as above.
Show: Triangle *PRS* congruent to triangle *ABC*.

Statement	Reason
1. Triangle *PRS* constructed as above.	1. Given
2. $\overline{PR} \cong \overline{AB}$	2. By construction.
3. $PS = AC$	3. Circle P_S was constructed with radii of length AC.
4. $\overline{PS} \cong \overline{AC}$	4. Def of segment congruence
5. $RS = BC$	5. Circle R_S was constructed with radius of length BC.
6. $\overline{RS} \cong \overline{AC}$	6. Def of segment congruence.
7. $\Delta PRS \cong \Delta ABC$	7. SSS Triangle Congruence Postulate

Method 2: Using the side-angle-side (SAS) triangle congruence postulate.

Construction:
Let triangle ABC be the given triangle and let ray PQ be the given ray (P the given point).
Construct an angle on ray PQ congruent to angle BAC.
Let r_1 denote the ray constructed.
Construct a point R on ray PQ so that PR $\cong$ AB.
Construct a point S on r_1 so that PS $\cong$ AC.
Claim: Triangle PRS is the desired triangle.

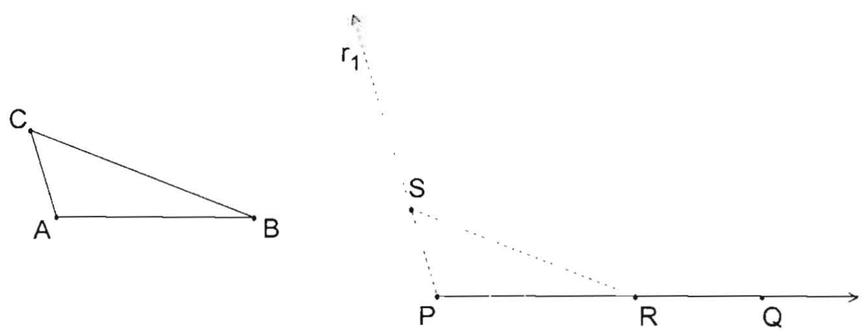


Figure 1.12

Justification:

Given: Triangle *PRS* constructed as above. **Show:** Triangle *PRS* congruent to triangle *ABC*.	
Statement	Reason
1. Triangle *PRS* constructed as above.	1. Given
2. $\overline{PR} \cong \overline{AB}$	2. By construction.
3. $\angle RPS \cong \angle BAC$	3. By construction.
4. $\overline{PS} \cong \overline{AC}$	4. By construction.
5. $\Delta PRS \cong \Delta ABC$	5. SAS Triangle Congruence Postulate

Method 3: Using the angle-side-angle (ASA) triangle congruence postulate.

Construction:
Let triangle ABC be the given triangle and ray PQ the given ray.
Construct point R on ray PQ so that PR ≅ AB.
Construct ray L_1 from P so that angle P is congruent to angle A.
Construct ray L_2 from R (on the same side of line PQ as L_1) so that angle formed on ray RP is congruent to angle B.
Let S denote the point of intersection of L_1 and L_2.
Claim: Triangle PRS is the desired triangle.

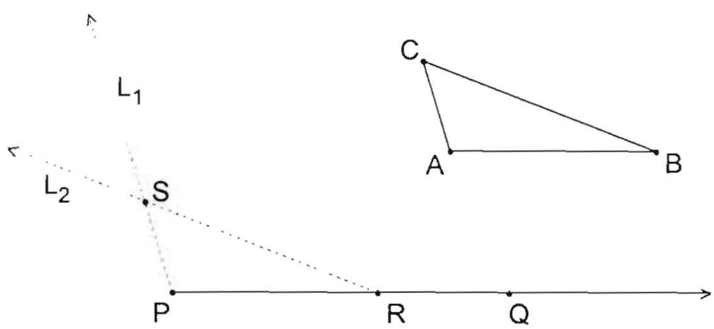


Figure 1.13

Justification:

Given: Triangle *PRS* constructed as above. **Show:** Triangle *PRS* congruent to triangle *ABC*.	
Statement	Reason
1. Triangle *PRS* constructed as above.	1. Given
2. $\overline{PR} \cong \overline{AB}$	2. By construction.
3. $\angle P \cong \angle A$	3. By construction.
4. $\angle R \cong \angle B$	4. By construction.
5. $\Delta PRS \cong \Delta ABC$	5. ASA Triangle Congruence Postulate

Though method 1 requires some creativity, the other two are a direct application of the postulates being used. Students may also suggest the SAA triangle congruence theorem, but when they attempt to use it, they will quickly see that it is not practical for this purpose. Another basic ability that we will want to have at our disposal is that of constructing a perpendicular line.

C.6. *To construct a perpendicular bisector of a given segment.*

Task:
Let AB be the given segment.
Construct a line that is perpendicular to AB and passes through its midpoint.

Hints:
Consider the construction for an equilateral triangle on this segment.
If you connect the two points of intersection of the circles it looks like it might be a perpendicular bisector to the segment. How could we justify this?
Consider the triangles formed.

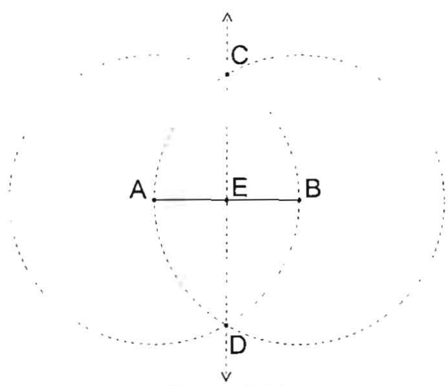


Figure 1.14

The construction and justification for C.6 are given as M.1 above. Notice that this construction would be very difficult for students to discover, but it is a good exercise in justification that requires using the characteristics of the larger triangles DAC and DBC to find properties of the smaller triangles, a common technique used in high school geometry proofs. With the technique to construct a perpendicular bisector of a segment established, the next two constructions follow easily.

C.7. *To construct a line perpendicular to a given line and through a point on the given line.*

Task:
Let L_1 be the given line and P the given point.
Construct a line perpendicular to L_1 and passing through P.

Hints:
What makes this construction different from C.4?

Construction:
Let L_1 be the given line and P the given point on L_1.
Choose another point A on L_1.

Construct circle P_A.
Let B denote the other point of intersection of circle P_A with L_1.
Construct the perpendicular bisector L_2 of segment AB.
Claim: L_2 is the desired line.

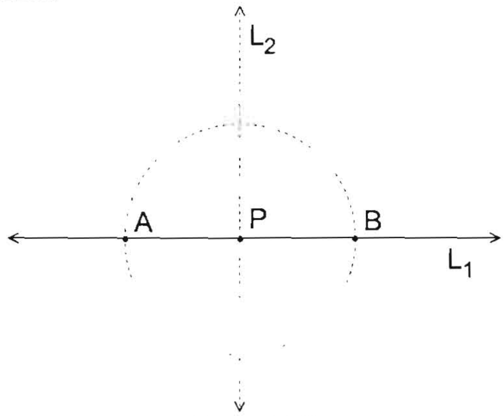


Figure 1.15

Justification:

Given: L_2 constructed as above.
Show: L_2 passes through point P and is perpendicular to L_1.

Statement	Reason
1. L_2 constructed as described.	1. Given
2. $\overline{PA} \cong \overline{PB}$	2. Radii of a circle are congruent.
3. $PA = PB$	3. Def of segment congruence.
4. P is the midpoint of AB	4. Def of midpoint.
5. L_2 is the perpendicular bisector of AB	5. By construction.
6. L_2 is perpendicular to L_1	6. Def of perpendicular bisector.
7. L_2 passes through P	7. Def of perpendicular bisector.

C.8. *To construct a line perpendicular to a given line and through a point not on the given line.*

Task:
Let L_1 be the given line and P the given point (not on L_1).
We want to construct a line that passes through P and is perpendicular to L_1.

Hints:
We know how to construct a perpendicular bisector.
How could we find two points on L_1 so that the perpendicular bisector of the segment formed would pass through P?

Construction:
Let L_1 be the given line and P the point not on the line.
Construct a circle P that intersects L_1 in two points.

Let R and Q denote these points of intersection of circle P and L_1.
Construct the perpendicular bisector of RQ, label it L_2.
Let M denote the point of intersection of L_2 and L_1.
Claim L_2 is the desired line.

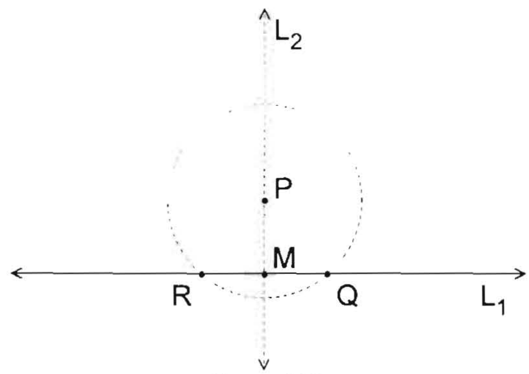


Figure 1.16

Justification:

Given: L_2 constructed as above.
Show: L_2 perpendicular to L_1 and passes through P.

Statement	Reason
1. L_2 constructed as above.	1. Given
2. L_2 is the perpendicular bisector of RQ.	2. By construction.
3. L_2 is perpendicular to L_1.	3. Def. of perpendicular bisector.
4. $\overline{RM} \cong \overline{QM}$	4. Def of perpendicular bisector.
5. $\overline{PR} \cong \overline{PQ}$	5. All radii of a circle are congruent.
6. $\overline{PM} \cong \overline{PM}$	6. Reflexive property of segment congruence.
7. $\Delta RMP \cong \Delta QMP$	7. SSS Triangle Congruence Postulate
8. $\angle RMP \cong \angle QMP$	8. Corresponding angles of congruent triangles congruent
9. $\angle RMP$ and $\angle QMP$ are a linear pair	9. Definition of linear pair.
10. $\angle RMP$ and $\angle QMP$ are right angles	10. Definition of right angle.
11. Line PM is perpendicular to L_1	11. Definition of perpendicular lines.
12. P is on L_2	12. Through a point on a line, there is exactly one perpendicular line passing through the point.

Once perpendiculars have been developed, we can move on to the construction of our second regular polygon, the square. Probably the most intuitive method is to construct perpendiculars to segment AB through both of its endpoints, and then to mark off a segment congruent to AB on one of these lines and create a perpendicular through this new point. A method that is slightly less obvious, but more practical for drawing [6], is to construct a perpendicular through one of the endpoints, say B, mark off a point C on this perpendicular so that BC congruent to AB and then to construct two circles passing through B with centers at A and C. The intersection of these two circles will provide the final vertex D for the square.

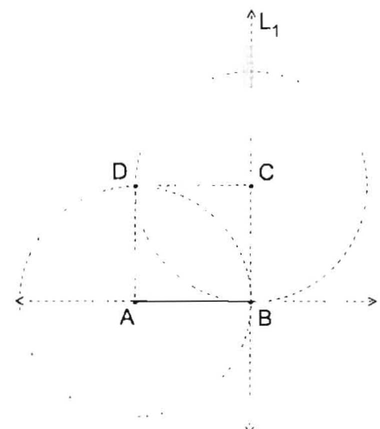


Figure 1.17

It may be worthwhile to let students work through the most obvious construction which requires three perpendiculars, and then to encourage them to find methods that are more economic for drawing (such as the one presented in detail below which requires two perpendiculars to be constructed or the even more concise method above.)

C.9. *To construct a square with a given segment as a side.*

Task:
Let AB be the given segment.
We wish to construct points C and D so that ABCD a square.

Hints:
What do we need to be true about figure ABCD so that it is a square?

Construction:
Let AB be the given segment.
Construct a line L_1 perpendicular to AB and passing through point A.
Construct a line L_2 perpendicular to AB and passing through point B.
Construct a point C on L_2 so that BC=AB.
Construct a point D on L_1 so that AD=AB.
Construct segment CD.
Claim: Figure ABCD is a square.

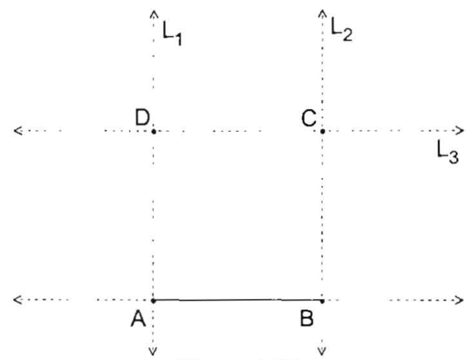


Figure 1.18

Justification:

Given: Figure $ABCD$ constructed as described.
Show: Figure $ABCD$ a square.

Statement	Reason
1. Figure $ABCD$ constructed as described.	1. Given
2. $\overline{AB} \cong \overline{BC}$, $\overline{AB} \cong \overline{AD}$	2. By construction.
3. $\overline{BC} \cong \overline{AD}$	3. Transitive property of segment congruence.
4. $L_1 \perp AB$, $L_2 \perp AB$	4. By construction.
5. $\angle DAB$ and $\angle CBA$ are right angles	5. Def of perpendicular lines.
6. $m\angle DAB = 90$, $m\angle CBA = 90$	6. Def. of right angles.
7. $m\angle DAB + m\angle CBA = 180$	7. Substitution
8. $\angle DAB$ is supplementary to $\angle CBA$	8. Def. of supplementary.
9. $L_1 \| L_2$	9. If two lines are crossed by a transversal and same-side interior angles are supplementary, then the lines are parallel.
10. Quadrilateral $ABCD$ a parallelogram.	10. In a quadrilateral, if a pair of opposite sides are both congruent and parallel, then the quadrilateral a parallelogram.
11. $\angle ABC \cong \angle ADC$, $\angle DAB \cong \angle DCB$	11. In a parallelogram, opposite angles are congruent.
12. $m\angle ABC = m\angle ADC$, $m\angle DAB = m\angle DCB$	12. Def of angle congruence.
13. $90 = m\angle ADC$, $90 = m\angle DCB$	13. Substitution
14. $\angle ADC$ and $\angle DCB$ are right angles	14. Def. of right angle

15. $\overline{AB} \cong \overline{CD}$	15. In a parallelogram, opposite sides are congruent.
16. $\overline{AD} \cong \overline{BC} \cong \overline{AB} \cong \overline{CD}$	16. Substitution.
17. Quadrilateral $ABCD$ is a square.	17. Def. of a square.

Notice that C.9 has also provided another way to construct parallel lines. This completes the first round of basic constructions that could be introduced after triangle congruence postulates were established. The next construction requires results on similar triangles.

C.10 *To cut a given segment into n congruent parts.*

Task:
Let AB be the given segment. We will first consider cutting it into five congruent parts. We will need to find 4 points equally spaced along AB.

Hints:
Consider a ray from A that is noncollinear with AB.
How could we find five equally spaced points along the ray?
Is there a way that we could use these points to find the desired points on AB?
Connect the last of these points to B. Now can you see a way?

Construction:
Let AB be the given segment.
Construct ray AX where X is not on line AB.
Choose a point C on ray AX.
Mark off the distance AC on ray CX. Let D denote this new point.
Mark off the distance AC on ray DX. Let E denote this new point.
Continue along ray EX, obtaining two more points F and G.
Construct segment GB.
Construct a line parallel to GB and passing through F.
Let H denote the point of intersection of this line and AB.
Construct parallel lines to GB through points F, E, D, and C.
Let H, I, J, and K respectively denote the intersections formed by these lines and segment AB.
Claim: The segments AK, KJ, JI, IH, and HB are the desired segments.

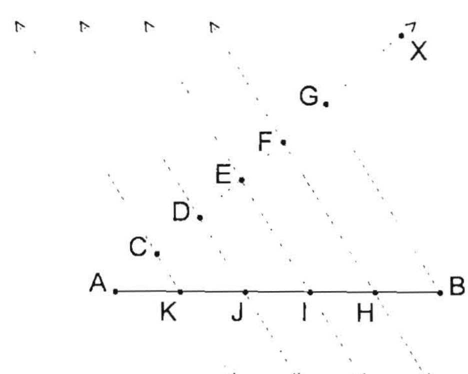


Figure 1.19

Justification:

Given: Segments AK, KJ, JI, IH, and HB constructed as above. **Show:** $\overline{AK} \cong \overline{KJ} \cong \overline{JI} \cong \overline{IH} \cong \overline{HB}$.	
Statement	Reason
1. $\overline{AK}$, $\overline{KJ}$, $\overline{JI}$, $\overline{IH}$, and $\overline{HB}$ constructed as described.	1. Given
2. Lines FH, EI, DJ, and CK are all parallel to line GB.	2. By construction
3. $AC = CD = DE = EF = FG$	3. By construction.
4. $AC : CD : DE : EF : FG = 1:1:1:1:1$	4. Def. of proportional.
5. $AK : KJ : JI : IH : HB = 1:1:1:1:1$	5. If three or more parallel lines intersect two transversals, they cut the transversals proportionally.
6. $AK = KI = JI = IH = HB$	6. Proportion Notation
7. $\overline{AK} \cong \overline{KI} \cong \overline{JI} \cong \overline{IH} \cong \overline{HB}$	7. Def. of segment congruence.

Though this construction cut the segment into five congruent segments, the same method can of course be used to cut a segment into any finite number of congruent pieces. The next construction follows easily from the result that a line tangent to a circle is perpendicular to the radius of the circle at the endpoint on the circle.

C.11 *To construct a line tangent to a given circle and passing through a given point on the circle.*

Task:
Let circle A be the given circle and let B be the given point on it.
We wish to construct a line through B that is tangent to circle A at B.

Hints:
What do we know about the relationship between a circle and its tangent lines?

Construction:
Let circle A be the given circle and point B the given point on the circle.
Construct ray AB.
Construct L_1 perpendicular to line AB and passing through point B.
Claim: L_1 is the desired line.

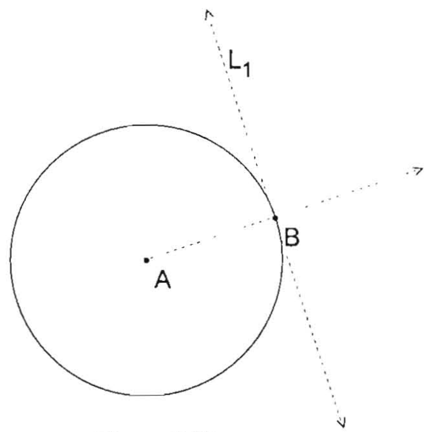


Figure 1.20

Justification:

Given: L_1 constructed as above. **Show**: L_1 tangent to circle A.	
Statement	Reason
1. L_1 constructed as described.	1. Given
2. L_1 perpendicular to line AB.	2. By construction.
3. L_1 is tangent to circle A.	3. If a line is perpendicular to the radius of a circle at the endpoint on the circle, then the line is a tangent to the circle.

Constructing a line tangent to a circle from a point not on the circle requires a little more ingenuity, but is still accessible with a few hints.

C.12 *To construct a line tangent to a given circle and passing through a given point not on the circle.*

Task:
Let B be the given circle and A the given point not on circle B.
We need to find a point on circle B so that the line passing through this point and point A is tangent to circle B.

Hints:
What do we know about the relationship between a circle and its tangent lines?
So what would we need to be true about the triangle with A, B, and the point of tangency as vertices?
In what situations have we found that we have a right triangle?
Where would the center of this circle need to be?
Construct this circle. What point(s) would complete the desired triangle?
So how can you now construct the tangent line(s)?

Construction:
Let B be the given circle and A the given point.
Construct segment BA.

Bisect segment BA, let C denote the midpoint.
Construct circle C_B.
Let D denote the point of intersection of circle C_B with circle B.
Construct line AD.
Claim: Line AD is the desired line.

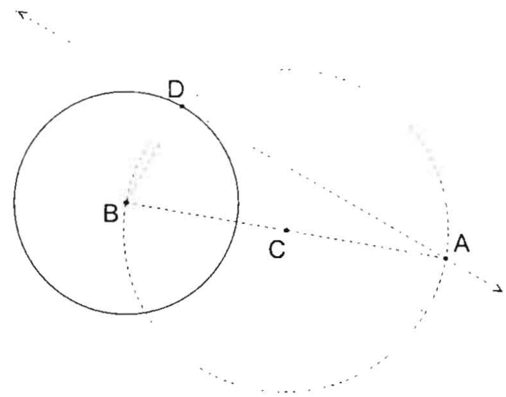


Figure 1.21

Justification:

Given: Line AD constructed as above.
Show: Line AD tangent to circle B.

Statement	Reason
1. Line AD constructed as described.	1. Given
2. C is the midpoint of AB	2. By construction.
3. $CB=CA$	3. Def. of midpoint.
4. Circle C_B contains A.	4. Def of circle.
5. Segment BA is a diameter of circle C_B.	5. Def. of diameter of a circle.
6. Arc BDA is a semicircle	6. Def of semicircle.
7. Angle BDA is a right angle	7. If an inscribed angle in a circle intercepts a semicircle, then the angle is a right angle.
8. Segment BD is perpendicular to DA	8. Def of right angle.
9. Line AD is tangent to circle B.	9. If a line is perpendicular to the radius of a circle at the endpoint on the circle, then the line is a tangent to the circle.

This concludes the basic toolkit of constructions that students will need to begin exploring applications. However, there is another construction that is instructional to students.

C.13 *To construct a triangle from three segments.*

Task:
Let AB, CD, and EF be the given segments.
We want to construct a triangle with sides of length AB, CD, and EF on ray PQ.

Hints:
Begin with one side of the triangle.
How can you find the third vertex?

Construction:
Let AB, CD, and EF be the given segments.
Mark off a distance EF from P on ray PQ.
Let R denote this point.
Construct a circle with center R and radius CD.
Construct a circle with center P and radius AB.
Let S denote one of the points of intersection of circle P and R.
Claim: PRS is the desired triangle.

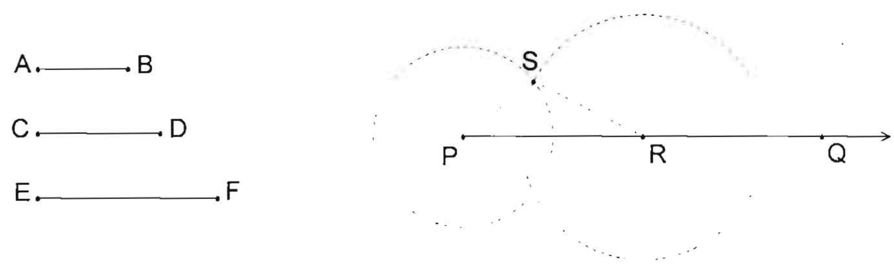


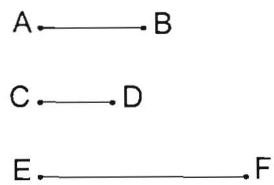
Figure 1.22

Justification:

Given: Triangle *PRS* constructed as above.
Show: *PR=EF, RS=CD*, and *SP=AB*.

Statement	Reason
1. Triangle *PRS* constructed as described.	1. Given
2. *PR=EF*	2. By construction.
3. *RS=CD*	3. Circle R_S was constructed with radius of length *CD*.
4. *SP=AB*	4. Circle P_S was constructed with radius of length *AB*.

Though this construction is quite simple, if we change the lengths of the given segments, we may certainly have a problem as you can see in figure 1.23 below where segment CD has been shortened. This is a simple illustration for students of why a more thorough development of geometry is sometimes desired.



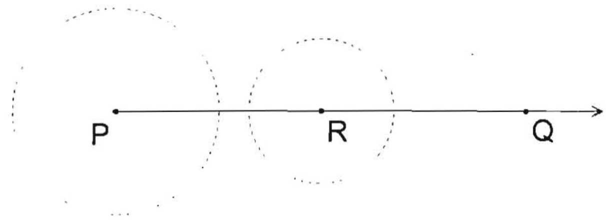


Figure 1.23

II. An Application: Gothic Tracery

Constructions with compass and straightedge have been widely used in art and design across many cultures. In this way, Greek geometry (primarily through Euclid's *Elements*) has influenced art and architecture across the world. The architecture of ancient Greece was designed around ideal geometric proportions that are still used in art today for their pleasing appearance. The stroke of the compass also arises in gothic architecture, Islamic tile work, paintings of the Renaissance, and even Buddhist sand mandalas; it is present almost anywhere we have sought to beautify the world around us with geometric design. The use of the compass by artists and artisans has caused the development of Euclidean geometry to become intertwined with the development of art and fine crafts. As translations of *Elements* arose in various cultures, it inspired artists and gave them a theory of the idealized shapes that were their basic tools; artists repaid their debt to mathematics by making creative advances in the techniques that begged mathematicians to examine the underlying geometry.

Geometric designs found in artistic masterpieces often lend themselves to wonderful problems of Euclidean geometry. They are visually interesting and complex, but based on relatively simple underlying principles. The designs found in gothic cathedrals are especially well suited for this purpose, as they are readily available for examination with countless texts and websites devoted to their documentation. Once students have mastered the basic constructions they can access endless possibilities in design.

Gothic architecture refers to a style of architecture that began in 12th century France and quickly spread throughout Europe where it remained the prominent style for cathedrals built during the next three-hundred years. Gothic architects utilized flying buttresses and the pointed arch to support soaring ribbed vault ceilings and large expanses of stained glass with delicate tracery. On the exterior, they used sharply pointed spires to further emphasize the vertical reach. Intricate carvings of circular designs, gargoyles, saints, and biblical scenes covered the stonework.

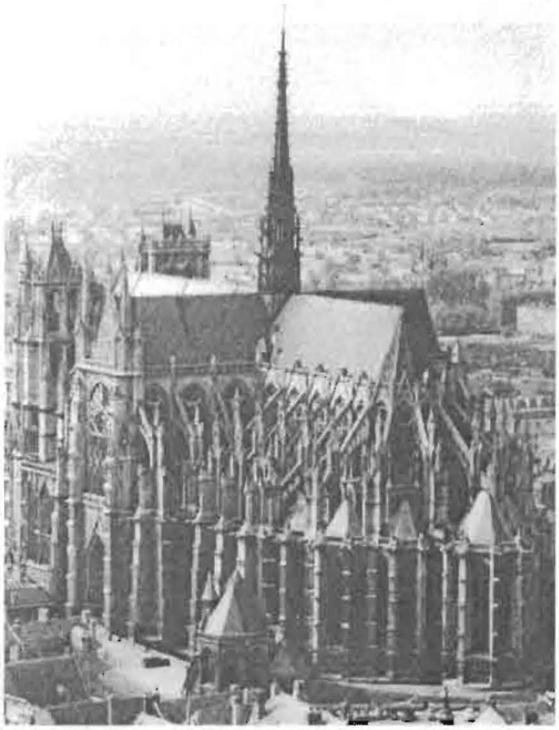


Figure 2.1. Photograph by Jacques Boulas.*

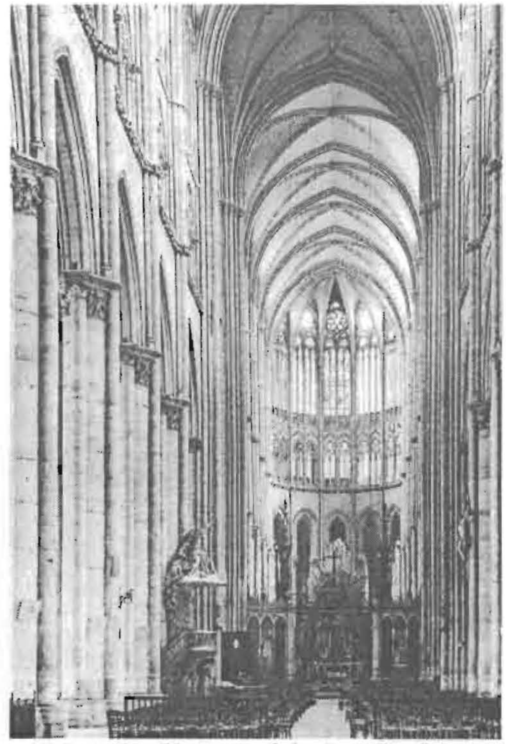


Figure 2.2. Photograph by Jean Roubier.*

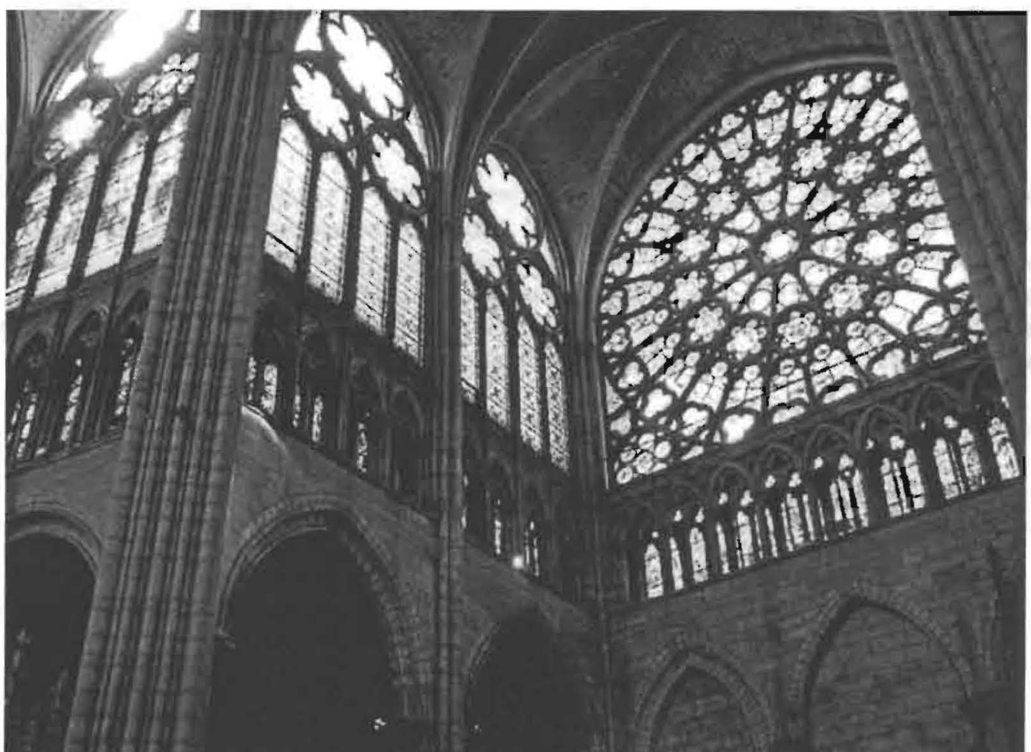


Figure 2.3. St. Denis Abbey, Paris. Photograph by Chuck LaChiusa**.

* *The Horizon Book of Great Cathedrals*, Jay Jacobs, editor in charge, American Heritage Publishing, 1968.

** *Architecture Around the World*, http://ah.bfn.org/. Used with permission.

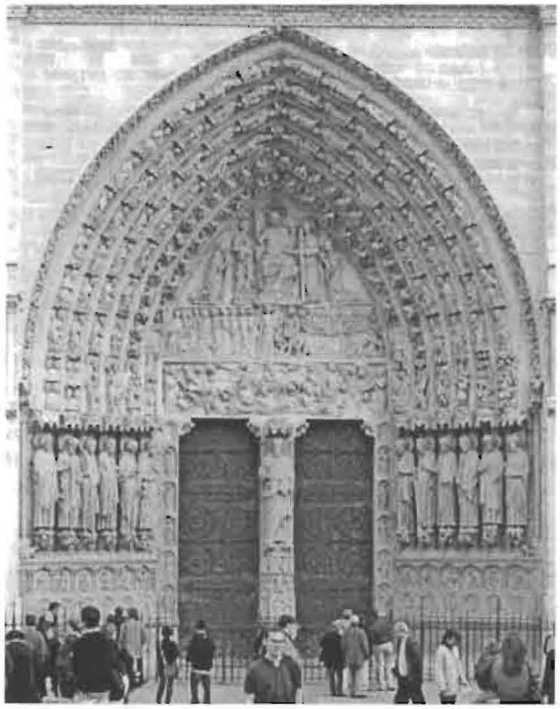


Figure 2.4. Notre Dame Cathedral, Paris.
Photograph by Chuck LaChiusa*.

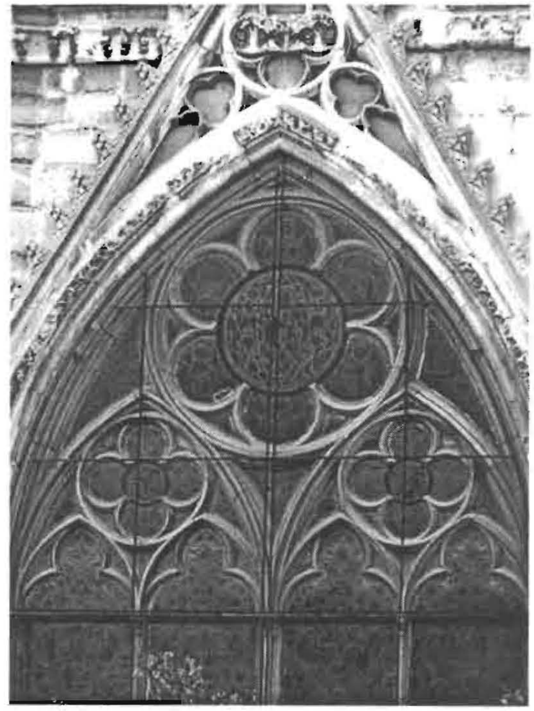


Figure 2.5. Notre Dame Cathedral, Paris.
Photograph by Chuck LaChiusa*.

An example

Let us examine the stone tracery of the window shown in Figure 2.5. Our challenge is to recreate this design with our compass and straightedge. It seems logical in an architectural application to begin with the window opening and work inward. The pointed arch outline of the window is one of the staples of gothic design, and a great example of the interaction between art and geometry. The gothic arch, or equilateral arch, is formed by the same method that we used for constructing an equilateral triangle on a segment. The arch is simply comprised of the arcs AC and BC as shown in figure 2.6 below.

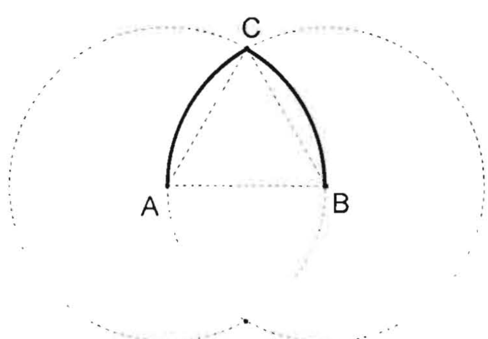


Figure 2.6

* *Architecture Around the World*, http://ah.bfn.org/. Used with permission.

Once the arch has been constructed, we wish to fill in the smaller equilateral arches along its base and the circle so that they are just touching one another and the outer arch. The following definition and result will be helpful.

Definition. We say that two circles or a line and a circle are *tangent*, if they intersect in exactly one point. This point of intersection is called the *point of tangency*.

Theorem. If circles with centers A and B pass thru a distinct point P that is on line AB, then the circles are tangent at the point P. [1]

Notice that the theorem does not specify whether or not the circles are exterior to one another, so it would apply to either of the arrangements shown in figures 2.7 and 2.8.

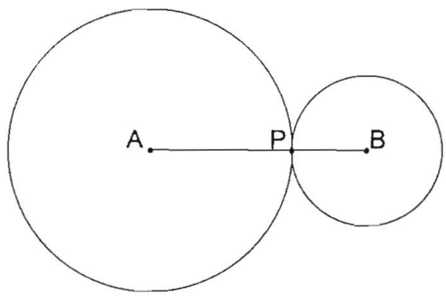


Figure 2.7

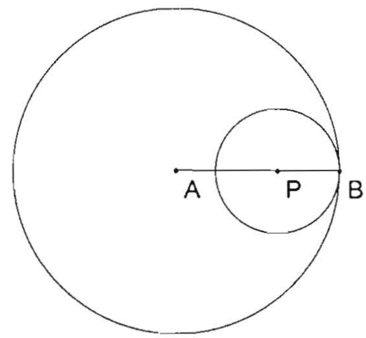


Figure 2.8

C.14 *To inscribe two congruent equilateral arches and a circle in a given equilateral arch as in figure 2.5 (so that each figure is tangent to the other two and to the larger arch).*

Hints
Construct the arches first.
How can we construct two congruent arches along this base line?

How can we construct a circle that just fills the space?
If D is the midpoint of segment AB, imagine circles A_D and A_B.
Now consider circles B_D and B_A.
What must be true about our desired circle? What do we know about the center point?
Now that we have found the center point, how do we find the radius?
Where will the desired circle be tangent to arc DE on circle A_D?

Construction:
Let ACB be the given equilateral arch with base AB.
Bisect segment AB.
Let D denote the midpoint.
Construct equilateral arches on AD and DB.
Bisect segments AD and DB.
Let G and H denote their respective midpoints.
Construct circle with center A and passing through H.
Construct circle with center B and passing through G.
Let K denote the intersection of circles A_H and B_G that is on the same side of line AB as C.
Construct segment AK.

Let I denote the point of intersection of segment AK with circle A_D.
Construct circle with center K and passing through I.
Claim: Circle K_I and equilateral arches AED and DFB are tangent to the larger equilateral arch and to each other.

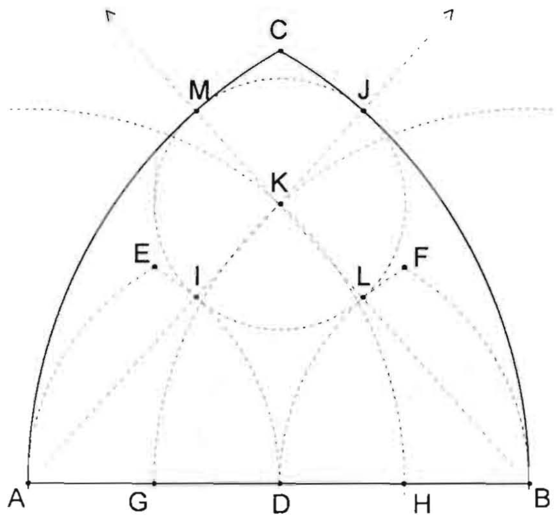


Figure 2.9

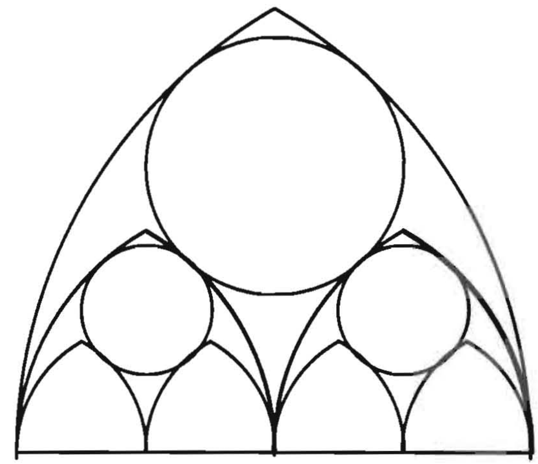


Figure 2.10

Justification:
Points A, D, B are collinear by construction of D, and D is on circles A_D and B_D by construction, so A_D is tangent to B_D at D by the above result. Thus the equilateral arches are tangent to one another.

Also G and H are collinear with A and B by their construction, and A is a point on circles G_A and B_A by their construction, so G_A is tangent to B_A at A. Similarly H_B is tangent to A_B at B. Thus the smaller arches are tangent to the larger arch.

Points A, I, K are collinear by construction of I, and I is on circles A_D and K_I by construction of I and K_I respectively, so A_D is tangent to K_I at I. Let L and M denote the points of intersection of ray BK with circle B_D and B_A respectively. We know that AH=AK as segments AH and AK are radii of the same circle. Also AD=AI as segments AD and AI are radii of the same circle. Subtracting AI from AK and subtracting AD from AH leaves us IK=DH. Similarly LK=DG. And DH=DG as halves of congruent segments are congruent. Thus IK=LK by transitivity, so L is on circle K_I. But L was also on circle B_D, and B, K, and L are collinear by construction of L, thus K_I is tangent to B_D at L. Therefore the circle is tangent to the two sub-arches.

Let J denote the intersection of ray AK and circle A_B. Reasoning similar to that in the previous argument gives us that, JK=BH and IK=DH. Since H is the midpoint of segment DB by construction, BH=DH. So by transitivity, JK=IK. Thus, J is on circle K_I. And A, K, J collinear by construction of J, so we have that K_I tangent to A_B at J. Similarly K_I is tangent to B_A at M. Therefore the circle is tangent to the larger arch. ■

As is common in the gothic windows, the same configuration is repeated within the smaller arches. With the basic structure in place, we turn our attention to the flower-like figures within the circles. Such designs, referred to as lobed or foiled figures, are present everywhere in gothic cathedrals. They are a major component of almost every window and adorn a variety of other surfaces, such as the marble floor shown in figure 2.11 and the iron gate in figure 2.12.

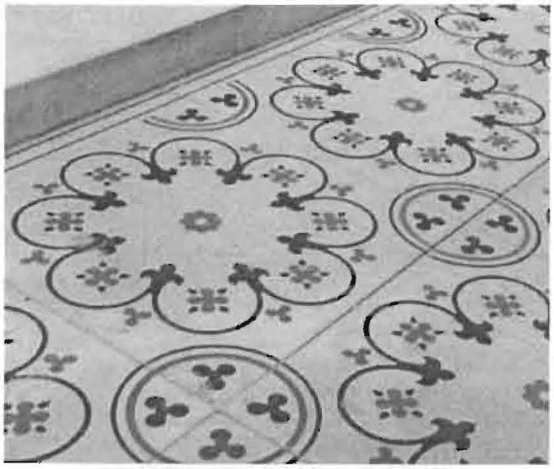


Figure 2.11. St. Denis Abbey, Paris.
Photo by Chuck LaChiusa*.

Figure 2.12. St. Denis Abbey, Paris.
Photo by Chuck LaChiusa*.

These lobed figures are formed by inscribing various numbers of congruent circles inside of a larger circle (around the perimeter) so that adjacent circles are tangent to one another and to the outer circle. To create an n-foil, we can think of slicing the larger circle into n congruent sectors and then inscribing a circle within each sector. We begin with a slightly simplified problem.

C.15 *To inscribe a circle in a given triangle.*

Hints:
Imagine such a circle. What would need to be true about the center of the circle?
Sketch such a circle in your triangle.
Sketch in the line segments that would show the perpendicular distance of the center of this circle from each of the sides of the triangle.
Consider the triangles formed by these segments, the sides of the triangle, and the segments connecting the vertices of the triangle to the center of the circle.
Would any of these triangles be congruent?
Starting again with the empty triangle, how could we construct these triangles in such a way that the appropriate pairs were congruent?

Construction:
Let triangle ABC be the given triangle.
Construct the angle bisectors of angles CAB and CBA.
Let D denote their point of intersection.
Construct a line perpendicular to line AB and passing thru point D.
Let E denote the intersection of this perpendicular with segment AB.
Construct circle with center D and passing through E.
Claim: Circle D_E is the desired circle.

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* *Architecture Around the World*, http://ah.bfn.org/. Used with permission.

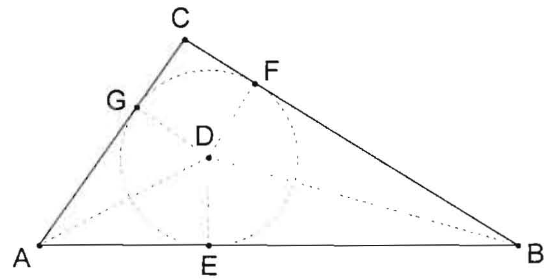

Figure 2.13

Justification:

Given: Circle D_E constructed as described.
Show: Segments AB, BC, and CA are tangent to the circle D_E.

Statement	Reason
1. Circle D_E constructed as described	1. Given
2. Construct the line perpendicular to BC that passes through D. Let F denote the intersection of this line with line BC.	2. Construction C.8.
3. Construct the line perpendicular to CA that passes through D. Let G denote the intersection of this line with CA.	3. Construction C.8.
4. $\overline{DE} \perp \overline{AB}$, $\overline{DF} \perp \overline{BC}$, $\overline{DG} \perp \overline{AC}$	4. By construction.
5. $\angle BFD$, $\angle BED$, $\angle DGA$, and $\angle DEA$ are right angles	5. Perpendicular lines form four right angles.
6. ΔEBD, ΔFBD, ΔEAD, and ΔGAD are right triangles	6. Definition of right triangle.
7. Ray BD bisects angle CBA. Ray AD bisects angle CAB.	7. By construction.
8. $\angle FBD \cong \angle EBD$, $\angle GAD \cong \angle EAD$	8. Definition of angle bisector.
9. $\overline{BD} \cong \overline{BD}$, $\overline{AD} \cong \overline{AD}$	9. Reflexive Property of Segment Congruence
10. $\Delta FBD \cong \Delta EBD$, $\Delta GAD \cong \Delta EAD$	10. Hypotenuse Leg Right Triangle Congruence Theorem
11. $\overline{ED} \cong \overline{FD}$, $\overline{GD} \cong \overline{ED}$	11. Corresponding segments in congruent triangles are congruent.

12. Points F and G are on circle D_E	12. Def. of circle.
13. $\overline{AB}$ is tangent to D_E at E, $\overline{BC}$ is tangent to D_E at F, $\overline{CA}$ is tangent to D_E at G	13. If a line is perpendicular to the radius of a circle at the endpoint on the circle, then the line is a tangent to the circle.

Now that we can inscribe a circle in an arbitrary triangle, our task of inscribing a circle in a sector becomes rather simple.

C.16 *To inscribe a circle in a given sector (of a given circle).*

Hints:
On what line would the center of the circle have to lie?
If this were a triangle, we would know what to do. Is there a triangle that we could use?

Construction:
Let A_B be the given circle and let the sector defined central angle BAC be the given sector.
Construct the angle bisector r_1 of angle BAC.
Let D denote the intersection of r_1 with arc BC.
Construct the tangent line to arc BC at D.
Let E and F denote the intersection of this tangent line with rays AB and AC respectively.
Inscribe a circle in triangle AEF.
Let G denote the center used in the construction of this circle.
Claim: Circle G is tangent to arc BC and segments AC and AB.

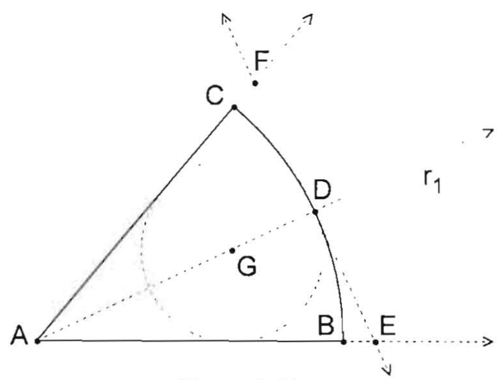

Figure 2.14

Justification:
Circle G_D is tangent to AF and AE by construction. So it is tangent to segments AC and AB. When constructing G_D in triangle AEF, the angle bisector AD of FAE had already been constructed, and its intersection with one of the remaining angles is used to find G. Thus G is collinear with A and D. Notice also that D is on A_B by construction of D as the intersection of r_1 and A_B. Thus G_D and A_B are tangent at D. ∎

With the ability to inscribe a circle in an arbitrary circle, we are ready to construct our first foiled figure. Let us begin with the 4-foil, also known as a quatrefoil.

C.17 *To inscribe four congruent circles in a given circle that are tangent to the larger circle and to each other.*

Hints:
How can we divide the circle into four congruent sectors?
The construction is simpler if you bisect two adjacent sectors and inscribe a circle in the sector bounded by the two bisectors. Why is this equivalent?
Once one of the circles is formed, how can we use this circle to more easily construct the others? (What determines a circle?)

Construction:
Let A_B be the given circle.
Construct line AB.
Construct a perpendicular to line AB through A.
Let E and D denote the points of intersection of this line with circle A_B.
Construct angle bisectors of EAC and EAB.
Let F and G respectively denote the points of intersection of these bisectors with circle A_B.
Inscribe a circle H in the sector of A_B with central angle GAF.
Construct circle A_H.
Let I, J, and K denote the intersections of A_H with rays AC, AD, and AB respectively.
Construct circles I_C, J_D, and K_B.
Claim: H, I_C, J_D, and K_B are the desired circles.

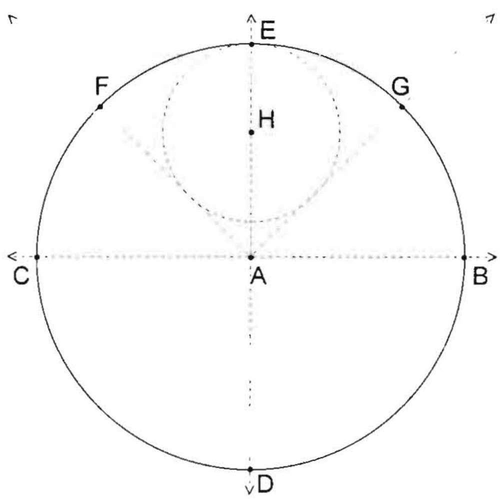

Figure 2.15

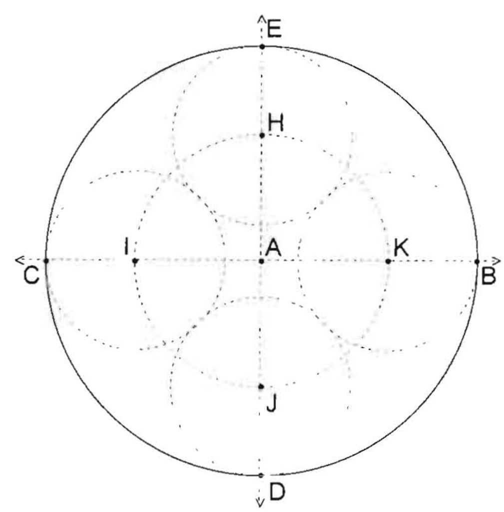

Figure 2.16

Justification:
All of the smaller circles are tangent to circle A_B by construction, so we only need to consider their tangency to one another. Without loss of generality, let us consider circles H_E and I_C. AE=AC as they are both radii of circle A_B by construction of E and C. Also AH=AI as radii of A_H. So by subtracting AH and AI from AE and AC respectively, we get HE=IC. In other words, H_E and I_C are congruent circles.

Circle H_E by construction is tangent to segment AF. Let us call this point of tangency L. Consider triangles HAL and IAL. By construction AF is the angle bisector of angle EAC, so angles HAL and IAL are congruent. As radii of the same circle HA=IA, and the segment AL is shared. Thus triangle HAL is congruent to triangle IAL. So IL=HL as IL and HL are corresponding parts of congruent triangles.

From above, the radius of circle I_C is equal to HL, so IL a radius of I_C.
Therefore L is on circle I_C. Also, HLA is a right angle since LA is tangent to circle H_E at L. Then by triangle congruence, ILA is also a right angle. Thus I, L, and H are collinear. Therefore I_C and H_E are tangent at point L. ■

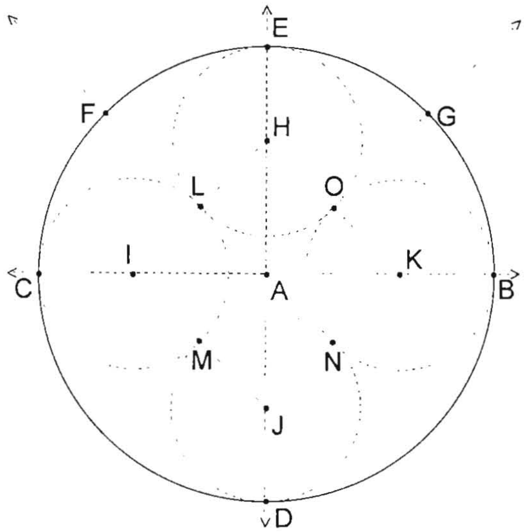

Figure 2.17

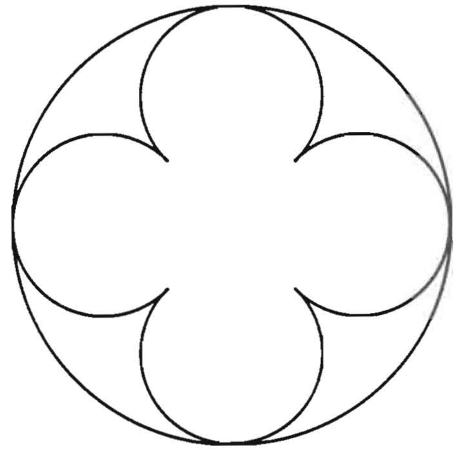

Figure 2.18

The other points of tangency can be found by connecting the centers of remaining consecutive circles pairs. We now have all of the curves and points needed to describe the quatrefoil as shown in figure 2.18 above. So we can now inscribe quatrefoils in the smaller circles of our basic window structure from figure 2.10 to further replicate the design of our example window. Other popular gothic designs that can be formed by the four inscribed circles are shown in figures 2.19 through 2.21 below.

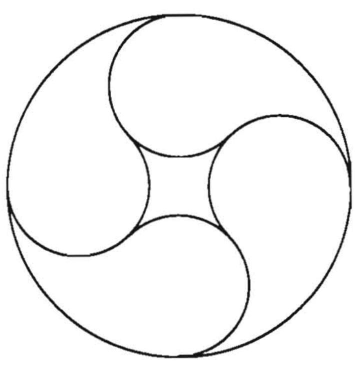

Figure 2.19

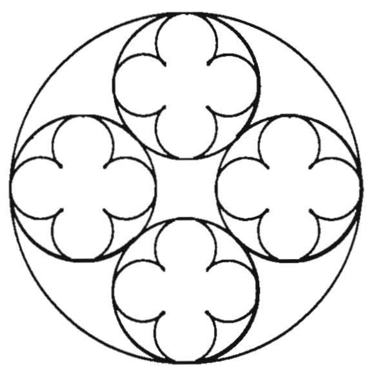

Figure 2.20

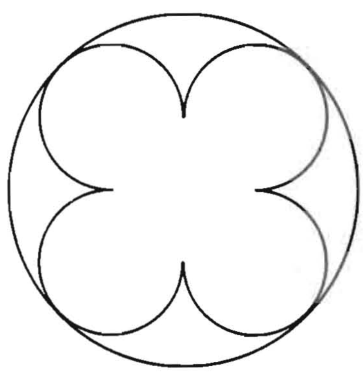

Figure 2.21

If we wish to construct an 8-foiled figure we can simply bisect all the angles between the diameters that we used to construct the 4-foil and then proceed in the same fashion as is illustrated in figures 2.22 and 2.23. An alternate design based on the eight inscribed circles is shown in figure 24. Notice that we could continue bisecting the sectors to create a 16-foil, then a 32-foil, etc.

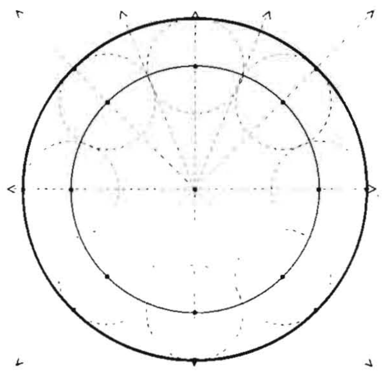

Figure 2.22

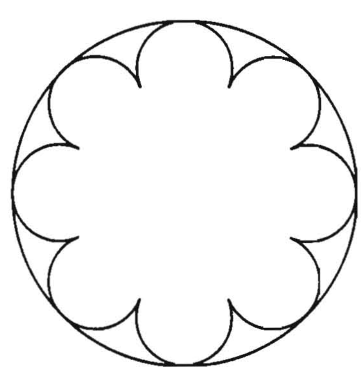

Figure 2.23

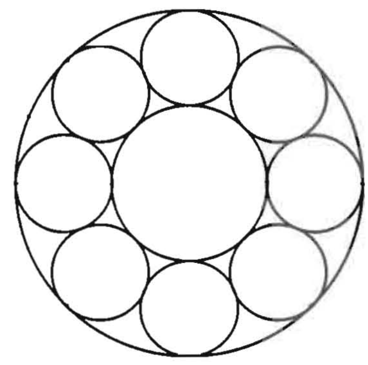

Figure 2.24

Notice that the tangency proof for the quatrefoil does not depend upon the measure of the central angle that determines the sectors; it only requires that the sectors are congruent, so we can generalize this method to create n-foils as long as we can devise a way to cut the circle into n congruent slices. One way to accomplish this is to inscribe a regular n-gon in the circle. The other lobed figure needed to complete our example window is the 6-foil, so let us turn our attention to the hexagon.

C.18 *To inscribe a regular hexagon in a circle.*

Hints:
Consider the desired regular hexagon; imagine six segments connecting the vertices to the center of the circle.
What do we know about these triangles?
What is the relationship between the various sides of these triangles and the circle?
How can we use this information to construct the vertices?

Construction:
Let A_B be the given circle.
Construct line AB.
Let C denote the other point of intersection of line AB with circle A_B.
Construct circles B_A and C_A.
Let D and E denote the intersections of B_A with A_B.
Let F denote the intersection of C_A and A_B that is on the same side of AB as D.
Let G denote the other point of intersection of C_A with A_B.
Construct segments BE, EG, GC, CF, FD, and DB.
Claim: Hexagon BEGCFD is the desired figure.

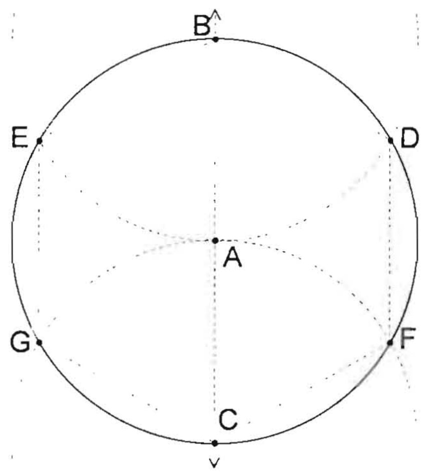

Figure 2.25

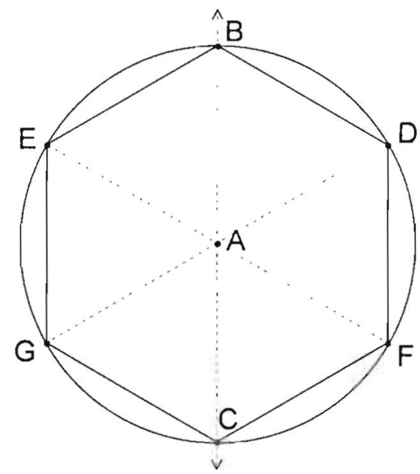

Figure 2.26

Justification:
Because they share radius AB, A_B and B_A are congruent circles. Thus AB, AD, AE, EB, and DB are all equal as they are the measures of radii of congruent circles. So triangles ABE and ABD are equilateral. Also, circle C_A is congruent to circle A_B, since they share radius AC. So AC, AF, AG, GC, and FC are all equal. Thus triangles ACG and ACF are equilateral. Notice that all four of these equilateral triangles are congruent as they all have side length equal to radius of A_B.

And equilateral triangles are also equiangular with all their angle measures equal to 60 degrees, so angles BAE, GAC, BAD, and FAC all have measure 60 degrees. The sum of the measures of angles BAE, EAG, and GAC is 180, because together they form a straight angle. Thus the measure of angle EAG is 60 degrees. Likewise, the sum of the measures of angles BAD, DAF, and FAC is 180. So the measure of angle DAF is 60 degrees. Therefore, triangles EAG and DAF are congruent to all of the other triangles by the Side-Angle-Side Triangle Congruence Postulate.

So all sides of hexagon BEGCFD are congruent as they are corresponding segments of congruent triangles, and all angles of hexagon BEGCFD are congruent (with measure 120). Thus BEGCFD is a regular hexagon. ■

We now have a way to slice the circle into 6 congruent sectors, so we can create a 6-foil just as we constructed the quatrefoil above - by inscribing a circle in each sector and connecting the centers of adjacent circles to find their points of tangency. The hexagon also allows us to construct another popular figure in gothic design, the 3-foil or tri-foil, by using every other vertex of the hexagon to divide our circle. We can also form a 12-foil, 24-foil, etc. by bisecting the central angles.

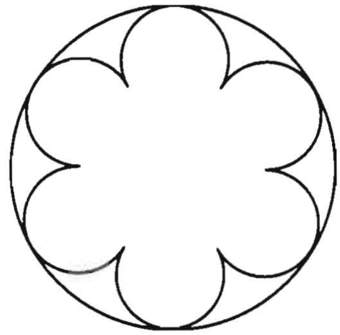

Figure 2.27

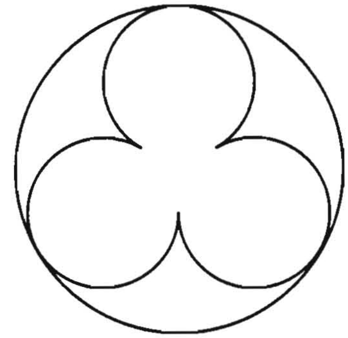

Figure 2.28

With the 6-lobed figures established, we can now complete the example window. The 3, 4, and 6-foil (and their multiples) are some of the most popular lobed figures in gothic design. Another common figure, the 5-foil requires a bit more work.

The pentagon

The problem of constructing the pentagon is equivalent to constructing a central angle of 72 degrees (or an inscribed angle of 36 degrees). This is accomplished by constructing an isosceles triangle whose base angles are twice the third angle, but we will first need to establish a few results.

C.19 *To cut a given segment so that the rectangle contained by the whole and one of the segments is equal (in area) to the square on the remaining segment.*
(To construct a point H on segment AB so that $(HB)(AB)=(AH)^2$.)

Construction:
Let AB be the given segment.
Construct square ABCD on segment AB.
Construct ray DA (extending side DA).
Construct the midpoint E of AD.
Construct circle E_B.
Let F denote the point of intersection of E_B with ray DA.
Construct square FAHG on FA so that H and G are on the same side of FA as B. (Notice that H will lie on segment AB.)
Claim: H is the desired point.

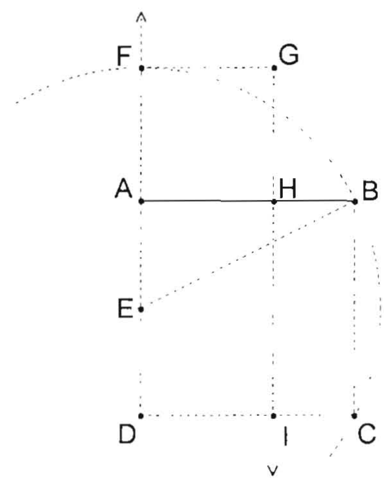

Figure 2.29

Justification:

Quadrilateral ABCD is a square by construction, thus angle DAB a right angle. So triangle EAB is a right triangle. The Pythagorean Theorem then gives us

$$(AE)^2+(AB)^2=(EB)^2.$$

Now E is the midpoint of segment AD by construction, so AE=½ (AD). Again because ABCD a square, AB=AD. From these two equalities we have

$$AE = \tfrac{1}{2}(AB).$$

Also, EB=EF as they are radii of the same circle, and EF=EA+AF, so EB=EA+AF. Substituting from our last result gives EB=½(AB)+AF. Also, quadrilateral FAHG is a square by construction, so AF=AH, which then gives us

$$EB= \tfrac{1}{2}(AB)+AH.$$

Now substituting into the equation that we obtained from the Pythagorean Theorem produces

$$(\tfrac{1}{2}(AB))^2 + (AB)^2 = (\tfrac{1}{2}(AB) + AH)^2$$

which becomes

$$(AB)(AB - AH) = (AH)^2$$

with simple algebraic manipulation. Thus

$$(AB)(HB) = (AH)^2 . \blacksquare$$

The ratio between the resulting segments is often referred to as the divine proportion or golden ratio. We will soon be utilizing this construction in the formation of our desired isosceles triangle. This construction was certainly not as intuitive as the previous constructions, but once the construction is complete, it is not too difficult to establish that it produces the desired result. Though this would not be an appropriate construction for high school students to develop, they could be given the construction and asked to provide a proof. This construction (along with the entire pentagon formation) would certainly be a good example for them of a more complicated construction that requires clever applications of the basic results of geometry. Before we move on to the construction of our triangle, we need one more result.

Theorem (E.III.32). The angle formed by a chord of a circle and a ray tangent to the circle at one of the endpoints of the chord is congruent to any inscribed angle that intercepts the arc that lies on the same side of the chord as the ray.

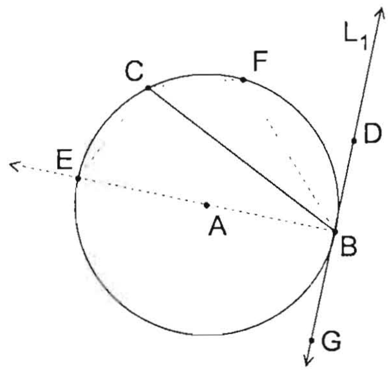

Figure 2.30

Proof. Let BC be a chord of circle A and let line L_1 be tangent to circle A at B. If CB is perpendicular to L_1, then it passes through A, and so cuts the circle into two semicircles. In this case any inscribed angle would also be a right angle, and would thus be congruent with the angle formed by L_1 and BC, which is what was required. Let us assume then that chord BC is not perpendicular to L_1. Thus it forms one acute angle and one obtuse angle with the opposite rays from B on L_1. Choose points D and G on L_1 so that D-B-G and D is on the ray of L_1 that forms the acute angle with BC. Choose a point F on minor arc BC. Construct ray BA and let E denote the other point of intersection of this ray with circle A. Angle BCE is a right angle because it is inscribed in a semicircle, thus angles CEB and CBE are complementary. Notice that DBC is also complementary to angle CBE because radius AB is perpendicular to tangent line L_1. Thus angles CEB and DBC are congruent, and all inscribed angles that intercept minor arc BC are congruent, so angle DBC is congruent to any arc that intercepts minor arc BC.

Now for the obtuse angle: Because angels CEB and BFC are opposite angles in an inscribed quadrilateral, they are supplementary. But we have established that angle CEB is congruent to angles DBC, so angles DBC and BFC are also supplementary. We also know that as a linear pair angles DBC and CBG are supplementary. Therefore angle BCF is congruent to angle CBG. ■

Now we are ready to construct our triangle. This construction is also not really appropriate for student discovery, but the proof is an excellent application of many familiar results. Due to the length of the proof, it would probably be best to present it to students (though they could provide the justification for each step).

C.20 *To construct an isosceles triangle whose base angles are equal to twice the third angle (with a given segment as one of the congruent sides).*

<u>Construction</u>:
Let AB be the given segment.
Construct circle A_B.
Cut segment AB at H so that $(HB)(AB)=(AH)^2$.
Construct a circle with center at B and radius equal to AH.
Let C denote one of the points of intersection of this circle with circle A_B.
Construct segments BC and CA.
Claim: BAC is the desired triangle.

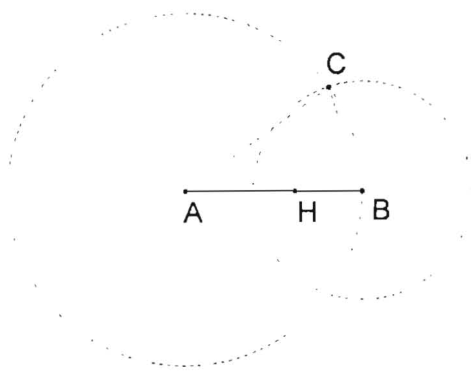

Figure 2.31

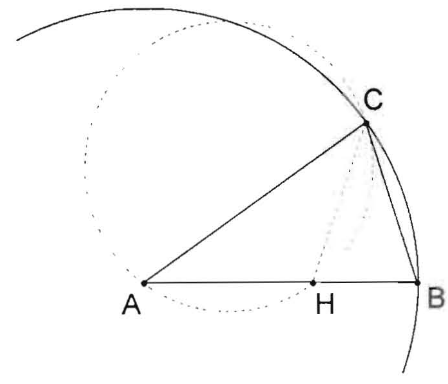

Figure 2.32

Justification:
(HB)(AB)=$(AH)^2$ by construction of H, and BC=AH by construction of C. Substitution then gives us (HB)(AB)=$(BC)^2$.

Construct segment HC, and consider the circle O through A, H, and C. Notice that BA is a secant segment to this circle and BC is a segment from an exterior point of the circle to a point on the circle, thus segment BC is tangent to circle O. (Euclid III.37: If from a point outside a circle a segment to a point on the circle and a secant segment to the circle is drawn and the square of the measure of the first segment is equal to the product of the measures of the secant segment and the external part of the secant segment, then the segment drawn to a point on the circle is tangent to the circle at that point.) Now by E.III.32 above, the measures of angles HCB and angle CAH are equal.

Since H is in the interior of angle ACB, $m\angle ACB = m\angle ACH + m\angle HCB$. Substitution then given us $m\angle ACB = m\angle ACH + m\angle CAH$. Using triangle HAC, $m\angle ACH + m\angle CAH = m\angle CHB$, because the measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles (). Thus the transitivity of equality gives us $m\angle ACB = m\angle CHB$.

Also, segment AC is congruent to AB as they are both radii of circle A_B. Thus triangle BAC is and isosceles triangle, so the measures of the base angles ACB and ABC are equal. So substitution in $m\angle ABC$ for $m\angle ACB$ results in $m\angle ABC = m\angle CHB$, thus triangle HCB is an isosceles triangle. In other words HC=BC. But BC=AH by construction of C, thus HC=AH. So triangle AHC is also isosceles, and so $m\angle ACH = m\angle CAH$.

Recall that $m\angle ACB = m\angle ACH + m\angle CAH$. Substituting in $m\angle CAH$ for $m\angle ACH$ yields $m\angle ACB = m\angle CAH + m\angle CAH = 2(m\angle CAH)$. Which means that $m\angle CBA = 2(m\angle CAH)$. Therefore triangle BAC is the desired isosceles triangle. ∎

Now that we have our triangle with the desired angle measurements, we simply need a way to inscribe it in a given circle. This construction is much more intuitive and its generation would be accessible to students.

C.21 *To inscribe a triangle in a given circle similar to a given triangle.*

Hints:
Could we choose any point on the circle for one of our vertices?

Consider the tangent line to the circle at this point.
How can we use this tangent line to inscribe angle ABC in the circle?
What about angle ACB?

Construction:
Let ABC be the given triangle and D_E the given circle.
Construct a tangent line L_1 to D_E at E.
Chose points G and F on L_1 so that G-E-F.
Construct a ray r_1 from E (on the same side of L_1 as D) so that the angle formed by r_1 and ray EF is congruent to angle ABC from the given triangle.
Let H denote the other point of intersection of r_1 and D_E.
Construct a ray r_2 from E (on the same side of L_1 as D) so that the angle formed by r_2 and ray EG is congruent to angle ACB from the given triangle.
Let I denote the other point of intersection of ray r_2 and D_E.
Construct segments EI, IH, and HE.
Claim: Figure EIH is the desired triangle.

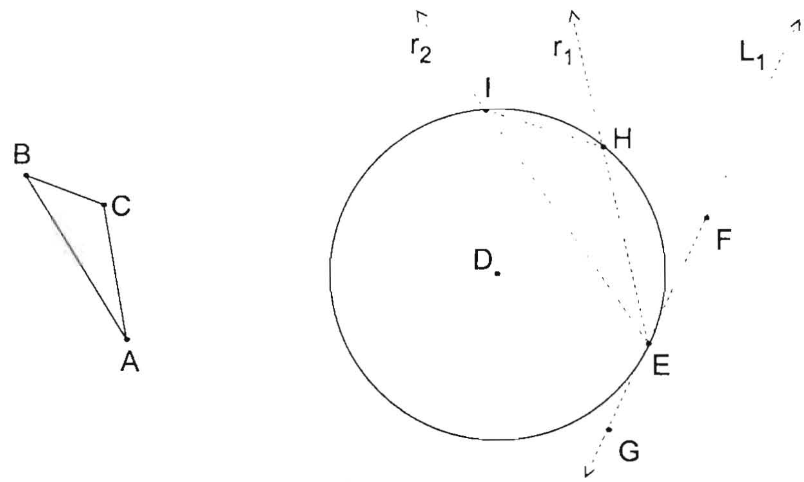

Figure 2.33

Justification:
Angle ABC and FEH are congruent by the construction of r_1, and angles ACB and GEI are congruent by the construction of r_1. Now by theorem (), angle FEH is congruent to angle EIH, and angle GEI is congruent to angle EHI. So by transitivity of angle congruence, angle ABC is congruent to angle EIH, and angle ACB is congruent to EHI. Thus, triangle ABC is similar to triangle EIH by the side-side triangle similarity theorem.
■

We now have our desired triangle and a method of inscribing it in a circle, so are ready to construct the regular pentagon.

C.22 *To construct a regular pentagon in a given circle.*

Hints:
Inscribe a triangle similar to the triangle from C.20 in the circle.
How can you determine where to place the other vertices?
Consider the arcs subtended by the angles of the triangle.

Construction:
Let A_B be the given circle.
Construct an isosceles triangle XYZ whose base angles (Y and Z) are twice its third angle (X).
Inscribe a triangle BEF in A_B that is similar to triangle XYZ.
Construct angle bisectors r_1 of BEF and r_2 of BFE.
Let G denote the other point of intersection of r_1 with A_B.
Let H denote the other point of intersection of r_2 with A_B.
Construct segments BH, HE, EF, FG, and GB.
Claim: Polygon BHEFG is the desired figure.

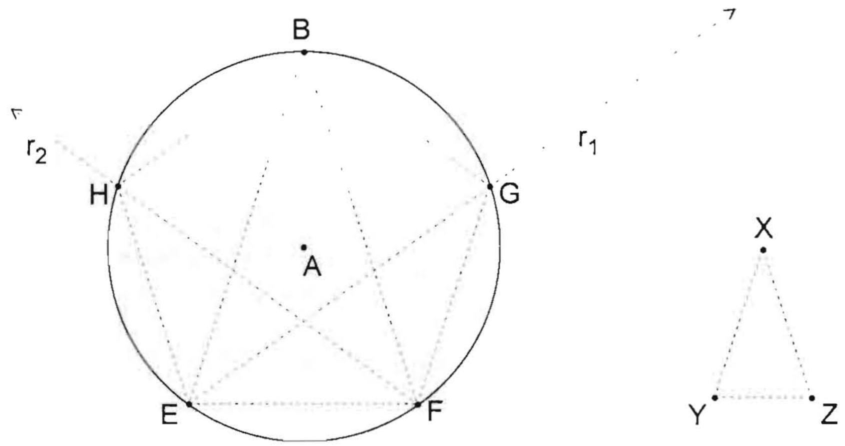

Figure 2.34

Justification:
By construction of isosceles triangle XYZ, the measures of angles X and Z are equal to half the measure of angle Y. Triangle BEF was constructed to be similar to triangle XYZ, so triangle BEF is also isosceles, and $m\angle EBF = \frac{1}{2}(m\angle BEF)$. Also, rays EG and FH were constructed to bisect angles BEF and BFE respectively, so $m\angle BEG = \frac{1}{2}(m\angle BEF)$, $m\angle GEF = \frac{1}{2}(m\angle BEF)$, $m\angle BFH = \frac{1}{2}(m\angle BFE)$, and $m\angle HFE = \frac{1}{2}(m\angle BFE)$. Since triangle BEF is isosceles, its base angles are congruent, i.e. $m\angle BFE = m\angle BEF$. So the last relationship implies that $m\angle BFH = \frac{1}{2}(m\angle BEF)$, and $m\angle HFE = \frac{1}{2}(m\angle BEF)$.

Now, we have that all five inscribed angles $\angle EBF$, $\angle BEG$, $\angle GEF$, $\angle BFH$, $\angle HFE$ are equal in measure to $\frac{1}{2}m\angle BEF$, thus they are all congruent to one another. Then the minor arcs of circle A that they intercept, EF, BG, GF, BH, and HE, are also congruent. Thus the corresponding cords of circle A are congruent, i.e. segments EF, BG, GF, BH, and HE are all congruent to one another.

Since the arcs that are cut by the vertices of pentagon BHEFG are congruent, any adjacent grouping of three of these arcs will also be congruent, i.e. major arcs HG, BE, HF, EG, and FB are all congruent to one another. Thus angles HBG, BHE, HEF, EFG, and FGB which subtend these arcs are congruent. Therefore, polygon BHEFG is a regular pentagon. ∎

With the pentagon established, the 5-foil can be constructed in the same manner as before. This also provides the 10 and 20 foil, etc., as well as the pentagram, or 5-pointed star.

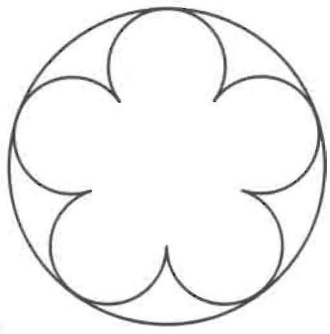

Figure 2.35

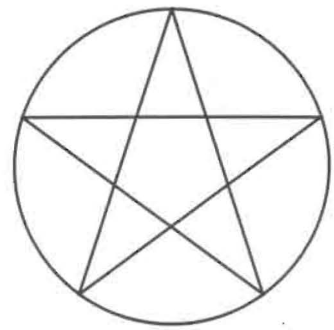

Figure 2.36

The rose window

The large round windows, called rose windows, that serve as the focal points in Gothic cathedral facades provide interesting geometric problems. Figure 2.37 shows the tracery of the rose window of the cathedral of Sens, France.

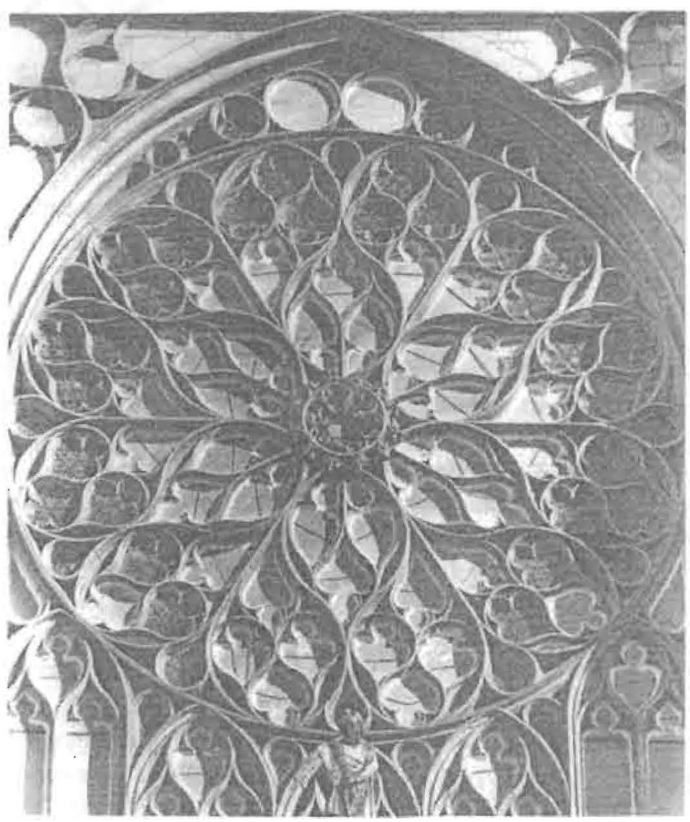

Figure 2.37. The cathedral of Sens [3].

Though this design is not composed directly of regular polygons and circles, these figures certainly underlie its basic structure. The outer ring is composed of twelve circles that are tangent to the larger circle and one another; they can be constructed using an inscribed hexagon as discussed above.

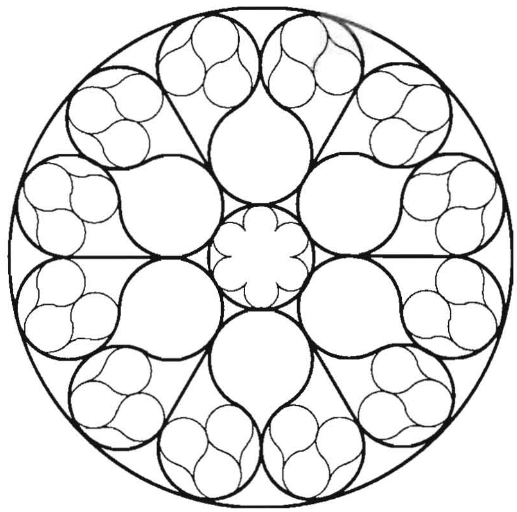

Figure 2.38

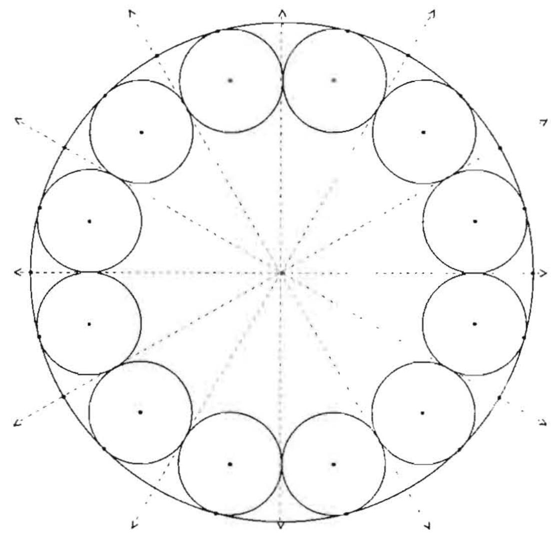

Figure 2.39

The next task is to construct the ring of six circles that lies within this ring of twelve.

C.23 *To construct 6 congruent circles that are tangent to one-another and to the ring of twelve circles in figure 2.39.*

Construction:
Construct segments connecting the centers of consecutive circles in the ring of twelve. Construct an equilateral triangle on one of these segments (towards the center of the larger circle) and then on alternating ones as you move around the circle as depicted in figure 2.40.

For each triangle, construct a circle with the inner vertex of the triangle as the center of the circle and passing thru the midpoint of one of the sides of the triangle that extends from this inner vertex (notice that it will then pass thru the midpoint of the other side extending from this vertex also).

Claim: These circles are tangent to one another and to the outer twelve circles.

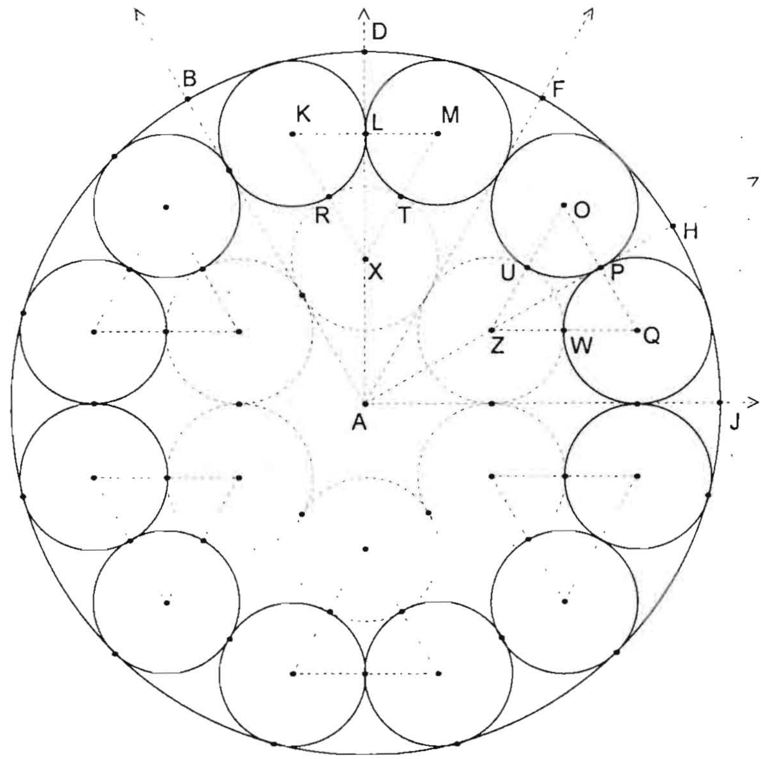

Figure 2.40

Justification:
Consider two consecutive circles, say X and Z constructed as above. Circles K, M, O, and Q are congruent by construction. By the prior discussion on constructing foiled figures, circles K and M are tangent to AD and each other at say L. Circles O and Q are tangent to AH and each other at say P. If two circles are tangent, then their centers are collinear with their point of tangency [1], so L is on KM and P is on OQ.

Triangles KXM and OZQ are equilateral by construction, and KM=OQ as they are both twice equal radii, so triangles KMX and OQZ are congruent. Since circles X and Z were constructed to have radii equal to half the side of the congruent equilateral triangles, they are also congruent to outer circles and to each other. Since circles X and K pass thru the same point R on KX by construction of circle X, they are tangent to each other at R. The radii of circles X and M are also half the length of XM, so they would be tangent to one another at T. Similarly, circle Z is tangent to circle O at U (on OZ) and to circle W (on ZQ).

Now to establish the tangency of circles X and Z:
X is on AL, and Z is on AP as each vertex of an equilateral triangle is on the perpendicular bisector of the opposite side. AK=AO by the construction of O on circle A_K. By the construction of K, $m\angle$KAL=30, and by the construction of P, $m\angle$OAP=30. Angles $\angle$KLA and $\angle$OPA are right angles, because they are formed by radii of circles and their respective tangent lines. Thus triangles KLA and OPA are congruent by the Angle-Side-Angle Triangle Congruence Postulate, so as corresponding segments AL=AP. Also, as corresponding altitudes of the congruent equilateral triangles, LX =PZ. So by subtraction of congruent segments AX=AZ, i.e. XAZ is an isosceles triangle. Since $m\angle$ XAZ=60, triangle XAZ is equilateral. So we have $m\angle$AXZ=60, and it is easy to see that

$m\angle$LXM=30, so $m\angle$MXZ=90 since the three angles together form a straight angle. Similarly $m\angle$OZX=90.

With XM and ZO being perpendicular to XZ, they are parallel to each other. Thus quadrilateral MXZO has a pair of opposite sides that are both congruent and parallel, so it is a parallelogram. Also, consecutive sides XM and MO are congruent, so MXZO a rhombus. We also know that XM=MO, so MXZO a square with sides twice the radii of the congruent circles. Therefore, circles X and Z both pass through the midpoint of segment XZ, and so are tangent to one-another there. ■

Other possibilities

So far, we have constructed figures based on squares, hexagons, and pentagons. What other regular n-gon's can we construct with compass and straightedge?

Gauss Wantzel Theorem. A regular n-gon is constructible with ruler and compass if and only if n is an integer greater than two such that the greatest odd factor of n is either 1 or a product of distinct Fermat primes. [4]

Definition. *Fermat primes* are odd primes of the form $F_n = 2^{2^n} + 1$.

There are 5 known Fermat primes: 3, 5, 17, 257, and 65537. So we have already established how to construct many of the n-gons that can be reasonably constructed by hand. Constructions of the 15-gon and 17-gon will not be included here, but can be found in [2] and [1] respectively.

Values of n<40 for which the n-gon is constructible with ruler and compass	
2^k	4,8,16,32,...
$2^k \cdot 3$	3,6,12,24,...
$2^k \cdot 5$	5,10,20,...
$2^k \cdot 17$	17, 34, ...
$2^k \cdot 3 \cdot 5$	15, 30,...

The Gauss Wantzel Theorem leads to an interesting question: Why would any figure not be constructible with compass and straightedge?

III. Constructability

Let us begin by describing what points we *can* construct with a compass and straightedge. Recall that in order to construct a line, we need two distinct points to already be present in our construction, and in order to construct a circle, we also need two distinct points (one as the center and the other a point on the circle), so we must have at least two points given. For simplicity, let us consider these given points as (0,0) and (1,0) in the x-y plane. From these, we can construct a line and two distinct circles. We then use the intersections of this line and these circles to get new points that we can use (in combination with our original points) to form new lines and circles, and so on.

Definition. We say that a point is a *constructible point* if it is the last of a finite sequence of points such that each point is in $\{(0,0),(1,0)\}$ or is

1) a point of intersection of two lines, each of which passes through two points that appear earlier in the sequence
2) a point of intersection of a line through two points that appear earlier in the sequence and a circle with an earlier point as center and passing through an earlier point
3) a point of intersection of two circles each of which have an earlier point in the sequence as center and pass through an earlier point in the sequence.

Let us refer to these lines and circles formed from points in the sequence as *ruler and compass lines (r-c lines)* and *ruler and compass circles (r-c circles)* respectively, but notice that not all points on a r-c line or circle are constructible (only the points formed by intersections of these lines/circles.) We'll say a number x is a *constructible number* if $(x,0)$ is a constructible point.

Now since the x-axis passes through (1,0) and (0,0), it is a r-c line. So its intersection with a circle centered at the origin and passing through (1,0), gives us that (-1,0) is also constructible. Continuing with circles of radius 1, we can get that $(m,0)$ is constructible for all integers m. We can also construct r-c lines through these points that are perpendicular to the x-axis (by using intersections of r-c circles to construct perpendicular bisectors), and these vertical lines intersect the circles just described to give us all points $(m,1)$. We can then continue to use circles of radius 1 to move up and down these vertical lines to construct all points (m,n) where m *and* n are integers as shown in figure 3.1 below.

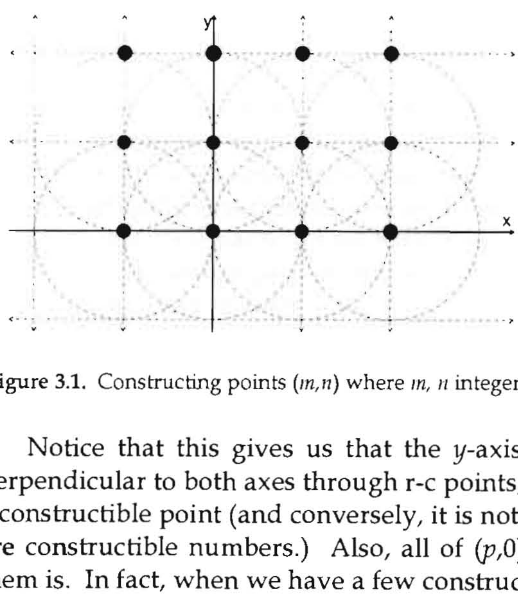

Figure 3.1. Constructing points (m,n) where m, n integers.

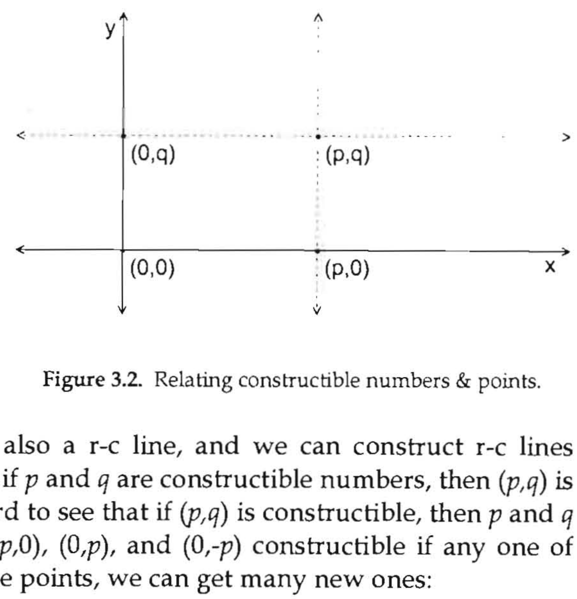

Figure 3.2. Relating constructible numbers & points.

Notice that this gives us that the y-axis is also a r-c line, and we can construct r-c lines perpendicular to both axes through r-c points, so if p and q are constructible numbers, then (p,q) is a constructible point (and conversely, it is not hard to see that if (p,q) is constructible, then p and q are constructible numbers.) Also, all of $(p,0)$, $(-p,0)$, $(0,p)$, and $(0,-p)$ constructible if any one of them is. In fact, when we have a few constructible points, we can get many new ones:

Theorem. The constructible numbers form a field, i.e. the constructible numbers contain 0 and 1, and for all constructible numbers a,b,c (c not zero), $a+b$, $a-b$, ab, a/c are also constructible.

Proof. First, we will consider $a+b$ and $a-b$ where a and b are constructible. Since a and b constructible, the point (a,b) constructible, so if we construct a circle with center at $(a,0)$ and passing through point (a,b) as in figure 3.3, the points where the r-c circle intersects the x-axis will give us points $(a\text{-}b,0)$ and $(a+b,0)$. So we have that $a\text{-}b$ and $a+b$ constructible numbers.

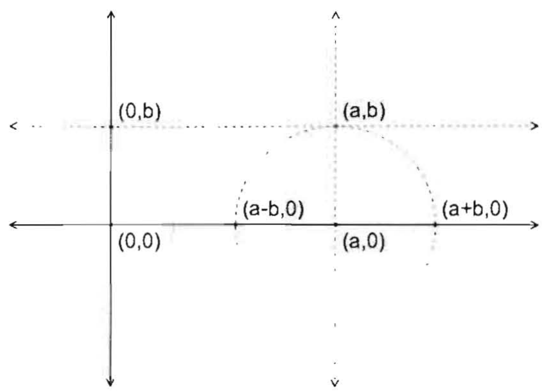

Figure 3.3. Constructing $a+b$ and $a\text{-}b$.

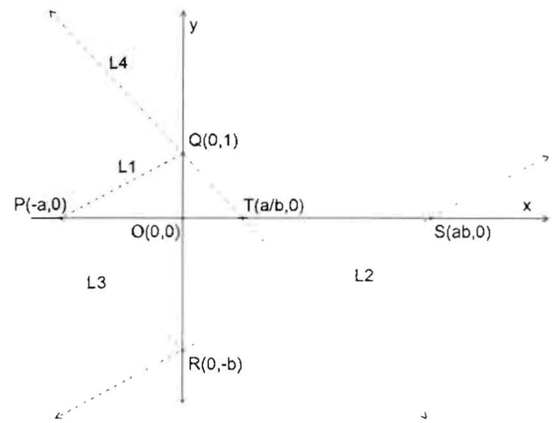

Figure 3.4 . Constructing ab and a/b.

Before we move on, notice that the constructions that we have previously discussed are made up of r-c lines and r-c circles and are thus constructible points since we have been following these same rules for creating lines and circles from previous points in our entire development of constructions. Thus we will now use these constructions to discover more constructible points without further justification.

Let us see how to construct ab. If a and b are constructible numbers, then $P(-a,0)$ and $R(0,-b)$ are certainly constructible points. And we already have $Q(0,1)$ constructible, so the line L1 in figure 3.4 a r-c line. We can construct line L2 through R parallel to L1, and it is also an r-c line (since C.4 produces an r-c line as we have just discussed). Thus the intersection point S of L2 with the x-axis is a constructible point. Considering the y-axis as a transversal, alternate interior angles RQP and QRS are congruent, so ΔOQP and ΔORS are similar right triangles (by the angle-angle similarity postulate) in ratio $1{:}b$. Which means that the length of line segment OS is ab, i.e. the coordinates of constructible point S are $(ab,0)$. So we have that ab also a constructible number.

A similar process can be used to construct a/b. This time we begin line L4 through $Q(0,1)$ and parallel to r-c line L3. L4 a r-c line, so its point of intersection T with the x-axis a constructible point. Notice that ΔOQT and ΔORP similar right triangles in ratio $1{:}b$, so the coordinates of constructible point T are $(a/b,0)$. Thus a/b is a constructible number. ∎

Since all integers are constructible, this last result implies that all rational numbers are constructible numbers, but are the rational numbers the only constructible numbers? The following result gives us the answer.

Theorem. If x constructible and $x>0$, then $\sqrt{x}$ is also constructible.

Proof. Since $(x,0)$ constructible, $P(0,-x)$ also constructible, and we can construct the midpoint M of $Q(0,1)$ and P. Now we can construct an r-c circle with center at M and passing through P. The point R, where this r-c circle intersects the x-axis is a constructible point. Notice that the distance from M to Q is $\frac{x+1}{2}$, and segment MR a radius of the same circle, so MR$=\frac{x+1}{2}$. Also

$$MO = MQ - OQ = \frac{x+1}{2} - 1 = \frac{x-1}{2}.$$

Using the Pythagorean theorem gives us

$$OR^2 + \left(\frac{x-1}{2}\right)^2 = \left(\frac{x+1}{2}\right)^2$$

$$OR^2 = \frac{\left(x^2+2x+1\right)-\left(x^2-2x+1\right)}{4}$$

$$OR = \sqrt{x}$$

Thus the coordinates of R are ($\sqrt{x}$,0), i.e. $\sqrt{x}$ a constructible number. ■

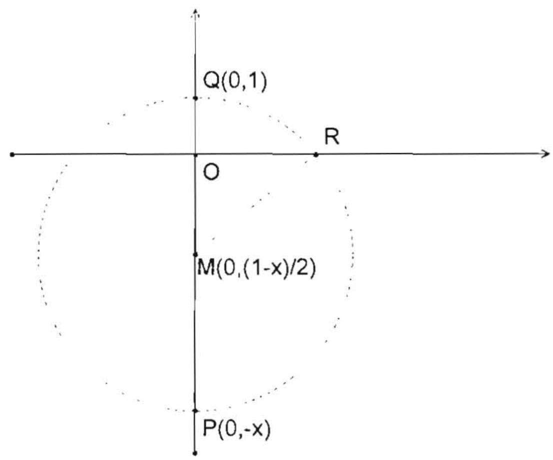

Figure 3.5 . Constructing $\sqrt{x}$.

We know that 2 and 3 are constructible numbers, so by this last result $\sqrt{2}$ and $\sqrt{3}$ are also constructible, but these are certainly not rational numbers. So the rational numbers do not give us all constructible numbers. Let us consider what numbers we must add to the rationals to get all constructible numbers.

Definition. If x a positive number in a field F, but $\sqrt{x}$ not in F, then

$$F(\sqrt{x}) = \{\, y + z\sqrt{x} \mid y \text{ and } z \text{ in } F \}$$

is called a *quadratic extension* of F.

It is not hard to verify that F($\sqrt{x}$) is also a field [4]. Let r be a rational number and let's consider $Q(\sqrt{r})$, a quadratic extension of the rational numbers. We will use the notation $Q(\sqrt{r_1}, \sqrt{r_2}, \sqrt{r_3}, \ldots, \sqrt{r_n})$ to denote a finite chain of quadratic extensions, meaning that the quadratic extension $Q(\sqrt{r_1})$ of the rationals is formed and then this new field is extended with $\sqrt{r_2}$ in the same manner, and so on. This is called an *iterated quadratic extension* of the rationals.

Let E denote the union of all iterated quadratic extensions of the field rationals. In other words, E is made up of all the possible numbers that you can get using the integers and the operations +,-,·,÷,√. We have just seen how these operations on the integers form ruler and compass numbers, so we have the following result.

Theorem. If x is in E, then x is a ruler and compass number.

Does E include all of the constructible numbers, or are there others? It turns out that the converse is also true.

Theorem. If x is a ruler and compass number, then x is in E.

Proof. Let x be a constructible number, then by definition $(x,0)$ is a constructible point. Notice that the coordinates of a constructible point can be determined algebraically by solving the system of two equations of the r-c objects (lines or circles) that are intersecting at this point. Solving these systems requires simply using the operations $+,-,\cdot,\div,\sqrt{}$. And the coefficients of these r-c lines and circles are formed by the operations $+,-,\cdot,\div$ from the coordinates of previous constructible points. Thus the coordinates of constructible points never leave E. So x is in E. ■

Thus E is exactly the field that we are looking for.

Theorem. Point P is a ruler and compass point if and only if the coordinates of P are in E. (Number x is a r-c number if and only if x is in the field E.)

The question, "What constructions are possible?" really becomes a question of what points can/cannot be constructed with a compass and straightedge, so we needed a very specific way to describe what points we could construct. It was helpful to consider our constructions in the x-y plane so that we could specify the coordinates of the points and use the notation and results of modern algebra. This notation and development was not available to the ancient Greeks and prevented them from having a full understanding of what constructions were possible. Three of the most famous constructions proposed by the ancient geometers have been found to be impossible in relatively modern times. The Greeks may have suspected their impossibility, but lacked the machinery to prove it. These three classical construction problems were

1) Given a cube, to construct a cube with twice its volume.
2) Given a circle, to construct a square with equal area.
3) Given an angle, to trisect it.

The first construction problem boils down to constructing a segment of the desired length to serve as the edge of the cube: for a given segment of unit length x, to construct a segment of length x_o such that $x_o^3 = 2x^3$. In other words, you would need to construct a segment of length $\sqrt[3]{2}$. The problem of "squaring the circle" requires constructing a segment of length y_o such that $y_o^2 = \pi y^2$ where y the radius of the given circle. In other words, you would need to construct a segment of length $\sqrt{\pi}$. Regarding the trisection of an arbitrary angle, some angles can be trisected (such as a right angle), but this can not be done in general. An angle measure of 60° will suffice for a counterexample. It can be shown that $\sqrt[3]{2}$, $\sqrt{\pi}$, and cos(20°) are not in E and thus not constructible numbers, thus the required segments can not be constructed with ruler and compass [1].

Concluding Remarks

There are countless geometric designs in Gothic architecture that would be interesting construction problems for students of Euclidean geometry. In the examples contained here, we have simply established the framework for these excursions and explored the potential of the compass and straightedge as instructional tools in teaching Euclidean geometry.

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