CATEGORICAL PROPERTIES OF ALGEBRAIC CONSTRUCTIONS

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ABSTRACT

CATEGORICAL PROPERTIES OF ALGEBRAIC CONSTRUCTIONS

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Master of Science in Mathematics
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This paper presents some fundamental ideas from category theory and abstract algebra. Several algebraic constructions are given and then interpreted as adjunction between categories. As the final result, the First Isomorphism Theorem for Groups is interpreted as an adjunction of categories.

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CHAPTER 0

INTRODUCTION

The intent of this thesis is to present some primary concepts of category theory and abstract algebra, and to develop a link between the two branches of mathematics. Much of the material presented in this paper is not new research; however this paper attempts to join the two areas in an interesting way. Moreover, although much of the material in Chapter 2 appears in print, often in an obscure way (for example, [1,pp.42-46] ,[13,p3-12]), it is the author's experience that this information as presented in Chapter 3 has not previously appeared in any standard text (see Bibliography). And, it is the author's conjecture that Chapter 4 is new material. Additionally, the intent of this thesis is to encourage at least some readers to a further study of these branches of mathematics.

Chapter 1 presents preliminary material referred to in later chapters. Chapter 2 gives three examples of standard algebraic constructions. Chapter 3 then interprets each algebraic construction of Chapter 2 as an adjunction of categories. Finally, Chapter 4 interprets the First

Isomorphism Theorem for Groups as an adjunction between categories.

CHAPTER 1

CATEGORICAL PRELIMINARIES

In this chapter, we develop the general notion of categories and adjunction between categories. For a more complete presentation of the ideas found in this chapter see any standard text on category theory. (for example [10])

A *graph* consists of:

(i) a class of objects a,b,c,....,

(ii) a class with equality of arrows f,g,h,...

(iii) two operations as follows.

Domain, which assigns to each arrow f an object $a=dom(f)$;

Codomain, which assigns to each arrow f an object $b=cod(f)$;

These operations on f can be indicated by showing f as an arrow from its domain to its codomain:

$$f:a\to b \quad \text{or} \quad a\xrightarrow{f}b$$

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These operations on f can be indicated by showing f as an arrow from its domain to its codomain:

$$f: a \to b \quad \text{or} \quad a \xrightarrow{f} b$$

Two examples of finite graphs are:

$$a \to b \to c \quad \text{and} \quad a \to b$$

A *category* is a graph with two additional operations:

Identity, which assigns to each object a an arrow $id_a : a \to a$;

Composition, which assigns to each pair (g, f) of arrows with dom(g) = cod(f) an arrow $g \circ f$ called their *composite*, with $f \circ g : \mathrm{dom}(f) \to \mathrm{cod}(g)$.

The following diagram illustrates the operation of composition.

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "g \circ f"'] & b \arrow[d, "g"] \arrow[dl, phantom, "\circ"] \\ & c \end{tikzcd}$$

These operations in a category are subject to the two following axioms:

Associativity. For given objects and arrows

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{k} d,$$

we have the equality

$$k \circ (g \circ f) = (k \circ g) \circ f.$$

Thus when the composites on either side of the equal sign in the above equation are defined, the associative law holds.

Unit law. For all arrows $f: a \to b$ and $g: b \to c$ composition with the identity arrow id_b gives

$$id_b \circ f = f \quad \text{and} \quad g \circ id_b = g$$

Thus the identity arrow id_b for each object b acts as an identity for the operation of composition. The above equation can be represented pictorially by the statement that the following diagram is commutative:

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[dr, "f"'] & b \arrow[d, "id_b"] \arrow[dr, "g"] & \\
 & b \arrow[r, "g"'] & c
\end{tikzcd}$$

The vertices of such a diagram represent the objects of a category and the edges represent the arrows of the same category. Such a diagram is *commutative* when, for each pair of vertices c and c', any two paths formed from directed edges leading from c to c' yield by composition of labels, equal arrows from c to c'.

Some examples of categories are listed below. The morphisms for all categories are functions (with certain restrictions) and composition is ordinary function composition.

SET = category of sets and functions.

SGP = category of semigroups and homomorphisms.

GRP = category of groups and homomorphisms.

AB = category of (additive) abelian groups and homomorphisms.

CRNG = category of commutative rings (with 1) and homomorphisms.

POS = category of posets and order preserving maps.

TOP = category of topological spaces and continuous functions.

A *functor* is a morphism of categories. In detail for categories $\mathcal{C}$ and $\mathcal{B}$, a functor $\mathbf{T}:\mathcal{C} \rightarrow \mathcal{B}$ with domain $\mathcal{C}$ and codomain $\mathcal{B}$ consists of two suitably related functions: the *object function* $\mathbf{T}$, which assigns to each object c of $\mathcal{C}$ an object Tc of $\mathcal{B}$; and the *arrow function*, also written $\mathbf{T}$, which assigns to each arrow $f:c \rightarrow c'$ of $\mathcal{C}$ an arrow $\mathbf{T}f:Tc \rightarrow Tc'$ of $\mathcal{B}$, in such a way that

$$\mathbf{T}(id_c) = id_{Tc}, \qquad \mathbf{T}(g \circ f) = \mathbf{T}g \circ \mathbf{T}f,$$

the latter holding whenever the composite $g \circ f$ is defined in $\mathcal{C}$.

A functor can be described as a function $\mathbf{T}$ from arrows f of $\mathcal{C}$ to arrows $\mathbf{T}f$ of $\mathcal{B}$, carrying each identity of $\mathcal{C}$ to an identity of $\mathcal{B}$ and each composable pair $\langle g, f\rangle$ in $\mathcal{C}$ to a composable pair $\langle \mathbf{T}g, \mathbf{T}f\rangle$ in $\mathcal{B}$, with $\mathbf{T}g \circ \mathbf{T}f = \mathbf{T}(g \circ f)$.

Some examples of functors are listed below.

EXAMPLE 1.1: *The identity functor* $\mathbf{ID}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. It maps each object of $\mathcal{C}$ to itself and each morphism of $\mathcal{C}$ to itself.

EXAMPLE 1.2: *Inclusion functors.* $\subset\!\to : \mathcal{C} \to \mathcal{B}$. Let $\mathcal{C}$ be a subcategory of $\mathcal{B}$. The inclusion functor from $\mathcal{C}$ to $\mathcal{B}$ maps each object and each morphism of $\mathcal{C}$ to itself.

EXAMPLE 1.3: *The powerset functor* $\mathbf{P}$:SET $\to$ SET. For any set A, let $\mathbf{P}(A)$ denote the set of all subsets of A. If $f:A \to B$ is any map, then $\mathbf{P}(f):\mathbf{P}(A) \to \mathbf{P}(B)$ sends each $S \subset A$, to its image $f^{\to}(S) \subset B$. This clearly defines a functor since both $\mathbf{P}(id_A) = id_{P(A)}$ and $\mathbf{P}(g \circ f) = \mathbf{P}(g) \circ \mathbf{P}(f)$.

EXAMPLE 1.4: *Forgetful functors.* Let the objects of $\mathcal{C}$ be sets with a certain structure (for example, groups, topological spaces etc.) and let the morphisms be structure preserving maps (homomorphisms, continuous functions, resp.). Then the forgetful functor $\mathbf{F}:\mathcal{C} \to$ SET assigns to each object its underlying set and to each morphism the corresponding set map. For example the forgetful functor $\mathbf{F}$:GRP $\to$ SET assigns to each group G the set $\mathbf{F}(G)$ of its elements (forgetting the multiplication and hence the group structure), and assigns to each morphism $f:G \to G'$ of groups the same function f, regarded just as a function between sets.

We now introduce the concept of adjunction. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An *adjunction* from $\mathcal{C}$ to $\mathcal{D}$, denoted by $(\mathbf{F},\mathbf{G},\alpha):\mathcal{C} \to \mathcal{D}$, is a triple where **F** and **G** are functors

$$\mathcal{C} \xrightarrow{\mathbf{F}} \mathcal{D} \qquad \mathcal{C} \xleftarrow[\mathbf{G}]{} \mathcal{D}$$

while α is a class of functions which assigns to each pair of objects $c \in \mathcal{C}$, $d \in \mathcal{D}$ a bijection

$$\alpha \equiv \alpha(c,d):\mathcal{D}(\mathbf{F}c,d) \cong \mathcal{C}(c,\mathbf{G}d)$$

which is *natural* in d and c. The naturality of the bijection means that for all $k:d\to d'$ and all $j:c\to c'$ both the diagrams:

$$\begin{tikzcd}
\mathcal{D}(\mathbf{F}c,d) \arrow[r, "\alpha"] \arrow[d, "k_*"'] & \mathcal{C}(c,\mathbf{G}d) \arrow[d, "G(k)_*"] \\
\mathcal{D}(\mathbf{F}c,d') \arrow[r, "\alpha"'] & \mathcal{C}(c,\mathbf{G}d')
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D}(\mathbf{F}c,d) \arrow[r, "\alpha"] \arrow[d, "F(j)^*"'] & \mathcal{C}(c,\mathbf{G}d) \arrow[d, "j^*"] \\
\mathcal{D}(\mathbf{F}c',d) \arrow[r, "\alpha"'] & \mathcal{C}(c',\mathbf{G}d)
\end{tikzcd}$$

will commute.

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In the above two diagram k_* represents composition with k on the left and j^* represents composition with j on the right.

In words the diagrams state:

$\forall\ c,c' \in \mathcal{C},\ \forall\ d,d' \in \mathcal{D},$

$\forall\ k{:}d \to d',\ \forall\ j{:}c \to c', \forall\ h{:}\mathbf{F}c \to d,$

$\alpha(c,d)(k \circ h) = \mathbf{G}(k) \circ \alpha(c,d)(h)$ and

$\alpha(c,d)(h \circ \mathbf{F}(j)) = \alpha(c,d)(h) \circ j$

If $(\mathbf{F},\mathbf{G},\alpha)$ is an adjunction, then $\mathbf{F}$ is said to be a *left adjoint* of $\mathbf{G}$ and $\mathbf{G}$ is said to be a *right adjoint* of $\mathbf{F}$. A functor is said to *have a right adjoint* if it is the left adjoint of a functor, and is said to *have a left adjoint* if it is the right adjoint of a functor.

An adjunction may also be described directly in terms of arrows. It is a bijection which assigns to each arrow $f{:}\mathbf{F}c \to d$ an arrow $\alpha(f) \equiv \mathrm{rad}(f){:}c \to \mathbf{G}d$, *the right adjunct of f,* in such a way that the naturality conditions of the above two diagrams

$$\alpha(f \circ \mathbf{F}h) = \alpha f \circ h, \qquad \alpha(k \circ f) = \mathbf{G}k \circ \alpha f,$$

hold for all $f{:}Fc \to d$ and all arrows $k{:}d \to d'$ and $h{:}c \to c'$. We now introduce the unit η and the counit ε by:

$$\eta = \alpha(\mathrm{id}) \qquad \text{and} \qquad \varepsilon = \alpha^{-1}(\mathrm{id}).$$

These definitions of adjunction, together with the notions of unit and counit, may be equivalently described as saying that the following two diagrams are always satisfied.

Major Diagram

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$
$$\mathcal{C} \xleftarrow[G]{} \mathcal{D}$$

$$\begin{tikzcd}
 & \mathcal{C} & \mathcal{D} \\
 & c \arrow[r, mapsto, "\mathbf{F}"] \arrow[dl, "f"'] \arrow[d, "\eta_c"] \arrow[dl, phantom, shift right=4, "\circ"] & \mathbf{F}c \arrow[dl, mapsto, "\mathbf{G}"'] \arrow[d, dashed, "\exists ! \bar{f}"] \\
\mathbf{G}d & \mathbf{GF}c \arrow[l, "\mathbf{G}\bar{f}"] & d
\end{tikzcd}$$

The Major Diagram is satisfied if the following quantified statements are true:

$$\forall\ c \in |\mathcal{C}|,\ \exists\ \eta_c : c \to \mathbf{GF}c \in \mathcal{C},$$

$$\forall\ d \in |\mathcal{D}|,\ \forall\ f : c \to \mathbf{G}d \in \mathcal{C},$$

$$\exists\,!\ \bar{f} : \mathbf{F}c \to d\ \in \mathcal{D},\ f = \mathbf{G}\bar{f} \circ \eta_c \in \mathcal{C}.$$

Minor Diagram

$$
\begin{tikzcd}
\mathcal{C} & & \mathcal{D} \\
c \arrow[r, "h"] \arrow[d, "\eta_c"'] \arrow[dr, phantom, "\circ"] & c' \arrow[d, "\eta_{c'}"] & \mathbf{F}c \arrow[d, "\exists\, !\,\overline{\eta_{c'} \circ h} = (\text{or} \equiv)\ \mathbf{F}h"] \\
\mathbf{GF}c \arrow[r, "\mathbf{G}(\overline{\eta_{c'} \circ h})"'] & \mathbf{GF}c' & \mathbf{F}c'
\end{tikzcd}
$$

To satisfy the Minor Diagram $\mathbf{F}:\mathcal{C} \to \mathcal{D}$ must be a functor such that the action of $\mathbf{F}$ on h, for $h: c \to c'$ in $\mathcal{C}$, is $\mathbf{F}h = \overline{\eta_{c'} \circ h}$. Note: The Major Diagram only uses the action of **F** on objects. Therefore, if **F** is only known on objects and the Major Diagram is satisfied, the Minor Diagram will stipulate the action of **F** on morphisms such that **F** becomes a functor and $\mathbf{F} \dashv \mathbf{G}$. Specifically, the left side and the right side of the Minor Diagram commuting implies $\mathbf{F}h = \overline{\eta_{c'} \circ h}$

CHAPTER 2

DEVELOPMENT OF STANDARD ALGEBRAIC CONSTRUCTIONS

In this chapter we give three examples of algebraic constuctions. Motivation for the following material can be found in Rotman [13] or Birkhoff and Maclane [1]. We start with booting certain semigroups up to groups.

SECTION 1. SEMIGROUPS TO GROUPS.

Let $(S,*)$ be a cancellation, abelian semigroup. Then $*$ is an operation from $S \times S$ into S such that the following properties hold;

(i) associativity: $\forall\ a,b,c \in S,\quad a*(b*c)=(a*b)*c$

(ii) cancellation: $\forall\ a,b,c \in S,\quad a*b=a*c \Rightarrow b=c$

(iii) commutativity: $\forall\ a,b \in S,\quad a*b=b*a$

$(S,*)$ need not be a group because it might lack identity and inverses.

Now let $(a,b) \in S \times S$. Think of (a,b) as $a*b^{-1}$. Then we define a relation on $S \times S$ by $(a,b) \approx (c,d) \Leftrightarrow a*d=b*c$.

LEMMA 2.1.1: $\approx$ is an equivalence relation on $S \times S$.

PROOF:

(i) *reflexivity*: Let $(a,b) \in S\times S$. $(a,b) \approx (b,a)$ since $a*b=b*a$ by commutivity

(ii) *symmetry*: Let $(a,b) \approx (c,d)$, i.e. $a*d=b*c$. Then $(c,d) \approx (a,b)$ by commutativity of $*$ and symmetry of "=".

(iii) *transitivity*: Let $(a,b) \approx (c,d) \approx (e,f)$. That means $a*d=b*c$ and $c*f=d*e$. Then by associativity and commutativity of $*$ we have:

$$(a*d)*f=(b*c)*f=b*(c*f)=b*(d*e)$$

So

$$(a*d)*f=b*(d*e)$$
$$a*(d*f)=b*(e*d)$$
$$a*(f*d)=(b*e)*d$$
$$(a*f)*d=(b*e)*d$$
$$a*f=b*e \quad \text{(by cancellation)}.$$

$\therefore \approx$ is an equivalence relation on $S\times S$. ∎

Now set $G=(S\times S)/\approx$, i.e. $G=\{[(a,b)]:(a,b)\in S\times S\}$. Define a binary operation $\circ$ on G to be:

$$[(a,b)]\circ[(c,d)]=[a*c,b*d].$$

LEMMA 2.1.2: The operation $\circ$ is well defined on G.

PROOF: Let $(\hat{a},\hat{b})\approx(a,b)$ and $(\hat{c},\hat{d})\approx(c,d)$. We must show

$$(\hat{a}*\hat{c},\hat{b}*\hat{d}) \approx (a*c,b*d).$$

By definition of $\approx$, it suffices to show

$$(\hat{a}*\hat{c})*(b*d)=(\hat{b}*\hat{d})*(a*c).$$

Since $(\hat{a},\hat{b}) \approx (a,b)$ and $(\hat{c},\hat{d}) \approx (c,d)$, then $\hat{a}*b=\hat{b}*a$ and $\hat{c}*d=\hat{d}*c$, respectively. Now by associativity and commutativity,

$$\begin{aligned}(\hat{a}*\hat{c})*(b*d) &= \hat{a}*(\hat{c}*b)*d\\ &= (\hat{a}*b)*(\hat{c}*d)\\ &= (\hat{b}*a)*(\hat{d}*c)\\ &= \hat{b}*(\hat{d}*a)*c\\ &= (\hat{b}*\hat{d})*(a*c)\end{aligned}$$

$\therefore$ $\circ$ is well defined. ∎

Lemma 2.1.3: The operation $\circ$ is associative in $(G,\circ)$.

Proof:

$$\begin{aligned}([(a,b)]\circ[(c,d)])\circ[(e,f)] &= [(a*c,b*d)]\circ[(e,f)]\\ &= [((a*c)*e,(b*d)*f)]\\ &= [(a*(c*e),b*(d*f))]\\ &= [(a,b)]\circ([(c*e,d*f)])\\ &= [(a,b)]\circ([(c,d)]\circ[(e,f)])\end{aligned}$$

$\therefore$ $\circ$ is associative. ∎

Lemma 2.1.4: $(G,\circ)$ has an identity.

Proof: Let $a \in S$ and let $(c,d) \in S\times S$. Then

$$[(a,a)]\circ[(c,d)]=[(a*c,a*d)]$$

and we know $[(a*c,a*d)] \approx [(c,d)]$ iff $(a*c)*d=(a*d)*c$.

Now $(a*c)*d = a*(c*d) = a*(d*c) = (a*d)*c$. Thus

$$[(a,a)] \circ [(c,d)] = [(c,d)]$$

So we have shown that given any $a \in S$, $[(a,a)]$ is the left identity element of $(G,\circ)$. It can be shown in a similar way that $[(a,a)]$ also acts as a right identity.

$\therefore$ $(G,\circ)$ has an identity. ∎

LEMMA 2.1.5: $(G,\circ)$ has the inverse property.

PROOF: Let $(a,b) \in S \times S$ and let $x \in S$. Then

$$[(a,b)] \circ [(b,a)] = [(a*b, b*a)],$$

and $[(a*b, b*a)] \approx [(x,x)]$ iff $(a*b)*x = (b*a)*x$. Since $*$ is commutative the latter is true. Thus the class $[(a,b)]$, has inverse $[(b,a)]$. $\therefore$ every $[(a,b)] \in (G,\circ)$ has an inverse. ∎

LEMMA 2.1.6: The operation $\circ$ is commutative in $(G,\circ)$.

PROOF: We need to show $\forall$ $[(a,b)], [(c,d)] \in (G,\circ)$,

$$[(a,b)] \circ [(c,d)] = [(c,d)] \circ [(a,b)].$$

Using the commutivity of $*$ we have:

$$\begin{aligned} [(a,b)] \circ [(c,d)] &= [a*c , b*d] \\ &= [c*a , d*b] \\ &= [(c,d)] \circ [(a,b)]. \end{aligned}$$

$\therefore$ $(G,\circ)$ is abelian. ∎

From Lemmas 2.1.1 through 2.1.6, we obtain the following theorem:

THEOREM 2.1.7: From $(S,*)$ we get an abelian group $(G,\circ)$.

We now examine the relationship between $(S,*)$ and $(G,\circ)$.

First, fix $d \in S$. Then $\forall\, a \in S$, we have the class $[(a*d,d)]$. Next, define a mapping $\eta_d : S \rightarrow G$ by

$$\eta_d(a)=[(a*d,d)].$$

LEMMA 2.1.8,: There is only one η; i.e., $\forall\, c \in S$, $\eta_c = \eta_d$.

PROOF: Let $a \in S$. We note $[(a*c,c)] \approx [(a*d,d)]$ iff

$(a*c)*d=c*(a*d)$. Since this is true by the commutativity and associativity of $*$ we have

$\eta_c(a)=[(a*c,c)] \approx [(a*d,d)]=\eta_d(a)$. $\therefore$ $\forall\, c \in S$, $\eta_c = \eta_d$. ∎

LEMMA 2.1.9: η is a semigroup homomorphism from $(S,*)$ to

$(G,\circ)$.

PROOF: Let $a,b \in S$. We must show $\eta(a*b) = \eta(a)\circ\eta(b)$. Observe

$$\eta(a*b) = [(a*b)*c,c)],$$

and

$$\eta(a)\circ\eta(b) = (a*c,c)]\circ[(b*c,c)]$$

$$= [((a*c)*(b*c),c*c)].$$

And $[((a*c)*(b*c),c*c)] = [((a*b)*c,c)]$ iff $(a*b)*c*(c*c) = c*(a*c)*(b*c)$. By commutativity and associativity of $*$ we have:

$$(a*b)*c*(c*c) = c*(a*b)*(c*c)$$

$$= c*a*(b*c)*c$$

$$= c*(a*c)*(b*c).$$

$\therefore\ \eta(a*b) = \eta(a)\circ\eta(b)$. ∎

Lemma 2.1.10: η is one to one.

Proof: Let $a,b \in S$, and let $\eta(a) = \eta(b)$. Then

$$\eta(a) = [(a*c,c)] = [(b*c,c)] = \eta(b).$$

But $[(a*c,c)] = [(b*c,c)]$ iff $a*c = b*c$ so, by cancellation $a = b$. $\therefore\ \eta$ is one to one. ∎

Lemmas 2.1.8 through 2.1.10 prove the following theorem.

Theorem 2.1.11: η is a homomorphic embedding of $(S,*)$ into $(G,\circ)$.

Note: At this point we will drop the inner parentheses when referring to a member of an equivalence class. This reduction of parentheses is only a shift of notation and does not represent a change of meaning.

Lemma 2.1.12: Let $(\hat{G},\hat{\circ})$ be a group and let $f:(S,*) \rightarrow (\hat{G},\hat{\circ})$ be a semigroup homomorphism. If $\overline{f}:(G,\circ) \rightarrow (\hat{G},\hat{\circ})$ is a homomorphism making the following triangle commute, then h is of the form

$$\overline{f}[a,b] = f(a)\hat{\circ}(-)f(b).$$

Where "-" indicates the inverse operation in $\hat{G}$.

$$\begin{tikzcd}
(S,*) \arrow[r, "\eta"] \arrow[dr, "f"'] & (G,\circ) \arrow[d, "\overline{f}"] \arrow[dl, phantom, "\circ"] \\
 & (\hat{G},\hat{\circ})
\end{tikzcd}$$

PROOF: Let $\overline{f}$ be a homomorphism that makes the triangle commute, and let $a \in S$. Then, given $[a*a,a] \in G$,

$$f(a)=(\overline{f}\circ\eta)(a)=\overline{f}(\eta(a))=\overline{f}[a*a,a].$$

Claim: $[a,b]=[a*a,a]\circ(-)[b*b,b]$. To justify this claim, we observe

$$\begin{aligned}
[a*a,a]\circ(-)[b*b,b] &= [a*a,a]\circ[b,b*b] \\
&= [(a*a)*b,a*(b*b)] \\
&= [a,b]
\end{aligned}$$

where the last "=" above follows from

$$a*(a*b*b) = b*(a*a*b),$$

which is true by associativity and commutativity of $*$. Now to finish the proof of the lemma, $\overline{f}$ is a homomorphism implies that;

$$\begin{aligned}
\overline{f}[a,b] &= \overline{f}([a*a,a]\circ(-)[b*b,b]) \\
&= \overline{f}[a*a,a]\hat{\circ}(-)\overline{f}[b*b,b] \\
&= f(a)\hat{\circ}(-)f(b)
\end{aligned}$$

$\therefore\ \overline{f}[a,b] = f(a)\hat{\circ}(-)f(b)$. ∎

Lemma 2.1.13: If $\overline{f}:(G,\circ) \to (\hat{G},\hat{\circ})$ is defined by $\overline{f}[a,b] = f(a)\hat{\circ}(-)f(b)$, then $\overline{f}$ is a homomorphism that makes the above triangle commute.

Proof: Let $[a,b],[c,d] \in G$, and let $\overline{f}:(G,\circ) \to (\hat{G},\hat{\circ})$ be defined by $\overline{f}[a,b] = f(a)\hat{\circ}(-)f(b)$. We first show $\overline{f}$ is a homomorphism. Note

$$\begin{aligned}
\overline{f}([a,b]\circ[c,d]) &= \overline{f}(a*c,b*d) \\
&= f(a*c)\hat{\circ}(-)f(b*d) \\
&= f(a)\hat{\circ}(-)f(c)\hat{\circ}(-)f(b)\hat{\circ}(-)f(d) \\
&= (f(a)\hat{\circ}(-)f(b))\hat{\circ}(f(c)\hat{\circ}(-)f(d)) \\
&= \overline{f}[a,b]\hat{\circ}\overline{f}[c,d].
\end{aligned}$$

Therefore, $\overline{f}$ is a homomorphism. We next show $\overline{f}$ makes the triangle commute.

$$\begin{aligned}
f(a) &= f(a)\hat{\circ}\hat{e} \\
&= f(a)\hat{\circ}(f(a)\hat{\circ}(-)f(a)) \\
&= f(a*a)\hat{\circ}(-)f(a) \\
&= \overline{f}[a*a,a] \\
&= \overline{f}(\eta(a))
\end{aligned}$$

$\therefore$ $\overline{f}$ is a homomorphism that makes the triangle commute. ∎

Lemmas 2.1.12 and 2.1.13 prove the following theorem.

Theorem 2.1.14: If $f:(S,*) \to (\hat{G},\hat{\circ})$ is a homomorphism then $\exists$ a unique homomorphism $\overline{f}:(G,\circ) \to (\hat{G},\hat{\circ})$ such that $f = \overline{f}\circ\eta$.

SECTION 2. SEMIRINGS TO RINGS.

All our work in Section 1 has concerned a set on which a single binary operation has been defined. In this section we look at a set on which two binary operations are defined. We now define an algebraic structure with two binary operations that we will call a *semiring*.

DEFINITION 2.2.1: $(S,+,\cdot)$ is a *semiring* iff $(S,+)$ is an abelian, cancellation semigroup, $(S,\cdot)$ is a semigroup, and $(S,+,\cdot)$ is distributive. We say $(S,+,\cdot)$ is a *commutative semiring* iff $(S,\cdot)$ is also abelian; and $(S,+,\cdot)$ is a *commutative semiring with unity* iff $(S,\cdot)$ further has identity 1.

DEFINITION 2.2.2: A *ring* $(R,+,\cdot)$ is a set R with two binary operations + and $\cdot$, which we call addition and multiplication, defined on R such that $(R,+)$ is an abelian group, multiplication is associative, and multiplication is left and right distributive over addition

We now look at booting a [commutative] semiring $(S,+,\cdot)$ to a [commutative] ring $(R,+,\cdot)$, and a commutative semiring with unity to a commutative ring with unity. In the construction of $(R,+,\cdot)$ from $(S,+,\cdot)$ the additive part is done the same as was shown for booting $(S,*)$ to $(G,\circ)$ in Section 1. Thus the construction of the equivalence classes of $(R,+)$ and the operation of addition are the same as defined above in Section 1. Therefore, only the multiplicative proofs will be shown here.

We now wish to define a multiplication on R and prove that the multiplication properties of $(S,+,\cdot)$ are preserved in $(R,+,\cdot)$. Let $a,b,c,d \in R$, and let $[a,b],[c,d]$ be members of $(R,+,\cdot)$. We define the operation of multiplication of the members of equivalence classes of $(R,+,\cdot)$ as:

$$[a,b][c,d] = [ac+bd,ad+bc],$$

(where '$\cdot$' is written as juxtaposition).

We start with showing that $(R,+,\cdot)$ is a ring.

Lemma 2.2.3: Multiplication is well defined in $(R,+,\cdot)$.

Proof: We must show that our definition of multiplication does not depend on the choice of representatives of equivalence classes. Let $[a,b],[c,d],[\hat{a},\hat{b}],[\hat{c},\hat{d}]$ be elements of $(R,+,\cdot)$, and let $(a,b) \approx (\hat{a},\hat{b})$ and $(c,d) \approx (\hat{c},\hat{d})$. We must show

$$[ac+bd,ad+bc] \approx [\hat{a}\hat{c}+\hat{b}\hat{d},\hat{a}\hat{d}+\hat{b}\hat{c}].$$

Using our definition of $\approx$, it suffices to show

$$(ac+bd)(\hat{a}\hat{d}+\hat{b}\hat{c}) = (ad+bc)(\hat{a}\hat{c}+\hat{b}\hat{d}).$$

By our definition of $\approx$ and by commutivity of multiplication in S, we know $a\hat{b}=b\hat{a}=\hat{a}b$ and $c\hat{d}=d\hat{c}=\hat{c}d$. Therefore,

$$\begin{aligned}(ac+bd)(\hat{a}\hat{d}+\hat{b}\hat{c}) &= ac\hat{a}\hat{d}+ac\hat{b}\hat{c}+bd\hat{a}\hat{d}+bd\hat{b}\hat{c}\\ &= a\hat{a}c\hat{d}+a\hat{b}c\hat{c}+\hat{a}bd\hat{d}+b\hat{b}\hat{c}d\\ &= a\hat{a}\hat{c}d+\hat{a}bc\hat{c}+a\hat{b}d\hat{b}+b\hat{b}c\hat{d}\\ &= a\hat{a}\hat{c}d+a\hat{b}d\hat{b}+\hat{a}bc\hat{c}+b\hat{b}c\hat{d}\\ &= ad\hat{a}\hat{c}+ad\hat{b}\hat{d}+bc\hat{a}\hat{c}+bc\hat{b}\hat{d}\\ &= (ad+bc)(\hat{a}\hat{c}+\hat{b}\hat{d}).\end{aligned}$$

$\therefore$ multiplication in $(R,+,\cdot)$ is well defined. ∎

Lemma 2.2.4: Multiplication in $(I,+,\cdot)$ is associative.

Proof:

$$\begin{aligned}([a,b][c,d])[e,f] &= [ac+bd,ad+bc][e,f]\\ &= [(ac+bd)e+(ad+bc)f,(ac+bd)f+(ad+bc)e]\\ &= [ace+bde+adf+bcf,acf+bdf+ade+bce]\\ &= [ace+adf+bcf+bde,acf+ade+bce+bdf]\\ &= [a(ce+df)+b(cf+de),a(cf+de)+b(ce+df)]\\ &= [a,b][ce+df,cf+de]\\ &= [a,b]([c,d][e,f])\end{aligned}$$

$\therefore$ multiplication in $(R,+,\cdot)$ is associative. ∎

Lemma 2.2.5: Multiplication distributes over addition on both sides in $(R,+,\cdot)$.

PROOF: We shall prove left distributivity. Right distributivity can be shown in a similar manner. Let [a,b], [c,d], [e,f] be elements of $(R,+,\cdot)$.

$$\begin{aligned}[e,f]([a,b]+[c,d]) &= [e,f][a+c,b+d]\\ &= [e(a+c)+f(b+d),e(b+d)+f(a+c)]\\ &= [ea+ec+fb+fd,eb+ed+fa+fc]\\ &= [ea+fb+ec+fd,eb+fa+ed+fc]\\ &= [ea+fb,eb+fa]+[ec+fd,ed+fc]\\ &= ([e,f]\,[a,b]) + ([e,f]\,[c,d])\end{aligned}$$

$\therefore$ multiplication distributes over addition on both sides in $(R,+,\cdot)$. ∎

THEOREM 2.2.6: If $(S,+,\cdot)$ is a [commutative] semiring, then $(R,+,\cdot)$ is a [commutative] ring.

PROOF: Lemmas 2.2.2 through 2.2.4 establish that $(R,+,\cdot)$ is a ring, thus we need only show that $(R,\cdot)$ is abelian. Let [a,b],[c,d] be elements of $(R,+,\cdot)$.

$$[a,b][c,d]=[ac+bd,ad+bc]=[ca+db,da+cb]=[c,d][a,b]$$

$\therefore$ $(R,\cdot)$ is abelian. ∎

THEOREM 2.2.7: If $(S,+,\cdot)$ is a [commutative] semiring with unity, then $(R,+,\cdot)$ is a [commutative] ring with unity.

PROOF: We claim that $\forall\ a \in S$, the identity element can be written as $[a+1_S,a]$. Let $c,d,a \in S$, and let $[c,d] \in R$, then

$$[c,d][a+1_S,a] = [c(a+1_S)+da,ca+d(a+1_S)]$$

$$= [ca+c+da, ca+da+d]$$

$$= [c+ca+da, d+ca+da]$$

$$= [c,d].$$

$\therefore\ \forall\ [c,d] \in R\ \ [c,d][a+1_S, a] = [c,d].$ ▮

Now let $a,b,c,d \in S$ and let $[a,b],[c,d] \in R$. Define a mapping $\eta : S \to R$ by $\eta(a)=[a+a,a]$.

LEMMA 2.2.8: η is a homomorphism with respect to addition and with respect to multiplication.

PROOF: Since the additive part follows immediately from the proof of Lemma 2.1.9 by replacing "*" with "+", we only show the multiplicative part:

$$\eta(ab) = [ab+ab, ab] = [2ab, ab],$$

and

$$\begin{aligned}\eta(a)\eta(b) &= [a+a,a][b+b,b] \\ &= [(2a)(2b)+ab, (2a)b+a(2b)] \\ &= [4ab+ab, 2ab+2ab] \\ &= [5ab, 4ab].\end{aligned}$$

By equivalence $[2ab, ab] \approx [5ab, ab]$ iff $2ab+4ab = ab+5ab$. Clearly, the latter is true. $\therefore$ η preserves multiplication. $\therefore$ η is a homomorphism with respect to addition and with respect to multiplication. ▮

Lemma 2.2.9: η is one to one.

Proof: This follows from the proof of Lemma 2.1.10 by replacing "*" with "+". $\therefore$ η is one to one. ∎

Lemma 2.2.10: η preserves unity if S has one.

Proof: Let 1_S be the multiplicative identity in S. Then $\eta(1_S)=[1_S+1_S, 1_S]$. But by the proof of lemma 2.2.5, $[1_S+1_S, 1_S]$ is an allowable form of the unity in R.

$\therefore$ η preserves unity in $(R,+,\cdot)$. ∎

Lemma 2.2.11: Let $(\hat{R},\hat{+},\hat{\cdot})$ be a ring and let $f:(S,+,\cdot) \to (\hat{R},\hat{+},\hat{\cdot})$ be a semiring homomorphism. If $\overline{f}:(R,+,\cdot) \to (\hat{R},\hat{+},\hat{\cdot})$ makes the following triangle commute, then $\overline{f}$ is of the form

$$\overline{f}[a,b] = f(a)\hat{\circ}(-)f(b).$$

Where "-" indicates the inverse operation in either R or $\hat{R}$.

$$\begin{tikzcd}
(S,+,\cdot) \arrow[r, "\eta"] \arrow[dr, "f"'] & (R,+,\cdot) \arrow[d, "\overline{f}"] \arrow[dl, phantom, "\circ"] \\
& (\hat{R},\hat{+},\hat{\cdot})
\end{tikzcd}$$

Proof: Since f and $\overline{f}$ satisfy the hypotheses of Lemma 2.1.12 with respect to the additions involved, then by Lemma 2.1.12 $\overline{f}$ has the desired form. ∎

Lemma 2.2.12: If $\overline{f}:(R,+,\cdot) \rightarrow (\hat{R},\hat{+},\hat{\cdot})$ is defined by $\overline{f}[a,b]= f(a)\hat{+}(-)f(b)$, then $\overline{f}$ is a homomorphism that makes the above triangle commute.

Proof: Since the additive part is the same as the proof of Lemma 2.1.13 only the proof of the multiplicative part will be shown. We first show $\overline{f}$ is a homomorphism with respect to multiplication:

$$
\begin{aligned}
\overline{f}([a,b][c,d]) &= \overline{f}[ac+bd,ad+bc]) \\
&= f(ac+bd)\hat{+}(-)f(ad+bc) \\
&= f(ac)\hat{+}f(bd)\hat{+}(-)f(ad)\hat{+}(-)f(bc) \\
&= f(a)f(c)\hat{+}f(b)f(d)\hat{+}(-)(f(a)f(d))\hat{+}(-)(f(b)f(c)) \\
&= f(a)f(c)\hat{+}(-)(f(a)f(d))\hat{+}(-)(f(b)f(c))\hat{+}f(b)f(d) \\
&= (f(a)\hat{+}(-)f(b))(f(c)\hat{+}(-)f(d)) \\
&= \overline{f}[a,b]\overline{f}[c,d].
\end{aligned}
$$

Finally, it follows from the proof of Lemma 2.1.13 that $\overline{f}$ makes the above triangle commute. $\therefore$ $\overline{f}$ is a ring homomorphism that makes the triangle commute. ∎

Lemmas 2.2.11 and 2.2.12 prove the following theorem.

Theorem 2.2.13: Let $(S,+,\cdot)$ be a [commutative] semiring and let $(\hat{R},\hat{+},\hat{\cdot})$ be a [commutative] ring. If $f:(S,+,\cdot) \rightarrow (\hat{R},\hat{+},\hat{\cdot})$ is a simiring homomorphism then $\exists$ a unique homomorphism $\overline{f}:(R,+,\cdot) \rightarrow (\hat{R},\hat{+},\hat{\cdot})$ such that $f = \overline{f}\circ\eta$.

The above theorem is also true for $(S,+,\cdot)$ a commutative semiring with unity and $(\hat{R},\hat{+},\hat{\cdot})$ a commutative ring with unity.

SECTION 3: Integral Domains to Fields.

In Section 2 we booted a semiring to a ring. This was done with respect to addition and we found that the multiplicative properties were undisturbed. In this section we start with an integral domain and boot to a field [1], but this time the booting is done with respect to multiplication. Our goal here is to develop multiplicative inverses while preserving the additive properties.

We begin this section with some definitions.

DEFINITION 2.3.1: $(I,+,\cdot)$ is an *integral domain* iff $(I,+,\cdot)$ is a commutative ring with unity such that the cancellation laws hold for multiplication: For every $a,b,c \in I$, if $ac=bc$ and $c\neq 0$, then $a=b$.

DEFINITION 2.3.2: A *field* $(F,+,\cdot)$ is a commutative ring with unity such that every nonzero element of $(F,+,\cdot)$ has a multiplicative inverse in $(F,+,\cdot)$.

Let $(I,+,\cdot)$ be an integral domain, let $I^{*}=(I-\{0\})$ and let (a,b) be an element in $I\times I^{*}$. Think of (a,b) as a/b. Now define a relation on $I\times I^{*}$ by $(a,b) \approx (c,d)$ iff $ad = bc$

Lemma 2.3.3: $\approx$ is an equivalence relation on $I\times I^{*}$.

Proof: This follows from the proof of Lemma 2.1.1 by replacing "$*$" with "$\cdot$". ∎

Now set $F = (I\times I^{*})/\approx$, i.e. $F = \{[(a,b)]: a \in I,\ b \in I^{*}\}$. (As was done in Section 1, we will drop the inner parentheses when referring to a member of an equivalence class). Define the operations of addition and multiplication on the equivalence classes of the relation $\approx$ in $I\times I^{*}$ thus:

$$[a,b]+[c,d] = [ad+bc,bd]$$

$$[a,b][c,d] = [ac,bd].$$

Lemma 2.3.4: The operations of multiplication and addition in $(F,+,\cdot)$ are well-defined.

Proof: If we replace "$\circ$" with "$\cdot$", in the proof of Lemma 2.1.2 the multiplicative part of the proof follows immediately. We now show that our definition of addition is well defined in $(F,+,\cdot)$. Let $[a,b],[c,d],[\hat{a},\hat{b}],[\hat{c},\hat{d}]$ be elements of $(R,+,\cdot)$, and let $(a,b) \approx (\hat{a},\hat{b})$ and $(c,d) \approx (\hat{c},\hat{d})$. We must show

$$[ad+bc,bd] \approx [\hat{a}\hat{d}+\hat{b}\hat{c},+\hat{b}\hat{d}].$$

Using our definition of $\approx$, it suffices to show

$$(ad+bc)(\hat{b}\hat{d}) = bd(\hat{a}\hat{d}+\hat{b}\hat{c}).$$

We know $a\hat{b}=b\hat{a}=\hat{a}b$ and $c\hat{d}=d\hat{c}=\hat{c}d$. Therefore,

$$\begin{aligned}(ad+bc)(\hat{b}\hat{d}) &= ad\hat{b}\hat{d}+bc\hat{b}\hat{d}\\ &= a\hat{b}d\hat{d}+c\hat{d}b\hat{b}\\ &= b\hat{a}d\hat{d}+d\hat{c}b\hat{b}\\ &= bd\hat{a}\hat{d}+bd\hat{b}\hat{c}\\ &= bd(\hat{a}\hat{d}+\hat{b}\hat{c}).\end{aligned}$$

$\therefore$ addition in $(F,+,\cdot)$ is well-defined. ∎

Lemma 2.3.5: Addition and multiplication in $(F,+,\cdot)$ are associative.

Proof: The multiplicative part of the proof is the same as the proof of Lemma 2.1.3 if we replace "$\circ$" with "$\cdot$". Let $[a,b],[c,d],[e,f] \in F$,

$$\begin{aligned}[a,b]+([c,d]+[e,f]) &= [a,b]+[cf+de,df]\\ &= [a(df)+b(cf+de),b(de)]\\ &= [adf+bcf+bde,bde]\\ &= [(ad+bc)f+bde,(bd)e]\\ &= [ad+bc,bd]+[e,f]\\ &= ([a,b]+[c,d])+[e,f]\end{aligned}$$

$\therefore$ multiplication and addition in $(F,+,\cdot)$ are associative. ∎

Lemma 2.3.6: Addition and multiplication in $(F,+,\cdot)$ are commutative.

Proof: The multiplicative part of the proof is analogous to the proof of Lemma 2.1.6 if we replace "$\circ$" with "$\cdot$". For the additive part let $[a,b],[c,d] \in F$:

$$\begin{aligned}[a,b]+[c,d] &= [ad+bc,bd]\\ &= [cb+da,db]\\ &= [c,d]+[a,b]\end{aligned}$$

$\therefore$ multiplication and addition in $(F,+,\cdot)$ are commutative. ∎

Lemma 2.3.7: Multiplication is left and right distributive over addition in $(F,+,\cdot)$.

Proof: Let $[a,b],[c,d],[e,f] \in F$.

$$\begin{aligned}[a,b]([c,d]+[e,f]) &= [a,b][cf+de,df]\\ &= [a(cf+de),bdf]\\ &= [acf+ade,bdf],\end{aligned}$$

and

$$\begin{aligned}[a,b][c,d]+[a,b][e,f] &= [ac,bd]+[ae,bf]\\ &= [acbf+bdae,bdbf].\end{aligned}$$

Now by our definition of equivalence

$$[acf+ade,bdf] \approx [acbf+bdae,bdbf] \text{ iff}$$
$$(acf+ade)(bdbf) = (bdf)(acbf+bdae).$$

Clearly, we can see that if we multiply both sides, these two are equal. And by the commutativity of multiplication, right distributivity follows. $\therefore$ multiplication in $(F,+,\cdot)$

distributes over addition. ∎

Lemma 2.3.8: [0,1] is the zero class of $(F,+,\cdot)$.

Proof: First we show $\forall$ [a,b] $\in (F,+,\cdot)$, {a,b]+[0,1] = [a,b]. Let [a,b] $\in (F,+,\cdot)$,

$$[a,b]+[0,1] = [a1+b0,b1] = [a,b].$$

Now we must show $\forall\ a\neq 0, b\neq 0 \in I$, [0,a] = [0,b], and this is obviously true since $0b = a0$. $\therefore$ [0,1] is the zero class of $(F,+,\cdot)$. ∎

Lemma 2.3.9: $\forall$ [a,b] $\in (F,+,\cdot)$, [a,b] is not the zero class of $(F,+,\cdot)$ iff $a\neq 0, b\neq 0$ in I.

Proof: Let $[a,b]\neq[0,1] \in (F,+,\cdot)$, then $a = a1 \neq b0 = 0$ so $a\neq 0, b\neq 0$. Now let $a\neq 0, b\neq 0$ in I, [a,b] $\approx$ [0,1] iff $a1 = b0$. Clearly this is not true. $\therefore$ [a,b] is not the zero class of $(F,+,\cdot)$. ∎

Lemma 2.3.10: $(F-\{[0,1]\},\cdot)$ is an abelian group and so has multiplicative inverses for each non-zero class.

Proof: Given Lemma 2.3.9, the reader can verify that $(F-\{[0,1]\},\cdot)$ is the abelian group produced by Section 1 from the abelian, cancellation semigroup $(I-\{0\},\cdot)$ with multiplicative identity. $\therefore$ $(F-\{[0,1]\},\cdot)$ has multiplicative inverses $\forall$ non-zero class. ∎

LEMMA 2.3.11: $(F,+,\cdot)$ has unity, namely $[1,1]$.

PROOF: $[1,1]$ is the identity from the group $(F-\{[0,1]\},\cdot)$ of Lemma 2.3.10. We need only show that it is the multiplicative identity of the zero class $[0,1]$:

$$[0,1][1,1] = [0,1]$$

is clearly true. $\therefore$ the unity of $(F,+,\cdot)$ is $[1,1]$. ∎

LEMMA 2.3.12: Every $[a,b]$ in $(F,+,\cdot)$ has an additive inverse in $(F,+)$.

PROOF: We claim that $\forall\ [a,b] \in (F,+,\cdot)$, the additive inverse of $[a,b]$ is $[-a,b]$ where $-a$ is the additive inverse of a in I. Let $[a,b] \in (F,+,\cdot)$, and let $-a$ be the additive inverse of a in I. We must to show $[a,b]+[-a,b] = [0,1]$.

$$\begin{aligned}[a,b]+[-a,b] &= [ab+b(-a),bb] \\ &= [ab+(-ab),bb] \\ &= [0,bb].\end{aligned}$$

And $[0,bb] = [0,1]$ since $0\cdot1 = bb\cdot0$. $\therefore\ \forall\ [a,b] \in (F,+,\cdot)$, the additive inverse of $[a,b]$ is $[-a,b]$, where $-a$ is the additive inverse of a in I. ∎

Lemma 2.3.4 through Lemma 2.3.12 establish the following theorem.

THEOREM 2.3.13: $(F,+,\cdot)$ is a field.

Now let $1 \in I$ and define a mapping $\eta : (I,+,\cdot) \to (F,+,\cdot)$ by $\forall\, a \in I,\ \eta(a) = [a,1]$.

Lemma 2.3.14: η is well defined and one-to-one.

Proof: Let $a,b \in I$. Then:

$$\eta(a) = \eta(b) \Leftrightarrow [a,1] = [b,1] \Leftrightarrow a1 = 1b \Leftrightarrow a = b\ .$$

$\therefore$ η is well defined and one-to-one. ▮

Lemma 2.3.15: η is an integral domain homomorphism with respect to addition and with respect to multiplication.

Proof: Let $a,b \in I$. For the additive part:

$$\eta(a)+\eta(b) = [a,1]+[b,1] = [a1+1b,1] = [a+b,1] = \eta(a+b).$$

Now for the multiplicative part:

$$\eta(a)\eta(b) = [a,1][b,1] = [ab,1] = \eta(ab).$$

$\therefore$ η preserves addition and multiplication. ▮

Lemma 2.3.16: η preserves both the multiplicative identity and the additive identity.

Proof: 1 is the multiplicative identity, and 0 is the additive identity in I. By our definition of η:

$$\eta(1) = [1,1] \quad \text{and} \quad \eta(0) = [0,1].$$

$\therefore$ η preserves both identities. ▮

$$\begin{tikzcd} (I,+,\cdot) \arrow[r, "\eta"] \arrow[dr, "f"'] & (F,+,\cdot) \arrow[d, "\overline{f}"] \\ & (\hat{F},\hat{+},\hat{\cdot}) \end{tikzcd}$$

Lemma 2.3.17: Let $(I,+,\cdot)$ be an integral domain, let $(\hat{F},\hat{+},\hat{\cdot})$ be a field and let $f:(I,+,\cdot) \rightarrow (\hat{F},\hat{+},\hat{\cdot})$ be an integral domain homomorphism. If $\overline{f}:(F,+,\cdot) \rightarrow (\hat{F},\hat{+},\hat{\cdot})$ is a homomorphism making the above triangle commute, then $\overline{f}$ is of the form

$$\overline{f}[a,b] = f(a)(f(b))^{-1},$$

(where "$^{-1}$" indicates the inverse operation in $\hat{F}$) and f is a monomorphism.

Proof: Let $\overline{f}$ be a homomorphism that makes the triangle commute, and let $a \in I$. Then, given $[a,1] \in F$,

$$f(a) = (\overline{f} \circ \eta)(a) = \overline{f}(\eta(a)) = \overline{f}[a,1].$$

In the sequel "$^{-1}$" refers to the inverse operation in F.

Claim: $[a,b]=[a,1]([b,1])^{-1}$. We justify this claim by noting

$$\begin{aligned} [a,1]([b,1])^{-1} &= [a,1][1,b] \\ &= [a1,1b] \\ &= [a,b]. \end{aligned}$$

Now, since $\overline{f}$ is a homomorphism:

$$\begin{aligned} \overline{f}[a,b] &= \overline{f}([a,1]([b,1])^{-1}) \\ &= \overline{f}[a,1]\overline{f}([b,1])^{-1}) \\ &= f(a)(f(b))^{-1}. \end{aligned}$$

To see how this forces f to be a monomorphism, the well-definedness of $\overline{f}$ implies that f(b) is invertible when $b \neq 0$, that is, $b \neq 0 \Rightarrow f(b) \neq 0$. This says $\ker(f) = \{0\}$, which implies f is injective. ∎

Lemma 2.3.18: If $f:(I,+,\cdot) \to (\hat{F},\hat{+},\hat{\cdot})$ is a monomorphism and $\overline{f}:(F,+,\cdot) \to (\hat{F},\hat{+},\hat{\cdot})$ is defined by $\overline{f}[a,b] = f(a)(f(b))^{-1}$, then $\overline{f}$ is a field homomorphism that makes the above triangle commute.

Proof: We first note that the injectivity of f implies $\ker(f)=\{0\}$, so that $b \neq 0$ implies $f(b) \neq 0$. It follows that f(b) is invertible and $\overline{f}$ is well-defined. Next we show that $\overline{f}$ is a homomorphism with respect to both addition and multiplication. Let $[a,b],[c,d] \in (F,+,\cdot)$. Then,

$$\begin{aligned}
\overline{f}([a,b]+[c,d]) &= \overline{f}(ad+bc,bd]) \\
&= f(ad+bc)(f(bd))^{-1} \\
&= (f(ad)+f(bc))(f(b)f(d))^{-1} \\
&= f(a)f(d)f(d)^{-1}f(b)^{-1}+f(b)f(c)f(d)^{-1}f(b)^{-1} \\
&= f(a)f(b)^{-1}+f(b)f(b)^{-1}f(c)f(d)^{-1} \\
&= f(a)f(b)^{-1}+f(c)f(d)^{-1} \\
&= \overline{f}[a,b]+\overline{f}[c,d].
\end{aligned}$$

And,

$$\begin{aligned}
\overline{f}([a,b][c,d]) &= \overline{f}[ac,bd] \\
&= f(ac)(f(bd))^{-1} \\
&= f(a)f(c)(f(b)f(d))^{-1} \\
&= f(a)f(c)f(d)^{-1}f(b)^{-1}
\end{aligned}$$

$$= f(a)f(b)^{-1}f(c)f(d)^{-1}$$
$$= \overline{f}[a,b]\overline{f}[c,d].$$

Therefore $\overline{f}$ is a homomorphism. Now we show that $\overline{f}$ makes the above triangle commute.

$$\begin{aligned} f(a) &= f(a)\hat{e} \\ &= f(a)f(1) \\ &= f(a)f(1)^{-1} \\ &= \overline{f}[a,1] \\ &= \overline{f}(\eta(a)) \end{aligned}$$

$\therefore$ $\overline{f}$ is a field homomorphism that makes the triangle commute. ▮

We can use the nature of field homomorphisms to derive the following lemma about integral domain homomorphisms from integral domains to fields.

Lemma 2.3.19: Let $(F,+,\cdot)$ and $(\hat{F},\hat{+},\hat{\cdot})$ be fields, and let $\overline{f}:F\to\hat{F}$ be a field homomorphism, then $\overline{f}$ is injective on F.

Proof: We must show that $\ker(\overline{f})=\{0\}$. Let $a\neq 0$ be in F. $a\neq 0$ implies a has a multiplicative inverse therefore $a\,a^{-1}=1$ and $\overline{f}(a\,a^{-1})=\overline{f}(1)$. So

$$\overline{f}(a)\,\overline{f}(a^{-1})=\overline{f}(a\,a^{-1})=\overline{f}(1)=\hat{1}\neq\hat{0}.$$

Now by the zero divisor law in $\hat{F}$, $\overline{f}(a)\neq\hat{0}$. Thus we conclude $a\notin\ker(\overline{f})$. ▮

Lemmas 2.3.17, 2.3.18 and 2.3.19 prove the following theorem.

THEOREM 2.3.20: If $f:(I,+,\cdot)\to(\hat{F},\hat{+},\hat{\cdot})$ is an integral domain homomorphism, then $\exists\,!$ monomorphism $\overline{f}:(F,+,\cdot)\to(\hat{F},\hat{+},\hat{\cdot})$ such that $f=\overline{f}\circ\eta$ iff f is a monomorphism.

CHAPTER 3

STANDARD ALGEBRAIC CONSTRUCTIONS INTERPRETED AS ADJUNCTION OF CATEGORIES

In Chapter 2 we showed three examples of algebraic constructions. We will now interpret each of these constructions as an adjunction between well known categories. Also we will use the adjunction to reveal more information about the construction than was possible in Chapter 2. The morphisms in each of the categories discussed in this chapter are functions with certain restrictions and the composition is ordinary function composition in SET.

SECTION 1. SEMIGROUPS AND ABELIAN GROUPS.

In this section we consider the adjunction between SEMIGRP and ABELGRP. A discussion of this adjunction can be found in Herrlich [8].

The objects of SEMIGRP, collectively denoted |SEMIGRP|, are the class of all cancellation, abelian semigroups, and the morphisms of SEMIGRP are homomorphisms between cancellation, abelian semigroups. Similarly, the objects of ABELGRP, collectively denoted |ABELGRP|, are the class of all abelian groups, and the morphisms of ABELGRP are homomorphisms between abelian groups.

Now we define the functors **G** and **V**, where **G**:SEMIGRP $\rightarrow$ ABELGRP and **V**:ABELGRP $\rightarrow$ SEMIGRP.

The action of **G** on objects of SEMIGRP is defined as the booting of a cancellation, abelian semigroup to an abelian group as in section 1 of chapter 2. Thus for (S,*) and (G,o) from Section 2.1, **G**((S,*)) = (G,o), and **G**(f), for f a semigroup morphism, will be stipulated later. The action of **V** on objects of ABELGRP is defined as **V**((G,o)) = (G,o) viewed as a cancellation abelian semigroup, and **V**(f) = f viewed as a semigroup homomorphism. Therefore, **V**:ABELGRP $\rightarrow$ SEMIGRP is a forgetful functor since **V** forgets the structure of (G,o).

The Major Diagram of Chapter 1 can be represented thus:

$$\begin{tikzcd}
 & \text{SEMIGRP} & & \text{ABELGRP} \\
 & \forall (S,*) \in |\text{SEMIGRP}| \arrow[rr, mapsto, "\mathbf{G}"] \arrow[dl, "f"'] \arrow[d, "\exists\,\eta"] & & \mathbf{G}(S,*) \in |\text{ABELGRP}| \arrow[dll, "\mathbf{V}"] \arrow[d, dashed, "\exists!\,\overline{f}"] \\
\mathbf{V}(\hat{G},\hat{\circ}) \arrow[r, phantom, "\circ"] & \mathbf{VG}(S,*) \in |\text{SEMIGRP}| \arrow[l, "\mathbf{V}(\overline{f}) \equiv \overline{f}"] & & (\hat{G},\hat{\circ}) \in |\text{ABELGRP}|
\end{tikzcd}$$

Now, given **V** defined above, **V** has a left adjoint whose action on objects coincides with **G**, and we see that Section 2.1 justifies the major diagram of Chapter 1. Then the minor diagram stipulates the action of **G** on the morphisms of SEMIGRP so that **G** becomes both a functor and the left adjoint of **V**. Specifically for f a semigroup homomorphism, $\mathbf{G}(f) = \overline{\eta \circ f}$.

We now give an illustration of adjunction between SEMIGRP and ABELGRP. Let $(N,+), (N_e,+) \in |\text{SEMIGRP}|$, where $(N,+)$ is the semigroup comprised of the natural numbers with the binary operation of addition; and $(N_e,+)$ is the abelian group comprised of the even natural numbers with the binary operation of addition. Let $(Z,+), (Z_e,+) \in |\text{ABELGRP}|$, where $(Z,+)$ is the abelian group comprised of the integers with the binary operation of addition; and $(Z_e,+)$ is the abelian group comprised of the even integers with the binary operation of addition. Now define a semigroup homomorphism $f:(N,+) \to (N_e,+)$ by $f(n) = 2n \ \forall\ n \in N$.

From Section 2.1 we have $\eta_1: N \to Z$ and $\eta_2: N_e \to Z_e$ defined to be mappings such that $\eta_1(a) = [a+a, a] = [2a, a]\ \forall\ a \in N$ and $\eta_2(b) = [b+b, b] = [2b, b]\ \forall\ b \in N_e$.

Consider this representation of the Minor Diagram:

$$\begin{tikzcd}
(N,+) \arrow[r, "f"] \arrow[d, "\eta_1"'] \arrow[dr, "\eta_2\circ f"] & (N_e,+) \arrow[d, "\eta_2"] & (Z,+) \arrow[d, "\exists!\ \overline{\eta_2\circ f}"] \\
(Z,+) \arrow[r, "\mathbf{V}(\eta_2\circ f)"'] & (Z_e,+) & (Z_e,+)
\end{tikzcd}$$

The above diagram stipulates $\mathbf{G}(f) = \mathbf{V}(\overline{\eta_2\circ f}) = \overline{\eta_2\circ f}$ such that $\overline{\eta_2\circ f}:(Z,+) \rightarrow (Z_e,+)$ is a mapping from the equivalence classes of the integers to the equivalence classes of the even integers. So given $[m,n] \in (Z,+)$ we expect $\overline{\eta_2\circ f}([m,n]) = 2[m,n] = [m,n] + [m,n] = [2m,2n]$. To check this, let $[m,n] \in (Z,+)$. Then

$$\begin{aligned}
\mathbf{V}(\overline{\eta_2\circ f})([m,n]) &= \overline{(\eta_2\circ f)}([m,n]) \\
&= [f(m)+f(m),f(m)]+[f(n)+f(n),f(n)]^{-1} \\
&= [2m+2m,2m]+[2n+2n,2n]^{-1} \\
&= [4m,2m]+[2n,4n] \\
&= [4m+2n,2m+4n].
\end{aligned}$$

And $[4m+2n,2m+4n] = [2m,2n]$ since $((4m+2n)+2n) = ((2m+4n)+2m)$.

SECTION 2. SEMIRINGS AND RINGS.

In this section we consider the adjunction between SEMIRNG and RNG.

The objects of SEMIRNG, collectively denoted |SEMIRNG|, are the class of all semirings, and the morphisms of SEMIGRP are homomorphisms between semirings. Similarly, the objects of RNG, collectively denoted |RNG|, are the class of all commutative rings, and the morphisms of RNG are homomorphisms between commutative rings.

Now we define the functors **G** and **V**, where **G**:SEMIRNG $\rightarrow$ RNG and **V**:RNG $\rightarrow$ SEMIRNG.

The action of **G** on objects of SEMIRNG is defined as the booting of a semiring to a ring as in Section 2.2. Thus for $(S,+,\cdot)$ and $(R,+,\cdot)$ from Section 2.2, $\mathbf{G}((S,+,\cdot)) = (R,+,\cdot)$, and $\mathbf{G}(f)$, for f a semiring morphism, will be stipulated later. The action of **V** on objects of RNG is defined as $\mathbf{V}((R,+,\cdot)) = (R,+,\cdot)$ viewed as a semiring, and $\mathbf{V}(f) = f$ viewed as a semiring homomorphism. Therefore, **V**:RNG $\rightarrow$ SEMIRNG is a forgetful functor since **V** forgets the structure of $(R,+,\cdot)$.

The Major Diagram of Chapter 1 can be represented thus:

$$
\begin{tikzcd}
 & \mathrm{SEMIRNG} & \mathrm{RNG} \\
 & \forall\ (S,+,\cdot) \in |\mathrm{SEMIRNG}| \arrow[dl, "f"'] \arrow[d, "\exists\eta"] \arrow[r, mapsto, "\mathbf{G}"] \arrow[dl, phantom, "\circ"] & \mathbf{G}(S,+,\cdot) \in |\mathrm{RNG}| \arrow[dl, "\mathbf{V}"'] \arrow[d, dashed, "\exists ! \overline{f}"] \\
\mathbf{V}(\hat{R},\hat{+},\hat{\cdot}) & \mathbf{VG}(S,+,\cdot) \in |\mathrm{SEMIRNG}| \arrow[l, "\mathbf{V}(\overline{f}) \equiv \overline{f}"] & (\hat{R},\hat{+},\hat{\cdot}) \in |\mathrm{RNG}|
\end{tikzcd}
$$

Now, given **V** defined above, **V** has a left adjoint whose action on objects coincides with **G**, and we see that Section 2.2 justifies the major diagram of Chapter 1. Then the minor diagram stipulates the action of **G** on the morphisms of SEMIRNG so that **G** becomes both a functor and the left adjoint of **V**. Specifically for f a semiring homomorphism, $\mathbf{G}(f) = \overline{\eta \circ f}$.

In addition to the adjunction between SEMIRNG and RNG seen above Theorem 2.2.13 allows for the categorical interpretation of the adjunction between the categories of commutative semirings and commutative rings, and the categories of commutative semirings with unity and commutative rings with unity.

Section 3. Integral Domains and Fields.

In this section we consider the adjunction between INTDOM and FIELD (see [8],[10]).

The objects of INTDOM, collectively denoted |INTDOM|, are the class of all integral domains, and the morphisms of INTDOM are monomorphisms between integral domains. Similarly, the objects of FIELD, collectively denoted |FIELD|, are the class of all fields, and the morphisms of FIELD are homomorphisms between fields.

Now we define the functors **G** and **V**, where **G**:INTDOM $\rightarrow$ FIELD and **V**:FIELD $\rightarrow$ INTDOM.

The action of **G** on objects of INTDOM is defined as the booting of an integral domain to a field as in section 3 of chapter 2. Thus for $(I,+,\cdot)$ and $(F,+,\cdot)$ from Section 2.3, **G**$((I,+,\cdot)) = (F,+,\cdot)$, and **G**$(f)$, for f an integral domain monomorphism, will be stipulated later. The action of **V** on objects of FIELD is defined as **V**$((F,+,\cdot)) = (F,+,\cdot)$ viewed as an integral domain, and **V**$(f) = f$ viewed as an integral domain monomorphism. Therefore, **V**:FIELD $\rightarrow$ INTDOM is a forgetful functor since **V** forgets the structure of $(F,+,\cdot)$.

The Major Diagram of Chapter 1 can be represented thus:

$$\begin{tikzcd}
 & \text{INTDOM} & \text{FIELD} \\
 & \forall\ (I,+,\cdot) \in |\text{INTDOM}| \arrow[r, mapsto, "\mathbf{G}"] \arrow[dl, "f"'] \arrow[d, "\exists\,\eta"] & \mathbf{G}(I,+,\cdot) \in |\text{FIELD}| \arrow[dl, "\mathbf{V}"'] \arrow[d, dashed, "\exists!\,\bar{f}"] \\
\mathbf{V}(\hat{F},\hat{+},\hat{\cdot}) & \mathbf{VG}(I,+,\cdot) \in |\text{INTDOM}| \arrow[l, "\mathbf{V}(\bar{f}) \equiv \bar{f}"] & (\hat{F},\hat{+},\hat{\cdot}) \in |\text{FIELD}|
\end{tikzcd}$$

Now, given **V** defined above, **V** has a left adjoint whose action on objects coincides with **G**, and we see that Section 2.3 justifies the major diagram of Chapter 1. Then the minor diagram stipulates the action of **G** on the morphisms of INTDOM so that **G** becomes both a functor and the left adjoint of **V**. Specifically for f an integral domain monomorphism, $\mathbf{G}(f) = \overline{\eta \circ f}$.

From Theorem 2.3.20 of Section 2.3 we have such an adjunction iff the morphisms of INTDOM are monomorphisms. But are there integral domain homomorphisms that are not monomorphisms, that is, is the restriction required by $\mathbf{G} \dashv \mathbf{V}$ significant?

Consider the following example. Let $(Z,+,\cdot) \in |\mathrm{INTDOM}|$, where $(Z,+,\cdot)$ is the integral domain comprised of the integers with the binary operations of addition and multiplication. Let $(Z_p,+,\cdot)$ be the field comprised of the congruence classes of the integers mod p, where p is prime, with the binary operations of addition and multiplication. Then the mapping $f:(Z,+,\cdot) \rightarrow (Z_p,+,\cdot)$ defined by $f(z)=z(\text{mod } p)$ $(\forall\ z \in Z)$ is a non-injective integral domain homomorphism.

The above two paragraphs seem to be an improvement on a remark of MacLane [10, p.56].

CHAPTER 4

THE FIRST ISOMORPHISM THEOREM INTERPRETED AS AN ADJUNCTION

In this chapter we interpret the First Isomorphism Theorem for Groups as an adjunction of categories.

We begin by stating the First Isomorphism Theorem for Groups. Our statement of this theorem is a combination of Theorem 2.9 and Theorem 3.1 from Fraleigh [5; p.148 and p.181].

THEOREM 4.1.1 (First Isomorphism Theorem for Groups): Let $(G,\circ)$ and $(\hat{G},\hat{\circ})$ be groups, and let $f:G \rightarrow \hat{G}$ be a group homomorphism with kernel K. Then f(G) is a group, and the map $\overline{f}:G/K \rightarrow f^{\rightarrow}(G)$ given by $\overline{f}(aK) = f(a)$ is an isomorphism. If $\eta:G \rightarrow G/K$ is the homomorphism given by $\eta(a) = aK$, then $\forall a \in G,\ f(a)=(\overline{f}\circ\eta)(a)$.

This theorem says that if $(G,\circ)$ and $(\hat{G},\hat{\circ})$ are groups, and if f is a homomorphism from G onto $\hat{G}$, where $K = \ker(f)$, then every element of G/K corresponds to one and only one element of $f^{\rightarrow}(G)\subset\hat{G}$ and $\overline{f}:G/K \rightarrow f^{\rightarrow}(G)$ is an isomorphism. Thus every group homomorphism with domain G gives rise to a factor group G/K, and every factor group G/K gives rise to a homomorphism mapping G into G/K.

THEOREM 4.1.2: $\forall$ group $(G,\circ)$, $\forall$ group $(\hat{G},\hat{\circ})$, $\forall$ $f:G \to \hat{G}$ with kernel K, $\exists$ $\eta:G \to G/K$, $\exists$ $\overline{f}:G/K \to \hat{G}$, such that $\forall a \in G$, $f(a)=(\overline{f}\circ\eta)(a)$. Furthermore, η is given by $\eta(a)=aK$, $\overline{f}$ is given by $\overline{f}(aK)=f(a)$, and the image of G under f is a group.

Theorem 4.1.2 is the quantified restatement of Theorem 4.1.1. We now present a theorem that is more appropriate for our categorical interpretation of The First Isomorphism Theorem for Groups.

THEOREM 4.1.3: $\forall$ group $(G,\circ)$, $\forall$ K, a normal subgroup of G, $\exists$ $\eta:G \to G/K$, $\forall$ group $(\hat{G},\hat{\circ})$, $\forall$ group homomorphism $f:G \to \hat{G}$ with $K=\ker(f)$, $\exists!$ $\overline{f}:G/K \to \hat{G}$, $f=\overline{f}\circ\eta$. Furthermore, η is given by $\eta(a)=aK$, $\overline{f}$ is given by $\overline{f}(aK)=f(a)$ and the image of G under f is a group.

METATHEOREM 4.1.4: Theorem 4.1.2 and Theorem 4.1.3 are equivalent.

PROOF: We first show that Theorem 4.1.2 implies Theorem 4.1.3. Let $(G,\circ)$ be a group, and let K be a normal subgroup of G. Define $\eta:G \to G/K$ by $\eta(a)=aK$, and let $(\hat{G},\hat{\circ})$ and

$f: G \to \hat{G}$ be given. Then by Theorem 4.1.2, $f^{\to}(G)$ is a group and $\exists\ \bar{f}: G/K \to f^{\to}(G)$ by $\bar{f}(aK) = f(a)$, and we claim $\bar{f}$ is the unique solution to $f = (\ ) \circ \eta$. To justify the claim that $\bar{f}$ is unique let $\hat{f}$ be another solution, then $f = \hat{f} \circ \eta$ and $f = \bar{f} \circ \eta$. Now let $a \in G$, then,

$$\begin{aligned} f(a) &= \hat{f} \circ \eta(a) \\ &= \hat{f}(\eta(a)) \\ &= \hat{f}(aK). \end{aligned}$$

And since we know that $\bar{f}(aK) = f(a)$, we conclude $\hat{f} = \bar{f}$. To show that Theorem 4.1.3 implies Theorem 4.1.2 is trivial since ker(f) is a normal subgroup of G, and Theorem 4.1.3 has the stronger ordering of the quantifiers. ∎

The importance of this metatheorem is that Theorem 4.1.1 is equivalent to the formally stronger Theorem 4.1.3, and this formal strength is needed to redescribe Theorem 4.1.1 by adjunction between categories.

The following diagram illustrates the above theorems.

$$\begin{tikzcd} G \arrow[r, "\eta"] \arrow[dr, "f"'] & G/K \arrow[d, "\bar{f}"] \arrow[dl, phantom, "\circ"] \\ & \hat{G} \end{tikzcd}$$

Let GROUP be the category whose objects are groups, and whose morphisms are group homomorphisms. Next we define the category QTGRP. The objects of QTGRP are the class of all ordered pairs of groups (G,H) such that H is a normal subgroup of G. Collectively we denote the objects as $|QTGRP|$. If we let (G,H), $(\hat{G},\hat{H}) \in |QTGRP|$, then $f:(G,H) \rightarrow (\hat{G},\hat{H})$ is a morphism of QTGRP if $f:G \rightarrow \hat{G}$ in GROUP and $f|_H:H \rightarrow \hat{H}$ in GROUP. Thus the morphisms of QTGRP are certain ordered pairs of homomorphisms between certain ordered pairs of groups. Composition of the morphisms of QTGRP is defined as ordinary function composition as from SET. Let $f:(G,H) \rightarrow (\hat{G},\hat{H})$ and $g:(\hat{G},\hat{H}) \rightarrow (\tilde{G},\tilde{H})$ be QTGRP morphisms. Then $g \circ f:G \rightarrow \tilde{G}$ and $(g \circ f)|_H:H \rightarrow \tilde{H}$.

We now interpret the First Isomorphism Theorem for Groups as an adjunction between the categories QTGRP and GROUP. To begin we justify the claim that QTGRP is a category. To show this is true we must show that QTGRP has a well-defined composition of its morphisms and has the identity morphism for each object. First, consider composition. Let $f:(G,H) \rightarrow (\hat{G},\hat{H})$ and $g:(\hat{G},\hat{H}) \rightarrow (\tilde{G},\tilde{H})$ be QTGRP morphisms. Then $g \circ f:G \rightarrow \tilde{G}$ is ordinary function composition as from SET, and $(g \circ f)|_H = g|_{\hat{H}} \circ f|_H:H \rightarrow \tilde{H}$ and both $g \circ f$ and $g|_{\hat{H}} \circ f|_H$ are homomorphisms. The commutative diagram below illustrates composition of QTGRP morphisms.

$$\begin{tikzcd} G \arrow[r, "f"] & \hat{G} \arrow[r, "g"] & \tilde{G} \\ H \arrow[u, hook] \arrow[r, "f|_H"'] \arrow[ur, phantom, "\circ"] & \hat{H} \arrow[u, hook] \arrow[r, "g|_{\hat{H}}"'] \arrow[ur, phantom, "\circ"] & \tilde{H} \arrow[u, hook] \end{tikzcd}$$

Now we need an identity morphism that maps every ordered pair in QTGRP to itself. For each $(G,H) \in |QTGRP|$ $id_{(G,H)} : (G,H) \rightarrow (G,H)$ is given by usual identity functions $id_G : G \rightarrow G$, and $(id_G)|_H : H \rightarrow H$. Thus we see that QTGRP is indeed a category.

We now describe a pair of functors, **P** and **Q**, where **P** maps the object of GROUP into the objects of QTGRP and the morphisms of GROUP into the morphisms of QTGRP; and **Q** maps the objects of QTGRP into the objects of GROUP and the morphisms of QTGRP into the morphisms of GROUP.

$$QTGRP \xrightarrow{Q} GROUP$$
$$QTGRP \xleftarrow[P]{} GROUP$$

The action of **P** on the objects of GROUP is defined as: $\mathbf{P}(G) = (G, \{e\})\ \forall\ G \in |QTGRP|$. Now let $f : G \rightarrow \hat{G}$ be a morphism of GROUP, then $\mathbf{P}(f) = f:(G,\{e\}) \to (\hat{G},\{\hat{e}\})$ where $f:G \to \hat{G}$ and $f|_{\{e\}}:\{e\} \to \{\hat{e}\}$. The action of $\mathbf{P}$ on morphisms of GROUP is illustrated in the following diagram.

$\mathbf{P}(f) = f$ viewed as:

$$\begin{tikzcd} G \arrow[r, "f"] & \hat{G} \\ \{e\} \arrow[u, hook] \arrow[r, "f|_{\{e\}}"'] & \{\hat{e}\} \arrow[u, hook] \end{tikzcd}$$

Let $(G,H) \in |\text{QTGRP}|$. The action of $\mathbf{Q}$ on (G,H) is described as $\mathbf{Q}((G,H)) = G/H$. The action of $\mathbf{Q}$ on the morphisms of QTGRP is not needed at this time and will be determined later. We will use the Minor Diagram from Chapter 1 to determine this action.

We are now in a position to show that $\mathbf{P}$ and $\mathbf{Q}$ satisfy the Major Diagram from Chapter 1, and thus constitute an adjunction of categories if the action of $\mathbf{Q}$ on morphisms of QTGRP is suitably chosen. To begin we give the following representation of the Major Diagram.

Major Diagram

$$\mathrm{QTGRP} \xrightarrow{\mathbf{Q}} \mathrm{GROUP}$$
$$\mathrm{QTGRP} \xleftarrow[\mathbf{P}]{} \mathrm{GROUP}$$

$$\begin{tikzcd}
\mathrm{QTGRP} & & \mathrm{GROUP} \\
& (G,H) \arrow[r, mapsto, "\mathbf{Q}"] \arrow[dl, "f"'] \arrow[d, "\eta"] \arrow[dl, phantom, shift left=3, "\circ"] & G/H \arrow[dl, "\mathbf{P}"'] \arrow[d, dashed, "\exists!\,\overline{f}"] \\
(\hat{G},\{\hat{e}\}) & (G/H,H) \arrow[l, "\overline{f}\equiv\mathbf{P}(\overline{f})"] & \hat{G}
\end{tikzcd}$$

In words the Major Diagram states:

$\forall\ (G,H) \in |\mathrm{QTGRP}|,\ \exists\,\eta : (G,H) \to (G/H,H) \in \mathrm{QTGRP},$

$\forall\ G/H \in |\mathrm{GROUP}|,\ \forall\ f : (G,H) \to (\hat{G},\{\hat{e}\}) \in \mathrm{QTGRP},$

$\exists\,!\ \overline{f} : G/H \to \hat{G} \in \mathrm{GROUP},\ f = \mathbf{P}(\overline{f}) \circ \eta \in \mathrm{QTGRP}.$

We use the First Isomorphism Theorem for Groups to prove that the Major Diagram is satisfied. Because of Metetheorem 4.1.4, we are allowed to use Theorem 4.1.3 as the First Isomorphism Theorem for Groups. We have such an η (which is a group homomorphism). Given $\hat{G}$ and $f : G \to \hat{G}$, where H is the kernel of f, by Theorem 4.1.3, $\exists\ !\ \overline{f} : G/H \to f^{\rightarrow}(G) \subset \hat{G}$. We can regard $\overline{f}$ as an isomorphism (and hence an arrow) of G/H into $\hat{G}$, i.e. $\overline{f} : G/H \to \hat{G}$ is a morphism of GROUP, and from Theorem 4.1.3, $f = \overline{f} \circ \eta$. Thus we see that the above representation of the major diagram is mathematically equivalent to Theorem 4.1.3

We now present the Minor Diagram and use it to determine the action of **Q** on the morphisms of QTGRP

Minor Diagram

$$
\begin{array}{ccc}
\text{QTGRP} & & \text{GROUP} \\
\begin{tikzcd}
(G_1,H_1) \arrow[r, "f"] \arrow[d, "\eta_1"'] \arrow[dr, phantom, "\circ"] & (G_2,H_2) \arrow[d, "\eta_2"] \\
\mathbf{PQ}((G_1,H_1)) \arrow[r, "\mathbf{P}(\overline{\eta_2 \circ f})"'] & \mathbf{PQ}((G_2,H_2))
\end{tikzcd}
&
&
\begin{tikzcd}
\mathbf{Q}((G_1,H_1)) \arrow[d, "\exists ! \, \overline{\eta_2 \circ f}"] \\
\mathbf{Q}((G_2,H_2))
\end{tikzcd}
\end{array}
$$

Since the Major Diagram is satisfied and the action of **Q** on objects of QTGRP is known the Minor Diagram stipulates that for $f:(G_1,H_1) \rightarrow (G_2,H_2)$ $\mathbf{Q}(f)= \overline{\eta_2 \circ f}$. Therefore, we see that $\mathbf{Q}$:QTGRP $\rightarrow$ GROUP is a functor whose action can be described as: $\mathbf{Q}$:|QTGRP| $\rightarrow$ |GROUP|; and $\mathbf{Q}$:(morphisms of QTGRP) $\rightarrow$ (morphisms of GROUP) such that $\mathbf{Q}(f \circ g) = \mathbf{Q}(f) \circ \mathbf{Q}(g)$ (where the first composition is in QTGRP and the second composition is in GROUP) and $\mathbf{Q}(id_{(G,H)})=id_{\mathbf{Q}((G,H))}$ (where $id_{(G,H)}$ is in QTGRP and $id_{\mathbf{Q}((G,H))}$ is in GROUP). We have both the Minor Diagram and the Major Diagram satisfied. Thus we have an adjunction between the categories QTGRP and GROUP. ∎

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