Quotient Spaces Generated by Thomae's Function over the Real Line

by

Chase Reiter

Submitted in Partial Fulfillment of the Requirements

for the Degree of

Master of Science

in the

Mathematics

Program

YOUNGSTOWN STATE UNIVERSITY

May 2023

Quotient Spaces Generated by Thomae's Function over the Real Line

Chase Reiter

I hereby release this thesis to the public. I understand that this thesis will be made available from the OhioLINK ETD Center and the Maag Library Circulation Desk for public access. I also authorize the University or other individuals to make copies of this thesis as needed for scholarly research.

Signature:

Chase Reiter, Student — Date

Approvals:

Jamal K. Tartir, Thesis Advisor — Date

Eric J. Wingler, Committee Member — Date

Stephen E. Rodabaugh, Committee Member — Date

Dr. Salvatore A. Sanders, Dean of Graduate Studies — Date

Table of Contents

Abstract 2

Chapter One: Introduction 3

Chapter Two: Separation Properties in $(\mathbb{R}/\sim)$ 6

Chapter Three: Compactness in $(\mathbb{R}/\sim)$ 11

Chapter Four: Topological Countability in $(\mathbb{R}/\sim)$ 32

Bibliography 34

Abstract

This thesis investigates a topological quotient space $(\mathbb{R}/\sim)$ where $\mathbb{R}$ typically carries the usual topology and the equivalence relation $\sim$ stipulates $x \sim y \iff t(x) = t(y)$ where t is Thomae's function from Thomae's 1875 work, *Einleitung in die Theorie der bestimmten Integrale* [1]. Various topological properties of $(\mathbb{R}/\sim)$ were examined, including compactness, separation axioms, and countability axioms. Under a few popular real-line topologies, the quotient space was often no more separable than T_0. The thesis devotes much of its attention toward a conjecture that postulates whether every real number can be the limit of a sequence of simplest form fractions with non-decreasing denominator. Despite many efforts and trials, a proof of the conjecture was not produced; instead, proofs of various propositions involving prime decomposition of integers were produced. Once this closer inspection of prime decomposition of integers was completed, a weaker form of the conjecture was proven and thereafter applied in a topological context. The thesis closes this topic by asserting observations about the space $(\mathbb{R}/\sim)$ that would follow if the conjecture could be proven true.

Chapter One
Introduction

1. Quotient Spaces

To introduce the thesis, we begin with a collection of widely known definitions in the topological world, most of which comprise elementary concepts regarding quotient topologies, quotient spaces, and the like thereof.

Definition 1.1. *Let X be a set and $\sim$ be an equivalence relation on X. Then the* ***equivalence class*** *$[x]$ of an element $x \in X$ is*

$$[x] = \{y \in X : x \sim y\}$$

Definition 1.2. *Let X be a set and $\sim$ be an equivalence relation on X. Then the set of all equivalence classes of X, denoted $(X/\sim)$, is called the* ***quotient set*** *of X.*

Definition 1.3. *Let X be a set and $\sim$ be an equivalence relation on X. A function $q : X \to (X/\sim)$ given by $q(x) = [x]$ is called a* ***quotient map*** *of X.*

Definition 1.4. *Let $(X, \mathfrak{T})f$ be a topological space. A subset $U \subset (X/\sim)$ is* ***open*** *in $(X/\sim)$ if and only if $\bigcup_{[x]\in U} [x]$ is open in X. Alternatively, we may also state that U is open in $(X/\sim) \iff q^{\leftarrow}(U)$ is open in X.*

Definition 1.5. *The topology induced by q is called the* ***quotient topology*** *$\mathfrak{T}$ on $(X/\sim)$ induced by q and explicitly is given by $\mathfrak{T} = \{U \subset (X/\sim) : q^{\leftarrow}(U) \text{ is open in } X\}$.*

2. Thomae's Function

Carl Johannes Thomae was a German mathematician during the nineteenth century who designed the real-valued function that mapped all simplest form fractions of denominator $n > 0$ to $\frac{1}{n}$, mapped all irrationals to 0, and mapped 0 to 1. It has sometimes been referred to as the popcorn function since, when graphed on a grid, the graph visually resembles the activity of popcorn kernels in a microwave. For the purposes of this endeavor, we will call such a function **Thomae's function**, t.

Definition 2.1. *Define* ***Thomae's function*** $t : \mathbb{R} \to \mathbb{R}$ *via the rule*

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } x = \frac{m}{n}, m \in \mathbb{Z}, n \in \mathbb{N}, \gcd(m, n) = 1 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Thomae's function [1] possesses some interesting properties, the most useful of which for this thesis is that it is periodic with period 1. Another interesting observations is that t is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous at every point of $\mathbb{Q}$, assuming $\mathbb{R}$ has the Euclidean topology. Though these latter two observations are potent, their influence did not affect the endeavors undertaken in this thesis.

3. The Purposes of the Thesis

Now that we have introduced quotient spaces and Thomae's function to the reader, we may introduce the purpose of the thesis. The goal and mission is as follows: Consider the real line $\mathbb{R}$ with some topology and endow $\mathbb{R}$ with the equivalence relation $\sim$ so that for each $x, y \in \mathbb{R}$, $x \sim y \iff t(x) = t(y)$. Thus, if $\frac{a}{n}$ and $\frac{b}{n}$ are simplest form with $n \in \mathbb{N}$, then $t(\frac{a}{n}) = t(\frac{b}{n}) = \frac{1}{n}$. Hence, $\frac{a}{n}, \frac{b}{n} \in [\frac{1}{n}]$. Similarly, if i_1, i_2 are irrational, then $t(i_1) = t(i_2) = 0$; as such, we write $i_1, i_2 \in [\sqrt{2}]$. Thus, $(\mathbb{R}/\sim)$ is the collection of equivalence classes induced by $\sim$. This set is equipped with the quotient topology, i.e. the topology satisfying $U \subset (\mathbb{R}/\sim)$ is open $\iff q^{\leftarrow}(U)$ is open in $\mathbb{R}$. From here, the endeavor undertaken was to investigate the topological properties of such a space. For the majority of this thesis, $\mathbb{R}$ is granted the usual topology; for the section on separation properties, however, four different topologies on $\mathbb{R}$ were tested.

Chapter Two
Separation Properties in $(\mathbb{R}/\sim)$

In this section, we investigate which topological separation axioms $(\mathbb{R}/\sim)$ will satisfy. This section is unique in that $\mathbb{R}$ was endowed with various topologies and the separation of $(\mathbb{R}/\sim)$ was investigated for each topology with which $\mathbb{R}$ was endowed. Before the results for separation are given, as a means of introduction to this topic, some examination of each $[\frac{1}{n}]$ is undertaken.

1. The Equivalence Classes Examined

Here, we investigate the equivalence classes of $\mathbb{R}$ more closely. Consider, for example, the class $[\frac{1}{3}]$. This class contains all irreducible, relatively prime multiples of $\frac{1}{3}$. Note that in terms of modular arithmetic, every numerator is equal to $0, 1,$ or $2 \mod 3$. Explicitly, this means

$$\left[\frac{1}{3}\right] = \left\{\frac{3m+1}{3} : m \in \mathbb{Z}\right\} \cup \left\{\frac{3m+2}{3} : m \in \mathbb{Z}\right\}$$

We can generalize this further for any denominator $n \in \mathbb{N}$. Observe that for any $n \in \mathbb{N}$, the class $[\frac{1}{n}]$ can be written in the following way

$$\left[\frac{1}{n}\right] = \bigcup_{\substack{\gcd(k,n)=1 \\ 1 \leq k < n}} \left\{\frac{n \cdot m + k}{n} : m \in \mathbb{Z}\right\}$$

2. Finding Separation

Below are the investigations into the separation of $(\mathbb{R}/\sim)$ under four well known real-line topologies. These topologies are as follows: the usual/Euclidean topology, the Michael line, the Sorgenfrey line, and the Discrete Rational Extension. For the first three topologies mentioned, it was discovered that $(\mathbb{R}/\sim)$ was only T_0. For the Discrete Rational Extension, Hausdorff (T_2) is achieved.

2.1. **Usual/Euclidean Topology.**

Recall that the usual topology $\mathfrak{T}_{\mathbb{R}}$ is generated by the basis comprised of all nonempty open intervals on the real line $\mathscr{B} = \{(a, b) : a, b \in \mathbb{R}\}$.

To investigate separability under the usual topology, we first need to investigate the open neighborhoods of the quotient space. We frequently utilize the fact that $U \subset (\mathbb{R}/\sim)$ is open $\iff \bigcup_{[x]\in U} [x]$ is open in $\mathbb{R}$. It follows as a corollary that $U \subset (\mathbb{R}/\sim)$ is closed $\iff \bigcup_{[x]\in U} [x]$ is closed in $\mathbb{R}$. We first investigate T_0.

Proposition 2.1. *If $\mathbb{R}$ has the usual topology, $(\mathbb{R}/\sim)$ is T_0.*

Proof.

To see that $(\mathbb{R}/\sim)$ is T_0, let $[x], [y] \in (\mathbb{R}/\sim)$ be nonequal. Then at least one of $[x]$ or $[y]$ is $[\frac{1}{n}]$ for some $n \in \mathbb{N}$. WLOG, $[x] = [\frac{1}{n}]$, then $[y] \in (\mathbb{R}/\sim) \setminus \{[\frac{1}{n}]\}$ which is open in $(\mathbb{R}/\sim)$ since the union of its classes is open in $\mathbb{R}$. Thus, $(\mathbb{R}/\sim)$ is T_0.

■

However, $(\mathbb{R}/\sim)$ is not T_1. A popular characterization of T_1 is that all singletons are closed. Note, however, that $\{[\sqrt{2}]\}$ is not closed in $(\mathbb{R}/\sim)$ since $\cup[\sqrt{2}] = \mathbb{R}\setminus\mathbb{Q}$ is not closed in $\mathbb{R}$ under the usual topology. Thus, $(\mathbb{R}/\sim)$ is T_0 and not T_1 under the usual topology.

2.2. **Michael Line.** Recall that the **Michael Line** topology $\mathfrak{T}_M$ on $\mathbb{R}$ is given by

$$\mathfrak{T}_M = \{U \cup I : U \in \mathfrak{T}_{\mathbb{R}}, I \subset \mathbb{R}\setminus\mathbb{Q}\}$$

Proposition 2.2. *If* $\mathbb{R}$ *has the Michael Line topology,* $(\mathbb{R}/\sim)$ *is* T_0*.*

Proof.

Case 1: Consider $[\frac{1}{m}], [\frac{1}{n}] \in (\mathbb{R}/\sim)$ with $m, n \in \mathbb{N}$. Then

$$\left[\frac{1}{n}\right] \in (\mathbb{R}/\sim)\setminus\left\{\left[\frac{1}{m}\right]\right\} \quad \text{and} \quad \left[\frac{1}{m}\right] \notin (\mathbb{R}/\sim)\setminus\left\{\left[\frac{1}{m}\right]\right\}$$

Case 2: Let $[\frac{1}{m}], [\sqrt{2}] \in (\mathbb{R}/\sim)$ for some $m \in \mathbb{N}$. Then $[\sqrt{2}] \in \{[\sqrt{2}]\}$ which is open since $\cup[\sqrt{2}]$ is open in $\mathbb{R}$; also, $[\frac{1}{m}] \notin \{[\sqrt{2}]\}$.

In either case, we have shown $(\mathbb{R}/\sim)$ is T_0. ■

As with the usual topology, $(\mathbb{R}/\sim)$ will not be T_1 when $\mathbb{R}$ is endowed with the Michael topology. Note that set $\{[\sqrt{2}]\}$ is open in $(\mathbb{R}/\sim)$ since the union of its members is open in the Michael line. Note that

$$\bigcup_{[x]\in\{[\sqrt{2}]\}^C} [x] = \bigcup_{n\in\mathbb{N}} \left[\frac{1}{n}\right] = \mathbb{Q}$$

which is not open in the Michael line. Thus, $[\sqrt{2}] = \mathbb{R} \setminus \mathbb{Q}$ is not closed in the Michael line and so $\{[\sqrt{2}]\}$ is not closed. Therefore, $(\mathbb{R}/\sim)$ is not T_1.

2.3. **Discrete Rational Extension.** The **discrete rational extension** is a topology $\mathfrak{T}_{\mathbb{Q}}$ on $\mathbb{R}$ given by

$$\mathfrak{T}_{\mathbb{Q}} = \{U \cup Q : U \in \mathfrak{T}_{\mathbb{R}}, Q \subset \mathbb{Q}\}$$

Proposition 2.3. *If $\mathbb{R}$ is endowed with the discrete rational extension, $(\mathbb{R}/\sim)$ will be T_2.*

Proof.

Case 1: Consider $[\frac{1}{m}]$ and $[\frac{1}{n}]$ for some $m, n \in \mathbb{N}$. Then $\{[\frac{1}{m}]\}$ and $\{[\frac{1}{n}]\}$ contain $[\frac{1}{m}]$ and $[\frac{1}{n}]$ respectively and they are disjoint.

Case 2: Consider $[\frac{1}{n}]$ and $[\sqrt{2}]$ for some $n \in \mathbb{N}$. Now, we can choose $U = \{[\frac{1}{n}]\}$ and $V = \{[\frac{1}{n}]\}^C$. U is open by the topology $\mathfrak{T}_{\mathbb{Q}}$ and V is open for the same reason. Clearly, U and V are disjoint. Thus, $(\mathbb{R}/\sim)$ is T_2. ■

2.4. **Sorgenfrey Line / Lower Limit Topology.** Recall that the **Sorgenfrey Line** or **Lower Limit Topology** $\mathfrak{T}_S$ on $\mathbb{R}$ is the topology generated by the basis $\mathscr{B} = \{[a, b) : a, b \in \mathbb{R}\}$, i.e. the topology generated by unions of all "half-open" intervals of $\mathbb{R}$. When the real line has the Lower Limit Topology, let us write $\mathbb{R}_S$.

Proposition 2.4. *Consider the space $\mathbb{R}_S$. then $(\mathbb{R}_S/\sim)$ is T_0.*

Proof.

To see this, let $[a],[b] \in (\mathbb{R}_S/\sim)$ be non-equal. Then at least one of $[a]$ or $[b]$ is equal to $[\frac{1}{n}]$ for some $n \in \mathbb{N}$. WLOG, suppose $[a] = [\frac{1}{n}]$. Then $[b] \in (\mathbb{R}/\sim) \setminus \{[\frac{1}{n}]\}$. Observe that $(\mathbb{R}_S/\sim) \setminus \{[\frac{1}{n}]\}$ is open in $(\mathbb{R}_S/\sim)$ and $[a] \notin (\mathbb{R}_S/\sim) \setminus \{[\frac{1}{n}]\}$. Hence, $(\mathbb{R}_S/\sim)$ is T_0. ■

However, $(\mathbb{R}_S/\sim)$ is not T_1. Observe $[\sqrt{2}] = \mathbb{R} \setminus \mathbb{Q}$. Now, $\{[\sqrt{2}]\}$ is closed in $(\mathbb{R}/\sim)$ if and only if $\bigcup_{[x]\in\{[\sqrt{2}]\}} [x]$ is closed in $\mathbb{R}$. Observe that $\bigcup[\sqrt{2}] = \mathbb{R} \setminus \mathbb{Q}$ is not closed in $\mathbb{R}_S$. Thus, $\{[\sqrt{2}]\}$ is not closed in $(\mathbb{R}_S/\sim)$. Therefore, $(\mathbb{R}_S/\sim)$ is not T_1.

Chapter Three
Compactness in $(\mathbb{R}/\sim)$

4. Investigations of Compactness

It is certainly interesting to determine whether $(\mathbb{R}/\sim)$ is compact. This, of course, may depend upon the topology that $\mathbb{R}$ is granted. In keeping with this, for the findings undertaken in this section, it will be assumed that $\mathbb{R}$ is endowed with the usual topology. The section is introduced by providing proofs of propositions related to compactness. Of exceptional wonder is the notion that $[\sqrt{2}]$ is contained in every nonempty open set of $(\mathbb{R}/\sim)$.

Proposition 4.1. *Every nonempty open set $U \subset (\mathbb{R}/\sim)$ contains $[\sqrt{2}]$.*

Proof.

To see this, suppose not. Suppose $U \subset (\mathbb{R}/\sim)$ does not contain $[\sqrt{2}]$ and is nonempty. Then for some $A \subset \mathbb{N}$,

$$U = \left\{ \left[\frac{1}{n}\right] : n \in A \right\}$$

Since U is open, by definition of the quotient topology, $q^{\leftarrow}(U)$ is open in $\mathbb{R}$. By definition, then the union

$$\bigcup_{[x]\in U} [x] = \bigcup_{n\in A} \left[\frac{1}{n}\right]$$

is open in $\mathbb{R}$. But note that for each $n \in A$, $[\frac{1}{n}] \subset \mathbb{Q}$. This implies

$$\bigcup_{k\in A}\left[\frac{1}{n}\right]\subset\mathbb{Q}$$

which is not open in $\mathbb{R}$. But this contradicts the fact that U is open. Thus, U contains $[\sqrt{2}]$.

■

Proposition 4.1 provides the groundwork for what will later be a proof of compactness of the quotient space. The significance of the ubiquity of the class $[\sqrt{2}]$ cannot be overstated as it grants insight into the structure of the open sets of $(\mathbb{R}/\sim)$. Equally as insightful into such structure is the notion that open subsets of $(\mathbb{R}/\sim)$ must be infinite.

Proposition 4.2. *Let $A\subset(\mathbb{R}/\sim)$ be nonempty. If A is open, then A is infinite.*

Proof.

Proceed by contraposition. Let A be a nonempty, finite subset of $(\mathbb{R}/\sim)$.

Case 1: $[\sqrt{2}]\in A$

If $[\sqrt{2}]\in A$, we write

$$A=\left\{[\sqrt{2}],\left[\frac{1}{n_1}\right],\left[\frac{1}{n_2}\right],\ldots,\left[\frac{1}{n_k}\right]\right\}$$

where $k\in\mathbb{N}\cup\{0\}$ ($k=0\iff A$ contains no $[\frac{1}{n_i}]$ classes). Note that A is open in $(\mathbb{R}/\sim)$ if and only if $\cup A$ is open in $\mathbb{R}$. Note that

$$\cup A = \bigcup\left\{[\sqrt{2}], \left[\frac{1}{n_1}\right], \left[\frac{1}{n_2}\right], \ldots, \left[\frac{1}{n_k}\right]\right\}$$

which is not open in $\mathbb{R}$.

<u>*Case 2:* $[\sqrt{2}] \notin A$</u>

If $[\sqrt{2}] \notin A$, we write

$$A = \left\{\left[\frac{1}{n_1}\right], \left[\frac{1}{n_2}\right], \ldots, \left[\frac{1}{n_k}\right]\right\}$$

Similarly to the previous case, it is observed that

$$\cup A = \bigcup\left\{\left[\frac{1}{n_1}\right], \left[\frac{1}{n_2}\right], \ldots, \left[\frac{1}{n_k}\right]\right\}$$

is closed and not open in $\mathbb{R}$. In either case, A is not open.

$\blacksquare$

Proposition 4.2 is essential in understanding the structure of the basic open sets as it places a constraint on set cardinality, although this is not necessary for showing compactness. We now present an essential proposition for showing compactness.

Proposition 4.3. *The sequence* $\{[\frac{1}{n}]\} : n \in \mathbb{N}\}$ *converges to* $[1]$*, i.e. every open nbhd of* $[1]$ *contains all but finitely many* $[\frac{1}{n}]$ *classes.*

Proof.

Let $U \subset (\mathbb{R}/\sim)$ be an open nbhd of $[1]$. Then since $q(0) = [1]$, it follows that $0 \in q^{\leftarrow}(\{[1]\})$. Since U is open, then $q^{\leftarrow}(U)$ is open in $\mathbb{R}$; thus, there is $\varepsilon > 0$ so that $0 \in (-\varepsilon, \varepsilon) \subset q^{\leftarrow}(U)$. Now, by the Archimedean Property, there exists some $M \in \mathbb{N}$ so that $0 < \frac{1}{M} < \varepsilon$. Then $\frac{1}{M} \in (-\varepsilon, \varepsilon)$. Similarly, if $n \geq M$, it's true that $\frac{1}{n} \leq \frac{1}{M} < \varepsilon$; hence,

$$\forall n \geq M, \frac{1}{n} \in (-\varepsilon, \varepsilon) \subset q^{\leftarrow}(U) \tag{1}$$

This implies that

$$q\left(\frac{1}{n}\right) = \left[\frac{1}{n}\right] \in q^{\rightarrow}[(-\varepsilon, \varepsilon)] \subset q^{\rightarrow}(q^{\leftarrow}(U)) \subset U \tag{2}$$

Thus, $\{[\frac{1}{n}]\}_{n \in \mathbb{N}}$ converges to $[1]$. Moreover, (1) implies that

$$\forall n \in \mathbb{N}, 1 < n < M, \frac{1}{n} \notin (-\varepsilon, \varepsilon) \subset q^{\leftarrow}(U)$$

Since ε was arbitrary, then when $n < M$, $\frac{1}{n} \notin q^{\leftarrow}(U)$ which means $[\frac{1}{n}] \notin U$. Thus, U contains all but finitely many $[\frac{1}{n}]$ classes.

$\blacksquare$

With the completion of the previous three propositions, we are now ready to show compactness of $(\mathbb{R}/\sim)$.

Theorem 4.1. *The space* $(\mathbb{R}/\sim)$ *is compact.*

Proof.

Let $\mathscr{U}$ be an open cover of $(\mathbb{R}/\sim)$. Since $[1] \in (\mathbb{R}/\sim)$, and $\mathscr{U}$ is an open cover, there is $U \in \mathscr{U}$ so that $[1] \in U$. By Proposition 4.3, U^C is finite. Again, since $\mathscr{U}$ covers the space, for each $[x] \in U^C$, there is $U_x \in \mathscr{U}$ so that $[x] \in U_x$. Let $\mathscr{V}$ be given by

$$\mathscr{V} = \{U\} \cup \{U_x : [x] \in U^C\}$$

Then $\mathscr{V}$ is a finite subcover of $\mathscr{U}$. Hence, $(\mathbb{R}/\sim)$ is compact.

■

Following the proof of compactness, the reader need be wary not to confuse the topology on $(\mathbb{R}/\sim)$ for the cofinite topology. It should be noted that not all subsets of $(\mathbb{R}/\sim)$ with finite complement are open. For example, Consider $\{[\frac{1}{n}] : n \in \mathbb{N}\}$. Observe that

$$\left\{\left[\frac{1}{n}\right] : n \in \mathbb{N}\right\}^c = \{[\sqrt{2}]\}$$

which, by earlier findings, is not closed; hence $\{[\frac{1}{n}] : n \in \mathbb{N}\}$ is not open.

5. The Convergence Conjecture

The previous section suggests the question of the convergence of sequences in $(\mathbb{R}/\sim)$. In particular, the work above begged the question of whether the space $(\mathbb{R}/\sim)$, when ordered as a sequence, converges to every one of its members. To investigate this, we pose the Conjecture 5.1 which is formalized below.

Conjecture 5.1. *Let* $x \in \mathbb{R}$ *and* $\varepsilon > 0$. *Then*

$$\exists\, M \in \mathbb{N}, \forall\, n \geq M, \exists\, k \in \mathbb{Z}, \gcd(k, n) = 1, \frac{k}{n} \in (x - \varepsilon, x + \varepsilon)$$

In simpler terms, the conjecture states that for any real number x, there is a sequence of simplest form fractions with non-decreasing denominator so that such a sequence converges to x. Verification of this conjecture would yield several results. Many of these results will be discussed in a later section. For the time being, turn attention to the notion that the verification of such a conjecture would imply that a sequence in $(\mathbb{R}/\sim)$ ordered by non-decreasing denominator, converges to every point of $(\mathbb{R}/\sim)$. To investigate such a conjecture, some understanding of number theory is required.

6. Filtering $\mathbb{N}$ by Coprimeness to the First k Primes

Since the conjecture includes an implication about all natural numbers n greater than or equal to some natural M and whether a relatively prime k can be found so that $\frac{k}{n}$ is arbitrarily close to x, inspection of the prime decomposition of natural numbers is appropriate.

As one moves along a natural number line, it is observed that natural numbers which are divisible by small primes such as 2, 3, and 5 are abundant. Therefore, it is significant to examine the distribution of the numbers not divisible by the first m primes; that is, we examine the set of numbers relatively prime to $p_1 p_2 \dots p_r$ where $p_1 p_2 \dots p_r$ is the product of the first r primes.

As the first prime is 2, the natural numbers relatively prime to 2 are the odd numbers:

$$1 \quad 3 \quad 5 \quad 7 \quad 9 \quad \ldots$$

Observe that the distance between two consecutive numbers in this sequence is no more than 2. Next, the natural numbers relatively prime to $2 \cdot 3 = 6$ begin as

$$1 \quad 5 \quad 7 \quad 11 \quad 13 \quad 17 \quad \ldots$$

where it is observed that the distance between two consecutive numbers in this sequence is no greater than 4. As a final example, the numbers relatively prime to $2 \cdot 3 \cdot 5 = 30$ begin as

$$1 \quad 7 \quad 11 \quad 13 \quad 17 \quad 23 \quad 29 \quad \ldots$$

where it can be observed that the distance between two consecutive numbers in this sequence is no greater than $2^3 = 8$.

As one continues this process, it might be postulated that the maximum distance between two numbers relatively prime to $p_1 \ldots p_m$ and are consecutive in this manner is 2^m. A result similar to this one has been formulated already. Let p_i be the i^{th} prime when ordered and suppose P_k is the product of the first k primes. There is a function $h(k)$ called Jacobstahl's function. Now, the value $h(k)$ is the smallest number m so that every sequence of m consecutive integers contains an integer relatively prime to P_k [2]. H.J. Kanold proved that $h(k) \leq 2^k$ in 1967. Formally, we state this as

Proposition 6.1. *Let $m \in \mathbb{N}$ be the smallest number so that the sequence of consecutive integers $\langle z_1, z_2, \ldots, z_m \rangle$ contains an integer relatively prime to P_k. Then $m \leq 2^k$ [3].*

From this result, it follows as a corollary then that

Corollary 6.1. *If $p_1 \ldots p_k$ is a product of the first k primes and r and s are consecutive in that both r and s are relatively prime to $p_1 \ldots p_n$, then $|r-s| \leq 2^k$. [2]*

Kanold's findings are convenient for this investigation as his findings put some constraints on where numbers relatively prime to $p_1 p_2 \ldots p_r$ are guaranteed to be found.

Suppose there are $a, b \in \mathbb{N}$ relatively prime to $p_1 p_2 \ldots p_m$ such that a and b are consecutive in this manner. If $|a - b| < \sqrt{p_1 p_2 \ldots p_r}$ how large does r have to be? Examine the finding below:

$$\begin{bmatrix} \sqrt{p_1 p_2 \ldots p_r} & \text{MAX GAP} \\ \sqrt{2} \approx 1.4 & 2 \\ \sqrt{2 \cdot 3} \approx 2.5 & 4 \\ \sqrt{2 \cdot 3 \cdot 5} \approx 5.5 & 8 \\ \sqrt{2 \cdot 3 \cdot 5 \cdot 7} \approx 14.5 & 16 \\ \sqrt{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11} \approx 48.0 & 32 \end{bmatrix}$$

Consider the table above. Select some natural m on the natural number line and consider the naturals that are greater than m but less than $m + \sqrt{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11}$. From this range, suppose one wanted to find some k so that k is relatively prime to $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. Then, we note that k is less than $m + 32$ by Kanold's findings. Following this, since $\sqrt{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11} \approx 48$, such a k is guaranteed to exist in this range. It is noted that the square root of the first n primes will grow much more quickly than 2^n; as such, $n \geq 2310$ are of particular interest in the context of proving Conjecture 6.1 below. Formalized, the conjecture reads:

Conjecture 6.1. *If $n \geq 2310$, then there is $k \in \mathbb{N}$ so that if $m \in \mathbb{N}$, we have $m < k < m + \sqrt{n}$ such that $\gcd(n, k) = 1$.*

We prove the case of $n = 2310$ in its own proposition.

Proposition 6.2. *If $m \in \mathbb{N}$, then there is $k \in \mathbb{N}$ so that $m < k < m + \sqrt{2310}$ with $\gcd(2310, k) = 1$.*

Proof.

Suppose $m \in \mathbb{N}$ and consider the natural numbers x satisfying $m < x < m + \sqrt{2310}$. Note that $48 < \sqrt{2310} < 49$. Thus, if we define A as

$$A := \{x \in \mathbb{N} : m < x < m + \sqrt{2310}\}$$

Then, in fact

$$A = \{m + 1, m + 2, m + 3, \ldots, m + 48\}$$

Observe that $2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = p_1 p_2 p_3 p_4 p_5$. By Proposition 6.1, if $j, k \in \mathbb{N}$ are relatively prime to $p_1 p_2 p_3 p_4 p_5$ and are consecutive in this manner, then $|j - k| \leq 2^5 = 32$. Since $\max(A) - \min(A) = 48 > 32$, then A contains at least one of j or k. WLOG, suppose $k \in A$. Then k satisfies

$$m < k < m + \sqrt{2310} \qquad \gcd(2310, k) = 1$$

■

It has been shown that proposition above holds for $n = 2310$. Ideally, it would be desirable to show that the proposition holds for $n \geq 2310$, i.e. the assertion of Conjecture 6.1, although doing so has proven to be difficult. Instead, a weaker variant of the case for $n \geq 2310$ was proven − given in Proposition 6.3 below.

Proposition 6.3. *Let $m \in \mathbb{N}$. If $n = p^\alpha \geq 2^3$ where p is prime and $\alpha \in \mathbb{N}$, then there is $k \in \mathbb{N}$ so that $m < k < m + \sqrt{n}$ and k is relatively prime to n. Moreover, if n_0 is such an n, then n_0^α also satisfies the proposition for $\alpha \in \mathbb{N}$.*

Proof.

Let $m \in \mathbb{N}$ and let $n = p^\alpha \geq 2^3$ where p is prime and $\alpha \in \mathbb{N}$. Consider the set

$$A = \{x \in \mathbb{N} : m < x < m + \sqrt{n}\}$$

Since $n \geq 2^3$, we can choose $a \in A$ for $a \neq \max(A)$. If $\gcd(a, n) = 1$, then we're done. So suppose $\gcd(a, n) \neq 1$. By the choice of n, $\gcd(a, n) \geq p$.

Now, $p \geq 2$. Thus, if $a_0 > a$ such that $\gcd(a_0, n) \geq p$ is minimal, then $|a_0 - a| \geq 2$. Hence,

$$\gcd(a+1, n) = 1$$

Choose $k = a+1$. Since $a \neq \max(A)$, then $k = a+1 \in A$, i.e.

$$m < k < m + \sqrt{n} \quad \text{and} \quad \gcd(k, n) = 1$$

Finally, suppose n_0 is such an n that satisfies the proposition. Then there is $k \in \mathbb{N}$ so that

$$m < k < m + \sqrt{n_0} \qquad \gcd(k, n_0) = 1$$

Then

$$m < k < m + \sqrt{n_0} < m + \sqrt{n_0^\alpha} \qquad \alpha \in \mathbb{N}$$

Again, $\gcd(k, n_0) = 1 \implies \gcd(k, n_0^\alpha) = 1$.

■

Now that the above proposition has been proven, it can be used to finally investigate Conjecture 5.1. Recall that Conjecture 5.1 asks whether the following is true:

$$\forall\, x \in \mathbb{R}, \forall\, \varepsilon > 0, \exists\, M \in \mathbb{N}, \forall\, n \geq M, \exists\, k \in \mathbb{N}, \gcd(k, n) = 1, \frac{k}{n} \in (x - \varepsilon, x + \varepsilon)$$

Now, it was shown Conjecture 6.1 holds for specific natural numbers, particularly, naturals which are powers of primes, i.e. $n = p^\alpha$ (i.e. Proposition 6.3). Since Conjecture 5.1 depends upon Conjecture 6.1, in this investigation, it was shown that Conjecture 5.1 holds for powers of primes $n = p^\alpha$. As such, the conjecture had to be weakened as formalized in the next section.

7. Proving a Weaker Version of Conjecture 5.1

Theorem 7.1. *Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. Then*

$$\exists\, \langle n_i \rangle_{i \in \mathbb{N}} \subset \mathbb{N}, \forall\, i \in \mathbb{N}, \exists\, k \in \mathbb{Z}, \gcd(k, n_i) = 1, \frac{k}{n_i} \in (x - \varepsilon, x + \varepsilon)$$

Proof.

Let $x \in \mathbb{R}$ with $\varepsilon > 0$.

<u>*Case 1:* Suppose $x - \varepsilon > 0$.</u>

Let $\frac{m}{n}$ be simplest form with $m \in \mathbb{Z}$ and with $n = p^\alpha \geq 2^3$ (p prime, $\alpha \in \mathbb{N}$) large enough so that

$$\left(\frac{m}{n}, \frac{m + \sqrt{n}}{n}\right) \subset (x - \varepsilon, x + \varepsilon) \tag{1}$$

Now, since $x - \varepsilon > 0$ and $\frac{m}{n} \in (x - \varepsilon, x + \varepsilon)$, then $m > 0$. Thus, by Proposition 6.3, there is $k \in \mathbb{N}$ so that

$$(2) \qquad m < k < m + \sqrt{n} \qquad \gcd(n, k) = 1$$

which implies that

$$(3) \qquad \frac{m}{n} < \frac{k}{n} < \frac{m + \sqrt{n}}{n}$$

Thus,

$$(4) \qquad \frac{k}{n} \in \left(\frac{m}{n}, \frac{m + \sqrt{n}}{n}\right) \subset (x - \varepsilon, x + \varepsilon)$$

Moreover, by (2), $\frac{k}{n}$ is simplest form.

Case 2: Suppose $x - \varepsilon < 0$.

Let $T \in \mathbb{N}$ satisfy $(x - \varepsilon) + T > 0$. By Proposition 6.3, there is $k \in \mathbb{N}$ so that

$$(5) \qquad \frac{k}{n} \in (x - \varepsilon + T, x + \varepsilon + T) \qquad \gcd(k, n) = 1$$

Observe also that since $\frac{k}{n} \in (x - \varepsilon + T, x + \varepsilon + T)$, then

$$(6) \qquad \left|x + T - \frac{k}{n}\right| < \varepsilon \iff \left|x - \left(\frac{k}{n} - T\right)\right| = \left|x - \frac{k - Tn}{n}\right| < \varepsilon$$

Now, since Thomae's function t is periodic with period 1 and $T \in \mathbb{N}$, then

$$(7) \qquad t\left(\frac{k}{n}\right) = \frac{1}{n} \implies t\left(\frac{k - Tn}{n}\right) = \frac{1}{n}$$

Hence, $\gcd(k - Tn, n) = 1$.

This concludes the demonstration of the existence of k so that $\frac{k}{n}$ is within ε of x for some $n \in \mathbb{N}$. From here, a sequence may be built.

Recall that $n = p^\alpha$ for p is prime and $\alpha \in \mathbb{N}$. Let $n_1 = p_1^{\alpha_1}$ be minimal so that (1) holds. Then by Proposition 6.3, we can choose $n_2 = p_2^{\alpha_2} > n_1$ so that (1) also holds. Proceeding in this manner, since $\mathbb{N}$ is unbounded above, we construct the sequence $\langle n_i \rangle_{i\in\mathbb{N}}$. Explicitly,

$$\langle n_i \rangle_{i\in\mathbb{N}} = \left\{ n = p^\alpha, p \text{ prime}, \alpha \in \mathbb{N} : \exists\, k \in \mathbb{N}, \gcd(k,n) = 1, \left| x - \frac{k}{n} \right| < \varepsilon \right\} \tag{8}$$

For each $i \in \mathbb{N}$, by choice of n_i, conclusions (1) through (7) will hold for n_i.

Thus, for every $x \in \mathbb{R}$ and every $\varepsilon > 0$, we have found a sequence $\langle n_i \rangle_{i\in\mathbb{N}}$ of natural numbers so that for each $i \in \mathbb{N}$, there is a $k \in \mathbb{Z}$ so that $\frac{k}{n_i}$ is simplest form and $\frac{k}{n_i} \in (x - \varepsilon, x + \varepsilon)$.

■

Following the completion of the proof of Theorem 7.1, we now have a partial result in the direction of answering the question begged by Conjecture 5.1. Recall Conjecture 5.1 which says: let $x \in \mathbb{R}$ and $\varepsilon > 0$. Then

$$\exists\, M \in \mathbb{N}, \forall\, n \geq M, \exists\, k \in \mathbb{N}, \gcd(k,n) = 1, \frac{k}{n} \in (x - \varepsilon, x + \varepsilon)$$

It is strongly suspected that this conjecture is true, but as mentioned previously, proving so is difficult and was not achieved. Nevertheless, if it should be proven eventually, below are some observations that would follow when regarding the space $(\mathbb{R}/\sim)$.

8. Observations if Conjecture 5.1 is True

Given the aforementioned suspicion that Conjecture 5.1 is true, it is worth investigating what results might come of the space $(\mathbb{R}/\sim)$ if the conjecture holds.

8.1. A Basis for $(\mathbb{R}/\sim)$.

If the conjecture holds, then we are granted insight into what a basis for $(\mathbb{R}/\sim)$ might look like. Let us expand upon this idea. To this end, if the conjecture holds, the first object of attention is that the quotient map $q : \mathbb{R} \to (\mathbb{R}/\sim)$ is an open mapping. We state this formally:

Proposition 8.1. *If Conjecture 5.1 holds, then the quotient map q is open.*

Proof.

Suppose the conjecture is true and let q be the quotient map. Now, let $U \subset \mathbb{R}$ be open. Since U is open, U is the union of open intervals, i.e.

$$U = \bigcup \{(a_\gamma, b_\gamma) : \gamma \in \Gamma, a_\gamma, b_\gamma \in \mathbb{R}\} \tag{1}$$

By Conjecture 5.1,

(2) there is $M \in \mathbb{N}$ so that for all naturals $n \geq M$, there is a simplest form $\frac{k}{n}$ in (a_γ, b_γ) for some $\gamma \in \Gamma$.

Now, examine $q^{\to}(U)$. Recall that $q(x) = [x]$, namely that $q(\frac{k}{n}) = [\frac{1}{n}]$. Then by (2), $q^{\to}(U)$ contains $[\frac{1}{n}]$ for all but finitely many $n \in \mathbb{N}$. Also, since U is open, U

contains irrationals; thus, $[\sqrt{2}] \in q^{\rightarrow}(U)$.

By our findings,

$$q^{\rightarrow}(U) = \left\{ \left[\frac{1}{n}\right] : n \geq M \right\} \cup \{[\sqrt{2}]\} = (\mathbb{R}/\sim) \setminus \bigcup_{n<M} \left\{ \left[\frac{1}{n}\right] \right\} \tag{3}$$

which is open since the union of its members is open in $\mathbb{R}$. Thus, q is an open mapping.

■

Note that though q is open, q need not be closed. Note that $\{\sqrt{2}\}$ is closed in $\mathbb{R}$ but $q^{\rightarrow}(\{\sqrt{2}\}) = \{[\sqrt{2}]\}$ which is not closed in $(\mathbb{R}/\sim)$. As such, q is not closed.

The second observation that should be noted is one that results from the first − particularly, that we can find a nice basis for $(\mathbb{R}/\sim)$. To give context, we reference a useful theorem for finding a basis for a given quotient space. The theorem asserts the following:

Theorem 8.1. *Let $(X, \mathfrak{T})$ be a topological space and let $\sim$ be an equivalence relation on $(X, \mathfrak{T})$ with quotient map q. Then $\mathscr{B} = \{q^{\rightarrow}(U) : U \in \mathfrak{T}\}$ is a basis for the quotient space $(\mathbb{R}/\sim)$ if and only if q is open.*

Recall that under the assumption of the conjecture, it was found that q was open. **Theorem 8.1** implies, therefore, that

$$\mathscr{B} = \{q^{\rightarrow}(U) : U \text{ is open in } \mathbb{R}\}$$

is a basis for $(\mathbb{R}/\sim)$; that is to say, the collection of images of $\mathbb{R}$-open sets via the map q is a basis for $(\mathbb{R}/\sim)$.

8.1.1. *Open Sets Have Finite Complements.* An immediate corollary of the verification of the conjecture is that open subsets would have finite complements.

Proposition 8.2. *If Conjecture 5.1 holds, then if $U \subset (\mathbb{R}/\sim)$ is open and nonempty, then U^c is finite.*

Proof.

Suppose the conjecture holds and let $U \subset (\mathbb{R}/\sim)$ be open and nonempty. Since U is open, $[\sqrt{2}] \in U$. Since U is open and nonempty, $q^{\leftarrow}(U)$ is open in $\mathbb{R}$ and $x \in q^{\leftarrow}(U)$ for some x. By openness, there is $\varepsilon > 0$ so that $x \in (x-\varepsilon, x+\varepsilon) \subset q^{\leftarrow}(U)$. By the conjecture, there is $M \in \mathbb{N}$ so that

$$(1) \qquad \forall\, n \geq M, \exists\, k \in \mathbb{Z}, \gcd(n,k) = 1, \frac{k}{n} \in (x-\varepsilon, x+\varepsilon) \subset q^{\leftarrow}(U)$$

Note that $q(\frac{k}{n}) = [\frac{1}{n}]$. Then by (1), $\frac{k}{n} \in q^{\leftarrow}(U)$ for all $n \geq M$ implies that

$$(2) \qquad \forall\, n \geq M, \left[\frac{1}{n}\right] \in U$$

Explicitly then, U is written

$$(3) \qquad \left\{[\sqrt{2}], \left[\frac{1}{M}\right], \left[\frac{1}{M+1}\right], \left[\frac{1}{M+2}\right], \ldots\right\}$$

And so

$$U^c = \left\{[1], \left[\frac{1}{2}\right], \left[\frac{1}{3}\right], \ldots, \left[\frac{1}{M-1}\right]\right\}$$

Thus, U^c is finite.

■

8.2. **Second Countability.** If the conjecture holds, the space $(\mathbb{R}/\sim)$ will be second countable. Toward this end, recall that if a topological space is countable as a set and is also first countable, then the space is second countable. We employ this method to illustrate second countability of the space $(\mathbb{R}/\sim)$; as stated, though, assumption of the conjecture is required. To lead into this, a lemma is posed.

Lemma 8.1. *If Conjecture 5.1 holds and $U \subset (\mathbb{R}/\sim)$ is open, then there is $M \in \mathbb{N}$ so that*

$$U = \left\{[\sqrt{2}], \left[\frac{1}{M}\right], \left[\frac{1}{M+1}\right], \dots\right\}$$

That is to say, U contains all $[\frac{1}{n}]$ classes when $n \geq M$ and does not contain all $[\frac{1}{n}]$ classes with $0 < n < M$.

Proof.

Let $U \subset (\mathbb{R}/\sim)$ be open and nonempty. Since U is open, $q^{\leftarrow}(U)$ is open and nonempty, then for some $x \in \mathbb{R}$, $x \in q^{\leftarrow}(U)$. Since $q^{\leftarrow}(U)$ is open, there is $\varepsilon > 0$ so that

$$x \in (x - \varepsilon, x + \varepsilon) \subset q^{\leftarrow}(U)$$

By Conjecture 5.1, there is $M \in \mathbb{N}$ so that whenever $n \geq M$, there is $k \in \mathbb{Z}$ so that

$$\frac{k}{n} \in (x - \varepsilon, x + \varepsilon) \qquad \gcd(n, k) = 1$$

But this means

$$\forall\, n \geq M,\ q\Big(\frac{k}{n}\Big) = \Big[\frac{1}{n}\Big] \in U$$

Thus, we may explicitly write

$$U = \Big\{[\sqrt{2}], \Big[\frac{1}{M}\Big], \Big[\frac{1}{M+1}\Big], \dots\Big\}$$

■

We are now ready to show that, under the assumption of the conjecture, $(\mathbb{R}/\sim)$ is first countable.

Proposition 8.3. *If Conjecture 5.1 holds, then the space* $(\mathbb{R}/\sim)$ *is first countable.*

Proof.

Suppose the conjecture holds and let $[x] \in (\mathbb{R}/\sim)$ with U is an open nbhd of $[x]$. Since U is open, there are finitely many members of $(\mathbb{R}/\sim)$ not contained in U. Let $B_1 = U$. Now, let $B_2 = B_1 \setminus \{[y]\}$ where $[y] \neq [x]$. Continue in this manner so that

$$B_n = B_{n-1} \setminus \{[y]\} \quad \text{where} \quad [y] \in B_{n-1},\ [y] \neq [x]$$

Thus, $B_1 \supset B_2 \supset B_3 \dots$ and $x \in B_i$ for every $i \in \mathbb{N}$. Lastly, let V be any open nbhd of $[x]$. By the conjecture and the previous lemma, there is $M \in \mathbb{N}$ so that

$$V = \Big\{\Big[\frac{1}{M}\Big], \Big[\frac{1}{M+1}\Big], \dots\Big\}$$

Then if $B_2 = V \setminus \{[y]\}$ where $[y] \neq [x]$, then $[x] \in B_2 \subset V$. Hence, the collection

$$\mathscr{B} = \{B_n : n \in \mathbb{N}\}$$

is a countable local basis at $[x]$. Thus, $(\mathbb{R}/\sim)$ is first countable.

■

Note now that the set $(\mathbb{R}/\sim)$ is itself a countable set and is also a first countable space. Thus, $(\mathbb{R}/\sim)$ is second countable.

Chapter Four
Topological Countability in $(\mathbb{R}/\sim)$

As seen in the previous section, it was noted that

$$(\mathbb{R}/\sim) \;\; = \{[\sqrt{2}]\} \cup \left\{ \left[\frac{1}{n} \right] \right\}_{n \in \mathbb{N}}$$

is itself a countable set. This is *not* sufficient to conclude that this space is second countable or first countable.

1. Second Countability and Corollaries

Without verification of Conjecture 5.1, it is uncertain whether the space $(\mathbb{R}/\sim)$ is first or second countable. Under the assumption of the conjecture (as detailed in the previous section), however, the space $(\mathbb{R}/\sim)$ would be second countable.

2. Regarding Separability of the Space

Observe that $(\mathbb{R}/\sim)$ is separable since it is countable. One could take $(\mathbb{R}/\sim)$ as a candidate for a countably dense subset of the space. More specifically, however there is a proper subset which is also dense in the space. Consider the set $\mathscr{N}$ as defined by

$$\mathscr{N} = \left\{ \left[\frac{1}{n} \right] : n \in \mathbb{N} \right\}$$

By definition of closure,

$$\overline{\mathscr{N}} = \left\{ [x] \in (\mathbb{R}/\sim) : \forall \text{ open } U, [x] \in U, U \cap \mathscr{N} \neq \varnothing \right\}$$

Since $\mathscr{N} \subset \overline{\mathscr{N}}$, then naturally $[\frac{1}{n}] \in \overline{\mathscr{N}}$. Now, $\cup\mathscr{N} = \mathbb{Q}$ which is neither open nor closed in $\mathbb{R}$ under the usual topology. Hence, $\mathscr{N} \neq \overline{\mathscr{N}}$, implying $\mathscr{N}$ is properly contained in its closure. But this means $[\sqrt{2}] \in \overline{\mathscr{N}}$. Therefore,

$$\overline{\mathscr{N}} = \left\{[\sqrt{2}], [1], \left[\frac{1}{2}\right], \left[\frac{1}{3}\right], \dots\right\} = (\mathbb{R}/\sim)$$

And so $\mathscr{N}$ is countable and dense in $(\mathbb{R}/\sim)$, illustrating the existence of such a properly contained subset in $(\mathbb{R}/\sim)$. Another, perhaps more obvious, example to this end would be $\{[\sqrt{2}]\}$. This is dense in $(\mathbb{R}/\sim)$ since $[\sqrt{2}]$ is contained in every nonempty open set.

Bibliography

[1] C.J. Thomae, *Einleitung in die Theorie der bestimmten Integrale*, Verlag von Louis Norbert (1875), Halle a/S, p.14, §20.

[2] F. Costello, P. Watts, *An Upper Bound On Jacobsthal's Function*, Mathematics of Computation 84:293 (2014), 1389-1399.

[3] H.J. Kanold, *Uber eine zahlentheoretische funktion von Jacobsthal*, Math. Annnalen 170(4) (1967), 314-326.