Functional Limits in Topology

by

Corey S. Chadman

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Approvals:

Jamal Tartir, Thesis Advisor	Date
Eric Wingler, Committee Member	Date
David Pollack, Committee Member	Date
Bryan W. Depoy, Dean of School of Graduate Studies & Research	Date

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ABSTRACT

The goal of this Thesis is to establish a firm foundation for for the theory of functional limits in a topological setting. We will generalize the definition of the functional limit from its traditional analytic setting to a topological setting. We will show that this generalized limit applies to sequences, functions, and even integrals. We will show its consistency with its analytic counterpart. We will expand the algebraic limit theorems over topological algebras and briefly discuss the concept of differentiation in a topological setting.

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Contents

Functional Limits in Topology

1 Definitions and Examples

Assume τ is a topology of open sets and $\langle\tau\rangle$ is a basis for the topology. Assume unless otherwise specified (X,τ) and (Y,τ') are topological spaces with $f,g:(X,\tau)\to(Y,\tau')$ functions. Also assume U_x is a neighborhood of x. Let τ_ϵ denote the Euclidean topology. If (B,τ) is a topological space and $A\subseteq B$ then (A,τ_A) is A with the subspace topology.

Definition. *We say $U'_x = U_x \setminus \{x\}$ is the punctured neighborhood of U_x.*

Definition. *We say $\lim\limits_{x\to c} f(x) = L$ if for any ,$U_L\in\tau'$, there exists a $U_c\in\tau$ with $U'_c\neq\varnothing$ such that $f(U'_c)\subseteq U_L$. We call L the functional limit of f as x approaches c. If we are examining the same function but with multiple topologies and we want to better specify the topologies we are using for the limit, say τ is a topology for X and τ' is a topology for Y. We will write $\overset{(\tau,\tau')}{\lim\limits_{x\to c}} f(x)$ in place of $\lim\limits_{x\to c} f(x)$ for $f:(X,\tau)\to(Y,\tau')$.*

Definition. *We say (X,τ) is non-isolated at $x\in X$ if $\{x\}\notin\tau$.*

We say (X,τ) is nowhere isolated if for any $x\in X$, $\{x\}\notin\tau$

Theorem 1.1. *Given $(X,\tau,\ c\in X)$ the following are equivalent:*

1) (X,τ) is non-isolated at c

2) For any $U_c,V_c\in\tau$ we have, $U_c\cap V_c\neq\{c\}$

3) For any $U_c,V_c\in\tau$ we have, $U'_c\cap V'_c\neq\varnothing$.

Proof. $1 \to 2$

Let (X, τ) be non-isolated at c. If there exists $U_c, V_c \in \tau$ such that $U_c \cap V_c = \{c\}$, then $\{c\} \in \tau$ a contradiction. Thus, $U_c \cap V_c \neq \varnothing$.

$2 \to 3$

If for any $U_c, V_c \in \tau$ we have, $U_c \cap V_c \neq \varnothing$. Then there exists $y \in X \setminus \{c\}$ such that $y \in U_c \cap V_c$. Thus, $y \in U_c' \cap V_c'$. Hence, $U_c' \cap V_c' \neq \varnothing$.

$3 \to 1$

Suppose for any $U_c, V_c \in \tau$ we have, $U_c' \cap V_c' \neq \varnothing$ and also assume $\{c\} \in \tau$. Then $U_c' \cap \{c\}' = U_c' \cap \varnothing = \varnothing$ a contradiction, Thus, (X, τ) is non-isolated at c. □

Example 1.1. *Let* $X = \mathbb{N} \cup \{\omega\}$ *where* ω *is the first infinite ordinal number. S for any* $n \in \mathbb{N}, n < \omega$. *Define* $\tau = \{(N, \omega]\} \mid N \in \mathbb{N} \cup \{0\} \cup \{\omega\}\}$. *Clearly,* τ *is a topology. Let* $f{:}(\mathbb{N}, \tau) \to (\mathbb{R}, \tau_\epsilon)$ *and define the sequence* $x_n = f(n)$. *If* $\lim_{n \to \omega} x_n = L$, *then given any* $(L - \epsilon, L + \epsilon)$, *there exists an* $(N, \omega]$ *such that* $f((N, \omega] \setminus \{\omega\}\} \subseteq (L - \epsilon, L + \epsilon)$. *A more familiar way to say this is that for every* $\epsilon > 0$ *there is an* $n_0 \in \mathbb{N}$ *such that for any* $n > n_0$, $x_n \in (L - \epsilon, L + \epsilon)$. *Similarly, if instead of* $(\mathbb{R}, \tau_\epsilon)$ *we have* (Y, τ') *then* $\lim_{x \to c} f(x) = L$ *means that for any* $U_L \in \tau'$, *there exists an* $N \in \mathbb{N}$ *such that for any* $n > N, x_n \in U_L$. *This defines sequential convergence in a topological space.*

Example 1.2. *Consider* $f{:}(\mathbb{R}, \tau_\epsilon) \to (\mathbb{R}, \tau_\epsilon)$. *If* $\lim_{x \to c} f(x) = L$ *then for any* $(L - \epsilon, L + \epsilon)$, *there exists* $(c - \delta, c + \delta)$, *such that* $f((c - \delta, c) \cup (c, c + \delta)) \subseteq (L - \epsilon, L + \epsilon)$. *Or in analysis for all* $\epsilon > 0$, *there will exist a* $\delta > 0$ *such that* $\mid x - c \mid < \delta$ $(x \neq c)$ *implies that* $\mid f(x) - L \mid < \epsilon$. *Let* $f : X \to Y$ *with* X *and* Y *metric spaces and* $\lim_{x \to c} f(x) = L$. *Then it follows from the definition that for any* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that for* $d_1(x, c) < \delta (x \neq c)$ *we have,* $d_2(f(x), L) < \epsilon$.

We will investigate the discrete metric in more detail later on. Metrics (such as the discrete metric) that generate isolated points cause something interesting to happen which we will investigate later.

Example 1.3. *Let $I = [a,b]$ and let $\mathcal{P} = \{\{x_0, x_1, ..., x_n\} \mid a = x_0 < x_1 < ... < x_n = b\} \bigcup \{I\}$ be the set of all Riemann Partitions and the entire interval. Let $f : I \to \mathbb{R}$. Now suppose $(P_a)_{a \in A}$ is an indexed set of elements from $\mathcal{P}$ and define $F_{(P_a)_{a\in A}} = \{P \in \mathcal{P} \mid \exists a \in A$ s.t. $P \supset P_a\}$. Now let $\tau = \{F_{(P_a)_{a\in A}}, \mathcal{P}, \varnothing \mid F_{(P_a)_{a\in A}}\}$ and notice τ is a topology on $\mathcal{P}$. A quick check,*

(1) $\mathcal{P}, \varnothing \in \tau$

(2) $F_{(P_a)_{a\in A}} \cup F_{(P_b)_{b\in B}}$

$= \{P \in \mathcal{P} \mid a \in A$ *or* $b \in B$ *such that* $P \supset P_a$ *or* $P \supset P_b\}$

Now create a new index set C such that, $\forall a \in A,\ b \in B$ there exists c, c_* such that $P_a = P_c$ and $P_b = P_{c_*}$. Thus,

$F_{(P_a)_{a\in A}} \cup F_{(P_b)_{b\in B}} = F_{(P_c)_{c\in C}}$. *The arbitrary union follows from* $\bigcup_{A \in \mathcal{A}} F_{(P_a)_{a\in A}} = F_{(P_a)_{a \in \bigcup_{A\in\mathcal{A}} A}}$

(3) $F_{(P_a)_{a\in A}} \cap F_{(P_b)_{b\in B}}$

$= \{P \in \mathcal{P} \mid$ *There will exist* $(a,b) \in A \times B$ *such that* $P \supset P_a$ *and* $P \supset P_b\}$

$= \{P \in \mathcal{P} \mid$ *there will exist* $(a,b) \in A \times B$ *such that* $P \supset P_a \cup P_b\}$

$= F_{(P_a \cup P_b)_{(a,b)\in A\times B}}$.

Notice $\{I\} \notin \tau$ since $F_{(P_a)_{a\in A}}$ has at least one partition, call it $P = \{x_0, x_1, ...x_n\}$. If $x_0 < x^* < x_1$ then $P^* = \{x_0, x^*, x_1, ..., x_n\} \in F_{(P_a)_{a\in A}}$. Thus, $F_{(P_a)_{a\in A}} \neq \{I\}$. Finally $F_{\{I\}} = \{P \in \mathcal{P} \mid P \supset I\} = \varnothing$. Thus we have, $I \notin \tau$. Thus we have τ is non-isolated at I. Now let $R(f,P) : (\mathcal{P} \setminus \{I\}, \tau) \to (\mathcal{R}, \tau_\epsilon)$ be the standard Riemann sum $\sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i)$, on f. Now let us consider if $\lim_{P\to I} R(f,P) = L$. It follows that for any $(L-\epsilon, L+\epsilon)$ there is an $F_{(P_a)_{a\in A}}$, $F'_{(P_a)_{a\in A}} \neq \varnothing$ and $R(f(F_{(P_a)_{a\in A}}) \subseteq (L-\epsilon, L+\epsilon)$. Without loss of *generality we can choose any* $P_a \in F_{(P_a)_{a\in A}}$ *Then we can say for all* $\epsilon > 0$, *there is a* $P_a \in \mathcal{P} \setminus \{I\}$ *such that for any* $P \supset P_a, | R(f,P) - L | < \epsilon$. *The number* L *is by definition the Riemann Integral.*

Example 1.4. *Let* $f : X \to Y$ *be a non-negative measurable function with respect to the measure* μ *Now let* ϕ *be a simple measurable function. Let* $\mathbb{X} = \{\phi \mid \phi \leq f$ *almost everywhere*$\} \cup \{f\}$. *Let* $(\phi_a)_{a\in A}$ *be a net and define* $F_{(\phi_a)_{a\in A}} = \{\phi \mid$ *there exists an* $a \in A$ *such that* $\phi_a < \phi \leq f$ *a.e.*$\} \cup \{f\}$. *Let* $\tau = \{X, \varnothing, F_{(\phi_a)_{a\in A}} \mid (\phi_a)_{a\in A}$ *is a net of simple measurable functions* $\}$. *I claim that* τ *is a topology on* $\mathbb{X}$. *Notice f is in every non-empty open set so f will be in the arbitrary union or finite intersection of open sets. So we need not worry about f, so without loss of generality we need only consider the simple measurable functions.*

(1) $\mathbb{X}, \varnothing \in \tau$.

(2) $F_{(\phi_a)_{a\in A}} \cup F_{(\phi_b)_{b\in B}} = \{\phi \mid \exists a \in A \; or; \; b \in B \; s.t. \; \phi_a < \phi \leq f \; a.e. \; or \; \phi_b < \phi \leq f \; a.e., f\}$.
Now create a new index C *such that for any* $(a,b) \in A \times B$ *there exist* c, c^* *such that* $\phi_a = \phi_c$ *and* $\phi_b = \phi_{c^*}$. *Thus,* $F_{(\phi_a)_{a\in A}} \cup F_{(\phi_b)_{b\in B}} = F_{(\phi_c)_{c\in C}}$.

Before we check finite intersections we need to notice something, for ϕ_a, ϕ_b *simple measurable functions,* $\phi_{(a,b)}(x) = max\{\phi_a(x), \phi_b(x)\}$ *is a simple measurable function. Let* $\phi_a = \sum_{i=1}^{n} a_i \chi_{A_i}$ *and* $\phi_b = \sum_{j=1}^{m} b_j \chi_{B_j}$. *Define* $h = \sum_{(i,j)\in\{1,2,\ldots,n\}\times\{1,2,..,m\}} max\{a_i, b_j\} \chi_{A_i \cap B_j}$.
Now h is clearly a simple function and notice that since A_i, B_j *are measurable we know* $A_i \cap B_j$ *is measurable. Hence, h is a simple measurable function. Now for* $x \in A_i \cap B_j$, *where* $(i,j) \in \{1,2,\ldots,n\} \times \{1,2,..,m\}$ *we have,*

$h(x) = max\{a_i, b_j\} \chi_{A_i \cap B_j}(x) = max\{a_i, b_j\} = \phi_{(a,b)}(x)$.

(3) $F_{(P_a)_{a\in A}} \cap F_{(P_b)_{b\in B}} = \{\phi \mid$ *there exists an* $a \in A$ *and* $b \in B$ *such that* $\phi_a < \phi \leq f$ *a.e. and* $f \wedge \phi_b < \phi \leq f$ *a.e.* $, f\} \bigcup \{F\}$. *We know,* $\phi \geq \max\{\phi_a, \phi_b\} = \phi_{(a,b)}$ *is a simple*

measurable function. Thus $F_{(P_a)_{a\in A}} \cap F_{(P_b)_{b\in B}} = F_{\{\phi_{(a,b)}\}_{(a,b)\in A\times B}}$.

Thus τ *is a topology on* $\mathbb{X}$. *Now define* $S : \mathbb{X} \to \mathbb{R}$ *by* $S(\phi) = \int_X \phi d\mu$.

Now if $\lim_{\phi\to f} S(f) = L$ *then for any* $(L-\epsilon, L+\epsilon)$, *there will exist an* $F_{(\phi_a)_{a\in A}}$, $F'_{(\phi_a)_{a\in A}} \neq \varnothing$ *such that* $S(F'_{(\phi_a)_{a\in A}}) \subseteq (L-\epsilon, L+\epsilon)$. *In analysis we can say for any* $\epsilon > 0$, *there is an* $F_{(\phi_a)_{a\in A}}$ *such that if* $\phi \in F_{(\phi_a)_{a\in A}}$, *then* $\mid L - S(\phi) \mid < \epsilon$. *Also, since* $\phi \leq f$ *for any* $\phi \in \mathbb{X}$ *we know,* $0 < L - S(\phi) < \epsilon$ *which gives us* $L - \epsilon < S(\phi)$. *That is, for any* $\epsilon > 0$, *there will exist a* $\phi \in \mathbb{X}$ *such that* $L - \epsilon < S(\phi)$ *which implies sup*$(S(\phi)) = L$. *The number* L *is the definition of the measure integral.*

Notice in examples 1, 3, and 4 that the domain is not T_2 and only non-isolated at the point where we are taking the limit. So we may construct a topology where we need only consider the limit at a particular point rather than the whole space.

2 Consistency and comparison with Analysis

Theorem 2.1. *Let* τ *be non-isolated at c and* τ' *be* T_2. *If* $\lim_{x\to c} f(x)$ *exists then it is unique.*

Proof. Assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} f(x) = L'$ with $L \neq L'$. Since τ' is T_2 there exists $U_L, U_{L'} \in \tau'$ such that $U_L \cap U_{L'} = \varnothing$. Since $\lim_{x\to c} f(x) = L$ we have for U_L a $U_c \in \tau$ such that $U'_c \neq \varnothing$ and $f(U'_c) \subseteq U_L$. Since $\lim_{x\to c} f(x) = L'$ we have for $U_{L'}$ there will be a $V_c \in \tau$ such that $V'_c \neq \varnothing$ and $f(V'_c) \subseteq U_{L'}$. Now $U'_c \cap V'_c \neq \varnothing$ since τ is non-isolated at c. Thus, $f(U'_c \cap V'_c) \subseteq U_L$ and $f(U'_c \cap V'_c) \subseteq U_{L'}$, hence $f(U'_c \cap V'_c) \subseteq U_L \cap U_{L'} = \varnothing$, a contradiction. Thus, the limit must be unique.

□

The following examples show the necessity of τ being non-isolated at c.

Example 2.1. *Define* $\langle\tau\rangle = \{(a,b),\{0\} \mid a,b \in \mathbb{R}, a < b\}$. *Clearly* τ *is a topology and is* T_2 *since* $\tau_\epsilon \subseteq \tau$. *Define* $f{:}(\mathbb{R},\tau) \to (\mathbb{R},\tau_\epsilon)$ *by* $f(x) = \begin{cases} x+1, & \text{if } x \geq 0; \\ x, & \text{if } x < 0. \end{cases}$

Consider $\lim_{x\to 0} f(x)$. *Let us first pick an open neighborhood of 1, say* $(1-\epsilon, 1+\epsilon)$, *and choose* $U_0 = \{0\} \cup (0,\epsilon) = [0,\epsilon) \in \tau$. *Now* $f(U_0') = f((0,\epsilon)) = (1, 1+\epsilon) \subseteq (1-\epsilon, 1+\epsilon)$; *thus* $\lim_{x\to c} f(x) = 1$. *On the other hand let us pick an open neighborhood of 0, say* $(-\epsilon,\epsilon)$, *and let us pick* $V_0 = \{0\} \cup (-\epsilon, 0) = (-\epsilon, 0] \in \tau$. *Then* $f(V_0') = f((-\epsilon,0)) = (-\epsilon,0) \subseteq (-\epsilon,\epsilon)$ *thus* $\lim_{x\to 0} f(x) = 0$.

A more extreme example of this comes from the range being discrete.

Example 2.2. *Notice that the discrete metric is isolated at every point. So let us consider* $D \subseteq \mathbb{R}$ *and consider* $f : (D, \mathbb{P}(D)) \to (\mathbb{R}, \tau_\epsilon)$ *with the condition that there exists* $y \in D \smallsetminus \{c\}$ *such that* $f(y) = L$, $y \neq c$. *Then we can say* $\lim_{x\to c} f(x) = L$. *Let* $(L-\epsilon, L+\epsilon)$ *be given. Choose* $\{c,y\} \in \mathbb{P}(D)$. *Then* $f(\{c,y\}') = f(\{y\}) = \{L\} \subseteq (L-\epsilon, L+\epsilon)$. *This tells us that for with the discrete metric topology defined on D we have* $\lim_{x\to c} f(x) = f(y)$ *for any* $y \in D \smallsetminus \{c\}$.

A few more definitions are needed before the next theorem.

Definition. *A filter* $\mathcal{F}$ *on a set* X *is a non-empty collection of subsets of* X *such that:*

(1) $\varnothing \notin \mathcal{F}$

(2) If $F, G \in \mathcal{F}$, *then* $F \cap G \in \mathcal{F}$

(3) If $F \in \mathcal{F}$ *and* $G \supset F$, *then* $G \in \mathcal{F}$

A subcollection $\mathcal{F}'$ is a filter base for $\mathcal{F}$, if $\mathcal{F} = \{F \subset X \mid F' \subset F$ for some $F' \in \mathcal{F}'\}$.

Definition. *A filter $\mathcal{F}$ is said to converge to c ($\mathcal{F} \to c$) if $\mathcal{F}$ if finer than any local neighborhood basis at c.*

Definition. *Let $\langle U_c \rangle$ be a neighborhood basis for $c \in \tau$ and define:*
$\langle U_c' \rangle = \{V_c' \mid V_c \in < U_c >\}$. Also, if $\mathcal{K} \subseteq \mathbb{P}(X)$ satisfies the definition of a filter basis, then the filter $\mathcal{F}_{\mathcal{K}}$ is the filter generated by $\mathcal{K}$.

It should also be mentioned that $\mathcal{F}_{\langle U_c' \rangle} \supseteq \mathcal{F}_{\langle U_c \rangle}$. If follows from the definition of filter convergence that $\mathcal{F}_{\langle U_c \rangle} \to c$; thus, $\mathcal{F}_{\langle U_c' \rangle} \to c$.

Theorem 2.2. *Given τ and τ'. If τ' is T_2 and τ non-isolated at c, then the following are equivalent:*

(1) $\lim_{x \to c} f(x) = L$

(2) For every $(x_a)_{a \in A}$ topological nets with for any $a \in A$, we have $x_a \neq c$. If $(x_a)_{a \in A} \to c$ then $(f(x_a))_{a \in A} \to L$.

(3) If $\mathcal{F}_{\langle U_c' \rangle} \supseteq \mathcal{F}$, then $f(\mathcal{F}) \to L$.

Proof. (1) $\Longrightarrow$ (2)

Let $\lim_{x \to c} f(x) = L$. Let $(x_a)_{a \in A} \to c$, $x_a \neq c$ for any $a \in A$. Now for U_L there is a $U_c \in \tau, U_c' \neq \varnothing$, such that $f(U_c') \subseteq U_L$. Since $(x_a)_{a \in A} \to c$ we know for U_c given, there exists $a_0 \in A$ such that for any $a \geq a_0, x_a \in U_c$, and $x_a \neq c$ gives $x_a \in U_c'$; thus, $f(x_a) \in U_L$. Hence, $(f(x_a))_{a \in A} \to L$.

(2) $\Longrightarrow$ (3)

It is enough to show that $f(\mathcal{F}_{\langle U_c' \rangle}) \to L$ since it would imply that $f(\mathcal{F}) \to L$. Define $A = \{(a, U_c') \mid a \in U_c',\ U_c' \in \langle V_c' \rangle\}$. For $(a, U_c'), (b, W_c') \in A$. Let $(a, U_c') \geq (b, W_c')$ if and only if $W_c' \supseteq U_c'$. Notice this makes A a directed set since it is the subset inclusion ordering. Now define $x_{(a,U_c')} = a$ and consider the net $(x_{(a,U_c')})_{(a,U_c')\in A}$.

Claim 1: $(x_{(a,U_c')})_{(a,U_c')\in A} \to c$. To show this let $U_c \in \langle V_c \rangle$. For $\langle V_c' \rangle$ there exists $x_{(a,U_c')}$(since τ non-isolated at c) and thus, for any $(b, W_c') \geq (a, U_c')$ implies that $x_{(b,W_c')} \in U_c' \subseteq U_c$. Thus, $x_{(b,W_c')} \in U_c$ which gives us $(x_{(a,U_c')})_{(a,U_c')\in A} \to c$.

Now by (2) we know $(f(x_{(a,U_c')}))_{(a,U_c')\in A} \to L$.

Thus, for $U_L \in \langle U_L \rangle$ there exists (a, U_c') such that for all $(b, W_c') \geq (a, U_c')$ we have $f(x_{(b,W_c')}) \in U_L$. Now notice for any $d \in U_c', (d, U_c') \geq (a, U_c')$, which implies $f(x_{(d,U_c')}) \in U_L$. Hence, $f(U_c') \subseteq U_L$. This gives us $f(\mathcal{F}_{\langle U_c' \rangle}) \supseteq \langle U_L \rangle$. Hence, $f(\mathcal{F}_{\langle U_c' \rangle}) \to L$.

(3) $\implies$ (1)

It is enough to show this for just $\mathcal{F}_{\langle U_c' \rangle}$ since $\mathcal{F}_{\langle U_c' \rangle} \supseteq \mathcal{F}$. Given $f(\mathcal{F}_{\langle U_c' \rangle}) \to L$ gives us $\mathcal{F}_{\langle U_L \rangle} \subseteq f(\mathcal{F}_{<U_c'>})$. We can also notice that any filter is its own basis. Thus, for any $U_L \in \tau'$ there is a $U_c \in \tau$, $U_c' \neq \varnothing$ such that $f(U_c') \in U_L$ and thus $\lim_{x\to c} f(x) = L$.

$\square$

Theorem 2.3. *Let $f : (X, \tau) \to (Y, \tau')$ be continuous at $c \in X$ and τ non-isolated at c. Then $\lim_{x\to c} f(x) = f(c)$.*

Proof. Let $U_{f(c)} \in \tau'$ be given. By continuity for any $U_{f(c)}$ there is a $U_c \in \tau$ such that $f(U_c) \subseteq U_{f(c)}$. Since τ is non-isolated at c we know that $\{c\} \notin \tau$ hence, $U_c' \neq \varnothing$ and thus, $f(U_c') \subseteq U_{f(c)}$. $\square$

The following example shows the necessity of non-isolated at c and is another example involving the discrete metric.

Example 2.3. *Let $f : (X, \mathbb{P}(X)) \to (X, \mathbb{P}(X))$ be defined by $f(x) = x$. This function is clearly continuous. Suppose we pick an arbitrary point c and try to show $\lim_{x\to c} f(x) = c$. Then using $\{c\} \in \mathbb{P}(X)$ we have $f^{-1}(\{c\}) = \{c\}$. Thus the only potential U_c is $\{c\}$,* *but* $U_c' = \varnothing$. *Thus, there does not exist a* U_c *such that* $U_c' \neq \varnothing$ *and* $f(U_c') \subseteq \{c\}$. *Thus the* $\lim_{x \to c} f(x) \neq f(c)$.

3 Modifying the Domain or Range Topologies

Theorem 3.1. *Suppose* τ *is non-isolated at* c. *If* $\lim_{x \to c} f(x) = L$ *and if* $S \subseteq \tau'$ *then for* $f : (X, \tau) \to (X, S)$ *we have* $\overset{(\tau,S)}{\lim_{x \to c}} f(x) = L$.

Proof. Let $U_L \in S$ be given, then $U_L \in \tau'$ thus there exists $U_c \in \tau$, $U_c' \neq \varnothing$ such that $f(U_c') \subseteq U_L$. Thus, $\overset{(\tau,S)}{\lim_{x \to c}} f(x) = L$. □

Notice this theorem applies only to manipulations of the range. We are not manipulating the topology of the domain, i.e. the domain topology is fixed.

Example 3.1. *Let* $f(x) = \begin{cases} x + 1 \ \mathit{if} \ x \geq 0 \\ \\ x \ \mathit{if} \ x < 0\} \end{cases}$ *and* $\langle S^- \rangle = \{[x, y) \mid x < y\}$. *We will consider* $\lim_{x \to 0} f(x)$ *in three cases.*
(1) $(\mathbb{R}, \langle S^- \rangle) \to (\mathbb{R}, \langle S^- \rangle)$
(2) $(\mathbb{R}, \langle S^- \rangle) \to (\mathbb{R}, \tau_\epsilon)$
(3) $(\mathbb{R}, \langle S^- \rangle) \to (\mathbb{R}\{\mathbb{R}, \varnothing\})$.

(1) For $[1, 1 + \epsilon)$ *in the range pick* $[0, 0 + \epsilon)$ *in the domain. Then* $f((0, 0 + \epsilon)) = (1, 1 + \epsilon) \subseteq [1, 1 + \epsilon)$. *Since* $\langle S^- \rangle$ *is non-isolated at 0 we know the limit is unique.*
(2) For $(1 - \epsilon, 1 + \epsilon)$ *we can use* $[0, 0 + \epsilon)$ *and since* $[1, 1 + \epsilon) \subset (1 - \epsilon, 1 + \epsilon)$ *and since the Euclidean topology is non-isolated at 1, we know the limit is unique. Just for fun lets look at say* $(-\epsilon, \epsilon)$ *with* $\epsilon < 1$ *and see that the only open sets that map into* $(-\epsilon, \epsilon)$ *are of the form* $[a, b)$ *with* $a, b < 0$ *but none of these are an open neighborhood of zero.*

(3) The only open set containing 1 is $\mathbb{R}$. We need only choose the same $[0,\epsilon)$ and since the indiscrete topology is clearly non-isolated at 0 we know the limit is unique by theorem 2.1.

For domains just the opposite happens.

Theorem 3.2. *Let $f:(X,\tau)\to(Y,\tau')$ and suppose $\lim_{x\to c} f(x)=L$. If $S\supseteq\tau$, and S is non-isolated at c, then for $f:(X,S)\to(Y,\tau')$ we have $\overset{(S,\tau')}{\lim_{x\to c}} f(x)=L$.*

Proof. For $U_L\in\tau'$ there will be a $U_c\in\tau$, $U_c'\neq\varnothing$ such that $f(U_c')\subseteq U_L$.
Since $U_c\in\tau$ it follows that $U_c\in S$. □

So domains can get finer and ranges can get courser. But going the opposite direction leads to trouble.

Example 3.2. *Consider $f(x)=\begin{cases} x+1 \text{ if } x\geq 0 \\ x \text{ if } x<0 \end{cases}$ in the Euclidean topology with the domain topologies,*

(1) $<S^->$

(2) τ_ϵ

(1) For $(1-\epsilon,1+\epsilon)$ pick $[0,0+\epsilon)$ and we have $f((0,0+\epsilon))=(1,1+\epsilon)\subseteq[1,1+\epsilon)$. Thus, $\lim_{x\to c} f(x)=1$ which we have already shown.
(2) Consider an open neighborhood of 0 say $(-\epsilon,\epsilon)$. For any value $x\in(-\epsilon,0)$ we know, $f(x)\notin[1,1+\epsilon)$. Thus the limit is not 1. We know from Real Analysis that in fact the limit does not exist.

Corollary. *If $f:\mathbb{R}\to\mathbb{R}$ in the standard topology with $\lim_{x\to c} f(x)=L$ then the left and right limits must agree.*

Proof. We know that $S^{+}, S^{-} \supset \tau_{\epsilon}$. We also know they are non-isolated at any point. Thus, by the prior theorem $\overset{(S^{*},\tau')}{\lim_{x\to c}} f(x) = L = \overset{(S,\tau')}{\lim_{x\to c}} (x)$. ∎

Theorem 3.3. *Given $f : (X,\tau) \to (Y,\tau')$, τ non-isolated at c and τ' is T_2. Suppose $\mathcal{S}$, $\mathcal{S}'$ are both finer topologies then τ' with $\overset{(\tau,\mathcal{S})}{\lim_{x\to c}} = L$ and $\overset{(\tau,\mathcal{S}')}{\lim_{x\to c}} = M$. If $L \neq M$ then $\lim_{x\to c} f(x)$ does not exist.*

Proof. Suppose $L \neq M$. Then since τ' is T_2 there exists $U_L, U_M \in \tau' \subseteq \mathcal{S} \cap \mathcal{S}'$ such that $U_L \cap U_M = \varnothing$. Since $\overset{(\tau,\mathcal{S})}{\lim_{x\to c}} f(x) = L$, there is a $U_c \in \tau, U_c' \neq \varnothing$ such that $f(U_c') \subseteq U_L$. Since $\overset{(\tau,\mathcal{S}')}{\lim_{x\to c}} f(x) = M$ there exists $V_c \in \tau, V_c' \neq \varnothing$ such that $f(V_c') \subseteq U_M$. Also $U_c \cap V_c \neq \varnothing$ and $(U_c \cap V_c)' = U_c' \cap V_c' \neq \varnothing$. Thus $f(U_c' \cap V_c') \subseteq U_L \cap U_M = \varnothing$, a contradiction. Thus, the limit cannot exist. ∎

Another very important result is the contrapositive of this.

Corollary. *Suppose $f : (X,\tau) \to (Y,\tau')$, τ is non-isolated at c and τ' is T_2. Suppose $\mathcal{S}, \mathcal{S}'$ are both finer topologies than τ' with $\overset{(\tau,\mathcal{S})}{\lim_{x\to c}} f(x) = L$ and $\overset{(\tau,\mathcal{S}')}{\lim_{x\to c}} f(x) = M$. If $\lim_{x\to c} f(x)$ exists, then $L = M$.*

Theorem 3.4. *Suppose $f : (X,\tau) \to (Y,\tau')$ with τ non-isolated at c, τ' is T_2. If for any $\mathcal{S}$ finer than τ' we have, $\overset{(\tau,\mathcal{S})}{\lim_{x\to c}} f(x) = L$ then $\lim_{x\to c} f(x) = L$.*

Proof. If $U_L \in \tau' \subseteq \mathcal{S}$, then there exists $U_c \in \tau$. Then $U_c' \neq \varnothing$ such that $f(U_c') \subseteq U_L$. ∎

4 Compositions of Functions

Suppose $g : (X,\tau) \to (Y,\tau')$, $\lim_{x\to c} g(x) = L$ and $f : (Y,\tau') \to (Z,\tau'')$, $\lim_{y\to L} f(y) = M$, and $h : (X,\tau) \to (Z,\tau'')$ is defined by $h(x) = f(g(x))$ When does $\lim_{x\to c} h(x)$

(1) exist

(2) equal M

To start a partial solution to part (1) we have the following,

Definition. *We define* $\lim_{x\to c}' h(x) = M$ *if for any* $U_M \in \tau''$ *there exists a* $U_c \in \tau, U_c' \neq \varnothing$ *such that* $g(U_c') \subseteq f^{-1}(U_M)$.

Theorem 4.1. *The* $\lim_{x\to c} h(x) = M$ *if and only if the* $\lim_{x\to c}' h(x) = M$. *Or in other words it is the same limit.*

Proof. $\Rightarrow$

Let $U_M \in \tau''$ be given. Since $\lim_{x\to c} h(x) = M$ there is a $U_c \in \tau, U_c' \neq \varnothing$ such that $h(U_c') = f(g(U_c')) \subseteq U_M$. Now apply f^{-1} to both sides and we get,

$g(U_c') \subseteq f^{-1}(f(g(U_c'))) \subseteq f^{-1}(U_M)$. Thus, for $U_M \in \tau''$ there exists $U_c \in \tau, U_c' \neq \varnothing$ such that $g(U_c') \subseteq f^{-1}(U_M)$, that is $\lim_{x\to c}' h(x) = M$.

$\Leftarrow$

Let $U_M \in \tau''$ be given. Since $\lim_{x\to c}' h(x) = M$ there exists a $U_c \in \tau, U_c' \neq \varnothing$ such that

$g(U_c') \subseteq f^{-1}(U_M)$. If we apply f to both sides we will get

$h(U_c') = f(g(U_c')) \subseteq f(f^{-1}(U_M)) \subseteq U_M$. Thus, $\lim_{x\to c} h(x) = M$. Hence, the two limits are equivalent. $\square$

Since the two limits are equivalent we will no longer distinguish $\lim'$ from $\lim$.

To give a partial answer to (2) I have two results.

Theorem 4.2. *If* f *is continuous at* L. *Then* $\lim_{x\to c} h(x) = M$.

Proof. Let $U_M \in \tau''$ be given. Since $\lim_{y\to L} f(y) = M$ and f continuous at L gives us for U_L there exists a $U_L \in \tau', U_L \neq \varnothing$ such that $f(U_L') \subseteq U_M$. Since $\lim_{x\to c} g(x) = L$, we know there exists a $U_c \in \tau, U_c' \neq \varnothing$ such $g(U_c') \subset U_L$. Hence, $h(U_c') = f(g(U_c')) \subseteq f(U_L) \subseteq U_M$. Thus, $\lim_{x \to c} h(x) = M$. □

The final condition I have found is a much stronger condition, but in some rare cases it allows you to work with f not being continuous.

Definition. *For $f : (X, \tau) \to (Y, \tau')$ we say $slim_{x \to c} f(x) = L$ if for each $U_L \in \tau'$ there is a $U_c \in \tau, U_c' \neq \varnothing$ such that $f(U_c') \subseteq U_L'$.*

The difference is very subtle between the lim and the slim. However notice that the $slim_{x \to 0} f(x) = \begin{cases} x \sin(x) \text{ if } x \neq 0, \\ \\ 0 \text{ if } x = 0 \end{cases}$ does not exist in the standard realm of analysis since 0 is in the image of every open neighborhood of 0, despite the fact that $f(x)$ is continuous at 0. Let us look at the definition in a standard analysis view.

Definition. *Given $f : \mathbb{R} \to \mathbb{R}$ we say $slim_{x \to c} f(x) = L$ if for any $\epsilon > 0$, there will exist $\delta > 0$ such that $\mid x - c \mid < \delta$, $(x \neq c)$ gives us that $0 < \mid f(x) - L \mid < \epsilon$.*

Clearly if the domain is non-isolated at c and the range is T_2 and the slim exists then the limit exists and they the same. This is because $f(U_c') \subseteq U_L' \subseteq U_L$. Thus, if the two limits exist they must be the same.

The next condition for compositions of limits to exist is given in the following theorem.

Theorem 4.3. *If $slim_{x \to c} g(x) = L$ then $\lim_{x \to c} h(x) = M$.*

Proof. Let $U_M \in \tau''$ be given. Since $\lim_{y \to L} f(x) = M$ there will exist a $U_L \in \tau', U_L' \neq \varnothing$ such that $f(U_L') \subseteq U_M$. Since $slim_{x \to c} g(x) = L$ it follows that $U_L \in \tau'$. $\exists U_c \in \tau, U_c' \neq \varnothing$ and $g(U_c') \subseteq U_L'$. Thus, $h(U_c') = f(g(U_c')) \subseteq f(U_L') \subseteq U_M$ giving the desired result.

□

A notable example about composition of limits is the following.

Example 4.1. *For a mapping from the reals to reals (under the traditional topology) consider* $f(x) = \begin{cases} \sin(\frac{1}{x}), & \text{if } x \neq 0 \\ x, & \text{if } x = 0 \end{cases}$ *and* $g(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$. *In both cases the limit as x approaches 0 does not exist. But $f(g(x)) = \sin(x)$ is a continuous function and as a result the limit exists as the x approaches zero.*

5 Discontinuities (of $\mathbb{R}$)

Next let us consider discontinuities of functions. In particular let us focus primarily on discontinuities of functions from $\mathbb{R}$ to $\mathbb{R}$. To do this we will use the extended real line as a tool in defining some discontinuities and discerning what are often the two cases of the essential discontinuity. One of these I will call an extended jump discontinuity to distinguish it from the other. But before we do that we should review the topology on the extended reals. The order topology on the extended reals has the sub-basis, $\{[-\infty, a), (b, \infty] \mid a, b \in \mathbb{R} \cup \{-\infty, \infty\}\}$. We will denote this by $(\mathbb{R}_{ext}, \tau_{ext})$. We will also need to examine the left and right limits very closely. One thing to notice is that the real line under the standard topology is a subspace of the extended real line.

So we can now consider the left hand and right hand limits of $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$.
(1) $\lim_{x \to c^-} f(x) = L$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that, $0 < x - c < \delta$ yields $\mid f(x) - L \mid < \epsilon$.
(2) $\lim_{x \to c^+} f(x) = L$ if for any $\epsilon > 0$ there will exist a $\delta > 0$ such that, $0 < c - x < \delta$ implies that $\mid f(x) - L \mid < \epsilon$. For our study of discontinuities at points we want to refer to both limits on the real line and limits on the extended real line. To remedy any confusion I will denote limits going to the extended real line by $\text{limext}_{x \to c} f(x)$ with a plus or

minus subscript on c indicating a left hand or right hand limit in the domain.

Theorem 5.1. *Let $L \in \mathbb{R}$. Then,* $\lim_{x \to c} f(x) = L$ *if an only if* $\operatorname{limext}_{x \to c} f(x) = L$ *and similarly for the left hand limit and right hand limit.*

Proof. Realize $\{(L - \epsilon, L + \epsilon) \mid \text{for } \epsilon \in (0, \infty)\}$ is a local basis for the extended reals as well as the reals.

□

When we are trying to show the extended limit is infinity we need to look at neighborhoods of the form $(\alpha, \infty], \alpha < \infty$ as the order topology would dictate. Similarly if we want to show our extended limit to be negative infinity we need to look at open sets sets of the form $[-\infty, \alpha), \alpha > -\infty$.

Definition. *We say $f : D \to \mathbb{R}$ has a removable discontinuity at c if*
$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) \neq f(c)$.

Definition. *We say $f : D \to \mathbb{R}_{ext}$ has an extended removed discontinuity at c if* $\operatorname{limext}_{x \to c^-} f(x) = \operatorname{limext}_{x \to c^+} f(x) \neq f(c)$.

Definition. *We say $f : D \to \mathbb{R}$ has a jump discontinuity at c if*
$\lim_{x \to c^-} f(x) = L$ *and* $\lim_{x \to c^+} f(x) = M$ *and* $L \neq M$.

Definition. *We say $f : D \to \mathbb{R}_{ext}$ has an extended jump discontinuity at c if* $\operatorname{limext}_{x \to c^-} f(x) = L$ *and* $\operatorname{limext}_{x \to c^+} f(x) = M$ *and* $L \neq M$.

Definition. *We say $f : D \to \mathbb{R}$ has an essential discontinuity if either $lim_{x \to c^-} f(x)$ or $lim_{x \to c^+} f(x)$ does not exist.*

In the case of functions from reals to reals any time a left hand or right hand limit is infinite then the function has an essential discontinuity. However, there are essential discontinuities that do not involve an infinite limit.

Definition. *Let* $f : (X, \tau) \to (Y, \tau')$ *and* $(c, L) \in X \times Y$. *We say f reverberates at (c,L) if for any* $U_c \in \tau$, $f(U_c') \cap \{L\} \neq \varnothing$.

Theorem 5.2. *Suppose* $f : (X, \tau) \to (Y, \tau'), \tau$ *is non-isolated at c and* τ' *is* T_2. *If f reverberates at (c,L) and (c,M)*$(L \neq M)$, *then* $\lim_{x \to c} f(x)$ *does not exist.*

Proof. Let $\langle G_c' \rangle$ be a local punctured neighborhood basis. We shall give $\langle G_c' \rangle$ the ordering, $U \leq V$ if and only if $V \subseteq U$. Under this ordering $\langle G_c' \rangle$ is a directed set. For all U_c' there exist $x, y \in U_c'$ such that $f(x) = L$ and $f(y) = M$. Label x and y as $x_{U_c'}$ and $y_{U_c'}$. Now we have constructed the nets, $(x_{U_c'})_{U_c' \in \langle G_c' \rangle}, (y_{U_c'})_{U_c' \in \langle G_c' \rangle}$ with both nets converging to c. However their images are both constant nets. Now by construction, $(f(x_{U_c'}))_{U_c' \in \langle G_c' \rangle} \to L$ and $(f(y)_{U_c'})_{U_c' \in \langle G_c' \rangle} \to M$. Thus the limit cannot exist. □

A function may have a jump discontinuity at a point c even if it reverberates. An example is the folowing:

Let $f(x) = \begin{cases} \{x \sin(\frac{1}{x}) \text{ if } x > 0 \\ x \sin(\frac{1}{x}) + 1 \text{ if } x < 0; 0, x = 0\} \end{cases}$ then this is just a jump discontinuity with f reverberating at $(0, 0)$ and $(0, 1)$. To give a relationship between essential discontinuity and reverberations we have the following theorem.

Theorem 5.3. *Let* $D \subseteq \mathbb{R}, f : (D, \tau_{\epsilon_D}) \to (\mathbb{R}, \tau_\epsilon)$ *with* f *reverberating at* $(c, L), (c, M)$, *and* (c, N), *where* $L \neq M, M \neq N$, *and* $N \neq L$. *Then* f *must have an essential discontinuity at c.*

Proof. We want to show that either the left hand limit or the right hand limit does not exist. Let $\{(c - \epsilon_n, c + \epsilon_n) \cap D\}_{n \in \mathbb{N}}$ be a countable basis with $(\epsilon_n) \to 0$. Now pick $x_n, y_n, z_n \in (c - \epsilon_n, c + \epsilon_n) \cap D$ such that $f(x_n) = L, f(y_n) = M, f(z_n) = N$. Now clearly $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}$, and $(z_n)_{n \in \mathbb{N}}$ converge to c but $(f(x_n))_{n \in \mathbb{N}} \to L, (f(y_n))_{n \in \mathbb{N}} \to$

M, and $(f(z_n))_{n\in\mathbb{N}} \to N$. Now $\{x_n\}_{n\in N} \cap ([c, c+\epsilon) \cap D)$ or $\{x_n\}_{n\in N} \cap ((c-\epsilon, c] \cap D)$ is infinite. If both are infinite, pick one and create the subsequence $(x_{n_l})_{l\in\mathbb{N}}$. Do the same process for (y_n) and (z_n) to obtain sequences $(y_{n_m})_{m\in\mathbb{N}}, (z_{n_k})_{k\in\mathbb{N}}$. Now at least two of these sequences must be contained within either $[c, c+\epsilon)$ or $(c-\epsilon, c]$. Without loss of generality we may assume $(y_{n_m})_{m\in\mathbb{N}}, (z_{n_k})_{k\in\mathbb{N}}$ are within $[c, c+\epsilon)$. Thus, by the above theorem, $\lim_{x\to c^-} f(x)$ does not exist. Thus, f has an essential discontinuity at c. $\square$

We should note you can mimic the above proof to get the same result for functions mapping to the extended real line.

6 Limits in Topological Algebras

We will allow $*$ to be an arbitrary binary operation.

Definition. *$(M, *)$ is a magma if for all $a, b \in M$ we have, $a * b \in M$.*

Definition. *$(X, *, \tau)$ is a Topological Magma if*

(1) X is a Magma

(2) The function $* : (X, \tau)^2 \to (X, \tau)$ by $*(x, y) = x * y$ is continuous

If X is understood to be a magma with continuous operations under τ, we write (X, τ) for short.

When the binary operation is understood we will suppress the binary operation and write $a * b$ as ab.

Some examples of topological magmas are the following.

Example 6.1. $[0, \infty)$ *with the euclidean topology.*

Example 6.2. $(\mathbb{N}, \tau)$ *where* τ *is the topology example 1.1.*

Definition. $(X, *, \tau)$ *is a topological group if*

(1) X *is a Group*

(2) The function $* : (X, \tau)^2 \to (X, \tau)$ *is a continuous function*

(3) The function $inv : (X, \tau) \to (X, \tau)$ *defined by* $inv(x) = x^{-1}$ *is continuous*

If X *is understood to be a group with continuous operations under* τ *we write* (X, τ) *for short.*

Clearly a topological group is a topological magma.

Definition. *Let* $(X, *, \tau)$ *be a topological group. Let* $'*, *' : (X, \tau)^2 \to (X, \tau)$ *be defined by:*

$' * (x, y) = x^{-1}y$ *and* $*'(x, y) = xy^{-1}$ *to be the left and right inverted binary operators.*

Theorem 6.1. *Let* $(X, *, \tau)$ *be a group then both* $(X, ' *, \tau)$ *and* $(X, *', \tau)$ *are Topological magmas.*

Proof. Clearly, both $'*$ and $*'$ are closed since the group is closed.

Notice that $' * (x, y) = x^{-1}y = x^{-1} * y = *(x^{-1}, y)$ and $*$ is continuous at (x^{-1}, y) for any $(x, y) \in X^2$.

Also similarly, $*'(x, y) = xy^{-1} = x * y^{-1} = *(x, y^{-1})$ and $*$ is continuous at (x, y^{-1}) for any $(x, y) \in X^2$.

Thus, $(X, ' *, \tau)$ and $(X, *', \tau)$ are Topological magmas. □

Theorem 6.2. AlgebraicLimitTheorem(ALT)

Let (Y, τ') *is a topological magma and* (X, τ) *non-isolated at c. If* $\lim_{x \to c} f(x) = L$ *and* $\lim_{x \to c} g(x) = M$*, then* $\lim_{x \to c} f(x)g(x) = LM$*.*

Proof. Let $U_{LM} \in \tau'$ be given. Since $*$ is continuous there exist $U_L, U_M \in \tau'$ such that $*(U_L \times U_M) \subseteq U_{LM}$.

Now $\lim_{x \to c} f(x) = L$ gives us for U_L given, there is $U_c \in \tau, U_c' \neq \emptyset$, and $f(U_c') \subseteq U_L$.

Also, $\lim_{x \to c} g(x) = M$ gives us for U_M given; there will be $V_c \in \tau, V_c' \neq \emptyset$, and $f(V_c') \subseteq U_M$.

Now $U_c \cap V_c \in \tau$ implies $U_c' \cap V_c' \neq \emptyset$ and a punctured neighborhood of c, thus

$f(U_c' \cap V_c')g(U_c' \cap V_c') = *(f(U_c' \cap V_c'), g(U_c' \cap V_c')) \subseteq *(f(U_c'), g(V_c')) \subseteq *(U_L, U_M) \subseteq U_{LM}$.

Hence, $\lim_{x \to c} f(x)g(x) = LM$. □

Corollary. *Let (Y, τ) be a topological magma with, $X \subseteq Y$, (X, τ) non-isolated at c and $\lim_{x \to c} f(x) = L$. Then*

(1) $\lim_{x \to c} kf(x) = kL$

(2) $\lim_{x \to c} f(x)k = Lk$

Proof. Let $h(x) = k$ be a constant function. Then $h(x)$ is continuous, giving us $\lim_{x \to c} h(x) = k$. Thus,

(1) $\lim_{x \to c} kf(x) = \lim_{x \to c} h(x)f(x) = kL$

(2) $\lim_{x \to c} f(x)k = \lim_{x \to c} f(x)h(x) = Lk$.

□

Theorem 6.3. *If (Y, τ') is a group, (X, τ) is non-isolated at c, and both $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = G$, then*

(1) $\lim_{x \to c} f(x)g(x) = LM$

(2) $\lim_{x \to c} f(x)g(x)^{-1} = LM^{-1}$

(3) $\lim_{x \to c} f(x)^{-1}g(x) = L^{-1}M$

Proof. This follows from Theorems 6.1 and 6.2 since $(Y, *, \tau'), (Y,' *, \tau')$ and $(Y, *', \tau')$ are topological magmas. $\square$

Definition. $(R, +, \times, \tau)$ *is a topological ring if*

(1) R is a ring (0 is the additive identity and 1 is the multiplicative identity if it is in the ring);

(2) $(R, +, \tau)$ is a topological group;

(3) $(R, \times, \tau)$ is a topological magma.

When R is understood to be a ring we write (R, τ) for short.

Definition. $(F, +, \times, \tau)$ *is a topological field if, (recall $F^\times = F \setminus \{0\}$)*

(1) F is a field;

(2) $(F, +, \times, \tau)$ is a topological ring;

(3) $(F^\times, \times, \tau)$ is a topological group;

(4) $F^\times \in \tau$.

When F is understood to be a field we write (F, τ).

Note that the general definition of a topological field need not include the third condition. The fourth condition guarantees for any $x \in F^\times$ and $U_x \in \tau$, there is a $V_x \in \tau$ such that $V_x \subseteq F^\times$ and $V_x \subseteq U_x$. Some authors assume that the topology is T_1 or T_2 while others assume no condition or separation axiom at all. This assumption is strictly weaker than T_1.

Theorem 6.4. *If (Y, τ') is a topological ring,(X, τ) is non-isolated at c, and both $\lim_{x \to c} f(x) = L$, and $\lim_{x \to c} g(x) = M$, then*

(1) $\lim_{x \to c} [f(x) + g(x)] = L + M$

(2) $\lim_{x \to c} [f(x) g(x)] = LM$

Example 6.3. $(\mathbb{R}, \tau_\epsilon)$. *We know* $\mathbb{R}$ *is a field, and* $inv_+(x) = -x$ *and* $inv_\times(x) = x^{-1}$ *are continuous so it is enough to show that (1)* $+(x,y) = x+y$ *and (2)* $\times(x,y) = xy$ *are continuous functions from* $\mathbb{R}^2$ *to* $\mathbb{R}$ *as well as to show that (3)* $\mathbb{R}^\times$ *is open.*

(1) For $(a,b) \in \mathbb{R}^2$ *and* $\epsilon > 0$ *let* $\delta < \epsilon/2$. *Then,* $\mid x-a \mid < \delta$ *and* $\mid y-b \mid < \delta$ *imply that*

$\mid +(x,y) - +(a,b) \mid = \mid x+y-a-b \mid = \mid (x-a)+(b-y) \mid$

$\leq \mid x-a \mid + \mid b-y \mid < \epsilon.$

(2) Let $(a,b) \in \mathbb{R}^2$ *and* $\epsilon > 0$. *If* $\mid x-a \mid < \frac{\epsilon}{2(|b|+1)}$ *and* $\mid y-b \mid < min\{1, \frac{\epsilon}{2(|a|+1)}\}$, *then* $\mid y \mid \leq \mid y-b \mid < \mid 1+b \mid$. *So*

$\mid \times(x,y) - \times(a,b) \mid = \mid xy-ab \mid = \mid xy-ay+ay-ab \mid \leq \mid xy-ay \mid + \mid ay-ab \mid$

$= \mid y \mid\mid x-a \mid + \mid a \mid\mid y-b \mid \leq \mid b+n \mid\mid x-a \mid + \mid a \mid\mid y-b \mid < \epsilon$

(3) Also $\{0\}$ *is closed in* $\mathbb{R}$*; thus* $\{0\}^c = \mathbb{R}^\times$ *is open. Thus,* $(\mathbb{R}, \tau_\epsilon)$ *is a topological field.*

Theorem 6.5. *If* (Y, τ') *is a topological field,* (X, τ) *is non-isolated at c, and both* $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = M$*, then*

(1) $\lim_{x\to c} kf(x) = kL$*, if* $X \subseteq \mathbb{R}$

(2) $\lim_{x\to c}(f(x)+g(x)) = L+M$

(3) $\lim_{x\to c}(f(x)g(x)) = LM$

(4) $\lim_{x\to c}(f(x)g(x)^{-1}) = LM^{-1}$ *provided* $M \neq 0$

Proof. Since $(Y,+,\tau'), (Y,\times,\tau')$ are topological magmas, (2) and (3) hold.

(4) $\times'(x,y) = xy^{-1} = \times(x, y^{-1})$ is continuous on $F \times F^\times$ so we can use the proof for the Algebraic Limit Theorem with $\times'$ instead of $\times$. □

Corollary. *If* (X, τ) *is non-isolated at c and* $f, g : (X,\tau) \to (\mathbb{R}, \tau_\epsilon)$ *with* $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = M$*, then*

(1) $\lim_{x\to c} kf(x) = kL$

(2) $\lim_{x\to c}[f(x) + g(x)] = L + M$

(3) $\lim_{x\to c}[f(x)g(x)] = LM$

(4) $\lim_{x\to c}[f(x)g(x)^{-1}] = LM^{-1}$ *provided,* $M \neq 0$

Proof. $\mathbb{R}$ is a topological field. □

Notice from examples 1.1,1.2,1.3, and 1.4 earlier that the algebraic limit theorems hold for sequences and functional limits as well as for the Riemann and measure integrals.

Definition. *Let R be a ring with one. Define*

$$R^{-1} = \{x \in R \mid \text{ there exists } y \in R \text{ such that } xy = yx = 1\}$$

.

We call R^{-1} the group of units.

Definition. *Let R be a ring with one. Then R is a division ring if $R^{-1} = R \setminus \{0\}$.*

Definition. (R, τ) *is a weak topological field if*

(1) R is a ring with 1;

(2) R is a topological ring;

(3) $(R^{-1}, \tau_{R^{-1}})$ is a topological group;

(4) $R^{-1} \in \tau$.

The fourth condition guarantees for any $x \in R^{-1}$ and $U_x \in \tau$ there is a $V_x \in \tau$ with $V_x \subseteq F^{\times}$ such that $V_x \subseteq U_x$.

Lemma 1. *Let $f_a : (X_a, \tau_a) \to (Y_a, \tau'_a)$ be continuous for all $a \in A$.*

Then $h : \prod_{a\in A}(X_a, \tau_a) \to \prod_{a\in A}(Y_a, \tau'_a)$ (where $\prod_{a\in A}(X_a, \tau_a)$ and $\prod_{a\in A}(Y_a, \tau'_a)$ have either the (1) Box topologies or (2) the product topology) defined by, $h(x) = (f_a(\pi_a(x)))_{a\in A}$ is continuous.

Proof. For all $(c_a)_{a\in A}$ we have $h((c_a)_{a\in A}) = f_a(\pi_a(c_a))_{a\in A} = \prod_{a\in a} f_a(c_a)$. We will show this is continuous at an arbitrary point $(c_a)_{a\in A}$. So for (1) consider, without loss of generality, the neighborhood $U_{(h((c_a)_{a\in A})} = \prod_{a\in A} U_{f_a(c_a)} \in \prod_{a\in A} \tau'_a$. Now since $f_a(x)$ is continuous at c_a, for $U_{f_a(c_a)} \in \tau'_a$, there exist $U_{c_a} \in \tau_a$ such that $f_a(U_{c_a}) \subseteq U_{f_a(c_a)}$. Now let $U_{(c_a)_{a\in A}} = \prod_{a\in A} U_{c_a}$. Then,

$$h(U_{(c_a)_{a\in A}}) = (f_a(\pi_a(U_{(c_a)_{a\in A}})))_{a\in A} = (f_a(U_{c_a}))_{a\in A} \subseteq (U_{f_a(c_a)})_{a\in A} = \prod_{a\in A} U_{f_a(c_a)} = U_{(h((c_a)_{a\in A})}$$

. Thus, the function is continuous at all points.

Now for (2) all but finitely many $a \in A$, we have $U_{f_a(c_a)} = Y_a$, so we let $U_{c_a} = X_a$. Thus for the product topology case we get the function to be continuous at all points as well. Thus in both cases h is a continuous function.

□

Theorem 6.6. *Let (X_a, τ_a) be a topological (1) magma, (2) group, or (3) ring. Then $(\prod_{a\in A} X_a, \tau)$, where τ is the box [product] topology, is a topological (1) magma, (2) group, or (3) ring*

Proof. (1) $(+(x, y) = (+_a(\pi_a(x, y))_{a\in A}$ and thus by above lemma continuous

(2) $inv(x) = (inv_a(\pi_a(x)))_{a\in A}$, so by the lemma is continuous.

(3) By (1), (2), the fact that $\prod_{a\in A}((X_a, +_a), \tau)$ is a topological group, and the fact that $\prod_{a\in A} *((X_a, +_a), \tau)$ is a topological magma, we have the desired result.

$\square$

Theorem 6.7. *Let* (X_a, τ_a) *be a weak topological field for any* $a \in A$. *Then* $\prod_{a\in A}(X_a, \tau_a)$ *is a weak topological field in the box topology.*

Proof. (i) $\prod_{a\in A} X_a$ has a one namely $(1_a)_{a\in A}$

(ii) Free from Theorem 6.6 (3)

(iii) $(X_a^{-1}, \times_a, \tau)$ is a topological group for all $a \in A$. Part (2) of Theorem 6.6 tells us that $(\prod_{a\in A}(X_a^{-1}, \times_a), \tau)$ is a topological group.

(iv) If $X_a^{-1} \in \tau_a$ for all $a \in A$, then $\prod_{a\in A} X_a^{-1} \in \prod_{a\in A} \tau_a$. So $\prod_{\imath nA} X^{-1}$ is in the box topology. $\square$

Notice that this is not possible in the infinite product topology since $X^{-1} \subset X$ Thus the arbitrary product of X^{-1} cannot possibly be open since it will not be X at all but finitely many places.

Theorem 6.8. *Let* (F_a, τ_a) *be topological fields. Then* $(\prod_{a\in A} F_a, \tau)$ *where* τ *is the box topology, is a weak topological field.*

Proof. Since (F_a, τ_a) is a topological field implies it is also a weak topological field for any $a \in A$. Now by theorem 6.7 (4) we get the desired result. $\square$

Example 6.4. *For any* $n \in \mathbb{N}$ *we have,* $\mathbb{R}^n$ *is a weak topological field.*
We know $\mathbb{R}^n$ *in the Euclidean metric is homeomorphic to* $\mathbb{R}^n$ *in the* d_1 *metric whose topology is homeomorphic to the product topology of* $\prod_{i=1}^n(\mathbb{R}, \tau_\epsilon)$, *which is the product of topological fields.*

Example 6.5. *Let* $D \subseteq \mathbb{R}$ *Now notice that* $(\mathbb{R}, \tau_\epsilon)^D$ *forms a weak topological field under the product topology.*

Theorem 6.9. ALT for Weak Topological Fields

Suppose (Y, τ') *is a weak topological field,* (X, τ) *is non-isolated at c,* $\lim_{x \to c} f(x) = L$, *and* $\lim_{x \to c} g(x) = M$. *Then*

(1) $\lim_{x \to c}(f(x) + g(x)) = L + M$

(2) $\lim_{x \to c}(f(x)g(x)) = LM$

(3) $\lim_{x \to c} kf(x) = kL$ *and* $\lim_{x \to c} f(x)k = Lk$, *if* $X \subseteq Y$

(4) $\lim_{x \to c}(f(x)g(x)^{-1}) = LM^{-1}$, *if* $M \in Y^{-1}$

(5) $\lim_{x \to c}(f(x)^{-1}g(x)) = L^{-1}M$, *if* $L \in Y^{-1}$.

Proof. Parts (1) and (2) come from $(Y, +, \tau')$ and $(Y, \times, \tau')$ being respectively a topological group and a topological magma. (3)Use (2) and the continuous function h(x) = k. For (4) and (5), we have $\times'(a, b), '\times(a, b)$ are continuous functions thus form topological magma's. Then apply the algebraic limit theorem for magma's. □

For a division ring R, R may only be a topological ring or it may be a weak topological field. A great example of a division ring that is a topological field are the quaternions with the traditional metric topology. Under this topology the set of quaternions is homeomorphic to $\mathbb{R}^4$ but not isomorphic. We can easily get this from the quaternions have only one non-invertible element while the topological weak field $\mathbb{R}^4$ has a non-invertible set of the form:

$\{(x_1, x_2, x_3, x_4) \mid x_1x_2x_3x_4 = 0\}$,

which is clearly an infinite set. Thus the multiplicative groups could not be isomorphic.

Example 6.6. *Notice C[0,1], the set of continuous functions on* $[0, 1]$ *to* $\mathbb{R}$ *with the sup metric, forms a weak topological field. The proofs that addition, subtraction, and multiplication are continuous binary operations are nearly identical to the proofs for* *the real line with the standard topology. We should also have an idea of what* $C[0,1]^{-1}$ *looks like.*

Suppose, $f \in C[0,1]^{\times}$*. Then* $f(x) \neq 0$ *for any* $x \in [0,1]$*. We have by continuity* $f(x) = \mid f(x) \mid$ *for all* $x \in [0,1]$ *or* $f(x) = -\mid f(x) \mid$ *for all* $x \in [0,1]$ *as well as* $f(x) \neq 0$ *for any* $x \in [0,1]$*. To confirm that this is sufficient let us assume there exists* $(g, x) \in C[0,1] \times [0,1]$ *such that* $\frac{g(x)}{f(x)}$ *is undefined. Then,*

(1) $f(x) = 0$ *implies* $f(x) \notin C[0,1]^{-1}$*.*

(2) $f(x) \neq 0$ *tells us that* $g(x)$ *is undefined which would imply* $g(x) \notin C[0,1]$*.*

Thus we can say $C[0,1]^{-1} = \{f(x) \in C[0,1] \mid f(x) = \mid f(x) \mid$ *or* $f(x) = -\mid f(x) \mid$ *for any* $x \in [0,1]$ *and* $\mid f(x) \neq 0$ *for any* $x \in [0,1]\}$*. From this we can see that the proof that* $inv_{\times}(f(x)) = \frac{1}{f(x)}$ *is continuous over* $C[0,1]^{\times}$ *is virtually identical to the proof that* $\frac{1}{x}$ *is continuous on* $\mathbb{R} \setminus \{0\}$*. There are two ways to see that* $C[0,1]^{-1}$ *is open. The first way is analytically, the second way is topologically.*

(1) Let us show that $C[0,1]^{-1}$ *is an open set. Let* $f \in C[0,1]^{-1}$ *and without loss of generality assume* $f > 0$*. Thus, since f is continuous on a compact set it has a local minimum,* $L > 0$*. Pick* $\epsilon \in (0, L)$*. Now for any* $x \in [0,1], f(x) - \epsilon > L - \epsilon > L - L > 0$*. Thus for all* $f \in C[0,1]^{-1}$ *there exists* ϵ_f *such that* $B(f, \epsilon_f) \subseteq C[0,1]^{-1}$*. Thus* $C[0,1]^{-1}$ *is open.*

(2) Let $f \in C[0,1]^{-1}$ *and* $g(x) = 0$*. Notice that* $g(x) \in C[0,1]$*. Now since any metric space is* T_2 *there will exist* U_f *and* U_g*, both open in* $C[0,1]$ *with the property* $U_f \cap U_g = \emptyset$*. Thus,* $U_f \subseteq C[0,1]^{-1}$ *which implies* $C[0,1]^{-1}$ *is open.*

Thus, we know that C[0,1] is a weak topological field. Hence, if $f, g : (X, \tau) \to C[0,1]$*,* τ *is non-isolated at c,* $\lim_{x \to c} f(x) = L$ *and* $\lim_{x \to c} g(x) = M$*, Then*

(1) $\lim_{x \to c}(f(x) + g(x)) = L + M$

(2) $\lim_{x \to c}(f(x)g(x)) = LM$

(3) $\lim_{x\to c} kf(x) = kL$ *and* $\lim_{x\to c} f(x)k = Lk$, *if* $X \subseteq Y$

(4) $\lim_{x\to c}(f(x)g(x)^{-1}) = LM^{-1}$, *if* $g(x) \in (\mathbb{R}^{[0,1]})^{-1}$

(5) $\lim_{x\to c}(f(x)^{-1}g(x)) = L^{-1}M$, *if* $L \in (\mathbb{R}^{[0,1]})^{-1}$.

7 Differentiability in a Topological Setting

Definition. *Let* $D \subseteq F$ *[ring with one] field and let* $f : (D,\tau) \to (F,\tau')$ *and* $c \in D$. *Define* $D_c : D - \{c\} \to F$ *by* $D_c(x) = [f(x) - f(c)][x - c]^{-1}$ *[when it exists] to be the difference quotient at c.*

Definition. *Given* (X,τ) *and* $(X - \{c\}, \tau_{X-\{c\}})$, *we say* $U_c \in \tau_{X\setminus\{c\}}$ *is an absent neighborhood of* $c \in X$ *if*

(1) There exists a $V_c \in \tau$ *such that* $U_c = V_c'$,

(2) $V_c' \neq \varnothing$.

The reason we need absent neighborhoods is to deal with the problem of taking the limit as x approaches c to a function who has $(x-c)^{-1}$ since this function clearly is not defined at c.

Notice if there is a non-empty U_c absent neighborhood then the original topology is non-isolated at c.

Theorem 7.1. *Let* $f : (X\ \tau) \to (Y,\tau')$ *and define* $plim_{x\to c} f(x) = L$ *if for all* $U_L \in \tau'$ *there exists a* $U_c \in \tau_{X-\{c\}}$ *such that* $U_c' \neq \varnothing$ *and* $f(U_c') \subseteq U_L$. *If* $\lim_{x\to c} f(x) = L$ *and* $plim_{x\to c} f(x) = M$ *we have,* $L = M$.

Proof. $U_c \in \tau_{X-\{c\}}$ if and only if there exists $V_c \in \tau$ such that $V_c' = U_c$. Thus, for the limit we would chose $V_c \in \tau$. Finally by definition, $V_c' \neq \varnothing$ and $f(V_c') = f(U_c') \subseteq U_L$.

□

For this reason,we will except the plim and the lim are the same operation when dealing with difference quotients as we will below. We need the plim because it allows us to define limits where they are not even defined in the domain. This is something we must deal with in difference quotients.

Definition. *Given Let F be a field [ring with one], $D \subseteq F$, and $f : (D, \tau) \to (F, \tau')$. We define the topological derivative at $c \in D$ to be $\frac{\tau'}{\tau x} f(c) = \lim_{x \to c} D_c(x)$[if it exists]. If f is differentiable on all of D, we denote the topological derivative of the function on D by $\frac{\tau'}{\tau x} f(x)$. When the two topologies are understood to be fixed we may instead write f' as the topological derivative and $f'(x)$ for the topological derivative at a point.*

Notice if limits are not unique then the derivative need not be a function.

Theorem. *If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at c (by the traditional analytic definition or the topological definition), then $\frac{d}{dx} f(x) = \frac{\tau_\epsilon}{\tau_\epsilon x} f(x)$.*

Proof. Suppose $\frac{\tau_\epsilon}{\tau_\epsilon x} f(c) = L$, which is by definition $\lim_{x \to c} D_c(x) = L$. Then,

(1) For any $U_L = (L - \epsilon, L + \epsilon), \epsilon > 0$,

(2) There will exist $U_c = (c - \delta, c + \delta), \delta > 0$,

(3)$U_c' = (c - \delta, c) \cup (c, c + \delta)$ and,

(4)$D_c(U_c') \subseteq U_L$.

From (1,2,3,4) we get, For any $\epsilon > 0$, there will exist a $\delta > 0$ such that $\mid x - c \mid < \delta (x \neq c)$ implies $\mid \frac{f(x) - f(x)}{x - c} - L \mid < \epsilon$.

Thus $\frac{d}{dx} f(c) = L$.

Conversely, let us assume $\frac{d}{dx} f(x) = L$ then,

(1) For all $\epsilon > 0$,

(2) There is a $\delta > 0$,

(3)such that $\mid x - c \mid < \delta, x \neq c$,

(4) Gives us that $\mid \frac{f(x)-f(c)}{x-c} - L \mid < \epsilon$. Now from (1,2,3,4) we get: For any $(L-\epsilon, L+\epsilon) \subseteq U_L$ there will exist a $U_c = (c-\delta, c+\delta)$ such that $U_c' \neq \varnothing$ and $D_c(U_c') \subseteq U_L$.

Hence, $\frac{\tau_\epsilon}{\tau_\epsilon x} f(c) = L$. Thus, the two are equivalent. □

Now we will present some interesting examples of how this may differ from the traditional derivative.

Example 7.1. *There exists a function $f : \mathbb{R} \to \mathbb{R}$ whose topological derivative is the floor function.*

Proof. Let $f : (\mathbb{R}, S^-) \to (\mathbb{R}, \tau_\epsilon)$ be defined by $f(x) = \lfloor x \rfloor x$. Now by the above theorem if the limit exists in a courser topology then the limit exists in the finer topology so long as the topology is non-isolated at that point. Recall that the lower limit topology is non-isolated everywhere. Thus, on intervals $(z, z+1)$, where $x \in \mathbb{Z}$, $f(x) = zx$ and thus,

$\frac{\tau_\epsilon}{S^- x} f(x) = \frac{d}{dx} f(x) = z = \lfloor x \rfloor$. Now to show $\frac{\tau_\epsilon}{S^- x} f(z) = z$ we must pick $(z-\epsilon, z+\epsilon) \in \tau_\epsilon$ (with out loss of generality we may assume $\epsilon < 1$). Now if we pick $[z, z+\epsilon) \in S^-$, then for $x = z + \delta \in [z, z+\epsilon)'$ we have

$[f(z+\delta) - f(z)][z+\delta-z]^{-1} = [z(z+\delta)-z^2]\delta^{-1} = [z^2+z\delta-z^2]\delta^{-1} = z\delta(\delta)^{-1} = z \in (z-\epsilon, z+\epsilon)$.

Thus, $\frac{\tau_\epsilon}{S^{-1}} f(z) = z = \lfloor x \rfloor$. Hence, $\frac{\tau_\epsilon}{S^{-1}} f(x) = \lfloor x \rfloor$. □

This examples show's that Darboux's property need not hold in a topological setting. Also realize this provides a counter example to the mean value property on the real line. Consider $[2, \frac{7}{2}]$ if the mean value theorem were to hold there would exist $c \in [2, \frac{7}{2}]$ such that $\lfloor c \rfloor = f'(c) = \frac{f(\frac{7}{2})-f(2)}{\frac{7}{2}-2} = \frac{3(\frac{7}{2})-2(2)}{\frac{7}{2}-2} = \frac{\frac{21}{2}-4}{\frac{3}{2}} = \frac{13}{3}$, which is impossible.

Example 7.2. *There exists a function and a domain and range topology such that the function is*

(1) not continuous at a point c and

(2) the function is differentiable at the point c.

Proof. Consider $\mathbb{Z}_3$ and define the following topologies on $\mathbb{Z}_3$: $\tau = \{\mathbb{Z}_3, \varnothing, \{0,2\}, \{1\}\}$, $\tau' = \tau$. Define $f : (\mathbb{Z}_3, \tau) \to (\mathbb{Z}_3, \tau')$ by $f(x) = x + 2$.

Claim 1: $f(x)$ is not continuous at 0

f(0) = 0 + 2 = 2. $f^{-1}(\{0,2\}) = \{1,2\} \notin \tau$. Thus, $f(x)$ is not continuous at 2.

Now let us compute $D_0(x) = \frac{f(x)-f(0)}{x-0} = \frac{x+2-2}{x} = \frac{x}{x} = 1$.

Claim 2: $\lim_{x\to 0} D_0(x) = 1$.

Let $U_1 = \{1\} \in \tau'$. Let $U_0 = \{0,2\}$. Then $U_0' = \{2\}$ and $D_0(\{2\}) = \{1\}$.

Thus, $\lim_{x\to 0} D_0(x) = 1$, which implies that $\frac{\tau}{\tau x} f(0) = 1$. Thus this function is discontinuous at 0 yet differentiable at 0. □

In fact this can be generalized to create functions that are everywhere differentiable yet nowhere continuous.

Example 7.3. *Let $D \subseteq F$, $D \neq \varnothing$, and $f : (D, \{\varnothing, D\}) \to (F, \{\varnothing, \{1\}, F\})$ is defined by $f(x) = x$ is nowhere continuous. For all $c \in D, D_c(x) = \frac{f(x)-f(c)}{x-c} = \frac{x-c}{x-c} = 1$. Thus, the derivative is 1 but one can easily see that the function is nowhere continuous.*

Example 7.4. *Here is another function that is everywhere differentiable but nowhere continuous on the real line with more efficient topologies. Let $c \in \mathbb{R}$ and let $f : (R, \tau_\epsilon) \to (R, S^-)$ be defined by $f(x) = x + c$. Clearly, f will be discontinuous everywhere because of the topologies, but the difference quotient is one. Now another set of topologies that does this is the indiscrete topology on the range space and the Euclidean topology on the domain space. If you let the discrete topology be the topology on the range space and you let the Euclidean topology be the topology on the domain space then you would have the same result.*

This shows us that in general differentiability and continuity are not entirely related. We will investigate under what conditions they are related, but first we need a couple of properties that are related to a single commutative binary operation.

Theorem 7.2. *Let G be an Abelian group under addition and let τ be a topology on G. If addition is separately continuous with respect to τ then we have the following property: If $U \in \tau$, then $U + c = \{x + c \mid x \in U\} \in \tau$.*

Proof. For any $c \in G$, Define $f_c : (G, \tau) \to (G, \tau)$ by $f_c(x) = x - c = x + (-c) = +(x, -c)$. Then f_c thus continuous by hypothesis. Thus, for any $U \in \tau, f_c^{-1}(U) = \{x + c \mid x \in U\} = U + c \in \tau$. □

Theorem 7.3. *Let G be an Abelian group under addition and let τ be a topology on G with addition is separately continuous with respect to τ. Let $f, g : (X, \tau) \to (G, \tau')$ with $g(x) = f(x) - L$ and $\lim_{x \to c} g(x) = 0$. Then $\lim_{x \to c} f(x) = L$.*

Proof. Let $U_L \in \tau'$ be given. Then $U_L - L$ is an open neighborhood of 0, which we will denote by U_0. Since $\lim_{x \to c} g(x) = 0$ there will exist a $U_c \in \tau$ such that $U_c' \neq \varnothing$ and $g(U_c') \subseteq U_0$. Now $f(U_c') = \{g(x) + L \mid x \in U_c'\} \subseteq U_0 + L = U_L$. □

Theorem 7.4. *Let F be a field, $D \subseteq F$, and let τ and τ' be topologies on D and F respectively. Suppose that $f : (D, \tau) \to (F, \tau')$ and τ' is jointly continuous under multiplication and separately continuous under addition. Then f is also continuous.*

Proof. $\lim_{x \to c}(f(x) - f(c)) = \lim_{x \to c}(\frac{f(x) - f(c)}{x - c}(x - c)) = \lim_{x \to c}(D_c(x)(x - c)) = f'(c)(0) = 0$. Hence, $\lim_{x \to c} f(x) - f(c) = 0$. Thus, $f(x) = f(c)$. □

This result does not require full continuity of addition, nor does it require continuity of inverses.