On formally undecidable propositions of Zermelo-Fraenkel set theory

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ABSTRACT

We present a demonstration of the Gödel's incompleteness phenomenon in the formal first-order axiomatization of the Zermelo-Fraenkel axioms (ZF) of set theory following the methods displayed in Gödel's famous 1931 paper, *Über formal unemtscheidbare Sätze der Principia Mathematica und verwandter Systeme I.* [1]

In dedication to Jeff Denniston,

whose insight,

patience,

and brilliance

guide me.

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Contents

1 Introduction

1.1 Outline of the Main Argument

In defining a formal theory, such as the first-order Zermelo-Fraenkel set theory (ZF), one observes that the meaningful formulae are merely finite strings of symbols that obey particular formation rules. Similarly, a proof-schema is a finite collection of formulae that obey particular formation rules given by the axioms and rules of inference. Using these rules one can, in a determinate way, decide whether a given string or a given ordered collection of formulae is indeed a formula or proof-schema, respectively.

From a metamathematical viewpoint the objects used as the primitive symbols are immaterial; it is merely their arrangement that serves purpose. We will instead use *natural numbers* as our primitive symbols[1] for the theory ZF. We use numbers because, on the one hand, metamathematical concepts of our system then become properties of natural numbers. It will be shown that concepts such as "is a formula," "is inferred by," "is a proof-schema," "is a provable formula," etc. are definable by certain number-theoretic functions and relations. On the other hand, the theory ZF is capable of expressing properties of natural numbers, i.e., ZF contains number theory. We are thus confronted with a most peculiar phenomena; namely, ZF can serve as its own metatheory.

We can now demonstrate, in outline, the existence of an undecidable proposition[2] in the theory ZF by a fixed-point argument. We will call a formula with one free variable a *class-property.* As will be shown, each formula is represented by a number

[1] Thus, a formula will be a specific natural number defined by a particular finite series of numbers and a certain proof-schema a specific natural number defined by a particular finite series of finite series of numbers.

[2] A proposition P of ZF is undecidable iff neither P nor $\neg P$ are provable within theory ZF

and thus can be ordered in a sequence by some relation R, where $R(n)$ denotes the n-th class-property. For a class-property ϕ, let $\phi(n)$ denote the sentence obtained by replacing each occurrence of the free variable in ϕ by the number n. Now, it can be shown that the concept "class-property," the relation R, and the relation $x = y(z)$ are all definable in ZF. Also, as stated above, the concept "is a provable formula" is definable as a number-theoretic relation $\mathrm{Thm}(x)$ which states "x is a provable formula." We now define a class Γ of natural numbers as follows:

$$n \,\epsilon\, \Gamma \iff \neg\mathrm{Thm}\big([R(n)](n)\big).$$

Thus the concept for "is a member of Γ," too, must be definable in ZF by some class-property, call it G. Since G is a class-property there must exist some number g such that $G = R(g)$.

We claim that the sentence $[R(g)](g)$ is undecidable, assuming ZF is consistent. If, on the one hand, $[R(g)](g)$ were provable then $\mathrm{Thm}\big([R(g)](g)\big)$ would hold. But this would imply $g \not\epsilon\, \Gamma$ and hence $\neg G(g)$. But $[R(g)](g)$ is $G(g)$, a contradiction. On the other hand, if $\neg[R(g)](g)$ were provable then $\neg G(g)$ would hold and, hence, $g \not\epsilon\, \Gamma$. But this means $\mathrm{Thm}\big([R(g)](g)\big)$ holds and $[R(g)](g)$ is provable, a contradiction.

Reflecting upon the content of the sentence $[R(g)](g)$, one sees that $[R(g)](g)$ essentially says, in a roundabout way, "I am unprovable." Now, $[R(g)](g)$ is indeed not provable in ZF, but is nevertheless a *true* sentence. Utilizing this fact we show that the consistency of ZF cannot be demonstrated within itself.

1.2 Metamathematical notation

The following logical notation will be used in metalogical strings:[3]

$$\neg \quad \& \quad \vee \quad \Rightarrow \quad \Leftrightarrow \quad \forall \quad \exists \quad \epsilon$$

representing negation, conjunction, disjunction, implication, biconditional, universal and existential quantifiers, and membership, respectively. We use $\equiv$ to represent definition of formulae where, in general, the lefthand argument is meant to stand as an abbreviation for the righthand argument.

We use the lowercase italicized english alphabet as metamathematical variables, *e.g.*, $x, y, z, x_1, ..., x_n, ...$, representing individual variables[4] ("sets") in Section 2, and representing natural numbers thenceforth. Similarly, we use the uppercase italicized english alphabet, *e.g.*, P, Q, R, to represent well-formed strings in Section 2, and representing number-theoretic relations thenceforth. Otherwise, use of such symbols will be specified in or understood by context.

[3] That is, strings outside the systems $\mathcal{L}_1$ and $\mathcal{L}_2$

[4] Or for terms in the case of using s and t

2 Zermelo-Fraenkel set theory

2.1 The language of set theory $\mathcal{L}_1$

We shall now define $\mathcal{L}_1$, our first-order language of set theory.[5] $\mathcal{L}_1$ consists of the following primitive symbols:

Logical symbols: $\sim\ \vee\ \forall\ (\ )$

Binary relations: $=\ \ \in$

Individual variables:[6] $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n, ...$

We shall call a finite series of primitive symbols a *string*. The meaningful strings of a logistic language are represented by *terms* and *formulae*. Since $\mathcal{L}_1$ contains no function symbols, the terms of $\mathcal{L}_1$ are precisely the variables.[7]

Definition 2.1. We define a *formula* by induction on the construction of a string as follows:

1. If s and t are terms, then $(s = t)$ is a formula.

2. If s and t are terms, then $(s \in t)$ is a formula.

3. If P is a formula, then so is $({\sim}P)$.

4. If P and Q are formulae, then so is $(P \vee Q)$.

5. If P is a formula and x a variable, then $(\forall x P)$ is a formula.

[5] We use the logic of $\mathcal{F}^=$ from [4], where it is shown that $\mathcal{F}^=$ is both sound and complete. It should be noted that we replace the word "well-formed formula" in [4] simply by "formula."

[6] We consider a denumerable collection of individual variables, one for each natural number ≥ 1.

[7] Although redundant for the formulae of $\mathcal{L}_1$, we use terms, instead of variables alone, in Definition 2.1 so we may use the very same definition in our construction of the formulae of $\mathcal{L}_2$, where the terms are nontrivial.

Definition 2.2. [4]

1. A *well-formed part* (wf part) of a formula P is a substring of P that is, itself, a formula.
2. An occurrence of a variable x in a formula P is said to be *bound* iff it is in a wf part of P of the form $\forall xQ$; otherwise x is said to be *free*. We use $P(x_1, ..., x_n)$ to denote that the free variables of P are among $x_1, ..., x_n$.[8]
3. A formula $R(x_1, ..., x_n)$ with precisely n free variables is called an n-ary *predicate*.
4. We call a 1-ary predicate $R(x)$ a *class-property*.
5. A 0-ary predicate, *i.e.* a formula with no free variables, is called a *sentence*.
6. If x is a variable and t is a term, we say that t is *free for x in a formula P* iff no free occurrence of x in P is in a wf part of P of the form $\forall yQ$, where y is a free variable of t.
7. Let P be a formula. Then $P\binom{x}{t}$ will denote the formula obtained by *substitution*, where each free occurrence of x in P is replaced by the term t, provided t is free for x in P.[9]

[8] Possibly some x_i are not free, or do not occur, in P. [5]

[9] If x is not free in P, then $P\binom{x}{t}$ is precisely P. To obtain the same desired result for multiple substitutions, and to avoid capture, we define $P\left(\begin{smallmatrix} a_1 & \cdots & a_n \\ t_1 & & t_n \end{smallmatrix} \right)$ as follows: Let

$$m = \max\{l \in \mathbb{N} : l = k \text{ for } a_i = \mathbf{x}_k,\ i = 1, ..., n \text{ or } l = k \text{ for some } \mathbf{x}_k \text{ free in } P\}.$$

Now, let $T_i = t_i\left(\begin{smallmatrix} a_1 & \cdots & a_n \\ \mathbf{x}_{m+(i-1)n+1} & & \mathbf{x}_{m+(i-1)n+n} \end{smallmatrix} \right)$ for each $i = 1, ..., n$ and let $Q = P\left(\begin{smallmatrix} a_1 \\ T_1 \end{smallmatrix} \right) \cdots \left(\begin{smallmatrix} a_n \\ T_n \end{smallmatrix} \right)$ by repeated substitution. Now, by repeated substitution, we define

$$P\left(\begin{smallmatrix} a_1 & \cdots & a_n \\ t_1 & & t_n \end{smallmatrix} \right) = Q\left(\begin{smallmatrix} \mathbf{x}_{m+1} \\ a_1 \end{smallmatrix} \right) \cdots \left(\begin{smallmatrix} \mathbf{x}_{m+n} \\ a_n \end{smallmatrix} \right) \cdots \left(\begin{smallmatrix} \mathbf{x}_{m+(i-1)n+1} \\ a_1 \end{smallmatrix} \right) \cdots \left(\begin{smallmatrix} \mathbf{x}_{m+(i-1)n+n} \\ a_n \end{smallmatrix} \right) \cdots \left(\begin{smallmatrix} \mathbf{x}_{m+n^2-n+1} \\ a_1 \end{smallmatrix} \right) \cdots \left(\begin{smallmatrix} \mathbf{x}_{m+n^2} \\ a_n \end{smallmatrix} \right).$$

The axiomatic structure of $\mathcal{L}_1$ includes the following two rules of inference and seven logical axioms. We shall also use the abbreviations:[10]

$$\wedge \qquad \rightarrow \qquad \leftrightarrow \qquad \exists$$

as well as omission of brackets.[11]

Rules of Inference:

(MP) **Modus Ponens:** From P and $P \rightarrow Q$ to infer Q.

(GN) **Generalization:** From P to infer $\forall x P$, where x is any variable.

Logical Axiom Schemata: [4]

(L1) $P \vee P \rightarrow P$

(L2) $P \rightarrow Q \vee P$

(L3) $(P \rightarrow Q) \rightarrow (R \vee P \rightarrow R \vee Q)$

(L4) $\forall x P \rightarrow P\binom{x}{t}$ where t is a term which is free for x in P.

(L5) $\forall x(P \vee Q) \rightarrow (P \vee \forall x Q)$ if x is not free in P.

(L6) $(\mathbf{x}_1 = \mathbf{x}_1)$

(L7) $x = y \rightarrow (P\binom{z}{x} \rightarrow P\binom{z}{y})$ for formula P.

Definition 2.3. A *proof* in a logistic system of a formula P is a sequence of formulae, each of which is either an axiom or is inferred from preceding formulae in the sequence by a rule of inference, with P as the last formula in the sequence. A *theorem* of a logistic system is a formula which has a proof in the system.

[10] Note: we replace "$\supset$" in [4] by "$\rightarrow$". The abbreviations are as follows:

$$\begin{array}{rcl@{\qquad}rcl} P \wedge Q & \equiv & \sim((\sim P) \vee (\sim Q)) & P \rightarrow Q & \equiv & ((\sim P) \vee Q) \\ P \leftrightarrow Q & \equiv & (P \rightarrow Q) \wedge (Q \rightarrow P) & \exists x P & \equiv & \sim(\forall x(\sim P)) \end{array}$$

[11] Connectives take precedence in the following order: $\sim$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$.

2.2 The theory ZF of $\mathcal{L}_1$

We can now provide a foundation for the first-order theory ZF, the axiomatic set theory of Zermelo-Fraenkel, which is obtained by adding the follow axioms and axiom schemata to the system $\mathcal{L}_1$. It is intended that the variables represent sets and predicates represent properties of sets.

Axioms of ZF:[12] [5]

(A1) $\forall \mathbf{x}_1 \forall \mathbf{x}_2 (\mathbf{x}_1 = \mathbf{x}_2 \leftrightarrow \forall \mathbf{x}_3 (\mathbf{x}_3 \in \mathbf{x}_1 \leftrightarrow \mathbf{x}_3 \in \mathbf{x}_2))$ [*Extensionality*]

(A2) $\forall \mathbf{x}_1 \forall \mathbf{x}_2 \exists \mathbf{x}_3 \forall \mathbf{x}_4 (\mathbf{x}_4 \in \mathbf{x}_3 \leftrightarrow (\mathbf{x}_4 = \mathbf{x}_1 \vee \mathbf{x}_4 = \mathbf{x}_2))$ [*Pairing*]

(A3)^P $\forall \mathbf{x}_1 \forall \mathbf{x}_2 \exists \mathbf{x}_3 \forall \mathbf{x}_4 (\mathbf{x}_4 \in \mathbf{x}_3 \leftrightarrow (\mathbf{x}_4 \in \mathbf{x}_1 \wedge P\left(\begin{smallmatrix} x & y \\ \mathbf{x}_4 & \mathbf{x}_2 \end{smallmatrix}\right)))$ [*Separation Schema*]

(A4) $\forall \mathbf{x}_1 \exists \mathbf{x}_2 \forall \mathbf{x}_3 (\mathbf{x}_3 \in \mathbf{x}_2 \leftrightarrow \exists \mathbf{x}_4 (\mathbf{x}_4 \in \mathbf{x}_1 \wedge \mathbf{x}_3 \in \mathbf{x}_4))$ [*Union*]

(A5) $\forall \mathbf{x}_1 \exists \mathbf{x}_2 \forall \mathbf{x}_3 (\mathbf{x}_3 \in \mathbf{x}_2 \leftrightarrow \mathbf{x}_3 \subseteq \mathbf{x}_1)$ [*Power Set*]

(A6) $\exists \mathbf{x}_1 (\emptyset \in \mathbf{x}_1 \wedge \forall \mathbf{x}_2 (\mathbf{x}_2 \in \mathbf{x}_1 \rightarrow \mathbf{x}_2 \cup \{\mathbf{x}_2\} \in \mathbf{x}_1))$ [*Infinity*]

(A7)^P $\forall \mathbf{x}_1 (\forall \mathbf{x}_2 \forall \mathbf{x}_3 \forall \mathbf{x}_4 (P\left(\begin{smallmatrix} x & y & z \\ \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_1 \end{smallmatrix}\right) \wedge P\left(\begin{smallmatrix} x & y & z \\ \mathbf{x}_2 & \mathbf{x}_4 & \mathbf{x}_1 \end{smallmatrix}\right) \rightarrow \mathbf{x}_3 = \mathbf{x}_4)$ [*Replacement*
$\rightarrow \forall \mathbf{x}_2 \exists \mathbf{x}_3 \forall \mathbf{x}_4 (\mathbf{x}_4 \in \mathbf{x}_3 \leftrightarrow \exists \mathbf{x}_5 (\mathbf{x}_5 \in \mathbf{x}_2 \wedge P\left(\begin{smallmatrix} x & y & z \\ \mathbf{x}_5 & \mathbf{x}_4 & \mathbf{x}_1 \end{smallmatrix}\right))))$ *Schema*]

(A8) $\forall \mathbf{x}_1 (\mathbf{x}_1 \neq \emptyset \rightarrow \exists \mathbf{x}_2 (\mathbf{x}_2 \in \mathbf{x}_1 \wedge \mathbf{x}_1 \cap \mathbf{x}_2 = \emptyset))$ [*Regularity*]

Note that both A3 and A7 are axiom schema, i.e. there are axioms A3^P and A7^P for each formula $P(x, y, z)$ in $\mathcal{L}_1$.

We now prove that ZF can demonstrate the existence of an *empty-set* $\emptyset$ and, given any set x, the *set-theoretical successor of* x, denoted $x \cup \{x\}$, is also a set.

[12] For clarity, we use the following abbreviations where s, t are terms and x, y are variables:

$$\begin{array}{rcl@{\qquad}rcl}
s \subseteq t & \equiv & \forall x (x \in s \rightarrow x \in t) & s \cap t = \emptyset & \equiv & \sim\exists x (x \in s \wedge x \in t) \\
\emptyset \in s & \equiv & \exists x \forall y (y \in s \wedge \sim(y \in x)) & s \cup \{s\} \in t & \equiv & \exists x (x \in t \wedge \forall y (y \in x \leftrightarrow (y = s \vee y \in s)) \\
s = \emptyset & \equiv & \forall x (\sim(x \in s)) & t \in s \cup \{s\} & \equiv & t \in s \vee t = s \\
s \neq \emptyset & \equiv & \exists x (x \in s) & t = s \cup \{s\} & \equiv & \forall x (x \in t \leftrightarrow (x \in s \vee x = s))
\end{array}$$

Proposition 2.4. *The following sentences are theorems of ZF:*

$$\text{(i) } \exists \mathbf{x}_1(\mathbf{x}_1 = \emptyset) \qquad\qquad \text{(ii) } \forall \mathbf{x}_1 \exists \mathbf{x}_2(\mathbf{x}_2 = \mathbf{x}_1 \cup \{\mathbf{x}_1\})$$

Proof. For the sake of clarity and brevity, we will provide an informal proof of the above claim.[13]

i. By A6, there exists a set x. Let $P(x) \equiv (x \neq x)$, then by A3^P there exists y, a subset of x, such that $z \in y$ iff $z \in x \wedge z \neq z$. But by L6 and substitution, $z = z$ for any z, thus y contains no elements. Hence $y = \emptyset$, making $\emptyset$ is a set.

ii. Let x be a set. Then $\{x, x\}$ is a set by A2, and $\{x\} = \{x, x\}$ by A1. Again, by A2, $\{x, \{x\}\}$ is a set. So there exists a set y, by A4, such that $z \in y$ iff $z \in x \vee z \in \{x\}$, i.e. $z \in y$ iff $z \in x \vee z = x$. $\square$

2.3 The language $\mathcal{L}_2$

Let $\mathcal{L}_2$ be the logistic system obtained by adding the constant $\mathbf{0}$ ("empty set") and the unary function symbol $\mathbf{s}$ ("successor") to the language of $\mathcal{L}_1$, in addition we define the terms of $\mathcal{L}_2$ inductively as follows:

1. $\mathbf{0}$ is a term.
2. If x is a variable, then x is a term.
3. If t is a term, then $\mathbf{s}t$ is a term.

The formulae of $\mathcal{L}_2$ are then defined by Definition 2.1.

[13] As with a few facts in this paper, the formal proof of this claim would require an obnoxious unpacking of abbreviations, but the author encourages the reader to enter the forest.

2.4 The theory Z of $\mathcal{L}_2$

The object of our discourse will be the theory Z of $\mathcal{L}_2$, which is the axioms of ZF combined with the following two axioms:

(A9) $(\mathbf{0} = \emptyset)$

(A10) $\forall \mathbf{x}_1(\mathbf{s}\mathbf{x}_1 = \mathbf{x}_1 \cup \{\mathbf{x}_1\})$

Axiom 9 defines the constant symbol $\mathbf{0}$ to be understood as the "empty-class," while axiom 10 defines the function symbol $\mathbf{s}$ on a set x to be understood as the set-theoretic successor function, *i.e.* $\mathbf{s}x$ is the class $x \cup \{x\}$. By Proposition 2.4, we see that $\mathbf{0}$ and $\mathbf{s}x$ are sets, and are therefore redundant and only meant to be used as convenient, albeit formal, definitions. It is readily apparent, then, that all theorems of ZF are theorems of Z,[14] i.e. Z is an *extension* of ZF. In fact, Z is a *conservative extension* of ZF which implies Z is consistent iff ZF is consistent.[15] Furthermore, one can see that for each formula P in $\mathcal{L}_2$ there is a formula P' in $\mathcal{L}_1$ such that $P \leftrightarrow P'$ is a theorem in Z using A9 and A10. That is, if P is provable in Z then P' is provable in ZF using Proposition 2.4. Thus, Z and ZF are, in a sense, equivalent. Therefore, when we refer to theory Z we are essentially referring to ZF.

For the contents of this paper, the key property of Z, aside from its expressive capabilities to serve as a foundation for a large part of mathematics, is that it can model elementary number theory, in particular the axioms of Peano Arithmetic[16] (PA).[17]

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[14] This is because $\mathcal{L}_1 \subset \mathcal{L}_2$ and ZF $\subset$ Z, so any proof in ZF is also a proof in Z.
[15] See [4] §25 **2502**.
[16] The axioms of PA are given by S1-S8 and axiom-schema S9 in [6].
[17] A clear exposition of this fact can be found in §5.1-2 in [7]

As is shown in [5], the natural numbers $\mathbb{N}$ are represented in Z by the set N, which is defined as the smallest inductive class.[18] In fact, $0 \equiv \mathbf{0}$, $1 \equiv \mathbf{s0}$, $2 = \mathbf{ss0}$, ... , $n \equiv \mathbf{s}...\mathbf{s0}$,[19] ... etc. As it turns out, N is equivalent to the first infinite ordinal ω, and the operations of addition and multiplication will be defined by ordinal addition and multiplication.[20] Using A8, along with other properties specific to N,we can derive a full principle of induction.[21]

2.5 Gödel-Numbering of Z

In defining the primitive symbols for the languages $\mathcal{L}_1$ and $\mathcal{L}_2$, one can see that the objects used to represent the symbols were immaterial with respect to the syntactic structure of the languages. That is, it would be no different if we decided to use "A" as the universal quantifier or "!" as a variable, or even reading strings from the right instead of left. We will instead use *numbers* as the primitive symbols of Z, and we will write these numbers using the arabic numerals in base 10. By defining a convenient correspondence between the symbols and numbers we can use the Fundamental Theorem of Arithmetic[22] to uniquely construct a number r that represents a particular string R of $\mathcal{L}_2$ by interpreting the numbers of the exponents[23] of the primes (in order of magnitude) of the prime factorization of r, and vice versa. We now set up such a

[18] A class C is inductive if $\emptyset \in C$ and if $x \in C$ then $x \cup \{x\} \in C$. By A6 there exists an inductive set X, so by A3 N is a set since it is a subclass of set X.

[19] Where there are n occurrences of $\mathbf{s}$ proceeding $\mathbf{0}$. Henceforth we will abbreviate this by s_n.

[20] Let α and β be ordinal numbers. As is defined in [5], $\alpha + \beta = \gamma$, where $\gamma \geq \alpha$ is the unique ordinal such that β is equipotent to $\gamma \setminus \alpha$; and $\alpha \cdot \beta = \gamma$ where γ is the unique ordinal equipotent to the cartesian product $\alpha \times \beta$.

[21] See 1.4-1.10 in [5].

[22] A theorem that can be demonstrated using the PA, and hence represented in Z.

[23] That is, the exponents *are* the primitive symbols.

one-to-one correspondence as follows:

$$\begin{array}{rcl|rcl|rcl}
1 & \equiv & \mathbf{0} & 3 & \equiv & \mathbf{s} & 5 & \equiv & \in \\
7 & \equiv & = & 9 & \equiv & \sim & 11 & \equiv & \vee \\
13 & \equiv & \forall & 15 & \equiv & (& 17 & \equiv &) \\
 & & & p_{n+7} & \equiv & \mathbf{x}_n & & & ,
\end{array}$$

where p_{n+7} is the n-th prime number ≥ 19, in order of magnitude. Now to each string a, i.e. finite series of symbols, there corresponds a finite series of **symbols** (numbers) $x_1...x_n$. We map each string a to the number $\Phi(a) = 2^{x_1} \cdot 3^{x_2} \cdot ... \cdot p_n^{x_n}$, and call such a number the **string** that corresponds to the series $x_1...x_n$ representing the string. Thus, a **string** that represents a term (or formula) is called a **term** (or **formula**); and similarly for each metamathematical concept, *e.g.* "variable", "negation", "generalization", "proof",[24] "theorem", etc., there corresponds a concept of the same name, in bold print, of a number that represents each concept, respectively. So, if $R(a_1, ..., a_n)$ is an n-ary predicate there will correspond an n-**ary predicate** $R'(x_1, ..., x_n)$ which holds for $x_1, ..., x_n$ iff $x_i = \Phi(a_i)$, for $i = 1, ..., n$ and $R(a_1, ..., a_n)$. **Theory Z** will, thus, denote the collection of **axioms** derived from applying Φ to each of the axioms of Z.

3 (Primitive) Recursive functions and relations

We shall now demonstrate elementary results in recursion theory that, at this stage, have no immediate connection to the formal system of Section 2.[25]

[24] In fact a **proof** is the product of the primes with exponents that are **formulae**, in the order that the formulae appear in the proof.

[25] By "functions" we mean "number-theoretic functions," where $\mathbb{N}$ stands the collection of natural numbers including 0.

3.1 Basics of recursion theory

Definition 3.1. [3][26] The functions $C_n : \mathbb{N} \to \mathbb{N}$, $S : \mathbb{N} \to \mathbb{N}$, and $P_i^m : \mathbb{N}^m \to \mathbb{N}$ are called *initial functions*, where $C_n(x) = n$, $S(x) = x + 1$,[27] and $P_i^m(x_1, ..., x_m) = x_i$; denoting the constant function (for $n \in \mathbb{N}$), the successor function, and the projective function (for $m, i \in \mathbb{N}$ and $1 \leq i \leq m$), respectively.

Definition 3.2. [1] A function $\phi : \mathbb{N}^n \to \mathbb{N}$ is said to be *recursively defined* by the functions $\psi : \mathbb{N}^{n-1} \to \mathbb{N}$ and $\mu : \mathbb{N}^{n+1} \to \mathbb{N}$, if for all $x_2, ..., x_n, k \in \mathbb{N}$ the following hold:[28]

$$\phi(0, x_2, ..., x_n) = \psi(x_2, ..., x_n)$$
$$\phi(k+1, x_2, ..., x_n) = \mu(k, \phi(k, x_2, ..., x_n), x_2, ..., x_n)$$

Definition 3.3. Let $\phi : \mathbb{N}^n \to \mathbb{N}$, $\mu : \mathbb{N}^m \to \mathbb{N}$, and $\psi_i : \mathbb{N}^{k_i} \to \mathbb{N}$ be functions, with $n, m, k_i \in \mathbb{N}$ and $1 \leq i \leq m$. Then ϕ is said to be *derived by substitution* from $\mu, \psi_1, ..., \psi_m$ if for all $\mathfrak{n} \in \mathbb{N}^n$ and $\mathfrak{n}_i \in \mathbb{N}^{k_i}$, where each value in $\mathfrak{n}_i$ is a number from $\mathfrak{n}$,

$$\phi(\mathfrak{n}) = \mu\left(\psi_1(\mathfrak{n}_1), ..., \psi_m(\mathfrak{n}_m)\right)$$

Definition 3.4. [1] A function ϕ is said to be *recursive*[29], if there is a finite series of functions $\phi_1, ..., \phi_n$, with $\phi_n = \phi$, satisfying the property that each ϕ_k is either an initial function, recursively defined by ϕ_i and ϕ_j, for $i, j < k$, or derived by any of the previous functions by substitution. The length of the shortest series of ϕ_i, which belong to a recursive function ϕ, is termed the *degree of* ϕ and is denoted $deg(\phi)$.

[26] Note that [3] uses "U_i^m" instead of "P_i^m"; as well as merely the zero function instead of C_n, which, like [1], we consider to be initial since it can easily be found to be primitive recursive.

[27] It should be noted that "$x+1$" or "$k+1$" denotes the successor of the number x or the number k, and is not meant to presume the existence of an addition function.

[28] If $n = 1$ then ψ will be a constant.

[29] We use "recursive" as is used in [1], but is commonly called "primitive recursive", as opposed to "general recursive", given by [3].

Proposition 3.5. *The addition and multiplication functions are recursive.*

Proof.

$$(i) \qquad \iota_1(x_1, x_2) = S(P^2_2(x_1, x_2))$$

$$(ii) \qquad \iota_2(x_1, x_2, x_3) = S(P^3_2(x_1, x_2, x_3))$$

$$(iii) \qquad {}_{0}+(0) = 0;$$

$$\qquad {}_{0}+(k+1) = \iota_1(k, {}_{0}+(k))$$

$$(iv) \qquad +(0, y) = {}_{0}+(y);$$

$$\qquad +(k+1, y) = \iota_2(k, +(k, y), y)$$

$$(v) \qquad \iota_3(x_1, x_2, x_3) = +(P^3_1(x_1, x_2, x_3), P^3_2(x_1, x_2, x_3))$$

$$(vi) \qquad \times(0, y) = C_0(y);$$

$$\qquad \times(k+1, y) = \iota_3(k, \times(k, y), y)$$

We see that (i) and (ii) are recursive by substitution of initial functions, (iii) and (iv) are recursively defined by (i), (ii), and (iii), respectively, hence recursive. Now, (v) is a substitution of initial functions in (iv), thus recursive. Finally, (vi) is recursively defined by a constant function and (v), and is, thus, recursive. Henceforth, we shall write (iv) and (vi) with the infix notation $x + y$ and $x \cdot y$. □

Definition 3.6. [1] A relation $R(x_1, ..., x_n)$ among natural numbers is said to be a *recursive relation* if there exists a recursive function ϕ such that, for all numbers $x_1, ..., x_n \epsilon \mathbb{N}$:

$$R(x_1, ..., x_n) \text{ holds iff } [\phi(x_1, ..., x_n) = 0] \text{ holds.}$$

Proposition 3.7. [1] *The following propositions hold:*[30]

I. *Every function (or relation) derived from recursive functions (or relations) by substitution of recursive functions in place of variables is recursive, so also is every function derived from recursive functions by recursive definition.*

II. *If R and S are recursive relations, then so also are $\neg R$ and $R \vee S$.*[31]

III. *If the functions $\phi(\mathfrak{n})$ and $\psi(\mathfrak{x})$ are recursive, then so is the relation: $\phi(\mathfrak{n}) = \psi(\mathfrak{x})$.*

IV. *If the function $\phi(\mathfrak{n})$ and the relation $R(x, \mathfrak{x})$ are recursive, then so are the relations S and T, as well as the function ψ, given by:*

$$S(\mathfrak{n}, \mathfrak{x}) \equiv (\exists x)[x \leq \phi(\mathfrak{n}) \;\&\; R(x, \mathfrak{x})]$$

$$T(\mathfrak{n}, \mathfrak{x}) \equiv (\forall x)[x \leq \phi(\mathfrak{n}) \Rightarrow R(x, \mathfrak{x})]$$

$$\psi(\mathfrak{n}, \mathfrak{x}) = (\varepsilon x)[x \leq \phi(\mathfrak{n}) \;\&\; R(x, \mathfrak{x})]$$

where $(\varepsilon x)F(x)$ means "the smallest number x such that $F(x)$ holds, and 0 if no such number exists."

Proof.

I. Suppose $\phi : \mathbb{N}^n \to \mathbb{N}$ is defined, as in Definition 3.3, by

$$\phi(\mathfrak{n}) = \mu\left(\psi_1(\mathfrak{n}_1), ..., \psi_m(\mathfrak{n}_m)\right),$$

where $\mu, \psi_1, ..., \psi_m$ are each recursive. By Definition 3.4, there exists a finite series of functions for each of $\mu, \psi_1, ..., \psi_m$. The combination of these series will remain finite and, with the addition of ϕ as the last function and using the same derivation rules therein, will satisfy Definition 3.4, making ϕ recursive. Similarly, let $R(\mathfrak{n})$ be an n-ary relation given by

$$R(\mathfrak{n}) \equiv S(\psi_1(\mathfrak{n}_1), ..., \psi_m(\mathfrak{n}_m)),$$

[30] By $\phi(\mathfrak{n})$ we mean ϕ is a function of n-tuple $\mathfrak{n}$, i.e. ϕ is an n-ary functions. We use a similar convention for relations $R(\mathfrak{x})$.

[31] And hence the remaining logical connectives.

where S is a recursive m-ary relation, and, $\psi_1, ..., \psi_m$ each recursive functions. By Definition 3.6, there exists a recursive function μ such that $S(x_1, ..., x_m)$ holds iff $[\mu(x_1, ..., x_m) = 0]$ holds. Thus $R(\mathfrak{n})$ holds iff $[\phi(\mathfrak{n}) = 0]$ holds (where ϕ is defined as above), but we have already shown such a ϕ to be recursive. Therefore $R(\mathfrak{n})$ is a recursive relation.

II-III. We now show there corresponds recursive functions to the connectives $\neg$, $\vee$, and $=$. Indeed,

$$\alpha(0) = 1;\ \alpha(x) = 0 \text{ for } x \neq 0$$

$$\beta(0, y) = \beta(x, 0) = 0;\ \beta(x, y) = 1 \text{ if } x = y = 1$$

$$\gamma(x, y) = 0 \text{ if } x = y;\ \gamma(x, y) = 1 \text{ if } x \neq y$$

are recursive. Let $R(\mathfrak{n})$ and $S(\mathfrak{x})$ be recursive, defined by $[\phi(\mathfrak{n}) = 0]$ and $[\psi(\mathfrak{x}) = 0]$, respectively. Then

$$\neg R(\mathfrak{n}) \equiv [\alpha(\phi(\mathfrak{n})) = 0]$$

$$R(\mathfrak{n}) \vee S(\mathfrak{x}) \equiv [\beta(\phi(\mathfrak{n}), \psi(\mathfrak{x})) = 0]$$

$$[\phi(\mathfrak{n}) = \psi(\mathfrak{x})] \equiv [\gamma(\phi(\mathfrak{n}), \psi(\mathfrak{x})) = 0].$$

By (I), each composition of functions is recursive, hence each relation defined by those connectives are recursive, and indeed all logical connectives can be shown to be recursive by proper use of α and β.

IV. Let $\phi(\mathfrak{n})$ and $R(x, \mathfrak{x})$ be a recursive function and relation, respectively. Then there is recursive $\rho(x, \mathfrak{x})$ such that $R(x, \mathfrak{x})$ holds iff $[\rho(x, \mathfrak{x}) = 0]$ holds. Now define a new function $\chi(x, \mathfrak{x})$ as follows

$$\chi(0, \mathfrak{x}) = 0; \qquad \chi(k + 1, \mathfrak{x}) = (k + 1) \cdot a + \chi(k, \mathfrak{x}) \cdot \alpha(a),$$

where

$$a = \alpha[\alpha(\rho(0, \mathfrak{x}))] \cdot \alpha[\rho(k + 1, \mathfrak{x})] \cdot \alpha[\chi(k, \mathfrak{x})].$$

It should be observed that, by definition of α, a can only take the values 0 or 1. In fact, we see that $\chi(k+1,\mathfrak{x}) = k+1$ iff $a = 1$ iff the following holds:

$$\neg R(0,\mathfrak{x}) \ \& \ R(k+1,\mathfrak{x}) \ \& \ [\chi(k,\mathfrak{x}) = 0],$$

otherwise $\chi(k+1,\mathfrak{x}) = \chi(k,\mathfrak{x})$. Thus, it is observed that $\chi(n,\mathfrak{x})$ remains 0 until that least value m for which $R(m,\mathfrak{x})$ holds, and for any $k \geq m$, $\chi(k,\mathfrak{x}) = m$. Hence

$$\psi(\mathfrak{n},\mathfrak{x}) = \chi(\phi(\mathfrak{n}),\mathfrak{x}) \qquad \text{and} \qquad S(\mathfrak{n},\mathfrak{x}) \equiv R[\psi(\mathfrak{n},\mathfrak{x}),\mathfrak{x}],$$

which are recursive by (I). For T, we build a new function χ', in a similar manner to χ, based on the recursive relation $\neg R$, as opposed to R. Then, similar to ψ and S, we define the new recursive function ψ' and recursive relation S' based on χ'. Hence

$$T(\mathfrak{n},\mathfrak{x}) \equiv \neg S'(\mathfrak{n},\mathfrak{x}),$$ [32]

and is recursive by (I). □

3.2 Recursion theory as the metalanguage of Z

Using Proposition 3.7, and following the direction of [1], the following list of functions and relations (1-50) are found to be recursive, with the exception of 51. This shows that the concepts " **is a formula**," " **is an axiom**," " **is inferred by** ," etc., are recursive.

1. $x^0 \equiv 1;\ x^{k+1} \equiv x \cdot (x^k)$

2. $x/y \equiv (\exists z)[z \leq x \ \& \ x = y \cdot z]$

   x is divisible by y.

[32] We see that

$$\begin{aligned}\neg S'(\mathfrak{n},\mathfrak{x}) &\equiv \neg(\exists x)[x \leq \phi(\mathfrak{n}) \ \& \ \neg R(x,\mathfrak{x})] \\ &\equiv \neg(\exists x)[\neg(\neg(x \leq \phi(\mathfrak{n})) \vee R(x,\mathfrak{x}))] \\ &\equiv \neg\neg(\forall x)[\neg(x \leq \phi(\mathfrak{n})) \vee R(x,\mathfrak{x})] \\ &\equiv (\forall x)[x \leq \phi(\mathfrak{n}) \Rightarrow R(x,\mathfrak{x})]\end{aligned}$$

3. $\mathrm{Prim}(x) \equiv \neg(\exists z)[z \leq x \ \& \ z \neq 1 \ \& \ z \neq x \ \& \ x/z] \ \& \ x > 1$

   x is a prime number.

4. $0 \ \mathrm{Pr} \ x \equiv 0;\ (k+1) \ \mathrm{Pr} \ x \equiv (\varepsilon y)[y \leq x \ \& \ \mathrm{Prim}(y) \ \& \ x/y \ \& \ y > k \ \mathrm{Pr} \ x]$

   $n \ \mathrm{Pr} \ x$ is the n-th prime factor contained in x, 0 otherwise.

5. $0! \equiv 1;\ (k+1)! \equiv (k+1) \cdot k!$

6. $\mathrm{Pr}(0) \equiv 0;\ \mathrm{Pr}(k+1) \equiv (\varepsilon y)[y \leq \mathrm{Pr}(k)! + 1 \ \& \ \mathrm{Prim}(y) \ \& \ y > \mathrm{Pr}(k)]$

   $\mathrm{Pr}(n)$ is the n-th prime number.

7. $n \ \mathrm{Exp} \ x \equiv (\varepsilon y)[y \leq x \ \& \ x/(n \ \mathrm{Pr} \ x)^y \ \& \ \neg(x/(n \ \mathrm{Pr} \ x)^{y+1})]$

   $n \ \mathrm{Exp} \ x$ is the n-th **symbol** assigned to x, i.e., the exponent of the n-th prime factor of x, 0 otherwise.

8. $l(x) \equiv (\varepsilon y)[y \leq x \ \& \ y \ \mathrm{Pr} \ x > 0 \ \& \ (y+1) \ \mathrm{Pr} \ x = 0]$

   $l(x)$ is the length of the series of **symbols** assigned to x, i.e., the number of prime factors of x.

9. $x * y \equiv (\varepsilon z)[z \leq \mathrm{Pr}(l(x) + l(y))^{x+y} \ \& \ (\forall n)[n \leq l(x) \Rightarrow n \ \mathrm{Exp} \ z = n \ \mathrm{Exp} \ x]$
   $\& \ (\forall n)[0 < n \leq l(y) \Rightarrow (n + l(x)) \ \mathrm{Exp} \ z = n \ \mathrm{Exp} \ y]]$

   $x * y$ is the concatenation of the **symbols** assigned to x and y.

10. $\mathrm{R}(x) \equiv 2^x$

    $\mathrm{R}(x)$ is the **string** consisting of the single **symbol** x.

11. $\mathrm{E}(x) \equiv \mathrm{R}(15) * x * \mathrm{R}(17)$

    $\mathrm{E}(x)$ corresponds to the operation of "bracketing" the **string** x.

12. $\mathrm{Var}(x) \equiv (x \geq 19)$ & $\mathrm{Prim}(x)$

x is a **variable**.

13. $\mathrm{Neg}(x) \equiv \mathrm{E}[\mathrm{R}(9) * x]$

$\mathrm{Neg}(x)$ is the **negation** of x.

14. $x \mathrm{\ Dis\ } y \equiv \mathrm{E}[x * \mathrm{R}(11) * y]$

x Dis y is the **disjunction** of x and y.

15. $x \mathrm{\ Gen\ } y \equiv \mathrm{E}[\mathrm{R}(13) * \mathrm{R}(x) * y]$

x Gen y is the **universal generalization** of y by means of the **variable** x (assuming x is a variable).

16. $x \mathrm{\ Imp\ } y \equiv [\mathrm{Neg}(x)] \mathrm{\ Dis\ } y$

x Imp y is the **implication** of x to y.

17. $x \mathrm{\ Con\ } y \equiv \mathrm{Neg}[\mathrm{Neg}(x) \mathrm{\ Dis\ Neg}(y)]$

x Con y is the **conjunction** of x and y.

18. $x \mathrm{\ Iff\ } y \equiv (x \mathrm{\ Imp\ } y) \mathrm{\ Con\ } (y \mathrm{\ Imp\ } x)$

x Iff y is the **biconditional** of x and y.

19. $x \mathrm{\ Ex\ } y \equiv \mathrm{Neg}[x \mathrm{\ Gen\ Neg}(y)]$

x Ex y is the **existential generalization** of y by means of the **variable** x, if x is a **variable**.

20. $0 \mathrm{\ S\ } x \equiv x;\ (k+1) \mathrm{\ S\ } x \equiv \mathrm{R}(3) * k \mathrm{\ S\ } x$

n S x corresponds to the prefixing of the **symbols** assigned to x, n occurrences of the **symbol** for “**s**”.

21. $\#(n) \equiv n \text{ S R}(1)$

$\#(n)$ is the **number** for the number n.

22. $x \text{ In } y \equiv \text{E}[x * \text{R}(5) * y]$

x In y is the **membership** of x in y.

23. $x \text{ Eq } y \equiv \text{E}[x * \text{R}(7) * y]$

x Eq y is the **equality** of x and y.

24. $\text{Term}(x) \equiv (\exists m, n)[m, n \leq x \;\&\; (m = 1 \vee \text{Var}(m)) \;\&\; x = n \text{ S R}(m)]$

x is a **term**.

25. $\text{Bf}(x) \equiv (\exists m, n)[\text{Term}(m) \;\&\; \text{Term}(n) \;\&\; (x = m \text{ In } n \vee x = m \text{ Eq } n)]$

x is a **basic formula**.

26. $\text{Op}(x, y, z) \equiv x = \text{Neg}(y) \vee x = y \text{ Dis } z \vee (\exists v)[v \leq x \;\&\; \text{Var}(v) \;\&\; x = v \text{ Gen } y]$

x is **obtained by logical operations** from y and z.

27. $$\text{SoF}(x) \equiv l(x) > 0 \;\&\; (\forall n)[0 < n \leq l(x) \Rightarrow (\text{Bf}(n \text{ Exp } x) \vee (\exists p, q)[0 < p, q < n \;\&\; \text{Op}(n \text{ Exp } x, p \text{ Exp } x, q \text{ Exp } x)])]$$

x is a **series of formulae** of which each is either a **basic formula** or is **obtained by logical operations** from those preceding.

28. $\text{Form}(x) \equiv (\exists n)[n \leq \text{Pr}(l(x)^2)^{x \cdot l(x)^2} \;\&\; \text{SoF}(n) \;\&\; x = (l(n) \text{ Exp } n)]$

x is a **formula**.[33]

[33] The motivation for the limitation $n \leq \text{Pr}(l(x)^2)^{x \cdot l(x)^2}$ is given as follows: The length of the shortest **series of formulae** belonging to x can at most be equal to the number of constituent *formulae* of x. There are at most $l(x)$ constituent **formulae** of length 1, $l(x) - 1$ of length 2, etc. and in hence at most $\frac{1}{2}l(x)[l(x) + 1] \leq l(x)^2$. The prime factors in n can therefore all be assumed to be smaller than $\text{Pr}(l(x)^2)$, of length smaller than $l(x)^2$ and with exponents less than x. [1]

29. $v \text{ Bd } n, x \equiv (\exists a, b, c)[a, b, c \leq x \ \& \ x = a * (v \text{ Gen } b) * c \ \& \ \text{Form}(b) \ \&$

$$l(a) + 1 \leq n \leq l(a) + l(v \text{ Gen } b)] \ \& \ \text{Var}(v) \ \& \ \text{Form}(x)$$

The **variable** v is **bound** at the n-th place in x.

30. $v \text{ Fr } n, x \equiv \text{Var}(v) \ \& \ \text{Form}(x) \ \& \ v = (n \text{ Exp } x) \ \& \ n \leq l(x) \ \& \ \neg(v \text{ Bd } n, x)$

The **variable** v is **free** at the n-th place in x.

31. $\text{Su } x\binom{n}{y} \equiv (\varepsilon z)[z \leq \Pr(l(x) + l(y))^{x+y} \ \& \ (\exists u, v)[u, v \leq x \ \& \ n = l(u) + 1$

$$\& \ x = u * \text{R}(n \text{ Exp } x) * v \ \& \ z = u * y * v]]$$

$\text{Su } x\binom{n}{y}$ derives from x by substituting y in place of the n-th **symbol** in x.

32. $0 \text{ Pl } v, x \equiv (\varepsilon n)[n \leq l(x) \ \& \ v \text{ Fr } n, x$

$$\& \ \neg(\exists p)[n < p < l(x) \ \& \ v \text{ Fr } p, x]]$$

$(k+1) \text{ Pl } v, x \equiv (\varepsilon n)[n < k \text{ Pl } v, x \ \& \ v \text{ Fr } n, x$

$$\& \ \neg(\exists p)[n < p < k \text{ Pl } v, x \ \& \ v \text{ Fr } p, x]]$$

$n \text{ Pl } v, x$ is the $(n+1)$-th place in x (counting from the end of **formula** x) at which v is **free** in x, 0 otherwise.

33. $\text{fo}(v, x) \equiv (\varepsilon n)[n \leq l(x) \ \& \ n \text{ Pl } v, x = 0]$

$\text{fo}(v, x)$ is the number of places at which v if **free** in x.

34. $\text{Sb}_0(x^v_y) \equiv x; \ \text{Sb}_{k+1}(x^v_y) \equiv \text{Su } [\text{Sb}_k(x^v_y)\binom{k \text{ Pl } v, x}{y}]$

35. $\text{Sub}(x^v_y) \equiv \text{Sb}_{\text{fo}(v,x)}(x^v_y)$

$\text{Sub}(x^v_y)$ is the concept of **substitution** as defined in Definition 2.2.7.[34]

[34]To evaluate multiple **substitutions**, i.e. $\text{Sub}(x \ {}^v_y \ {}^u_z)$, we proceed in a similar manner to substitutions in Z to avoid capture, or simple evaluate iteratively, i.e. $\text{Sub}(\text{Sub}(x^v_y) \ {}^u_z)$, if capture is not an issue.

36. $\mathrm{FF}(x, v, y) \equiv \neg(\exists m, n, w)[m \leq l(x) \;\&\; n \leq l(y) \;\&\; w \leq x$

$$\& \; w = m \text{ Exp } x \;\&\; w \text{ Bd } n, y \;\&\; v \text{ Fr } n, y]$$

x is **free for** v **in** y, as in Definition 2.2.6.

Now, for the axioms L6, A1,A2, A4, A5, A6, A8, A9 and A10 there corresponds determinate numbers $l_6, z_1, z_2, z_4, z_5, z_6, z_8, z_9, z_{10}$, respectively, determined by Φ.[35] However, since the remaining axioms are schemata, we provide the following

37. $L_1(x) \equiv (\exists p)[p \leq x \;\&\; \mathrm{Form}(p) \;\&\; x = (p \text{ Dis } p) \text{ Imp } p]$

38. $L_2(x) \equiv (\exists p, q)[p, q \leq x \;\&\; \mathrm{Form}(p) \;\&\; \mathrm{Form}(q) \;\&\; x = p \text{ Imp } (q \text{ Dis } p)]$

39. $L_3(x) \equiv (\exists p, q, r)[p, q, r \leq x \;\&\; \mathrm{Form}(p) \;\&\; \mathrm{Form}(q) \;\&\; \mathrm{Form}(r)$

$$\& \; x = (p \text{ Imp } q) \text{ Imp } ([r \text{ Dis } p] \text{ Imp } [r \text{ Dis } q])]$$

40. $L_4(x) \equiv (\exists v, p, t)[v, p, t \leq x \;\&\; \mathrm{Var}(v) \;\&\; \mathrm{Form}(p) \;\&\; \mathrm{Term}(t) \;\&\; \mathrm{FF}(t, v, p)$

$$\& \; x = (v \text{ Gen } p) \text{ Imp } \mathrm{Sub}(p^v_t)]$$

41. $L_5(x) \equiv (\exists v, p, q)[v, p, q \leq x \;\&\; \mathrm{Var}(v) \;\&\; \mathrm{Form}(p) \;\&\; \mathrm{Form}(q) \;\&\; \mathrm{fo}(v, p) = 0$

$$\& \; x = (v \text{ Gen } [p \text{ Dis } q]) \text{ Imp } (p \text{ Dis } [v \text{ Gen } q])]$$

42. $L_7(x) \equiv (\exists v, w, z, p)[v, w, z, p \leq x \;\&\; \mathrm{Var}(v) \;\&\; \mathrm{Var}(w) \;\&\; \mathrm{Var}(z) \;\&\; \mathrm{Form}(p)$

$$\& \; x = (v \text{ Eq } w) \text{ Imp } [\mathrm{Sub}(p^z_v) \text{ Imp } \mathrm{Sub}(p^z_w)]]$$

43. $L - Ax(x) \equiv L_1(x) \vee L_2(x) \vee L_3(x) \vee L_4(x) \vee L_5(x) \vee x = l_6 \vee L_7(x)$

x is an instance of an **logical axiom**

[35]For instance,

$$l_6 = 2^{15} \cdot 3^{19} \cdot 5^7 \cdot 7^{19} \cdot 11^{17} = 171428179428817764838319852874486162191747865600000000$$

44. $Z_3(x) \equiv (\exists p, v, w)[p, v, w \leq x \;\&\; \text{Form}(p) \;\&\; \text{Var}(v) \;\&\; \text{Var}(w) \;\&\; x =$
$19 \text{ Gen } 23 \text{ Gen } 29 \text{ Ex } 31 \text{ Gen } (31 \text{ In } 29 \text{ Iff } [31 \text{ In } 19 \text{ Con Sub}(p \;{}^{v}_{\mathrm{R}(31)}\;{}^{w}_{\mathrm{R}(23)})])]$

45. $Z_7(x) \equiv (\exists p, u, v, w)[p, u, v, w \leq x \;\&\; \text{Form}(p) \;\&\; \text{Var}(u) \;\&\; \text{Var}(v) \;\&\; \text{Var}(w)$
$\&\; x = 19 \text{ Gen } [23 \text{ Gen } 29 \text{ Gen } 31 \text{ Gen } (\text{Sub}(p \;{}^{u}_{\mathrm{R}(23)}\;{}^{v}_{\mathrm{R}(29)}\;{}^{w}_{\mathrm{R}(19)})$
$\text{Con Sub}(p \;{}^{u}_{\mathrm{R}(23)}\;{}^{v}_{\mathrm{R}(31)}\;{}^{w}_{\mathrm{R}(19)}) \text{ Imp R}(29) \text{ Eq R}(31)) \text{ Imp } 23 \text{ Gen } 29 \text{ Ex}$
$31 \text{ Gen } (31 \text{ In } 23 \text{ Iff } 37 \text{ Ex } [37 \text{ In } 23 \text{ Con Sub}(p \;{}^{u}_{\mathrm{R}(37)}\;{}^{v}_{\mathrm{R}(29)}\;{}^{w}_{\mathrm{R}(19)})])]]$

46. $Z - Ax(x) \equiv x = z_1 \lor x = z_2 \lor Z_3(x) \lor x = z_4 \lor x = z_5$
$\lor\; x = z_6 \lor Z_7(x) \lor x = z_8 \lor x = z_9 \lor x = z_{10}$

x is an instance of a **set-theoretical axiom**

47. $\text{Ax}(x) \equiv L - Ax(x) \lor Z - Ax(x)$

x is an **axiom of theory Z**.

48. $\text{Inf}(x, y, z) \equiv y = z \text{ Imp } x \lor (\exists v)[v \leq x \;\&\; \text{Var}(v) \;\&\; x = v \text{ Gen } y]$

x is **inferred by** y and z.

49. $\text{Pf-S}(x) \equiv l(x) > 0 \;\&\; (\forall n)[0 < n \leq l(x) \Rightarrow \text{Ax}(n \text{ Exp } x)$
$\lor\; (\exists p, q)[0 < p, q < n \;\&\; \text{Inf}(n \text{ Exp } x, p \text{ Exp } x, q \text{ Exp } x)]]$

x is a **proof-schema**, i.e., a finite series of **formulae**, of which each is either an **axiom** or **inferred by** two previous ones.

50. $x \text{ Pf } y \equiv \text{Pf-S}(x) \;\&\; [l(x)] \text{ Exp } x = y$

x is a **proof** of **formula** y

51. $\text{Thm}(x) \equiv (\exists y)[y \text{ Pf } x]$

x is a **theorem**, i.e. x is a **provable formula**.[36]

[36]Note that Thm is not justifiably recursive. In fact, Theorem 4.2 will show Thm is not recursive.

3.3 Representability of recursiveness in Z

Above we have shown that recursion theory is capable of serving as the metalanguage for the **theory Z**, which, again, is essentially Z with numbers as the primitive symbols. We now demonstrate the peculiar fact that recursion theory can be represented in **Z**.

Definition 3.8. [2] Let $\phi(x_1, ..., x_n)$ be a function. The **predicate** r, with **free variables**[37] $u_1, ..., u_{n+1}$ *formally represents* $\phi(x_1, ..., x_n)$ iff for any numbers $x_1, ..., x_n, k \,\epsilon\, \mathbb{N}$

$$[\phi(x_1, ..., x_n) = k] \Rightarrow \text{Thm}\left[\text{Sub}\left(r \; {}^{u_1}_{\#(x_1)} \cdots {}^{u_n}_{\#(x_n)} \; {}^{u_{n+1}}_{\#(k)}\right)\right] \tag{1}$$

$$[\phi(x_1, ..., x_n) \neq k] \Rightarrow \text{Thm}\left[\text{Neg Sub}\left(r \; {}^{u_1}_{\#(x_1)} \cdots {}^{u_n}_{\#(x_n)} \; {}^{u_{n+1}}_{\#(k)}\right)\right] \tag{2}$$

We say r is a **recursive predicate** if it formally represents a recursive function.

Proposition 3.9. [2] [6](Gödel's β-function) *Let β be the function defined by*

$$\beta(n, d, k) \equiv n \text{ mod } (1 + (k + 1)d)$$

for each $n, d, k \,\epsilon\, \mathbb{N}$.

I. *β is a recursive function.*

II. *Let σ be a sequence of natural numbers where $\sigma(i)$ denotes the i-th term of the sequence. Then for each $k \,\epsilon\, \mathbb{N}$ there exists $n, d \in \mathbb{N}$ such that $\beta(n, d, i) = \sigma(i)$ for each $i \leq k$.*

III. *β is formally representable by a **recursive** 4-**ary predicate** b.*

Proof.

I. By Propositions 3.5 and 3.7.IV, the function

$$x \text{ mod } (y) \equiv (\varepsilon n)[n < y \;\&\; (\exists d, r)[r < y \;\&\; x = dy + r]]$$

is recursive. Thus $\beta(n, d, k) \equiv n \text{ mod } (1 + (k + 1)d)$ is recursive by substitution.

[37] i.e. primes ≥ 19

II. Let $k \epsilon \mathbb{N}$ and $l = \max\{k, \sigma(0), ..., \sigma(k-1)\}$. We claim that, for any $i, j < k$ with $i \neq j$, the numbers $1 + (i+1)l!$ and $1 + (j+1)l!$ are relatively prime. If we suppose the contrary, then there are numbers $j < i < k$ and prime $p > 1$ such that $p \mid (1 + (i+1)l!)$ and $p \mid (1 + (j+1)l!)$. Thus p will divide their difference, yielding $p \mid (i-j)l!$. But this implies $p \mid l!$, since both $i, j < k$ making $i - j < k \leq l$, which forces $i - j$ to be a factor $l!$. However $p \mid (1 + (i+1)l!)$, which implies that $p \not\mid l!$, a contradiction.[38] Now, since $1 + (i+1)l!$ and $1 + (j+1)l!$ are relatively prime for any $i \neq j$, by the Chinese Remainder Theorem (*CRT*), there is a smallest n such that $\sigma(i) \equiv n \bmod (1 + (i+1)l!)$ for each $i < k$. Let $d = l!$, then $\beta(n, d, i) = \sigma(i)$ for each $i < k$, and we are done.

III. The proof of this claim, upon reflection, offers no difficulty in principle[39] since Z models PA and PA serves as a foundation for elementary number theory; hence there is a constructive proof in Z of the *CRT* called CRT. We shall omit the formal demonstration since it is outside the scope of this paper, but one need only formalize the proof of the CRT given in any text on elementary number theory.[40] Once this proof has been formalized, a predicate $B(x, y, z, w)$ can be constructed in Z so that $B(\begin{smallmatrix} x & y & z & w \\ s_n & s_d & s_k & s_m \end{smallmatrix})$ holds when and only when $\beta(n, d, k) = m$ holds. Using this we can prove **CRT** in **theory Z** and construct **predicate** $b \equiv \Phi(B)$ formally representing β; i.e. by formally mimicking the proofs for $B(\begin{smallmatrix} x & y & z & w \\ s_n & s_d & s_k & s_m \end{smallmatrix})$ and $\sim B(\begin{smallmatrix} x & y & z & w \\ s_n & s_d & s_k & s_m \end{smallmatrix})$ as **proofs** in **Z**. In the same fashion, let l denote the **predicate** representing the predicate $L(x, y)$, where $L(\begin{smallmatrix} x & y \\ s_n & s_m \end{smallmatrix})$ is provable precisely when the relation $n < m$ holds. $\square$

[38] This is an elementary fact of number theory. If $p \mid n+1$ and $p \mid n$, then $p \mid (n+1-n) = 1$, a contradiction. So if $p \mid (1 + (i+1)l!)$ then $p \not\mid (i+1)l!$, hence $p \not\mid (i+1)$ and, in particular, $p \not\mid l!$.

[39] See Prop. 3.21 in [6] for a more rigorous exposition.

[40] Such as [8].

Proposition 3.10. [2] *Every recursive function is formally represented by a* ***recursive relation****.*

Proof. Let ϕ be a recursive function. We proceed by complete induction on $deg(\phi)$. If $deg(\phi) = 1$ then ϕ must be an initial function.

i. $S(x) = y$ is given by $r \equiv \Phi(\mathbf{sx}_1 = \mathbf{x}_2) = (1 \text{ S } 19)$ Eq 23.

ii. $P_i^n(x_1, ..., x_n) = y$ is given by $r \equiv \Phi(\mathbf{x}_i = \mathbf{x}_{n+1}) = \text{Pr}(i)$ Eq $\text{Pr}(n+1)$.

iii. $C_n(x) = y$ is given by $r \equiv \Phi(\mathbf{x}_2 = s_n\mathbf{0}) = 23$ Eq $\#(n)$.

Since Z models PA, (i)-(iii) are all demonstrable in the system Z, i.e. conditions (1) and (2) hold, and the base case is satisfied.

Now, suppose the hypothesis holds for all recursive functions of degree less than ϕ. Thus ϕ must be derived from recursive functions of lesser degree by either (iv) substitution or (v) recursive definition.

iv. Suppose ϕ is derived by substitution, where for each $x_1, ..., x_{n-1} \in \mathbb{N}$

$$\phi(x_1, ..., x_{n-1}) = \mu\left(\psi_1(x_1, ..., x_{n-1}), ..., \psi_{m-1}(x_1, ..., x_{n-1})\right),$$

where $\mu, \psi_1, ..., \psi_{m-1}$ are recursive functions, each with degree less than ϕ. By the inductive hypothesis there are **predicates** $r_\mu, r_{\psi_1}, ..., r_{\psi_{m-1}}$ that satisfy conditions (1) and (2). Observe that the **free variables** amongst the r_{ψ_i}'s are shared, of which there are n-many, and there are m **free variables** in r_μ. We may assume, without loss of generality, that the **free variables** shared by the r_{ϕ_i}'s are the first n prime numbers (≥ 19) and those of r_μ are the proceeding m prime numbers. Let $N = n + m + 1$ and p_i denote the i-th prime number ≥ 19. We define **predicate** r (with **free variables**

$19, ..., p_n)$ as follows:[41]

$$r \equiv p_N \text{ Ex } ... \; p_{N+m-1} \text{ Ex } p_{N+m} \text{ Ex } \Bigg[\text{Sub}\left(r_\mu \; {}^{p_{n+1}}_{\mathrm{R}(p_N)} \cdots {}^{p_m}_{\mathrm{R}(p_{N+m})} \right)$$
$$\text{Con Sub}\left(r_{\phi_1} \; {}^{p_n}_{\mathrm{R}(p_N)} \right) \text{ Con } ... \text{ Con Sub}\left(r_{\phi_{m-1}} \; {}^{p_n}_{\mathrm{R}(p_{N+m-1})} \right) \Bigg].$$

It is clear now that, for given numbers $x_1, ..., x_{n-1}, k$,

$$\phi(x_1, ..., x_{n-1}) = k \Leftrightarrow \mu\left(\psi_1(x_1, ..., x_{n-1}), ..., \psi_{m-1}(x_1, ..., x_{n-1})\right) = k,$$

that is, iff there are numbers $y_1, ..., y_{m-1}$ such that $\psi_i(x_1, ..., x_{n-1}) = y_i$, for each $i = 1, ..., m-1$, and $\mu(y_1, ..., y_{m-1}) = k$. If such is the case, then by condition (1),

$$\text{Sub}\left(r_{\psi_i} \; {}^{19}_{\#(x_1)} \cdots {}^{p_{n-1}}_{\#(x_{n-1})} \; {}^{p_n}_{\#(y_i)} \right)$$

for each $i = 1, ..., m-1$, and

$$\text{Sub}\left(r_\mu \; {}^{p_n+1}_{\#(y_1)} \cdots {}^{p_{n+m-1}}_{\#(y_{m-1})} \; {}^{p_{n+m}}_{\#(k)} \right)$$

are **provable**. So the repeated **conjunction** of the above **sentences** would also be **provable**, and thus the **existential quantification**, with the y_i's removed, is **provable**. Hence

$$\phi(x_1, ..., x_{n-1}, k) \Rightarrow \text{Thm}\left[\text{Sub}\left(r \; {}^{19}_{\#(x_1)} \cdots {}^{p_{n-1}}_{\#(x_{n-1})} \; {}^{p_n}_{\#(k)} \right) \right]$$

and condition (1) is satisfied. By a similar argument, (2) is also satisfied.

v. Suppose ϕ is derived from recursive definition by

$$\phi(0, x_2, ..., x_{n-1}) = \psi(x_2, ..., x_{n-1})$$
$$\phi(k+1, x_2, ..., x_{n-1}) = \mu(k, \phi(k, x_2, ..., x_{n-1}), x_2, ..., x_{n-1})$$

where ψ and μ are recursive functions with degree less than ϕ. By the inductive

[41] Intuitively, but informally, r is understood as follows:

$$r(x_1, ..., x_n) \equiv \exists y_1 ... \exists y_m \exists z \big(r_\mu(y_1, ..., y_m, z) \wedge r_{\psi_1}(x_1, ..., x_n, y_1) \wedge ... \wedge r_{\psi_m}(x_1, ..., x_n, y_m) \big)$$

hypothesis there are predicates r_ψ (with **free variables** $23, ..., p_n$) and r_μ (with **free variables** $v_1, v_2, 23, ..., p_n$) that satisfy conditions (1) and (2). We define **predicate** r (with **free variables** $19, ..., p_n$) as follows:[42]

$$
\begin{aligned}
r \;\equiv\; q_1 \text{ Ex } q_2 \text{ Ex } \Big[q_3 \text{ Ex } \Big[& \text{Sub}\left(b \;{\scriptstyle u_1 \atop \mathrm{R}(q_1)} \;{\scriptstyle u_2 \atop \mathrm{R}(q_2)} \;{\scriptstyle u_3 \atop \#(0)} \;{\scriptstyle u_4 \atop \mathrm{R}(p_3)} \right) \text{ Con Sub} \left(r_\psi \;{\scriptstyle p_n \atop \mathrm{R}(q_3)} \right) \Big] \\
& \text{Con } \Big[q_4 \text{ Gen } q_5 \text{ Ex } \Big[(\mathrm{R}(q_5) \text{ Eq } \mathrm{R}(3) * \mathrm{R}(q_4)) \\
& \qquad \text{Con } \left(\text{Sub} \left(l \;{\scriptstyle u_1 \atop \mathrm{R}(q_4)} \;{\scriptstyle u_2 \atop \mathrm{R}(p_n)} \right) \right. \\
& \qquad \text{Imp } \Big(q_6 \text{ Ex } q_7 \text{ Ex } \Big(\\
& \qquad\quad \text{Sub} \left(b \;{\scriptstyle u_1 \atop \mathrm{R}(q_1)} \;{\scriptstyle u_2 \atop \mathrm{R}(q_2)} \;{\scriptstyle u_3 \atop \mathrm{R}(q_4)} \;{\scriptstyle u_4 \atop \mathrm{R}(q_6)} \right) \\
& \qquad\quad \text{Con Sub} \left(b \;{\scriptstyle u_1 \atop \mathrm{R}(q_1)} \;{\scriptstyle u_2 \atop \mathrm{R}(q_2)} \;{\scriptstyle u_3 \atop \mathrm{R}(q_4)} \;{\scriptstyle u_4 \atop \mathrm{R}(q_7)} \right) \\
& \qquad\quad \text{Con Sub} \left(r_\mu \;{\scriptstyle v_1 \atop \mathrm{R}(q_4)} \;{\scriptstyle v_2 \atop \mathrm{R}(q_6)} \;{\scriptstyle p_n \atop \mathrm{R}(q_7)} \right) \Big) \Big) \Big) \Big] \Big] \Big]
\end{aligned}
$$

where b and l (with **free variables** among $u_1, ..., u_4$) are given by Proposition 3.9, and $u_1, ..., u_7$ are the first **variables** (primes) that do not occur **free** amongst r_ψ, r_μ, b and l. The proof that r satisfies conditions (1) and (2) requires a richer development of the syntactic structure of $\mathcal{L}_2$. A complete exposition can be found in [6].[43] □

[42] Intuitively, albeit informally, we can describe construction as follows:
To determine the value of $\phi(k, \mathfrak{n})$ one needs a sequence σ of length k, where $\sigma(i) = \phi(i, \mathfrak{n})$ for each $i < k$. This allows us evaluate μ without directly using ϕ, which in turn will also ensure that the construction of r will be finite and not self-referential. For a given k, Proposition 3.9 provides us with an $m, d \,\epsilon\, \mathbb{N}$ where the β-function is precisely σ. That is $\phi(0, \mathfrak{n}) = \psi(\mathfrak{n}) = \beta(m, d, 0)$ and $\phi(i+1, \mathfrak{n}) = \mu(i, \beta(m, d, i), \mathfrak{n})$. So essentially

$$r(x_1, \mathfrak{n}, k) \equiv \exists m \exists d \big[r_\psi(\mathfrak{n}, \beta(m, n, 0)) \wedge \forall i [i < k \to r_\mu(i, \beta(m, d, i), \mathfrak{n}, \beta(m, d, i+1))] \big],$$

but due to our formalization, it looks more like

$$
\begin{aligned}
r(x_1, \mathfrak{n}, k) \equiv (\exists m, d) \big[(\exists z)(b(m, d, 0, z) \wedge r_\psi(\mathfrak{n}, z)) \wedge (\forall i)(l(i, k) \to \\
(\exists u)(\exists v)[b(m, n, i, u) \wedge b(m, n, i+1, v) \wedge r_\mu(i, u, \mathfrak{n}, v)]) \big]
\end{aligned}
$$

[43] Specifically Prop. 3.23 from [6].

Proposition 3.11. *To each recursive relation $R(x_1, ..., x_n)$ there corresponds a* ***recursive** n**-ary predicate*** *r (with **free-variables** $u_1, ..., u_n$) such that for every n-tuple of numbers $(x_1, ..., x_n)$ the following hold:*

$$R(x_1, ..., x_n) \Rightarrow \text{Thm}\left[\text{Sub}\left(r \begin{smallmatrix} u_1 & \cdots & u_n \\ \#(x_1) & \cdots & \#(x_n) \end{smallmatrix}\right)\right] \tag{3}$$

$$\neg R(x_1, ..., x_n) \Rightarrow \text{Thm}\left[\text{Neg Sub}\left(r \begin{smallmatrix} u_1 & \cdots & u_n \\ \#(x_1) & \cdots & \#(x_n) \end{smallmatrix}\right)\right] \tag{4}$$

Proof. Let $\phi(x_1, ..., x_n)$ be the recursive function such that $R(x_1, ..., x_n)$ holds precisely when $[\phi(x_1, ..., x_n) = 0]$. By Proposition 3.10, there is a **relation** r_ϕ that satisfies (1) and (2) of Definition 3.8. Let $r \equiv \text{Sub}(r_\phi \begin{smallmatrix} u_{n+1} \\ \#(0) \end{smallmatrix})$. Hence r is as required. □

4 The Incompleteness of Z

Let C be any collection of **formulae**.[44] By Ded(C) we mean the smallest collection of **formulae** that contains all the **axioms of Z** and **formulae** of C, and is closed with respect to the relation "**inferred by**" (Inf).

Definition 4.1. Let C be a collection of **formulae**.

1. C is said to be **incomplete** iff there is a **sentence** p such that neither p nor Neg(p) are in Ded(C). Otherwise C is **complete**.

2. C is said to be **inconsistent** iff there is a **sentence** p where both p and Neg(p) are contained in Ded(C). Otherwise C is **consistent**.

3. C is said to be ω**-inconsistent** iff there is a **class-property** r such that

$$(\forall n)[\text{Sub}(r \begin{smallmatrix} v \\ \#(n) \end{smallmatrix}) \in \text{Ded}(C)] \ \& \ [\text{Neg}(v \text{ Gen } r)] \in \text{Ded}(C), \tag{5}$$

[44] Including infinite or empty collections.

where v is the **free-variable** in the **class-property** r. Otherwise C is ω-**consistent**.[45]

We now demonstrate two truly remarkable results, known as Gödel's First Incompleteness Theorem and Gödel's Second Incompleteness Theorem, respectively.

Theorem 4.2. [1] *Let C be any ω-**consistent** recursive collection of **formulae**. Then C is **incomplete**, in particular there corresponds a **recursive class-property** r such that neither v Gen r nor Neg(v Gen r) belong to* Ded(C).

Proof. Let C be a recursive ω-**consistent** collection of **sentences**. We now define the following relations specific to C:

$$\begin{aligned}\text{Pf-S}_C(x) \equiv (\forall n)[n \leq l(x) \Rightarrow \text{Ax}(n \text{ Exp } x) \vee (n \text{ Exp } x) \in C \\ \vee\ (\exists p, q)[0 < p, q < n \ \&\ \text{Inf}(n \text{ Exp } x, p \text{ Exp } x, q \text{ Exp } x)]] \ \&\ l(x) > 0\end{aligned} \tag{6}$$

$$x \text{ Pf}_C\ y \equiv \text{Pf-S}_C(x) \ \&\ [l(x) \text{ Exp } x = y] \tag{7}$$

$$\text{Thm}_C(y) \equiv (\exists x)[x \text{ Pf}_C\ y] \tag{8}$$

By Proposition 3.7, both (6) and (7) are recursive. Now, it is clear that[46]

$$(\forall y)[\text{Thm}(y) \Rightarrow \text{Thm}_C(y)] \tag{9}$$

and, by definition of Ded(C),

$$(\forall y)[\text{Thm}_C(y) \Leftrightarrow y \in \text{Ded}(C)]. \tag{10}$$

We now define the following relation:

$$Q(x, y) \equiv \neg \left[x \text{ Pf}_C \text{ Sub}\left(y \ {}^{23}_{\#(y)}\right)\right]. \tag{11}$$

[45]Note that every ω-**consistent** C is **consistent**, but the converse does not hold. C is said to be recursive if the relation $x \in C$ is a recursive relation.

[46]Ded($\emptyset$) $\subseteq$ Ded(C), where Ded($\emptyset$) is the collection of all **provable sentences** of **theory Z**.

It follows that $Q(x,y)$ is recursive by Proposition 3.7 and the fact that Pf_C , Sub, and # are recursive. By Proposition 3.11 and (9), there is a **recursive** 2**-ary predicate** q, with **free variables** 19 and 23, such that

$$\neg\left[x\ \mathrm{Pf}_C\ \mathrm{Sub}\left(y\ {}^{23}_{\#(y)}\right)\right] \quad\Rightarrow\quad \mathrm{Thm}_C\left[\mathrm{Sub}\left(q\ {}^{19}_{\#(x)}\ {}^{23}_{\#(y)}\right)\right] \tag{12}$$

$$x\ \mathrm{Pf}_C\ \mathrm{Sub}\left(y\ {}^{23}_{\#(y)}\right) \quad\Rightarrow\quad \mathrm{Thm}_C\left[\mathrm{Neg\ Sub}\left(q\ {}^{19}_{\#(x)}\ {}^{23}_{\#(y)}\right)\right] \tag{13}$$

Define the number g as

$$g \equiv 19\ \mathrm{Gen}\ q. \tag{14}$$

Thus g is a **recursive class-property** with **free-variable** 23. Next, we define the number r as

$$r \equiv \mathrm{Sub}\left(q\ {}^{23}_{\#(g)}\right) \tag{15}$$

where it is readily seen that r is **recursive** by Proposition 3.7 and is a **class-property** with **free-variable** 19. Observe that, by (14) and (15),

$$\begin{aligned} \mathrm{Sub}\left(g\ {}^{23}_{\#(g)}\right) &= \mathrm{Sub}\left([19\ \mathrm{Gen}\ q]\ {}^{23}_{\#(g)}\right) \\ &=^{47} 19\ \mathrm{Gen\ Sub}\left(q\ {}^{23}_{\#(g)}\right) \\ &= 19\ \mathrm{Gen}\ r \end{aligned} \tag{16}$$

Furthermore,

$$\begin{aligned} \mathrm{Sub}\left(q\ {}^{19}_{\#(x)}\ {}^{23}_{\#(g)}\right) &=^{48} \mathrm{Sub}\left(\mathrm{Sub}\left(q\ {}^{23}_{\#(g)}\right)\ {}^{19}_{\#(x)}\right) \\ &= \mathrm{Sub}\left(r\ {}^{19}_{\#(x)}\right) \end{aligned} \tag{17}$$

By replacing y with g, as well as applying (16) and (17) to the antecedent and consequent, respectively, in (12) and (13) we obtain

$$\neg\left[x\ \mathrm{Pf}_C\ (19\ \mathrm{Gen}\ r)\right] \quad\Rightarrow\quad \mathrm{Thm}_C\left[\mathrm{Sub}\left(r\ {}^{19}_{\#(x)}\right)\right] \tag{18}$$

[47] In this case Sub preserves Gen since $\#(g)$ is **free for** 19 in q.

[48] Since both $\#(g)$ is **free for** 19 and $\#(x)$ is **free for** 23, there will be no unwanted capturing in the **substitution** and, therefore, does not affect the order of **substitutions**.

$$x \text{ Pf}_C \text{ (19 Gen } r) \quad \Rightarrow \quad \text{Thm}_C \left[\text{Neg Sub} \left(r \; {}^{19}_{\#(x)} \right) \right] \tag{19}$$

We can now demonstrate that neither 19 Gen r nor Neg(19 Gen r) are C**-provable**.[49]

Case 1. 19 Gen r *is not* C***-provable:***

Suppose, *a contrario*, that (19 Gen r) ϵ Ded(C). By (10), Thm_C(19 Gen r) holds so there is a **proof-schema** n given by (8) such that n Pf_C (19 Gen r). But by (16), 19 Gen $r = \text{Sub}\left(g \; {}^{23}_{\#(g)}\right)$, hence n Pf_C $\text{Sub}\left(g \; {}^{23}_{\#(g)}\right)$ holds, which by (19) yields

$$\text{Thm}_C \left[\text{Neg Sub} \left(r \; {}^{19}_{\#(n)} \right) \right].$$

Thus Neg Sub$\left(r \; {}^{19}_{\#(n)}\right)$ is C-**provable**. However, since 19 Gen r is C-**provable** it follows then that $\text{Sub}\left(r \; {}^{19}_{\#(n)}\right)$, by L_4 and Inf, is also C-**provable**.[50] Thus C is not **consistent**, hence it is not ω-**consistent**. *Reductio ad absurdum.*

Case 2. Neg(19 Gen r) *is not* C***-provable:***

Suppose, *a contrario*, that Thm_C[Neg(19 Gen r)] holds. Since we have already determined that 19 Gen r is not C**-provable**, it must be the case that no number is a **proof-schema** for 19 Gen r. That is, for any number n, $\neg\,[n \text{ Pf}_C \text{ (17 Gen } r)]$. It follows from (18) that

$$(\forall n) \left[\text{Thm}_C \left[\text{Sub} \left(r \; {}^{19}_{\#(n)} \right) \right] \right].$$

By (10) this means

$$(\forall n) \left[\text{Sub} \left(r \; {}^{19}_{\#(n)} \right) \; \epsilon \; \text{Ded}(C) \right].$$

Hence C is ω-**inconsistent**, *reductio ad absurdum.*

Therefore, neither 19 Gen r nor Neg(19 Gen r) are in Ded(C), so C is **incomplete**.

□

[49] "x is C-**provable**" means that x ϵ Ded(C), which by (10) is equivalent to $\text{Thm}_C(x)$.

[50] That is, by axiom L4 and rule of inference MP.

Theorem 4.3. [1] *Let C be a* ***consistent*** *recursive collection of* ***formulae****. Then the* ***sentence*** *which asserts that C is* ***consistent*** *is not C-****provable****.*

Proof. The proposition $\mathrm{Con}(C)$, denoting "C is **consistent**," is defined as follows:

$$\mathrm{Con}(C) \equiv (\exists x)[\mathrm{Form}(x) \ \& \ \neg\mathrm{Thm}_C(x)].$$

Observe that, in Case 1 of Theorem 4.2, only the **consistency** of C was exploited to show the **sentence** 19 Gen r was not C-**provable**. Hence

$$\begin{aligned} \mathrm{Con}(C) &\Rightarrow \neg\mathrm{Thm}_C(19 \text{ Gen } r) & \\ &\Rightarrow (\forall x)[\neg[x \ \mathrm{Pf}_C \ (19 \text{ Gen } r)]] & \text{by (8)} \\ &\Rightarrow (\forall x)\left[\neg\left[x \ \mathrm{Pf}_C \ \mathrm{Sub}\left(g \ {}^{23}_{\#(g)}\right)\right]\right] & \text{by (16).} \end{aligned}$$

Thus, by (11),

$$\mathrm{Con}(C) \Rightarrow (\forall x)Q(x, g). \tag{20}$$

We proceed by formalizing previous results, in particular Theorem 4.2, *within* **theory ZF**. We will assume, without demonstration, that this has been carried out.[51] That is, assume all contents from Section 2.5 up to (20), including the collection C, have been formalized in **theory Z**. Let c, then, be the **sentence** representing $\mathrm{Con}(C)$. We see that, in this formalization, the relation $Q(x, y)$ will be represented by **relation** q, class-property $Q(x, g)$ by **class-property** r, and the sentence $(\forall x)Q(x, g)$ by **sentence** 19 Gen r. By **Theorem 4.2**, the formalization of Theorem 4.2, there corresponds an analogue to (20), namely

$$c \ \mathrm{Imp} \ (19 \text{ Gen } r).$$

Now if c were C-**provable**, then so would 19 Gen r, contradicting our assumption of **consistency**. □

[51] The results hitherto provided utilize only methods of classical mathematics; methods for which ZF provides foundation. However, the actual demonstration of this claim would contain, for the most part, the entirety of this paper.

5 Closing remarks

We first note that the assumption that collection C being ω-**consistent** can be replaced simply by **consistent**.[52]

The introduction of the recursive collection C in Theorem 4.2 was essential in securing the absoluteness of Gödel's result. When C is empty one clearly sees that **theory Z** is **incomplete**. But for C nonempty implies that even the resulting system obtained by the addition of arbitrary, yet recursive, **formulae** will still remain **incomplete**. Introducing the Axiom of Choice, the Generalized Continuum Hypothesis, or any other controversial axiom to our theory will not yield a completeness result unless that theory is inconsistent. Furthermore, by Theorem 4.3, one *cannot* carry out, in a consistent theory, a proof of that theory's consistency.

It is also worth reflecting upon the content of the **sentence** 19 Gen r, which by (16) is equivalent to Sub $\left(g^{23}_{\#(g)}\right)$. Now, 19 Gen r represents, in **theory Z**, the sentence $(\forall x)Q(x, g)$ which essential states:

"Sub $\left(g^{23}_{\#(g)}\right)$ is **not provable**."

By Theorem 4.2, 19 Gen r, and hence Sub $\left(g^{23}_{\#(g)}\right)$, is **not provable.** Therefore the above claim is nevertheless a **true** statement, yet it's formally undecidable in ZF.

[52]Theorem II in [9].

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