

Topological Function Spaces

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ABSTRACT

This thesis is an introduction to function spaces endowed with the topology of pointwise convergence, abbreviated C_p -theory, and a look at a small portion of the open questions in the field. In the introductory chapters, several different cardinal functions are presented and then used to characterize properties of C_p -spaces. The final chapters explore several open questions pertaining to Lindelöfness and meta-compactness of C_p -spaces.

Contents

Index	1
1 Basics	1
1.1 History	1
1.2 Terminology	1
1.3 Cardinal Invariants	3
1.4 Compactness Properties	8
2 Introduction to $C_p(X)$	10
2.1 Definitions	10
2.2 Properties	11
2.3 Examples	14
3 $C_p(X)$ and Lindelöfness	18
4 Lindelöfness of $C_p(X)$ Examples	20
4.1 X has a Single Non-Isolated Point	20
4.2 X is a Michael Space	21
4.3 X is a Mrówka Space	22
5 $C_p(X)$ and Metacompactness	24
References	26

Chapter 1

Basics

This thesis focuses on function spaces with the topology of pointwise convergence or C_p -theory for short. The beginning sections provide a brief recollection of concepts from general topology needed to begin studying C_p -theory. The second chapter provides a basic understanding of the properties of a space $C_p(X)$. The final three chapters of the thesis consider open problems in C_p -theory and explore partial solutions.

1.1 History

The study of sets, or spaces, of functions began with the works of Ascoli, Arzelà and Hadamard. Their work proposed questions that helped to shape the work of mathematicians such as Fréchet, Riesz, Weyl and Hausdorff.

While the contributions of all of the above mentioned mathematicians have helped develop the field of functional analysis, it was Tychonoff who, in 1935 with *Über einen Funktionenraum* [12], first began placing a topology on spaces of functions. Merging together ideas from general topology, algebraic topology, and functional analysis, mathematicians began studying C_p -theory [13].

Without question the first mathematician to realize the significance of studying C_p -theory for its own sake was Alexander Arhangel'skii. Arhangel'skii was the first to propose open questions and wrote one of the few books [1] devoted entirely to the subject.

Since then, many mathematicians have studied C_p -theory and made significant contributions. However, a great deal of C_p -theory remains new and it is difficult to avoid finding open questions.

1.2 Terminology

In this section, terminology and concepts will be introduced that play an important role in studying and understanding C_p -theory. We begin by recalling some separation

axioms, as all spaces throughout this thesis will be considered Tychonoff. Why this is the case will be described more in the next chapter.

Definition 1.2.1. A topological space X is *completely regular* if whenever $E \subseteq X$ is closed and $x \notin E$, there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in E$. That is, $E \subseteq f^{-1}(1)$.

Definition 1.2.2. A completely regular T_1 -space is called *Tychonoff* or $T_{3\frac{1}{2}}$.

Definition 1.2.3. A topological space X is *normal* if whenever E and F are disjoint, closed sets, there are disjoint open sets, U and V , with $E \subseteq U$ and $F \subseteq V$.

Definition 1.2.4. A space is called *Lindelöf* if every open cover has a countable subcover.

The following definitions will be seen throughout later chapters to describe properties of spaces. These properties are used to show results towards solving some open problems in the theory of topological function spaces.

Definition 1.2.5. Let (X, \mathcal{T}) be a topological space. Then \mathcal{B} is a *base* for \mathcal{T} if for all $U \subseteq \mathcal{T}$, there exists $\mathcal{B}' \subseteq \mathcal{B}$ such that $U = \cup \mathcal{B}'$.

Definition 1.2.6. Let (X, \mathcal{T}) be a topological space and $x \in X$. Then $\mathcal{B} \subseteq \mathcal{T}$ is called a *local base at x* if

- i. $x \in B$ for all $B \in \mathcal{B}$; and
- ii. whenever $x \in U \in \mathcal{T}$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Definition 1.2.7. A topological space with a countable base is called *second countable*.

Definition 1.2.8. A topological space X is *first countable* if for every $x \in X$, there is a countable local base at x .

Definition 1.2.9. A topological space is *separable* if it has a countable dense subset.

Definition 1.2.10. A topological space (X, \mathcal{T}) is *metrizable* if there is a metric d on X such that $\mathcal{T} = \mathcal{T}_d$.

Definition 1.2.11. A topological space X is *scattered* if for every closed subset C of X , the set of isolated points of C is dense in C .

Definition 1.2.12. A subset of a topological space is G_δ if it is the intersection of a countable collection of open sets.

Definition 1.2.13. A subset of a topological space is F_σ if it is the union of a countable collection of closed sets.

This last set of definitions will briefly remind the reader of the product topology by first recalling the topology on $X \times Y$, then introducing infinite products.

Definition 1.2.14. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. Then $\{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ is a base for $X \times Y$ and generates a topology called the *product topology* on $X \times Y$.

Definition 1.2.15. Suppose Λ is a set and X_α is a set for all $\alpha \in \Lambda$. Then

$$\prod_{\alpha \in \Lambda} X_\alpha = \{f : \Lambda \rightarrow \bigcup_{\alpha \in \Lambda} X_\alpha \mid f(\alpha) \in X_\alpha \text{ for all } \alpha \in \Lambda\}.$$

Definition 1.2.16. A *basic open set* in $\prod_{\alpha \in \Lambda} X_\alpha$ is of the form $\prod_{\alpha \in \Lambda} U_\alpha$ where each U_α is open in X_α and $U_\alpha = X_\alpha$ for all but finitely many α . The *topology* on $\prod_{\alpha \in \Lambda} X_\alpha$ is the topology generated by this base.

1.3 Cardinal Invariants

The next definitions are an introduction to cardinal invariants. These will be used in the same manner as the previous definitions to describe various spaces in theorems in later chapters. Cardinal invariants are useful in many proofs and provide some different properties than the definitions given in the previous section. The first cardinal invariants that will be seen are useful for describing a topological space as a whole.

Definition 1.3.1. The *Lindelöf number* is the cardinal

$$l(X) = \min\{\kappa : \text{every open cover of } X \text{ has a subcover of power } \leq \kappa\}.$$

Definition 1.3.2. Let (X, \mathcal{T}) be a topological space. The *Suslin number* of X is the cardinal

$$c(X) = \sup\{|\mathcal{U}| : \mathcal{U} \subseteq \mathcal{T} \text{ is disjoint}\}.$$

Definition 1.3.3. The *weight* of a space X is the cardinal

$$w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base of } X\}.$$

Since the weight is the minimal cardinality of a base of X , if $w(X) = \omega$ then the space X is second countable.

Definition 1.3.4. A *network* in a space (X, \mathcal{T}) is a family \mathcal{N} of subsets of the set X such that for any open $U \in \mathcal{T}$, there is $\mathcal{N}' \subseteq \mathcal{N}$ with $\cup \mathcal{N}' = U$.

Definition 1.3.5. The *network weight* is the cardinal

$$nw(X) = \min\{|\mathcal{N}| : \mathcal{N} \text{ is a network of } X\}.$$

Definition 1.3.6. A *condensation* is a bijective continuous map.

Definition 1.3.7. The *i -weight* of X is the cardinal

$$iw(X) = \min\{|\kappa| : \text{there is a condensation of } X \text{ onto a space of weight } < \kappa\}.$$

Definition 1.3.8. The *density* of a space X is the cardinal

$$d(X) = \min\{|A| : A \text{ is dense in } X\}.$$

A space X is separable if $d(X) = \omega$.

Definition 1.3.9. The *spread* of a space X is the cardinal

$$s(X) = \sup\{|D| : D \text{ is a discrete subset of } X\}.$$

Definition 1.3.10. The *extent* of a space X is the cardinal

$$\text{ext}(X) = \sup\{|D| : D \subseteq X \text{ is both closed and discrete}\}.$$

The next set of cardinal invariants helps to describe topological spaces at a given point. These are sometimes referred to as point cardinal invariants.

Definition 1.3.11. Let κ be an infinite cardinal number and $A \subseteq X$. Let $[A]_\kappa = \cup\{\overline{B} : B \subseteq A \text{ and } |B| \leq \kappa\}$. Then the *tightness* of X is defined as

$$t(X) = \min\{\kappa : \overline{A} = [A]_\kappa \text{ for every } A \subseteq X\}.$$

Definition 1.3.12. Let (X, \mathcal{T}) be a topological space, $A \subseteq X$ and

$$\psi(A, X) = \min\{|\mathcal{U}| : \mathcal{U} \subseteq \mathcal{T} \text{ and } \cap \mathcal{U} = A\}.$$

Then $\psi(A, X)$ is the *pseudocharacter* of A in X . If the set $A = \{x\}$, we use the notation $\psi(x, X)$.

Notice that G_δ -sets have a countable pseudocharacter.

Definition 1.3.13. The cardinal $\psi(X) = \sup\{\psi(x, X) : x \in X\}$ and is called the *pseudocharacter* of X .

Definition 1.3.14. Let (X, \mathcal{T}) be a topological space and $A \subseteq X$. Then a family $\mathcal{B} \subseteq \mathcal{T}$ is an *outer base* of A in X if $\mathcal{B} \neq \emptyset$, $A \subseteq \cap \mathcal{B}$, and for every $U \in \mathcal{T}$ with $A \subseteq U$, there is $V \in \mathcal{B}$ such that $V \subseteq U$.

Notice that if the subset A is a singleton, then an outer base is a local base.

Definition 1.3.15. Let X be a topological space and $A \subseteq X$. The minimum cardinality of the outer bases of A in X is called the *character* of A in X and is denoted by $\chi(A, X)$. If $A = \{x\}$, the notation $\chi(x, X)$ is used.

Definition 1.3.16. The cardinal $\chi(X) = \sup\{\chi(x, X) : x \in X\}$ and is called the *character* of X .

Definition 1.3.17. Let X be a space. Then the set $\Delta_X = \{(x, x) : x \in X\} \subseteq X \times X$ is called the *diagonal of X* . The cardinal $\Delta(X) = \psi(\Delta_X, X \times X)$ is called the *diagonal number of X* .

Now that several cardinal invariants have been defined, it will be useful to see the relationships between them. The following lemmas and theorem illustrate some of the most basic relationships between the previously defined cardinal invariants. The proofs in this section are adapted from [9].

Lemma 1.3.18. If a space X is second countable and A is an infinite set, then for any $Y \subseteq X^A$, we have that $w(Y) \leq |A|$.

Proof. Fix a countable base \mathcal{B} in X and let $a_1, \dots, a_n \in A$ and $O_1, \dots, O_n \in \mathcal{B}$ be given. Let $[a_1, \dots, a_n; O_1, \dots, O_n] = \{x \in X^A : x(a_i) \in O_i \text{ for all } i \leq n\}$. Then the family $\mathcal{C} = \{[a_1, \dots, a_n; O_1, \dots, O_n] : n \in \mathbb{N}, a_i \in A \text{ and } O_i \in \mathcal{B} \text{ for all } i \leq n\}$ is a base for X^A . Also, $|\mathcal{C}| \leq |A|$ which implies that $w(X^A) \leq |A|$. Now let $\mathcal{C}' = \{U \cap Y : U \in \mathcal{C}\}$. Then \mathcal{C}' forms a base for Y and $|\mathcal{C}'| \leq |\mathcal{C}| \leq |A|$. Thus $w(Y) \leq |A|$. ■

Lemma 1.3.19. There exists a set $A \subseteq C(X, [0, 1])$ with $|A| \leq nw(C(X, [0, 1])) = \kappa$ such that, for any distinct $x, y \in X$, there is $f \in A$ for which $f(x) \neq f(y)$.

Proof. Let \mathcal{N} be a network of X . Define a pair $p = (N_1, N_2) \in \mathcal{N} \times \mathcal{N}$ as being *marked* if there is a function $f_p \in C(X, [0, 1])$ such that $f_p(N_1) \subseteq [0, \frac{1}{3}]$ and $f_p(N_2) \subseteq [\frac{2}{3}, 1]$. Let $A = \{f_p : p \text{ is a marked pair}\}$. Now by construction, we have that $|A| \leq |\mathcal{N} \times \mathcal{N}| \leq \kappa$. Let $x, y \in X$ such that $x \neq y$. Since X is Tychonoff (and thus completely regular), there is $g \in C(X, [0, 1])$ such that $g(x) = 0$ and $g(y) = 1$. Let $U = g^{-1}([0, \frac{1}{3}))$ and $V = g^{-1}([\frac{2}{3}, 1])$. Then $x \in U \in \mathcal{T}_X$ and $y \in V \in \mathcal{T}_X$. Since \mathcal{N} is a network, this implies there exist $N_1, N_2 \in \mathcal{N}$ such that $x \in N_1 \in U$ and $y \in N_2 \in V$. Now $g(N_1) \subseteq [0, \frac{1}{3}]$ and $g(N_2) \subseteq [\frac{2}{3}, 1]$. Thus $p = (N_1, N_2)$ is a marked pair and so $f_p = g \in A$. Hence, $f_p(x) \leq \frac{1}{3}$ and $f_p(y) \geq \frac{2}{3}$ and so $f_p(x) \neq f_p(y)$. ■

Theorem 1.3.20. For any space X ,

1. $c(X) \leq d(X) \leq nw(X) \leq w(X)$;
2. $c(X) \leq s(X)$ and $ext(X) \leq l(X) \leq nw(X)$;
3. $\psi(X) \leq \chi(X)$ and $\psi(X) \leq iw(X) \leq nw(X)$;
4. $t(X) \leq \chi(X) \leq w(X)$ and $t(X) \leq nw(X)$.

Proof. (1) Let (X, \mathcal{T}) be a topological space. Let $D \subseteq X$ be dense and let $\mathcal{U} \subset \mathcal{T}$ be a nonempty disjoint family. Choose $y(U) \in U \cap D$ for any $U \in \mathcal{U}$. Consider the mapping from $U \rightarrow y(U)$. Since \mathcal{U} is pairwise disjoint, this is a one-to-one correspondence from \mathcal{U} to D . Hence, $c(X) \leq d(X)$. Now let \mathcal{N} be a network of X . Choose $x(N) \in N$ for some $N \in \mathcal{N}$ and consider $\{x(N) : N \in \mathcal{N}\}$. Clearly, this set is dense in X . Thus, $d(X) \leq nw(X)$. Finally, by definition a base is also a network resulting in $nw(X) \leq w(X)$.

(2) Let $\mathcal{U} \subseteq \mathcal{T}$ be a disjoint family and choose $y(U) \in U$ for each $U \in \mathcal{U}$. Then the subspace $D = \{y(U) : U \in \mathcal{U}\}$ is discrete and $|D| = |\mathcal{U}|$. Thus $c(X) \leq s(X)$.

Now, assume $l(X) \leq \kappa$ and let $D \subseteq X$ be closed and discrete. For each $d \in D$, choose $U_d \in \mathcal{T}$ such that $d \in U_d$ with $U_d \cap D = \{d\}$. Let $\mathcal{U} = \{U_d : d \in D\} \cup \{X \setminus D\}$. Clearly, \mathcal{U} is an open cover of X . Since any subcover \mathcal{V} of X would need to contain $\{U_d : d \in D\}$, we see that $|\mathcal{V}| = |\mathcal{U}|$. Thus, by our assumption that $l(X) \leq \kappa$, we see that $|D| = |\mathcal{U}| \leq \kappa$, and so $ext(X) \leq l(X)$. Finally, let \mathcal{N} be a network of X , $|\mathcal{N}| = \kappa$ and $\mathcal{U} \subseteq \mathcal{T}$ be an open cover of X . Let $\mathcal{N}' = \{N \in \mathcal{N} : \text{there exists } U(N) \in \mathcal{U} \text{ with } N \subseteq U(N)\}$. Then $\mathcal{V} = \{U(N) : N \in \mathcal{N}'\}$ is a subcover of \mathcal{U} with cardinality $< \kappa$. To see that \mathcal{V} is a cover of X , let $x \in X$. Then there is $U \in \mathcal{U}$ with $x \in U$ and, since \mathcal{N} is a network, there is $N \in \mathcal{N}$ with $x \in N \subseteq U$. But this implies that $N \in \mathcal{N}'$. Thus, $x \in N \subseteq U(N)$, and so \mathcal{V} is a cover of X . Hence, $l(X) \leq nw(X)$.

(3) Let \mathcal{B} be a local base at a point $x \in X$. Then since X is a Tychonoff space, for any $y \neq x$, $X \setminus \{y\}$ is an open neighborhood of x . Thus there is $U \in \mathcal{B}$ such that $x \in U \subseteq X \setminus \{y\}$ and so, $\cap \mathcal{B} = \{x\}$ which gives $\psi(X) \leq \chi(X)$.

Let X, Y be topological spaces and $f : X \rightarrow Y$ be a condensation such that $w(Y) \leq \kappa$. Then by definition and previously proven properties, we have $\psi(Y) \leq \chi(Y) \leq w(Y) \leq \kappa$. Now, for $x \in X$, let $y = f(x)$ and fix a family $\mathcal{V} \subseteq \mathcal{T}_Y$ such that $\cap \mathcal{V} = \{y\}$. Let $\mathcal{V}' = \{f^{-1}(U) : U \in \mathcal{V}\}$. Then $\mathcal{V}' \subseteq \mathcal{T}_X$ and $\cap \mathcal{V}' = \{x\}$. Hence, $\psi(X) \leq iw(X)$.

Let \mathcal{N} be a network of X such that $|\mathcal{N}| = nw(X) = \kappa$. Since X is regular, we see that for $N \in \mathcal{N}$ we can take the closure of each N and $\{\overline{N} : N \in \mathcal{N}\}$ is a network as well. Thus, we can consider all the elements of \mathcal{N} to be closed. Now, let $A \subseteq C(X, [0, 1])$ with $|A| \leq \kappa$. Let $\phi : X \rightarrow [0, 1]^A$ be defined by $\phi(x)(f) = f(x)$ for all $x \in X$ and $f \in A$. Let $\pi_f : [0, 1]^A \rightarrow [0, 1]$ be the natural projection mapping for $f \in A$. Then $\pi_f \circ \phi = f$ is a continuous map on X and so, ϕ is a continuous map. Let $Y = \phi(X)$. Then by Lemma 1.3.18, we have $w(Y) \leq |A| \leq \kappa$. To see that ϕ is one-to-one, let $x, y \in X$ with $x \neq y$. Then by Lemma 1.3.19, there exists $f \in A$ with $f(x) \neq f(y)$. Thus, $\phi(x)(f) = f(x) \neq f(y) = \phi(y)(f)$. Therefore, $iw(X) \leq nw(X)$.

(4) To see that $t(X) \leq \chi(X)$, suppose that $\chi(X) \leq \kappa$ for some infinite cardinal κ . Let $A \subseteq X$ such that $x \in \overline{A}$. Let \mathcal{B} be a local base at x such that $|\mathcal{B}| \leq \kappa$. For each $U \in \mathcal{B}$ pick $x_U \in U \cap A$. Then, let $B = \{x_U : U \in \mathcal{B}\}$. Clearly, $B \in A$ and $|B| \leq \kappa$. Now, let $x \in V \in \mathcal{T}$. Since \mathcal{B} is a local base at x , there is $U \in \mathcal{B}$ such

that $U \subseteq V$ and so, $x_U \in V \cap A$. Thus, $V \cap A \neq \emptyset$ for any $V \in \mathcal{T}$ with $x \in V$. Thus, $x \in \overline{B}$ and $t(X) \leq \chi(X)$.

To see that $\chi(X) \leq w(X)$, let \mathcal{B} be a base of X . Then for any $x \in X$, the set $\mathcal{B}_x = \{U \in \mathcal{B} : x \in U\}$ forms a local base at x . Since $\mathcal{B}_x \subseteq \mathcal{B}$, we have that $|\mathcal{B}_x| \leq |\mathcal{B}|$, proving $\chi(X) \leq w(X)$.

Suppose $nw(X) \leq \kappa$ for some infinite cardinal κ . Let $A \subseteq X$ and $x \in \overline{A}$. Let \mathcal{N} be a network of X and define $\mathcal{N}' = \{N \in \mathcal{N} : N \cap A \neq \emptyset\}$. Choose $a_N \in N \cap A$ for each $N \in \mathcal{N}'$. The let $B = \{a_N : N \in \mathcal{N}'\}$. Clearly, $B \subseteq A$. Also, $|B| \leq |\mathcal{N}'| \leq \kappa$. To see that $x \in \overline{B}$, suppose that it is not. Then there is a $U \in \mathcal{T}$ with $x \in U$ such that $U \cap B = \emptyset$. Now, $B \cap A \neq \emptyset$ since $x \in \overline{A}$. Choose $a \in A \cap U$ and $N \in \mathcal{N}$ such that $a \in N \subseteq U$. Then $N \in \mathcal{N}'$. But this implies that $a \in B \cap U$, a contradiction. Thus, $x \in \overline{B}$ and $t(X) \leq nw(X)$. ■

1.4 Compactness Properties

The following section visits several types of compactness. The goal of this section is to familiarize the reader with these types of compactness as they will be used often in the remainder of the thesis. In addition to these, more specific types will be introduced throughout later chapters as necessary.

Definition 1.4.1. Let X be a topological space and $A \subseteq X$. Then A is *compact* if every open cover of A has a finite subcover.

Definition 1.4.2. Let X be a topological space and $A \subseteq X$. Then A is *countably compact* if every countable open cover of A has a finite subcover.

Definition 1.4.3. If \mathcal{U} and \mathcal{V} are covers of X , we say \mathcal{U} *refines* \mathcal{V} , and write $\mathcal{U} < \mathcal{V}$, if and only if each $U \in \mathcal{U}$ is contained in some $V \in \mathcal{V}$. We say \mathcal{U} is a *refinement* of \mathcal{V} .

Definition 1.4.4. A collection \mathcal{U} of subsets of X is *locally finite* if and only if each $x \in X$ has a neighborhood meeting only finitely many $U \in \mathcal{U}$.

Definition 1.4.5. A collection \mathcal{U} of subsets of X is *point finite* if and only if each $x \in X$ belongs to only finitely many $U \in \mathcal{U}$.

Definition 1.4.6. A Hausdorff space X is *paracompact* if and only if each open cover of X has an open locally finite refinement.

Definition 1.4.7. A space is *metacompact* if and only if each open cover has an open point finite refinement.

Definition 1.4.8. A space is called *σ -compact* (*σ -countably compact*) if it is the union of a countable set of compact (countably compact) subspaces.

Definition 1.4.9. A space is called *pseudocompact* if every real-valued continuous function on it is bounded.

Chapter 2

Introduction to $C_p(X)$

The goal of this chapter is to introduce the reader to the basic ideas and concepts relating to C_p -theory. As mentioned in Chapter 1, all spaces will be taken to be Tychonoff. The reason being that Tychonoff is needed to separate points.

2.1 Definitions

The following section contains several terms that provide the reader with an introduction to function spaces and various topologies that can be placed on these spaces [1].

Definition 2.1.1. The set of all continuous maps from a space X into a space Y is denoted by $C(X, Y)$.

Note that we will write $C(X, \mathbb{R})$ as $C(X)$.

Definition 2.1.2. The set of all functions, $f \in C(X)$, such that f is bounded, is denoted by $C^0(X)$.

Definition 2.1.3. $C_p(X)$ and $C_p^0(X)$ are used to represent $C(X)$ and $C^0(X)$ endowed with the topology of pointwise convergence.

Definition 2.1.4. Let \mathcal{E} be the family of all finite subsets of X . Then $\mathcal{T}_{\mathcal{E}}$ is the *topology of pointwise convergence*. $(C(X, Y), \mathcal{T}_{\mathcal{E}}) = C_p(X, Y)$.

Definition 2.1.5. Let \mathcal{E} be the family of all compact subsets of X . Then $\mathcal{T}_{\mathcal{E}}$ is the *compact-open topology*. $(C(X, Y), \mathcal{T}_{\mathcal{E}}) = C_c(X, Y)$.

Definition 2.1.6. Let \mathcal{E} be the family of all bounded subsets in X . Then

$$(C(X, Y), \mathcal{T}_{\mathcal{E}}) = C_0(X, Y).$$

2.2 Properties

In this section we will examine some basic properties of $C_p(X)$. These properties will familiarize us with $C_p(X)$ spaces and provide a better understanding of the way in which these spaces work. The proofs in this section are modified versions of those found in [9] and [1].

Proposition 2.2.1. Let X and Z be topological spaces and $Y \subseteq Z$, then the topology of $C_p(X, Y)$ coincides with the topology on $C(X, Z)$ induced from $C_p(X, Y)$ and so $C_p(X, Y) \subseteq C_p(X, Z)$.

Proof. Let

$$A = \{x_1, x_2, \dots, x_n; U_1, U_2, \dots, U_n\} = \{f \in C_p(X, Z) : f(x_i) \in U_i \text{ for all } i \leq n\}$$

be a basic open set in $C_p(X, Z)$. Then

$$A \cap C_p(X, Y) = \{f \in C_p(X, Y) : f(x_i) \in U_i \cap Y \text{ for all } i \leq n\}$$

and $A \cap C_p(X, Y)$ is an open set in $C_p(X, Y)$.

Let

$$W = \{x_1, \dots, x_n; U_1, \dots, U_n\} = \{f \in C_p(X, Y) : f(x_i) \in U_i \text{ for all } i \leq n\}$$

be a basic open set of $C_p(X, Y)$. For each $i \leq n$, let $O_i \in \mathcal{T}_Z$ such that $U_i = O_i \cap Y$.

Let

$$W' = \{x_1, \dots, x_n; O_1, \dots, O_n\} = \{f \in C_p(X, Z) : f(x_i) \in O_i \text{ for all } i \leq n\}.$$

Then $W' = W \cap C_p(X, Y)$ and W' is a basic open set in $C_p(X, Z)$. ■

Proposition 2.2.2. If X, Z are topological spaces and $Y \subseteq Z$ is closed, then $C_p(X, Y)$ is a closed subspace of $C_p(X, Z)$.

Proof. Let $f \in C_p(X, Z) \setminus C_p(X, Y)$. Then there exists an $x \in X$ such that $f(x) \in Z \setminus Y$. Let $U_f = \{f \in C_p(X, Z) : f(x) \in Z \setminus Y\}$. Then U_f is open in $C_p(X, Z)$. Furthermore, $f \in U_f \subseteq C_p(X, Z) \setminus C_p(X, Y)$. Therefore $C_p(X, Z) \setminus C_p(X, Y) = \bigcup \{U_f : f \in C_p(X, Z) \setminus C_p(X, Y)\}$ which is open. Thus, $C_p(X, Y)$ is a closed subset. ■

The topological product \mathbb{R}^X is the set of all maps from X into \mathbb{R} endowed with the topology of pointwise convergence. By definition of $C_p(X)$, we see that $C_p(X) \subseteq \mathbb{R}^X$. Furthermore, $C_p(X)$ is everywhere dense in \mathbb{R}^X .

The following theorems and propositions illustrate how properties of a space X are characterized by topological properties of $C_p(X)$.

Theorem 2.2.3. For any infinite space X ,

$$|X| = \chi(C_p(X)) = w(C_p(X)).$$

Proof. Let $\mathcal{B} = \{[x_1, \dots, x_k; U_1, \dots, U_k] : k \in \mathbb{N}, x_i \in X, \text{ and } U_i \text{ is a rational interval for } i \leq k\}$ be a base for $C_p(X)$. Then $|\mathcal{B}| \leq |X|$, since \mathcal{B} is a base for $C_p(X)$ and the topology on $C_p(X)$ is created from the finite subsets of X . Thus, we have that $w(C_p(X)) \leq |X|$.

Now, let $h : X \rightarrow \mathbb{R}$ be defined by $h(x) = 0$ for all $x \in X$, and let \mathcal{C} be a local base at h in $C_p(X)$. For any $V \in \mathcal{C}$, there is a basic open set $U(V)$ with $h \in U(V) \subseteq V$. Then $\{U(V) : V \in \mathcal{C}\}$ also forms a local at h . Thus, it may be assumed without loss of generality that the elements of \mathcal{C} are basic. Given $V = [x_1, \dots, x_k; U_1, \dots, U_k] \in \mathcal{C}$, let $\text{supp}(V) = \{x_1, \dots, x_k\}$ and let $Y = \cup\{\text{supp}(V) : V \in \mathcal{C}\}$, where $\text{supp}(V)$ is the support. If $x \in X \setminus Y$, then there is $V = [x_1, \dots, x_k; U_1, \dots, U_k] \in \mathcal{C}$ with $V \subseteq W = [x; (-1, 1)]$. However, $\text{supp}(V) = \{x_1, \dots, x_k\} \subseteq Y \subseteq X \setminus \{x\}$. So there exists a function $f \in C_p(X)$ such that $f(x_i) \in U_i$ and $f(x) = 1$. Then $f \in V \setminus W$, which gives a contradiction. Thus, $X = Y$ and so $|X| = |Y| \leq \chi(C_p(X))$. Also, by Theorem 1.3.20, we have that $w(C_p(X)) \leq |X| \leq \chi(C_p(X)) \leq w(C_p(X))$. Therefore, $|X| = \chi(C_p(X)) = w(C_p(X))$. \blacksquare

Theorem 2.2.4. For a space X , $nw(X) = nw(C_p(X))$.

Proof. Let \mathcal{N} be a network of the topological space X such that $|\mathcal{N}| = nw(X)$. For any collection $N_1, \dots, N_k \in \mathcal{N}$ and any open intervals with rational end points I_1, \dots, I_k , let $M(N_1, \dots, N_k; I_1, \dots, I_k) = \{f \in C_p(X) : f(N_j) \subseteq I_j \text{ for all } j \leq k\}$. Now, let $\mathcal{M} = \{M(N_1, \dots, N_k; I_1, \dots, I_k) : n \in \mathbb{N}, N_1, \dots, N_k \in \mathcal{N}, \text{ and } I_j \text{ is a rational interval for all } j \leq k\}$. Clearly, $|\mathcal{M}| \leq nw(X)$. To see that \mathcal{M} is a network of $C_p(X)$, let $f \in C_p(X)$ and $f \in U \in \mathcal{T}_{(C_p(X))}$. There is a canonical open set $V = [x_1, \dots, x_k; O_1, \dots, O_k]$ with $f \in V \subseteq U$. Without loss of generality, consider x_1, \dots, x_k distinct. Since f is continuous, we can choose disjoint open sets U_1, \dots, U_k such that $x_i \in U_i$ for each $i \leq k$ and rational intervals I_1, \dots, I_k such that $f(U_j) \subseteq I_j \subseteq O_j$ for all $j \leq k$. Then there exist $N_1, \dots, N_k \in \mathcal{N}$ such that $x_j \in N_j \subseteq U_j$ for all $j \leq k$. Thus, $f \in M(N_1, \dots, N_k; I_1, \dots, I_k) \subseteq V \subseteq U$, and \mathcal{M} is a network of $C_p(X)$. Since $|\mathcal{M}| \leq nw(X)$, we have $nw(C_p(X)) \leq nw(X)$ for any space X . Thus, $nw(C_p(C_p(X))) \leq nw(C_p(X)) \leq nw(X)$ for any space X . But, X can be embedded in $C_p(C_p(X))$ which tells us $nw(X) \leq nw(C_p(C_p(X)))$. Combining these inequalities, we have that $nw(X) = nw(C_p(X))$. \blacksquare

Theorem 2.2.5. $d(X) = \psi(C_p(X)) = \Delta(C_p(X)) = iw(C_p(X))$.

Proof. Let Y be a topological space and $\gamma \subseteq \mathcal{T}_{Y \times Y}$ be a family such that $\Delta_Y = \bigcap \gamma$. Fix $y \in Y$ and for $U \in \gamma$, choose $O_U \in \mathcal{T}_Y$ with $y \in O_U$ such that $O_U \times O_U \subseteq U$. Then $\bigcap \{O_U : U \in \gamma\} = \{y\}$. Therefore, $\psi(Y) \leq \Delta(Y)$.

Suppose that $f : Z \rightarrow Y$ is a condensation and \mathcal{B} is a base for Y such that $|\mathcal{B}| = \kappa = iw(X)$. Then $l((Y \times Y) \setminus \Delta_Y) \leq w((Y \times Y) \setminus \Delta_Y) \leq \kappa$. For each $z \in (Y \times Y) \setminus \Delta_Y$, choose $U_z \in \mathcal{T}_{Y \times Y}$ with $z \in U_z$ such that $\overline{U_z} \cap \Delta_Y = \emptyset$. Let $\mathcal{U} = \{U_z : z \in (Y \times Y) \setminus \Delta_Y\}$ be an open cover of the $(Y \times Y) \setminus \Delta_Y$ with a subcover \mathcal{V} such that $|\mathcal{V}| \leq \kappa$. Then, $(Y \times Y) \setminus \Delta_Y = \bigcup \{\overline{U} : U \in \mathcal{V}\}$ is a union of less than or equal to κ closed set. Thus, $\psi(\Delta_Y, Y \times Y) \leq \kappa$. Since f is a condensation, $\psi(\Delta_Z, Z \times Z) \leq \kappa$ and so, $\Delta(Z) \leq iw(Z)$.

Given a basic set $V = [x_1, \dots, x_n; O_1, \dots, O_n] \in \mathcal{T}_{C_p(X)}$, let $supp(V) = \{x_1, \dots, x_n\}$. Let γ be any subset of $\mathcal{T}_{C_p(X)}$ such that $\{u\} = \bigcap \gamma$, where $u \equiv 0$ and $|\gamma| = \kappa$. For each $U \in \gamma$ let $O_U = [x_1, \dots, x_n; O_1, \dots, O_n]$ be a basic open set such that $u \in O_U \subseteq U$. Then $Y = \bigcup \{supp(O_U) : U \in \gamma\}$ and $|Y| \leq \kappa$. If $X \neq \overline{Y}$, choose $x \in X \setminus \overline{Y}$ and $f \in C_p(X)$ such that $f(x) = 1$ and $f(Y) \subseteq \{0\}$. Then $f \in \bigcap \gamma$, a contradiction to $f \neq u$ and $\bigcap \gamma = \{u\}$. Thus Y is dense in X and $d(X) \leq |Y| \leq \kappa$. Therefore, $d(X) \leq \psi(C_p(X))$, and so we have $d(X) \leq \psi(C_p(X)) \leq \Delta(C_p(X)) \leq iw(C_p(X))$.

To see that $iw(C_p(X)) \leq d(X)$, fix $Y \subseteq X$ such that $|Y| = d(X)$ and $X = \overline{Y}$. Let $\pi_Y : C_p(X) \rightarrow Z = \pi_Y(C_p(X)) \subseteq C_p(Y)$ be the restriction map ($\pi_Y(f) = f|_Y$). Then π_Y is a condensation and $w(Z) \leq w(C_p(Y)) = |Y| = d(X)$. Therefore, $iw(C_p(X)) \leq d(X)$. ■

Using the definitions of our cardinal invariants, and the result proved above, we can summarize the previous theorem as $C_p(X)$ condenses onto a second countable space if and only if X is separable.

Theorem 2.2.6. $iw(X) = d(C_p(X))$

Proof. Note that X embeds $C_p(C_p(X))$. So, we have that $iw(X) \leq iw(C_p(C_p(X))) \leq d(C_p(X))$ by that and the result proved in the previous theorem.

To see that $d(C_p(X)) \leq iw(X)$, suppose $iw(X) \leq \kappa$. Then take a condensation $f : X \rightarrow Y$ such that $w(Y) \leq \kappa$. Then the dual mapping f^* embeds $C_p(Y)$ into $C_p(X)$ as a dense subspace. Let $Z = f^*(C_p(Y))$. Then $d(Z) \leq nw(Z) = nw(C_p(Y)) = nw(Y) \leq w(Y) \leq \kappa$. Therefore, $d(C_p(X)) \leq d(Z) \leq \kappa$, and so $d(C_p(X)) \leq iw(X)$. ■

Using the definitions and the result of the preceding theorem, we can summarize that $C_p(X)$ is separable if and only if X condenses onto a second countable space.

2.3 Examples

In this section, we will look at three familiar spaces - the real line, the Sorgenfrey line and the Moore-Niemytzki Plane - and some of the properties of their corresponding C_p -spaces. The proofs in this section are modified versions of the ones in [9].

Theorem 2.3.1. The set of all increasing functions is closed in $C_p(\mathbb{R})$.

Proof. Let $\mathcal{I} \subseteq C_p(\mathbb{R})$ be the set of all of increasing functions. For $f \in C_p(\mathbb{R}) \setminus \mathcal{I}$ there are $x, y \in X$ such that $x < y$ and $f(x) > f(y)$. Let $d = f(x) - f(y)$. Let $U = [x, y; (f(x) - \frac{d}{3}, f(x) + \frac{d}{3}), (f(y) - \frac{d}{3}, f(y) + \frac{d}{3})]$. Then U is open in $C_p(\mathbb{R})$ and $f \in U$. Let $g \in U$. Then $g(y) < f(y) + \frac{d}{3} < f(x) - \frac{d}{3} < g(x)$. Thus $g(y) < g(x)$ and so g is not increasing. Also, $U \cap \mathcal{I} = \emptyset$. Since each $f \in C_p(\mathbb{R}) \setminus \mathcal{I}$ has a neighborhood that is also contained in $C_p(\mathbb{R}) \setminus \mathcal{I}$, $C_p(\mathbb{R}) \setminus \mathcal{I}$ is open and \mathcal{I} is closed. ■

Using a nearly identical proof as the theorem above, we get the following result.

Corollary 2.3.2. The set of all decreasing functions is closed in $C_p(\mathbb{R})$.

The following theorem will provide a further look into $C_p(\mathbb{R})$, comparing density and network weight.

Theorem 2.3.3. There exists a countable $A \subseteq C_p(\mathbb{R})$ such that $\overline{A} = C_p(\mathbb{R})$.

Proof. Let $\mathcal{B} = \{\{O_1, \dots, O_n\} : n \in \mathbb{N}, O_i = (a_i, b_i) \text{ is a rational open interval for all } i \leq n \text{ and } [a_i, b_i] \cap [a_j, b_j] = \emptyset \text{ if } i \neq j\}$. Given that $O = \{O_1, \dots, O_n\} \in \mathcal{B}$, let $m(O) = n$. If $q = (q_1, \dots, q_n)$ is an n -tuple of rational numbers, fix $f_{O,q} \in C_p(\mathbb{R})$ such that $f_{O,q}(O_i) = \{q_i\}$ for all $i \leq n$. Then the set $A = \{f_{O,q} : O \in \mathcal{B} \text{ and } q \text{ is an } m(O)\text{-tuple of rationals}\}$ is countable. To see that $\overline{A} = C_p(\mathbb{R})$, given a basic open set $W = [x_1, \dots, x_n; U_1, \dots, U_n]$ choose $q_i \in U_i \cap \mathbb{Q}$ for all $i \leq n$ and let $q = (q_1, \dots, q_n)$. Then, there exists $O = \{O_1, \dots, O_n\} \in \mathcal{B}$ such that $x_i \in O_i$ for all $i \leq n$. Then $f_{O,q} \in W \cap A$. Thus, $\overline{A} = C_p(\mathbb{R})$. ■

Another way of looking at the preceding theorem is that

$$d(C_p(\mathbb{R})) \leq nw(C_p(\mathbb{R})) \leq nw(\mathbb{R}) = \aleph_0.$$

The following two corollaries focus on subsets of the reals, namely, the rationals and natural numbers.

Corollary 2.3.4. $C_p(\mathbb{Q})$ has a countable base.

Proof. Let \mathcal{U} be the family of all open intervals in \mathbb{R} that have rational endpoints. Consider the family \mathcal{B} of all sets $[x_1, \dots, x_n; O_1, \dots, O_n]$ where $n \in \mathbb{N}$, $x_i \in \mathbb{Q}$ and $O_i \in \mathcal{U}$ for all $i \leq n$. Then \mathcal{B} forms a countable base for $C_p(\mathbb{Q})$. ■

Corollary 2.3.5. $C_p(\mathbb{N})$ has a countable base.

Proof. Let \mathcal{U} be the family of all open intervals in \mathbb{R} that have rational endpoints. Consider the family \mathcal{B} of all sets $[x_1, \dots, x_n; O_1, \dots, O_n]$ where $n \in \mathbb{N}$, $x_i \in \mathbb{N}$ and $O_i \in \mathcal{U}$ for all $i \leq n$. Then \mathcal{B} forms a countable base for $C_p(\mathbb{N})$. ■

The following examples deal with two common spaces: the Sorgenfrey line and the Moore-Niemytzki Plane.

Definition 2.3.6. The real line with the topology generated by $\{[a, b) : a, b \in \mathbb{R}, a < b\}$ is called the *Sorgenfrey line*. We will denote the Sorgenfrey line by the notation (S, \mathcal{T}) .

Proposition 2.3.7. Let X be a separable space with $ext(X) \geq \mathfrak{c}$. Then X is not normal.

Proof. Let $S \subseteq X$ be closed discrete with $|S| = \mathfrak{c}$. Since X is separable, we know there is a countable dense $D \subseteq X$. Then $|\mathcal{P}(D)| = \aleph_0 < |\mathcal{P}(S)| = \mathfrak{c}$. Thus, by Jones' Lemma, X is not normal. ■

Proposition 2.3.8. The space $S \times S$ has a closed discrete subspace of cardinality continuum.

Proof. Let $D = \{(t, -t) : t \in \mathbb{R}\} \subseteq S \times S$. Now for any $d = (t, -t) \in D$, the set $U = [t, t+1) \times [-t, -t+1)$ is open in $S \times S$. Also, $U \cap D = \{d\}$, and so D is discrete. Now if $z = (x, y) \notin D$, then there is $\varepsilon = |x + y| > 0$. Thus, $W = [x, x + \frac{\varepsilon}{2}) \times [y, y + \frac{\varepsilon}{2})$ is an open neighborhood of $S \times S$ with $W \cap D = \emptyset$. Thus, D is closed in $S \times S$ and $ext(S \times S) = \mathfrak{c}$. ■

Proposition 2.3.9. Let X, Y be topological spaces. Given a continuous map $r : X \rightarrow Y$, define the dual map $r^* : C_p(X) \rightarrow C_p(Y)$ by $r^*(f) = f \circ r$ for any $f \in C_p(Y)$. If $r(X) = Y$, then $r^*(C_p(Y))$ is dense in $C_p(X)$ if and only if r is a condensation.

Proof. Suppose $r : X \rightarrow Y$ is a condensation. Let $f \in C_p(X)$, $x_1, \dots, x_n \in X$ and $\varepsilon > 0$ be given. Then there exists $g \in C_p(Y)$ such that $g(r(x_i)) = f(x_i)$ for all $i \leq n$. Then $r^*(g) = g \circ r$ is an element of the basic open neighborhood $O_X(f, x_1, \dots, x_n, \varepsilon) = \{h \in C_p(X) : |h(x_i) - f(x_i)| < \varepsilon \text{ for all } i \leq n\}$. Thus, $f \in r^*(C_p(Y))$ for each $f \in C_p(X)$. Thus, $r^*(C_p(Y)) = C_p(X)$.

To see that if $r^*(C_p(Y))$ is dense in $C_p(X)$ then r is a condensation, suppose r is not one-to-one. Let $x, y \in X$ with $x \neq y$ such that $r(x) = r(y)$. Then $U_X = [x, y; (0, 1), (2, 3)]$ is a non-empty open set in $C_p(X)$. However, $U_X \cap r^*(C_p(Y)) = \emptyset$. Thus $r^*(C_p(Y))$ is not dense in $C_p(X)$. ■

Theorem 2.3.10. $C_p(S)$ is not normal.

Proof. Let $r : \mathbb{R} \rightarrow \mathbb{R}$ be the identity mapping for all $t \in \mathbb{R}$. Then $r : S \rightarrow \mathbb{R}$ is a condensation. Thus, $r(C_p(\mathbb{R}))$ is a dense subspace of $C_p(S)$ which is homeomorphic to $C_p(\mathbb{R})$. If A is a countable dense subspace of $C_p(\mathbb{R})$, then $r^*(A)$ is a countable dense subspace of $C_p(S)$. Hence $C_p(S)$ is separable.

Let $[a, b] \subseteq S$. Let $f(t) = t - a$ and $g(t) = \frac{1}{b-a} \cdot t$ for all $t \in \mathbb{R}$. Then $h = g \circ f : S \rightarrow S$ is a homeomorphism. Also, $h([a, b]) = [0, 1]$. Thus, any two clopen intervals in S are homeomorphic subspaces. Now $S = \cup\{A_k : k \in \omega\}$ where $A_{2i} = [-i - 1, -i]$ and $A_{2i+1} = [i, i + 1]$ for all $i \in \omega$. Thus, since S can be represented as a union of a family $\{A_k : k \in \omega\}$ of pairwise disjoint open subsets of S , then S is homeomorphic to $\bigoplus\{A_k : k \in \omega\}$. Notice that similarly, we can write $[0, 1] = \cup\{[p_i, q_i] : i \in \omega\}$ where $p_i = \frac{i}{i+1}$ and $q_i = \frac{i+1}{i+2}$ for all $i \in \omega$. Thus, $[0, 1] = \bigoplus\{B_k : k \in \omega\}$ where $B_k = [p_k, q_k]$. This implies that S is homeomorphic to any of its clopen intervals. Let us take $S' = [0, 1]$ for the remainder of the proof.

Now, for each $t \in (0, 1]$, let $f_t(x) = 1$ if $0 \leq x < t$ and $f_t(x) = 0$ for all $x \in [t, 1]$. Then $f_t \in C_p(S')$ for all $t \in [0, 1]$.

Claim 1. The set $F = \{f_t : t \in (0, 1]\}$ is closed in $C_p(S')$.

Proof. Given that $f \notin F$, suppose $f(x) \notin \{0, 1\}$ for some $x \in S'$. Then for $\varepsilon = \min\{|f(x)|, |f(x) - 1|\}$ the set $[x; (f(x) - \varepsilon, f(x) + \varepsilon)]$ is an open neighborhood of f that does not meet F . If $f(S') \subseteq \{0, 1\}$ and $f \in \overline{F}$, then if $f(x) = 1$ then $f(y) = 1$ for $y < x$. For if $f(y) = 0$ for some $y < x$, then $[y, x : (-\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{3}{2})]$ is an open neighborhood of f that does not meet F since all functions from F are non-increasing. By similar reasoning, $f(x) = 0$ tells us $f(y) = 0$ for $y > x$. Then, there exists $t \in S'$ such that $f(x) = 1$ for all $x < t$ and $f(x) = 0$ for all $x > t$. If $f(t) = 1$, then f is discontinuous, so $f(t) = 0$ and $f = f_t \in F$. Therefore, F is closed in $C_p(S')$.

Claim 2. The map $\phi : S' \rightarrow F$ defined by $\phi(s) = f_{1-s}$ is a homeomorphism.

Proof. Clearly, ϕ is one-to-one and onto. To prove the continuity of ϕ , it suffices to show that $h_t = \pi_t \circ \phi$ is continuous for each π_t defined by $\pi_t(f) = f(t)$ for all $f \in C_p(S')$ and $t \in [0, 1)$, since $C_p(S')$ is a subspace of $\mathbb{R}^{S'}$. Since $h_t(s) = f_{1-s}(t) = 1$ for $s < 1 - t$ and $h_t(s) = f_{1-s}(t) = 0$ for $s \geq 1 - t$, h_t is continuous for all $t \in S'$. Finally, to see that ϕ is a homeomorphism, it need only be shown that ϕ is open. Since the clopen intervals form a base of S' , it suffices to show that $\phi(U)$ is open in F for any $U = [a, b) \subseteq S'$. Now, $\phi(U) = \{f_t : 1 - b < t \leq 1 - a\} = [1 - b, 1 - a; (\frac{1}{2}, \frac{3}{2}), (-\frac{1}{2}, \frac{1}{2})] \cap F$ which is open in F . Thus, the mapping ϕ is a homeomorphism.

Claim 3. The spaces $C_p(S')$ and $C_p(S') \times C_p(S')$ are homeomorphic.

Proof. Now $S' = [0, \frac{1}{2}) \cup [\frac{1}{2}, 1)$ and since S' is homeomorphic to every arrow, we know that the space S' is homeomorphic to $S' \oplus S'$. Then $C_p(S')$ is homeomorphic to $C_p(S' \oplus S')$. But, $C_p(S' \oplus S')$ is homeomorphic to $C_p(S' \times S')$ and so $C_p(S')$ is homeomorphic to $C_p(S' \times S')$.

From Claims 1 and 2, we see that F is a closed subspace of $C_p(S')$ homeomorphic to S' . And so, $C_p(S)$ has some closed subspace T that is homeomorphic to S . Then, the space $T \times T$ is closed subspace of $C_p(S) \times C_p(S)$. Now, we know there is a closed discrete $D \subseteq T \times T$ with $|D| = \mathfrak{c}$. Then D is a closed discrete subspace of $C_p(S) \times C_p(S)$ as well. By Claim 3, we know that $C_p(S)$ is homeomorphic to $C_p(S) \times C_p(S)$. Thus, $C_p(S)$ has a closed discrete subspace with cardinality of the continuum and so is not normal. ■

Definition 2.3.11. Let $\Gamma = \mathbb{R} \times [0, \infty)$ and $S = \mathbb{R} \times \{0\}$. Also, let d denote the usual metric on $\mathbb{R} \times \mathbb{R}$. For each $x \in \Gamma \setminus S$, let $\mathcal{B}_x = \{B_d(x, r) : r < d(x, S)\}$. Now suppose that $x = (s, 0) \in S$. Then $\mathcal{B}_x = \{B_d((s, r), r) \cup \{x\} : r > 0\}$. Let $\mathcal{B} = \cup_{x \in \Gamma} \mathcal{B}_x$. Let \mathcal{T} be the topology on Γ generated by \mathcal{B} . Then (Γ, \mathcal{T}) is called the *Moore-Niemytzki Plane*.

The proof of the following theorem can be found in full in [9].

Theorem 2.3.12. $C_p(\Gamma)$ is not normal.

Chapter 3

$C_p(X)$ and Lindelöfness

In this chapter, we will explore two open questions in C_p -theory originally posed by Arhangel'skii in [1]. The results provided in this chapter are the work of many mathematicians. The first question:

Question. Does there exist a natural topological property of $C_p(X)$ which would characterize whether the space X is Lindelöf?

The second question is again a Lindelöf question, however, this one pertains to the product topology:

Question. Suppose $C_p(X)$ is Lindelöf. Is it true that the space $C_p(X) \times C_p(X)$ is Lindelöf?

Unfortunately, the answers to these questions for general spaces do not exist. In the following chapter, we will take a look at specific spaces in regards to the above mentioned questions.

For the theorems we will be discussing in the remaining chapters, we will often need to leave ZFC and explore other models of set theory. We begin by mentioning a consequence of the Continuum Hypothesis, namely, *Luzin's Axiom* which states that $2^{\omega_1} > \mathfrak{c}$.

The following theorem and corollary were proven by Tkachenko and Tkachuk and can be found in [11] with complete proofs in [8].

Theorem 3.0.13. If X is Lindelöf and $w(X) < \mathfrak{c}$, then under Luzin's Axiom every compact continuous image of $C_p(X)$ is metrizable.

Corollary 3.0.14. Under Luzin's Axiom, if X is a Lindelöf first countable space, then every compact continuous image of $C_p(X)$ is metrizable.

In ZFC, the above theorem and corollary is not obtained, even if the conditions are changed to include a compact first countable space.

Next we look at a theorem which can be found in [9].

Theorem 3.0.15. $t(C_p(X)) = \sup\{l(X^n) : n \in \mathbb{N}\}$. In particular, the tightness of $C_p(X)$ is countable if and only if X^n is a Lindelöf space for any $n \in \mathbb{N}$.

This theorem is the motivation behind a question first asked by Arhangel'skii in [2].

Question. Let X be a Lindelöf space. Is it true that $C_p(X)$ condenses onto a space of countable tightness?

Going back to the second question presented in the chapter, we have the following two theorems proved by Okunev [6] and Reznichenko [1], respectively. Notice that both theorems are not ZFC, and instead $\text{MA} + \neg\text{CH}$ which is taken to be Martin's Axiom with the negation of the Continuum Hypothesis.

Definition 3.0.16. A space X is said to be ω -monolithic if $nw(\overline{A}) \leq \omega$ for every $A \subseteq X$ such that $|A| \leq \omega$.

Theorem 3.0.17. ($\text{MA} + \neg\text{CH}$) If X^n is Lindelöf, for each $n \in \omega$, then every compact subspace of $C_p(X)$ is ω -monolithic.

Theorem 3.0.18. ($\text{MA} + \neg\text{CH}$) If X is compact, and $C_p(X)$ is Lindelöf, then $C_p(X)$ is ω -monolithic.

This section will end with a conjecture made by Arhangel'skii in [2].

Conjecture. If X is compact, and $C_p(X)$ is Lindelöf, then $C_p(X) \times C_p(X)$ is also Lindelöf.

Chapter 4

Lindelöfness of $C_p(X)$ Examples

In the previous chapter, a general look was taken at two specific open questions originally asked by Arhangel'skii. In this chapter, we will consider more specific questions by putting constraints on the space X . The three spaces we will look at occur when X has a single non-isolated point, when X is a Michael space, and finally, when X is a Mrówka space [5].

4.1 X has a Single Non-Isolated Point

In this section, we take a look at the case when X is a space with all but one of its points isolated. We begin by trying to find properties of this space X that will make $C_p(X)$ Lindelöf. We begin with the following theorem and corollary proven by Malykhin and Leiderman in [4].

Theorem 4.1.1. Let X be a space with a single non-isolated point. The following properties are equivalent:

1. X is Lindelöf and for any finite n the tightness of X^n is countable.
2. $C_p(X)$ is Lindelöf.
3. $C_p(X, M)$ is Lindelöf for any separable metrizable space M .

A corollary to the preceding theorem considers the countable product of $C_p(X)$.

Corollary 4.1.2. Let X be a topological space with a single non-isolated point. If $C_p(X)$ is Lindelöf, then the countable product $C_p(X)^\omega$ is also Lindelöf.

Finally, we consider the possibility of $C_p(X) \times C_p(Y)$ where X and Y are different spaces. To do so, we use different models of set theory besides ZFC.

The following theorem assuming the model of ZFC obtained by adding one Cohen real can also be found in [4].

Theorem 4.1.3. There are two spaces with a single non-isolated point X and Y such that both $C_p(X)$ and $C_p(Y)$ are Lindelöf but the product $C_p(X) \times C_p(Y)$ is not Lindelöf.

Our final theorems for this space involve the PFA or proper forcing axiom model of set theory. Both theorems are attributed to Todorćević and can be found in [5].

Theorem 4.1.4. (PFA) Let X and Y be two spaces with at most one non-isolated point. If the spaces $C_p(X)$ and $C_p(Y)$ are Lindelöf, then the product $C_p(X) \times C_p(Y)$ is Lindelöf.

Theorem 4.1.5. (PFA) Let X be a space with a single non-isolated point. The following properties are equivalent:

1. X is Lindelöf and $t(X)$ is countable.
2. $C_p(X)$ is Lindelöf.

4.2 X is a Michael Space

The following specific case that we will examine considers a space X which is a Michael space.

Definition 4.2.1. Let X be a topological space and $A \subseteq X$. Let $X_{(A)}$ be obtained by retaining the topology at each point of A and by taking the points of $X \setminus A$ isolated. Then $X_{(A)}$ is referred to as a *Michael space*.

The following are a few properties of Michael spaces that are required before considering questions of Lindelöfness in $C_p(X)$.

Proposition 4.2.2. Let A and B be two subsets of a topological space X . Then, there exists a one-to-one mapping $i : X_{(A)} \rightarrow X_{(B)}$, where i is the identity mapping on X . If $A \subseteq B$, this mapping is continuous.

Proposition 4.2.3. The diagonal in $X_{(A)} \times X_{(B)}$ is closed and homeomorphic to $X_{(A \cap B)}$.

The following definitions are necessary to understand the findings for Michael spaces.

Definition 4.2.4. Let X be a topological space and $A \subseteq X$. Then, A is a *holding* if the countable power $(X_{(A)})^\omega$ is Lindelöf.

Definition 4.2.5. A separable completely metrizable topological space is called a *Polish space*. Equivalently, a Polish space is a topological space which is homeomorphic to a complete metric space with a countable dense subset.

In [7], Okunev and Tamano proved the following:

Theorem 4.2.6. If A is a set in a Polish space X such that $|X \setminus A| < 2^\omega$, then A is a holding.

Theorem 4.2.7. Let X be an uncountable Polish space. Then there exists a family of pairwise disjoint holding subsets $\{A_\alpha : \alpha < 2^\omega\}$.

The constructions of the previous two theorems would lead Okunev and Tamano to the following results:

Theorem 4.2.8. There exist separable, scattered, σ -compact spaces X and Y such that $C_p(X)^\omega$ and $C_p(Y)^\omega$ are Lindelöf but the product $C_p(X) \times C_p(Y)$ is not Lindelöf.

Theorem 4.2.9. (CH) Let $D = \{0, 1\}$ denote the discrete space consisting of two points. There is a separable, scattered, σ -compact space X such that for any finite n $C_p(X, D)^n$ is Lindelöf, but $C_p(X, D)^\omega$ is not Lindelöf.

4.3 X is a Mrówka Space

The final example we will look at in this chapter revolves around $C_p(X)$ when X is a Mrówka space. To begin, we recall the following definitions.

Definition 4.3.1. Two countable infinite sets are said to be *almost disjoint* if their intersection is finite.

Definition 4.3.2. A *maximal almost disjoint family* is a maximal element in the family of all almost disjoint families ordered by inclusion.

With the previous definitions in mind, we are able to define a Mrówka space.

Definition 4.3.3. Let \mathcal{A} be a maximal family of pairwise, almost disjoint, infinite subsets of \mathbb{N} . We place a topology on the union $\mathbb{N} \cup \mathcal{A}$ by letting sets of the form

$$U_n(A) = (A \setminus \{1, 2, \dots, n\}) \cup \{A\}, \quad n \in \mathbb{N},$$

be basic neighborhoods of $A \in \mathcal{A}$ and declaring points of \mathbb{N} to be isolated. The resulting space $\mathbb{N} \cup \mathcal{A}$ is called a Ψ -space or *Mrówka space* and is denoted $\Psi(\mathcal{A})$.

Some properties of the space $X = \Psi(\mathcal{A})$ are considered below.

Proposition 4.3.4. For every almost disjoint family \mathcal{A} , the space $X = \Psi(\mathcal{A})$ is a zero-dimensional, locally compact, first countable space. Additionally, \mathcal{A} is a closed discrete subspace of $\Psi(\mathcal{A})$ and ω is dense.

Theorem 4.3.5. $\Psi(\mathcal{A})$ is not normal if \mathcal{A} is a maximal almost disjoint family.

An important result with Mrówka spaces was proven in a 2005 paper by Hrusak, Szeptycki and Tamariz- Mascarua [3]. The following theorem is stated assuming CH.

Theorem 4.3.6. There is a maximal almost disjoint family \mathcal{A} such that for $X = \Psi(\mathcal{A})$ the finite power $C_p(X, D)^n$ is Lindelöf for every finite n , but the countable product $C_p(X, D)^\omega$ is not normal.

Chapter 5

$C_p(X)$ and Metacompactness

In this chapter we will look at the spaces $C_p(X)$ and the property of metacompactness. Several open questions on the topic are presented in this section. However, despite the numerous questions presented on this subject, there are very few answers available to date. We begin with some of these unanswered questions.

Question. Does metacompactness of $C_p(X)$ imply that $C_p(X)$ is Lindelöf?

This question, first asked by Arhangel'skii in [1], is often considered to be a part of C_p -theory folklore. While we do not have a complete answer, there is a potential one due to a theorem by Reznichenko [2]. First we begin with a definition.

Definition 5.0.7. A space X is called *collectionwise normal* if, for any discrete $\{F_t : t \in T\}$ of closed subsets of X , there exists a discrete family $\{U_t : t \in T\}$ of open subsets of X such that $F_t \subseteq U_t$ for each $t \in T$.

Theorem 5.0.8. If the space $C_p(X)$ is normal, then it is collectionwise normal.

This previous theorem, gives an important result [1]:

Theorem 5.0.9. If $C_p(X)$ is normal and metacompact, then $C_p(X)$ is Lindelöf.

The next question follows naturally from the above, and questions the necessity of $C_p(X)$ being normal [10].

Question. (open reference) Suppose that $C_p(X)$ is metacompact. Must it be Lindelöf?

In a contradiction to the conjecture from [2] stated at the end of Chapter 3, Tkachuk in [11] believes that there are counterexamples that exist to that conjecture. He does however, believe that the Lindelöf property of $C_p(X)$ effects $C_p(X) \times C_p(X)$ as illustrated in the question below.

Question. Suppose that $C_p(X)$ is Lindelöf. Must $C_p(X) \times C_p(X)$ be metacompact?

The following question naturally follows from the preceding question, but first we must define metalindelöf [10].

Definition 5.0.10. A space X is *metalindelöf* if every open cover \mathcal{U} of X has a point-countable open refinement.

Question. Suppose $C_p(X)$ is Lindelöf. Must $C_p(X) \times C_p(X)$ be metalindelöf?

Finally, we end with an open question that involves putting more constraints on the space X [10].

Question. Suppose that X is compact $C_p(X)$ is Lindelöf. Must the space $C_p(X) \times C_p(X)$ be metacompact?

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