Existence and Uniqueness of Solutions to Positive Bounded Below Operator Equations

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Abstract

We investigate solving a differential equation boundary value problem. Using "variational" or "energy" methods, we transform the problem into one of minimizing the value of a certain functional expression involving a definite integral. Finally, we show the existence and uniqueness of solutions to positive bounded below operator equations.

Contents

Introduction

The purpose of this thesis is to show the existence and uniqueness of solutions to positive bounded below operator equations. The operator equations studied here are applicable to ordinary and partial differential equation boundary value problems arising in solid mechanics.

Problems of this type were analyzed in detail by Soviet mathematicians during the 1930's and 1940's. One of the most notable works of the Soviet school is that of S.G. Mikhlin's *Variation Methods in Mathematical Physics*. [3]

The theory described in this thesis helped lay the foundation for several approximate solution procedures that are known as "variational" or "energy" methods. The theory transforms the problem of solving a differential equation boundary value problem into the problem of minimizing the value of a certain functional expression involving a definite integral. Variational approaches to solving boundary value problems were instrumental in the development of finite element methods for approximating the solution of boundary value problems that arise in structural analysis and a variety of other areas of engineering analysis.

Fundamental Notions

Definition 1: A **vector space V** is said to be **normed** if we can assign to every element x of V a value $\|x\|$, the norm of x, with the following properties.

1) $\|x\| \geq 0$
2) $\|x\| = 0 \; iff \; x = 0$
3) $\|kx\| = |k|\|x\|$ for any scalar k
4) $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in V$

Definition 2: A sequence of vectors $\{\Phi^k\}$ in a normed vector space V is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there exists an integer $\boldsymbol{N}$ (which may depend on ε) $\ni$ $\|\Phi^n - \Phi^m\| < \varepsilon$, whenever $n > N, m > N$.

Definition 3: A normed vector space V is said to be a **complete vector space** if every Cauchy sequence in V has a limit in V. Complete normed vector spaces are also called Banach spaces.

Definition 4: Let V be a real vector space,. Suppose to each pair of vectors $u, v \in V$ there is assigned a real number, denoted by $\langle u, v \rangle$. This function is called a (real) inner product on V, if it satisfies the following axioms.

(i.) Bilinearity property: $\langle au_1 + bu_2, v \rangle = a\langle u_1,v \rangle + b\langle u_2,v \rangle$.

Also, $(u, cv_1 + dv_2) = c(u, v_1) + d(u, v_2)$

(ii.) Symmetry property: $\langle u, v \rangle = \langle v, u \rangle$

(iii.) Positive definite property: $\langle u, u\rangle \geq 0, and\ \langle u, u\rangle = 0 \text{ iff } u = 0$

The vector space with a (real) inner product is called a (real) **inner product space**.

Definition 5: An inner product space in which $\|\boldsymbol{u}\| = (\boldsymbol{u}, \boldsymbol{u})^{1/2}$ for all $\boldsymbol{u}$ in the space is a Euclidean space.

Definition 6: A **Hilbert Space** is a complete Euclidean space.

An example of a Hilbert space is the space $L_2(\Omega)$ of square integrable real fuctions of a single variable with inner product $(f, g) = \int_\Omega f(x)g(x)\, dx_1 \dots dx_n$ and norm $\|f\|^2 = \int_\Omega [f(x)]^2\, dx$. (See the appendix for the inner product and norm of vector valued L_2 functions.) It is shown in many standard references (see, for example, [6]) that $L_2(\Omega)$ is a complete Euclidean space. (It is proved in Theorem 3 of this thesis that every normed vector space "can be completed".)

Definition 7: Let V be a vector space and S be a subset of V; i.e. if $x \in S$, then $x \in V$. If S is a vector space with the same operations as those in V, then S is called a **subspace** of V.

Theorem 1: (Cauchy-Schwartz Inequality) Let x and y be any vectors in a real Euclidean space V. Then

$$|(\boldsymbol{x}, \boldsymbol{y})| \leq \|\boldsymbol{x}\|\|\boldsymbol{y}\|.$$

Proof:

Since the norm is defined by the inner product, for any real number α,

$$0 \le \|\boldsymbol{x} + \alpha \boldsymbol{y}\|^2 = (\boldsymbol{x} + \alpha \boldsymbol{y}, \boldsymbol{x} + \alpha \boldsymbol{y})$$

$$= \|\boldsymbol{x}\|^2 + 2\alpha(\boldsymbol{x}, \boldsymbol{y}) + \alpha^2 \|\boldsymbol{y}\|^2 \equiv g(\alpha)$$

which must hold for all real α. If the discriminant of the quadratic expression $g(\alpha)$ is negative, then quadratic equation $g(\alpha) = 0$ has no solution. If the discriminant is zero, then $g(\alpha)$ has exactly one root. Therefore, $g(\alpha) \ge$ implies

$$[2(\boldsymbol{x}, \boldsymbol{y})]^2 - 4\|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|^2 \le 0.$$

Hence,

$$4(\boldsymbol{x}, \boldsymbol{y})^2 - 4\|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|^2 \le 0, (x, y)^2 \le \|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|^2, \quad \text{and}$$

$$|(\boldsymbol{x}, \boldsymbol{y})| \le \|\boldsymbol{x}\| \|\boldsymbol{y}\| .$$

If f and g are elements of the Hilbert space $L_2(\Omega)$, then $|(f, g)| \le \|f\| \|g\|$, or

$$\left| \int_\Omega f(x) g(x) \; dx \right| \le \sqrt{\int_\Omega [f(x)]^2 \, dx \bullet \int_\Omega [g(x)]^2 \, dx} \, .$$

Theorem 2: (Triangle Inequality) For any vectors x and y in a Euclidean space V, $\|\boldsymbol{x} + \boldsymbol{y}\| \le \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$.

Proof:

$$\|\boldsymbol{x} + \boldsymbol{y}\|^2 = (\boldsymbol{x} + \boldsymbol{y}, \boldsymbol{x} + \boldsymbol{y})$$

$$= \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + (\boldsymbol{x}, \boldsymbol{y}) + (\boldsymbol{y}, \boldsymbol{x})$$

$$\le \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2|(\boldsymbol{x}, \boldsymbol{y})|$$

$$\le \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\|\boldsymbol{x}\| \|\boldsymbol{y}\| = (\|\boldsymbol{x}\| + \|\boldsymbol{y}\|)^2$$

Therefore, $\|\boldsymbol{x} + \boldsymbol{y}\| \le \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$

If f and g are elements of $L_2(\Omega)$, the triangle inequality implies that

$$\left(\int_\Omega [f(x)+g(x)]^2\,dx\right)^{1/2} \le \left(\int_\Omega [f(x)]^2\,dx\right)^{1/2} + \left(\int_\Omega [g(x)]^2\,dx\right)^{1/2}$$

The Space L_2

Definition 8: The space of all square integrable functions on a domain Ω in $\boldsymbol{R}^n$ is denoted as $\boldsymbol{L_2}(\Omega)$.

$$u(x) \in L_2(\Omega) \;\Rightarrow \left[\int_\Omega \left(\sum_{i=1}^{m} u_i^z(x)\right) dx_1, \dots, dx_n\right]^{\frac{1}{2}} < \infty\},\quad \text{where}$$

$$\boldsymbol{u}(x) = [u_1(\boldsymbol{x}), \dots, u_m(\boldsymbol{x})]$$

The zero function in $L_2(\Omega)$ is defined to be the equivalence class of functions $V(x)$ such that $\|V\| = 0$.

Convergence in L_2 and Completeness of L_2

Theorem 3 [5] Every normed vector space V can be completed to form a complete normed vector space B such that V is dense in B. If V is Euclidean, B is a Hilbert space.

Proof:

To establish the theorem we will form a new space B and show that there is a subspace B_1 of B that has all the properties of V. We then identify B_1 with V.

If $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences of elements of V which satisfy $\lim_{n\to\infty}(x_n - y_n) = 0$, then the two sequences will be called equivalent. The class of all Cauchy sequences in V that are equivalent to $\{x_n\}$ we will denote by $[\{x_n\}]$. We now define the space B to be the set of all classes $[\{x_n\}]$. That is, B is a set of equivalence classes of Cauchy sequences.

In order to show that B is a vector space, we define our various vector operations in the following way.

1) $[\{x_n\}] = [\{y_n\}]$ if $\{x_n\}$ is equivalent to $\{y_n\}$.
2) $k[\{x_n\}] = [\{kx_n\}]$ for any scalar k
3) $[\{x_n\}] + [\{y_n\}] = [\{x_n + y_n\}]$.

It is not hard to confirm that these operations are independent of the sequences chosen. For example, suppose $\{x_n\}$ is equivalent to $\{x'_n\}$ and $\{y_n\}$ is equivalent to $\{y'_n\}$. Then

$$[\{x_n\}] = [\{x'_n\}]\ ; \quad [\{y_n\}] = [\{y'_n\}]\ ,$$

$$[\{x_n\}] + [\{y_n\}] = [\{x'_n\}] + [\{y'_n\}]$$

Therefore

$$[\{x_n\}] + [\{y_n\}] = [\{x'_n + y'_n\}]$$

and

$$[\{kx_n\}] = k[\{x'_n\}],$$

The zero vector in B will be denoted $\mathbf{0}$. It is defined to be $[\{\mathbf{0}\}]$ where $\{\mathbf{0}\}$ is the Cauchy sequence of vectors in V, all of whose elements are zero vectors. Therefore, B is a vector space.

Now for every element x in V, the sequence $\{x_n\}$ with $x_n = x$, for every n, is a Cauchy sequence. Consequently, if we set up the correspondence:

$$x \leftrightarrow [\{x_n\}], \quad x_n = x \tag{1}$$

we see that this sets up a one-to-one correspondence between elements of V and some subset B_1 of B. The correspondence maintains the operations of multiplication by a scalar and addition. That is, if

$$y \leftrightarrow [\{y_n\}], \tag{2}$$

then it follows that

$$x + y \leftrightarrow [\{x_n\}] + [\{y_n\}]$$

$$ky \leftrightarrow k[\{y_n\}]$$

If the norm on V is defined as an inner product norm, we will define the inner product of two elements of B by

$$([\{x_n\}], [\{y_n\}]) = \lim_{n\to\infty}(x_n, y_n). \tag{3}$$

That this limit does indeed exist can be seen as follows (using bilinearity of the inner product, the triangle inequality, and the Cauchy-Schwartz inequality):

$$\left|(x_i, y_i) - (x_j, y_j)\right| \leq \left|(x_i, y_i - y_j)\right| + \left|(x_i - x_j, y_j)\right| \leq \|x_i\|\left\|y_i - y_j\right\| + \left\|x_i - x_j\right\|\left\|y_j\right\|.$$

Since the sequences $\{x_i\}$ and $\{y_i\}$ are Cauchy sequences, it follows that the right-hand side tends to zero as i and j tend to infinity. The limit exists because Cauchy sequences of real numbers converge, and the limit is independent of the sequences chosen from $[\{x_i\}]$ and $[\{y_i\}]$. We can show, too, that the operation of taking the inner product is preserved under the correspondence (1), and we have that

$$([\{x_n\}], [\{y_n\}]) = \lim_{n\to\infty}(x_n, y_n) = (x, y)\,.$$

If the norm in V is not defined in terms of an inner product, we define instead,

$$\|[\{x_n\}]\| = \lim_{n\to\infty}\|x_n\| \tag{4}$$

The existence of this limit follows from the observation that

$$\|x_n\| = \|x_m + x_n - x_m\| \leq \|x_m\| + \|x_n - x_m\|.$$

Also,

$$\|x_m\| = \|x_n + x_m - x_n\| \leq \|x_n\| + \|x_m - x_n\|.$$

Therefore,

$$|\|x_n\| - \|x_m\||^2 \;\leq\; \|x_n - x_m\|^2$$

Since $\{x_n\}$ is a Cauchy sequence, the existence of the limit follows. A similar inequality establishes that (4) is independent of the choice of sequence from $[\{x_n\}]$. To see this, suppose that $[\{x_n\}] = [\{y_n\}]$. Then

$$0 = \lim_{n\to\infty} \|x_n - y_n\| \geq \lim_{n\to\infty} |\|x_n\| - \|y_n\||\,,$$

which implies that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\|\,.$$

Also we see that the operation of taking this norm is preserved under the correspondence (1) since $x_n = x$ implies

$$\|[\{x_n\}]\| = \lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|x\| = \|x\|.$$

The norm just defined satisfies the requirements expected of a norm. For example, the triangle inequality follows from the following observation.

$$\|[\{x_n\}] + [\{y_n\}]\| = \lim_{n\to\infty} \|x_n + y_n\| \leq \lim_{n\to\infty} \|x_n\| + \lim_{n\to\infty} \|y_n\|$$

Therefore, we have shown that B_1 has exactly the same properties as V, and we say that V is embedded in B.

We now show that B_1 is dense in B. To this end let $[\{x_n\}]$ be any element in B. Define $z_{ij} = x_j$ for every i. Therefore, for fixed j, $[\{z_{ij}\}]$ is in B_i. We shall show that

$$[\{x_n\}] = \lim_{j\to\infty} [\{z_{ij}\}]$$

and so establish that B_1 is dense in B.

For any fixed j, the sequence $\{x_i - x_j\}$ is a Cauchy sequence. We have shown, above, that the limit of norms of the elements of a Cauchy sequence exists. Therefore

$$\lim_{i\to\infty}\|x_i - x_j\| < \varepsilon \quad \text{for} \quad j \geq N \quad \text{exists.}$$

That is

$$\|[\{x_i\}] - [\{z_{ij}\}]\| = \|x_i - z_{ij}\| < \varepsilon \quad \text{for} \quad j \geq N,$$

and we have established that B_1 is dense in B.

Finally, we must show that B is complete. Let $[\{x_{ni}\}], i = 1,2, \ldots$, be a Cauchy sequence in B. Then if $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that when $i, j \geq N(\epsilon)$,

$$\|[\{x_{ni}\}] - [\{x_{nj}\}]\| = \lim_{n\to\infty}\|x_{ni} - x_{nj}\| \leq \varepsilon.$$

Therefore, for each $i \geq N(\varepsilon)$ and $j \geq N(\varepsilon)$ there is an integer $M(i,j)$ such that

$$\|x_{ni} - x_{nj}\| \leq 2\varepsilon \quad \text{for} \quad n \geq M(i,j).$$

Since, by definition, $\{x_{ni}\}$ is a Cauchy sequence for fixed i, there is a number $Q(i) > i$ such that

$$\|x_{ni} - x_{mi}\| < \varepsilon \quad \text{for} \quad n \geq Q, m \geq Q.$$

The sequence $\{y_i\} = \{x_{Q(i)i}\}$ is also a Cauchy sequence because whenever $i \geq N(\varepsilon), j \geq N(\varepsilon)$ we can always choose a number $m \geq \max\{Q(i), Q(j), M(i,j)\}$ which will ensure that

$$\|x_{Q(i)i} - x_{Q(i)j}\| \leq \|x_{Q(i)i} - x_{mi}\| + \|x_{mi} - x_{mj}\| + \|x_{mj} - x_{Q(j)j}\|$$

$$\leq \ \varepsilon + 2\varepsilon + \varepsilon = 4\varepsilon.$$

The completeness of B will be established if we can show that

$$[\{y_n\}] = \lim_{i\to\infty}[\{x_{ni}\}].$$

This requires that we show, for $\varepsilon > 0$ and sufficiently large i, that:

$\|[\{y_n\}] - [\{x_{ni}\}]\| = \lim_{n\to\infty} \|y_n - x_{ni}\| \leq \varepsilon.$

Fix $i \geq N(\varepsilon),\ \ n > Q(i),\ \text{and}\ \ j > \max\{Q(n), Q(j), M(n,i)\}$, then

$$\begin{aligned}\|y_n - x_{ni}\| &= \|x_{Q(n)n} - x_{ni}\| \\ &\leq \|x_{Q(n)n} - x_{jn}\| + \|x_{jn} - x_{ji}\| + \|x_{ji} - x_{ni}\| \\ &\leq \varepsilon + 2\varepsilon + \varepsilon = 4\varepsilon.\end{aligned}$$

Therefore for $i \geq N(\varepsilon),\ \lim_{n\to\infty}\|y_n - x_{ni}\| \leq 4\varepsilon.$

Functionals and Linear Operators

Definition 9: Let H be a Hilbert Space and D_T be a subset of H. If to each u in D_T there corresponds a real number $T(u)$, then T is called a (real) **functional** with domain D_T.

Definition 10: A functional T is a **bounded functional** if there exists a positive number *c* such that

$$|T(u)| \leq c\|u\|$$

for all u in D_T. The smallest *c* for which this inequality holds is the **norm** of *T*, denoted as $\|T\|$, and given by

$$\|T\| = \sup \frac{|T(u)|}{\|u\|};\ u \in D_T; u \neq 0$$

Hence, $|T(u)| \leq \|T\|\|u\|$. A functional T is **continuous at u_0** if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $\|u_0 - u\| < \delta$, then $|T(u_0) - T(u)| < \epsilon$. If T is continuous at each point of D_T, then T is **continuous on D_T**.

Definition 11: A functional T is said to be a **linear functional** if

1. its domain D_T is a linear (vector) space
2. $T(u^1 + u^2) = T(u^1) + T(u^2)$
3. $T(\alpha u) = \alpha T(u) \quad \forall\, u^1, u^2 \in D_{T,} \forall\, \alpha \in R$

(Note if T is linear, then $T(0) = 0$)

Theorem 4: If a linear functional T is continuous at 0, then T is continuous on its entire domain D_T.

Proof:

Let u be in the domain of T. Since T is continuous at **0**, there is a $\delta > 0$ such that if $\|u_n - u\| < \delta$, then $|T(u_n - u)| < \epsilon$. By the linearity of *T*, we have $\|u_n - u\| < \delta$ implies that $|T(u_n) - T(u)| < \epsilon$. Hence *T* is continuous at *u*.

Theorem 5: [2] A linear functional is continuous if and only if it is bounded.

Proof:

Suppose T is bounded. Let $\varepsilon > 0$, and let $\delta = \frac{\varepsilon}{\|T\|}$. Then $\|u\| < \delta$ implies

$$|T(u)| \leq \|T\|\|u\| < \|T\| \frac{\varepsilon}{\|T\|} = \epsilon.$$

Therefore, *T* is continuous at **0**. Now, suppose T is continuous at **0** and T is not bounded. Then for each natural number n, there is a vector u_n such that $|T(u_n)| \geq n\|u_n\|$. The sequence $\{v_n\}$, where $v_n = \frac{u_n}{n\|u_n\|}$, approaches zero as $n \to \infty$. We note that

$$T(v_n) = \frac{1}{n\|u_n\|} T(u_n) \geq 1 \quad \text{for all } n,$$

contradicting the continuity of T at zero. Therefore, T must be bounded.

Example: Let *H* be a Hilbert space and let *f* be an element of *H*. Now define the functional *T* by

$$T(u) = (f, u)$$

for all *u* in H. It follows from the bilinearity of the inner product that *T* is linear. By the Cauchy-Schwartz Inequality,

$$|T(u)| = |(f, u)| \leq \|f\|\|u\| \ .$$

We will show that $\|T\| = \|f\|$. By the definition of bounded functional, $\|T\| \leq \|f\|$.

Suppose that $\|T\| < \|f\|$. Letting $u = f$, we have

$$T(f) = (f, f) = \|f\|\|f\| \geq \|T\|\|f\|,$$

which is a contradiction. Hence, $\|T\| = \|f\|$. By Theorem 5, *T* is also continuous.

Theorem 6: [2] (Riesz Representation) Every continuous linear functional T on a Hilbert space H can be expressed in the form $T(u) = (u, f)$, where f is in H. Furthermore, f is unique.

Proof:

Let N be the null space of T; i.e., $N = \{u: T(u) = 0\}$. Then N is a closed subspace of H. If $N = H$, then select $f = 0$. If $N \neq H$, then write $H = N + N^{\perp}$, where $N^{\perp}$ is the set of all vectors that are orthogonal to each vector in N. Since $N \neq H, N^{\perp}$ contains a nonzero element; say, f_0. By normalization, set $\|f_0\| = 1$. Let $v = T(u)f_0 - uT(f_0)$, where u is arbitrary. Clearly, v is in N, since $T(v) = T(u)T(f_0) - T(f_0)T(u) = 0$. Thus, with f_0 in $N^\perp$ and v in N, we have $(f_0, v) = 0$.

Now,

$$(v, f_0) = (T(u)f_0 - uT(f_0), f_0) = (T(u)f_0, f_0) - (T(f_0)u, f_0)$$
$$= T(u)(1) - T(f_0)(u, f_0)$$
$$= 0$$

Therefore, for any u in H,

$$T(u) = T(f_0)(u, f_0) = (u, T(f_0)f_0)$$

Therefore,

$$T(u) = (u, f) \text{ for all } u \text{ in } H, \text{ with } f = T(f_0)f_0.$$

To prove uniqueness, suppose $T(u) = (u, f) = (u, g)$, for all u in H. Then $(u, f - g) = 0$ for all u. In particular, for $= f - g$, we obtain $\|f - g\|^2 = 0$, which implies $f = g$.

<u>Definition 12:</u> Let V be a vector space and A an operator that assigns to each element u in a linear subspace D_A of V a vector g (i.e. $A_u = g$) such that for any $u, v \in D_A$ and $\alpha, \beta \in R$,

$A(\alpha u + \beta v) = \alpha Au + \beta Av$. Then A is a **linear operator** with domain D_A and range R_A.

That is,

$$R_A = \{g: Au = g, \text{ for every } u \in D_A\}.$$

<u>Definition 13:</u> A linear operator A with Domain D_A is said to be a **bounded linear operator** if there exists a real number c such that

$$\|Au\| \le c\|u\|, \qquad \forall\, u \in D_A$$

(the smallest c that satisfies the above inequality is called the norm of the linear operator A, denoted $\|A\|$. Then

$$\|A\| = \sup \frac{\|Au\|}{\|u\|} \quad u \in D_A, u \neq 0$$

A linear operator is continuous if and only if it is bounded. Further, if it is continuous at 0, then it is continuous on the entire domain. (See [6])

<u>Definition 14:</u> A linear operator A, with domain D_A and range R_A that are subspaces of a Hilbert space H, is said to be a **symmetric linear operator** if

$$(Au, v) = (u, Av) \text{ for all } u \text{ and } v \text{ in } D_A$$

<u>Definition 15:</u> Let A be a symmetric linear operator that is defined on a dense subspace D_A of a Hilbert Space H, with range in H. Then A is said to be a **positive definite linear operator** if $(Au, u) \ge 0$ and $(Au, u) = 0$ if and only if $u = 0$. A is <u>positive bounded below</u> if there exists a constant $c > 0$ such that $(Au, u) \ge c(u, u) = c\|u\|^2$ for all u in D_A.

<u>Example: Positive Bounded Below Operator</u>

Let $Au = -\frac{d^2u}{dx^2}$, where $D_A = \{u(x) \in C^2[0,1] |\, u(0) = u(1) = 0\}$.

To show that A is symmetric, we note that $(Au, v) = \int_0^1 v \frac{d^2u}{du^2} dx$, and integrate by parts twice, making use of the boundary conditions on u and v.

Next we show that A is positive bounded below. For each u (using integration by parts),

$$(Au, u) = \int_0^1 u'^2 \, dx$$

$u(x) = \int_0^x u'(\xi)\, d\xi$. The Cauchy-Schwartz inequality

implies that $[u(x)]^2 = \left(\int_0^x \underbrace{1}_{f} \cdot \underbrace{u'(\xi)}_{g} \, d\xi \right)^2 \leq$

$x \int_0^x {u'}^2(\xi)\, d\xi = x\|u'\|^2.$

Therefore

$\int_0^1 [u(x)^2]\, dx \ \leq \int_0^1 x\|u'\|^2\, dx$, which implies

$$\|u\| \ \leq \frac{1}{\sqrt{2}} \|u'\|.$$

Therefore,

$(Au, u) = \|u'\|^2 \ \geq \ \sqrt{2}\|u\|$

Therefore, A is positive bounded below.

Convergence in Energy

Definition 16: If A is a positive definite linear operator, the quantity (Au, v) defines an inner product on D_A called the **energy inner (or scalar) product** denoted by

$$[u, v]_A \equiv (Au, v) = \int_\Omega vAu \, d\Omega$$

To show this defines an inner product, note first that for any real α,

$[\alpha u, v]_A = (A\alpha u, v) = (\alpha Au, v) = \alpha[u, v]_A$ and

$[u_1 + u_2, v]_A = [A(u_1 + u_2), v] = [Au_1, v] + [Au_2, v] = [u_1, v]_A + [u_2, v]_A$

Next, since A is symmetric

$$[u, v]_A = (Au, v) = (u, Av) = [v, u]_A.$$

Finally, since A is positive definite

$$[u, v]_A = (Au, u) > 0, u \neq 0 \text{ and } [u, v] = 0 \Leftrightarrow u = 0.$$

The quantity $\|u\|_A \equiv (Au, u)^{1/2}$ is called the **energy norm** of the function u.

Definition 17: Let u and u_n be in $D_A, n = 1,2, \ldots$ for a positive definite operator A. The sequence u_n is said to **converge in energy** to u if $\lim_{n\to\infty} \|u_n - u\|_A = 0$, denoted by $u_n(x) \overset{E}{\rightarrow} u(x)$.

Theorem 7: [2] If the operator A is positive bounded below and the sequence $\{u_n\}$ in D_A converges to *u* in energy, then $\lim_{n\to\infty} \|u_n - u\| = 0$.

Proof:

There is a positive number *c* such that $(Au, u) \geq c\|u\|^2$ for all u in D_A. Therefore, $\|u\|^2 \leq \frac{1}{c}\|u\|_A^2$. Therefore, $\|u_n - u\|^2 \leq \frac{1}{c}\|u_n - u\|_A^2$. Therefore, if $\|u_n - u\|_A \rightarrow 0$, then $\|u_n - u\| \rightarrow 0$.

The Minimum Principle for Operator Equations

Let A be a positive definite linear operator and consider the problem of finding a solution u of

$$Au = f \qquad (5)$$

By this is meant, find a function u in D_A that satisfies this equation. The fact that u is in D_A implies that it satisfies the boundary conditions of the problem. In our problem, the boundary conditions are homogeneous. Thus D_A is a dense subspace of $L_2(\Omega)$, where Ω is a bounded, open subset of $\boldsymbol{R}^n$. Also, f belongs to $L_2(\Omega)$. For example, let $A = -\frac{d^2u}{dx^2}$ and

$$D_A = \{u(x) \in C^2[0,1] |\, u(0) = u(1) = 0\}.$$

Theorem 8: [2] (Minimum Functional Theorem**)** Let A be a positive definite linear operator with domain $D_A$that is dense in $L_2(\Omega)$. If $Au = f$ has a solution, then the solution is unique and minimizes the energy functional

$$F(u) = (Au, u) - 2(u, f) = \int_{\Omega} (u\, Au - 2uf)\, dx \qquad (6)$$

over all $u \in D_A$. Conversely, if there exists a function u in D_A that minimizes $F(u)$, then it is the unique solution of $Au = f$.

Proof:

(i.) (uniqueness) Assume there are two solutions, u and v, of $Au = f$ in D_A. Then, $u - v$ satisfies $A(u - v) = 0$. Thus, $\big(A(u - v), (u - v)\big) = 0$

Since A is positive definite, $u - v = 0$. Hence, $u = v$.

(ii.) Next let u_0 in D_A so that $Au_0 = f$. Substitution for f in $F(u)$ yields

$$\begin{aligned} F(u) &= (Au, u) - 2(Au_0, u) \\ &= [u, u]_A - 2[u_0, u]_A \\ &= [u - u_0, u - u_0]_A - [u_0, u_0]_A \\ &= \|u - u_0\|_A^2 - \|u_0\|_A^2 \end{aligned}$$

It is clear that $F(u)$ assumes its minimum value if and only if $u = u_0$.

(iii.) Now suppose that there exists a function u_0 in D_A that minimizes the functional F. Let $v(x)$ be an arbitrary function from D_A and let α be an arbitrary real number. Then $F(u_0 + \alpha v) - F(u_0) \geq 0$. Using symmetry of the operator A, we obtain

$F(u_0 + \alpha v) - F(u_0) = 2\alpha(Au_0 - f, v) + \alpha^2(Av, v) \geq 0$. As a function of α, the function $F(u_0 + \alpha V) - F(u_0)$ takes on its minimum value of zero at $\alpha = 0$. Thus, its derivative with respect to α at $\alpha = 0$ must be zero; i.e., $2(Au_0 - f, v) = 0$, for all v in D_A. Since D_A is dense in $L_2(\Omega)$ it follows that $Au_0 - f = 0$.

The minimum functional theorem provides a rigorous proof of the principle of minimum total potential energy and allows the problem of solving a differential equation under specified boundary conditions to be replaced by the problem of seeking a function that minimizes the functional F in the preceding theorem.

Definition 18: Let A be a positive bounded below operator with domain D_A. (D_A is dense in L_2.) The **energy space** H_A is the completion of D_A in the energy norm. That is, every element of D_A is an element of H_A and every sequence of vectors in D_A that is Cauchy in the energy norm, converges in the energy norm to a point of H_A. We may also refer to H_A as the "space of functions of finite energy."

The preceding definition makes sense by virtue of Theorem 3, proved earlier. Also, from the proof of Theorem 3 we can define the inner product and norm on H_A in the following way. Given vectors u and v in H_A, there are Cauchy sequences $\{u_n\}$ and $\{v_n\}$ in D_A such that u and v are the respective limits. Then

$$\|u\|_A = \lim_{n\to\infty}\|u_n\|_A \tag{7}$$

$$[u, v]_A = \lim_{n\to\infty}[u_n, v_n]_A \tag{8}$$

We note that H_A is, by definition, a Hilbert space. The next theorem tells us that H_A is a subspace of L_2.

<u>Theorem 9:</u> Let *A* be a positive bounded below operator, defined on D_A, where D_A is a dense subspace of a Hilbert space H. Let H_A be the completion of D_A in the energy norm (as per Definition 18). Then $H_A \subseteq H$.

<u>Proof:</u>

Let u be a vector in H_A. Then there is a sequence $\{u_n\}$ in D_A such that u_n converges to *u* in the energy norm. By Theorem 7, u_n converges to u in the norm of *H*. Hence, *u* is an element of *H*.

<u>Definition 19:</u> For a positive bounded below operator A on a domain D_A, the energy functional $F(u) = (Au, u) - 2(u, f)$ can be extended to all of the energy space H_A as

$$F(u) = \|u\|_A^2 - 2(u, f).$$

We will now prove an important existence and uniqueness result.

Theorem 10: [2] In the energy space H_A of a positive bounded below operator A, for each f in $L_2(\Omega)$, there exists one and only one element for which the energy functional attains a minimum.

Proof:

Since A is positive bounded below, $\|u\|_A^2 = [Au, u] \geq c\|u\|^2$ for some $c > 0$ and all u in H_A.

Consider the linear functional $l(u) = (u, f)$, defined on $L_2(\Omega)$. Then $|l(u)| = |(u, f)| \leq \|u\|\|f\| \leq \sqrt{c}\|f\|\|u\|_A$, which implies that l is a continuous linear functional on H_A. By the Riesz representation theorem, there exists a unique element u_0 in H_A such that $(u, f) = [u, u_0]_A$, for all u in H_A. The energy functional F can now be written as $F(u) = \|u\|_A^2 - 2[u, u_0]_A = \|u - u_0\|_A^2 - \|u_0\|_A^2$, from which it follows that the minimum of F is attained at u_0.

Generalized Solutions of the Equation $Au = f$

Definition 20: We will say that u_0 is a generalized solution to $Au = f$, if u_0 is in H_A and $(u_0, v)_A - (f, v) = 0$ for every v in H_A.

We can now establish existence and uniqueness of generalized solutions to the positive bounded below operator equation $Au = f$.

Theorem 11: In the energy space H_A of a positive bounded below operator A, for each f in $L_2(\Omega)$, there exists a unique generalized solution to $Au = f$.

Proof:

(i.) (uniqueness) Let u_0 and v_0 both be generalized solutions to $Au = f$. Then $(u_0, v)_A - (v_0, v)_A = 0$ and $(u_0 - v_0, v)_A = 0$ for all v in H_A. Therefore, $u_0 = v_0$ (by the lemma in the appendix).

(ii.) (existence) By Theorem 10, there is an element u_0 in H_A that minimizes the energy functional $F(u) = \|u\|_A^2 - 2[u, f]$. Now, let v be any arbitrary element of H_A. Then, $F(u_0 + \alpha v) - F(u_0) \geq 0$. Using the symmetry of A we have, for any real α,

$F(u_0 + \alpha v) - F(u_0) = \|u_0 + \alpha v\|_A^2 - 2[u_0 + \alpha v, f] - (\|u_0\|^2 - 2[u_0, f]) = [u_0 + \alpha v, u_0 + \alpha v]_A - 2\alpha[v, f] - [u_0, u_0]_A = \alpha^2[v, v]_A + 2\alpha[v, u_0] - 2\alpha[v, f] = \alpha^2[v, v]_A + 2\alpha([v, u_0]_A - [v, f]) \geq 0.$

The expression $F(u_0 + \alpha v) - F(u_0)$ is a function of the real number α. As such, this expression takes its minimum value of zero when $\alpha = 0$. Therefore, its derivative with respect to α at $\alpha = 0$ must be zero i.e.

$$2([v, u_0]_A - [v, f]) = 0$$

for all v in H_A.

Appendix

Let Ω be an open set in $\mathbf{R}^n$. A vector valued function $\boldsymbol{u}(\cdot)$ of a vector variable $\boldsymbol{x}$**,** defined on Ω, is denoted as

$$\boldsymbol{u}(\boldsymbol{x}) = [u_1(\boldsymbol{x}), \dots, u_m(\boldsymbol{x})]^T \;; \quad \boldsymbol{x} = [x_1, x_2, \dots, x_n]^t \in \Omega$$

The zero vector vector is $\quad \mathbf{0}(x) = [0,0,\dots,0]^T$

The Hilbert space $L_2(\Omega)$ is the space of square integrable functions (in the Lebesque sense), with domain Ω, and inner product and norm defined as follows.

$$(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{u}(\boldsymbol{x})^T \boldsymbol{v}(\boldsymbol{x}) \; dx_1, \dots, dx_n \qquad \|u\|^2 = \int_\Omega \boldsymbol{u}(\boldsymbol{x})^T \boldsymbol{u}(\boldsymbol{x}) \; dx_1, \dots, dx_n$$

Properties:

i. $(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{v}, \boldsymbol{u})$

ii. $(\boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u}, \boldsymbol{v}) + (\boldsymbol{u}, \boldsymbol{w})$

iii. $(\boldsymbol{\alpha u}, \boldsymbol{v}) = \boldsymbol{\alpha}(\boldsymbol{u}, \boldsymbol{v})$

iv. $(\boldsymbol{u}, \boldsymbol{u}) \geq 0$

v. $(\boldsymbol{u}, \boldsymbol{u}) = 0$ only if $\boldsymbol{u} = 0$

Lemma Let *V* be an inner product space. If $[u, v] = 0$ for all v in V,then $u = 0$.

Proof:

Since $[u, v] = 0$ for all v in V, then $[u, u] = \sqrt{\|u\|} = 0$. Hence, $\|u\| = 0$, and by the properties of norms it follows that $u = 0$.

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