On the Trajectories of Particles in Solitary Waves

by

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Submitted in Partial Fulfillment of the Requirements

for the Degree of

Masters of Science

in the

Mathematics

Program

YOUNGSTOWN STATE UNIVERSITY

May, 2011

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ABSTRACT

Across the country, school students learn that ocean waves cause water particles to form looping paths, traveling in circles which become smaller as you look deeper underwater.

In this paper, we investigate the approximations which are used to make this claim. Furthermore, we investigate closer approximation techniques which show that these looping paths actually propogate forward with the wave's motion.

Finally, we investigate the specific case of the soliton, which causes particles underneath to travel in a forward-moving arc, with no looping motion at all.

With this background, we examine the recent work of A. Constantin and collaborators, specifically his conclusion that our results for the soliton hold for any solitary wave.

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1 Introduction

In this paper, we investigate the trajectories of neutrally bouyant particles in inviscid fluids associated with traveling gravity waves. First, we take a brief look at the history of the field, and some of the influences that led to its development. From the broad field of differential equations, we discuss some analytical and numerical tools which we will use frequently for the remainder of this document.

Next, we formally establish the governing equations of water waves. (For convenience, we use “water” and “fluid” interchangeably when discussing the medium on which the wave propogates, and “air” when discussing the medium with which it interfaces at the free boundary.)

We begin our quest with the most simple approximation – linear water waves. From this approximation, we further linearize to establish the equations used by general science textbooks to justify describing trajectories as circles or elipses, looping around and around with each passing peak. Using the XPPAUT numerical integration package instead of the latter approximation, we show instead that linear waves produce a net forward drift of particles with the wave’s motion.

Moving on, we consider the nonlinear Korteweg–de Vries approximation for the free surface boundary. Under this approximation, we consider the special case of the soliton – a solitary traveling wave, which gives particles forward-arcing trajectories with no looping behavior at all.

Finally, following the recent work of Constantin and collaborators ([4], [5], and [6]), we prove that our observations from the numerical analysis of solitons hold for all solitary water waves.

1.1 The Euler Equations and the KdV Equation

Leonhard Euler's equations of fluid motion were first presented in 1752, and first published in 1757 in *"Me'moires de l'Academie des Sciences de Berlin"* [11]. The Euler Equations are among the earliest written partial differential equations, possibly preceeded only by the D'Alembert Equations (1749). [3]

The Euler Equations describe the flow of an inviscid fluid – one for which we assume that the particles of the fluid can flow past one another with a negligible amount of friction. This assumption provides a surprisingly good approximation to many physical systems. Everything from the flow of water on Earth to the motion of galaxies have been modeled using the Euler Equations.

Among the most mathematically challenging class of problems found in the Euler Equations are free boundary problems. Free boundary problems involve a system of two distinct fluids in contact with one another. As the name implies, the boundary between the two fluids is not rigidly defined, but is free to change as the fluids move against each other. This phenomenon is most commonly seen in our daily lives where water and air meet; from the largest boundaries – the surface of the ocean rippling up and down as waves pass by, to the smallest – drops of water building up and falling from a leaky faucet. Simplifications by approximation of free boundary problems lead to other famous equations of fluids, such as the Korteweg–de Vries equation. [7]

The Korteweg–de Vries equation, usually referred to as the KdV equation for convenience, was named for the authors who studied it extensively in an 1895 paper [14]. Their paper builds on work by other mathematicians, such as Boussinesq's paper from 1871. The KdV equation is one of the simplest partial differential equation which exhibits both the dispersion and the nonlinearity of the Euler equations. Waves are called dispersive when the

the velocity at which the wave travels depends on wavelength (or its reciprocal, the wave number). This feature causes waves to naturally spread themselves out, or disperse, with time. Beyond the increased mathematical difficulty of nonlinear equations, nonlinearity is also significant because we cannot use the superposition principle – linear combinations of known solutions to nonlinear equations do not necessarily produce new solutions. [10] We examine one derivation of the KdV equation in Section 6.

1.2 Solitary Waves and Solitons

Solitary waves were first observed by J. Scott Russell, who recorded in his 1844 "Report on Waves" [13] his observation of the phenomenon on the Edinburgh-Glasgow canal:

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. ... rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height.

Solitary waves such as the one observed by Russell can be described by the Korteweg–de Vries equation. Such waves have a profile described by the square of the hyperbolic secant function. We will discuss this mathematical formulation in depth in Section 7.

One of the more interesting discoveries of solitary waves was first reported by Zabusky and Kruskal in 1965 [18]. Using numerical computations, they showed that, for a certain class of mathematical models of solitary waves, two such waves can pass through each other, and emerge in the same shape as when they started. However, the collision has an interesting effect on the two waves – the entire wave profile is shifted after the collision as compared to where it would be if the wave had not been disturbed. Comparing this phenomenon to the collisions of elementary particles in physics such as protons and electrons, Zabusky and Kruskal called waves that interact in this way 'solitons.' [10]

2 Ordinary and Partial Differential Equations

2.1 Harmonic Functions

Definition 2.1. In two dimensions, a second-order **Linear Differential Operator**, L, acting on a funtion $u : \mathfrak{D} \longrightarrow \mathbb{R}$ is of the form

$$L[u(x,y)] = A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu,$$

where the paramaters A, B, C, D, E, and F are each functions of x and y. A linear differential operator is called **elliptic** on an open domain, $\mathfrak{D}$, iff its discriminant, $B^2 - AC$, is negative everywhere on $\mathfrak{D}$.

In this paper, we will work with the most basic linear elliptic differential operator, the cartesian Laplacian operator, ∇^2. In two-dimensions, the Laplacian operator is given by

$$\nabla^2 u(x,y) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Definition 2.2 (Harmonic Function)**.** A function, $u : \mathfrak{D} \longrightarrow \mathbb{R}$, is called **harmonic** on an open domain, $\mathfrak{D}$, iff it is a solution to Laplace's Equation,

$$\nabla^2 u = 0, \tag{2.1}$$

$\forall \vec{x} \in \mathfrak{D}$.

Definition 2.3 (Subharmonic Function)**.** A function, $u : \mathfrak{D} \longrightarrow \mathbb{R}$, is called **subharmonic** on an open domain, $\mathfrak{D}$, iff $\nabla^2 u \geqslant 0 \; \forall \; \vec{x} \in \mathfrak{D}$.

Definition 2.4 (Superharmonic Function)**.** A function, $u : \mathfrak{D} \longrightarrow \mathbb{R}$, is called **superharmonic** on an open domain, $\mathfrak{D}$, iff $\nabla^2 u \leqslant 0 \; \forall \; \vec{x} \in \mathfrak{D}$.

Proposition 2.1. *A function is harmonic iff it is subharmonic and it is superharmonic.*

2.2 Solving Laplace's Equation

We employ separation of variables to solve Laplace's equation with periodic boundary condition given as

$$u(x+\lambda, y) = u(x,y) \qquad \text{for some } \lambda, \text{ constant} \tag{2.2}$$
$$\frac{\partial u}{\partial y} = 0 \qquad \text{when } y = 0. \tag{2.3}$$

Let $u(x,y) = X(x)Y(y)$. Then Laplace's Equation (2.1) gives us that

$$\frac{X''}{X} = \frac{-Y''}{Y} = -k^2,$$

for some $k \in \mathbb{R}$. Splitting the equations, first,

$$X'' + k^2 X = 0, \tag{2.4}$$

which, along with the periodic BC (2.2), has solution

$$X(x) = C_1 \cos(kx) + C_2 \sin(kx),$$

with C_1 and C_2 arbitrary constants, when

$$k = \frac{2\pi}{\lambda}.$$

For convenience, we want $x = 0$ to correspond to a peak, so we let $C_2 = 0$, giving a final solution

$$X(x) = C_1 \cos(kx). \tag{2.5}$$

Next, we consider

$$Y'' - k^2 Y = 0.$$

This ordinary differential equation has solutions of the form

$$Y(y) = B_1 \sinh(ky) + B_2 \cosh(ky),$$

for arbitrary constants B_1 and B_2. The Neumann Boundary Condition (2.3) gives us that $Y'(0) = 0$, so

$$\begin{aligned} 0 &= B_1 \cosh(0) + B_2 \sinh(0) \\ &= B_1. \end{aligned}$$

Thus,

$$Y(y) = B_1 \cosh(ky). \tag{2.6}$$

And so, combining equations (2.5) and (2.6),

$$u(x, y) = X(x)Y(y) \tag{2.7}$$

$$= A \sin(kx) \cosh(ky), \tag{2.8}$$

for some constant, $A = B_1 C_1$.

2.3 The Maximum Principle

Theorem 2.2 (The Maximum Principle for Harmonic Functions)**.** *Let $u(\vec{x})$ be a harmonic function on some open domain, $\mathfrak{D}$, and continuous on its closure, $\overline{\mathfrak{D}}$. Then the maximum and the minimum values of u are attained on the boundary, $\partial\mathfrak{D}$. The maximum and the minimum are attained nowhere on the interior, $\mathfrak{D}$, unles $u(\vec{x})$ is constant $\forall \vec{x} \in \overline{\mathfrak{D}}$.*

In other words, $\exists\, \vec{x}_m, \vec{x}_M \in \partial\mathfrak{D}$ s.t. $u(\vec{x}_m) \leqslant u(\vec{x}) \leqslant u(\vec{x}_M)\ \forall \vec{x} \in \mathfrak{D}$, and one equality holds iff the other does.

We prove a version of this restricted to two dimensions, as presented in [15]:

Corollary 2.2.1. *Let $u(x, y)$ be a harmonic function on some open domain, $\mathfrak{D} \subseteq \mathbb{R}^2$, and continuous on its closure, $\overline{\mathfrak{D}}$. Then the maximum and the minimum values of u are attained on the boundary, $\partial\mathfrak{D}$.*

Proof. We consider a maximum point and note that the proof holds identically for a minimum, given the mapping $v = -u$.

Recall, from Calculus, the second derivative test gives us that, at a maximal point,

$$\frac{\partial^2 u}{\partial x^2} \leqslant 0 \text{ and } \frac{\partial^2 u}{\partial y^2} \leqslant 0. \tag{2.9}$$

Hence,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \leqslant 0. \tag{2.10}$$

But, u is harmonic, so, by equation (2.1),

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{2.11}$$

Equations (2.9) and (2.11) can only be consistent if

$$\frac{\partial^2 u}{\partial x^2} = 0 = \frac{\partial^2 u}{\partial y^2}, \tag{2.12}$$

otherwise we have a contradiction.

Let $\varepsilon > 0$ and define

$$v(\vec{x}) = u(\vec{x}) + \varepsilon \left|\vec{x}\right|^2 = u(\vec{x}) + \varepsilon\left(x^2 + y^2\right). \tag{2.13}$$

Note that, since u is continuous, v is also continuous. Differentiating (2.13), we get

$$\nabla^2 v = \nabla^2 u + \varepsilon \nabla^2 \left[x^2 + y^2\right] \tag{2.14}$$

$$= 0 + \varepsilon(2 + 2) \quad \forall\, \vec{x} \in \mathfrak{D}$$

$$= 4\varepsilon$$

$$> 0 \quad \forall\, \vec{x} \in \mathfrak{D}. \tag{2.15}$$

Thus, in light of (2.10), v has no maximum in $\mathfrak{D}$. However, since v is continuous, it must have a maximum somewhere in $\overline{\mathfrak{D}}$. Thus, $\exists\, \vec{x}_\circ \in \partial\mathfrak{D}$ s.t. $v(\vec{x}_\circ) = \max u(\vec{x})$. And so,

$\forall \vec{x} \in \mathfrak{D}$, we have

$$\begin{aligned} u(\vec{x}) &\leqslant v(\vec{x}) \\ &\leqslant v(\vec{x}_\circ) \\ &\equiv u(\vec{x}_\circ) + \varepsilon \left|\vec{x}_\circ\right|^2 \\ &\leqslant \max_{\partial \mathfrak{D}} u + \varepsilon \ell^2, \end{aligned} \tag{2.16}$$

where

$$\ell = \max_{\mathfrak{D}} \left|\vec{x}\right|.$$

And, since $\varepsilon > 0$ in (2.16) is chosen abitrarily, we have that

$$u(\vec{x}) \leqslant \max_{\partial \mathfrak{D}} u \qquad \forall\, \vec{x} \in \mathfrak{D}. \tag{2.17}$$

Finally, since u is continuous, this maximum exists; that is, $\exists\, \vec{x}_M \in \partial \mathfrak{D}$ s.t. $u(\vec{x}_M) = \max u$ on $\overline{\mathfrak{D}}$.

■

Theorem 2.3 (Hopf's Maximum Principle)**.** *Let $u(\vec{x})$ be a function wth continuous second derivatives that satisfies $L[u] \geqslant 0$ for a linear elliptic differential operator, L, on some open domain, $\mathfrak{D}$. If u attains its maximum value at some point $\vec{x}_\circ \in \mathfrak{D}$, then $u(\vec{x}) = u(\vec{x}_\circ)\,\forall\, \vec{x} \in \mathfrak{D}$.*

From Courant and Hilbert's *Methods of Mathematical Physics* [8]. Stated without proof.

Corollary 2.3.1 (A special case)**.** *Let $u(x, y)$ be a subharmonic function defined on some open domain, $\mathfrak{D} \subseteq \mathbb{R}^2$. If u attains its maximum value, M at some point $(x_\circ, y_\circ) \in \mathfrak{D}$, then $u(x, y) \equiv M$ throughout $\mathfrak{D}$.*

Equivalently, let $u(x, y)$ be a superharmonic function defined on some open domain, $\mathfrak{D} \subseteq \mathbb{R}^2$. If u attains its minimum value, m at some point $(x_\circ, y_\circ) \in \mathfrak{D}$, then $u(x, y) \equiv m$ throughout $\mathfrak{D}$.

2.4 Planar Systems of Differential Equations

Definition 2.5 (Planar Systems of Differential Equations)**.** A planar system is a parametric curve given by $(x(t), y(t))$ in $\mathbb{R}^2$, defined as solutions to the system of first order ordinary differential equations

$$x'(t) = f(x, y) \tag{2.18a}$$

$$y'(t) = g(x, y), \tag{2.18b}$$

for some functions, $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

Definition 2.6 (Hamiltonian System)**.** A planar system of differential equations is called a **Hamiltonian system** iff there exists a continuously differentiable function $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$, called the **Hamiltonian Energy Function**, such that

$$x'(t) = \frac{\partial H}{\partial y} \tag{2.19a}$$

$$y'(t) = -\frac{\partial H}{\partial x}, \tag{2.19b}$$

Proposition 2.4. *A parametric curve, $(\xi(t), \zeta(t))$ is a solution to a Hamiltonian system only if the energy function along that curve, $H(t) \equiv H(\xi(t), \zeta(t))$, is constant.*

Taken from [12].

3 Frames of Reference

In this document, we will concern ourselves with a two-dimensional invariant wave or waves propogating at constant speed, $c > 0$, in a fluid of constant depth, $h > 0$ above a flat bed. To this end, we define two separate frames of reference which will be convenient in different applications: one affixed to the flat bed, and another traveling along with the waves at speed c. These two coordinate systems are depicted in Figure 1 on page 70.

3.1 Fixed Frame of Reference

In the fixed frame of reference, we use a position vector, $\vec{X} = (X, Y) \in \mathbb{R}^2$, and time coordinate, $T \in \mathbb{R}$. We define $Y = 0$ to be the flat bed, and orient the axes such that the wave propogates in the $+X$ direction. We take $T = 0$ to be a time when a wave crest is located at $X = 0$.

Define a function $\eta^*(X, T)$ such that $Y = \eta^*(X, T)$ gives the shape of the air-water interface, called the **free surface** of the wave, at position X and time T.

3.2 Moving Frame of Reference

Often, we find it more convenient to work in a frame of reference moving along with the waveform. In the moving frame of reference, we use a position vector, $\vec{x} = (x, y) \in \mathbb{R}^2$ and time coordinate, $t \in \mathbb{R}$. This coordinate system travels along the undisturbed water surface in the $+X$ direction with the wave at speed c. In particular, we define the position of the wave crest located at $(X = 0, T = 0)$ to be $x = 0$ for all time. Together, we have:

$$x = X - cT, \tag{3.1a}$$

$$y = Y - h, \tag{3.1b}$$

$$t = T. \tag{3.1c}$$

This gives us the family of partial derivatives,

$$\frac{\partial x}{\partial Y} = \frac{\partial y}{\partial X} = \frac{\partial y}{\partial T} = \frac{\partial t}{\partial X} = \frac{\partial t}{\partial Y} = 0,$$

$$\frac{\partial x}{\partial X} = \frac{\partial y}{\partial Y} = \frac{\partial t}{\partial T} = 1, \qquad \frac{\partial x}{\partial T} = -c.$$

Using the chain rule, the differential operators under the change of coordinates are

$$\begin{aligned} \frac{\partial}{\partial X} &= \frac{\partial x}{\partial X}\frac{\partial}{\partial x} + \frac{\partial y}{\partial X}\frac{\partial}{\partial y} + \frac{\partial t}{\partial X}\frac{\partial}{\partial t} \\ &= \frac{\partial}{\partial x}. \end{aligned} \tag{3.2a}$$

Similarly,

$$\frac{\partial}{\partial Y} = \frac{\partial}{\partial y}, \tag{3.2b}$$

and

$$\begin{aligned} \frac{\partial}{\partial T} &= \frac{\partial}{\partial t} - c\frac{\partial}{\partial x} \\ \text{or} \qquad c\frac{\partial}{\partial X} + \frac{\partial}{\partial T} &= \frac{\partial}{\partial t}. \end{aligned} \tag{3.2c}$$

Again, we define the waveform function, $\eta(x)$, with $y = \eta(x)$ giving the height of a wave at position x. Notice that, unlike in the fixed frame, this function is independent of time in the moving frame of reference. Using the change of coordinates given in equation (3.1),

$$\eta^*(X, T) = \eta(X - cT) + h.$$

3.3 Regions of Interest

There are three regions that we will be interested to study. First, we define the set of points on the flat bed,

$$\mathcal{B} = \left\{ \vec{x} \in \mathbb{R}^2 | y = -h \right\}, \tag{3.3}$$

and the set of points on the free surface,

$$\mathcal{S} = \left\{ \vec{x} \in \mathbb{R}^2 | y = \eta(x) \right\}. \tag{3.4}$$

Finally, the open domain,

$$\mathfrak{D} = \left\{ \vec{x} \in \mathbb{R}^2 | -h < y < \eta(x) \right\}, \tag{3.5}$$

is the set of all points between these two boundaries. The boundary of this domain is $\partial\mathfrak{D} = \mathcal{S} \cup \mathcal{B}$, and its closure is $\overline{\mathfrak{D}} = \mathfrak{D} \cup \mathcal{S} \cup \mathcal{B}$.

In the fixed frame of reference, these sets change with time. Thus, we define the equivalent sets:

$$\mathcal{B}^* = \left\{\vec{X} \in \mathbb{R}^2 | Y = 0\right\},$$
$$\mathcal{S}_T^* = \left\{\vec{X} \in \mathbb{R}^2 | Y = \eta^*(X,T)\right\},$$
$$\mathfrak{D}_T^* = \left\{\vec{X} \in \mathbb{R}^2 | 0 < Y < \eta^*(X,T)\right\}.$$

3.4 Velocity Vector Fields

In the fixed frame of reference, we define the vector field,

$$\vec{U}(\vec{X},T) = (U(X,Y,T), V(X,Y,T)), \tag{3.6}$$

to be the velocity of a particle of water located at position $\vec{X}$ at time T.

In the moving frame of reference, we define an associated vector field,

$$\vec{u}^*(\vec{x}) = (u^*(x,y), v^*(x,y)), \tag{3.7}$$

by the relationship

$$\vec{U}(X,Y,T) = \vec{u}^*(X - cT, Y - h). \tag{3.8}$$

Note that, as with the waveform function, the velocity field in the moving frame of reference, $\vec{u}^*$, does not change with respect to time, while the velocity field in the fixed frame of reference $\vec{U}$ has an explicit time dependence.

Notice also that the vector field $\vec{u}^*$ is defined at a point given with respect to the moving coordinate system, but the velocities are still given with respect to the fixed coordinate system. That is to say, a particle with velocity $\vec{u}^* = 0$ is still in motion with respect to the moving coordinate system – it has a relative speed c in the $-x$ direction.

We define a second velocity vector field for the moving coordinate system, $\vec{u}(\vec{x}) = (u(x,y), v(x,y))$, as follows

$$u(x,y) = u^*(x,y) - c \tag{3.9a}$$

$$v(x,y) = v^*(x,y). \tag{3.9b}$$

With this definition, the velocity of a particle given by coordinates in the moving frame of reference is now considered with respect to the same moving coordinate system. Notice that, in all cases, derivatives of u and v are equal to those of u^* and v^*, since they differ by only a constant.

4 The Governing Equations of Water Waves

4.1 The Euler Equations for Two Dimensional Water Waves

In two dimensions, the Euler Equations of Fluid Dynamics are given as

$$\frac{\partial U}{\partial T} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{\partial P^*}{\partial X} \tag{4.1a}$$

$$\frac{\partial V}{\partial T} + U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} = -\frac{\partial P^*}{\partial Y} - g, \tag{4.1b}$$

where $P^*(X, Y, T)$ denotes pressure and g is the gravitational acceleration. [4]

We seek to change to the moving coordinate system. First, we map $P(x, y, t) = P^*(X, Y, T)$

using the change of coordinates given in equation (3.1). Next, starting with equation (4.1a):

$$\left(\frac{\partial}{\partial T}\right)U + U\left(\frac{\partial}{\partial X}\right)U + V\left(\frac{\partial}{\partial Y}\right)U = -\left(\frac{\partial}{\partial X}\right)P,$$

we substitute in our change of coordinates from equations (3.2) and (3.8) to get

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u^* + u^*\left(\frac{\partial}{\partial x}\right)u^* + v^*\left(\frac{\partial}{\partial y}\right)u^* = -\left(\frac{\partial}{\partial x}\right)P,$$

which reduces to

$$\frac{\partial u^*}{\partial t} + (u^* - c)\frac{\partial u^*}{\partial x} + v^*\frac{\partial u^*}{\partial y} = -\frac{\partial P}{\partial x}. \tag{4.2a}$$

Likewise, from (4.1b), we find that

$$\frac{\partial v^*}{\partial t} + (u^* - c)\frac{\partial v^*}{\partial x} + v^*\frac{\partial v^*}{\partial y} = -\frac{\partial P}{\partial y} - g. \tag{4.2b}$$

Working instead with the velocity field described in equations (3.9), equations (4.2) become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} \tag{4.3a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} - g, \tag{4.3b}$$

which, conveniently, are identical to the equations for the fixed frame of reference, (4.1).

4.1.1 Invariant form

Particularly in dealing with solitary waves, it is useful to work with waves that are of invariant form. Thus, we may assume that the velocity field remains steady with respect to the moving coordinate system. Mathematically, this means that

$$\frac{\partial \vec{u}^*}{\partial t} = 0, \tag{4.4}$$

so equations (4.2) become

$$(u^* - c)\frac{\partial u^*}{\partial x} + v^*\frac{\partial u^*}{\partial y} = -\frac{\partial P}{\partial x} \tag{4.5a}$$

$$(u^* - c)\frac{\partial v^*}{\partial x} + v^*\frac{\partial v^*}{\partial y} = -\frac{\partial P}{\partial y} - g. \tag{4.5b}$$

4.2 Incompressibility

For our approximation techniques, we will assume that the fluid has constant density, so we use the equation of conservation of mass, which reduces to

$$\vec{\nabla} \cdot \vec{u} = 0.$$

In the two-dimensional system we are working in, this gives us that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{4.6}$$

4.3 Irrotationality

For the water waves discussed in this paper, we may assume irrotationality:

$$\vec{\nabla} \times \vec{u} = 0 \quad \text{throughout } \mathfrak{D}.$$

In the two-dimensional system we are working in, this gives us that

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \tag{4.7}$$

4.4 Boundary Conditions for Inviscid Water Waves

First, on the free surface, the only pressure is the atmospheric pressure, which we take to be constant:

$$P(\vec{x}) = P_0 \quad \forall \vec{x} \in \mathcal{S}. \tag{4.8}$$

Next, we assume that the particles in the fluid can neither escape into the air through the free surface nor appear at the free surface from the air – at the surface, all motion is along the waveform:

$$v^*(\vec{x}) = \frac{\partial \eta}{\partial t} + (u^* - c)\frac{\partial \eta}{\partial x} \quad \forall \vec{x} \in \mathcal{S}. \tag{4.9}$$

Likewise, particles cannot burrow into the flat bed, nor do they appear from the bed:

$$v(\vec{x}) = 0 \quad \forall \vec{x} \in \mathcal{B}. \tag{4.10}$$

4.5 Boundary Conditions for Solitary Waves

In addition to the governing restrictions for all inviscid water waves described in section 4.4, solitary waves require the following conditions:

$$\text{As } |x| \longrightarrow \infty, \qquad u^* \longrightarrow 0, \tag{4.11a}$$

$$v^* \longrightarrow 0, \tag{4.11b}$$

$$\text{and} \quad \eta(x) \longrightarrow 0. \tag{4.11c}$$

4.6 Bernoulli's Law

From (4.2a),

$$\frac{\partial u^*}{\partial t} + (u^* - c)\frac{\partial u^*}{\partial x} + v^*\frac{\partial u^*}{\partial y} = -\frac{\partial P}{\partial x}.$$

By integrating,

$$-\int \frac{\partial P}{\partial x}\,\mathrm{d}x = \int \left[\frac{\partial u^*}{\partial t} + (u^* - c)\frac{\partial u^*}{\partial x} + v^*\frac{\partial u^*}{\partial y}\right]\mathrm{d}x. \tag{4.12}$$

Since we assumed irrotational flow in (4.7), equation (4.12) becomes

$$\begin{aligned}
-\int \frac{\partial P}{\partial x}\,\mathrm{d}x &= \int \frac{\partial u^*}{\partial t}\,\mathrm{d}x + \int (u^* - c)\frac{\partial u^*}{\partial x}\,\mathrm{d}x + \int v^*\frac{\partial v^*}{\partial x}\,\mathrm{d}x \\
-\int \mathrm{d}P &= \int \frac{\partial u^*}{\partial t}\,\mathrm{d}x + \int (u^* - c)\,\mathrm{d}u^* + \int v^*\,\mathrm{d}v^* \\
-P - k(y) &= \int \frac{\partial u^*}{\partial t}\,\mathrm{d}x + \frac{1}{2}(u^* - c)^2 + \frac{1}{2}v^{*2},
\end{aligned} \tag{4.13}$$

for some unknown function, $k(y)$. Differentiating now with respect to y, we get

$$-\frac{\partial P}{\partial y} - \frac{\mathrm{d}k}{\mathrm{d}y} = (u^* - c)\frac{\partial u^*}{\partial y} + v^*\frac{\partial v^*}{\partial y} + \frac{\partial}{\partial y}\left[\int \frac{\partial u^*}{\partial t}\,\mathrm{d}x\right]. \tag{4.14}$$

By the condition of irrotationality (4.7), equation (4.14) becomes

$$\begin{aligned}
-\frac{\partial P}{\partial y} - \frac{\mathrm{d}k}{\mathrm{d}y} &= (u^* - c)\frac{\partial u^*}{\partial y} + v^*\frac{\partial v^*}{\partial y} + \int \frac{\partial}{\partial t}\frac{\partial u^*}{\partial y}\,\mathrm{d}x \\
&= (u^* - c)\frac{\partial v^*}{\partial x} + v^*\frac{\partial v^*}{\partial y} + \frac{\partial}{\partial t}\left[\int \frac{\partial v^*}{\partial x}\,\mathrm{d}x\right] \\
&= (u^* - c)\frac{\partial v^*}{\partial x} + v^*\frac{\partial v^*}{\partial y} + \frac{\partial v^*}{\partial t}.
\end{aligned} \tag{4.15}$$

But, by comparing (4.15) to (4.2b):

$$\frac{\partial v^*}{\partial t} + (u^* - c)\frac{\partial v^*}{\partial x} + v^*\frac{\partial v^*}{\partial y} = -\frac{\partial P}{\partial y} - g,$$

we see that $\frac{\mathrm{d}k}{\mathrm{d}y} = g$. Therefore, $k(y) = gy - \kappa$, for some arbitrary constant, κ. Substituting this into (4.13) and solving for κ, we have

$$\kappa = \frac{1}{2}(u^* - c)^2 + \frac{1}{2}v^{*2} + P + gy + \int \frac{\partial u^*}{\partial t}\,\mathrm{d}x. \tag{4.16}$$

This is Bernoulli's Law.

4.6.1 Bernoulli's Law for Solitary Waves

Working in the case of solitary waves, we can evaluate κ. First, we take advantage of the assumption that solitary waves are of invariant form (4.4). Thus, equation (4.16) becomes

$$\kappa = \frac{1}{2}(u^* - c)^2 + \frac{1}{2}v^{*2} + P + gy. \tag{4.17}$$

Now, consider the boundary $y = \eta(x)$. We know that κ is constant, so its value on this boundary will be its value everywhere. Inserting the values from the surface boundary conditions, (4.8) and (4.9), into the equation, we get that

$$\kappa = (u^* - c)^2\left[\frac{1}{2} + \frac{1}{2}\left(\frac{\partial \eta}{\partial x}\right)^2\right] + P_\circ + g\eta(x).$$

As $|x| \to \infty$, $\eta(x) \to 0$ and $u^*, v^* \to 0$ by the solitary wave boundary conditions, (4.11a), (4.11b), and (4.11c). Since $v^* = (u^* - c)\frac{\partial \eta}{\partial x}$ by the kinematic boundary condition (4.9), as

$|x| \to \infty$, we also have $\frac{\partial\eta}{\partial x} \to 0$. Thus,

$$\kappa = \frac{1}{2}c^2 + P_\circ.$$

Finally, substituting this value for κ back into equation (4.17), we obtain the Bernoulli equation for solitary waves,

$$\frac{1}{2}(u^* - c)^2 + \frac{1}{2}v^{*2} + P + gy = \frac{1}{2}c^2 + P_\circ. \tag{4.18}$$

4.7 The Velocity Potential Function

We have assumed irrotational flow, $\vec{\nabla} \times \vec{u} = \vec{0}$, and there exists a potential function, call it φ, such that

$$\vec{\nabla}\varphi = \vec{u}, \tag{4.19}$$

or

$$\frac{\partial\varphi}{\partial x} = u \quad \text{and} \quad \frac{\partial\varphi}{\partial y} = v.$$

Combining equation (4.19) with equation (4.6), we see that φ satisfies Laplace's Equation

$$\nabla^2\varphi = 0. \tag{4.20}$$

4.7.1 Boundary Conditions for the Velocity Potential

From the boundary condition on the flat bed, (4.10), we get

$$\frac{\partial\varphi}{\partial y} = 0 \qquad \text{when } y = -h. \tag{4.21}$$

From the kinematic surface boundary condition, (4.9), we get

$$\frac{\partial \eta}{\partial t} + \frac{\partial \varphi}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \varphi}{\partial y} \qquad \text{when } y = \eta(x,t). \tag{4.22}$$

Finally, by way of Bernoulli's Law (4.16), the dynamic boundary condition, (4.8) becomes

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right] = -g\eta \qquad \text{when } y = \eta(x,t), \tag{4.23}$$

4.8 The Stream Function

The stream function is defined as:

$$\frac{\partial \psi}{\partial y} = u^* - c \quad \text{and} \quad \frac{\partial \psi}{\partial x} = -v^*. \tag{4.24}$$

Consider the second derivatives of ψ:

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial u^*}{\partial y} \quad \text{and} \quad \frac{\partial^2 \psi}{\partial x^2} = -\frac{\partial v^*}{\partial x}.$$

When taken with the condition of irrotationality (4.7), this gives us that ψ is harmonic, or

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \forall\, (x,y) \in \mathfrak{D}. \tag{4.25}$$

Recall that, from (4.9),

$$v^* = (u^* - c)\frac{\partial \eta}{\partial x} \quad \text{on } \mathcal{S},$$

so

$$\frac{\partial \eta}{\partial x} = \frac{v^*}{u^* - c}.$$

On $\mathcal{S}$, $\psi(x, y) = \psi(x, \eta(x))$, so

$$\begin{aligned}\frac{\mathrm{d}\psi}{\mathrm{d}\xi} &= \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\partial\eta}{\partial x} \\ &= -v^* + (u^* - c)\left(\frac{v^*}{u^* - c}\right) \\ &= -v^* + v^* = 0.\end{aligned}$$

Thus, ψ is constant on $\mathcal{S}$. Define

$$\psi|_{\mathcal{S}} \equiv 0. \tag{4.26}$$

Since Lemma A.1, cited in the appendix, gives us that ψ decreases as y increases, we can say that $\psi|_{\mathcal{S}} < \psi|_{\mathcal{B}}$.

On $\mathcal{B}$, $\frac{\partial\psi}{\partial x} = -v^* = 0$ by definition of the stream function and Boundary Condition (4.10) (and y does not change), so ψ is constant. Define

$$\psi|_{\mathcal{B}} \equiv m > 0. \tag{4.27}$$

For each $\alpha \in (0, m]$, consider the level curve

$$\psi(x, y) = \alpha.$$

By the Implicit Function Theorem, each such curve defines a smooth function

$$y = h_\alpha(x). \tag{4.28}$$

From our earlier discussion, $h_0(x) = \eta(x)$ and $h_m(x) \equiv 0$. We exclude $\alpha = 0$ from consideration because a wave of greatest height has a corner of angle $2\pi/3$ at the point $(0, \eta(0))$.

[4], [6]

Proposition 4.1. *For $\alpha < \beta$, $h_\beta(x) < h_\alpha(x)\ \forall\ x$.*

This claim logically follows from the preceeding discussion, and is stated without further proof.

4.9 Particle Trajectories

Consider a particular particle located at $(\Xi_\circ, Z_\circ)$ at time $T = 0$ with respect to the fixed coordinate system (X, Y, T). The path this particle travels is a parametric curve, $(\Xi(T), Z(T))$, defined as the solution to the system of differential equations:

$$\Xi'(T) = U(\Xi(T), Z(T)) \tag{4.29a}$$

$$Z'(T) = V(\Xi(T), Z(T)). \tag{4.29b}$$

In the moving coordinate system, (x, y, t), this particle corresponds to the point $(\xi_\circ, \zeta_\circ)$ at time $t = 0$; and it follows the path given by the parametric curve $(\xi(t), \zeta(t))$. Under the change of coordinates defined in equations (3.1) and (3.2), we see that

$$\begin{aligned}\xi'(t) &= \Xi'(t) - c \\ &= U(\Xi(t), Z(t)) - c \\ &= u^*(\xi(t), \zeta(t)) - c, \end{aligned} \tag{4.30a}$$

and

$$\begin{aligned}\zeta'(t) &= Z'(t)\\ &= V(\Xi(t), Z(t))\\ &= v^*(\xi(t), \zeta(t)).\end{aligned} \tag{4.30b}$$

But, by the definition of the stream function (4.24), this is the Hamiltonian system

$$\xi'(t) = \frac{\partial\psi}{\partial y}(\xi, \zeta) \tag{4.31a}$$

$$\zeta'(t) = -\frac{\partial\psi}{\partial x}(\xi, \zeta), \tag{4.31b}$$

with Hamiltonian energy function $H(t) = \psi(\xi(t), \zeta(t))$.

5 A First Approximation – Linear Waves

We first examine the relatively simple case of linear water waves, following loosely the work of Constantin and Villari [5].

5.1 Linearized Boundary Conditions

To simplify the problem, we linearize the free surface boundary conditions (4.22) and (4.23), and approximate the surface by $y = 0$. This gives us

$$\frac{\partial\eta}{\partial t} = \frac{\partial\varphi}{\partial y} \qquad \text{when } y = 0, \tag{5.1}$$

and

$$\frac{\partial \varphi}{\partial t} = -g\eta \qquad \text{when } y = 0. \tag{5.2}$$

Now, we can eliminate η by combining the equations. Differentiating (5.2) with respect to t gives us

$$\frac{\partial^2 \varphi}{\partial t^2} = -g\frac{\partial \eta}{\partial t}.$$

Substituting (5.1) into this, we get

$$\frac{\partial^2 \varphi}{\partial t^2} + g\frac{\partial \varphi}{\partial y} = 0 \qquad \text{when } y = 0. \tag{5.3}$$

Finally, the waves are moving in the $+x$ direction with speed $c > 0$. The waves are periodic, so there is a $\lambda > 0$ called the wavelength such that

$$\varphi(x, y, t) = \varphi(x + \lambda, y, t). \tag{5.4}$$

Likewise, the wave has a period, given by $\tau = \frac{\lambda}{c}$ such that

$$\varphi(x, y, t) = \varphi(x, y, t + \tau). \tag{5.5}$$

Using separation of variables, as shown in Section 2.2, φ under these boundary conditions is of the form

$$\varphi(x, y, t) = A\sin(kx - \omega t)\cosh(k(y + h)), \tag{5.6}$$

where $k = \frac{2\pi}{\lambda}$ and $\omega = ck$. Finally, since $\vec{u} = \vec{\nabla} \cdot \varphi$, we have

$$u(x, y, t) = M\cos(kx - \omega t)\cosh(k(y + h)) \tag{5.7a}$$

$$v(x, y, t) = M\sin(kx - \omega t)\sinh(k(y + h)), \tag{5.7b}$$

where $M = kA$.

5.2 Using the Free Surface Boundary

The linearized free surface boundary condition, (5.3), gives

$$\begin{aligned} 0 &= \left.\frac{\partial^2\varphi}{\partial t^2}\right|_{y=0} + g\left.\frac{\partial\varphi}{\partial y}\right|_{y=0} \\ &= -\omega^2 A\sin(kx - \omega t)\cosh(kh) + gkA\sin(kx - \omega t)\sinh(kh) \\ &= A\sin(kx - \omega t)\left[-\omega^2\cosh(kh) + gk\sinh(kh)\right]. \end{aligned}$$

By the zero product rule,

$$A = 0, \quad \text{or} \quad \sin(kx - \omega t) = 0,$$

both of which give uninteresting solutions, or

$$\begin{aligned} \omega^2\cosh(kh) &= gk\sinh(kh) \\ \tanh(kh) &= \frac{\omega^2}{gk} \\ &= \frac{kc^2}{g}. \end{aligned} \tag{5.8}$$

This is the dispersion relation for linear waves. The speed of a wave, c, is related to its wavelength, given as the wave number, k. It indicates that waves of different wavelengths will travel at different speeds.

From equation (5.2), we have the expression for the free surface

$$\begin{aligned} \eta(x,t) &= \frac{-1}{g}\left.\frac{\partial\varphi}{\partial t}\right|_{y=0} \\ &= \frac{\omega}{g}A\cos(kx-\omega t)\cosh(kh) \\ &= \varepsilon h\cos(kx-\omega t), \end{aligned} \tag{5.9}$$

where

$$\varepsilon = \frac{\omega A}{gh}\cosh(kh). \tag{5.10}$$

Our approximation of applying the free surface boundary condition at $y = 0$ works only under the assumption that this ε, and thus the wave amplitude, is small.

Rearranging equation (5.10),

$$A = \frac{\varepsilon gh}{\omega\cosh(kh)},$$

so

$$\begin{aligned} M &= kA \\ &= \frac{k\varepsilon gh}{\omega\cosh(kh)} \\ &= \frac{k\varepsilon gh}{ck\cosh(kh)} \\ &= \frac{\varepsilon gh}{c\cosh(kh)}. \end{aligned}$$

To take advantage of the dispersion relation (5.8), we introduce a factor of sinh(kh), and we get

$$\begin{aligned} M &= \frac{\varepsilon g h}{c} \frac{1}{\sinh(kh)} \tanh(kh) \\ &= \frac{\varepsilon g h}{c \sinh(kh)} \frac{kc^2}{g} \\ &= \frac{\varepsilon k c h}{\sinh(kh)} \\ &= \frac{\varepsilon \omega h}{\sinh(kh)}. \end{aligned} \tag{5.11}$$

5.3 Particle Trajectories and Non-dimensional Coordinates

Summarizing the previous sections, we have that

$$\begin{aligned} u(x, y, t) &= M \cos(kx - \omega t) \cosh(k(y + h)) \\ v(x, y, t) &= M \sin(kx - \omega t) \sinh(k(y + h)), \end{aligned}$$

where

$$\begin{aligned} M &= \frac{\varepsilon \omega h}{\sinh(kh)}, \\ k &= \frac{2\pi}{\lambda}, \\ \text{and} \qquad \omega &= \frac{2\pi}{\tau} = \frac{2\pi c}{\lambda}. \end{aligned}$$

Trajectories of a particle moving in a velocity field are given by

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{u}.$$

And so, the particle trajectories in a linear two-dimensional fluid wave are given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = M\cos(kx - \omega t)\cosh(k(y+h)) \tag{5.12a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = M\sin(kx - \omega t)\sinh(k(y+h)). \tag{5.12b}$$

5.3.1 A change of coordinates

Let $\lambda = \alpha h$ for some new paramater, α. Then

$$\begin{aligned} M &= \frac{\varepsilon\omega h}{\sinh(kh)} \\ &= \frac{\varepsilon\frac{2\pi c}{\alpha h}h}{\sinh\left(\frac{2\pi h}{\alpha h}\right)} \\ &= c\frac{2\pi\varepsilon}{\alpha\sinh\left(\frac{2\pi}{\alpha}\right)}, \end{aligned} \tag{5.13}$$

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= M\cos(kx - \omega t)\cosh(k(y+h)) \\ &= M\cos\left(\frac{2\pi x}{\alpha h} - \frac{2\pi ct}{\alpha h}\right)\cosh\left(\frac{2\pi(y+h)}{\alpha h}\right) \\ &= M\cos\left(\frac{2\pi}{\alpha}\frac{x}{h} - \frac{2\pi}{\alpha}\frac{ct}{h}\right)\cosh\left(\frac{2\pi}{\alpha}\frac{y}{h} + \frac{2\pi}{\alpha}\right), \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= M\sin(kx - \omega t)\sinh(k(y+h)) \\ &= M\sin\left(\frac{2\pi x}{\alpha h} - \frac{2\pi ct}{\alpha h}\right)\sinh\left(\frac{2\pi(y+h)}{\alpha h}\right) \\ &= M\sin\left(\frac{2\pi}{\alpha}\frac{x}{h} - \frac{2\pi}{\alpha}\frac{ct}{h}\right)\sinh\left(\frac{2\pi}{\alpha}\frac{y}{h} + \frac{2\pi}{\alpha}\right). \end{aligned} \tag{5.15}$$

This leads to a natural change to the dimensionless coordinates

$$\begin{aligned} X^* &= \frac{x}{h} \\ Y^* &= \frac{y}{h} + 1 \\ T^* &= \frac{ct}{h}. \end{aligned}$$

Under these coordinates,

$$\begin{aligned} \frac{\mathrm{d}X^*}{\mathrm{d}T^*} &= \frac{\mathrm{d}t}{\mathrm{d}T^*}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{x}{h}\right] \\ &= \frac{h}{c}\frac{1}{h}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{1}{c}\frac{\mathrm{d}x}{\mathrm{d}t}, \end{aligned}$$

and substituting in for $\frac{\mathrm{d}x}{\mathrm{d}t}$,

$$\begin{aligned} \frac{\mathrm{d}X^*}{\mathrm{d}T^*} &= \frac{1}{c}M\cos\left(\frac{2\pi}{\alpha}(X^* - T^*)\right)\cosh\left(\frac{2\pi}{\alpha}Y^*\right) \\ &= \frac{1}{c}c\frac{2\pi\varepsilon}{\alpha\sinh\left(\frac{2\pi}{\alpha}\right)}\cos\left(\frac{2\pi}{\alpha}(X^* - T^*)\right)\cosh\left(\frac{2\pi}{\alpha}Y^*\right) \\ &= \frac{2\pi\varepsilon}{\alpha\sinh\left(\frac{2\pi}{\alpha}\right)}\cos\left(\frac{2\pi}{\alpha}(X^* - T^*)\right)\cosh\left(\frac{2\pi}{\alpha}Y^*\right). \end{aligned} \tag{5.16}$$

Likewise,

$$\begin{aligned} \frac{\mathrm{d}Y^*}{\mathrm{d}T^*} &= \frac{1}{c}\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= \frac{2\pi\varepsilon}{\alpha\sinh\left(\frac{2\pi}{\alpha}\right)}\sin\left(\frac{2\pi}{\alpha}(X^* - T^*)\right)\sinh\left(\frac{2\pi}{\alpha}Y^*\right). \end{aligned} \tag{5.17}$$

Finally, we introduce the parameter $\kappa = \frac{2\pi}{\alpha}$, and thus have the trajectory

$$\frac{\mathrm{d}X^*}{\mathrm{d}T^*} = \frac{\kappa\varepsilon}{\sinh(\kappa)} \cos\left(\kappa(X^* - T^*)\right) \cosh\left(\kappa Y^*\right) \tag{5.18a}$$

$$\frac{\mathrm{d}Y^*}{\mathrm{d}T^*} = \frac{\kappa\varepsilon}{\sinh(\kappa)} \sin\left(\kappa(X^* - T^*)\right) \sinh\left(\kappa Y^*\right). \tag{5.18b}$$

5.4 Linearized Trajectories

Since the derivation of the system of ODE's in equation (5.18) began by linearizing the surface boundary conditions – equations (5.1) and (5.2), it would at first seem appropriate to linearize this system as well. However, we will see in section 5.5 that this additional approximation weakens the connection to the behavior of physical water waves. For clarity, we will refer to the result in equations (5.18) as trajectories for 'general linear waves,' or simply 'linear waves.' We will refer to the result of the following derivation as 'linearized linear waves,' or simply 'linearized waves.'

Near some initial point, $(X^*_\circ, Y^*_\circ)$, the particle trajectory described by equations (5.18) can be approximated by the series expansion

$$\begin{aligned}\frac{\mathrm{d}X^*}{\mathrm{d}T^*} = \frac{\kappa\varepsilon}{\sinh(\kappa)} \cos\left(\kappa(X^*_\circ - T^*)\right) \cosh\left(\kappa Y^*_\circ\right) + \\ (X^* - X^*_\circ)\frac{\mathrm{d}}{\mathrm{d}X^*}\left[\frac{\kappa\varepsilon}{\sinh(\kappa)} \cos\left(\kappa(X^* - T^*)\right) \cosh\left(\kappa Y^*\right)\right] + \\ (Y^* - Y^*_\circ)\frac{\mathrm{d}}{\mathrm{d}Y^*}\left[\frac{\kappa\varepsilon}{\sinh(\kappa)} \cos\left(\kappa(X^* - T^*)\right) \cosh\left(\kappa Y^*\right)\right] + \cdots\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}Y^*}{\mathrm{d}T^*} = &\frac{\kappa\varepsilon}{\sinh(\kappa)}\sin\left(\kappa(X^*_\circ - T^*)\right)\sinh\left(\kappa Y^*_\circ\right) + \\ &(X^* - X^*_\circ)\frac{\mathrm{d}}{\mathrm{d}X^*}\left[\frac{\kappa\varepsilon}{\sinh(\kappa)}\sin\left(\kappa(X^* - T^*)\right)\sinh\left(\kappa Y^*\right)\right] + \\ &(Y^* - Y^*_\circ)\frac{\mathrm{d}}{\mathrm{d}Y^*}\left[\frac{\kappa\varepsilon}{\sinh(\kappa)}\sin\left(\kappa(X^* - T^*)\right)\sinh\left(\kappa Y^*\right)\right] + \cdots\end{aligned}$$

If we assume that the motion of the particle is on the order of the wave height, that is, $\exists\, C > 0$ s.t. $X^* - X^*_\circ < C\varepsilon$ and $Y^* - Y^*_\circ < C\varepsilon$, then

$$\begin{aligned}\frac{\mathrm{d}X^*}{\mathrm{d}T^*} &= \frac{\kappa\varepsilon}{\sinh(\kappa)}\cos\left(\kappa(X^*_\circ - T^*)\right)\cosh\left(\kappa Y^*_\circ\right) + \boldsymbol{O}\left(\varepsilon^2\right) \\ \frac{\mathrm{d}Y^*}{\mathrm{d}T^*} &= \frac{\kappa\varepsilon}{\sinh(\kappa)}\sin\left(\kappa(X^*_\circ - T^*)\right)\sinh\left(\kappa Y^*_\circ\right) + \boldsymbol{O}\left(\varepsilon^2\right).\end{aligned}$$

And so, for sufficiently small ε,

$$\frac{\mathrm{d}X^*}{\mathrm{d}T^*} \simeq \frac{\kappa\varepsilon}{\sinh(\kappa)}\cos\left(\kappa(X^*_\circ - T^*)\right)\cosh\left(\kappa Y^*_\circ\right) \tag{5.19a}$$

$$\frac{\mathrm{d}Y^*}{\mathrm{d}T^*} \simeq \frac{\kappa\varepsilon}{\sinh(\kappa)}\sin\left(\kappa(X^*_\circ - T^*)\right)\sinh\left(\kappa Y^*_\circ\right). \tag{5.19b}$$

Now we can solve these differential equations by direct integration:

$$X^* \simeq X^*_\circ - \frac{\varepsilon}{\sinh(\kappa)}\sin\left(\kappa(X^*_\circ - T^*)\right)\cosh\left(\kappa Y^*_\circ\right) \tag{5.20a}$$

$$Y^* \simeq Y^*_\circ + \frac{\varepsilon}{\sinh(\kappa)}\cos\left(\kappa(X^*_\circ - T^*)\right)\sinh\left(\kappa Y^*_\circ\right). \tag{5.20b}$$

Rearranging these equations gives

$$\frac{X^* - X^*_\circ}{\cosh(\kappa Y^*_\circ)} \simeq \frac{-\varepsilon}{\sinh(\kappa)} \sin(\kappa(X^* - T^*)) \tag{5.21a}$$

$$\frac{Y^* - Y^*_\circ}{\sinh(\kappa Y^*_\circ)} \simeq \frac{\varepsilon}{\sinh(\kappa)} \cos(\kappa(X^* - T^*)). \tag{5.21b}$$

Finally, squaring, then adding the two equations gives us

$$\frac{(X^* - X^*_\circ)^2}{\cosh^2(\kappa Y^*_\circ)} + \frac{(Y^* - Y^*_\circ)^2}{\sinh^2(\kappa Y^*_\circ)} = \frac{\varepsilon^2}{\sinh^2(\kappa)}, \tag{5.22}$$

which is the equation for an elipse centered at $(X^*_\circ, Y^*_\circ)$.

5.5 Numerical Solutions to Particle Trajectories

Returning to the general linear waves we discussed in Section 5.3, particle trajectories are the solutions to the system of differential equations (5.18)

$$\frac{dX^*}{dT^*} = \frac{\kappa\varepsilon}{\sinh(\kappa)} \cos\left(\kappa(X^* - T^*)\right) \cosh\left(\kappa Y^*\right)$$
$$\frac{dY^*}{dT^*} = \frac{\kappa\varepsilon}{\sinh(\kappa)} \sin\left(\kappa(X^* - T^*)\right) \sinh\left(\kappa Y^*\right),$$

with $\kappa = \frac{2\pi}{\alpha}$. The trajectories are dependent on two parameters, α and ε. Using the XPPAUT software package, we can solve this system of ODE's numerically.

5.6 Plots of Trajectories in Linear Waves

All figures are printed in Appendix C. Plots of particle trajectories in linear waves appear starting on page 71. The graphs of linearized trajectories were created by plotting the parametric equations (5.20) in gnuplot. The more general graphs were created in XPPAUT using the `linear.ode` script listed in appendix B.

For each graph, we set $\alpha = 10$. Using $\varepsilon = 0.3$, Figure 2 is the image commonly seen in general science textbooks, showing how the loops of four distinct water particles get progressively smaller as you look deeper under the surface of the water. In Figure 3, we look at particles starting at the same four points on the Y^*-axis – namely $Y^* = 0.8$, $Y^* = 0.4$, $Y^* = 0.2$, and $Y^* = 0.1$. Using the more general approximation, we see how each particle travels forward with each passing wave. One interesting feature is that the particles near the surface travel further with each successive wave than particles near the flat bed.

To see how ε affects the trajectories, we simplify the picture, only looking at the curves starting at $Y^* = 0.8$ and $Y^* = 0.2$. We also look only at the particle's motion forward in time, ignoring the path it followed before crossing the Y^*-axis. In Figures 4 through 8, ε drops from 0.5 to 0.1, and we see the spacing between each successive loop grow smaller. In figure 9, we switch back from the progressing loops to the closed loops of the linearized trajectory. Contrasting Figures 8 and 9 shows again the information lost by the linearized approximation.

In Figures 10 and 11, we look at the general and linearized trajectories when $\varepsilon = 0.01$. With this small value for ε, you can see that even after approximately 30 waves pass over the particle, it has not moved a significant distance compared to the closed loop of the linearized path.

Finally, in Figures 12 through 15, we fix again $\varepsilon = 0.3$, this time investigating the effect of changing α. Specifically, we look at $\alpha = 1, 10, 50$, and 100.

6 The Korteweg–de Vries Approximation

Next, we look at the Korteweg–de Vries equation, a more involved approximation of the Euler equations. This derivation follows the structure presented in Whitham's "Linear and Nonlinear Waves" [17].

6.1 Solutions to Laplace's Equation

To solve Laplace's equation under the boundary conditions from section 4.7.1, we should "clearly" assume that φ is of the form

$$\varphi(x, y, t) = \sum_{n=0}^{\infty} (y + h)^n f_n(x, t),$$

for some sequence of functions $f_n : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Apply the boundary condition at $y = -h$:

$$\begin{aligned} \frac{\partial \varphi}{\partial y} &= f_1 + \sum_{n=2}^{\infty} n(y + h)^{(n-1)} f_n(x, t), \\ \text{so} \qquad 0 &= \left.\frac{\partial \varphi}{\partial y}\right|_{y=-h} \\ &= f_1 + \sum_{n=2}^{\infty} 0 \\ &= f_1. \end{aligned} \tag{6.1}$$

Now let us turn our attention once again to Laplace's Equation. First,

$$\frac{\partial^2 \varphi}{\partial x^2} = \sum_{n=0}^{\infty} (y+h)^n \frac{\partial^2 f_n}{\partial x^2},$$

and

$$\begin{aligned}\frac{\partial^2 \varphi}{\partial y^2} &= \sum_{n=2}^{\infty} n(n-1)(y+h)^{(n-2)} f_n(x,t) \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)(y+h)^n f_{n+2}(x,t).\end{aligned}$$

Thus,

$$\begin{aligned}0 &= \nabla^2 \varphi \\ &= \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \\ &= \left[\sum_{n=0}^{\infty} (y+h)^n \frac{\partial^2 f_n}{\partial x^2}\right] + \left[\sum_{n=0}^{\infty} (n+2)(n+1)(y+h)^n f_{n+2}(x,t)\right] \\ &= \sum_{n=0}^{\infty} (y+h)^n \left[\frac{\partial^2 f_n}{\partial x^2} + (n+2)(n+1) f_{n+2}\right].\end{aligned}$$

This is a polynomial in $(y+h)$, and so the equation holds true for all y only if

$$0 = \frac{\partial^2 f_n}{\partial x^2} + (n+2)(n+1) f_{n+2}$$

$$\text{or} \quad f_{n+2} = \frac{-1}{(n+1)(n+2)} \frac{\partial^2 f_n}{\partial x^2}. \tag{6.2}$$

This relationship gives us two cases to consider. First, if n is even, then $\exists\, k \in \mathbb{N}$ s.t. $n = 2k$. So, equation (6.2) gives

$$f_{2k} = \frac{(-1)^k}{(2k)!} \frac{\partial^{2k} f_0}{\partial x^{2k}}. \tag{6.3}$$

If, on the other hand, n is odd, then $\exists\, k \in \mathbb{N}$ s.t. $n = 2k + 1$ and we have

$$f_{2k+1} = \frac{(-1)^{k+1}}{(2k+1)!} \frac{\partial^{2k+1} f_1}{\partial x^{2k+1}} = 0 \tag{6.4}$$

by the result in equation (6.1).

For simplicity, define $f = f_0$. Then,

$$\varphi(x, y, t) = \sum_{k=0}^{\infty} \frac{(-1)^k (y+h)^{2k}}{(2k)!} \frac{\partial^{2k} f}{\partial x^{2k}}. \tag{6.5}$$

6.2 Non-Dimensional Coordinates

Define the parameters

$$\alpha = \frac{a}{h} \text{ and } \beta = \frac{h^2}{\ell^2},$$

where h is the water depth, a is the amplitude of the wave, and ℓ is a length scale in the x-direction.

We denote with primes the variables used in this section up to this point, and their unprimed counterparts represent the nondimensional variables that will be used in the rest of this section.

$$x = \frac{x'}{\ell} \qquad y = \frac{y'+h}{h} \qquad t = \sqrt{gh}\,\frac{t'}{\ell} \qquad \eta = \frac{\eta'}{a} \qquad \varphi = \sqrt{\frac{h}{g}}\,\frac{\varphi'}{\ell a}, \tag{6.6}$$

where g is the acceleration of gravity. The factor of $\sqrt{gh}$ which appears in the t and φ conversions is a representative speed for the wavecrest traveling in the $+x$-direction. Note that we are now interested in solutions on $0 < y < 1 + \alpha\eta(x, t)$.

For convenience, we absorb constants into a 'new' f, defined as

$$f = \sqrt{\frac{h}{g}}\,\frac{cf'}{\ell a}.$$

So, we now have that

$$\varphi(x, y, t) = \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!}\,\frac{\partial^{2k} f}{\partial x^{2k}}\beta^k, \tag{6.7}$$

with free surface boundary conditions

$$\frac{\partial\eta}{\partial t} + \alpha\frac{\partial\varphi}{\partial x}\frac{\partial\eta}{\partial x} - \frac{1}{\beta}\frac{\partial\varphi}{\partial y} = 0 \tag{6.8}$$

$$\text{and } \eta + \frac{\partial\varphi}{\partial t} + \frac{\alpha}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{1}{2}\frac{\alpha}{\beta}\left(\frac{\partial\varphi}{\partial y}\right)^2 = 0 \tag{6.9}$$

$$\text{when} \quad y = 1 + \alpha\eta(x, t).$$

6.3 Approximation

We approximate to order $\alpha\beta, \beta^2$ when we evaluate φ on the free surface boundary. Notice the $\frac{1}{\beta}$ factor on the $\frac{\partial\varphi}{\partial y}$ term. First, evaluate the derivatives

$$\begin{aligned}\frac{\partial\varphi}{\partial x} &= \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!}\,\frac{\partial^{2k+1} f}{\partial x^{2k+1}}\beta^k \\ &\simeq \frac{\partial f}{\partial x} - \frac{y^2}{2}\frac{\partial^3 f}{\partial x^3}\beta + \boldsymbol{O}\left(\beta^2\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial \varphi}{\partial y} &= \sum_{k=1}^{\infty} \frac{(-1)^k y^{2k-1}}{(2k-1)!} \frac{\partial^{2k} f}{\partial x^{2k}} \beta^k \\ &\simeq -y \frac{\partial^2 f}{\partial x^2}\beta + \frac{y^3}{6}\frac{\partial^4 f}{\partial x^4}\beta^2 + \boldsymbol{O}\left(\beta^3\right),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \varphi}{\partial t} &= \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!} \frac{\partial^{2k+1} f}{\partial x^{2k}\partial t} \beta^k \\ &\simeq \frac{\partial f}{\partial t} - \frac{y^2}{2}\frac{\partial^3 f}{\partial x^2\,\partial t}\beta + \boldsymbol{O}\left(\beta^2\right).\end{aligned}$$

Substituting these into the boundary conditions, we have

$$\begin{aligned}0 &= \left[\frac{\partial \eta}{\partial t} + \alpha \frac{\partial \varphi}{\partial x}\frac{\partial \eta}{\partial x} - \frac{1}{\beta}\frac{\partial \varphi}{\partial y}\right]\Bigg|_{y=1+\alpha\eta} \\ &\simeq \left[\frac{\partial \eta}{\partial t} + \alpha \frac{\partial \eta}{\partial x}\left(\frac{\partial f}{\partial x} - \frac{1}{2}y^2\frac{\partial^3 f}{\partial x^3}\beta\right) - \frac{1}{\beta}\left(-y\frac{\partial^2 f}{\partial x^2}\beta + \frac{y^3}{6}\frac{\partial^4 f}{\partial x^4}\beta^2\right) + \boldsymbol{O}\left(\beta^2\right)\right]\Bigg|_{y=1+\alpha\eta} \\ &\simeq \left[\frac{\partial \eta}{\partial t} + \alpha \frac{\partial f}{\partial x}\frac{\partial \eta}{\partial x} + y\frac{\partial^2 f}{\partial x^2} - \frac{y^3}{6}\frac{\partial^4 f}{\partial x^4}\beta + \boldsymbol{O}\left(\alpha\beta, \beta^2\right)\right]\Bigg|_{y=1+\alpha\eta} \\ &= \frac{\partial \eta}{\partial t} + \alpha \frac{\partial f}{\partial x}\frac{\partial \eta}{\partial x} + (1+\alpha\eta)\frac{\partial^2 f}{\partial x^2} - \frac{(1+\alpha\eta)^3}{6}\frac{\partial^4 f}{\partial x^4}\beta + \boldsymbol{O}\left(\alpha\beta, \beta^2\right) \\ &\simeq \frac{\partial \eta}{\partial t} + \alpha \frac{\partial f}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 f}{\partial x^2} + \alpha\eta\frac{\partial^2 f}{\partial x^2} - \frac{1}{6}\frac{\partial^4 f}{\partial x^4}\beta + \boldsymbol{O}\left(\alpha\beta, \beta^2\right) \\ &= \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left[(\alpha\eta + 1)\frac{\partial f}{\partial x}\right] - \frac{1}{6}\frac{\partial^4 f}{\partial x^4}\beta + \boldsymbol{O}\left(\alpha\beta, \beta^2\right),\end{aligned} \tag{6.10}$$

and

$$
\begin{aligned}
0 &= \left[\eta + \frac{\partial\varphi}{\partial t} + \frac{\alpha}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{1}{2}\frac{\alpha}{\beta}\left(\frac{\partial\varphi}{\partial y}\right)^2\right]\Bigg|_{y=1+\alpha\eta} \\
&\simeq \left[\eta + \left(\frac{\partial f}{\partial t} - \frac{y^2}{2}\frac{\partial^3 f}{\partial x^2\,\partial t}\beta\right) + \frac{\alpha}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{2}y^2\frac{\partial^3 f}{\partial x^3}\beta\right)^2 + \right. \\
&\qquad \left.\frac{1}{2}\frac{\alpha}{\beta}\left(-y\frac{\partial^2 f}{\partial x^2}\beta + \frac{y^3}{6}\frac{\partial^4 f}{\partial x^4}\beta^2\right)^2 + \boldsymbol{O}\left(\beta^2\right)\right]\Bigg|_{y=1+\alpha\eta} \\
&\simeq \left[\eta + \frac{\partial f}{\partial t} - \frac{y^2}{2}\frac{\partial^3 f}{\partial x^2\,\partial t}\beta + \frac{\alpha}{2}\left[\left(\frac{\partial f}{\partial x}\right)^2 - y^2\frac{\partial f}{\partial x}\frac{\partial^3 f}{\partial x^3}\beta\right] + \frac{\alpha}{2}\left[y^2\left(\frac{\partial^2 f}{\partial x^2}\right)^2\beta\right] + \boldsymbol{O}\left(\beta^2\right)\right]\Bigg|_{y=1+\alpha\eta} \\
&\simeq \left[\eta + \frac{\partial f}{\partial t} - \frac{y^2}{2}\frac{\partial^3 f}{\partial x^2\,\partial t}\beta + \frac{\alpha}{2}\left(\frac{\partial f}{\partial x}\right)^2 + \boldsymbol{O}\left(\alpha\beta,\beta^2\right)\right]\Bigg|_{y=1+\alpha\eta} \\
&= \eta + \frac{\partial f}{\partial t} - \frac{(1+\alpha\eta)^2}{2}\frac{\partial^3 f}{\partial x^2\,\partial t}\beta + \frac{\alpha}{2}\left(\frac{\partial f}{\partial x}\right)^2 + \boldsymbol{O}\left(\alpha\beta,\beta^2\right) \\
&\simeq \eta + \frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^3 f}{\partial x^2\,\partial t}\beta + \frac{\alpha}{2}\left(\frac{\partial f}{\partial x}\right)^2 + \boldsymbol{O}\left(\alpha\beta,\beta^2\right)
\end{aligned}
\tag{6.11}
$$

To clean upthe notation, first differentiate equation (6.11) with respect to x:

$$
0 = \frac{\partial\eta}{\partial x} + \frac{\partial^2 f}{\partial x\,\partial t} - \frac{\beta}{2}\frac{\partial^4 f}{\partial x^3\partial t} + \alpha\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x^2}. \tag{6.12}
$$

Now, define the function $w = \frac{\partial f}{\partial x}$. Substitute w into equations (6.10) and (6.12) to get

$$
0 = \frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}\left[w(1+\alpha\eta)\right] - \frac{\beta}{6}\frac{\partial^3 w}{\partial x^3} + \boldsymbol{O}\left(\alpha\beta,\beta^2\right) \tag{6.13}
$$

$$
0 = \frac{\partial\eta}{\partial x} + \frac{\partial w}{\partial t} - \frac{\beta}{2}\frac{\partial^3 w}{\partial x^2\,\partial t} + \alpha w\frac{\partial w}{\partial x} + \boldsymbol{O}\left(\alpha\beta,\beta^2\right). \tag{6.14}
$$

6.4 The KdV Equation

Consider the equations at the free surface boundary, (6.13) and (6.14). To first order approximation, these equations give

$$\frac{\partial w}{\partial x} = -\frac{\partial \eta}{\partial t} + \boldsymbol{O}(\alpha, \beta)$$
$$\frac{\partial w}{\partial t} = -\frac{\partial \eta}{\partial x} + \boldsymbol{O}(\alpha, \beta)$$

If we accept that, to first order,

$$\frac{\partial \eta}{\partial t} = -\frac{\partial \eta}{\partial x} + \boldsymbol{O}(\alpha, \beta), \tag{6.15}$$

then solutions to this system are of the form

$$w = \eta + \boldsymbol{O}(\alpha, \beta).$$

To find a more general expression for w, we look for some functions of η and its derivatives with respect to x, called A and B, such that w is of the form

$$w = \eta + \alpha A + \beta B + \boldsymbol{O}\left(\alpha^2, \alpha\beta, \beta^2\right).$$

Then we have

$$0 = \frac{\partial \eta}{\partial t} + \left(\frac{\partial \eta}{\partial x} + \alpha\frac{\partial A}{\partial x} + \beta\frac{\partial B}{\partial x}\right) + 2\alpha\eta\frac{\partial \eta}{\partial x} - \frac{\beta}{6}\frac{\partial^3 \eta}{\partial x^3} + \boldsymbol{O}\left(\alpha^2, \alpha\beta, \beta^2\right) \tag{6.16}$$
$$0 = \frac{\partial \eta}{\partial x} + \left(\frac{\partial \eta}{\partial t} + \alpha\frac{\partial A}{\partial t} + \beta\frac{\partial B}{\partial t}\right) - \frac{\beta}{2}\frac{\partial^3 \eta}{\partial x^2\,\partial t} + \alpha\eta\frac{\partial \eta}{\partial x} + \boldsymbol{O}\left(\alpha^2, \alpha\beta, \beta^2\right). \tag{6.17}$$

Group the α and β terms to get

$$0 = \left(\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}\right) + \alpha\left(\frac{\partial A}{\partial x} + 2\eta\frac{\partial \eta}{\partial x}\right) + \beta\left(\frac{\partial B}{\partial x} - \frac{1}{6}\frac{\partial^3 \eta}{\partial x^3}\right) + \boldsymbol{O}\left(\alpha^2, \alpha\beta, \beta^2\right) \tag{6.18}$$

$$0 = \left(\frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial t}\right) + \alpha\left(\frac{\partial A}{\partial t} + \eta\frac{\partial \eta}{\partial x}\right) + \beta\left(\frac{\partial B}{\partial t} - \frac{1}{2}\frac{\partial^3 \eta}{\partial x^2\,\partial t}\right) + \boldsymbol{O}\left(\alpha^2, \alpha\beta, \beta^2\right). \tag{6.19}$$

These two equations are consistent only if the α terms and the β terms both agree. Since we are multiplying these by α and β respectively, we need only approximate to first order. Since A and B are functions of η and its x derivatives, we may use equation (6.15) to find that

$$\begin{aligned}\frac{\partial A}{\partial t} &= \frac{\partial A}{\partial \eta}\frac{\partial \eta}{\partial t} + \ldots \\ &\simeq -\frac{\partial A}{\partial \eta}\frac{\partial \eta}{\partial x} + \ldots + \boldsymbol{O}(\alpha, \beta) \\ &= -\frac{\partial A}{\partial x} + \boldsymbol{O}(\alpha, \beta),\end{aligned} \tag{6.20a}$$

and likewise,

$$\frac{\partial B}{\partial t} = -\frac{\partial B}{\partial x} + \boldsymbol{O}(\alpha, \beta). \tag{6.20b}$$

Taking advantage of equation (6.20), the α terms of equations (6.18) and (6.19) give us

$$\begin{aligned}\frac{\partial A}{\partial x} + 2\eta\frac{\partial \eta}{\partial x} &\simeq \frac{\partial A}{\partial t} + \eta\frac{\partial \eta}{\partial x} + \boldsymbol{O}(\alpha, \beta) \\ &= -\frac{\partial A}{\partial x} + \eta\frac{\partial \eta}{\partial x} + \boldsymbol{O}(\alpha, \beta),\end{aligned}$$

so

$$2\frac{\partial A}{\partial x} = -\eta\frac{\partial \eta}{\partial x} + \boldsymbol{O}(\alpha, \beta),$$

and, by integrating,

$$\begin{aligned} A &= \frac{-1}{2}\int \eta \frac{\partial \eta}{\partial x}\,\mathrm{d}x + \boldsymbol{O}(\alpha,\beta) \\ &= \frac{-1}{2}\int \eta\,\mathrm{d}\eta + \boldsymbol{O}(\alpha,\beta) \\ &= \frac{-1}{4}\eta^2 + \kappa_A + \boldsymbol{O}(\alpha,\beta). \end{aligned}$$

Likewise, the β terms give us

$$\begin{aligned} \frac{\partial B}{\partial x} - \frac{1}{6}\frac{\partial^3 \eta}{\partial x^3} &= \frac{\partial B}{\partial t} - \frac{1}{2}\frac{\partial^3 \eta}{\partial x^2\,\partial t} + \boldsymbol{O}(\alpha,\beta) \\ \frac{\partial B}{\partial x} - \frac{1}{6}\frac{\partial^3 \eta}{\partial x^3} &= -\frac{\partial B}{\partial x} + \frac{1}{2}\frac{\partial^3 \eta}{\partial x^3} + \boldsymbol{O}(\alpha,\beta) \\ 2\frac{\partial B}{\partial x} &= \frac{2}{3}\frac{\partial^3 \eta}{\partial x^3} + \boldsymbol{O}(\alpha,\beta) \\ B &= \frac{1}{3}\frac{\partial^2 \eta}{\partial x^2} + \kappa_B + \boldsymbol{O}(\alpha,\beta), \end{aligned}$$

for some constants of integration, κ_A and κ_B. We may assume that both of these are zero, up to our approximation.

Thus, we have the expression for w

$$w = \eta - \frac{\alpha}{4}\eta^2 + \frac{\beta}{3}\frac{\partial^2 \eta}{\partial x^2} + \boldsymbol{O}\left(\alpha^2, \alpha\beta, \beta^2\right). \tag{6.21}$$

And so, the free surface boundary conditions become a single equation

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} + \frac{3}{2}\alpha\eta\frac{\partial \eta}{\partial x} + \frac{1}{6}\beta\frac{\partial^3 \eta}{\partial x^3} + \boldsymbol{O}\left(\alpha^2, \alpha\beta, \beta^2\right) = 0. \tag{6.22}$$

This is the normalized form of the KdV Equation.

6.5 Particle Velocity

Consider

$$\begin{aligned} u &\equiv \frac{\partial \varphi}{\partial x} \\ &\simeq \frac{\partial}{\partial x}\left[f - \frac{y^2}{2}\beta\frac{\partial^2 f}{\partial x^2} + \boldsymbol{O}\left(\beta^2\right)\right] \\ &= \frac{\partial f}{\partial x} - \frac{y^2}{2}\beta\frac{\partial^3 f}{\partial x^3} + \boldsymbol{O}\left(\beta^2\right) \\ &= w - \frac{y^2}{2}\beta\frac{\partial^2 w}{\partial x^2} + \boldsymbol{O}\left(\beta^2\right). \end{aligned}$$

And so,

$$\frac{\partial \varphi}{\partial x} = \eta - \frac{\alpha}{4}\eta^2 + \beta\left(\frac{1}{3} - \frac{y^2}{2}\right)\frac{\partial^2 \eta}{\partial x^2} + \boldsymbol{O}\left(\alpha^2, \alpha\beta, \beta^2\right). \tag{6.23}$$

7 Solitons

7.1 Using the KdV Approximation

The free surface of a soliton is given by [17]

$$\eta(x,t) = \operatorname{sech}^2\left[\sqrt{\frac{3\alpha}{4\beta}}\left[x - \left(1 + \frac{\alpha}{2}\right)t\right]\right]. \tag{7.1}$$

It may also be useful to describe a soliton using coordinates of physical dimension. Doing so, the free surface is given by

$$\eta'(x',t') = a\operatorname{sech}^2\left[\sqrt{\frac{3a}{4h^3}}\left(x' - Ut'\right)\right], \tag{7.2}$$

where

$$U = \sqrt{gh}\left(1 + \frac{1}{2}\frac{a}{h}\right),$$

is the speed at which the wave travels in the $+x$-direction.

7.2 Particle Velocities

To ease notation, define

$$\theta(x,t) = \sqrt{\frac{3\alpha}{4\beta}}\left[x - \left(1 + \frac{\alpha}{2}\right)t\right], \tag{7.3}$$

and note that

$$\frac{\partial\theta}{\partial x} = \sqrt{\frac{3\alpha}{4\beta}}.$$

Consider equation(6.23):

$$\frac{\partial\varphi}{\partial x} = \eta - \frac{\alpha}{4}\eta^2 + \beta\left(\frac{1}{3} - \frac{y^2}{2}\right)\frac{\partial^2\eta}{\partial x^2} + \boldsymbol{O}\left(\alpha^2, \alpha\beta, \beta^2\right).$$

Thus,

$$\frac{\partial^2\varphi}{\partial x\,\partial y} = -\beta y\frac{\partial^2\eta}{\partial x^2} + \boldsymbol{O}\left(\alpha^2, \alpha\beta, \beta^2\right),$$

so

$$\frac{\partial\varphi}{\partial y} = -\beta y\frac{\partial\eta}{\partial x} + \kappa(y) + \boldsymbol{O}\left(\alpha^2, \alpha\beta, \beta^2\right),$$

where $\kappa(y)$ is some constant of integration with respect to x, possibly a function of y.

From equation (7.1),

$$
\begin{aligned}
\frac{\partial\eta}{\partial x} &= \frac{\partial}{\partial x}\left[\operatorname{sech}^2(\theta)\right] \\
&= -2\operatorname{sech}^2(\theta)\tanh(\theta)\frac{\partial\theta}{\partial x} \\
&= -\sqrt{\frac{3\alpha}{\beta}}\operatorname{sech}^2(\theta)\tanh(\theta),
\end{aligned}
$$

so

$$
\begin{aligned}
\frac{\partial^2\eta}{\partial x^2} &= \left(-2\frac{\partial\theta}{\partial x}\right)\frac{\partial}{\partial x}\left[\operatorname{sech}^2(\theta)\tanh(\theta)\right] \\
&= \left(-2\frac{\partial\theta}{\partial x}\right)\left[\operatorname{sech}^2(\theta)\operatorname{sech}^2(\theta)\frac{\partial\theta}{\partial x} + \tanh(\theta)\left(-2\operatorname{sech}^2(\theta)\tanh(\theta)\frac{\partial\theta}{\partial x}\right)\right] \\
&= 2\left(\frac{\partial\theta}{\partial x}\right)^2\left[2\operatorname{sech}^2(\theta)\tanh^2(\theta) - \operatorname{sech}^4(\theta)\right] \\
&= \frac{3\alpha}{2\beta}\operatorname{sech}^2(\theta)\left[2\tanh^2(\theta) - \operatorname{sech}^2(\theta)\right] \\
&= \frac{3\alpha}{2\beta}\operatorname{sech}^2(\theta)\left[2 - 2\operatorname{sech}^2(\theta) - \operatorname{sech}^2(\theta)\right] \\
&= \frac{3\alpha}{2\beta}\operatorname{sech}^2(\theta)\left[2 - 3\operatorname{sech}^2(\theta)\right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{\partial\varphi}{\partial y} &\simeq -\beta y\frac{\partial\eta}{\partial x} \\
&= \beta y\sqrt{\frac{3\alpha}{\beta}}\operatorname{sech}^2(\theta)\tanh(\theta) \\
&= y\sqrt{3\alpha\beta}\operatorname{sech}^2(\theta)\tanh(\theta), \qquad (7.4)
\end{aligned}
$$

and

$$\begin{aligned}
\frac{\partial \varphi}{\partial x} &\simeq \eta - \frac{\alpha}{4}\eta^2 + \beta\left(\frac{1}{3} - \frac{y^2}{2}\right)\frac{\partial^2 \eta}{\partial x^2} \\
&= \operatorname{sech}^2(\theta) - \frac{\alpha}{4}\operatorname{sech}^4(\theta) + \beta\left(\frac{1}{3} - \frac{y^2}{2}\right)\left[\frac{3\alpha}{2\beta}\operatorname{sech}^2(\theta)\left[2 - 3\operatorname{sech}^2(\theta)\right]\right] \\
&= \operatorname{sech}^2(\theta) - \frac{\alpha}{4}\operatorname{sech}^4(\theta) + \alpha\left(\frac{1}{2} - \frac{3}{4}y^2\right)\left(\operatorname{sech}^2(\theta)\left[2 - 3\operatorname{sech}^2(\theta)\right]\right) \\
&= \operatorname{sech}^2(\theta) - \frac{\alpha}{4}\operatorname{sech}^4(\theta) + \left(\frac{\alpha}{2} - \frac{3\alpha}{4}y^2\right)\left(2\operatorname{sech}^2(\theta) - 3\operatorname{sech}^4(\theta)\right) \\
&= \operatorname{sech}^2(\theta) - \frac{\alpha}{4}\operatorname{sech}^4(\theta) + \alpha\operatorname{sech}^2(\theta) - \frac{3\alpha}{2}y^2\operatorname{sech}^2(\theta) + \\
&\qquad - \frac{3\alpha}{2}\operatorname{sech}^4(\theta) + \frac{9\alpha}{4}y^2\operatorname{sech}^4(\theta) \\
&= (1+\alpha)\operatorname{sech}^2(\theta) - \left(\frac{3\alpha}{2} + \frac{\alpha}{4}\right)\operatorname{sech}^4(\theta) - \frac{3\alpha}{2}y^2\left(\operatorname{sech}^2(\theta) - \frac{3}{2}\operatorname{sech}^4(\theta)\right) \\
&= \left(1 + \alpha - \frac{7\alpha}{4}\operatorname{sech}^2(\theta)\right)\operatorname{sech}^2(\theta) - \frac{3\alpha}{4}y^2\left(2 - 3\operatorname{sech}^2(\theta)\right)\operatorname{sech}^2(\theta) \\
&= \frac{1}{4}\operatorname{sech}^2(\theta)\left[4 + \alpha\left(4 - 7\operatorname{sech}^2(\theta)\right) - 3\alpha y^2\left(2 - 3\operatorname{sech}^2(\theta)\right)\right].
\end{aligned} \tag{7.5}$$

7.3 Particle Trajectories

In the change to non-dimensional coordinates (6.6), we defined our new coordinate y in terms of the water depth, h, but defined x and t in terms of a vaguely defined length scale, ℓ. While this does not prevent us from determining particle trajectories, the different scales make interpreting our results difficult.

Thus, we change to a new set of coordinates for plotting purposes. We define $(\hat{x}, \hat{y}, \hat{t})$ by

$$\begin{aligned}
\hat{x} &= \frac{x'}{h} = \frac{\ell}{h}x = \frac{x}{\sqrt{\beta}},\\
\hat{y} &= \frac{y'}{h} = y,\\
\hat{t} &= \sqrt{\frac{g}{h}}t' = \frac{\ell}{h}t = \frac{t}{\sqrt{\beta}}.
\end{aligned}$$

Under these coordinates, the horizontal velocity is given as

$$\hat{u} = \frac{\mathrm{d}\hat{x}}{\mathrm{d}\hat{t}} = \frac{1}{\sqrt{gh}}\frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{u'}{\sqrt{gh}} = \frac{1}{\sqrt{gh}}\frac{\partial\varphi'}{\partial x'}.$$

But, using the non-dimensional coordinates from the previous chapter,

$$\frac{\partial\varphi'}{\partial x'} = \sqrt{\frac{g}{h}}\,a\frac{\partial\varphi}{\partial x},$$

and so

$$\hat{u} = \frac{a}{h}\frac{\partial\varphi}{\partial x} = \alpha\frac{\partial\varphi}{\partial x}.$$

Likewise,

$$\hat{v} = \frac{\alpha}{\sqrt{\beta}}\frac{\partial\varphi}{\partial y}.$$

Using these coordinates, first equation (7.3) becomes

$$\begin{aligned}
\theta &= \sqrt{\frac{3\alpha}{4\beta}}\left[\sqrt{\beta}\,\hat{x} - \left(1 + \frac{\alpha}{2}\right)\sqrt{\beta}\,\hat{t}\right]\\
&= \sqrt{\frac{3}{4}\alpha}\left[\hat{x} - \left(1 + \frac{\alpha}{2}\right)\hat{t}\right].
\end{aligned}$$

Since the coordinate variables do not appear explicitly in equations (7.5) or (7.4), we can now say that the trajectories of particles are given by the solutions to the system

$$\frac{d\hat{x}}{d\hat{t}} = \frac{\alpha}{4}\operatorname{sech}^2(\theta)\left[4+\alpha\left(4-7\operatorname{sech}^2(\theta)\right)-3\alpha y^2\left(2-3\operatorname{sech}^2(\theta)\right)\right] \tag{7.6a}$$

$$\frac{d\hat{y}}{d\hat{t}} = y\alpha\sqrt{3\alpha}\operatorname{sech}^2(\theta)\tanh(\theta). \tag{7.6b}$$

It is interesting that, while the KdV equations are a two parameter system, this system for solitons depend only on a single parameter – the wave amplitude, α.

7.4 Plots of Trajectories of Solitons

All figures are printed in Appendix C. Plots of particle trajectories in solitons appear starting on page 78. Each graph was created in XPPAUT using the `KdV.ode` script listed in Appendix B.

Figures 16 through 21 show the trajectories of particles for α ranging from 0.2 through 0.7. In each graph, we follow the trajectory of two particles, located at $\hat{y} = 0.8$ and $\hat{y} = 0.2$ when $\hat{x} = 0$ and $\hat{t} = 0$, and graph its movement in forwards and backwards time. (It moves to the right in forward time and to the left in backwards time.)

In each figure, we see that the particle moves in a single arc. The obvious feature of the graphs is that for larger α's, the particle travels higher than it does for smaller α's. Less obvious at first inspection, the larger α's also cause the particle to travel further in the $\hat{x}$ direction. Most clearly evident for the larger values of α and for the trajectory starting at $\hat{y} = 0.2$, the arcs flatten out near the tails.

8 General Results for Solitary Waves

We now examine some general theorems on the trajectories of particles in solitary waves, many of which were first discussed by Constantin and Escher [6].

We make the following observations using only the Euler equations (4.2) along with the boundary conditions for solitary waves listed in sections 4.4 and 4.5. First, Craig and Sternberg [9] observe that

Proposition 8.1. *All solitary waves are of positive elevation above their asymptotic limit, and with a strictly monotone wave profile on either side of this crest.*

The horizontal component of velocity, $u(x, y)$, and the free surface, $\eta(x)$, are symmetric with respect to the wave crest, $x = 0$. The vertical component of velocity, $v(x, y)$, is antisymmetric with respect to the crest.

Then, from the work of Amick and Toland [1], we have that

Proposition 8.2. *As $|x| \to \infty$, $u^*, v^* \to 0$ exponentially fast. Moreover, $v^* \nearrow 0$ as $x \to -\infty$ and $v^* \searrow 0$ as $x \to +\infty$.*

We state these propositions here without proof, but will build on them in the following lemmas.

Lemma 8.3. *At any time, t, the vertical component of the velocity, v^*, is determined by the following: $v^* = 0$ when $x = 0$ and on $\mathcal{B}$. For all $\vec{x} \in \mathfrak{D}$, $v^* > 0$ if $x > 0$ and $v^* < 0$ if $x < 0$.*

Proof. First, the boundary condition on the flat bed, (4.10), explicitly gives us that $v^* = 0$ on $\mathcal{B}$. Since solitary waves are symmetric by Proposition 8.1, when $x = 0$, we have $v^* = 0$. This also gives us that $v^* > 0 \; \forall \; x > 0$ iff $v^* < 0 \; \forall \; x < 0$. So, without loss of generality, we need only consider $x < 0$, and show that $v^* < 0$.

By Proposition 8.2, $v^* \nearrow 0$ as $x \to -\infty$.

Recall, the definition of the stream function, (4.24),

$$\frac{\partial \psi}{\partial y} = u^* - c < 0.$$

When $x < 0$, $y = \eta(x)$ is strictly increasing by Proposition 8.2, so $\frac{\partial \eta}{\partial x} > 0$. Thus, $\forall \vec{x} \in \mathcal{S}$ s.t. $x < 0$, the kinematic boundary condition, (4.9) gives us:

$$\begin{aligned} v^* &= (u^* - c)\frac{\partial \eta}{\partial x} \\ &< 0 \end{aligned} \tag{8.1}$$

We have just shown that $v^* \leqslant 0$ on $x = 0$, $y = -h$, $y = \eta(x)$, and $x \to -\infty$. Since v^* is harmonic, by the maximum principle, Theorem 2.2, $v^* < 0$ in the region between these boundaries – where $-\infty < x < 0$ and $0 < y < \eta(x)$. ■

Lemma 8.4. *At any time, t, the horizontal component of the velocity, u^*, is positive $\forall \vec{x} \in \mathfrak{D}$.*

Proof. By Hopf's Maximum Principle, Theorem 2.3, the minimum of P on $\overline{\mathfrak{D}}$ must be attained on $\partial\mathfrak{D} = \mathcal{S} \cup \mathcal{B}$. Starting from the special case of Bernoulli's Law for solitary waves, (4.18), we have

$$P(x, y) = P_\circ - gy + \frac{1}{2}\left[c^2 - v^{*2} - (u^* - c)^2\right]. \tag{8.2}$$

As $|x| \to \infty$, u^* and $v^* \to 0$ by the boundary conditions on solitary waves, (4.11a) and

(4.11b), so

$$\begin{aligned} P(x,y) \to & P_\circ - gy + \frac{1}{2}\left[c^2 - (c)^2\right] \\ = & P_\circ - gy \\ \geqslant & P_\circ. \end{aligned} \tag{8.3}$$

Recall, from the dynamic boundary condition, (4.8),

$$P(\vec{x}) = P_\circ \; \forall \; \vec{x} \in \mathcal{S}.$$

Thus, we have at $|x| \to \infty$, $\min P = P_\circ$, which is attained on $\mathcal{S}$.

Again using the boundary condition on the flat bed, (4.10), $v^*(x,0) = 0$. Thus, on $\mathcal{B}$, v^* is constant with respect to x; or, $\frac{\partial v^*}{\partial x} = 0$. When combined with the Euler Equations (4.5), we get that

$$\frac{\partial P}{\partial y} = -g < 0 \quad \forall \; \vec{x} \in \mathcal{B}. \tag{8.4}$$

Or, in other words, as y increases above the flat bed, the pressure decreases. Thus, $\min P$ cannot occur on the flat bed, $\mathcal{B}$.

And so, we have

$$\min_{\vec{x} \in \overline{\mathfrak{D}}} P(\vec{x}) = P_\circ, \tag{8.5}$$

which occurs $\forall \vec{x} \in \mathcal{S}$. Furthermore, since P is superharmonic (as shown in Appendix A.1), Hopf's Maximum Principle gives us the strict inequality

$$P(\vec{x}) > P_\circ \quad \forall \; \vec{x} \in \mathfrak{D}. \tag{8.6}$$

Next, consider the free surface, $\mathcal{S}$, where $y = \eta(x)$. When $x < 0$, $v^* < 0$ by lemma 8.3. As cited in Appendix A.2, $u^* - c < 0 \ \forall (x, y) \in \mathfrak{D}$. The kinematic boundary condition, (4.9) gives us $v^* = (u^* - c)\frac{\partial \eta}{\partial x}$, so we have that $\frac{\partial \eta}{\partial x} > 0 \ \forall \ x < 0$. Hence, $0 < \eta(x) < \eta(x + \delta x)$ for an infinitesimal positive step, δx. That is to say, the point $(x + \delta x, \eta(x))$ is between the boundaries. By our previous statement by the maximum principle, $P(x, \eta(x)) < P(x + \delta x, \eta(x))$. Therefore,

$$\left.\frac{\partial P}{\partial x}\right|_{y=\eta(x);\ x<0} > 0. \tag{8.7}$$

And, by a similar argument using the symmetry of $\eta(x)$,

$$\left.\frac{\partial P}{\partial x}\right|_{y=\eta(x);\ x>0} < 0. \tag{8.8}$$

Recall, (4.5a):

$$(u^* - c)\frac{\partial u^*}{\partial x} + v^*\frac{\partial u^*}{\partial y} = -\frac{\partial P}{\partial x}.$$

Combining this with the kinematic boundary condition, (4.9), we find that on the free surface,

$$\frac{\partial P}{\partial x} = -(u^* - c)\left[\frac{\partial u^*}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial u^*}{\partial y}\right] \tag{8.9}$$

$$= (c - u^*)\frac{\mathrm{d}}{\mathrm{d}x}\left[u^*(x, \eta(x))\right]. \tag{8.10}$$

As before, $c - u^* > 0$, so we get that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[u^*(x, \eta(x))\right] > 0 \qquad \text{when } x < 0 \tag{8.11a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[u^*(x, \eta(x))\right] < 0 \qquad \text{when } x > 0. \tag{8.11b}$$

That is to say, on the free surface, u^* is strictly increasing for $x < 0$ and u^* is strictly decreasing for $x > 0$.

Since the boundary condition on the flat bed, (4.10) gives us that $v^* = 0$ on $\mathcal{B}$ for all x, $\frac{\partial v^*}{\partial x} = 0$ on $\mathcal{B}$. Also, by the condition of irrotationality (4.7) $\frac{\partial u^*}{\partial y} = \frac{\partial v^*}{\partial x}$, so $\frac{\partial u^*}{\partial y} = 0$ when $y = -h$. Since u^* is harmonic, the first derivative test and the maximum principle (Theorem 2.2) ensure that neither $\max u^*$ nor $\min u^*$ occur on the flat bed.

Since $u^* \to 0$ as $|x| \to \infty$ by solitary wave boundary condition (4.11a), the monotonicity of u^* from (8.11) gives us that $u^* > 0 \; \forall \vec{x} \in \mathfrak{D}$. Furthermore, (8.11) also gives us that $\max u^*$ is attained on the wave crest, $(0, \eta(0))$. ■

Lemma 8.5. *For a wave of greatest height,* $c - u^*(x, \eta(x)) \simeq \boldsymbol{O}\left(\sqrt{x}\right)$ *as* $x \searrow 0$.

Proof. Starting with Bernoulli's Law, (4.16):

$$\frac{1}{2}(u^* - c)^2 + \frac{1}{2}v^{*2} + P + gy = \frac{1}{2}c^2 + P_\circ.$$

On $\mathcal{S}$, $P = P_\circ$ by the dynamic boundary condition, (4.8), so we can rearrange the terms in the above equation to get

$$(u^* - c)^2 + v^{*2} = c^2 - 2g\eta(x). \tag{8.12}$$

For a wave of greatest height, $\eta(0) = c^2/2g$. Also, $\eta(x)$ may be approximated as linear as $x \to 0$. Hence, for some positive constant κ, we have

$$\begin{aligned}(u^* - c)^2 + (v^*)^2 &\to c^2 - c^2(1 - \kappa|x|)\\ &= c^2\kappa|x|\\ &\simeq \boldsymbol{O}(x).\end{aligned} \tag{8.13}$$

Define $\theta(x_\circ)$ to be the angle measured counterclockwise from the wave profile ($y = \frac{\partial\eta}{\partial x}(x_\circ)$) to the horizontal ($y = \eta(x_\circ)$). Since a wave of greatest height has an angle of $2\pi/3$ at $(0, \eta(0))$ and the wave profile is symmetric about $x = 0$,

$$\lim_{x\to 0^-} \theta(x) = \frac{\pi}{6} \quad \text{and} \quad \lim_{x\to 0^+} \theta(x) = -\frac{\pi}{6}. \tag{8.14}$$

However, we also know that

$$\tan(\theta(x)) \equiv \frac{\partial\eta}{\partial x} = \frac{v^*(x, \eta(x))}{u^*(x, \eta(x)) - c}, \tag{8.15}$$

whenever $x \neq 0$. As $|x| \to 0$, $\tan(\theta(x)) \to \left|\frac{\sqrt{3}}{3}\right|$. And so (8.15) gives us that

$$u^* - c \to \left|\sqrt{3}\, v^*\right| \text{ and } v^* \to \left|\frac{u^* - c}{\sqrt{3}}\right| \text{ on } \mathcal{S} \text{ as } x \to 0. \tag{8.16}$$

Finally, combining equation (8.16) with equation (8.13) gives us that

$$(u^* - c)^2 \simeq \boldsymbol{O}(x) \text{ and } v^{*2} \simeq \boldsymbol{O}(x) \text{ on } \mathcal{S} \text{ as } x \to 0,$$

or

$$u^* - c \simeq \boldsymbol{O}\left(\sqrt{x}\right) \text{ and } v^* \simeq \boldsymbol{O}\left(\sqrt{x}\right) \text{ on } \mathcal{S} \text{ as } x \to 0. \tag{8.17}$$

■

Lemma 8.6. *For each particle, $\exists\, t_\circ \in \mathbb{R}$ s.t. for $t = t_\circ$, the particle is exactly below the wave crest. For $t < t_\circ$, the particle is ahead of the wave crest, and for $t > t_\circ$, the particle is behind the wave crest.*

Proof. We start with the system of differential equations from (4.30):

$$\xi'(t) = u^*(\xi(t), \zeta(t)) - c,$$
$$\zeta'(t) = v^*(\xi(t), \zeta(t)).$$

From Theorem A.4, $u^*(x, y) \leqslant c \ \forall (x, y) \in \overline{\mathfrak{D}}$.

If we are not dealing with a wave of greatest height, then we have the strict inequality

$$u^*(x, y) < c \qquad \forall\, (x, y) \in \overline{\mathfrak{D}}. \tag{8.19}$$

Thus $\exists\, \varepsilon > 0$ s.t. $u^* \leqslant c - \varepsilon \ \forall (x, y) \in \overline{\mathfrak{D}}$. Hence, for $t > 0$,

$$\begin{aligned}
\xi(t) &= \xi_\circ + \int_0^t \xi'(s)\,\mathrm{d}s \\
&= \xi_\circ + \int_0^t \left[u^*(\xi, \zeta) - c\right]\,\mathrm{d}s \\
&\leqslant \xi_\circ + \int_0^t [(c - \varepsilon) - c]\,\mathrm{d}s \\
&= \xi_\circ - \varepsilon \int_0^t \mathrm{d}s \\
&= \xi_\circ - \varepsilon t.
\end{aligned} \tag{8.20}$$

If $t < 0$, we instead get

$$\begin{aligned}
\xi(t) &= \xi_\circ - \int_t^0 \xi'(s)\,\mathrm{d}s \\
&= \xi_\circ - \int_t^0 \left[u^*(\xi,\zeta) - c\right]\,\mathrm{d}s \\
&\geqslant \xi_\circ - \int_t^0 [(c-\varepsilon) - c]\,\mathrm{d}s \\
&= \xi_\circ + \varepsilon \int_t^0 \mathrm{d}s \\
&= \xi_\circ - \varepsilon t \\
&= \xi_\circ + \varepsilon\,|t|\,. \qquad (8.21)
\end{aligned}$$

Without loss of generality, if $\xi(t_\circ) = 0$, a shift in the time coordinate gives us our result.

If, instead, we do have a wave that is of greatest height, work still needs to be done. In this case $u = c$ at $(0, \eta(0))$. Since ψ is the Hamiltonian function for the system by definition 2.6, we have

$$\xi'(t) \equiv \frac{\partial \psi}{\partial y} = u^* - c \leqslant 0 \quad \forall\, t. \qquad (8.22)$$

Note that equality holds iff the particle is on the crest of the wave at some time $t_\circ$; that is, $(\xi(t_\circ), \zeta(t_\circ)) = (0, \eta(0))$

Case 1: $(\xi(t), \zeta(t))$ *never approaches* $(0, \eta(0))$. Then equation (8.22) becomes a strict inequality, and the theorem holds exactly as shown for a wave not of greatest height.

Case 2: $(\xi(t), \zeta(t)) = (0, \eta(0))$ *for some finite or infinite time, t.*

Since $u \leqslant c$, we need only consider particles with $\xi_\circ > 0$ in positive time.

Claim 1: The particle reaches the wave crest in finite time.

Clearly, a particle on the free surface must remain on the free surface. Thus, we have that

$$\zeta(t) = h_0(\xi(t)) = \eta(\xi(t)),$$

which implies that

$$(\xi(t), \zeta(t)) \in \mathcal{S} \quad \forall\, t.$$

Since the wave crest is always moving in the $+x$-direction as fast or faster than each particle in $\overline{\mathfrak{D}}$, we may assume that $\xi(0) > 0$.

From equation (4.30a),

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = u^*(\xi(t), \zeta(t)) - c,$$

so

$$1 = \frac{1}{u^*(\xi, \zeta) - c}\frac{\mathrm{d}\xi}{\mathrm{d}t}.$$

To determine the total amount of time, T, it takes a particle located at $(\xi_\circ, \zeta_\circ)$ at time $t = 0$ to reach $(0, \eta(0))$, consider

$$\begin{aligned} T &= \int_0^T \mathrm{d}t \\ &= \int_0^T \frac{1}{u(\xi, \eta(\xi)) - c}\frac{\mathrm{d}\xi}{\mathrm{d}t}\,\mathrm{d}t \\ &= \int_{\xi(0)}^{\xi(T)} \frac{1}{u(\xi, \eta(\xi)) - c}\,\mathrm{d}\xi \\ &= \int_{\xi_\circ}^{0} \frac{\mathrm{d}x}{u(x, \eta(x)) - c} \\ &= \int_0^{\xi_\circ} \frac{\mathrm{d}x}{c - u(x, \eta(x))}. \end{aligned} \tag{8.23}$$

By Lemma 8.5, the denominator of the integrand in equation (8.23) goes to zero as $\sqrt{x}$ and is clearly non-zero and finite elsewhere. Thus, for $\delta\xi$ sufficiently small,

$$\begin{aligned} T &= \int_0^{\xi_\circ} \frac{\mathrm{d}x}{c - u(x,\eta(x))} \\ &\simeq \int_{\delta\xi}^{\xi_\circ} \frac{\mathrm{d}x}{c - u(x,\eta(x))} + \int_0^{\delta\xi} \frac{\mathrm{d}x}{\sqrt{x}} \\ &< \infty. \end{aligned}$$

Thus, any particle which drifts onto the crest of the wave will do so in finite time.

Claim 2: The particle rests at the wave crest for a infinitesimal period of time.

Physical reasoning tells us that particles do not 'build up' at the wave crest. However, we know from the arguments presented in Claim 1 that particles are continually arriving at the wave crest from the $+x$ region of $\mathcal{S}$. Thus, particles must be continually *leaving* the wave crest.

Furthermore, because $u^* < c$, particles cannot move in the $+x$-direction. Thus, particles leaving the wave crest move from $x = 0$ to a region where $x < 0$, which is the result we want. ■

Main Result

With these lemmas, we are now ready to discuss the main result, which gives restrictions on how a particle moves as a solitary wave passes overhead. As an observer in the fixed frame of reference, we see the particle move up and to the right as the wave crest is approaching a particle from the left, until the point at which the crest catches up to the position of the particle. The path of the particle peaks here, at the point directly below the wave crest. Then, as the wave continues past it, the particle continues moving to the right, now falling back down to the height at which it started.

With this information, we can define two curves which we will use to describe the motion of the particle more clearly. Trajectories have a horizontal asymptote, which we will call $y = \ell$, for both $x \longrightarrow +\infty$ and $x \longrightarrow -\infty$. We will show that particle trajectories always remain above this asymptote.

In Section 4.8, we defined streamlines, denoted $y = h_\alpha(x)$, as the curve on which the stream function is a constant, namely $\psi(x, y) = \alpha$. Now we look specifically at the streamline defined by the location of the particle when it reaches the peak of its trajectory. We will show that everywhere else in the path of the particle, it is located below this streamline.

Theorem 8.7. *For solitary waves, the following properties hold:*

A. *A particle on the flat bed moves in a straight line to the right at a positive speed.*

B. *Any particle located at $(\xi(t), \zeta(t))$ above the bed reaches at some instant, $t_\circ$, a location $(0, \zeta_\circ)$ below the wave crest at $(0, \eta(0))$. Let $\alpha = \psi(0, \zeta_\circ)$. For $t \neq t_\circ$, we have the strict inequality $\zeta < h_\alpha(\xi)$.*

C. *The particle path has a horizontal asymptote $y = \ell(\zeta_\circ)$ at $\xi = \pm\infty$. For all t and for all x, $\zeta > \ell(\zeta_\circ)$.*

Proof of A. By Boundary Condition (4.10), a particle on the flat bed stays on the flat bed. By Lemma 8.6, the particle moves to the left with respect to the wave crest, but with speed $0 < u < c$. Thus, with respect to a fixed point on the flat bed, the particle moves to the right with positive speed.

Proof of B. Without loss of generality, we may assume that $t_\circ = 0$. By Lemma 8.6, the particle is located at $x = 0$ at time $t = 0$.

By symmetry, $\zeta < h_\alpha(\xi)$ for $t > 0$ iff $\zeta < h_\alpha(\xi)$ for $t < 0$. So, without loss of generality, assume that $t > 0$.

For convenience, we move back to the fixed coordinate system, (X, Y, T). Define the function $\Psi(X, Y) = \psi(x - ct, y + h)$. Along particle paths, the stream function becomes a one-variable function,

$$\Psi(T) \equiv \Psi(X(T), Y(T)). \tag{8.24}$$

We wish to examine how the stream function evolves with time. Thus, we consider

$$\begin{aligned}\frac{\mathrm{d}\Psi}{\mathrm{d}T} &= \frac{\partial\Psi}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}T} + \frac{\partial\Psi}{\partial Y}\frac{\mathrm{d}Y}{\mathrm{d}T} \\ &= [-V][U] + [U - c][V] \\ &= -cV.\end{aligned}$$

By definition, $c > 0$. Since we assume that $t > 0$, $x < 0$ by Lemma 8.6, and so $v^* < 0$ by Lemma 8.3. Thus,

$$\frac{\mathrm{d}\Psi}{\mathrm{d}T} > 0. \tag{8.25}$$

And so, for $t > 0$,

$$\begin{aligned}\psi(\xi(t), \zeta(t) &= \Psi(\Xi(T), Z(T)) \\ &> \Psi(\Xi(0), Z(0)) \\ &= \psi(0, \zeta_\circ) \\ &\equiv \alpha.\end{aligned} \tag{8.26}$$

Thus, by Proposition 4.1, for $t > 0$, the particle trajectory $(\xi(t), \zeta(t))$ must lie below $h_\alpha(\xi)$.

Proof of C. The streamlines $y = h_\alpha(x)$ are monotone increasing for $x \in (-\infty, 0)$ and are monotone decreasing for $x \in (0, \infty)$.

Starting with equation (8.26) from the proof of B:

$$\psi\,(\xi(t), \zeta(t)) > \alpha \qquad \forall\, t > 0,$$

and so, by symmetry, we also have that

$$\psi\,(\xi(t), \zeta(t)) > \alpha \qquad \forall\, t < 0.$$

Thus,

$$\lim_{x\to\pm\infty} \psi(x, y) = \lim_{t\to\pm\infty} \psi(\xi, \zeta) > \alpha,$$

so

$$\lim_{t\to\pm\infty} \zeta(t) < h_\alpha.$$

Hence, there exists some number, $\ell(\zeta_\circ)$, such that

$$\lim_{t\to\pm\infty} \zeta(t) \leqslant \ell(\zeta_\circ). \tag{8.27}$$

Since ψ is the Hamiltonian Energy Function for (ξ, ζ),

$$H(t) = \psi(\xi(t), \zeta(t)),$$

and

$$H_\circ \equiv H(0) = \psi(0, \zeta_\circ) = \alpha.$$

Recall, from Proposition 2.4, a parametric curve is a solution to a Hamiltonian system only if $H(t)$ is constant along that curve. Thus,

$$\alpha = \psi(\xi(t), \zeta(t)) \qquad \forall\, t.$$

And so,

$$\begin{aligned} \ell(\zeta_\circ) &= \lim_{x\to\pm\infty} h_\alpha(x) \\ &\leqslant \lim_{t\to\pm\infty} h_\alpha(\xi(t)) \\ &= \lim_{t\to\pm\infty} \zeta(t). \end{aligned} \tag{8.28}$$

Finally, by combining equations (8.27) and (8.28), we see that

$$\ell(\zeta_\circ) = \lim_{t\to\pm\infty} \zeta(t),$$

or that ℓ gives the asymptotic limit of the curve (ξ, ζ). ■

9 Discussion

Should educators now run off to change general science texts in light of the information discussed in this document? Probably not. However, it does serve as a reminder that we often use simplified models to introduce concepts, especially for younger students. Used correctly, as this document shows, approximations can be a powerful tool to aid understanding. By considering successively more general approaches to the problem, we are able to focus on adding small pieces of the puzzle with each step.

We first looked at linear water waves. The further linearized trajectories give us closed loop paths. Closed loops are relatively easy to understand, and can begin to allow conjecture into the mechanics of the oceans; for example, we may ask 'does this allow for mixing dissolved materials between deep and shallow waters?'

However, in experiments such as the one photographed by Wallet and Ruellan [16], we see that the paths of particles are not quite closed (Figure 22). To explain these findings, we looked at the system of differential equations for linear waves directly. Numerically calculating the trajectories described by the system without further approximation, we generated forward-drifting, looping paths. This gives us other information that the linearized approximation had obscured; for example, particles near the flat bed propogate more slowly than particles near the surface. Now, we may ask 'Does this phenomenon have an effect on the Earth's ocean currents?'

Amick and Toland [2] showed that with increasing wavelengths, periodic waves converge towards a solitary wave limit. Our numerical investigation of the soliton under the KdV approximation shows that the particle trajectories show no looping behavior, but instead travel in a single arc in the direction in which the wave propogates. This result is then shown to hold in general for solitary waves.

It would be interesting to further investigate the behaviors of trajectories as wavelength increases. We looked briefly at linear waves with long wavelengths in Figures 13 through 15. This series of graphs gives a hint of a possible convergence to a single path in which the particle swings back and forth. Unfortunately, the tools available allow us to make no conclusions in this regard.

Appendix

A Additional Proofs and Propositions

A.1 Proof that Pressure is Superharmonic

Proof. Starting from the Euler Equations, (4.5), Differentiating (4.5a) with respect to x and (4.5b) with respect to y, we get

$$-\frac{\partial^2 P}{\partial x^2} = (u^* - c)\frac{\partial^2 u^*}{\partial x^2} + \left(\frac{\partial u^*}{\partial x}\right)^2 + v^* \frac{\partial^2 u^*}{\partial x\, \partial y} + \frac{\partial v^*}{\partial x}\frac{\partial u^*}{\partial y}, \tag{A.1a}$$

$$-\frac{\partial^2 P}{\partial y^2} = (u^* - c)\frac{\partial^2 v^*}{\partial x\, \partial y} + \frac{\partial u^*}{\partial y}\frac{\partial v^*}{\partial x} + v^* \frac{\partial^2 v^*}{\partial y^2} + \left(\frac{\partial v^*}{\partial y}\right)^2. \tag{A.1b}$$

But, substituting the stream function (4.24) into (A.1) gives us

$$-\frac{\partial^2 P}{\partial x^2} = \frac{\partial \psi}{\partial y}\frac{\partial^3 \psi}{\partial x^2 \partial y} + \left(\frac{\partial^2 \psi}{\partial x \partial y}\right)^2 - \frac{\partial \psi}{\partial x}\frac{\partial^3 \psi}{\partial x \partial y^2} + \frac{\partial^2 \psi}{\partial x^2}\frac{\partial^2 \psi}{\partial y^2}, \tag{A.2a}$$

$$-\frac{\partial^2 P}{\partial y^2} = -\frac{\partial \psi}{\partial y}\frac{\partial^3 \psi}{\partial x^2 \partial y} - \frac{\partial^2 \psi}{\partial y^2}\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial x}\frac{\partial^3 \psi}{\partial x \partial y^2} + \left(-\frac{\partial^2 \psi}{\partial x \partial y}\right)^2. \tag{A.2b}$$

By adding equations (A.2a) and (A.2b), and taking advantage of cancellations, we get

$$\begin{aligned} -\nabla^2 P &\equiv -\frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial y^2} \\ &= 2\left(\frac{\partial^2 \psi}{\partial x \partial y}\right)^2 - 2\frac{\partial^2 \psi}{\partial x^2}\,\frac{\partial^2 \psi}{\partial y^2}. \end{aligned} \tag{A.3}$$

Finally, ψ is harmonic by equation (4.25). Thus, from (A.3),

$$\nabla^2 P = -2\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2 - 2\left(\frac{\partial^2\psi}{\partial x^2}\right)^2$$
$$\leqslant 0, \tag{A.4}$$

which is the definition of superharmonic. ■

A.2 Cited Theorems on Stokes Waves

These claims are all stated and proven in [4].

Lemma A.1.

$$\frac{\partial\psi}{\partial y} \equiv u^* - c$$
$$< 0 \quad \forall\, (x, y) \in \mathfrak{D}.$$

Lemma A.2. *For a Stokes wave, we have*

$$\frac{\partial\psi}{\partial x} < 0 \quad \forall\, (x, y) \in \overline{\mathfrak{D}} \text{ s.t. } 0 < x < \pi \tag{A.5}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[u^*(x, \eta(x)\right] < 0 \quad \text{for } 0 < x < \pi. \tag{A.6}$$

Lemma A.3. *For a Stokes wave, the horizontal component of velocity, $u^*(x, y)$ is strictly decreasing along the broken-line path from $(0, \eta(0))$ through $(0, -h)$ and $(\pi, -h)$ to $(\pi, \eta(\pi))$.*

Theorem A.4. *For a Stokes wave, for all $(x, y) \in \overline{\mathfrak{D}}$, $u^*(x, y) \leqslant c$; equality holds only for a Stokes wave of greatest height, and then only at the crest $(0, \eta(0))$.*

B Numerical Scripts

linear.ode

```
# linear.ode: Particle Trajectories of Linearized Water Waves

# Parameters
param a=10
param e=0.5

# Derived Constants
!k=2*pi/a
!c=k*e/sinh(k)

# Differential Equations
x'=c*cosh(k*y)*cos(k*(x-t))
y'=c*sinh(k*y)*sin(k*(x-t))

# Default initial conditions
init x=0
init y=0.8

# Numerical settings
@ XP=x,YP=y,XLO=-0.5,XHI=4.5,YLO=0,YHI=1
@ TOTAL=1000,DT=0.05,MAXSTOR=10000000

done
```

KdV.ode

```
# KdV.ode: Particle Trajectories of Solitons under the KdV approximation

# Parameters
param a=0.1

# Derived Constants
!k=sqrt(3*a/4)
!w=k*(1+a/2)

# Useful Functions
tq(x,t)=tanh(k*x-w*t)
sqs(x,t)=cosh(k*x-w*t)^(-2)

# Differential Equations
x'=a/4*sqs(x,t)*(4+a*(4-7*sqs(x,t))-3*a*y^2*(2-3*sqs(x,t)))
y'=a*sqrt(3*a)*y*sqs(x,t)*tq(x,t)

# Default initial conditions
init x=0
init y=0.8

#Numerical settings
@ XP=x,YP=y,XLO=-3,XHI=3,YLO=0,YHI=1
@ TOTAL=2000,MAXSTOR=1000000

done
```

C Figures

Figure 1: The two primary frames of reference.

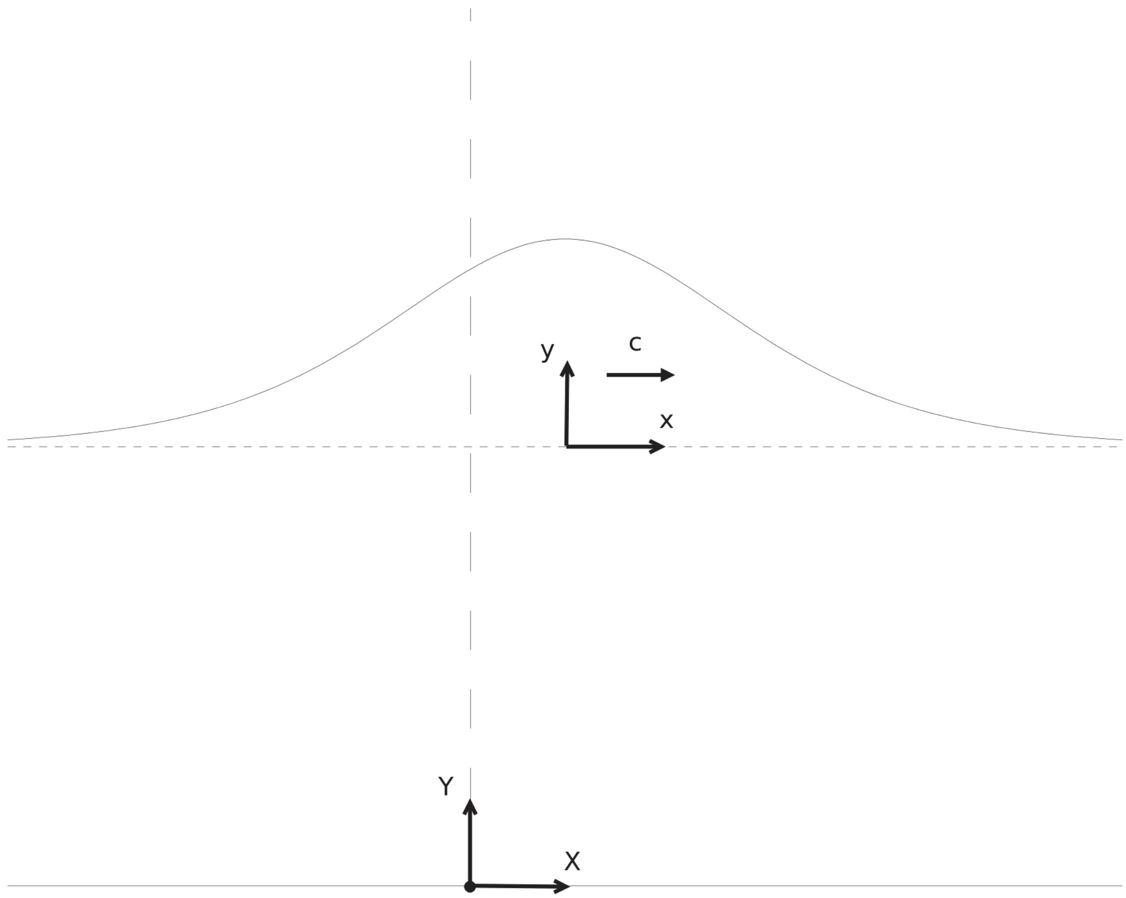

Figure 2: Linearized Linear Trajectory, $\varepsilon = 0.3$, $\alpha = 10$.

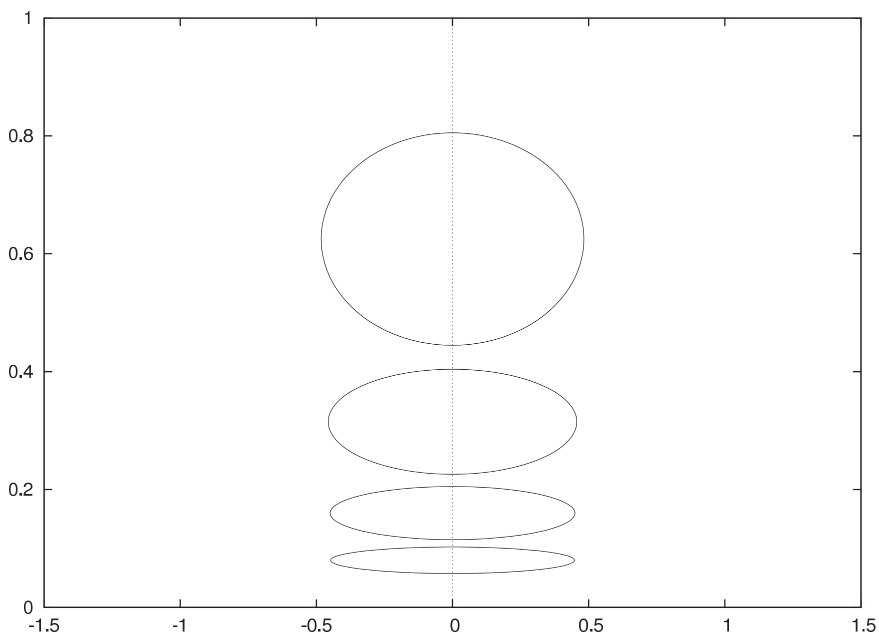

Figure 3: Linear Trajectory, $\varepsilon = 0.3$, $\alpha = 10$.

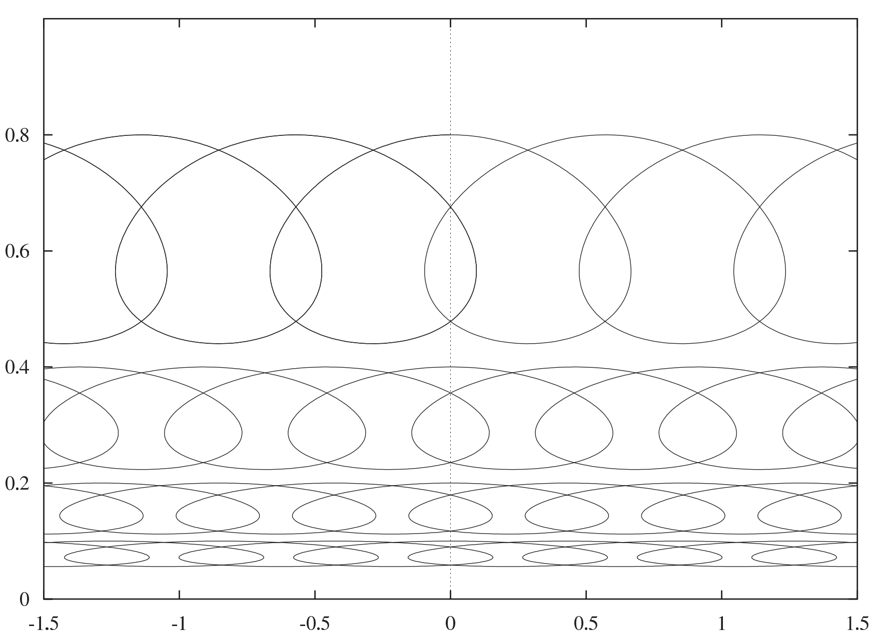

Figure 4: Linear Trajectory, $\varepsilon = 0.5$, $\alpha = 10$.

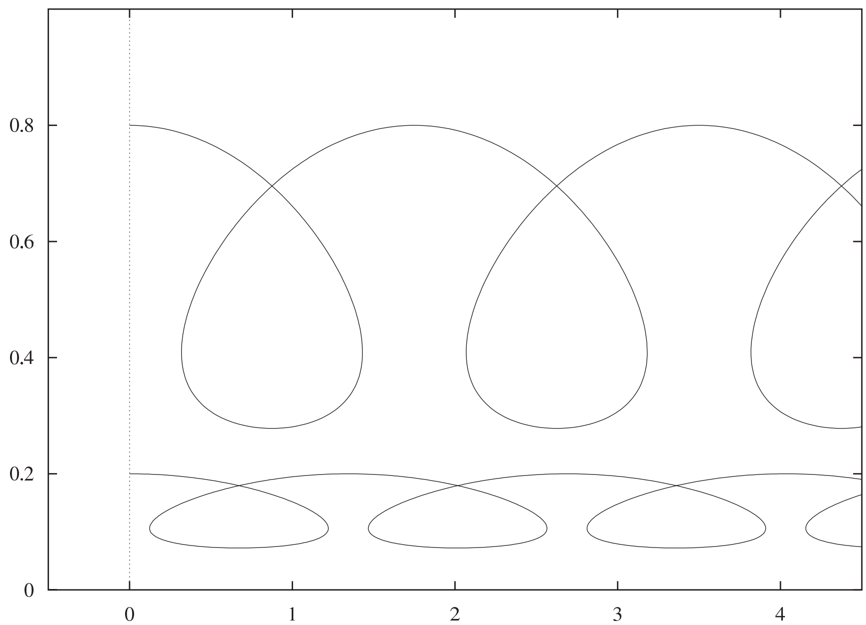

Figure 5: Linear Trajectory, $\varepsilon = 0.4$, $\alpha = 10$.

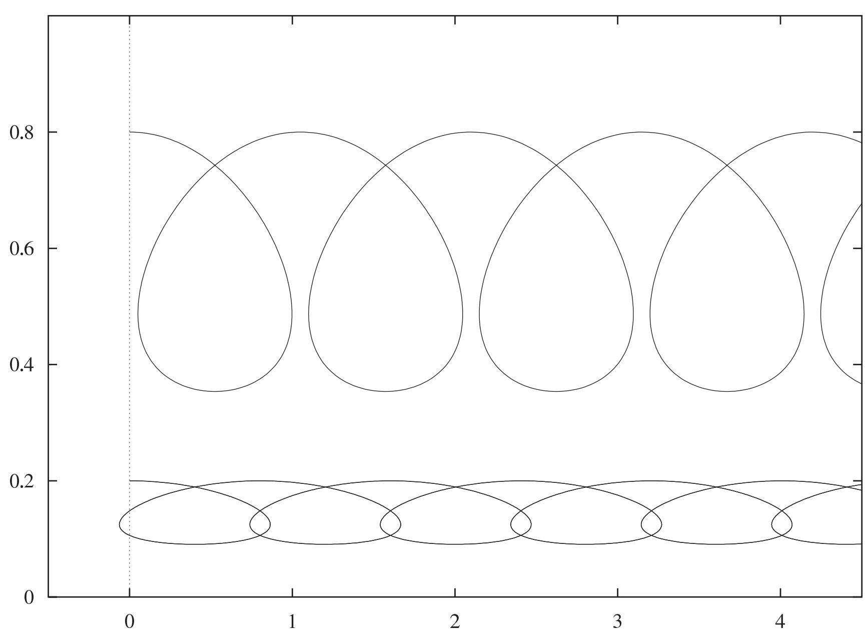

Figure 6: Linear Trajectory, $\varepsilon = 0.3$, $\alpha = 10$.

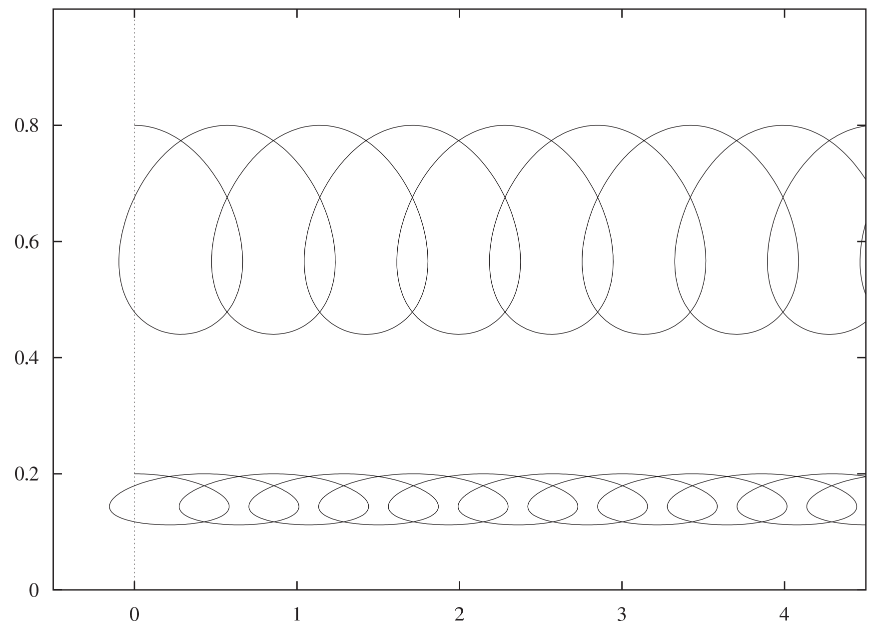

Figure 7: Linear Trajectory, $\varepsilon = 0.2$, $\alpha = 10$.

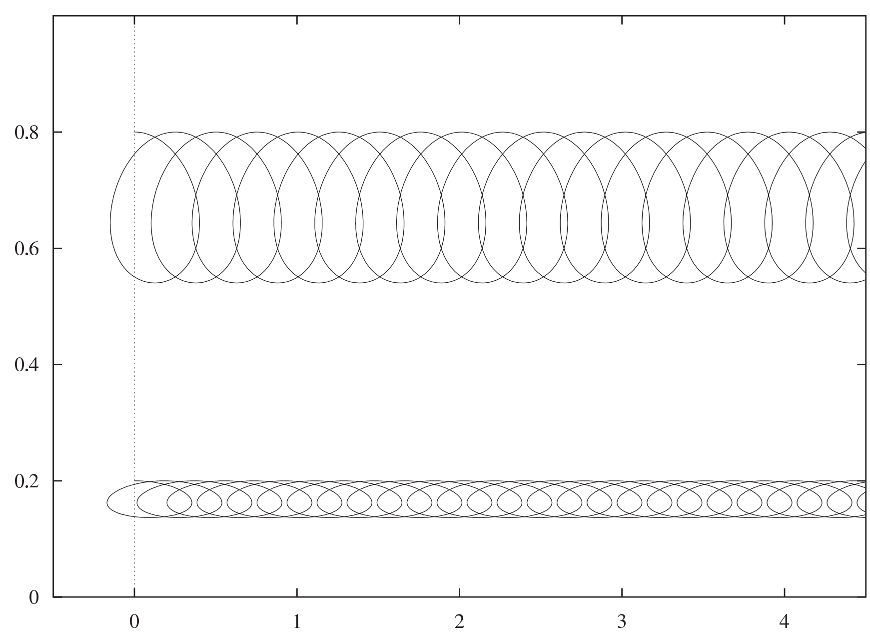

Figure 8: Linear Trajectory, $\varepsilon = 0.1$, $\alpha = 10$.

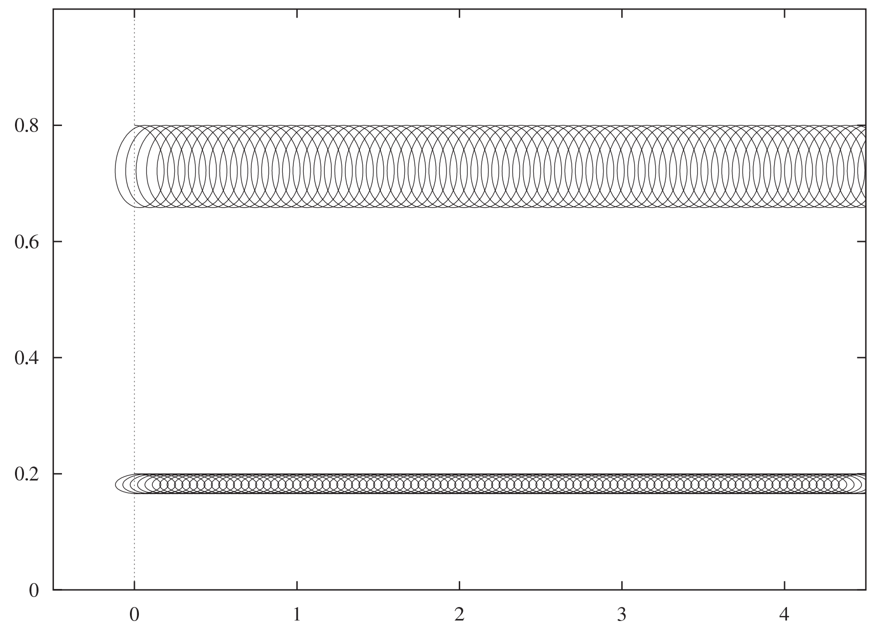

Figure 9: Linearized Linear Trajectory, $\varepsilon = 0.1$, $\alpha = 10$.

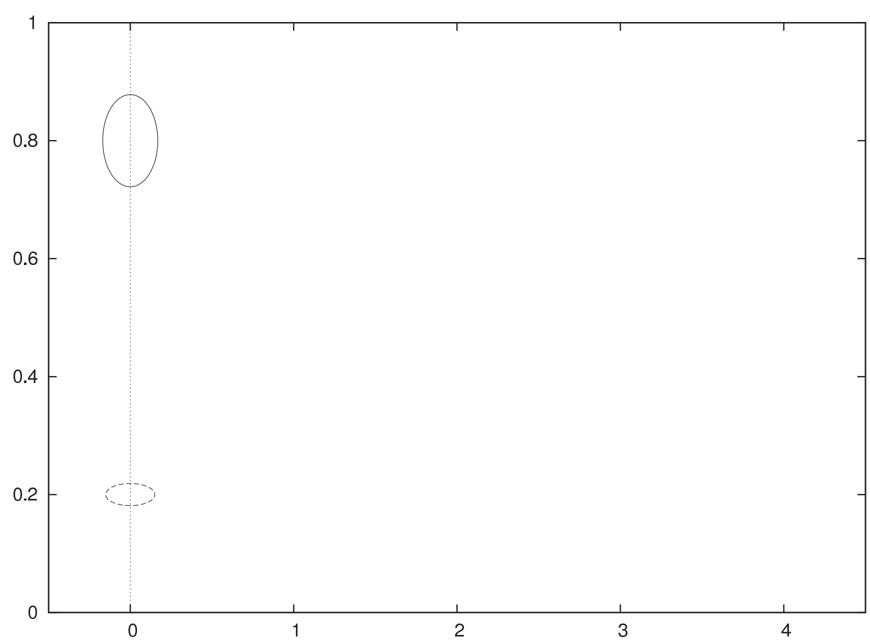

Figure 10: Linear Trajectory, $\varepsilon = 0.01$, $\alpha = 10$.

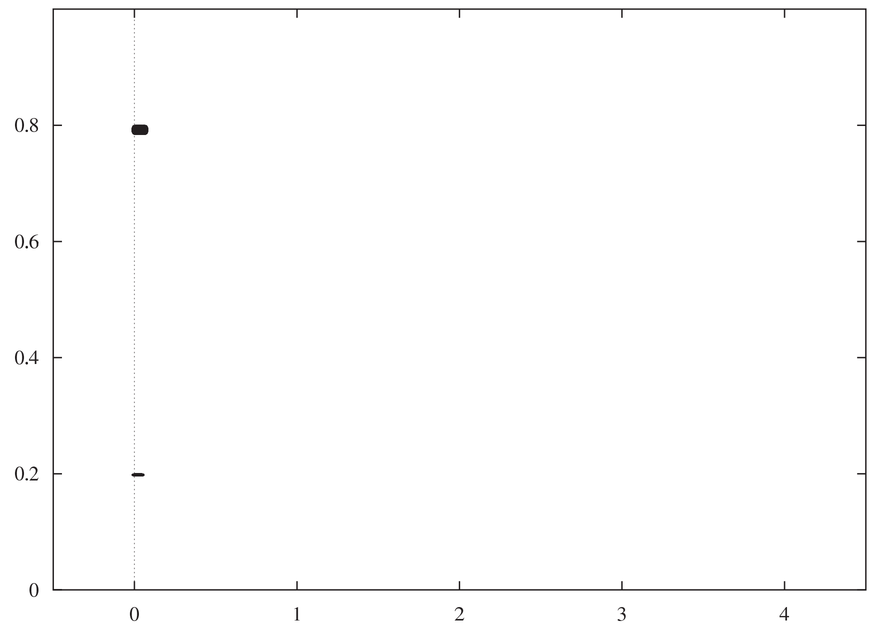

Figure 11: Linearized Linear Trajectory, $\varepsilon = 0.01$, $\alpha = 10$.

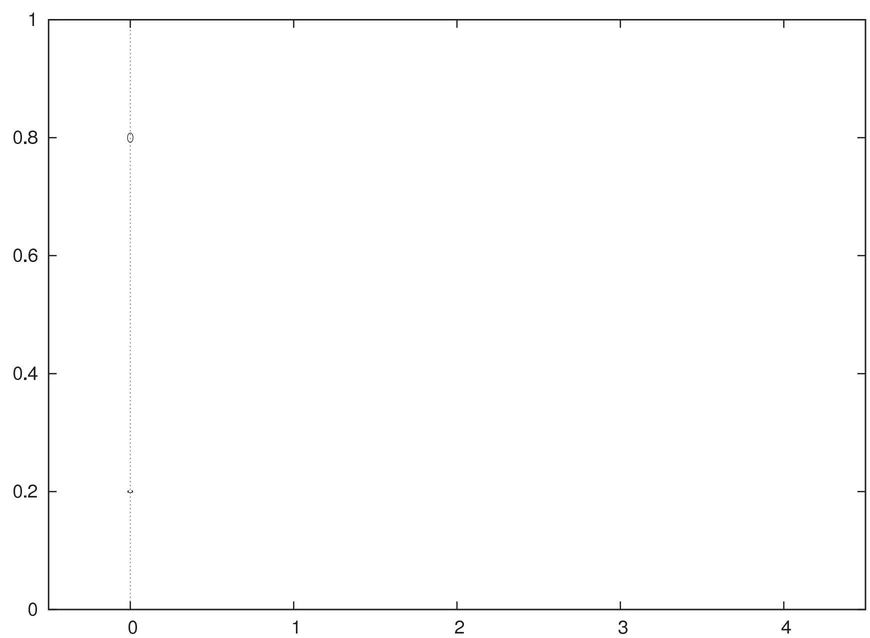

Figure 12: Linear Trajectory, $\varepsilon = 0.3$, $\alpha = 1$.

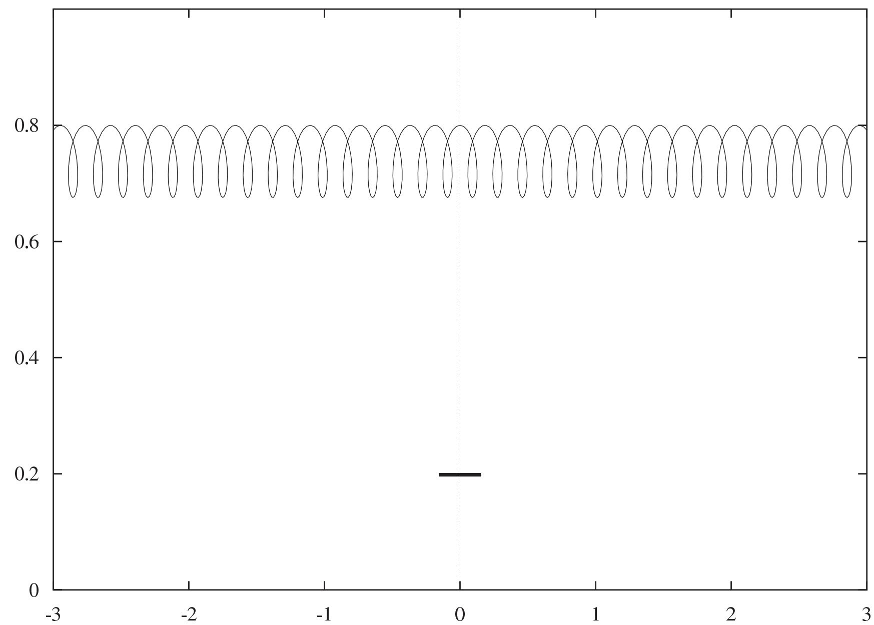

Figure 13: Linear Trajectory, $\varepsilon = 0.3$, $\alpha = 10$.

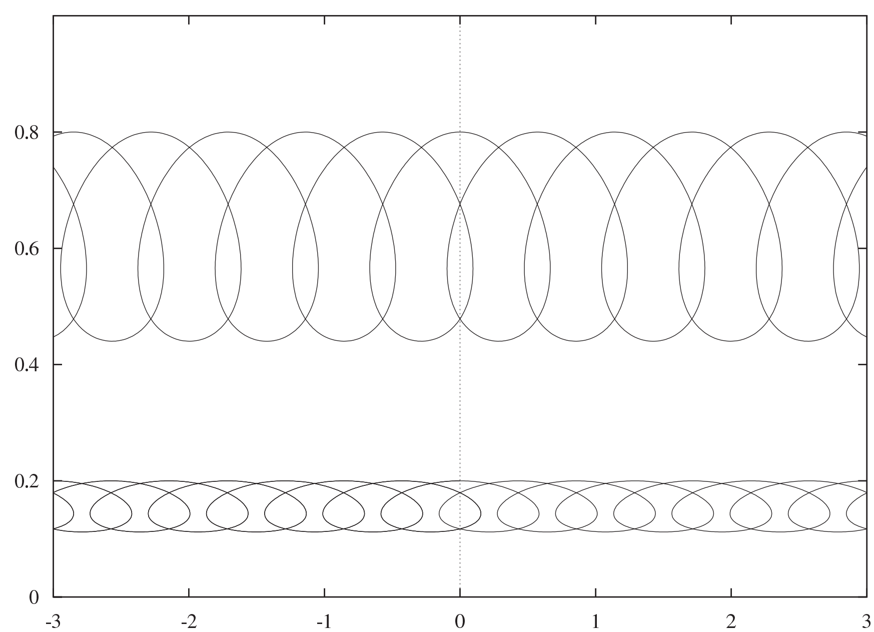

Figure 14: Linear Trajectory, $\varepsilon = 0.3$, $\alpha = 50$.

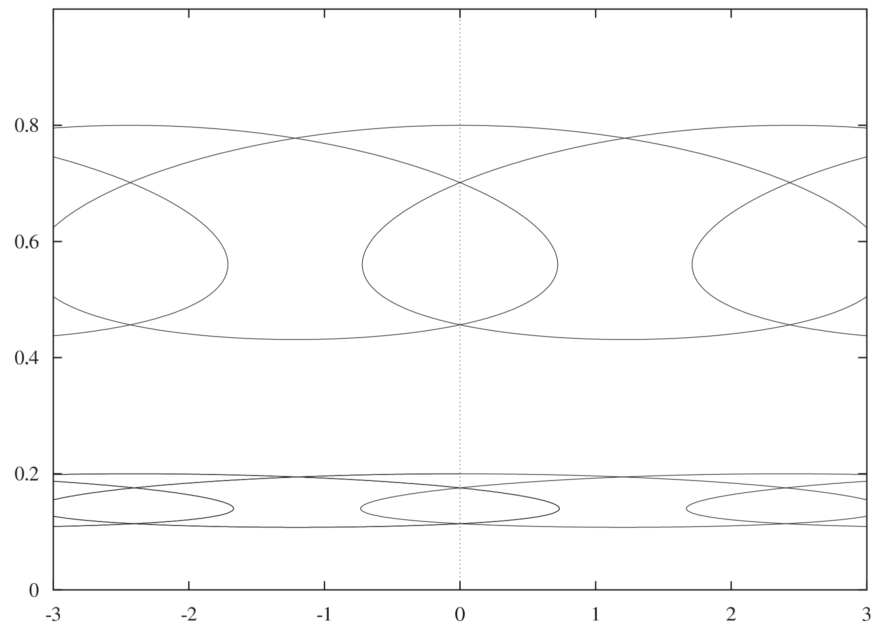

Figure 15: Linear Trajectory, $\varepsilon = 0.3$, $\alpha = 100$.

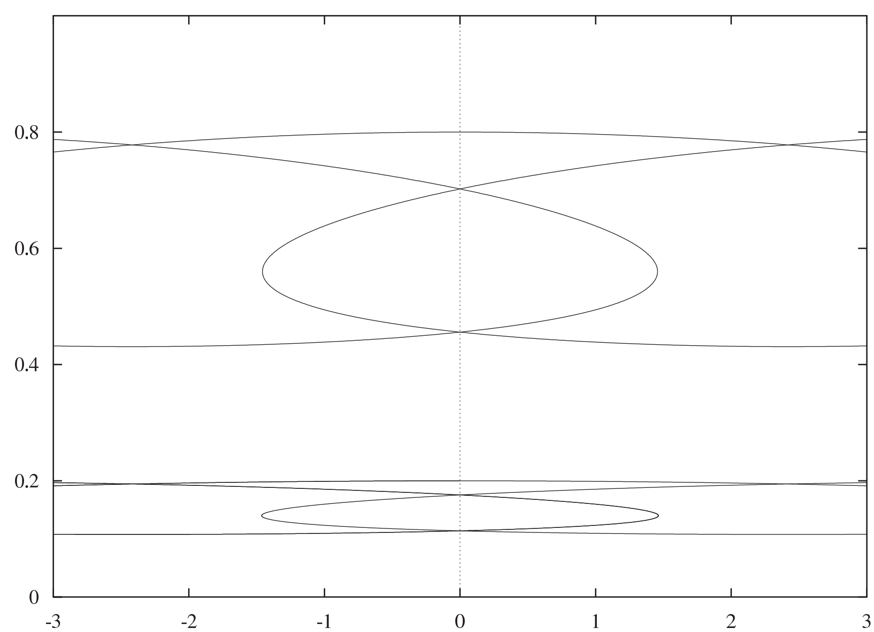

Figure 16: Trajectory in a Soliton, $\alpha = 0.2$.

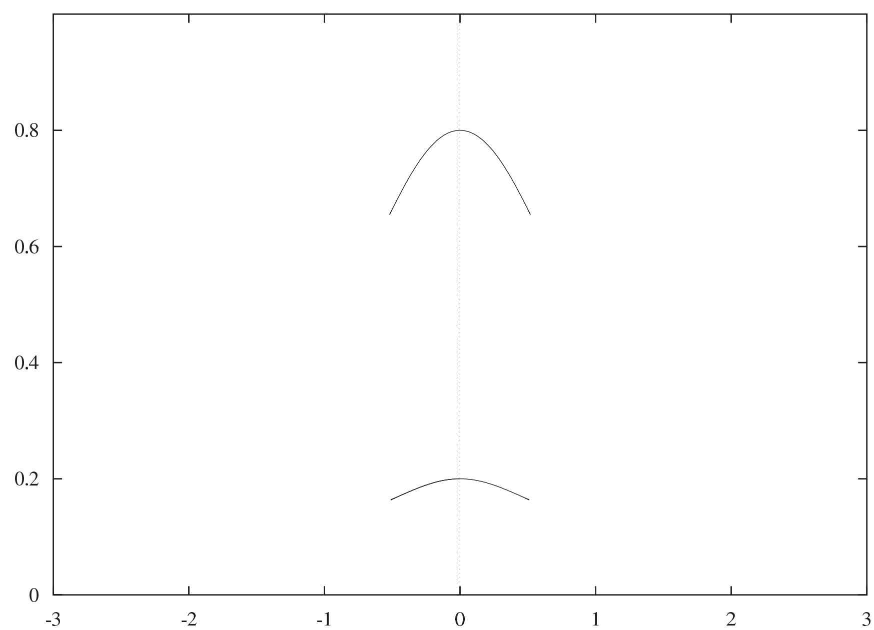

Figure 17: Trajectory in a Soliton, $\alpha = 0.3$.

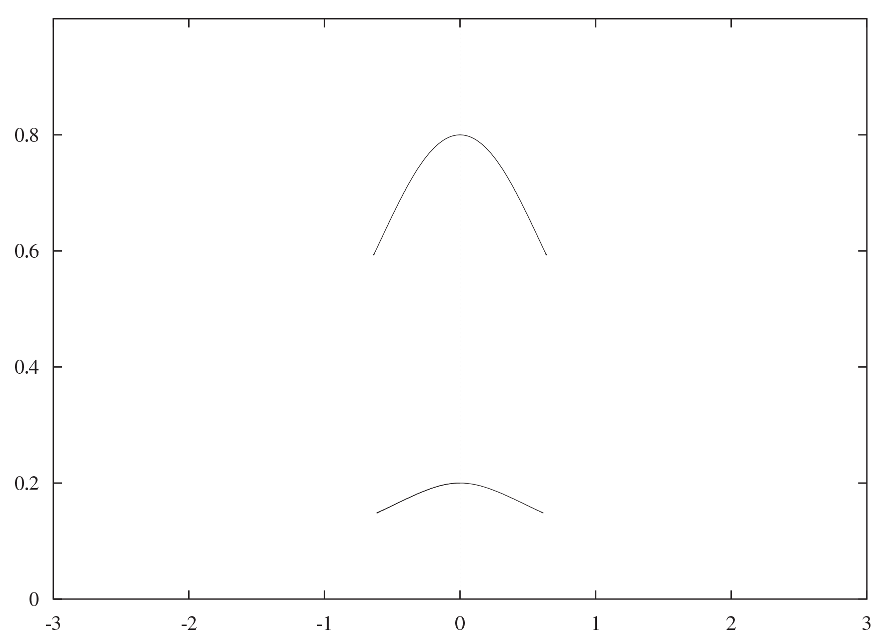

Figure 18: Trajectory in a Soliton, $\alpha = 0.4$.

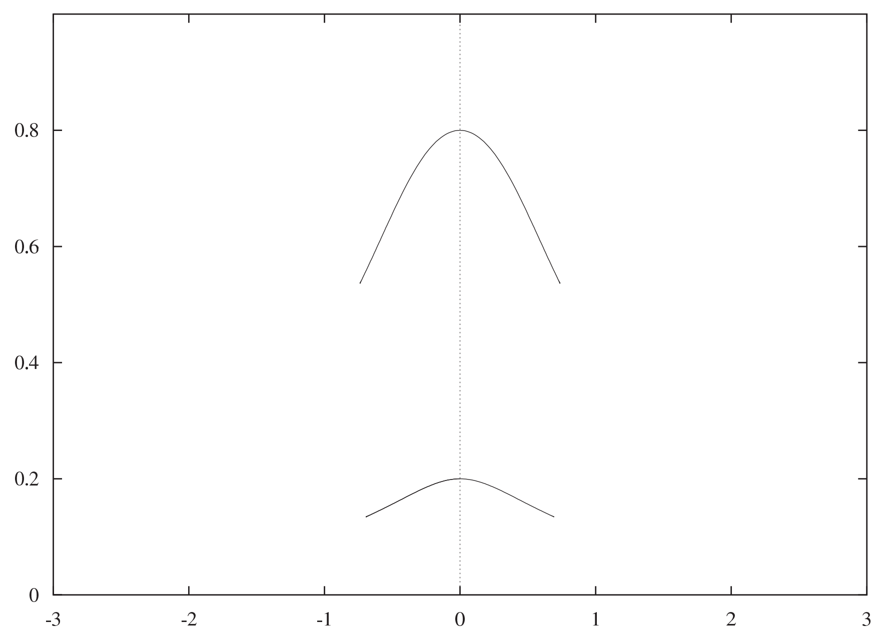

Figure 19: Trajectory in a Soliton, $\alpha = 0.5$.

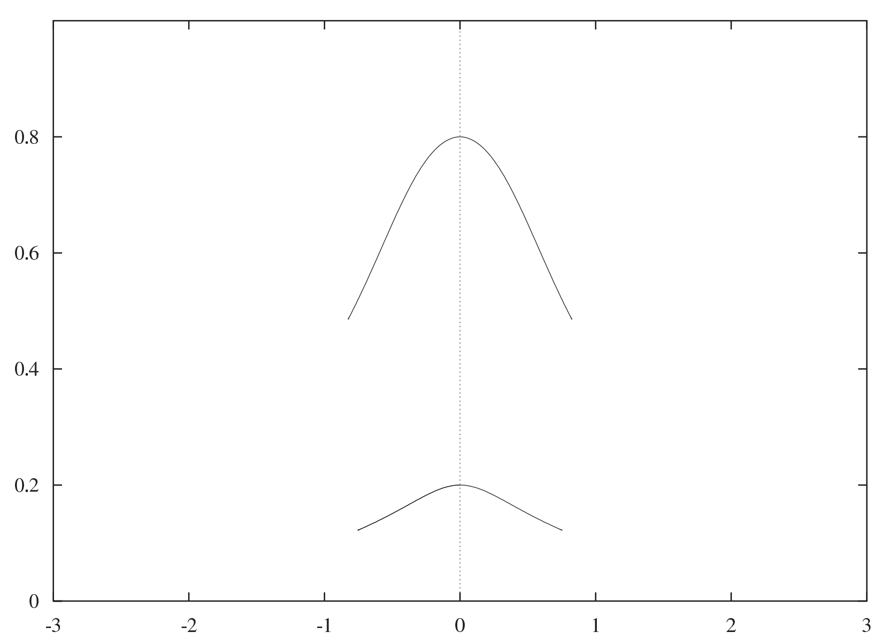

Figure 20: Trajectory in a Soliton, $\alpha = 0.6$.

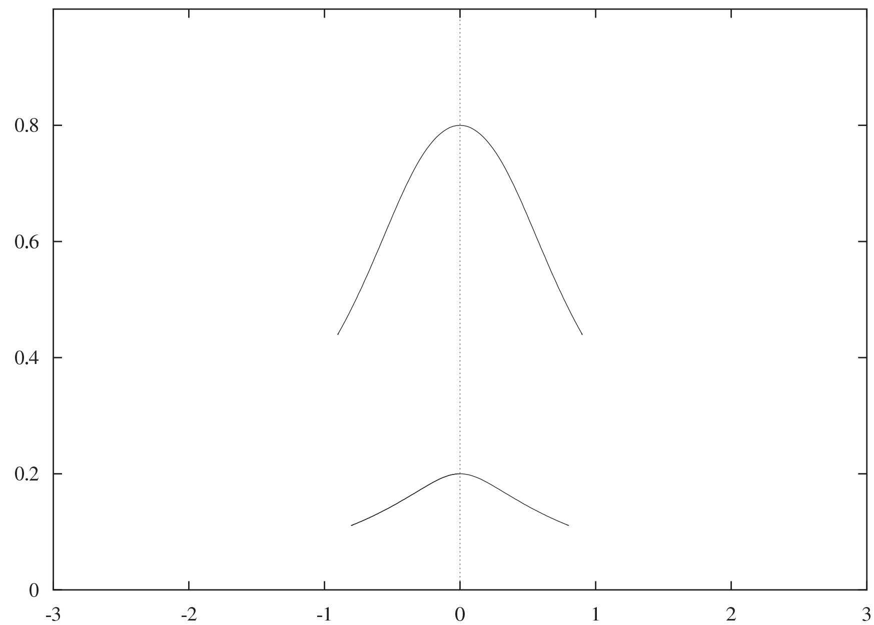

Figure 21: Trajectory in a Soliton, $\alpha = 0.7$.

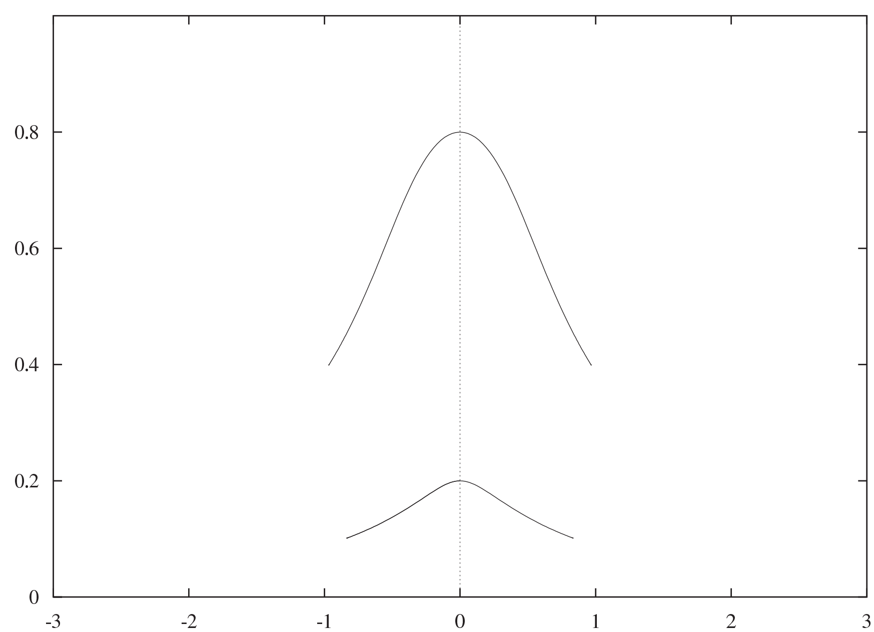

Figure 22: Photograph of particle motion in water waves through one period.
Wavelength is 4.54 times water depth, and wave amplitude is 0.18 times water depth.

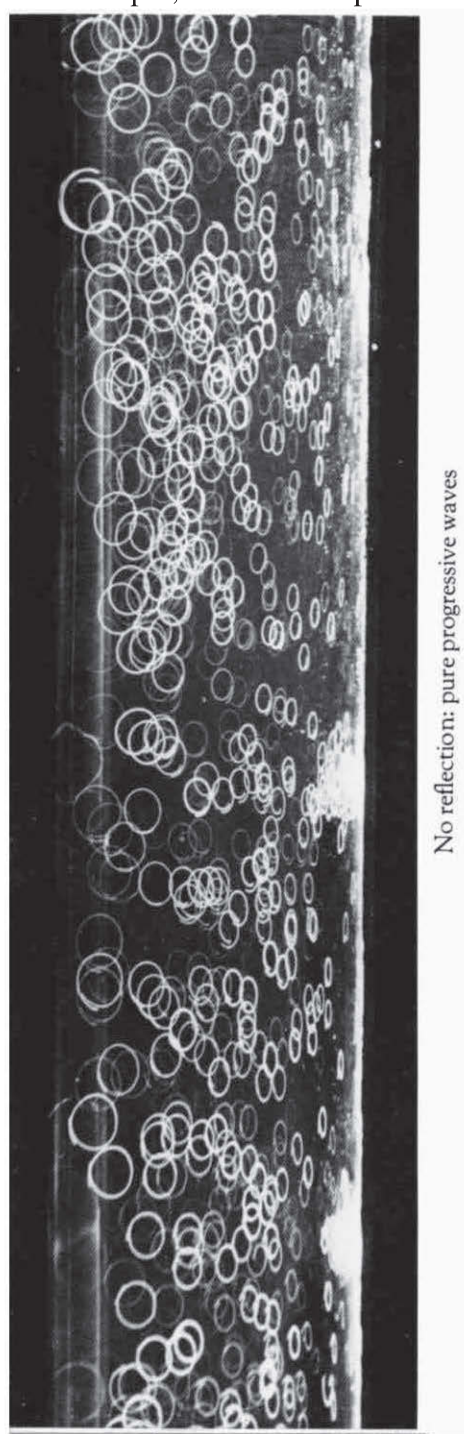

Photo credit: Wallet and Ruellan, [16].

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