On Volterra Spaces

by

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ABSTRACT

Continuity is one of the most important concepts in Mathematics. A. Cauchy was one of the first to define the continuity of a function. Here is Cauchy's condition of the continuity of a function:

> " . . . We also say that the function $f(x)$ is a continuous function of x in the neighborhood of a particular value assigned to the variable x as long as it (the function) is continuous between those two limits of x, no matter how close together, which enclose the value in question . . . "(see [31]).

This concept was refined by K. Weierstrass, which is the definition of continuity that we use today. For a historical account on how the notation of continuity has evolved see [31].

The study of continuity usually begins in calculus, where we study continuous functions. Questions on how a set of points of continuity of a given real-valued function of real variable look are very important. I will discuss these sets for real-valued functions of real variable in chapter 1. In chapter 2, I will continue to look at real-valued functions and examine their points of continuity. This time, the functions will be defined on metric spaces. Furthermore, I will examine the set of points of continuity of real-valued functions defined on topological spaces in chapter 4. In chapter 3, important topological concepts are introduced that will be used from chapters 4 through 6.

The analysis of the existing proofs on the Volterra theorem, which I will begin to discuss in chapter 1, led to the class of spaces known as *Volterra spaces*. In chapter 5, I will explain the concept of a Volterra space and explain various properties of these spaces. Finally, in chapter 6, I will list original and recent research results from various articles to illustrate the progress on Volterra spaces since their introduction in [15].

Throughout my thesis, I will illustrate various concepts with the help of diagrams and examples, some of I created. In addition, I will further explain topics from classical sources in detail to make the material easy to follow and more understandable to the reader.

ACKNOWLEDGMENTS

There are no words to describe how grateful I am to have had Dr. Piotrowski as my thesis advisor. I first learned about Volterra spaces, “his baby”, during his Topology II course. After expressing an interest in this subject, he has provided me with almost every article and book in my reference page, and taught me the various proof methods and terminology pertaining to the subject. I have learned a great deal of history and interesting concepts under his guidance. He has influenced me to specialize in topology, and I greatly thank him for working with me in topology for three semesters.

In addition, I would like to thank Dr. Tartir for allowing me to use his problem in section 6.3 pertaining to the composition of spaces being Volterra. Moreover, I would like to thank Dr. Wingler for providing me with the necessary historical information about the development of continuity that I used in my abstract.

Finally, I thank all the members of my thesis committee for their comments, criticisms and suggestions.

Contents

[1]Section 6.2 consists of advanced research level material taken directly from recent research results.

Chapter 1

Continuity

Continuity plays an important role in the study of topology and analysis. A. Cauchy was the first to give the modern definition and to focus attention on the subject. This concept will be used throughout my thesis since it is closely related to Volterra spaces. I will begin by examining continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

1.1 Continuity on the Real Line

Definition 1.1.1. Given a function $f : \mathbb{R} \to \mathbb{R}$, f is continuous at x_0 provided that for every $\varepsilon > 0$, there is a $\delta > 0$ for all $x \in \mathbb{R}$ such that

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

We say that f is continuous if it is continuous at every point in its domain.

The above condition is know as Cauchy's condition or the "$\varepsilon - \delta$" condition.

Let us now look at the sequential or Heine's condition of continuity. If we are given a function $f : \mathbb{R} \to \mathbb{R}$, then f is continuous at x_0 provided that for every $\{x_n\} \subset \mathbb{R}$,

$$\lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} f(x_n) = f(x_0).$$

We will now show that f satisfies the Cauchy condition at a point x_0 if and only if it satisfies the Heine condition at this point.

Proof. (Cauchy $\Rightarrow$ Heine)

Suppose f satisfies the Cauchy condition and $\lim_{n\to\infty} x_n = x_0$. Then, there is a k such that if $n > k$, then

$$|x_n - x_0| < \delta.$$

Hence, by the Cauchy condition

$$|f(x_n) - f(x_0)| < \varepsilon.$$

This in turn implies $\lim_{n\to\infty} f(x_n) = f(x_0)$.

(Heine $\Rightarrow$ Cauchy)

In order to show that Heine implies Cauchy we will prove by contradiction. Suppose to the contrary that Cauchy's condition does not hold, that is, there exists an $\varepsilon > 0$ such that for every $\delta > 0$, there exists an $x \in \mathbb{R}$ where

$$|x - x_0| < \delta \text{ and } |f(x) - f(x_0)| \geq \varepsilon.$$

In particular, assume $\delta = \frac{1}{n}$. By the Axiom of Choice, we can select a sequence $x_1, x_2, x_3, \ldots$ of points, such that:

$$|x_n - x_0| < \frac{1}{n} \text{ and } |f(x_n) - f(x_0)| \geq \varepsilon.$$

However, $|x_n - x_0| < \frac{1}{n}$ implies $\lim_{n\to\infty} x_n = x_0$. Hence, by Heine's condition of continuity at x_0, we get $\lim_{n\to\infty} f(x_n) = f(x_0)$. The latter equality contradicts the inequality:

$$|f(x_n) - f(x_0)| \geq \varepsilon.$$

Hence, the assumption that Cauchy's condition does not hold here led us to a contradiction. Thus, Heine implies Cauchy. ∎

The following examples illustrate how a function is continuous or not continuous at a point using Cauchy's condition and Heine's condition.

Example 1.1.1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^2.$$

We will show that $f(x)$ is continuous at $x_0 = 1$ using the Cauchy condition.

Proof. Let $\varepsilon > 0$. Notice that

$$|f(x) - f(1)| = |x^2 - 1| = |x + 1||x - 1|.$$

If we insist that $|x - 1| < 1$, then $0 < x < 2 \Rightarrow |x + 1| < 3$. Thus, if we choose $\delta = \min\{\frac{\varepsilon}{3}, 1\}$, then for all x such that $|x - 1| < \delta$,

$$\begin{aligned} |f(x) - f(1)| &= |x^2 - 1|, \\ &= |x + 1||x - 1|, \\ &< 3|x - 1|, \\ &\leq 3\delta, \\ &= \varepsilon. \end{aligned}$$

Therefore, $f(x)$ is continuous at $x_0 = 1$. ∎

To use the Heine condition to show that $f(x)$ is continuous at $x_0 = 1$ we have to show that for every sequence $\{x_n\} \to 1$, $\lim_{n\to\infty} f(x_n) = f(1) = 1$.

Proof. Let $\{x_n\}$ be a sequence and suppose that $\lim_{n\to\infty} x_n = 1$.

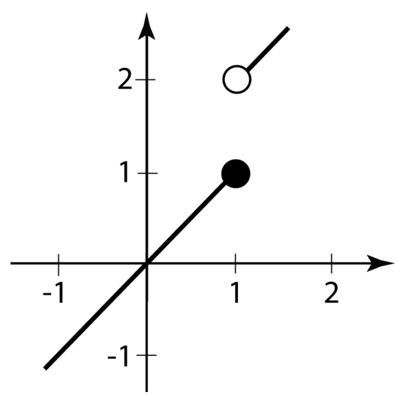
Notice that

$$\begin{aligned}
\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} x_n^2, \\
&= \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} x_n, \\
&= 1 \cdot 1, \\
&= 1^2, \\
&= f(1).
\end{aligned}$$

Therefore, $f(x)$ is continuous at $x_0 = 1$. ■

Example 1.1.2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \le 1 \\ x+1 & \text{if } x > 1. \end{cases}$$



We will first show that $f(x)$ is discontinuous at $x_0 = 1$ using Cauchy's condition.

Proof. By negating the definition of the Cauchy condition of continuity, $f(x)$ is not continuous at x_0 if there exists an $\varepsilon > 0$ such that for every $\delta > 0$, there exists an $x \in \mathbb{R}$ such that

$$|x - x_0| < \delta \text{ and } |f(x) - f(x_0)| \ge \varepsilon.$$

Thus, $f(x)$ is discontinuous at $x_0 = 1$ if there exists an $\varepsilon > 0$ such that for every $\delta > 0$, there exists an $x \in \mathbb{R}$ such that

$$|x - 1| < \delta \text{ and } |f(x) - 1| \ge \varepsilon.$$

Now, let $\varepsilon = \frac{1}{2}$ and x be any point from $(1, 1+\delta)$ for $\delta > 0$. Then, we have $|x-1| < |(1+\delta)-1| = \delta$, and

$$|f(x)-1| > |2-1| = |1| \geq \frac{1}{2} = \varepsilon.$$

Therefore, $f(x)$ is discontinuous at $x_0 = 1$. ■

Now, we will show that $f(x)$ is discontinuous at $x_0 = 1$ by using the Heine condition. By negating Heine's condition, $f(x)$ is discontinuous at x_0 if there is a sequence $\{x_n\} \subset \mathbb{R}$ such that $\lim_{n\to\infty} x_n = x_0$ and it is not true that $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Proof. Let $x_0 = 1$. Notice that there exists a sequence $\{x_n\} = \{1 + \frac{1}{n}\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} x_n = x_0 = 1.$$

However, it is not true that $\lim_{n\to\infty} f(x_n) = f(x_0)$. Clearly,

$$\begin{aligned}
\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)+1\right], \\
&= \lim_{n\to\infty} \left(2+\frac{1}{n}\right), \\
&= 2, \\
&\neq 1, \\
&= f(1).
\end{aligned}$$

Therefore, $f(x)$ is discontinuous at $x_0 = 1$. ■

1.2 Open and Closed Subsets on the Real Line

To further explore the investigations of the continuity of functions on the real line, we can look at what is called the *oscillation* of a function. The following definitions will play an important role in the next section.

Definition 1.2.1. The set $A \subset \mathbb{R}$ is <u>dense</u> in $\mathbb{R}$ if for each open interval $(a,b) \subset \mathbb{R}$, the set $A \cap (a,b) \neq \varnothing$.

Example 1.2.1. Clearly, $\mathbb{Q}$, the set of rational numbers, is dense in $\mathbb{R}$. Furthermore, $\mathbb{R} \setminus \mathbb{Q}$, the set of irrational numbers, is also dense in $\mathbb{R}$.

Definition 1.2.2. The set $A \subset \mathbb{R}$ is co-dense in $\mathbb{R}$ if the complement of A is dense in $\mathbb{R}$.

Definition 1.2.3. Let $E \subset \mathbb{R}$. Any point x that belongs to E is said to be an interior point of E provided that there exists a $\delta > 0$ such that

$$(x - \delta, x + \delta) \subset E.$$

Definition 1.2.4. Let $E \subset \mathbb{R}$. Then E is said to be open if every point of E is also an interior point of E.

Definition 1.2.5. Let $E \subset \mathbb{R}$. Any point x (not necessarily in E) is said to be an accumulation point of E provided that for every $\delta > 0$, the intersection

$$(x - \delta, x + \delta) \cap E$$

contains infinitely many points.

Definition 1.2.6. Let $E \subset \mathbb{R}$. Then E is said to be closed provided that every accumulation point of E belongs to the set E.

Definition 1.2.7. Let $E \subset \mathbb{R}$. A point $x \in E$ is said to be an isolated point of E provided that for some interval $(x - \delta, x + \delta)$ with $\delta > 0$,

$$(x - \delta, x + \delta) \cap E = \{x\}.$$

Definition 1.2.8. The subset $A \subset \mathbb{R}$ is said to be a G_δ-set if it can be expressed as a countable intersection of open sets, that is, if there exist open sets $G_1, G_2, \ldots$ such that

$$A = \bigcap_{k=1}^{\infty} G_k.$$

Definition 1.2.9. The subset $A \subset \mathbb{R}$ is said to be an F_σ-set if it can be expressed as a countable union of closed sets, that is, if there exist closed sets $F_1, F_2, \ldots$ such that

$$A = \bigcup_{k=1}^{\infty} F_k.$$

Using DeMorgan's laws, it is easy to see that the complement of any G_δ-set is an F_σ-set and vice versa.

1.3 The Oscillation of a Function

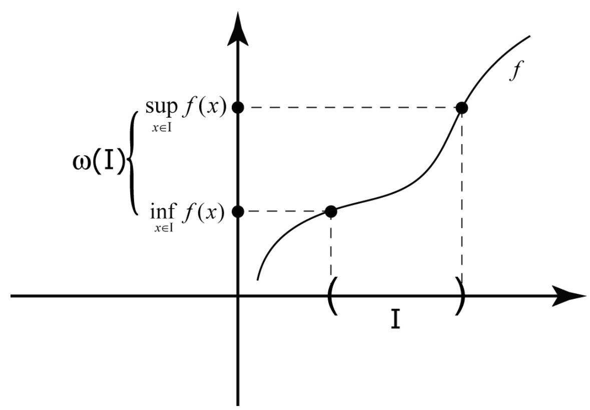
Having the required terminology (see section 1.2), we are now ready to begin looking at the oscillation of a function.

Definition 1.3.1. ([28], p. 31) Let f be a real-valued function on $\mathbb{R}$. For any interval I, the quantity

$$\omega(I) = \sup_{x\in I} f(x) - \inf_{x\in I} f(x)$$

is called the oscillation of f on I.

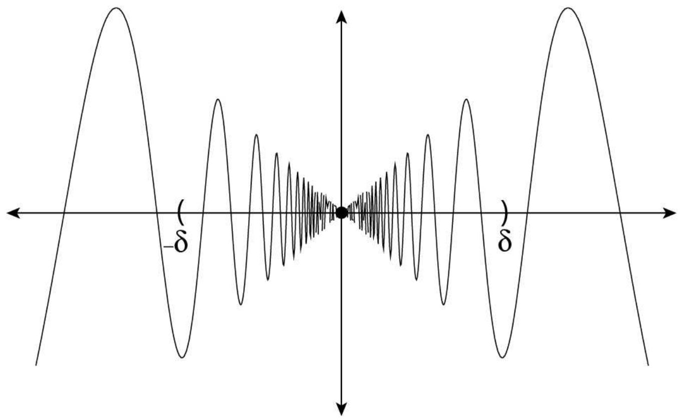
We have the following diagram to illustrate the oscillation of f on I.



Example 1.3.1. ([34], p. 1) Consider

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

and assume $x_0 = 0$.



Notice that if you shrink δ, the oscillation $\omega((-\delta, \delta))$ decreases.

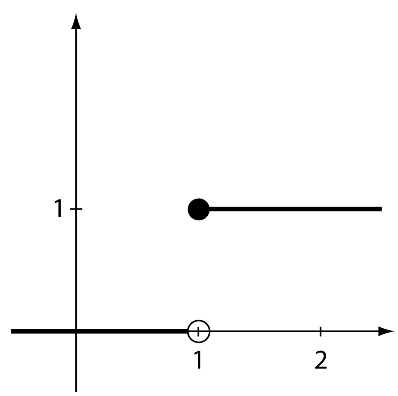
Definition 1.3.2. ([28], p. 31) Given a function f, let x belong to the domain of f. Then, the function ω defined by

$$\omega(x) = \lim_{\delta \to 0} \omega((x - \delta, x + \delta)),$$

is called the <u>oscillation of f at x</u> (see **Definition 1.3.1**).

Evidently, $\omega(x_0) = 0$ if and only if f is continuous at x_0. If $\omega(x_0) \neq 0$, then $\omega(x_0)$ is a measure of the size of discontinuity of f at x_0.

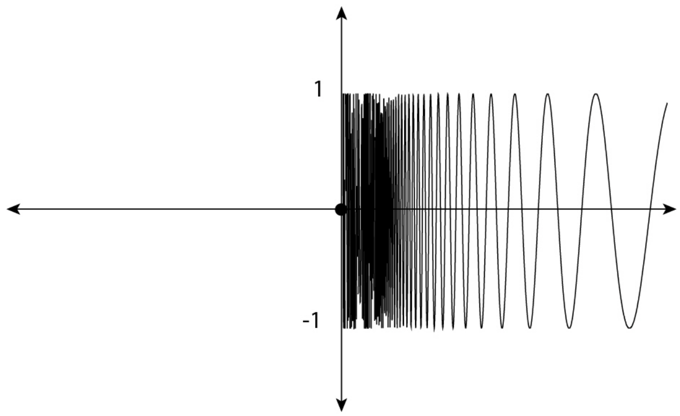
Example 1.3.2. Consider the graph of the following function of $f(x)$.



Notice that $\omega(x_0) = 0$ for all $x_0 \neq 1$. However, since the function has a jump discontinuity at 1, $\omega(1) \neq 0$. This is true because for every open interval I around 1, $\sup_{x \in I} f(x) = 1$ and $\inf_{x \in I} f(x) = 0$, which implies that $\omega(1) = 1 - 0 = 1$.

Example 1.3.3. ([34], p. 2) Consider

$$g(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

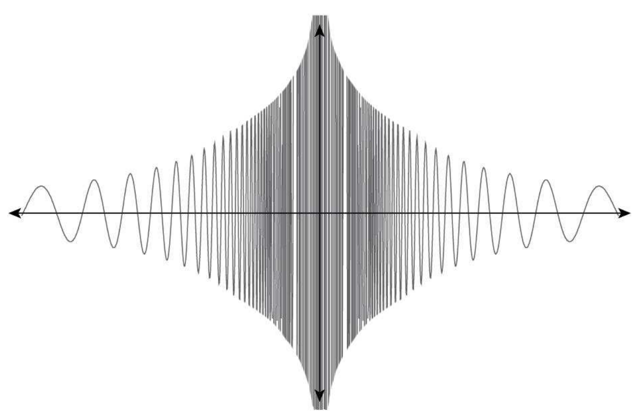


Notice that for any $x_0 > 0$, $\omega(x_0) = 0$. However, at $x_0 = 0$, the oscillation of $g(x)$ is 2. This is true because for every open interval I around 0, $\sup_{x \in I} g(x) = 1$ and $\inf_{x \in I} g(x) = -1$, which implies that $\omega(0) = 1 - (-1) = 2$. Hence, $g(x)$ is not continuous at $x_0 = 0$.

While studying the properties of oscillations, we will be working with the extended set of reals, $\mathbb{R}$, i.e., including $-\infty$ and $+\infty$. In fact, we have the following example.

Example 1.3.4. ([34], p. 3) Consider the function

$$h(x) = \frac{1}{x} \sin\left(\frac{1}{x}\right).$$



At $x = 0$, $\omega(0) = \infty$. Notice that $h(x)$ is undefined at zero.

If $\omega(x_0) < \varepsilon$, then $\omega(x) < \varepsilon$ for all x in a neighborhood of x_0. Hence, the set

$$\{x : \omega(x) < \varepsilon\}$$

is open. Notice that the set $D(f)$ of all points at which f is discontinuous can be represented in the form

$$D(f) = \bigcup_{n=1}^{\infty} \left\{ x : \omega(x) \geq \frac{1}{n} \right\}.$$

Thus, $D(f)$ is an F_σ-set. This leads us to the our next theorem.

Theorem 1.3.1. ([28], p. 31) If f is a real-valued function on $\mathbb{R}$, then the set of points of discontinuity is an F_σ-set.

Actually, we can obtain the converse of **Theorem 1.3.1**. This will be proved in Section 2.3.

Given a function f, the set of points of continuity will be denoted by $C(f)$. Since $C(f)$ is the complement of $D(f)$, we obtain the following corollary by using DeMorgan's laws.

Corollary 1.3.1. If f is a real-valued function on $\mathbb{R}$, then the set of continuity points is a G_δ-set.

This corollary can be strengthened to the following: A is a G_δ-subset of $\mathbb{R}$ if and only if there exists a real-valued function f that has A as its set of points of continuity, i.e., there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that $A = C(f)$ (see section 2.3).

Even though the set of all continuity points is a G_δ-set, it can also be empty. Keep in mind that the empty set $\varnothing$ is a G_δ-set: $\varnothing \cap \varnothing \cap \ldots$. We will now give an example of when $C(f)$ is empty.

Example 1.3.5. Let A and B be any two dense, co-dense subsets of $\mathbb{R}$. Then, define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \chi_{A(x)} = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

This function is called the *characteristic function* of A. We claim the set $C(f)$ is empty. We will now prove this claim.

Proof. Let x_0 be any real number. Without loss of generality, assume $x_0 \in A$. Then $f(x_0) = 1$. However, since the set B is dense in $\mathbb{R}$, there is a sequence of points $b_1, b_2, \ldots$ in B that converges to x_0.

Hence, $\lim_{n\to\infty} f(b_n) = 0 \neq 1 = f(x_0)$. Thus, $f(x)$ does not satisfy the Heine condition of continuity at x_0. Similar arguments can be provided for a point $x_0 \in B$.

Therefore, $C(f)$ is empty. ■

Remark 1.3.1. If $A = \mathbb{Q}$, the function defined in **Example 1.3.5** will be called the *"salt and pepper" function.*

1.4 Volterra Theorem for Real-Valued Functions of a Real Variable

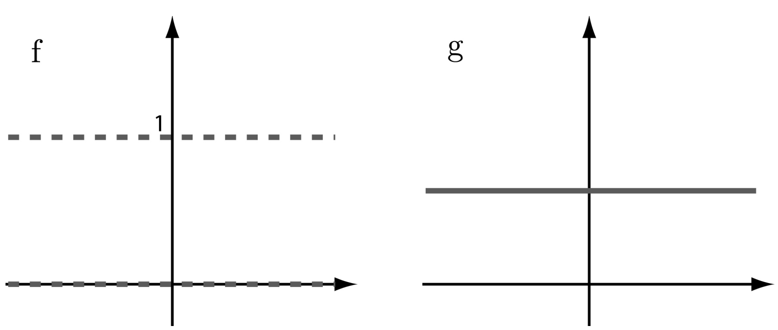
First Approach

Let $A \subset \mathbb{R}$. We are going to investigate if there exist two functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$A = C(f) = D(g) \qquad \text{and} \qquad \mathbb{R} \setminus A = D(f) = C(g). \tag{1.1}$$

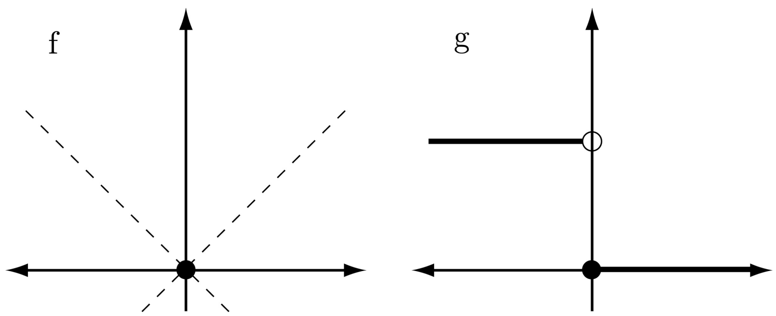
Example 1.4.1. ([36], p. 1) **If A is the empty set**, then (1.1) holds. Let $g(x)$ be *any continuous function* on $\mathbb{R}$ and let $f(x)$ be the "salt and pepper function", i.e.,

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$



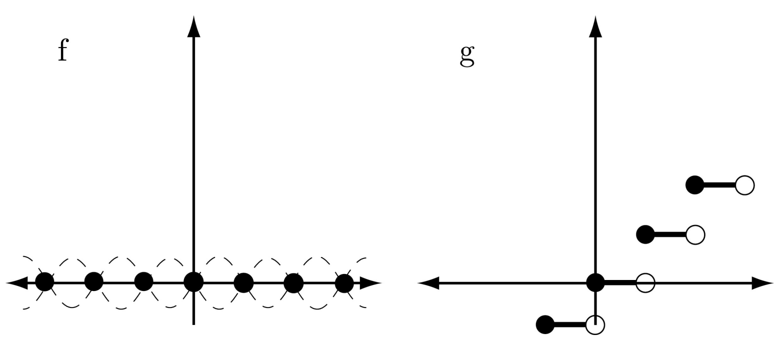
Example 1.4.2. ([36], p. 1) **If A is a singleton**, (1.1) holds. Let $g(x)$ be a function with a jump discontinuity at $x = 0$ and

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$



Example 1.4.3. ([36], p. 2) **If A is a countably infinite discrete set**, (1.1) holds. Let $g(x) = \lfloor x \rfloor$ and

$$f(x) = \begin{cases} \sin(\pi x) & \text{if } \sin(\pi x) \in \mathbb{Q} \\ -\sin(\pi x) & \text{if } -\sin(\pi x) \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

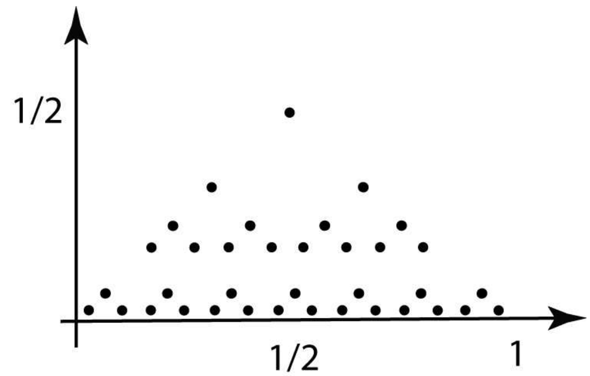


Furthermore, (1.1) holds if A is dense and $\mathbb{R} \setminus A$ is a finite set. See **Example 1.4.2**.

But what happens if A and $\mathbb{R} \setminus A$ are dense?

Can we find a function $f : \mathbb{R} \to \mathbb{R}$ such that $C(f)$ is the set of irrationals and $D(f)$ is the set of rationals? We actually **have** such a function. It is known as the "small" Riemann function. It is also known as the Dirichlet function in [6] and the Thomae function in [4]. The function is defined on $[0, 1]$ by:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, \text{ where } \gcd(p,q) = 1, q > 0. \end{cases}$$



We will prove that f is continuous at every irrational number and discontinuous at every rational number. We will assume that $x \in (0,1)$.

Proof. First, observe that the number of points x in $(0,1)$ for which $f(x) > \frac{1}{q}$ is finite. We will show that for every $x_0 \in (0,1)$ we have

$$\lim_{x \to x_0} f(x) = 0.$$

Let $\varepsilon > 0$. If q is a rational number such that $\frac{1}{q} < \varepsilon$, then there is a $\delta > 0$ such that there are no points x for which $f(x) \geq \frac{1}{q}$, where $x \in (x_0 - \delta, x_0 + \delta)$, except possibly the point x_0 itself.

Now,

$$f(x) < \frac{1}{q} < \varepsilon$$

for all x in $(x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$. This shows that

$$\lim_{x \to x_0} f(x) = 0.$$

■

Remark 1.4.1. ([4], p. 531) It can be shown that the "small" Riemann function is not differentiable on the irrationals.

We will now illustrate how this was shown in [4]. This derives from the following fact: for all $a \in \mathbb{R} \setminus \mathbb{Q}$, and for each $n \in \mathbb{N}$, there exists a $b_n \in \mathbb{Z}$ such that $|b_n/n - a| \leq \frac{1}{n}$. By definition, $f(\frac{b_n}{n}) \geq \frac{1}{n}$. Thus, it follows that

$$\frac{|f(b_n/n) - f(a)|}{|b_n/n - a|} = \frac{f(b_n/n)}{|b_n/n - a|} \geq 1 \quad \text{for all } n.$$

Since $\frac{b_n}{n} \to a$ as $n \to \infty$, this rational approximation of a yields that the derivative cannot be zero. However, the irrational approximation must be zero.

This proof relies on the fact that f sends $\frac{m}{n}$ to $\frac{1}{n}$, making the approximation of $\frac{b_n}{n}$ to a sufficiently close. If, for example, $\frac{m}{n}$ is sent to $\frac{1}{n^2}$, the approximation of $\frac{b_n}{n}$ to a is no longer close enough to ensure that the function is not differentiable at a.

Second Approach

A curious student might ask the following question:

"Since there is a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous on the irrationals and discontinuous on the rationals, is there a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous on the rationals and discontinuous on the irrationals?"

Actually, the answer is **no**. This comes from a result by Vito Volterra in 1881. He showed that there are no pointwise discontinuous functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $C(f) = D(g)$ and $D(f) = C(g)$. Volterra was not even 20 years old when he proved this result.

We need the following definition and property in order to prove Volterra's result.

Definition 1.4.1. ([25], p. 105) A function is pointwise discontinuous if its set of points of discontinuity is of first category (see section 3.6).

It will be shown in **Theorem 4.1.1(2)** and in **Theorem 4.2.1** that every real-valued pointwise discontinuous function of real variable has a dense set $C(f)$ for its continuity points.

Proposition 1.4.1. (Nested Interval Property of $\mathbb{R}$)
If

$$[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots$$

is a sequence of nested closed intervals in $\mathbb{R}$ such that $diam([a_n, b_n]) \to 0$ as $n \to \infty$, then there is a unique point p such that

$$\bigcap_{n=1}^{\infty} [a_n, b_n] = \{p\}.$$

We will now state and prove Volterra's theorem. The following proof is due to W. Dunham in [10].

Theorem 1.4.1. ([10], p. 235) There do not exist pointwise discontinuous functions f and g defined on an interval (a, b) for which the continuity points of one are the discontinuity points of the other, and vice versa.

Proof. Suppose there exist two such functions f and g. Thus, $C(f)$ and $C(g)$ partition (a, b) into disjoint subsets. Let x_0 be any point in $C(f)$ and let $\varepsilon = 1$.

By the definition of continuity, there exists a $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset (a, b)$ and $|f(x) - f(x_0)| < \frac{1}{2}$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

Now, we can choose $a_1 < b_1$ such that $[a_1, b_1]$ is a closed subinterval of $(x_0 - \delta, x_0 + \delta)$. Then, for any x and y in $[a_1, b_1]$,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(x_0)| + |f(x_0) - f(y)| \\ &< \tfrac{1}{2} + \tfrac{1}{2} \\ &= \varepsilon. \end{aligned}$$

Pointwise discontinuity now yields a continuity point of g in the open interval (a_1, b_1), and by the preceding argument, there exists $a_1' < b_1'$ with $[a_1', b_1'] \subset (a_1, b_1)$ and with

$$|g(x) - g(y)| < 1 \quad \text{for all } x \text{ and } y \text{ in } [a_1', b_1'].$$

To summarize, then, for all x and y in $[a_1', b_1'] \subset (a, b)$,

$$|f(x) - f(y)| < 1 \qquad \text{and} \qquad |g(x) - g(y)| < 1,$$

as well.

Now, repeat this argument by starting with the open interval (a_1', b_1') and the oscillation $\varepsilon = \frac{1}{2}$, then $\varepsilon = \frac{1}{4}$, and, generally, $\varepsilon = \frac{1}{2^n}$. This generates a strictly descending sequence of closed intervals

$$(a, b) \supset [a_1', b_1'] \supset (a_1', b_1') \supset \cdots \supset [a_n', b_n'] \supset (a_n', b_n') \supset \cdots$$

such that for all x and y in $[a_n', b_n']$, we have

$$|f(x) - f(y)| < \tfrac{1}{2^n} \qquad \text{and} \qquad |g(x) - g(y)| < \tfrac{1}{2^n}.$$

By pointwise discontinuity, there are infinitely many points of continuity in every open subinterval. However, by the Nested Interval Property, there is exactly one point p contained in all the closed subintervals above. Thus, both f and g are continuous at p.

This implies that, $C(f) \cap C(g) \neq \varnothing$, which is a contradiction.

Therefore, No such functions f and g exist. ■

1.5 Around J. Fabrykowski's Problem

In [11], J. Fabrykowski formulated the problem of finding a sequence of functions continuous on $[0,1]$ whose pointwise limit is finite on the rationals and infinite on the irrationals. To analyze a proposed solution to the problem, he required several facts from continued fractions. However, it turned out that such a function is closely related to the "small" Riemann function.

The following solution is due to Gelbaum and Olmstead in [16]. They give a sequence of continuous functions converging pointwise to the "small" Riemann function;

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, \text{ where } \gcd(p,q) = 1, q > 0. \end{cases}$$

Taking reciprocals yields a sequence of continuous functions solving Fabrykowski's problem. However, the construction in [16] makes no use of continued fractions or other number-theoretical devices.

For each arbitrary positive integer n, define $f_n(x)$ as follows: According to each point $\left(\frac{p}{q}, \frac{1}{q}\right)$, where $1 \le q < n$, $0 \le p \le q$, in each interval of the form $\left(\frac{p}{q} - \frac{1}{2n^2}, \frac{p}{q}\right)$ define

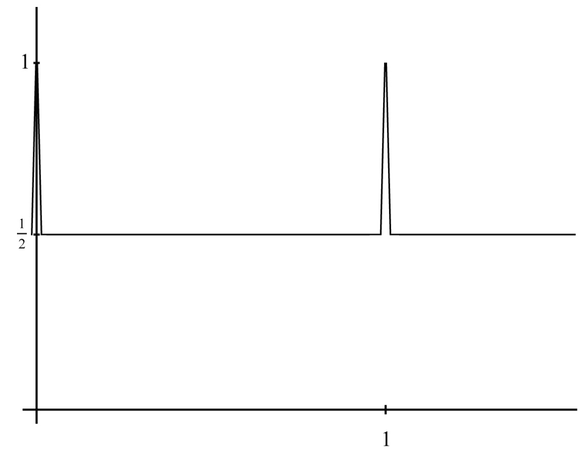
$$f_n(x) = \max\left(\frac{1}{n}, \frac{1}{q} + 2n^2\left(x - \frac{p}{q}\right)\right);$$

in each interval of the form $\left(\frac{p}{q}, \frac{p}{q} + \frac{1}{2n^2}\right)$ define

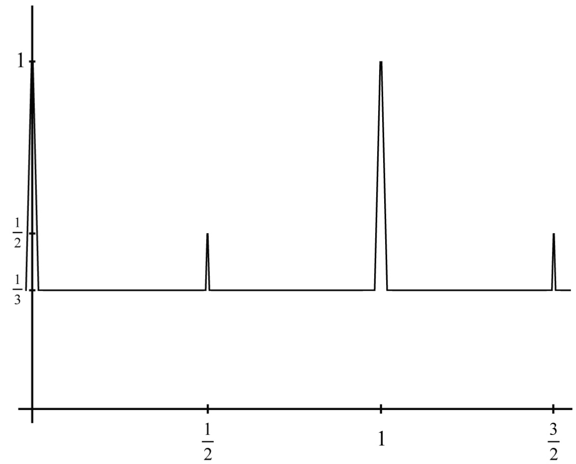
$$f_n(x) = \max\left(\frac{1}{n}, \frac{1}{q} - 2n^2\left(x - \frac{p}{q}\right)\right);$$

and at every point x of $[0,1]$ at which $f_n(x)$ has not already been defined, let $f_n(x) = \frac{1}{n}$.

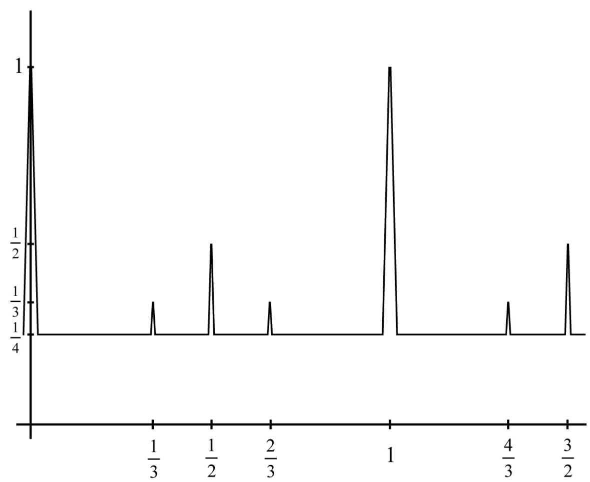
Note, the above $f_n(x)$ can be combined. For each x such that $|x - \frac{p}{q}| < \frac{1}{2n^2}$, we can let $f_n(x) = \max(\frac{1}{n}, \frac{1}{q} - 2n^2|x - \frac{p}{q}|)$.



f$_2$



f$_3$



f$_4$

The graph of $f_n(x)$ consists of an infinite polygonal arc made up of segments that either lie along the horizontal line $y = \frac{1}{n}$ or rise with a slope of $\pm 2n^2$ to the isolated points of the graph of f.

As n increases, these "spikes" sharpen and the base approaches the x-axis. Thus, for each $x \in \mathbb{R}$ and $n = 1, 2, \ldots,$

$$f_n(x) \geq f_{n+1}(x), \qquad \text{and} \qquad \lim_{n\to\infty} f_n(x) = f(x).$$

Each function f_n is everywhere continuous, but the limit function f is discontinuous on $\mathbb{Q}$.

Remark 1.5.1. A function f is said to be of the first class of Baire if it can be represented as the limit of an everywhere convergent sequence of continuous functions (see [28], p. 32). This construction shows that the "small" Riemann function is of first class of Baire.

Remark 1.5.2. Another construction of a sequence of continuous functions that is a solution to Fabrykowski's problem is due to G. Myerson in [27].

Myerson's construction is similar, and arguably even a bit simpler, to the above construction. The construction goes as follows: Enumerate the rationals in $(0, 1)$ as $r_1, r_2, \ldots$, and let f_n be the piecewise linear function whose graph passes through the points $(0, 0), (r_1, 1), \ldots, (r_n, n)$, and $(1, 0)$.

If x is rational, then x is 0, 1 or r_k for some k, and $\lim_{n\to\infty} f_n(x)$ is 0, 0 or k, respectively. In any event, $\lim_{n\to\infty} f_n(x)$ is finite.

If x is irrational, then for n sufficiently large, the set $\{r_1, r_2, \ldots, r_n\}$ contains elements less than x and elements greater than x since the rationals are dense in $[0, 1]$.

So, let $r_{l(n)}$ be the greatest of the elements less than x and $r_{g(n)}$ be the least of the elements greater than x. Then $f(x) > \min\{l(n), g(n)\}$. Now, $l(n)$ and $g(n)$ both go to infinity as n goes to infinity since the rationals are dense in $[0, 1]$. Therefore $\lim_{n\to\infty} f_n(x) = \infty$.

Chapter 2

Extensions of the Volterra Theorem

Volterra proved that if two functions from $\mathbb{R}$ to $\mathbb{R}$ are continuous on dense subsets of $\mathbb{R}$, then the set of common points of continuity is dense in $\mathbb{R}$. D. Gauld and V. Rădulescu gave a generalization of this result in [12] and [38], respectively. More precisely, Rădulescu proved that if (X, d_1) is a complete metric space and (Y, d_2) is a metric space and $f, g : X \to Y$ are continuous on dense subsets of X, then the set of their common points of continuity is dense in X. Gauld provided a routine proof for real-valued functions of real variables, but he indicated that it can be generalized to compact metric spaces for the domain space. Furthermore, both proved that the intersection of the two dense sets of points of continuity is uncountable.

I will begin by introducing metric spaces, complete metric spaces and compact metric spaces. Then, I will provide proofs of Gauld and Rădulescu's generalizations.

2.1 Introduction to Metric Spaces

Definition 2.1.1. Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}$ is called a metric, or a distance function on X if for any $x, y, z \in X$,

(M1) $d(x, y) = 0 \Leftrightarrow x = y$,

(M2) $d(x, y) = d(y, x)$,

(M3) $d(x, y) \leq d(x, z) + d(z, y)$.

If d is a metric on a set X, then the ordered pair (X, d) is called a metric space. Furthermore, if $x, y \in X$, then $d(x, y)$ is the distance from x to y.

$(M3)$ is called the *triangle inequality* due to the fact that the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides.

Example 2.1.1. Let X be a nonempty set and define $d : X \times X \to \mathbb{R}$, where $x, y \in X$, as follows:

$$\mathcal{D}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Then, $\mathcal{D}$ is a metric on X, and is called the discrete metric.

Example 2.1.2. The function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d(x, y) = |x - y|$ is a metric and is called the usual metric on $\mathbb{R}$. Furthermore, the function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$, is a metric and is called the usual metric on $\mathbb{R}^2$.

Example 2.1.3. Let $n \in \mathbb{N}$ and define a function $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}$ by

$$\mathcal{E}(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, is a metric on $\mathbb{R}^n$. It is called the Euclidean metric on $\mathbb{R}^n$.

2.2 Open and Closed Subsets in a Metric Space

We shall first consider open balls in order to define open and closed sets.

Definition 2.2.1. Let (X, d) be a metric space, let x_o be an arbitrary point of X and let r be an arbitrary nonnegative number. An open ball centered at x_o having radius r is the following set

$$B(x_0, r) = \{x \in X | d(x, x_0) < r\}.$$

Definition 2.2.2. Let (X, d) be a metric space, let x_o be an arbitrary point of X and let r be an arbitrary nonnegative number. The closed ball centered at x_o having radius r is the following set

$$\overline{B}(x_0, r) = \{x \in X | d(x, x_0) \leq r\}.$$

Definition 2.2.3. A point x_0 in a metric space (X, d) is called an interior point of a set $A \subset X$ if there is an $r > 0$ such that $B(x_0, r) \subset A$.

Furthermore, the set of all the interior points of $A \subset X$ is called the interior of A and is denoted $int(A)$.

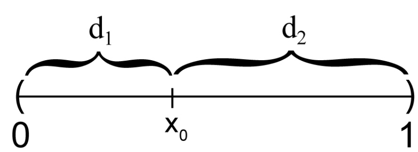
Definition 2.2.4. A subset G of a metric space (X, d) is called an open set if all of its points are interior points.

It follows from **Definition 2.2.4** that a set A in (X, d) is open if and only if $int(A) = A$.

Example 2.2.1. The interval $(0, 1)$ is open in $(\mathbb{R}, \mathcal{E})$. We will prove why this is true.

Proof. Let x_0 be an arbitrary point in $(0, 1)$ and consider the following distances on $\mathbb{R}$:

$$d_1 = |x_0 - 0| = x_0 \qquad \text{and} \qquad d_2 = |1 - x_0| = 1 - x_0.$$



Now, let $r = \min\{d_1, d_2\} > 0$. Then $B(x_0, r) \subset (0, 1)$ for every $x_0 \in (0, 1)$.

Therefore, $(0, 1)$ is open in $(\mathbb{R}, \mathcal{E})$. ■

Now, we will look into closed subsets of metric spaces.

Definition 2.2.5. A subset A of a metric space (X, d) is closed if its complement is open.

Definition 2.2.6. Let A be a subset of a metric space (X, d). A point x_0 is an accumulation point of A if for every $r > 0$,

$$(B(x_0, r) \cap A) \setminus \{x_0\} \neq \varnothing,$$

in other words, if every open ball about x_0 contains a point $p \neq x_0$ such that $p \in A$.

Furthermore, the set of all accumulation points of A is called the derived set of A. It is denoted by A'.

Definition 2.2.7. Let (X, d) be a metric space, and $A \subset X$. The closure of A is

$$\overline{A} = A' \cup A.$$

Example 2.2.2. Consider the interval $(0, 1)$ in $(\mathbb{R}, \mathcal{E})$. Notice that $A' = [0, 1]$ since every open ball about every number in $[0, 1]$ contains another point in $(0, 1)$. Also $\overline{(0, 1)} = [0, 1] \cup (0, 1) = [0, 1]$.

Furthermore, it can be shown that:

Theorem 2.2.1. A subset A of a metric space (X, d) is closed if and only if $A = \overline{A}$.

We can also use the closure operator to define dense subsets in metric spaces.

Definition 2.2.8. A subset A of a metric space (X, d) is dense (in X) if $\overline{A} = X$.

Note that $(\mathbb{R}, \mathcal{E})$ is a metric space and dense subsets of $\mathbb{R}$ using **Definition 1.2.1** are **the same** as those using **Definition 2.2.8**.

Example 2.2.3. In $(\mathbb{R}, \mathcal{E})$, the rationals $\mathbb{Q}$ are dense since $\overline{\mathbb{Q}} = \mathbb{R}$.

Definition 2.2.9. A subset A of a metric space (X, d) is dense-in-itself if it has no isolated points.

Example 2.2.4. The subset of irrationals in the $(\mathbb{R}, \mathcal{E})$ is dense-in-itself because every neighborhood of an irrational number contains at least one other irrational number.

2.3 Resolvable Spaces

Definition 2.3.1. A space X is resolvable if $X = A \cup B$, where A and B are disjoint, dense subsets of X.

A space with isolated points cannot be resolvable.

Example 2.3.1. The set of real numbers $\mathbb{R}$ is resolvable since $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$, and the set of rationals and irrationals are dense and disjoint in $\mathbb{R}$.

In fact, every dense-in-itself metric space is a resolvable space. The following proof is based on a result by S. Kim in [23], but it is a classical result.

Theorem 2.3.1. ([23], p. 258) If (X, d) is a nonempty metric space without isolated points, then X has a dense subset A whose complement is also dense in X.

Proof. Let $S \subset X$. S is called an ε-net if

(1) $d(x, y) \geq \varepsilon$ for any two distinct points x, y in S, and

(2) S is maximal with respect to (1).

Kuratowski-Zorn's Lemma (see Appendix A) yields that this ε-net exists for every $\varepsilon > 0$.

Now, suppose we have disjoint sets $S_1, S_2, \ldots S_k$ where each S_i, $i = 1, 2, \ldots k$, is an $(\frac{1}{i})-$net. Then, the complement of $S_1 \cup S_2 \cup \cdots \cup S_k$ is nonempty, and has no isolated points. Therefore, there is an S_{k+1}, disjoint from $S_1 \cup S_2 \cup \cdots \cup S_k$, which is an $(\frac{1}{k+1})-$net.

Then, $A = \bigcup_{n=1}^{\infty} S_{2n}$ and $B = \bigcup_{n=1}^{\infty} S_{2n-1}$ are disjoint, and both are dense in X. ■

Many Ph.D students have done research on resolvable spaces and topological groups under W. W. Comfort at Wesleyan University.

In [20], E. Hewitt studied resolvability for more abstract spaces.

Definition 2.3.2. We say X is a k-space if a set A is closed in X if and only if $A \cap K$ is closed for every compact $K \subset X$ (see section 3.8).

N.V. Velichko proved that dense-in-themselves k-spaces are resolvable in [40].

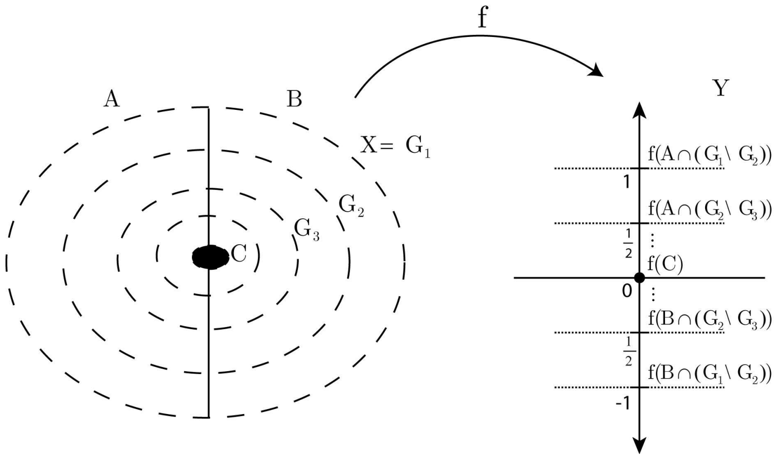
We will now prove the converse of **Theorem 1.3.1** using the fact that dense-in-themselves metric spaces are resolvable. W. Sierpiński was the first to publish this construction in [39]. We will now provide the proof given by S. Kim and S. Plewik in [24].

Theorem 2.3.2. ([24], p. 7) Let X be a resolvable space. Then, for any F_σ-set E, there exists a bounded, real-valued function f having E for its set of points of discontinuity.

Proof. Let $X = A \cup B$ be a resolvable space, where A and B are dense and disjoint subsets of X. Let f be a function such that $f : X \to Y$, where $Y = \{0\} \cup \{\frac{1}{n} : n = 1, 2, \ldots\} \cup \{-\frac{1}{n} : n = 1, 2, \ldots\}$.

Now, suppose $C = \bigcap_{n=1}^{\infty} G_n$ is the intersection of a decreasing sequence of open sets $G_n \subset X$ with $G_1 = X$. Then define f as follows:

1. if $x \in C$, let $f(x) = 0$,
2. if $x \in A \cap (G_n \setminus G_{n+1})$, let $f(x) = \frac{1}{n}$,
3. if $x \in B \cap (G_n \setminus G_{n+1})$, let $f(x) = -\frac{1}{n}$.



Since the oscillation of f on C (and nowhere else) is 0, the set C is the set of points of continuity of f. Thus, $X \setminus C$ is the set of points of discontinuity of f. Furthermore, since C is a G_δ-set, $X \setminus C$ is an F_σ-set.

Therefore, $X \setminus C$ is an F_σ-set and there is a bounded function f in which $X \setminus C$ is the set of discontinuity points for f. Notice, if we started with any F_σ-set E, then C can be taken to be $X \setminus E$. ■

In section 4.1 , we will prove that a second countable (see section 3.4), dense-in-itself Baire space is resolvable.

2.4 Complete and Compact Metric Spaces

In order to define a complete metric space, we first have to define what it means for a sequence to be Cauchy.

Definition 2.4.1. A sequence $\{x_n\}$ of points of a metric space is a <u>Cauchy sequence</u> if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $n, m \geq N$,

$$d(x_n, x_m) < \varepsilon.$$

Example 2.4.1. Consider the sequence $\{x_n\}$, where $x_n = \frac{1}{n}$, $n \in \mathbb{N}$ defined in $(\mathbb{R}^+, \mathcal{E})$, the Euclidean metric on the positive real line. To show this is a Cauchy sequence, let $\varepsilon > 0$. We are to find an index N depending on ε.

We claim that $N = [\frac{1}{\varepsilon}] + 1$ is such an index.

If $m > n \geq N$, then

$$\begin{aligned} \left|\frac{1}{n} - \frac{1}{m}\right| &< \frac{1}{n} \\ &\leq \frac{1}{N} \\ &= \frac{1}{[\frac{1}{\varepsilon}] + 1} \\ &< 1/(1/\varepsilon) \\ &= \varepsilon. \end{aligned}$$

This shows that $\{x_n\}$ is Cauchy.

Observe that $\{x_n\}$ is not convergent in $(\mathbb{R}^+, \mathcal{E})$ since

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \tfrac{1}{n} = 0 \notin \mathbb{R}^+.$$

Example 2.4.2. Consider the sequence $\{a_n\}$, where

$$a_1 = 1.4, \quad a_2 = 1.41, \quad a_3 = 1.414, \quad a_4 = 1.4142, \quad a_5 = 1.41421, \quad \ldots$$

of finite decimals converging to $\sqrt{2}$. The sequence $\{a_n\}$ is a Cauchy sequence. Although it does not converge in $\mathbb{Q}$, it is convergent to $\sqrt{2}$ in $\mathbb{R}$.

To establish a relationship between Cauchy sequences and convergent sequences, we can use the following theorem.

Theorem 2.4.1. Every convergent sequence is a Cauchy sequence.

Proof. Let $\{a_n\}$ be a convergent sequence. That is, let $a_n \to p$. Furthermore, let $\varepsilon > 0$. Then, there exists an $N \in \mathbb{N}$ such that for every $k \geq N$

$$d(a_k, p) < \frac{1}{2}\varepsilon.$$

Hence, by the triangle inequality,

$$\begin{aligned} n, m \geq N \Rightarrow d(a_n, a_m) &\leq d(a_n, p) + d(a_m, p) \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon \\ &= \varepsilon. \end{aligned}$$

Therefore, $\{a_n\}$ is a Cauchy sequence. ■

We now have the terminology to define a complete metric space.

Definition 2.4.2. A metric space (X, d) is called complete if every Cauchy sequence $x_1, x_2, \ldots$ of points in X is convergent in (X, d).

Example 2.4.3. $(\mathbb{Q}, \mathcal{E})$, the set of rationals in the Euclidean metric, is not a complete metric space since the Cauchy sequence of rational numbers $\{a_n\}$ in **Example 2.4.2** does not converge in $\mathbb{Q}$.

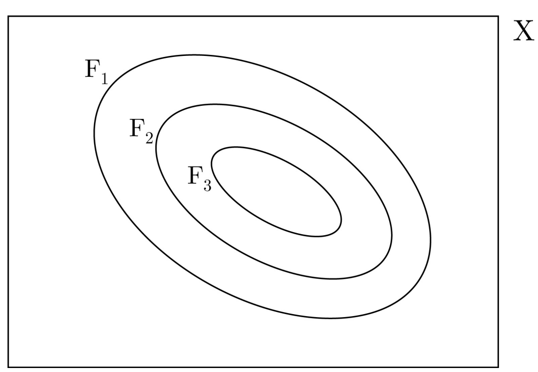
Example 2.4.4. $(\mathbb{R}, \mathcal{E})$ is a complete metric space since every Cauchy sequence of real numbers converges to a real number.

The following theorem, which is known as Cantor's theorem, can also be used to determine whether or not a metric space is complete.

Theorem 2.4.2. ([35], p. 2) A metric space (X, d) is complete if and only if every sequence of nonempty subsets $\{F_n\}$ of X satisfying:

1. $F_1 \supset F_2 \supset \ldots F_n \supset \ldots$,
2. $F_i = \overline{F_i}$ for all $i = 1, 2, \ldots$, and
3. $\lim_{n\to\infty} diam(F_n) = 0$,

has a nonempty intersection. More precisely, there exists the unique point $p \in X$ such that $\bigcap_{n=1}^{\infty} F_n = \{p\}$.



Proof. Assume that (X, d) is complete and let p_n be any point from F_n. First, we will show that a such constructed sequence $\{p_n\}$ is Cauchy.

By (3), for every $\varepsilon > 0$ there is an N such that for $n \geq N$, $diam(F_n) < \varepsilon$. By (1), if $n \geq k$, then $p_n \in F_n \subset F_k$. Hence, for $n \geq k$ we get p_n, $p_k \in F_k$, which in turn gives

$$d(p_n, p_k) \leq diam(F_k) < \varepsilon$$

if $k \geq N$. Hence, $\{p_n\}$ is Cauchy.

Since (X, d) is complete, $\{p_n\}$ is convergent. So, let $p = \lim_{n\to\infty} p_n$.

Now, for every n, the terms $p_1, p_2, \ldots$ of $\{p_n\}$, belong to F_n, except for at most the first n - 1 terms. However, F_n is closed, so it contains the limits of all sequences of its elements; in particular, $p \in F_n$, n = 1,2, ...

Hence, $p \in \bigcap_{n=1}^{\infty} F_n$. This part of the proof shows that $\bigcap_{n=1}^{\infty} F_n \neq \varnothing$ since p is in there.

Now, we will prove the uniqueness part of the proof, that is, p is the only point of $\bigcap_{n=1}^{\infty} F_n$. So, suppose there is another point q in the intersection. Since $p \neq q$, $d(p, q) = \alpha > 0$.

Note that if $q \in \bigcap_{n=1}^{\infty} F_n$, then both p and q are in F_k for all k. So,

$$\alpha = d(p, q) \leq diam(F_k)$$

for all k. This implies that $\alpha = 0$, which leads to a contradiction since we defined α to be greater than 0. This proves that there cannot be another point q in the intersection. This proof takes care of one of the implications in Cantor's theorem.

Conversely, assume (X, d) is a metric space in which every decreasing sequence of nonempty closed sets satisfying condition (2) has a nonempty intersection and let $\{x_n\}$ be a Cauchy sequence in (X, d).

Let us form a monotonically decreasing sequence of closed sets

$$F_n = \overline{\{x_n, x_{n+1}, \ldots\}}, \quad n = 1, 2, \ldots$$

having a nonempty intersection (see (1) and (2)). Let $x_0 \in \bigcap_{n=1}^{\infty} F_n$. Then,

$$d(x_n, x_0) \leq diam(F_n) \to 0.$$

Hence, the sequence $\{x_n\}$ converges to x_0, which proves the completeness of (X, d). ■

We will now define compact metric spaces.

Definition 2.4.3. A metric space (X, d) is compact if every sequence $\{x_n\}$ of points of X contains a subsequence which converges to an element of X.

Example 2.4.5. $(\mathbb{R}, \mathcal{E})$ is not a compact metric space. In fact, take $x_n = n$. Note that $\{x_n\}$ is divergent (as is any of its subsequences).

Example 2.4.6. $((0,1), \mathcal{E})$ is not a compact metric space. For instance, let $x_1 = \frac{1}{2}, x_2 = \frac{1}{3}, x_3 = \frac{1}{4}, \ldots x_n = \frac{1}{n+1}$. Then $\lim_{k\to\infty} x_{n_k} = 0 \notin (0,1)$ for all subsequences $\{x_{n_k}\}$ of $\{x_n\}$

Example 2.4.7. It can be shown ([29], p. 132, example 1d) that $([0,1], \mathcal{E})$ is a compact metric space.

Remark 2.4.1. It can be shown that every compact metric space is a complete metric space.

2.5 Volterra's Theorem for Complete Metric Spaces

The concept of a continuous mapping in a metric space will be useful in the proof of Volterra's result for complete metric spaces.

Definition 2.5.1. Let (X, d_1) and (Y, d_2) be metric spaces. A mapping $f : X \to Y$ is continuous if for every $x \in X$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d_1(x, y) < \delta \;\Rightarrow\; d_2(f(x), f(y)) < \varepsilon.$$

We will now state and prove Volterra's result for complete metric spaces.

Theorem 2.5.1. ([38], p. 9) Let (X, d_1) be a complete metric space such that X is uncountable and (Y, d_2) be a metric space. Consider two maps $f, g : X \to Y$ and denote by $C(f), C(g)$ the sets of their points of continuity. If $C(f)$ and $C(g)$ are dense subsets of X, then $C(f) \cap C(g)$ is an uncountable dense subset of X.

Proof. Let $C = \{c_1, c_2, \ldots\}$ be an arbitrary countable subset of X and

$$\begin{aligned} B(a, r) &= \{x \in X : d_1(x, a) < r\}, \\ \overline{B}(a, r) &= \{x \in X : d_1(x, a) \leq r\}, \end{aligned}$$

where $a \in X$ and $r > 0$ are arbitrarily chosen. It is enough to prove that

$$B(a, r) \cap C(f) \cap C(g) - C \neq \varnothing \tag{2.1}$$

to deduce our conclusion. If $B(a, r) \cap C(f) \cap C(g) \neq \varnothing$, it follows that $C(f) \cap C(g)$ is a dense subset of X. Furthermore, (2.1) implies that $C(f) \cap C(g)$ is uncountable because if it were not, then we would take $C = C(f) \cap C(g)$, which contradicts $B(a, r) \cap C(f) \cap C(g) - C \neq \varnothing$.

We define, by induction, two sequences $\{a_n\}_{n \geq 0} \subset X$ and $\{r_n\}_{n \geq 0} \subset \mathbb{R}^+$, the positive reals, with the following properties:

(i) $a_0 = a$, $r_0 = r$ and $B(a_n, r_n) \supset \overline{B}(a_{n+1}, r_{n+1})$, for every $n \geq 0$,

(ii) $c_n \notin B(a_n, r_n)$, for every $n \geq 1$,

(iii) if n is odd, then $d_2(f(x), f(y)) < \frac{1}{n}$ for each $x, y \in B(a_n, r_n)$,

(iv) if n is even then $d_2(g(x), g(y)) < \frac{1}{n}$ for each $x, y \in B(a_n, r_n)$.

Notice that our induction process has already been started at 0. So, suppose we have defined a_k and r_k for each $k < n$. If n is odd, we choose $a_n \in B(a_{n-1}, r_{n-1}) \cap C(f)$. Such an element exists because $C(f)$ is dense in X.

Since f is continuous at a_n, there is a $\delta > 0$ such that

$$d_2(f(x), f(a_n)) < \frac{1}{2n}$$

if $d_1(x, a_n) < \delta$.

Now, choose $r_n > 0$ so that

$$r_n < \min\left\{\delta, \frac{r}{n}, r_{n-1} - d_1(a_n, a_{n-1})\right\}.$$

It follows that if $x, y \in B(a_n, r_n)$ we have

$$\begin{aligned} d_2(f(x), f(y)) &\leq d_2(f(x), f(a_n)) + d_2(f(a_n), f(y)), \\ &< \tfrac{1}{n}. \end{aligned}$$

Furthermore, $\overline{B}(a_n, r_n) \subset B(a_{n-1}, r_{n-1})$. Moreover, if $d_1(x, a_n) \leq r_n$, then

$$\begin{aligned} d_1(x, a_{n-1}) &\leq d_1(x, a_n) + d_1(a_n, a_{n-1}), \\ &\leq r_n + d_1(a_n, a_{n-1}), \\ &< r_{n-1}. \end{aligned}$$

If n is even, we construct a_n and r_n similarly, replacing f by g and $C(f)$ by $C(g)$.

Since X is complete and $\{r_n\}$ is a sequence converging to 0, by the Cantor theorem, there exists a point b such that

$$b \in \bigcap_{n \geq 0} \overline{B}(a_n, r_n).$$

Since $\overline{B}(a_{n+1}, r_{n+1}) \subset B(a_n, r_n)$, it follows that $b \in B(a_n, r_n)$ for each $n \geq 0$.

We will now show that $b \in C(f) \cap C(g)$, that is, f and g are continuous at b. If $\varepsilon > 0$, we choose an odd integer n_0 such that $\frac{1}{n_0} < \varepsilon$. By (iii), it follows that if $\delta < r_{n_0} - d_1(b, a_{n_0})$, then

$$d_2(f(x), f(b)) < \frac{1}{n_0} < \varepsilon$$

if $d_1(x, b) < \delta$. Similarly, $b \in C(g)$.

Since $b \in B(a_n, r_n)$ and $c_n \notin B(a_n, r_n)$ for each $n \geq 1$, it follows that $b \notin C$.

Therefore, $B(a, r) \cap C(f) \cap C(g) \neq \varnothing$. ■

2.6 Volterra's Theorem for Compact Metric Spaces

We will now provide yet another proof of Volterra's theorem. Although the statement of **Theorem 2.6.1** pertains to real-valued functions of real variables, it can be appropriately generalized to compact metric spaces for the domain space.

Theorem 2.6.1. ([12], p. 246) If $f, g : \mathbb{R} \to \mathbb{R}$ are two functions and $C(f)$ and $C(g)$ are dense in $\mathbb{R}$, then $C(f) \cap C(g)$ is an uncountable dense subset of $\mathbb{R}$.

Proof. Let (a_0, b_0) be an interval in $\mathbb{R}$ and let $C = \{c_n : n = 1, 2, \ldots\}$ be a countable subset of $\mathbb{R}$. It will be shown that

$$(a_0, b_0) \cap C(f) \cap C(g) - C \neq \varnothing,$$

from which the conclusion will then be deduced.

Use induction to define two sequences $\{a_n\}$ and $\{b_n\}$ with $\{a_n\}$ strictly increasing and $\{b_n\}$ strictly decreasing so that for all n,

(i) $a_n < b_n$,

(ii) $c_n \notin (a_n, b_n)$,

(iii) if n is odd, then $|f(x) - f(y)| < \frac{1}{n}$ for each $x, y \in (a_n, b_n)$,

(iv) if n is even, then $|g(x) - g(y)| < \frac{1}{n}$ for each $x, y \in (a_n, b_n)$.

Induction has already been started at 0. So, suppose there is an m such that a_n and b_n have been defined for each $n < m$.

If m is odd, then since $C(f)$ is dense in $\mathbb{R}$, there is a number $a_m \in (a_{m-1}, b_{m-1}) \cap C(f)$. We can assume that if $c_m < b_{m-1}$, then $a_m > c_m$. Since f is continuous at a_m, there is a $\delta > 0$ so that if $|x - a_m| < \delta$ then $|f(x) - f(a_m)| < \frac{1}{2m}$.

Now, choose b_m so that $a_m < b_m < b_{m-1}$ and $b_m < a_m + \delta$. Then for each $x, y \in (a_m, b_m)$, we have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(a_m)| + |f(a_m) - f(y)|, \\ &< \tfrac{1}{2m} + \tfrac{1}{2m}, \\ &= \tfrac{1}{m}. \end{aligned}$$

If m is even, then construct a_m and b_m similarly, replacing $C(f)$ by $C(g)$ and f by g.

Notice that the intervals $[a_n, b_n], n = 0, 1, \ldots$, are closed, nested intervals. Thus, by the Nested Interval Property, there is an $a \in \mathbb{R}$ such that $a \in [a_n, b_n]$ for each n. Since $a_n < a_{n+1} \leq a \leq b_{n+1} < b_n$, it follows that $a_n < a < b_n$ for all n; in particular $a \in (a_0, b_0)$.

Furthermore, the point a is in $C(f)$; i.e., f is continuous at a. To show this is true, let $\varepsilon > 0$ and let n be an odd integer with $n \geq \frac{1}{\varepsilon}$. Then, let δ be the smaller of $a - a_n$ and $b_n - a$. By (iii), if $|x - a| < \delta$, then $|f(x) - f(a)| < \frac{1}{n} \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that f is continuous at a. Thus, $a \in C(f)$. Similarly, $a \in C(g)$.

Finally, $a \neq c_n$ for each n since $a \in (a_n, b_n)$ but $c_n \notin (a_n, b_n)$ by (ii). Thus,

$$(a_0, b_0) \cap C(f) \cap C(g) - C \neq \varnothing.$$

Since $(a_0, b_0) \cap C(f) \cap C(g) \neq \varnothing$, it follows that $C(f) \cap C(g)$ is dense in $\mathbb{R}$. Furthermore, $C(f) \cap C(g)$ cannot be countable because if it was, then we could take $C = C(f) \cap C(g)$ and obtain a contradiction since it has been shown that $C(f) \cap C(g) - C \neq \varnothing$. ■

Chapter 3

Topological Preliminaries

Throughout my thesis, I will need many important topological concepts. In this chapter, these concepts will be reviewed.

3.1 Topological Spaces

Definition 3.1.1. Let X be a nonempty set. A collection $\mathfrak{T}$ of subsets of X is a topology on X if it satisfies the following properties:

1. $\varnothing \in \mathfrak{T}$ and $X \in \mathfrak{T}$;
2. If $U \in \mathfrak{T}$ and $V \in \mathfrak{T}$, then $U \cap V \in \mathfrak{T}$;
3. If $U_\alpha \in \mathfrak{T}$ for each α in an indexing set Γ, then $\bigcup_{\alpha \in \Gamma} U_\alpha \in \mathfrak{T}$.

The members of $\mathfrak{T}$ are called open sets, and X together with $\mathfrak{T}$, i.e., the pair $(X, \mathfrak{T})$, is called a topological space.

Example 3.1.1. Let X be a nonempty set, and let $\mathfrak{T} = \{\varnothing, X\}$. Then $\mathfrak{T}$ is a topology on X, and it is called the trivial topology on X.

Example 3.1.2. Let $\mathcal{E}$ denote the set of all open sets of real numbers (as defined previously in section 2.1). Then $\mathcal{E}$ is a topology on $\mathbb{R}$, called the Euclidean topology on $\mathbb{R}$. We will denote this topological space as $(\mathbb{R}, \mathcal{E})$. Similarly, the class $\mathcal{E}$ of all open sets in the $\mathbb{R}^2$ plane is a topology called the Euclidean topology on $\mathbb{R}^2$. We will denote this topological space as $(\mathbb{R}^2, \mathcal{E})$.

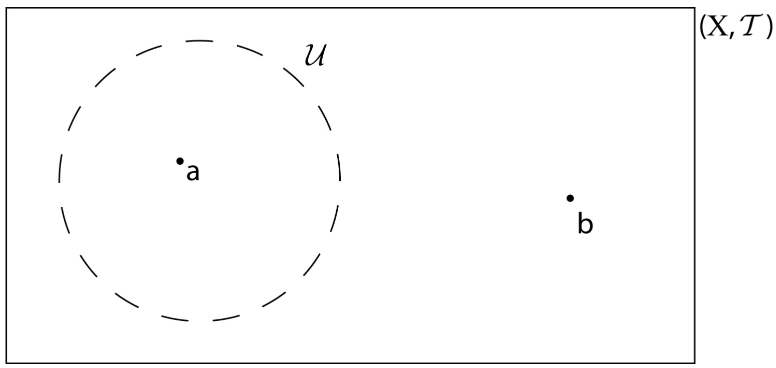
Example 3.1.3. Let $\mathcal{D}$ denote the set of all subsets of X. This is a topology on X called the discrete topology. X together with its discrete topology, $(X, \mathcal{D})$, is called a discrete topological space.

Example 3.1.4. Let $\mathcal{C}$ denote the set of all subsets of X whose complements are finite together with the empty set $\varnothing$. Then, this class $\mathcal{C}$ is a topology on X, called the cofinite topology.

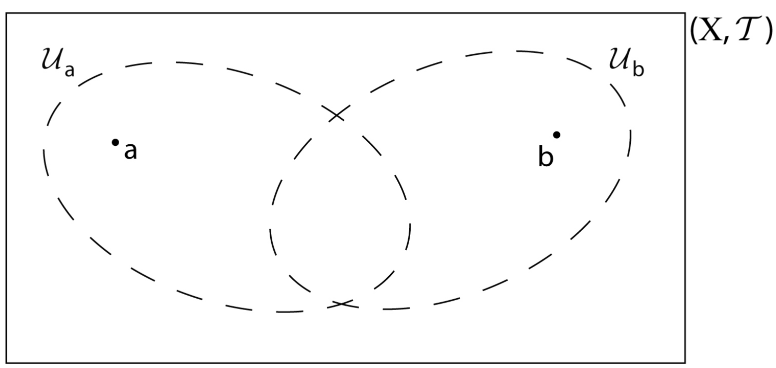
3.2 The Separation Axioms

We will only define up to T_2-spaces and regular spaces.

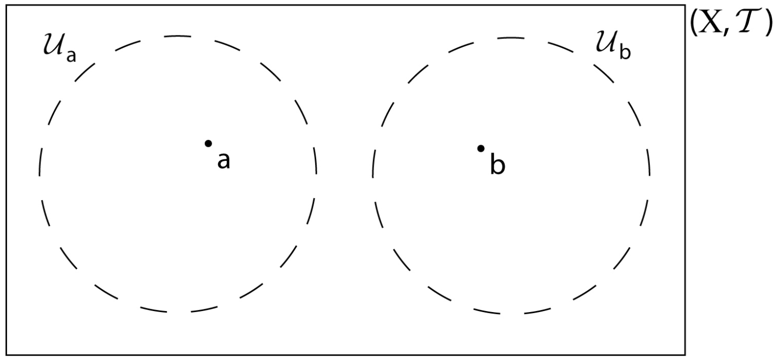
Definition 3.2.1. A topological space $(X, \mathfrak{T})$ is a T_0-space if for $a, b \in X$ there is an open set $U \in \mathfrak{T}$ such that either $a \in U$ and $b \notin U$ or $b \in U$ and $a \notin U$.



Definition 3.2.2. A topological space $(X, \mathfrak{T})$ is a T_1-space if for $a, b \in X$ there are open sets $U_a, U_b \in \mathfrak{T}$ containing a and b respectively such that $b \notin U_a$ and $a \notin U_b$.



Definition 3.2.3. A topological space $(X, \mathfrak{T})$ is a T_2-space or a Hausdorff space if there are disjoint open sets U_a and U_b containing a and b respectively.



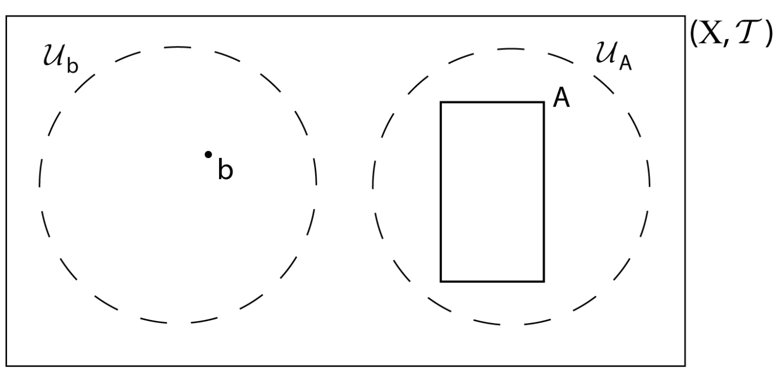
Example 3.2.1. If we equip the reals $\mathbb{R}$ (or any infinite set) with the cofinite topology $\mathcal{C}$, it can be shown that $(\mathbb{R}, \mathcal{C})$ is not Hausdorff. It is actually T_1. To show why this is true, let a and b be two different arbitrary points of $\mathbb{R}$. Any open set U containing a is of the form $\mathbb{R} \setminus A$, where $A \subset \mathbb{R}$ is finite. Similarly, any open set V containing b is of the form $\mathbb{R} \setminus B$, where $B \subset \mathbb{R}$ is finite. Thus,

$$U \cap V = (\mathbb{R} \setminus A) \cap (\mathbb{R} \setminus B) = \mathbb{R} \setminus (A \cup B) \neq \varnothing,$$

since there are infinitely many points in $\mathbb{R}$.

Example 3.2.2. Every metric space is Hausdorff. If x and y are distinct points in a metric space (X, d), then $d(x, y) > 0$. Let $r = \frac{1}{2}d(x, y)$. Then, $B(x, r)$ and $B(y, r)$ are two disjoint open sets containing x and y respectively.

Definition 3.2.4. A topological space $(X, \mathfrak{T})$ is a <u>regular space</u> if given a closed set A and a point $b \notin A$, there are disjoint open sets U_A and U_b containing A and b respectively.



3.3 Kuratowski's Closure Operator

Recall that if "U is an open set such that $x \in U$" then we say "U is an *open neighborhood* of x."

Also, let A be a subset of a topological space $(X, \mathfrak{T})$ and $x \in X$. Then $x \in \overline{A}$, the *closure* of A, if and only if every neighborhood of X has a nonempty intersection with A.

We will list important properties pertaining to the closure operator.

Theorem 3.3.1. Let A and B be subsets of a topological space $(X, \mathfrak{T})$. Then,

1. $\overline{\varnothing} = \varnothing$,
2. $\overline{\overline{A}} = \overline{A}$,
3. $\overline{A} \subset \overline{B}$ whenever $A \subset B$,
4. $\overline{A \cup B} = \overline{A} \cup \overline{B}$
5. $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

A subset A of $(X, \mathfrak{T})$ is *closed* if $A = \overline{A}$. Also recall that the *interior* of A, denoted $int(A)$, is as follows:

$$int(A) = X \setminus \overline{X \setminus A}.$$

Furthermore, it can be shown that A is an *open set* in a topological space if and only if $A = int(A)$.

3.4 Bases

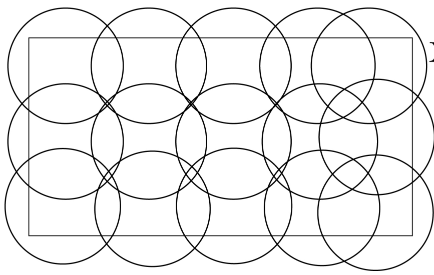
Definition 3.4.1. Let $(X, \mathfrak{T})$ be a topological space. A base $\mathfrak{B}$ for the topology $\mathfrak{T}$ is a subcollection of $\mathfrak{T}$ with the property that if $U \in \mathfrak{T}$, then $U = \varnothing$ or there is a subcollection $\mathfrak{B}'$ of $\mathfrak{B}$ such that

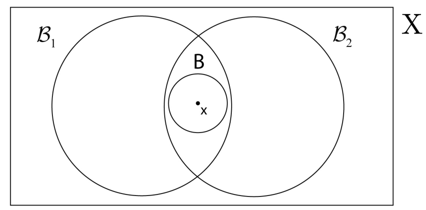
$$U = \bigcup \{B : B \subset \mathfrak{B}'\}.$$

Furthermore, we can use the following theorem to find a base for a topological space.

Theorem 3.4.1. A collection $\mathfrak{B}$ of subsets of X is a base for some topology $\mathfrak{T}$ on X if and only if

1. $X = \bigcup\{B : B \subset \mathfrak{B}\}$,
2. if $B_1, B_2 \subset \mathfrak{B}$ and $x \in B_1 \cap B_2$, then there exists a $B \in \mathfrak{B}$ such that $x \in B$ and $B \subset B_1 \cap B_2$.





Example 3.4.1. In $(\mathbb{R}^2, \mathcal{E})$, the subcollection of all open balls having rational coordinates for their centers and rational radii is a base for $(\mathbb{R}^2, \mathcal{E})$.

Definition 3.4.2. Given a topological space $(X, \mathfrak{T})$, and a point $a \in X$, a subcollection $\mathfrak{B}_a$ of $\mathfrak{T}$ is a <u>local base at a</u> provided that

1. if $B \in \mathfrak{B}_a$, then $a \in B$,
2. if $U \in \mathfrak{T}$ and $a \in U$, then there exists a $B \in \mathfrak{B}_a$ such that $a \in B \subset U$.

Example 3.4.2. In $(\mathbb{R}, \mathcal{E})$ let $a \in \mathbb{R}$. Then, the subcollection

$$\mathfrak{B}_a = \left\{ \left(a - \frac{1}{n}, a + \frac{1}{n} \right) : n \in \mathbb{N} \right\}$$

is a local base at a.

Definition 3.4.3. A topological space $(X, \mathfrak{T})$ is called <u>first countable</u> or <u>satisfies the first axiom of countability</u> if there is a countable local base at each point of X.

In fact, every metric space is first countable. In a metric space (X, d), let $x \in X$ and consider the subcollection $\mathfrak{B}_x = \left\{ B\left(x, \frac{1}{n}\right) : n \in \mathbb{N} \right\}$.

Definition 3.4.4. We say that $(X, \mathfrak{T})$ is second countable or satisfies the second axiom of countability provided there is a countable base for $\mathfrak{T}$.

Theorem 3.4.2. Every second countable space is first countable.

Proof. Let $\mathfrak{B}$ be a countable base for $\mathfrak{T}$ and let $p \in X$.

Then the subcollection $\mathfrak{B}'$ of $\mathfrak{B}$ consisting of those members of $\mathfrak{B}$ which contain p is a countable local base at p. ■

However, the converse does not hold.

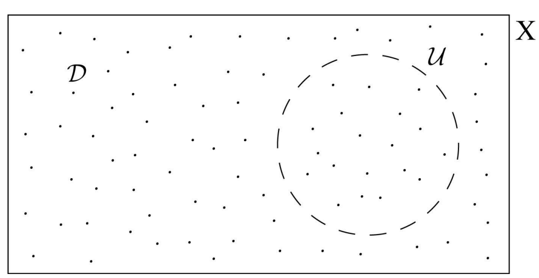
Example 3.4.3. Consider the reals with the discrete topology $(\mathbb{R}, \mathcal{D})$ and consider the subcollection $\mathfrak{B}_x = \{\{x\} : x \in \mathbb{R}\}$. This is an example of a first countable space which is not second countable.

This is true because we cannot obtain the reals as a countable union of the elements from $\mathfrak{B}_x$ since the reals have uncountably many points.

3.5 Denseness

Definition 3.5.1. Given a topological space X, a subset D of X is called everywhere dense, or simply dense, if it intersects every nonempty open subset U of X, i.e.,

$$\forall\, U_{\text{open},\neq\varnothing} \subset X, \;\; D \cap U \neq \varnothing.$$



Using the closure operator, D is dense if and only if

$$\overline{D} = X.$$

Example 3.5.1. In $(\mathbb{R}^2, \mathcal{E})$, the set D, where

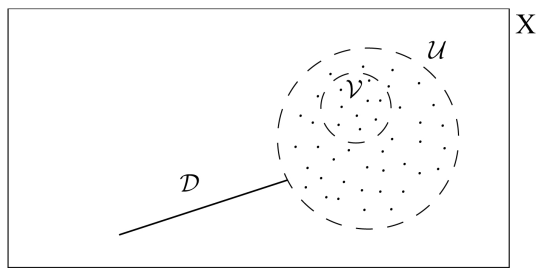
$$D = \{(x, y) : x, y \in \mathbb{Q}\}$$

is dense.

Example 3.5.2. In $(\mathbb{R}, \mathcal{E})$, the set $\mathbb{Q}$ of the rational numbers is dense. Furthermore, so are the irrational numbers, $\mathbb{R} \backslash \mathbb{Q}$.

Definition 3.5.2. ([19], p. 6) A subset D of X is called <u>somewhere dense</u> if there is an open, nonempty subset U of X in which D is dense, i.e.,

$$\exists\, U_{\text{open},\neq\varnothing} \text{ such that } \forall\, V_{\text{open},\neq\varnothing} \subset U, \;\; D \cap V \neq \varnothing.$$



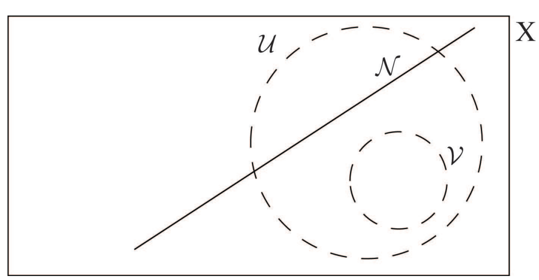
Using the closure operator, D is somewhere dense if and only if

$$int(\overline{D}) \neq \varnothing.$$

Example 3.5.3. The union of the set of all rational numbers in $(0, 1)$ together with a finite set in $\mathbb{R} \setminus (0, 1)$ is somewhere dense.

Definition 3.5.3. A subset N of X is called <u>nowhere dense</u> if there is no open, nonempty set U in which N is dense, i.e.,

$$\forall\, U_{\text{open},\neq\varnothing} \subset X \;\; \exists\, V_{\text{open},\neq\varnothing} \subset U \text{ such that } N \cap V = \varnothing.$$



Using the closure operator, N is nowhere dense if and only if

$$int(\overline{D}) = \varnothing.$$

Example 3.5.4. The set of all integers $\mathbb{Z}$ in the reals $\mathbb{R}$ with the Euclidean topology $\mathcal{E}$ is nowhere dense.

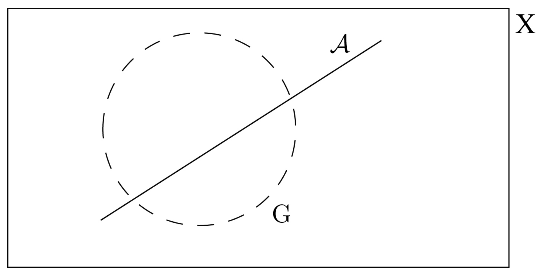
Definition 3.5.4. ([25], p. 66) A subset B of a topological space X is boundary if the complement of B is dense in X. These sets are also known as co-dense sets.

Using the closure operator, B is boundary if and only if

$$\overline{X \setminus B} = X.$$

The proof of the following lemma is a part of topological folklore.

Lemma 3.5.1. Every closed and boundary set A in X is nowhere dense.



Proof. Since A is closed and boundary in X, its complement $X \setminus A$ is open and dense in X. This shows that A contains no nonempty open set. Otherwise, it would intersect $X \setminus A$, which is dense in X.

Thus, if G is an arbitrary open, nonempty set, then

$$G \setminus \overline{A} = G \setminus A$$

is an open, nonempty set which is disjoint from A.

Therefore, A is nowhere dense. ■

3.6 First and Second Category

Definition 3.6.1. A set is of first category if it is the countable union of nowhere dense sets.

Example 3.6.1. In $(\mathbb{R}, \mathcal{E})$, the subset $\mathbb{Q}$ of all rational numbers is of first category. In fact

$$\mathbb{Q} = \bigcup_{i=1}^{\infty} \{q_i\},$$

where each $\{q_i\}$ is nowhere dense.

We have the following important lemma pertaining to sets of first category. This lemma will be used later in the thesis.

Lemma 3.6.1. The countable union of sets of first category is also of first category.

Definition 3.6.2. A set is of second category if it is not of first category.

Example 3.6.2. The set of irrational numbers is of second category.

3.7 F_σ and G_δ Sets

In section 1.2, we defined G_δ and F_σ-sets. We will now look at properties of these types of sets on topological spaces.

Example 3.7.1. Every closed set is an F_σ-set and every open set is a G_δ-set.

Example 3.7.2. The set $\mathbb{Q}$ of rationals is an F_σ-set of the reals $\mathbb{R}$. We can express $\mathbb{Q}$ as

$$\mathbb{Q} = \bigcup_{q_i \in \mathbb{Q}} \{q_i\},$$

where each $\{q_i\}$ is a closed set, since singletons are closed in $\mathbb{R}$.

Example 3.7.3. The set $\mathbb{R} \setminus \mathbb{Q}$ of irrationals is a G_δ-set of $\mathbb{R}$. We can express $\mathbb{R} \setminus \mathbb{Q}$ as

$$\mathbb{R} \setminus \mathbb{Q} = \bigcap_{q_i \in \mathbb{Q}} \mathbb{R} \setminus \{q_i\},$$

where each $\mathbb{R} \setminus \{q_i\}$ is an open set since the complement of the closed set of the singleton $\{q_i\}$ is an open set.

It can be show that the set of irrational numbers is **not** F_σ by using the Baire Category theorem for complete metric spaces (see section 4.2).

Proposition 3.7.1. ([33], p. 1) The set of irrationals is not an F_σ-set.

Proof. Suppose

$$\mathbb{R} \setminus \mathbb{Q} = \bigcup_{i=1}^{\infty} F_i,$$

where $F_i = \overline{F_i}$.

Then, we have two cases.

For our first case, suppose there is an i such that F_i is dense in some interval. By being dense, it would contain this interval, i.e., the rational numbers would belong. This would be impossible since $F_i \subset \mathbb{R} \setminus \mathbb{Q}$. So, we have the other case.

Suppose there is no i for which F_i is dense in an interval. This means that each F_i is nowhere dense, i.e., $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{i=1}^{\infty} N_i$ where each N_i is nowhere dense.

However, $\mathbb{Q}$, the set of all rationals, is of first category. This set is a countable union of singletons $\{q_i\}$, where $q_i \in \mathbb{Q}$.

This would mean that the set $\mathbb{R}$ of all real numbers would be the union of two sets of first category, hence of first category by **Lemma 3.6.1**. This is a contradiction since $(\mathbb{R}, \mathcal{E})$ is a complete metric space and of second category by the Baire Category theorem.

Therefore, $\mathbb{R} \setminus \mathbb{Q}$ is not an F_σ-set. ■

Lemma 3.7.1. ([32], p. 11) Every boundary F_σ-set is of first category.

Proof. If $\bigcup_{n=1}^{\infty} F_n$ is a boundary set, then each F_n is boundary.

Since each F_n is closed, they are nowhere dense by **Lemma 3.5.1**.

Therefore, $\bigcup_{n=1}^{\infty} F_n$ is of first category. ■

3.8 Topological Compactness

Definition 3.8.1. A collection Γ of subsets of a topological space $(X, \mathfrak{T})$ is called a cover, or covering of a set $B \subset X$ if

$$B \subset \bigcup_{A \in \Gamma} A.$$

We say that Γ is an open covering if each member of Γ is open.

Definition 3.8.2. A subset A of a topological space $(X, \mathfrak{T})$ is compact if every open cover has a finite subcover.

The concept of compactness is a generalization of the Heine-Borel theorem.

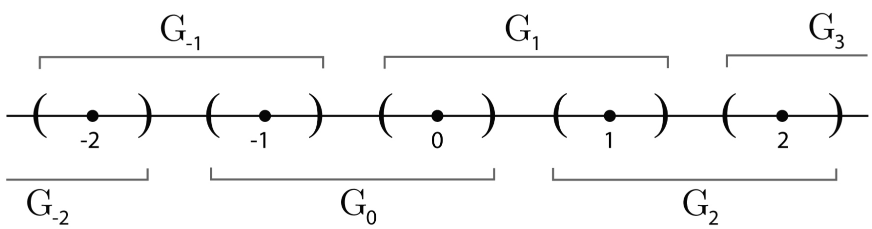
Heine-Borel Theorem. If a and b are real numbers with $a < b$ and $\mathcal{O}$ is a collection of open intervals such that $[a, b] \subset \bigcup\{O : O \in \mathcal{O}\}$, then there is a finite subset $\{O_1, O_2, \ldots O_N\}$ of $\mathcal{O}$ such that $[a, b] \in \bigcup_{n=1}^{N} O_n$.

In the Euclidean space $\mathbb{R}^n$, it can be shown that compact subspaces, in the sense of the sequential definition (**Definition 2.4.3**), *coincide* with compact subspaces in the sense of the above covering definition (**Definition 3.8.2**).

Example 3.8.1. In $(X, \mathfrak{T})$, if X is finite, it is a compact space.

Example 3.8.2. The set of real numbers $\mathbb{R}$ is not compact. In fact, you cannot find a finite subcover from the countable open covers $\{G_n : n \in \mathbb{Z}\}$, where

$$G_n = \left(n - \frac{4}{3}, n + \frac{1}{3}\right).$$



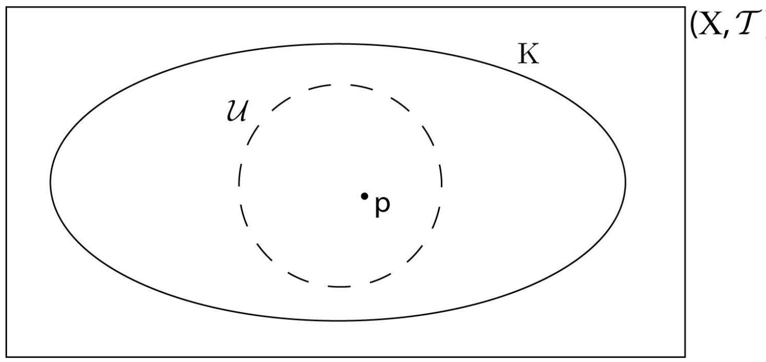
For each finite collection $\{G_{n_i} : i = 1, 2, \ldots, k\}$, there is a largest element n_j of $\{n_1, n_2, \ldots, n_k\}$. If $x > n_j + 1$, then $x \notin \bigcup_{i=1}^{k} G_{n_i}$. Hence, $\mathbb{R}$ is not a compact space.

Example 3.8.3. The closed interval [0, 1] with the Euclidean topology is compact. Let $\mathcal{U}$ be an open cover of $[0, 1]$. For each $x \in [0, 1]$, let $U_x \in \mathcal{U}$ such that $x \in U_x$. Then, there is an open interval I_x such that $x \in I_x \subset U_x$. By the Heine-Borel theorem, there is a finite subcollection $\{I_{x_1}, I_{x_2}, \ldots, I_{x_n}\}$ of $\{I_x : x \in [0, 1]\}$ such that $[0, 1] \subset \bigcup_{i=1}^{n} I_{x_i}$. The collection $\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}$ of members of $\mathcal{U}$ that corresponds to $I_{x_1}, I_{x_2}, \ldots, I_{x_n}$ is a finite subcollection of $\mathcal{U}$ that covers $[0, 1]$. Therefore, $[0, 1]$ is compact.

Example 3.8.4. $(\mathbb{R}, \mathcal{C})$, the cofinite topology on $\mathbb{R}$, is a compact space. If we have a collection of open sets covering the real line $\mathbb{R}$, any one of the sets will cover all but a finite number of points of $\mathbb{R}$, say n points of $\mathbb{R}$. We can choose n other sets of the collection, one for each point, and together, these $n + 1$ open sets will constitute a finite subcover of $\mathbb{R}$.

3.9 Local Compactness

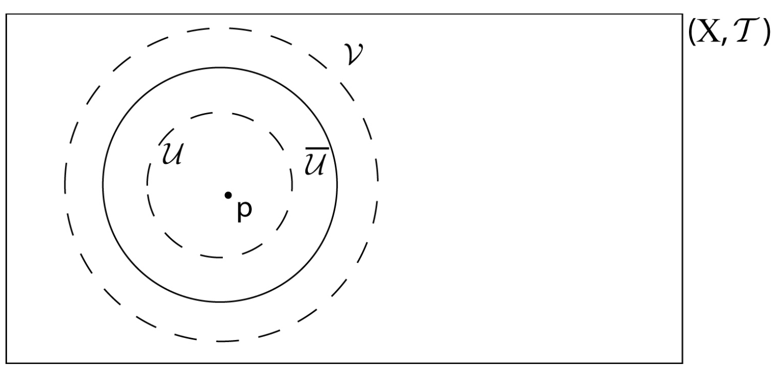
Definition 3.9.1. A topological space $(X, \mathfrak{T})$ is called locally compact at a point p if there is an open set U and there is a compact subspace K of X such that $p \in U$ and $U \subset K$.



Definition 3.9.2. A topological space $(X, \mathfrak{T})$ is locally compact if it is locally compact at each of its points.

We can also determine local compactness from the following theorem.

Theorem 3.9.1. Assuming $(X, \mathfrak{T})$ is Hausdorff, a space $(X, \mathfrak{T})$ is locally compact if and only if for each point $p \in X$ and for each open neighborhood V of p there is a neighborhood U of p such that $\overline{U}$ is compact and $\overline{U} \subset V$.



Example 3.9.1. $(\mathbb{R}, \mathcal{E})$ is locally compact.

Chapter 4

Generalized Volterra Theorem

To prove the generalized Volterra theorem, I need to consider Baire spaces. I will define what Baire spaces are and state equivalent conditions for Baireness. Then, I will show that complete metric spaces are Baire spaces using the Baire Category theorem. After that, I will show that every locally compact Hausdorff space is a Baire space by using another version of the Baire Category theorem.

4.1 Baire Spaces

Definition 4.1.1. A Baire space is a topological space such that every nonempty open subset is of second category.

In the next theorem, several equivalent conditions for being a Baire space are explained.

Theorem 4.1.1. ([19], p. 11) The following are equivalent for a space X:

1. X is a Baire space.
2. The intersection of any (monotone decreasing) sequence of dense open sets is dense in X.
3. The complement of any set of first category in X is dense in X.
4. Every countable union of closed sets with no interior points in X has no interior point in X.

Proof. (1) implies (2): Suppose that $\{D_i\}$ is a sequence of dense open sets and that U is an open subset of X that does not intersect $\bigcap_{i=1}^{\infty} D_i$. That is, $\bigcap_{i=1}^{\infty} D_i$ is not dense in X.

Then,

$$U = U - \bigcap_{i=1}^{\infty} D_i = \bigcup_{i=1}^{\infty} (U - D_i)$$

and $U - D_i$ is nowhere dense in X.

Therefore, U is of first category.

(2) implies (3): For each i, let N_i be a closed, nowhere dense subset of X and let U be a nonempty open set that does not intersect

$$X - \bigcup_{i=1}^{\infty} N_i = \bigcap_{i=1}^{\infty} (X - N_i).$$

Thus, the complement of the set $\bigcup_{i=1}^{\infty} N_i$, which is of first category, is not dense in X.

For each n, define

$$V_n = \bigcap_{i=1}^{n} (X - N_i).$$

Therefore, $\{V_n\}$ is a sequence of dense open sets whose intersection is not dense in X.

(3) implies (4): Let A be the countable union of closed sets with no interior points in X.

Suppose A contains an interior point in X, then $X - A$ would not be dense in X. This would result in a contradiction.

(4) implies (1): Suppose that for each i, N_i is nowhere dense in X and $\bigcup N_i$ is open, and hence, not a Baire space.

Then, each $\overline{N_i}$ has no interior points, but $\bigcup_{i=1}^{\infty} \overline{N_i}$ does have an interior point. ■

We will need the following known proposition, which can be found in [5].

Proposition 4.1.1. ([5], p. 432) In a T_1 dense-in-itself Baire space, the intersection of a dense G_δ-set A and a dense, open set B is uncountable.

Now, we will prove the following theorem, which deals with Baire and resolvable spaces. It is a part of topological folklore.

Theorem 4.1.2. Let X be a T_1, second countable, dense-in-itself Baire space. Then X is resolvable.

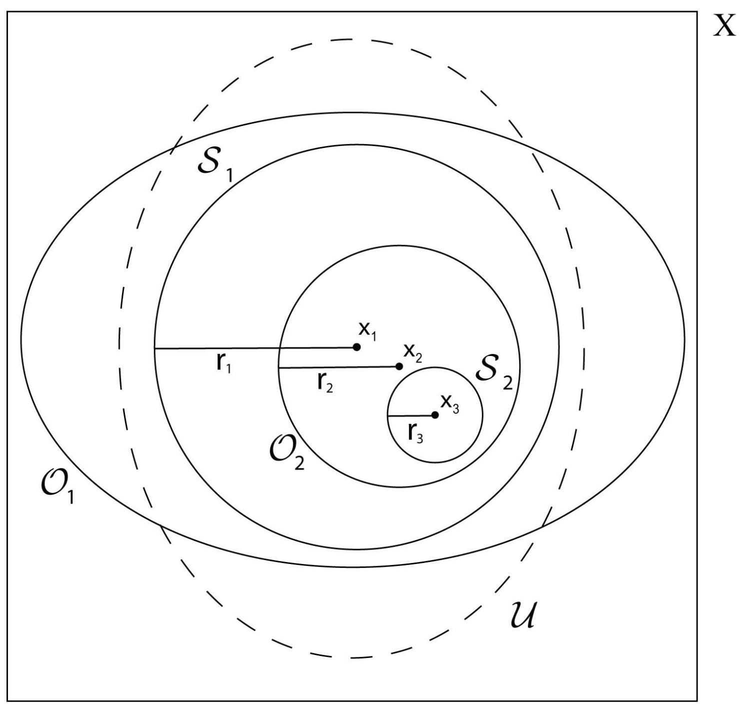
Proof. Let $B_1, B_2, \ldots B_n$ be a countable base for X. Since X is a second countable, dense-in-itself Baire space, each B_i, $i = 1, 2, \ldots$, is uncountable by **Proposition 4.1.1**

Now, we are only taking one point from every B_i. From each B_i, pick a point b_i. Let $D = \bigcup_{i=1}^{\infty} \{b_i\}$. Clearly D is dense. Furthermore, $X \setminus D$ is also dense. ■

4.2 Baire Category Theorem for Complete Metric Spaces

We showed that X being a Baire space is equivalent to the intersection of any sequence of dense open sets being dense in X. In the Baire Category theorem, this equivalent statement is used to show that every complete metric space is a Baire space.

Theorem 4.2.1. Let X be a complete metric space and $\{O_n\}$ be a countable collection of dense open subsets of X. Then, $\bigcap_{n=1}^{\infty} O_n$ is dense.



Proof. Given an open set U, let x_1 be a point of $O_1 \cap U$ and S_1 an open ball of radius r_1 centered at x_1 and contained in $O_1 \cap U$.

Since O_2 is dense, there exists a point $x_2 \in O_2 \cap S_1$.

Since O_2 is open, there exists an open ball $S_2(x_2, r_2)$ centered at x_2 and contained in O_2, and we may take the radius r_2 of S_2 to be smaller than $\frac{1}{2}r$ and smaller than $r_1 - d(x_1, x_2)$.

Then, $\overline{S_2} \subset S_1$.

Proceeding inductively, we obtain a sequence $\{S_n\}$ of open balls such that $\overline{S_n} \subset S_{n-1}$, and $S_n \subset O_n$, whose radii $\{r_n\}$ tend to zero.

Let $\{x_n\}$ be the sequence of centers of these open balls. Then, for $n, m \geq N$ we have $x_n \in S_N$ and $x_m \in S_N$. Therefore, $d(x_n, x_m) \leq 2r_N$, and $\{x_n\}$ is a Cauchy sequence since $r_n \to 0$.

By the completeness of X, there exists a point x such that $x_n \to x$.

Now, since $x_n \in S_{N+1}$ for $n > N$, we have

$$x \in \overline{S_{N+1}} \subset S_N \subset O_N.$$

Thus, $x \in \bigcap_{n=1}^{\infty} O_n$ and $x \in U$.

Therefore, Since U was an arbitrary open set, $\bigcap O_n$ is dense in X. ■

Thus, we have shown that every complete metric space is a Baire space. Consider the following examples.

Example 4.2.1. $(\mathbb{R}, \mathcal{E})$ is a Baire space since we showed in **Example 2.4.4** that $(\mathbb{R}, \mathcal{E})$ is a complete metric space.

Example 4.2.2. $(\mathbb{Q}, \mathcal{E})$ is a not Baire space since we showed in **Example 2.4.3** that $(\mathbb{Q}, \mathcal{E})$ is not a complete metric space.

Notice, there are Baire space that are not metric spaces. However,

$$\bigcap_{i=1}^{\infty} (\mathbb{Q} \setminus \{q_i\}) \neq \varnothing.$$

So, $\mathbb{Q}$ is not Baire since it does not satisfy the second condition of **Theorem 4.1.1**.

4.3 Baire Category Theorem for Locally Compact Hausdorff Spaces

Along with complete metric spaces, locally compact Hausdorff spaces are Baire. Yet, for these spaces, many of the results of the Baire Category theorem follow directly from local compactness.

Theorem 4.3.1. Let X be a locally compact Hausdorff space and $\{D_n\}$ a countable collection of dense open subsets of X. Then, $\bigcap\{D_n\}$ is dense.

Proof. Let $D_1, D_2, \dots$ be open dense sets. We must show that

$$U \cap \bigcap_{i=1}^{\infty} D_i \neq \varnothing$$

for each open $U \subset X$.

Since $U \cap D_1 \neq \varnothing$ (because D_1 is dense), by the definition of local compactness, there is a nonempty compact open B_1 such that $\overline{B_1} \subset U \cap D_1$.

With B_1 and D_2, we find, for the same reason, that there is a nonempty compact open B_2 such that $\overline{B_2} \subset B_1 \cap D_2$.

Proceeding by induction, we obtain a sequence $\{B_n\}$ of nonempty open sets such that

$$\overline{B_n} \subset B_{n-1} \cap D_n \quad \text{for all } n.$$

Then, the sets $\overline{B_n}$ are closed in the compact $\overline{B_1}$ and we have the finite intersection property. So,

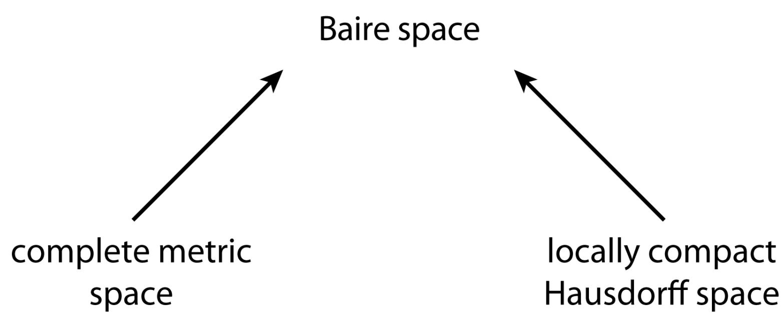
$$\bigcap_{n=1}^{\infty} \overline{B_n} \neq \varnothing.$$

Since

$$\bigcap_{n=1}^{\infty} \overline{B_n} \subset U \cap \bigcap_{n=1}^{\infty} D_n$$

because $\overline{B_1} \subset U \cap D_1$ and $\overline{B_n} \subset D_n$ for all n, the proof is complete. ■

The following diagram illustrates the results of the Baire Category theorems.



The condition that a locally compact space X be Hausdorff is essential in determining that X is a Baire space. Consider the following example where X is not Hausdorff.

Example 4.3.1. Consider the cofinite topology on the rationals $\mathbb{Q}$. Notice that this topology is both compact and T_1 (see **Examples 3.2.1** and **3.8.4**). Now, enumerate $\mathbb{Q}$ as

$$\mathbb{Q} = \{q_1, q_2, \dots, \}.$$

Then, observe that $\mathbb{Q} \setminus \{q_i\}$ is open and dense. Also,

$$\bigcap_{q_1 \in \mathbb{Q}} \mathbb{Q} \setminus \{q_i\} = \varnothing.$$

Thus, $(\mathbb{Q}, \mathcal{C})$ is not a Baire space.

4.4 The Generalized Volterra Theorem

To prove a stronger version of Volterra's result using Baire spaces, we need the following lemma.

Lemma 4.4.1. ([32], p. 11) Let X be a Baire space. If $G_1, G_2, \dots$ are dense G_δ-sets, then so is the set $G_1 \cap G_2 \cap \dots$.

Proof. Each of the sets $X \setminus G_n$ is a boundary, F_σ-set. Hence, they are of first category by **Lemma 3.7.1**.

The union

$$(X \setminus G_1) \cup (X \setminus G_2) \cup (X \setminus G_3) \cup \dots$$

is also of first category by **Lemma 3.6.1**.

Since X is a Baire space, the complement $G_1 \cap G_2 \cap \dots$ is dense. ∎

The assumption that $G_1, G_2, \ldots$ are dense G_δ-sets is necessary. To illustrate, consider $G_1 = \mathbb{Q}$ and $G_2 = \mathbb{R} \setminus \mathbb{Q}$. G_2 is a G_δ-set, but G_1 is an F_σ-set, not a G_δ-set. Notice that their intersection is empty and not a dense G_δ-set.

Now we can prove the generalization of Volterra's result with Baire spaces.

Theorem 4.4.1. ([37], p. 23) Let X be a nonempty Baire space and let $f : X \to \mathbb{R}$ be a function for which $C(f)$ and $D(f)$ are dense. Then there is no function $g : X \to \mathbb{R}$ such that

$$C(f) = D(g) \qquad \text{and} \qquad D(f) = C(g).$$

Proof. Suppose that $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are two functions and that $C(f)$ and $D(f)$ are dense.

Then:

(1) $C(f) \cup D(f) = X$ and $C(f) \cap D(f) = \varnothing$,

(2) $C(g) \cup D(g) = X$ and $C(g) \cap D(g) = \varnothing$.

If a function g were to exist and satisfy

$$C(f) = D(g) \qquad \text{and} \qquad D(f) = C(g),$$

then in view of (1) and (2), this implies:

$$C(f) \cap C(g) = \varnothing. \tag{3}$$

We shall prove that (3) cannot happen. In fact, since $C(f)$ and $C(g)$ are both sets of continuity points of f and g, we know that they are dense by our assumption and G_δ by **Corollary 1.3.1**.

Furthermore, since X is Baire, $C(f)$ and $C(g)$ have to intersect on a dense G_δ-subset by **Lemma 4.4.1**.

Now, for this dense G_δ-subset

$$C(f) \cap C(g) \neq \varnothing.$$

This contradicts condition (3). ■

The assumption that X is a Baire space in **Theorem 4.4.1** cannot be dropped. To illustrate, consider the following example.

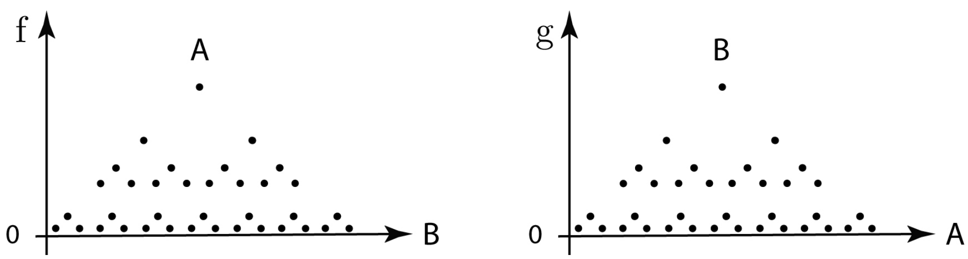
Example 4.4.1. ([15], p. 212) Let $\mathbb{Q}$ denote the set of rational numbers and let

$$A = \left\{ \frac{p}{q} : q \text{ is odd} \right\} \qquad \text{and} \qquad B = \left\{ \frac{p}{q} : q \text{ is even} \right\}$$

where $\gcd(p, q) = 1$.

Notice that A and B are disjoint dense G_δ-subsets of $\mathbb{Q}$. Thus, $\mathbb{Q}$ is not a Baire space (see **Lemma 4.4.1**). Now, enumerate the elements of A and B respectively, i.e., $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$. Then, define the functions $f : \mathbb{Q} \to \mathbb{R}$ and $g : \mathbb{Q} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} \frac{1}{i} & \text{if } x = a_i, a_i \in A \\ 0 & \text{if } x \in B, \end{cases} \qquad g(x) = \begin{cases} \frac{1}{i} & \text{if } x = b_i, b_i \in B \\ 0 & \text{if } x \in A. \end{cases}$$



Clearly, $D(f) = C(g)$ and $C(f) = D(g)$.

Chapter 5

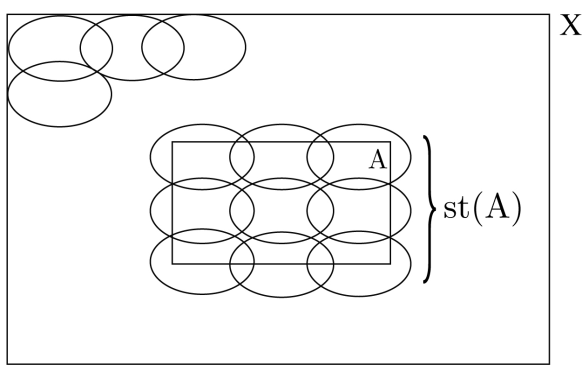
Volterra and Weakly Volterra Spaces

The idea and generalizations of Volterra's theorem led to the idea of Volterra and weakly Volterra spaces. The class of Volterra spaces was first introduced in [15] by Z. Piotrowski and D. Gauld. First, I will look at developable spaces as range spaces of functions. Finally, I will explore some of the behaviors of these spaces.

5.1 Developable Spaces

In [15], Piotrowski and Gauld generalized the classical Volterra theorem to certain "nice" topological spaces. Such an example of these types of spaces are developable spaces. They defined the notion of generalized oscillation for these spaces. The covers of developable spaces will constitute "the smallness" of the images under a function.

Definition 5.1.1. If $A \subset X$ and $\mathfrak{U}$ is a collection of subsets of X, then $st(A, \mathfrak{U}) = \cup\{U \in \mathfrak{U} : U \cap A \neq \varnothing\}$. $st(A, \mathfrak{U})$ is called the star of A.



Definition 5.1.2. A sequence $\{\mathfrak{G}_n\}$ of open covers of X is called a development of X if for every $x \in X$, the set $\{st(x, \mathfrak{G}_n) : n \in \mathbb{N}\}$ is a base at x.

A space which has a development is called a developable space.

Remark 5.1.1. All metric spaces have developments.

In fact, let $f : X \to Y$ be a function and $\mathfrak{C}$ be an open cover for Y. Then, define

$$\Omega(f, \mathfrak{C}) = \{x \in X : \exists \text{ open } U \text{ containing } x \text{ and } \exists V \in \mathfrak{C} \text{ with } f(U) \subset V\}.$$

The sets $\Omega(f, \mathfrak{C})$ are clearly open, as they are the union of open sets. Furthermore, notice that $C(f) \subset \Omega(f, \mathfrak{C})$. Now, let $\{\mathfrak{C}_n\}$ be a sequence of covers of Y. Define $\Omega(f, \{\mathfrak{C}_n\}) = \bigcap_{n=1}^{\infty} \Omega(f, \mathfrak{C}_n)$. Clearly, $\Omega(f, \{\mathfrak{C}_n\})$ is a G_δ-set.

This leads us to the following proposition.

Proposition 5.1.1. ([15], p. 210) Let $f : X \to Y$ be a function, where Y is a developable space with a development $\{\mathfrak{G}_n\}$. Then $C(f) = \bigcap_{n=1}^{\infty} \Omega(f, \mathfrak{G}_n)$.

Definition 5.1.3. A regular developable space is called a Moore space.

In [15], the above generalized oscillation (see **Remark 5.1.1** and **Proposition 5.1.1**) has been extended even further to weakly developable spaces. Weakly developable spaces were introduced by J. Calbrix and B. Alleche in [3]. It was proved in [2] that a completely regular space is weakly developable if and only if it is a p-space with a G_δ-diagonal. L. Holá and Z. Piotrowski showed in [21] that to have a G_δ-diagonal is not sufficient to guarantee that the set $C(f)$ of continuity points of every function into such a space is a G_δ set.

Example 5.1.1. ([21], p. 153, example 3.6) Let Y be the Michael line (the real line with the isolated irrationals and the rationals having their usual neighborhoods) and $X = \mathbb{R}$. Let $f : X \to Y$ be the identity mapping. Then, $C(f) = \mathbb{Q}$, the set of rational numbers, i.e., $C(f)$ is not a G_δ-set. The Michael line is a submetrizable non-developable space ([17], p. 428-430).

5.2 Introduction to Volterra and Weakly Volterra Spaces

Definition 5.2.1. ([14], p. 169) A topological space X is called Volterra if for each pair $f, g : X \to \mathbb{R}$ of functions such that $C(f)$ and $C(g)$ are both dense in X, the set $C(f) \cap C(g)$ is dense in X. If the latter condition is changed to

$$C(f) \cap C(g) \neq \varnothing,$$

then such a space is called weakly Volterra.

Proposition 5.2.1. ([14], p. 170) For any topological space X, the following are equivalent:

1. X is Volterra;
2. for each pair A, B of dense G_δ-subsets of X, the set $A \cap B$ is dense;
3. for each pair $C, D \subset X$ of boundary F_σ-subsets of X, the set $C \cup D$ is boundary;
4. for each pair Y, Z of developable spaces and each pair $f : X \to Y$ and $g : X \to Z$ of functions for which $C(f)$ and $C(g)$ are dense in X, the set $C(f) \cap C(g)$ is dense.

Recall that a topological space is Baire if the intersection of countably many dense open sets is a dense set. Thus, every Baire space is Volterra.

Example 5.2.1. Any Baire space, i.e., any complete metric space or locally compact Hausdorff space, is a Volterra space. For instance,

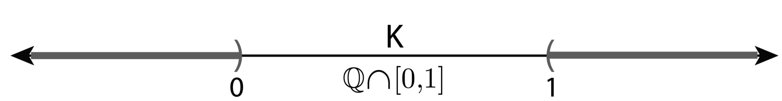
$$\mathbb{R}, \mathbb{R}^n \text{ and } [0, 1]$$

are all Volterra spaces.

Proposition 5.2.2. ([14], p. 170) For any nonempty topological space X, the following are equivalent:

1. X is weakly Volterra;
2. for each pair A, B of dense G_δ-subsets of X, the set $A \cap B \neq \varnothing$;
3. for each pair $C, D \subset X$ of F_σ-subsets such that $C \cup D = X$, either $int(C)$ or $int(D)$ is nonempty;
4. for each pair Y, Z of developable spaces and each pair $f : X \to Y$ and $g : X \to Z$ of functions for which $C(f)$ and $C(g)$ are dense in X, the set $C(f) \cap C(g)$ is nonempty.

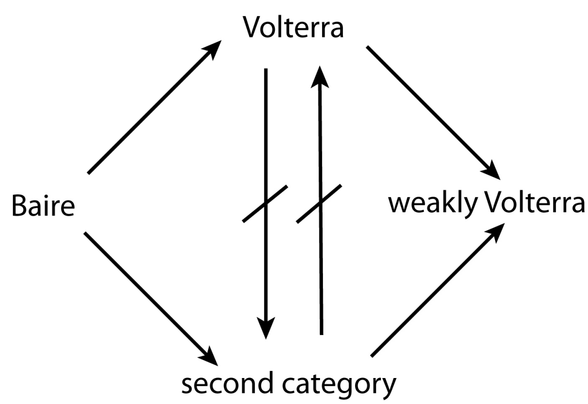
Example 5.2.2. Let $X = (-\infty, 0) \cup (\mathbb{Q} \cap [0, 1]) \cup (1, \infty)$.



Let f and g be any pointwise discontinuous functions defined on $(-\infty, 0) \cup (1, \infty)$. This is the Volterra subspace of X. Now, decompose K, the rationals in $[0, 1]$, into two sets A and B by using the method from **Example 4.4.1** for f and g, respectively.

Over K, the intersection $C(f)$ with $C(g)$ is empty. Notice that the intersection of $C(f)$ with $C(g)$ is dense outside of K. Thus, X is weakly Volterra, but not Volterra.

We have the following diagram relating Volterra, weakly Volterra, Baire and second category spaces.



It is important to realize that Volterra spaces are not always Baire. We will look into the class of spaces in which Volterra spaces are Baire in chapter 6.

5.3 Properties

I will present a number of examples and propositions which illustrate some of the behaviors of Volterra and weakly Volterra spaces. The following two examples show that the converse implications of the diagram in the previous section do not hold.

Example 5.3.1. ([14], p. 172, example 3.5) A space which is not of second category, hence not Baire, but is Volterra.

Let $X = [0, \infty)$ with a topology whose base consists of sets of the form:

$$\{[a, \infty) - F : a \in X \text{ and } F \text{ is a finite subset of } X\}.$$

X is not second category because it is the union of countably many closed, nowhere dense sets $\{[0, n) : n \in \mathbb{N}\}$. However, each dense G_δ-set is of the form $[a, \infty) - C$, where $a \in X$ and $C \subset X$ is a countable subset. Furthermore, the intersection of any pair of such sets is also dense. Hence, X is Volterra. Thus, Volterra $\not\Rightarrow$ Baire.

Example 5.3.2. ([14], p. 172, example 3.6) A space which is not Volterra, hence not Baire, but is of second category, hence weakly Volterra.

Let $X = \mathbb{R}^- \cup \mathbb{Q}^+$ with the topology inherited from the reals, where $\mathbb{R}^-$ denotes the non-positive reals and $\mathbb{Q}^+$ denotes the non-negative rationals.

X is not Volterra. To show this, let $\mathbb{Q}_o^+ = \left\{\frac{p}{q} : q \text{ is odd}\right\}$ and $\mathbb{Q}_e^+ = \left\{\frac{p}{q} : q \text{ is even,}\right\}$, where $\gcd(p, q) = 1$. Then, $\mathbb{R}^- \cup \mathbb{Q}_o^+$ and $\mathbb{R}^- \cup \mathbb{Q}_e^+$ are dense G_δ-subsets of X whose intersection is not dense. Therefore, X is not Volterra. On the other hand, X is weakly Volterra because if $f, g : X \to \mathbb{R}$ exist such that $C(f)$ and $C(g)$ are dense in X, then $C(f|_{\mathbb{R}^-})$ and $C(g|_{\mathbb{R}^-})$ are dense in the weakly Volterra space $\mathbb{R}^-$. Hence their intersection is nonempty. Thus, weakly Volterra $\not\Rightarrow$ Volterra.

We will now give examples of Volterra and weakly Volterra spaces.

Example 5.3.3. ([14], p. 171, example 3.1) The discrete and indiscrete spaces are Baire spaces. Hence, they are both Volterra.

Example 5.3.4. ([14], p. 171, example 3.2) A space which is not weakly Volterra.

Let $\mathbb{Q}$ denote the set of rational numbers and let

$$\mathbb{Q}_o = \left\{\frac{p}{q} : q \text{ is odd}\right\} \qquad \text{and} \qquad \mathbb{Q}_e = \left\{\frac{p}{q} : q \text{ is even,}\right\}$$

where $\gcd(p, q) = 1$.

Notice that $\mathbb{Q}_o$ and $\mathbb{Q}_e$ are disjoint dense G_δ-subsets of $\mathbb{Q}$. Thus, the space $\mathbb{Q}$ is not weakly Volterra.

These next three examples illustrate the dynamics of Volterra and weakly Volterra spaces.

Example 5.3.5. ([14], p. 171, example 3.3) A space which is not weakly Volterra but has disjoint dense G_δ-subspaces which are Volterra.

Let $X = \mathbb{N}$ and let O denote the odd positive integers and E the even positive integers.

For each pair (m, n) of positive integers, let

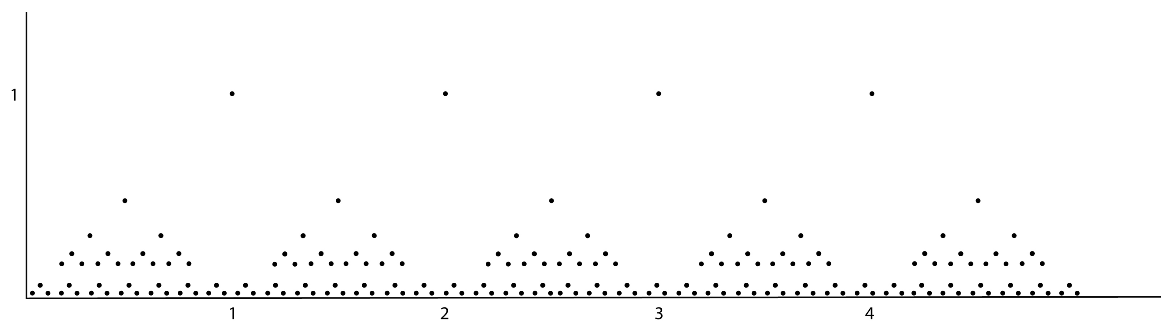
$$U_{m,n} = \{x \in X : \text{either } x \geq 2m - 1 \text{ and } x \in O \text{ or } x \geq 2n \text{ and } x \in E\}.$$

Then $\{\varnothing\} \cup \{U_{m,n} : m, n \in \mathbb{N}\}$ is a topology on X. Furthermore, O and E are dense G_δ-subsets. Thus, as subspaces, they are Volterra. However, since $O \cap E = \varnothing$, X is not weakly Volterra.

Example 5.3.6. ([14], p. 173, example 3.8) A Volterra space whose subspace is not weakly Volterra.

Consider the countable cloud space. Assume $p, q \in \mathbb{Z}$, where $\gcd(p, q) = 1$ and $q \neq 0$. The countable cloud space X is the subspace of the plane given by $X = Y \cup Z$, where

$$Y = \{(x, 0) : x \in \mathbb{Q}\} \qquad \text{and} \qquad Z = \left\{ \left(\frac{p}{q}, \frac{1}{q} \right) : \frac{p}{q} \in \mathbb{Q} \right\}.$$



Since it contains a dense G_δ set of isolated points, the countable cloud space is a Baire space, hence it is Volterra. Notice that it contains a closed subspace which is not weakly Volterra, that is, the rationals in the x-axis.

Example 5.3.7. Using the same notation from **Example 5.3.6**, consider the projection from X onto Y. Note that X is Volterra since it is a metric Baire space. However, Y is not Volterra, hence not Baire. Furthermore, this projection is continuous, closed and perfect. See [9] for any undefined terms.

Finally, I will look at the preservation of Volterra spaces under certain types of functions. More results on these preservations can be found in [14].

I will begin by stating what it means for a function to be open.

Definition 5.3.1. A function $f : X \to Y$ is open if for any open set $U \in X$, the image $f(U)$ is open in Y.

Now, let X be a space which is not weakly Volterra. As noted in **Example 5.3.3**, when X is re-topologized with either the discrete or indiscrete topology, X becomes Volterra. Thus, the identity function from X with the discrete (respectively the indiscrete) topology to X with the original topology is a continuous (relatively open) function from a Volterra space to a space which is not weakly Volterra. Thus, it follows that the image of a Volterra space under either a continuous function or an open function need not be Volterra.

In **Example 5.3.7**, we showed that continuous, closed and perfect functions do not preserve Volterra spaces. In order to explain which functions preserve Volterra spaces, we must consider the following definition.

Definition 5.3.2. ([14], p. 177) Let $f : X \to Y$ be a function from X onto Y. Then, f is feebly open if for each nonempty, open set $U \in X$, $int(f(U)) \neq \varnothing$.

Theorem 5.3.1. ([14], p. 177) Suppose that $f : X \to Y$ is continuous and feebly open. If X is Volterra and f is surjective, then Y is Volterra.

Proof. Let A and B be two dense G_δ-subsets of Y. Then, $f^{-1}(A)$ and $f^{-1}(B)$ are G_δ-subsets of X.

Furthermore, if $U \subset X$ is nonempty and open, then so is $int(f(U)) \subset Y$, so that $int(f(U)) \cap A \neq \varnothing$. Hence,

$$U \cap f^{-1}(A) \neq \varnothing.$$

Thus, $f^{-1}(A)$ is dense. Similarly, $f^{-1}(B)$ is dense. Now, since X is Volterra, $f^{-1}(A) \cap f^{-1}(B)$ is dense in X, which is equivalent to $f^{-1}(A \cap B)$ being dense in X. It follows that $f(f^{-1}(A \cap B)) = A \cap B$ is dense in Y since continuous surjective functions preserve dense sets.

Therefore, Y is Volterra. ■

We have the following table illustrating our results.

Function Image Table				
	Continuous	Open	Continuous, Open	Closed, Perfect
Weakly Volterra	No **Ex 5.3.3**	No **Ex 5.3.3**	Yes **Thm 5.3.1**	No **Ex 5.3.7**
Volterra	No **Ex 5.3.3**	No **Ex 5.3.3**	Yes **Thm 5.3.1**	No **Ex 5.3.7**

Chapter 6

Current Research Results

In this chapter, I will explore recent results of Volterra spaces from various research articles. Then, I will state unanswered questions pertaining to properties of Volterra spaces.

6.1 Volterra Spaces: Current State of Investigations

When Volterra spaces were introduced in [15], it was easily shown that every Baire space is a Volterra space. However, the converse does not necessarily hold. I will now exhibit classes of spaces X such that X is Baire if and only if it is Volterra.

In fact, the first natural class would be the class of all metric spaces. This question was first asked by Piotrowski. In 2000, G. Gruenhage and D. Lutzer positively answered Piotrowski's problem in [18], and they provided a larger class of spaces in which Volterra spaces are Baire.

The following theorem gives Gruenhage and Lutzer's results. Part b. of this theorem includes all metric spaces, which answers Piotrowski's problem.

Theorem 6.1.1. ([18], p. 3118) A Volterra space X is Baire if X belongs to any one of the following classes:

a. X has a dense subspace Y that is a strongly collectionwise Hausdorff, sequential, and has a relatively σ-closed discrete dense subset;

b. X has a dense metrizable subspace;

c. X is a Lasnev space, i.e., a closed continuous image of a metric space;

d. X is a metacompact sequential space that has a σ-closed discrete dense set;

e. X is a metacompact Moore space or, more generally, a metacompact semistratifiable sequential space;

f. X is separable and sequential.

In addition to proving **Theorem 6.1.1**, Gruenhage and Lutzer provided the following examples:

Example 6.1.1. ([18], p. 3119, example 3.1) There is a countable regular space that is Volterra but not Baire.

Example 6.1.2. ([18], p. 3119, example 3.2) There is a Lindelöf, hereditarily paracompact, linearly ordered topological space that is Volterra but not Baire.

Example 6.1.3. ([18], p. 3119, example 3.3) There is a first countable, completely regular, paracompact space that is a Volterra space and is not a Baire space.

Since every metric space is a Moore space, Gruenhage and Lutzer asked if it was true that any Volterra Moore space must be a Baire space.

Definition 6.1.1. A regular space is stratifiable if one can assign a sequence of open sets $\{G(n,H) : n \in \mathbb{N}\}$ to each closed set $H \subset X$ such that

$$H = \bigcap_{n\in\mathbb{N}} G(n,H) = \bigcap_{n\in\mathbb{N}} \overline{G(n,H)},$$

and $H \subset K$ implies that $G(n,H) \subset G(n,K)$.

In 2007, Cao and Junnila proved an even stronger result in [7] using stratifiable spaces. Every Moore space is a stratifiable space (see [17], p. 459), and using this result, Cao and Junnila proved that if X is a stratifiable space, then X is Volterra if (and only if) X is Baire. In addition, they constructed a Hausdorff topological group that is Volterra but not Baire.

There is also a connection between the Volterra theorem and the Banach Category theorem (see Appendix B). In [8], J. Cao and S. Greenwood gave a strengthened version of the Banach Category theorem, namely:

Theorem 6.1.2. ([8], p. 260) In any topological space $(X, \mathfrak{T})$, the union of any family of open non-weakly Volterra subspaces is not weakly Volterra.

Recall that:

Definition 6.1.2. A topological space X is a P-space if every G_δ-set is open.

Definition 6.1.3. A topological space X is an almost P-space if every nonempty G_δ-set has a nonempty interior.

Ph.D. student S. Spadaro, who is currently working on his Ph.D. thesis under G. Gruenhage at Auburn University, proved that every almost P-space is Volterra.

6.2 Outline of the Proofs of Gruenhage and Lutzer, and Cao and Junnila[1]

The cited references within the outlined material can be found within the respected articles and will not be formally listed in my reference page.

Outline of Gruenhage and Lutzer's Method in [18]

Lemma 6.2.1. (E.G. Pytkeev, Trudy M.I.S (1983)) Every dense-in-itself subspace of a sequential space is resolvable.

Lemma 6.2.2. Let M be a dense subset of a regular T_1 space Y, such that every point of M is G_δ in Y. Suppose that $\{p\} \cup M$ is homeomorphic to a subspace of a T_3 sequential space S (p not necessarily belongs to Y), and $p \in cl_S(M)$. Then, there is a countable subset $C(p, M)$ of M with $p \in cl_S(C(p, m))$ and such that $C(p, M)$ is G_δ in Y.

Sketch of the Proof. For a subset A of S let $seq(A)$ be the set of all limits in the space S of convergent sequences $\{a_n : n \in \omega\} \subset A$. Then, define $A^0 = A$, $A^{\alpha+1} = seq(A^\alpha)$ and $A^\beta = \bigcup\{A^\alpha : \alpha < \beta\}$ if β is a limit ordinal.

Recall that S being sequential means $cl_S(A) = \bigcup\{A^\alpha : \alpha < \omega_1\}$. Let α be the least such that $p \in M^\alpha$. Call α the "sequential order of p w.r.t M" and note that α is a successor.

If $\alpha = 1$, then there is a sequence $J = \{m_k : k \in \omega\}$ of points in M converging to p. Since J is relatively discrete in M, it follows that since Y is T_3 there is a disjoint collection and $C(p, M) - J$.

In the rest of the proof, the authors show $C(p, M)$ is G_δ in Y. ■

Pytkeev's result is needed in view of the following property inherited by subspaces:

Any dense-in-itself subspace of a sequential space is resolvable.

If one replaces "sequential" with "k-space" in the above result, it is false since any completely regular space is a subspace of a compact space, hence of a k-space.

Lemma 6.2.3. Suppose $\mathfrak{U}$ is a point-finite collection of open subsets of a space X and that for each $U \in \mathfrak{U}$ we have a G_δ-subset $G(U) \subset U$. Then $S = \bigcup\{G(U) : U \in \mathfrak{U}\}$ is a G_δ-subset of X.

Lemma 6.2.4. Suppose X is regular, points are G_δ, and X has a dense subspace $D = \bigcup\{D_n : n \geq 1\}$ satisfying:

(a) D is homeomorphic to a subspace of a T_2 sequential space, and

(b) for each $n \geq 1$, there is a collection $\{V(d,n) : d \in D_n\}$ of open subsets of X that is point-finite in X and has $\{d\} = V(d,n) \cap D_n$.

If X is of the first category in itself, then D contains a subspace E that is dense in X and is a G_δ-subset of X.

Theorem 6.2.1. Suppose X is regular and has a dense subspace $D = \bigcup\{D_n : n \geq 1\}$ satisfying:

(a) D is homeomorphic to a subspace of a Hausdorff sequential space;

(b) for each $n \geq 1$ there is a collection $\{V(d,n) : d \in D_n\}$ of open subsets of X that is point-finite in X and has $V(d,n) \cap D_n = \{d\}$ for each $d \in D_n$.

Then X is a Baire space if and only if X is Volterra.

Proof. Any Baire space is Volterra, so it is enough to prove the converse. Suppose X is Volterra and yet there is a sequence $\{G_n : n \geq 1\}$ of dense open subsets of X such that $G_{n+1} \subset G_n$ and $\bigcap\{G_n : n \geq 1\}$ is not dense. Then, there is a nonempty open subset $Y \subset X$ such that $Y \cap \bigcap\{G_n : n \geq 1\} = \varnothing$.

Observe that the set $D \cap Y$ is dense in Y and satisfies both (a) and (b) above. Replacing X by its subspace Y if necessary, we may assume that $\bigcap\{G_n : n \geq 1\} = \varnothing$. It follows that X has no isolated points. Hence, neither does the dense subspace D.

Apply **Lemma 6.2.1** to D to find two disjoint, dense subspaces D_1, D_2 of D. Apply **Lemma 6.2.4** to each D_i to find a subspace $E_i \subset D_i$ that is dense in X and is a G_δ-subset of X. But then, we have two disjoint dense G_δ-subsets of X, and that is impossible because X is Volterra. ∎

Corollary 6.2.1. A Volterra space X is Baire if X belongs to any one of the following classes:

a. X has a dense subspace Y that is a strongly collectionwise Hausdorff, sequential, and has a relatively σ-closed discrete dense subset;

b. X has a dense metrizable subspace;

c. X is a Lasnev space, i.e., a closed continuous image of a metric space;

d. X is a metacompact sequential space that has a σ-closed discrete dense set;

e. X is a metacompact Moore space or, more generally, a metacompact semistratifiable sequential space;

f. X is separable and sequential.

Outline of Cao and Junnila's Method in [7]

Notation: M^d will denote the derived set of M. Recall that a subset M of X is called simultaneously separated in X if each point $x \in M$ has an open neighborhood U_x in X such that $\{U_x : x \in M\}$ is a pairwise disjoint family in X. Define $\lambda(M)$ by:

$$\lambda(M) = \bigcup \{A^d : A \subset M \text{ and } A \text{ is simultaneously separated in } X\}.$$

Lemma 6.2.5. (P.L. Sharma, S. Sharma (1988)) Let X be a dense-in-itself Hausdorff space. If $\lambda(X) = X$, then X is resolvable.

A space X is said to be monotonically normal (G. Gruenhage (1984)) if there exists a map

$$G : \{x \in G(x, U) : x \in U \in T(X)\} \to T(X)$$

such that $x \in G(x, U)$ and $G(x, U) \cap G(y, V) \neq \varnothing$ implies that either $y \in U$ or $x \in V$.

It is well-known that every stratifiable space is monotonically normal.

Lemma 6.2.6. (Dow-Tkachenko-Tkachuk-Wilson (2002)) Let X be a dense-in-itself, monotonically normal Hausdorff space. Then $\overline{M} = \lambda(M)$ for any $M \subset X$.

Corollary 6.2.2. Every dense-in-itself monotonically normal Hausdorff space is resolvable.

Lemma 6.2.7. (G. Gruenhage, D. Lutzer (2000)) Suppose $\mathfrak{U}$ is a point-finite collection of open sets in X and that each $U \in \mathfrak{U}$ contains a G_δ-set $G(U)$ of X. Then:

$$\mathcal{S} = \{G(U) : U \in \mathfrak{U}\}$$

is a G_δ-set in X.

Theorem 6.2.2. Every stratifiable Volterra space is Baire.

Proof. Suppose X is not Baire. We will show that X is not Volterra. Let G be a nonempty open set in X which is of first category. We are done if we can show that G is not weakly Volterra, since a space is Volterra if and only if all of its nonempty open subspaces are weakly Volterra.

Without loss of generality, let $G = X$. Then, let $\{G_n : n \in \mathbb{N}\}$ be a sequence of dense open subsets of X such that $\bigcap_{n\in\mathbb{N}} G_n = \varnothing$.

We may assume $G_{n+1} \subset G_n$ for all $n \in \mathbb{N}$. This implies that X is dense-in-itself and the family $\{G_n : n \in \mathbb{N}\}$ is point-finite in X.
Now, by the above corollary, it is resolvable. So, X has a resolution (D, E). In the sequel, we shall construct two dense G_δ-subsets D' and E' of X such that $D' \subset D$ and $E' \subset E$. By the stratifiability of X, there is a σ-discrete network $\mathcal{N}$.

Let $\mathcal{N}_0 = \{N \in \mathcal{N} : N \cap D \neq \varnothing\}$ and for each $N \in \mathcal{N}_0$ select a point $x_N \in N \cap D$. It can be checked that $\{x_N : N \in \mathcal{N}_0\}$ is a σ-discrete and dense subset of X. We may also assume that D itself is σ-discrete, since $D = \bigcup_{n\in\mathbb{N}} D_n$, where each D_n is discrete in X.

For each $n \in \mathbb{N}$ and $d \in D_n$ choose an open subset $V(d, n)$ of X such that:

$$D_n \cap V(d, n) = \{d\}.$$

Lemma 6.2.7 can be used to show that $K(d, n)$ is G_δ.

Now, let $H_n = \bigcup_{d\in D} K(d, n)$. Then, $D' = \bigcup_{n\in\mathbb{N}} H_n$ is a G_δ-set in X. Similarly, we can construct a dense G_δ-set $E' \subset E$.

So, concluding, since X contains two disjoint dense G_δ-sets D' and E', it is not a weakly Volterra space. ■

6.3 Problems I Wish I Could Solve

The following two questions were asked by Z. Piotrowski.

Problem 1. Let X be a Volterra space and let $Y = [0, 1]$. Must the cartesian product $X \times Y$ be Volterra?

Problem 2. If the answer to the previous problems is "yes", then let Y be a Volterra space having a countable base. Must $X \times Y$ be Volterra?

This next question was asked by J. Tartir as a follow-up to the previous two problems.

Problem 3. Let X be a Volterra space and let $Y = \{y : y = \frac{1}{n}, n = 1, 2, \ldots\} \cup \{0\}$. Must $X \times Y$ be Volterra?

The next two problems are due to Z. Piotrowski in view of **Theorem 5.3.1**. However, before considering the fourth problem we need the following definition.

Definition 6.3.1. (see [22]) Let $f : X \to Y$ be a function from X onto Y. Then, f is quasi-continuous if for every open subset $V \subset Y$, $f^{-1}(V) \subset \overline{int(f^{-1}(V))}$.

Problem 4. Are Volterra spaces preserved by quasi-continuous, feebly open surjections?

Just recently, the fourth problem was solved. See J. Dalbec and Z. Piotrowski "Mappings of Generalized Baire and Volterra Spaces" (in preparation).

For any topological space X, it is known that for any two dense, open subsets U and V of X, their intersection $U \cap V$ is dense. Furthermore, if X is a Volterra space, then by definition we know that given any two dense G_δ-subsets U and V of X, their intersection $U \cap V$ is dense. This led Z. Piotrowski to ask the following question.

Problem 5. What are the topological spaces X such that when given any two dense $G_{\delta\sigma}$-subsets U and V of X, their intersection $U \cap V$ is dense?

Problems 6 and 7 were asked by G. Gruenhage and D. Lutzer in [18].

Problem 6. Is it true that a space X must be a Baire space provided X is Volterra and has a dense subspace that is developable and metacompact?

Problem 7. Suppose X is a σ-space and first category in itself. Must every dense-in-itself subset D of X contain a dense subset E which is G_δ in X?

Since every almost P-space is a Volterra space, S. Spadaro asked the following question.

Problem 8. Does there exist an almost P-space with a dense non-Volterra subspace?

In view of the discussion proceeding **Example 5.1.1**, we have the following problem.

Problem 9. What is the natural class $\mathcal{S}$ of spaces having G_δ-diagonal, and containing all weakly developable spaces such that for any topological space X, any space Y from $\mathcal{S}$ and any function $f : X \to Y$, the set $C(f)$ of points of continuity of f is a G_δ-set?

Appendix A

Kuratowski-Zorn Lemma

We will state and prove the Kuratowski-Zorn Lemma, which is used in the proofs of many mathematical theorems. This theorem was first proved by K. Kuratowski in 1922 and its importance in applications was first demonstrated by M. Zorn in 1936. The following terminology is required for the Kuratowski-Zorn Lemma. All the theorems, definitions and lemmas were taken directly from [30].

Definition A.0.2. A relation $R \subset X \times X$ is called an ordering relation on X if it is reflexive, antisymmetric and transitive, i.e.,

1. for every $x \in X, xRx$,
2. for every $x, y \in X, (xRy \text{ and } yRx) \Rightarrow x = y$,
3. for every $x, y, z \in X, (xRy \text{ and } yRz) \Rightarrow xRz$.

Instead of xRy, we write $x \leq y$ which is read "x is contained in y" or "y contains x." We also say that R orders X, and the ordered pair (X, R) is called an *ordered set*.

Definition A.0.3. Let $\leq$ be an order relation on X. Define the relation $\prec$ as follows:

$$x \prec y \text{ if and only if } x \leq y \text{ and } x \neq y, \text{ for every } x, y \in X.$$

$x \prec y$ is read "x precedes y".

Definition A.0.4. An ordering relation is called a linear ordering if it satisfies the following connectivity condition, i.e., for every $x, y \in X$,

$$x \leq y \text{ or } y \leq x.$$

If $\leq$ is a linear ordering on X, we say that $\leq$ linearly orders X. The ordered pair $(X, \leq)$ is called a linearly ordered set (l.o.s.), or a chain.

Definition A.0.5. Let $A \subset X$, and let $(X, \leq)$ be an ordered set. An element $x_0 \in X$ is called an <u>upper bound of the set A</u> if for every $x \in A$, $x \leq x_0$.

Definition A.0.6. Let $(X, \leq)$ be an ordered set. An element $x_0 \in X$ is called <u>maximal</u> if it does not precede any element. That is, there is no $x \in X$ such that $x_0 \prec x$.

Definition A.0.7. A binary relation $\leq$ on X which establishes a linear ordering is called a <u>well-ordering</u> if for every nonempty set $A \subset X$, the linearly ordered set $(A, \leq|_A)$ has a first element.

Then, we also say that $\leq$ <u>well-orders</u> X, and the ordered pair $(X, \leq)$ is called a <u>well-ordered set</u>.

We will now state the theorem on transfinite induction. This process will be used to prove the Kuratowski-Zorn lemma.

Theorem on Transfinite Induction. Let $(X, \leq)$ be any well-ordered set. If $P(x)$ is a propositional function which ranges over X and satisfies the following conditions:

1. the first element of X satisfies the propositional function $P(x)$,
2. for every $y \in X$, if every $z \in X$, such that $z \leq y$ and $z \neq y$ satisfies the propositional function $P(x)$, then y also satisfies $P(x)$,

then every element of X satisfies the propositional function $P(x)$.

Definition A.0.8. The order types of well-ordered sets are called <u>ordinal numbers</u>.

The ordinal types include the order type of the empty set, denoted by 0. The concept of a well-ordered set and an ordinal number is due to G. Cantor (1883). The ordinal number of a well-ordered set of n elements is denoted by $\mathfrak{n}$.

Definition A.0.9. We say that the sets X and Y are <u>equipotent</u> (or have the same <u>cardinality</u>) if there is a bijective function from X onto Y.

With every element of a collection of equipotent sets, we may associate an object, called *the cardinal number* of X, denoted by $card(X)$. We shall also say that X *has the cardinality* $card(X)$.

In the case of a finite set, its cardinal number is the number of its elements. Cardinal numbers of finite sets begin with 0, which is the cardinal number of the empty set. The cardinal number of countable infinite sets is denoted by $\aleph_0$.

We can now state and prove the Kuratowski-Zorn lemma.

The Kuratowski-Zorn Lemma. Let $(X, \leq)$ be an ordered set. If every chain $A \subset X$ has an upper bound in X, then X has a maximal element. More precisely, for every $x_0 \in X$, there is a maximal element x such that $x_0 \leq x$.

Proof. Let $(X, \leq)$ be an ordered set that satisfies the assumptions made. Assume $card(X) = m$. Suppose, that for an element x_0 in X, there is no maximal element in X such that $x_0 \leq x$. It follows from this and from the definition of a maximal element that for every element y of X such that $x_0 \leq y$, there is an element $z \in X$ such that $y \leq z$ and $y \neq z$.

Let us now use transfinite induction to define a sequence $\{z_\beta\}_{\beta<\alpha}$, where α is the order type of the set $Z(m)$ of all ordinal numbers of the power not greater than m.

The inductive definition goes as follows:

1. $z_0 = x_0$,
2. for every isolated ordinal number $\beta = \gamma + 1, \beta \leq \alpha$, the term $z_\beta \in X$ is such that $z_\gamma \neq z_\beta$ and $z_\gamma \leq z_\beta$,
3. for every limit ordinal number $\beta < \alpha$, the term $z_\beta \in X$ is an upper bound of the set $\{z_\gamma\}_{\gamma<\beta}$ if that upper bound exists, and $z_\beta = x_0$ otherwise.

We will show that $z_\gamma \prec z_\xi$, i.e., $z_\gamma \leq z_\xi$, and $z_\gamma \neq z_\xi$ for $\gamma < \xi < \alpha$. Let Z be the set of all ordinal numbers $\beta < \alpha$, such that $\gamma < \xi \leq \beta$. Then, the above condition is satisfied.

Obviously, 0 is in Z. Assume that $Z(\beta) \subset Z$. If $\beta = \delta + 1$, then we infer from (2) that $z_\delta \prec z_\beta$.

Since $\delta < \beta$, we have $\delta \in Z(\beta)$ and accordingly $\delta \in Z$. It follows that if $\gamma < \xi \leq \delta$, then $z_\gamma \prec z_\xi$. We infer from this and from $z_\delta \prec z_\xi$ that $\beta \in Z$.

If β is a limit ordinal number and $Z(\beta) \subset Z$, then for any ordinal numbers γ and ξ such that $\gamma < \beta$ and $\xi < \beta$, one of the following conditions is met:

$$\gamma < \xi, \quad \gamma = \xi, \quad \xi < \gamma,$$

i.e., the set $\{z_\gamma\}_{\gamma<\beta}$ is a chain.

In fact, since $\xi, \gamma \in Z$, one of the conditions $z_\gamma \prec z_\xi, z_\gamma = z_\xi, z_\xi \prec z_\gamma$ is satisfied.

It follows that $z_\gamma \leq z_\xi$ or $z_\xi \leq z_\gamma$ which proves that $\{z_\gamma\}_{\gamma<\beta}$ is a chain.

Now, it follows from (3) and the assumption made in the theorem that z_β is an upper bound of this chain, so that $z_\gamma \neq z_\beta$ for every $\gamma < \beta$. We infer from this and from the fact that $Z(\beta) \subset Z$ and β is a limit ordinal number that $z_\gamma \prec z_\beta$ for every $\gamma < \beta$.

Since the condition $\gamma < \xi < \beta$ implies $z_\gamma \prec z_\xi$, we infer that $\beta \in Z$. By the theorem on transfinite induction, we infer that every ordinal number $\beta < \alpha$ is in Z. This proves that $z_\gamma \prec z_\xi$ for $\gamma < \xi < \alpha$. Thus, $(z_\beta)_{\beta<\alpha}$ is a one-to-one sequence. Hence, the set $(z_\beta)_{\beta<\alpha}$ is equipotent with the set $Z(\alpha)$.

Accordingly,

4. $card((z_\beta)_{\beta<\alpha}) = card(Z(\alpha)) = card(\alpha)$.

Also, since $(z_\beta)_{\beta<\alpha} \subset X$, we have

5. $card((z_\beta)_{\beta<\alpha}) \leq card(X) = m$.

We now infer from (4) and (5) that the assumption concerning the ordinal number α and the formula $\aleph(m) = card(Z(m))$ that

$$\aleph(m) = card(z(m)) = card(\alpha) \leq card(X) = m,$$

so that $\aleph(m) \leq m$. So, for every cardinal number m, the power of the set of all ordinals α such that $card(\alpha) \leq m$ is neither less than nor equal to the cardinal number m. Thus, $\aleph(m) > m$. ■

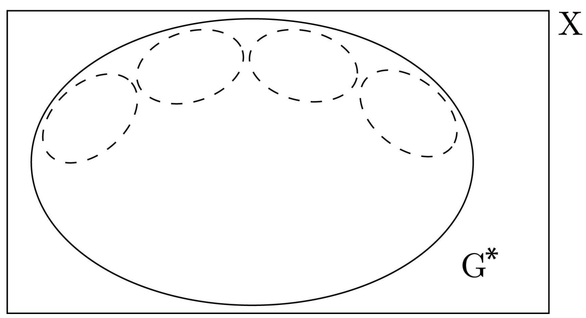
Appendix B

The Banach Category Theorem

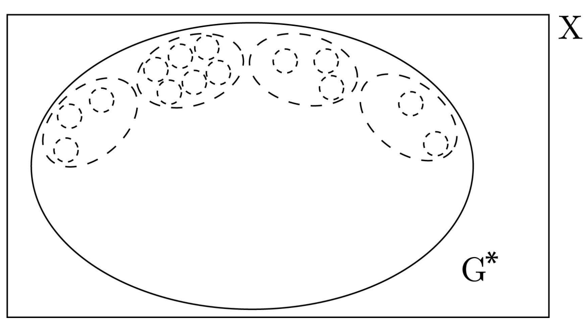
This proof of the Banach Category theorem is based on the proof of the Banach Category theorem in [28].

Theorem B.0.1. In a topological space X, the union of any collection of open sets of first category is of first category.

Proof. Let G^* be the union of a collection G of nonempty open sets of first category.

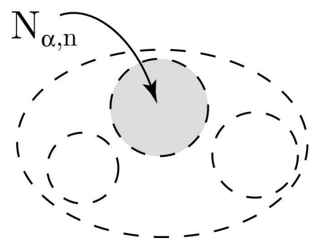


Let $\mathcal{F} = \{U_\alpha : \alpha \in A\}$ be a maximal collection of disjoint nonempty open sets such that each is contained in some member of G.

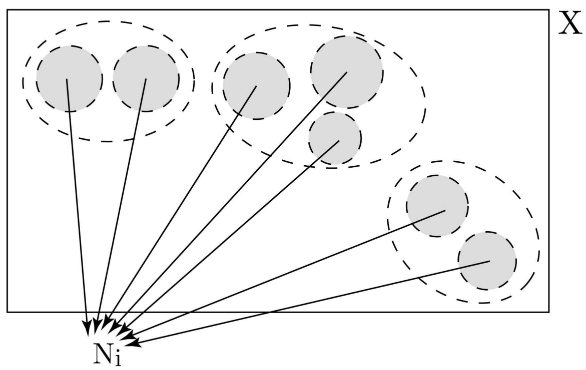


Observe that due to the maximality of $\mathcal{F}$, the closed set $\overline{G^*} \setminus \bigcup \mathcal{F} = \overline{G^*} \setminus \mathcal{F}^*$ is nowhere dense.

As a subset of a respective member of G, every set U_α is of first category. Thus, U_α can be represented as the countable union of nowhere dense sets, say $U_\alpha = \bigcup_{n=1}^{\infty} N_{\alpha,n}$.



Now, let us define $N_n = \bigcup_{\alpha \in A} N_{\alpha,n}$.

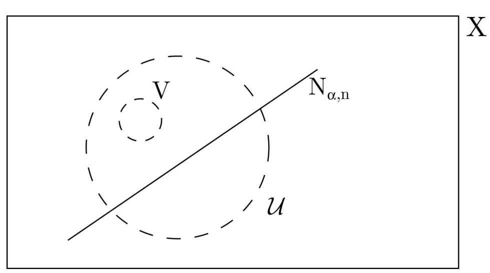


Observe that

$$\bigcup_{n=1}^{\infty} \bigcup_{\alpha \in A} N_{\alpha,n} = \bigcup_{\alpha \in A} U_\alpha = \bigcup_{n=1}^{\infty} N_n.$$

Now, if an open, nonempty set U intersects N_n, then there is an index α such that:

$$U \cap N_{\alpha,n} \neq \varnothing.$$



However, since $N_{\alpha,n}$ is nowhere dense, there is an open nonempty set V such that

$$V \subset (U \cap U_\alpha) \setminus N_{\alpha,n}.$$

Hence, $V \subset U \setminus N_n$ and so, N_n is nowhere dense. Thus,

$$\begin{aligned} G^* &\subset (\overline{G^*} \setminus \mathcal{F}^*) \cup \bigcup_{\alpha \in A} U_\alpha \\ &= (\overline{G^*} \setminus \mathcal{F}^*) \cup \bigcup_{n=1}^{\infty} N_n \end{aligned}$$

is of first category. ■

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