AN EXPLORATION OF THE ERDÖS-MORDELL INEQUALITY

by

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Submitted in Partial Fulfillment of the Requirements

for the Degree of

Master of Science

in the

Mathematics

Program

YOUNGSTOWN STATE UNIVERSITY

August, 2010

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Abstract

We investigate the Erdös-Mordell Inequality for triangles through the literature: proving the result in its original form, modifying the result, looking at applications of the result, providing other inequalities resembling the Erdös-Mordell Inequality, and finding a comparable inequality for quadrilaterals.

Table of Contents

1 Background

In the "Advanced Problems" section of the June-July 1935 issue of *The American Mathematical Monthly*, noted mathematician Paul Erdös posed exactly what is written below **[ERD]**

> *3740. Proposed by Paul Erdös, The University, Manchester, England.*
> From a point O inside a given triangle ABC the perpendiculars OP, OQ, OR are drawn to its sides. Prove that
>
> $$OA+OB+OC \geq 2(OP+OQ+OR)\ .$$

The first published proof of this solution would be given in the April 1937 issue of *The American Mathematical Monthly*, offered by L. J. Mordell **[EMB]**. In **[MOR]**, Mordell explains how the solution came into existence. Apparently, Erdös mentioned his conjecture to Mordell around 1937. Mordell proved the result, and Erdös sent the solution into the *Monthly* for publication. Thus, we have the Erdös-Mordell Inequality today.

This one problem, dealing with an interior point of a triangle, has given rise to a number of publications. Various mathematicians have devised alternate proofs, have determined a myriad of consequences, or have investigated similar inequalities that arise when considering an interior point of a triangle. We explore these items in this paper.

Throughout this paper, we adopt a common notation when considering this problem and its extensions. Thus, we will consider the problem in the following way:

> Given $\triangle A_1A_2A_3$ and interior point P of $\triangle A_1A_2A_3$, let p_i denote the distance from P to the side of $\triangle A_1A_2A_3$ opposite vertex A_i, for each $1 \leq i \leq 3$, as shown in Figure 1.1. Then the following result holds.
>
> $$PA_1+PA_2+PA_3 \geq 2(p_1+p_2+p_3)$$

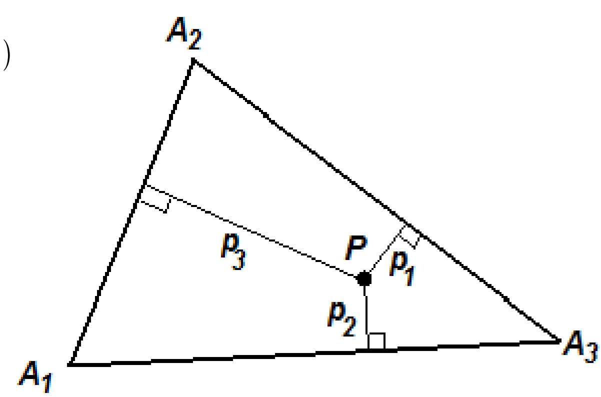

Figure 1.1

We now begin our journey into the world of the Erdös-Mordell Inequality!

2 Preliminaries

Before proving the Erdös-Mordell Inequality, we need to establish a few results.

Theorem 2.1. **Pappus's Theorem.** **[KAD; KAN pg 84]**

Given $\triangle ABC$, let *ABDE* and *ACFG* be two parallelograms, of which either both or neither lies entirely outside of $\triangle ABC$. Let *H* be the point where the extensions of $\overline{DE}$ and $\overline{FG}$ intersect, and let parallelogram *BCKL* be where $\overline{CK}$ is a translate of $\overrightarrow{AH}$. Then the sum of the areas of *ABDE* and *ACFG* is equal to the area of *BCKL*.

Proof of Theorem 2.1. Based on **[KAN pg 84]**

By cases.

Case 1: Both *ABDE* and *ACFG* lie entirely outside of $\triangle ABC$, as shown in Figure 2.1.

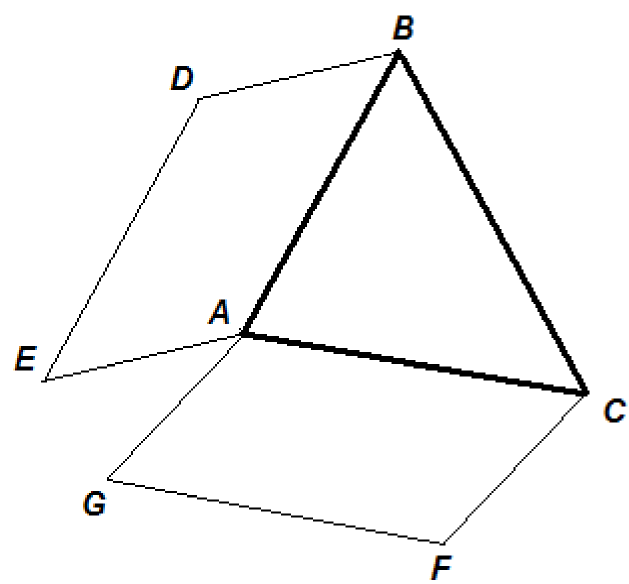

Figure 2.1

We first extend $\overline{DE}$ and $\overline{FG}$ to their point of intersection, which we will call *H*. This point exists since $\overline{DE} \| \overline{AB}$, $\overline{GF} \| \overline{AC}$, and $\overline{AB}$ intersects $\overline{AC}$. Moreover, it will intersect outside of $\triangle ABC$ on the same side of $\overline{BC}$ as *A*. This is shown in Figure 2.2.

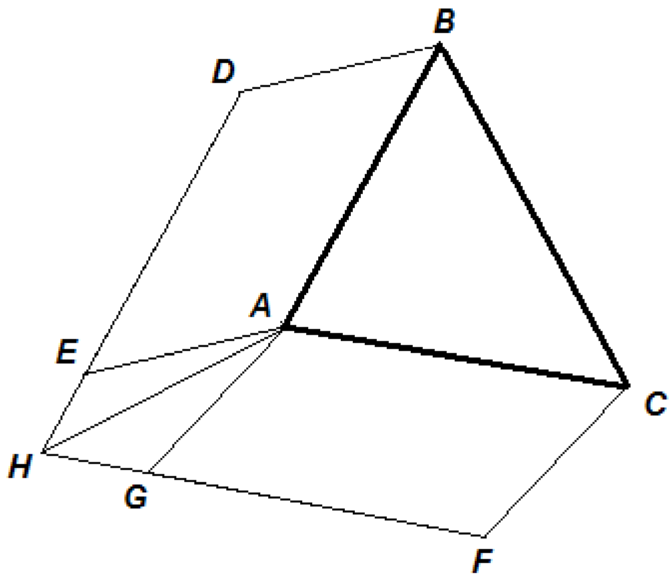

Figure 2.2

Next, we let D_1 be on the line containing $\overline{DE}$ and F_1 be on the line containing $\overline{FG}$ such that $\overline{BD_1} \| \overline{AH}$ and $\overline{CF_1} \| \overline{AH}$, as shown in Figure 2.3.

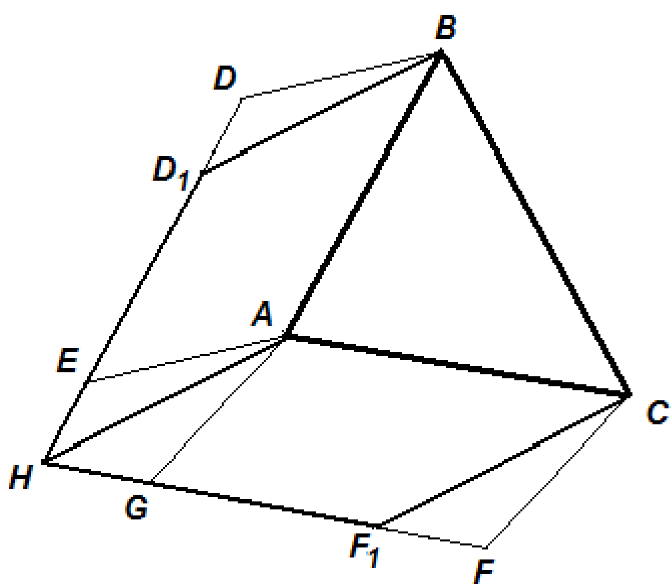

Figure 2.3

From there, we construct parallelogram *BCKL* so that $\overline{CK} \| \overline{AH}$ and $CK = AH$, to meet the given description in the statement of Pappus's Theorem. We draw *BCKL* outside of $\triangle ABC$ for clarity and so that both *K* and F_1 are distinct. This is shown in Figure 2.4.

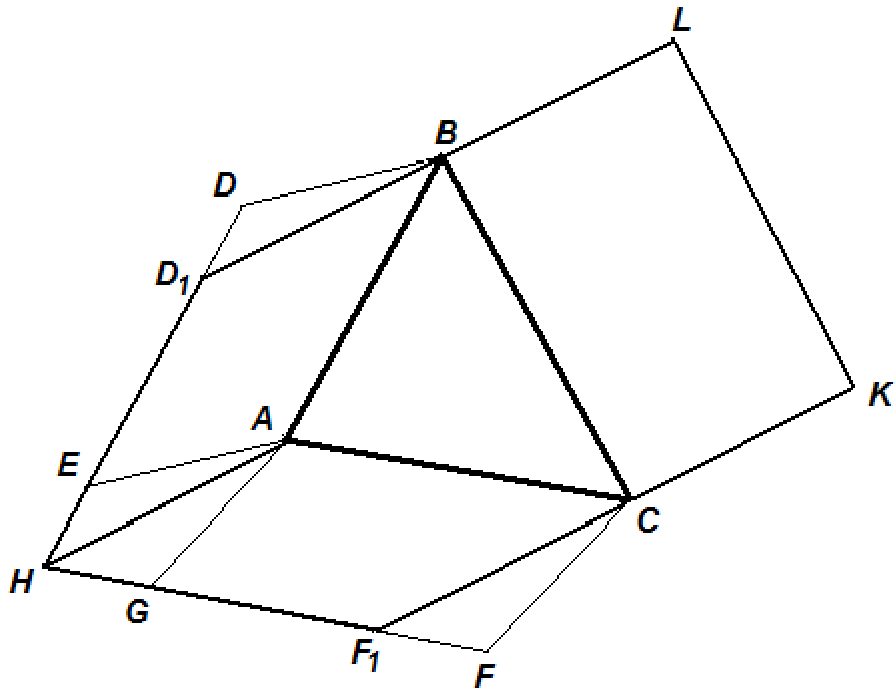

Figure 2.4

Now, we note the area of ABD_1H is the same as the area of $ABDE$, as they share $\overline{AB}$ as a base, and the distance between $\overline{AB}$ and $\overline{DE}$ is the same as the distance between $\overline{AB}$ and $\overline{D_1H}$ since $\overline{DE}$ and $\overline{D_1H}$ are part of the same line which is parallel to $\overline{AB}$.

Similarly, the area of ACF_1H is the same as the area of $ACFG$.

Next, we extend $\overline{AH}$ in order to make some additional observations. We let *M* be the point where this extension intersects $\overline{BC}$, and we let *N* be the point where this extension intersects $\overline{LK}$.

We note that points *M* and *N* must exist as described, as *H* must be on the opposite side of $\overline{AB}$ as *C*, since $ABDE$ is a parallelogram completely outside $\triangle ABC$ and *H* must be on the opposite side of $\overline{AC}$ as *B*, since $ACFG$ is also a parallelogram completely outside $\triangle ABC$. Further, since $\overline{BD_1} \| \overline{AH} \| \overline{CF_1}$ and $\overline{BL} \| \overline{AH} \| \overline{CK}$, it follows that *B* is on $\overline{LD_1}$ and *C* is on $\overline{KF_1}$. From this, we gather that the extension of $\overline{AH}$ must intersect $\overline{BC}$ and $\overline{LK}$, as shown in Figure 2.5.

Now, by the choice of *K*, we know $AH = CK$. Since ACF_1H is a parallelogram, we know $AH = CF_1$. This gives us that $CF_1 = CK$.

Since $BCKL$ is a parallelogram, $AH = CK = BL$. Using that ABD_1H is also a parallelogram, we gather that $AH = BD_1$. This gives us that $BD_1 = BL$.

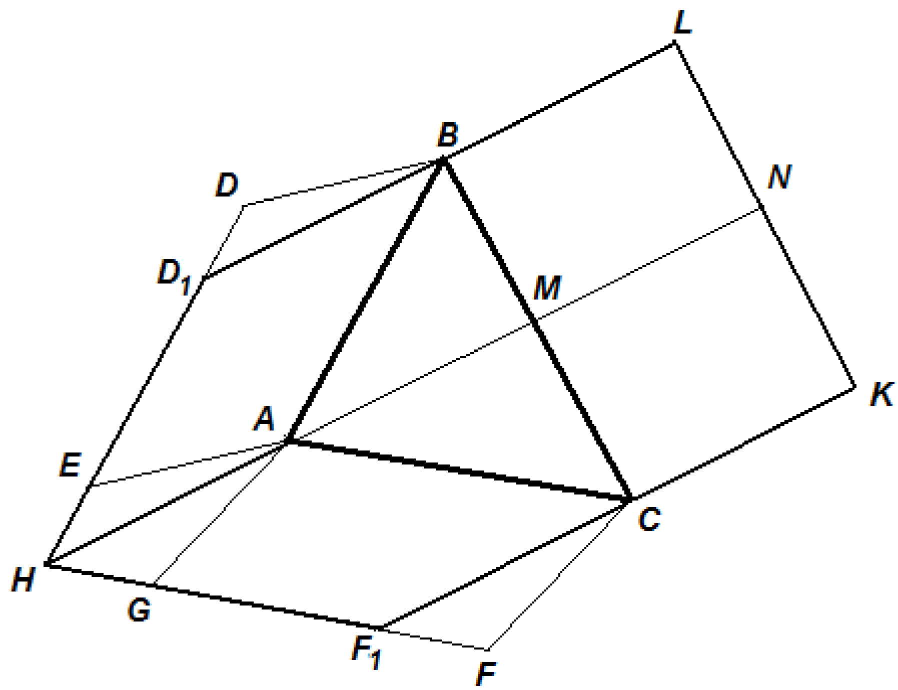

Figure 2.5

At this point we notice that $\overline{KF_1} \| \overline{HN}$. Since $\overline{AH}$ and $\overline{MN}$ are both subsets of $\overline{HN}$ as well as $\overline{CF_1}$ and $\overline{CK}$ are both subsets of $\overline{KF_1}$, it follows that the distance between $\overline{AH}$ and $\overline{CF_1}$ is equal to the distance between $\overline{MN}$ and $\overline{CK}$, for which we will use the notation $d(\overline{AH}, \overline{CF_1}) = d(\overline{MN}, \overline{CK})$.

Similarly, $d(\overline{AH}, \overline{BD_1}) = d(\overline{MN}, \overline{BL})$.

So, we have the following:

$$\text{Area } ABDE + \text{Area } ACFG = \text{Area } ABD_1H + \text{Area } ACF_1H$$

Which, based on the formula for area of parallelograms is

$$= BD_1 \cdot d(\overline{AH}, \overline{BD_1}) + CF_1 \cdot d(\overline{AH}, \overline{CF_1})$$

But since $BD_1 = BL$ and $CF_1 = CK$, this is

$$= BL \cdot d(\overline{AH}, \overline{BD_1}) + CK \cdot d(\overline{AH}, \overline{CF_1})$$

Now using $d(\overline{AH}, \overline{BD_1}) = d(\overline{MN}, \overline{BL})$ and $d(\overline{AH}, \overline{CF_1}) = d(\overline{MN}, \overline{CK})$, this becomes

$$= \ BL \cdot d(\overline{MN}, \overline{BL}) \ + \ CK \cdot d(\overline{MN}, \overline{CK})$$

Using *BL* = CK, this becomes

$$= \ BL \cdot d(\overline{MN}, \overline{BL}) \ + \ BL \cdot d(\overline{MN}, \overline{CK})$$

Factoring out *BL*, we get

$$= \ BL \cdot [d(\overline{MN}, \overline{BL}) + d(\overline{MN}, \overline{CK})]$$

Now, using the properties of parallelogram *BCKL*, we get

$$= \ BL \cdot d(\overline{BL}, \overline{CK})$$

= Area *BCKL*.

Thus, we have established

Area *ABDE* + Area *ACFG* = Area *BCKL*,

and Case 1 holds.

Case 2: Neither *ABDE* nor *ACFG* lie entirely outside of $\triangle ABC$, as shown in Figure 2.6.

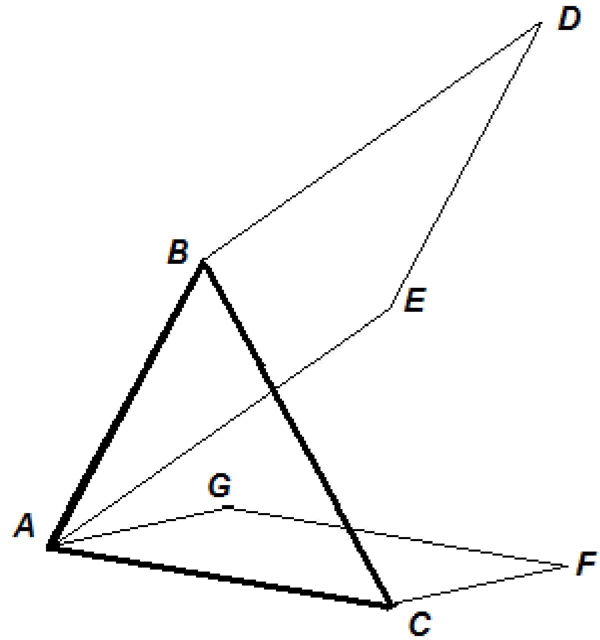

Figure 2.6

We first extend $\overline{DE}$ and $\overline{FG}$ to their point of intersection, which we will call *H*, as shown in Figure 2.7. This point exists since $\overline{DE} \| \overline{AB}$, $\overline{GF} \| \overline{AC}$, and $\overline{AB}$ intersects $\overline{AC}$.

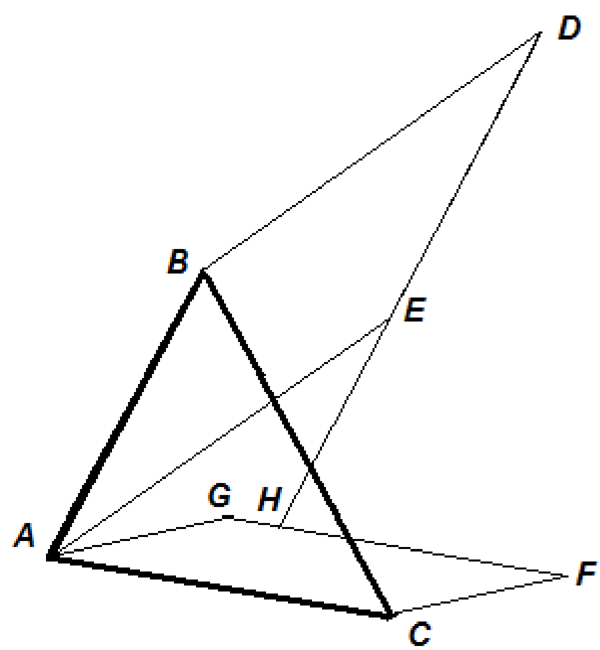

Figure 2.7

Next, we let D_1 be on the line containing $\overline{DE}$ and F_1 be on the line containing $\overline{FG}$ such that $\overline{BD_1} \| \overline{AH}$ and $\overline{CF_1} \| \overline{AH}$, as shown in Figure 2.8.

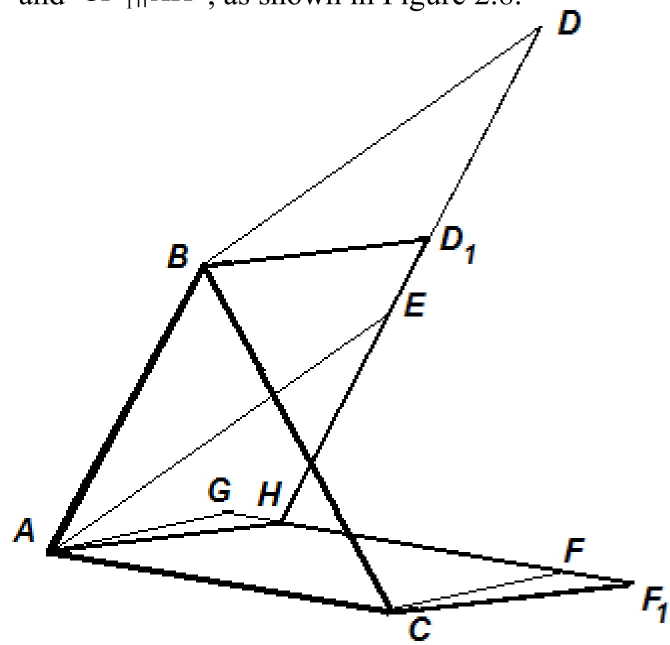

Figure 2.8

From there, we construct parallelogram *BCKL* so that $\overline{CK} \| \overline{AH}$ and *CK* = *AH*, to meet the given description in the statement of Pappus's Theorem. We draw *BCKL* so that it is not entirely outside of $\triangle ABC$. This is done in Figure 2.9.

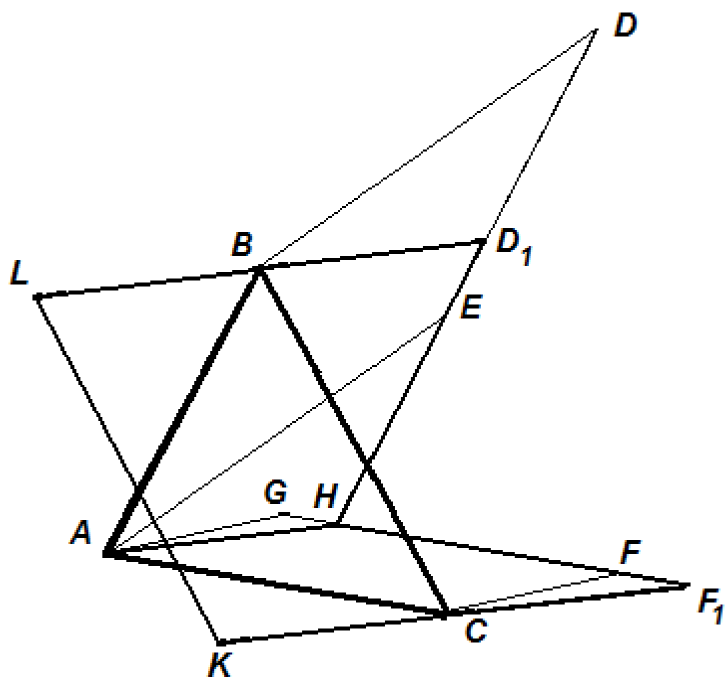

Figure 2.9

For the same reasons as in Case 1, we note that:

$$\text{Area } ABD_1H = \text{Area } ABDE \quad \text{and} \quad \text{Area } ACF_1H = \text{Area } ACFG$$

Next, we extend $\overline{AH}$. We let *M* be the point where this extension intersects $\overline{BC}$, and we let *N* be the point where this extension intersects $\overline{LK}$.

We note that points *M* and *N* must exist as described, as *H* must be on the same side of $\overline{AB}$ as *C*, since $ABDE$ is a parallelogram opening inside $\triangle ABC$ and *H* must be on the same side of $\overline{AC}$ as *B*, since $ACFG$ is also a parallelogram opening inside $\triangle ABC$. Further, since $\overline{BD_1} \| \overline{AH} \| \overline{CF_1}$ and $\overline{BL} \| \overline{AH} \| \overline{CK}$, it follows that *B* is on $\overline{LD_1}$ and *C* is on $\overline{KF_1}$. From this, we gather that the extension of $\overline{AH}$ must intersect $\overline{BC}$ and $\overline{LK}$. This is shown in Figure 2.10.

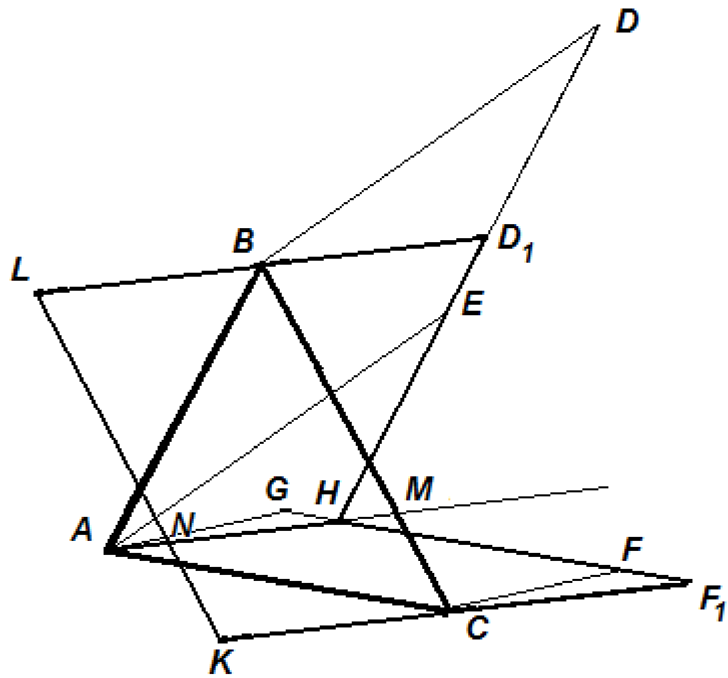

Figure 2.10

Following the same rationale as described in Case 1, we get each of the following results:

$CF_1 = CK$; $BD_1 = BL$; $d(\overline{AH}, \overline{CF_1}) = d(\overline{MN}, \overline{CK})$; $d(\overline{AH}, \overline{BD_1}) = d(\overline{MN}, \overline{BL})$.

Which yields the same process as we had in Case 1, where the rationale is identical for each step, so it will not be repeated again.

$$
\begin{aligned}
\text{Area ABDE} + \text{Area ACFG} &= \text{Area ABD}_1\text{H} + \text{Area ACF}_1\text{H} \\
&= BD_1 \cdot d\,(\overline{AH}, \overline{BD_1}) + CF_1 \cdot d\,(\overline{AH}, \overline{CF_1}) \\
&= BL \cdot d\,(\overline{AH}, \overline{BD_1}) + CK \cdot d\,(\overline{AH}, \overline{CF_1}) \\
&= BL \cdot d\,(\overline{MN}, \overline{BL}) + CK \cdot d\,(\overline{MN}, \overline{CK}) \\
&= BL \cdot d\,(\overline{MN}, \overline{BL}) + BL \cdot d\,(\overline{MN}, \overline{CK}) \\
&= BL \cdot [\, d\,(\overline{MN}, \overline{BL}) + d\,(\overline{MN}, \overline{CK})\,] \\
&= BL \cdot d\,(\overline{BL}, \overline{CK}) \\
&= \text{Area } BCKL.
\end{aligned}
$$

Thus, we have established

$$\text{Area } ABDE + \text{Area } ACFG = \text{Area } BCKL,$$

so Case 2 holds.

By Case 1 and Case 2, Pappus's Theorem holds.

■

It is worth noting that the key to Pappus's Theorem is that both of the parallelograms *ABDE* and *ACFG* must either be both completely outside the original triangle or both not completely outside the original triangle. If one of them was completely outside the triangle and the other wasn't, we would be unable to guarantee the properties of *M* and *N* that make the proof work.

Lemma 2.2. **[ALT pg 53]**

Given $\triangle ABC$, let D be the point on $\overline{BC}$ so that $\overline{AD}$ is an altitude of $\triangle ABC$, and let O be the center of the circumscribed circle of $\triangle ABC$. Then, the bisector of $\angle BAC$ is also the bisector of $\angle DAO$.

Proof of Lemma 2.2. Based on **[ALT pg 53]**

Let E be the such that $\overline{AE}$ is a diameter of the circumcircle, and let F be such that $\overline{AF}$ bisects $\angle BAC$.

Case 1: $\angle ABC$ is acute.

We see a diagram of the situation in Figure 2.11.

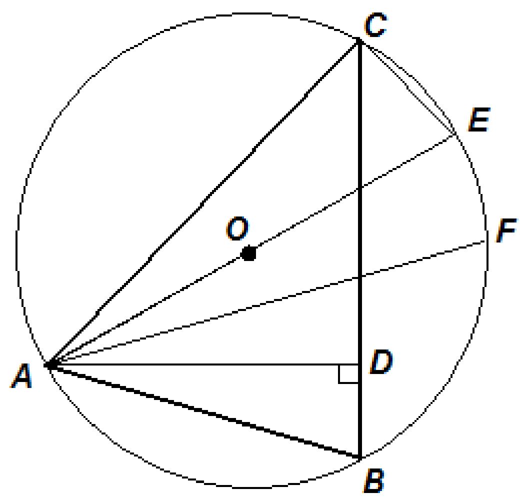

Figure 2.11

First, notice that $\angle ABC$ and $\angle AEC$ are both inscribed angles in the circumcircle intercepting arc *AC*. Thus, we conclude that $m\angle ABC = m\angle AEC$.

Looking at $\triangle ABD$, we conclude that

$$m\angle BAD = 90° - m\angle ABD .$$

Looking at $\triangle AEC$, we notice that $m\angle ACE = 90°$ since $\angle ACE$ is inscribed in the circumcircle and it intercepts arc *ABE*, which is a semicircle. Further, we conclude that

$$m\angle CAE = 90° - m\angle AEC .$$

Thus, we conclude $m\angle BAD = m\angle CAE$.

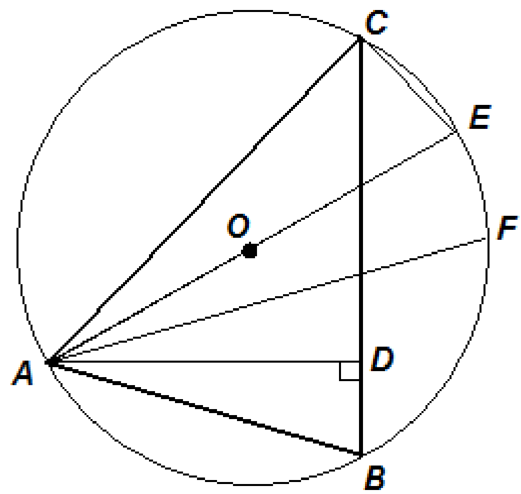

Figure 2.12

Next, since $\overline{AF}$ bisects $\angle BAC$, we get

$$m\angle BAF = m\angle CAF.$$

We also have

$$m\angle BAF = m\angle BAD + m\angle DAF \quad \text{and} \quad m\angle CAF = m\angle CAE + m\angle EAF.$$

Thus, since $m\angle BAF = m\angle CAF$, we have:

$$m\angle BAD + m\angle DAF = m\angle CAE + m\angle EAF.$$

When substituting $m\angle BAD = m\angle CAE$, we get:

$$m\angle BAD + m\angle DAF = m\angle BAD + m\angle EAF,$$

which, when subtracting, gives us

$$m\angle DAF = m\angle EAF.$$

Equivalently,

$$m\angle DAF = m\angle OAF,$$

which means that $\overline{AF}$ bisects $\angle DAO$, and Lemma 2.2 holds for Case 1.

Case 2: $\angle ABC$ is a right angle.

We see a diagram of the situation in Figure 2.13.

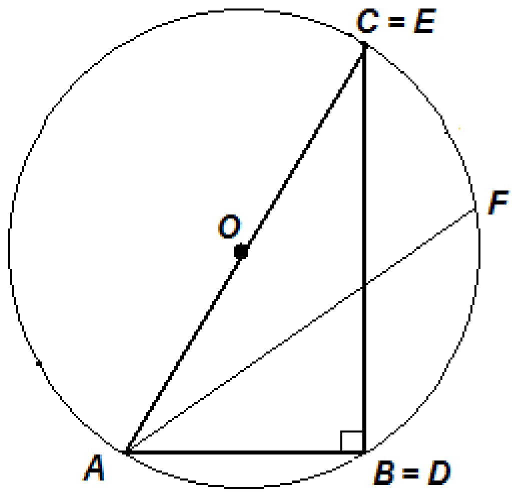

Figure 2.13

This case is trivial, as $\angle DAO = \angle BAC$.

Case 3: $\angle ABC$ is an obtuse angle.

We see a diagram of the situation in Figure 2.14.

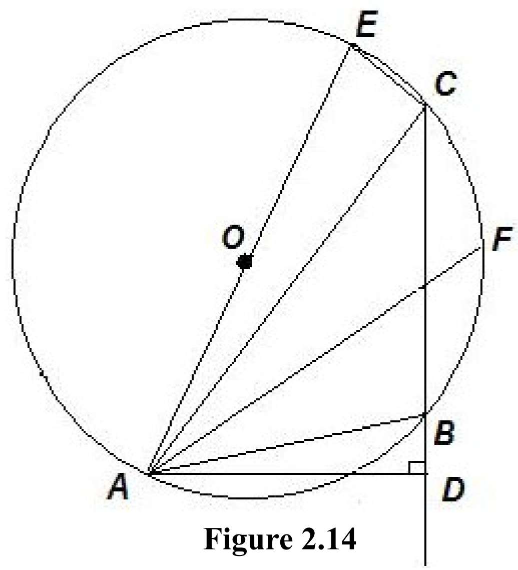

Figure 2.14

First, notice that $\angle ABC$ intercepting arc AEC and $\angle AEC$ intercepting arc ABC are both inscribed angles in the circumcircle. Thus, we have

$$m\angle ABC + m\angle AEC = 180°$$

so we conclude

$$m\angle ABC = 180° - m\angle AEC .$$

Since $\angle ABD$ and $\angle ABC$ are supplementary, we have

$$m\angle ABD = m\angle AEC .$$

Looking at $\triangle ABD$, we get

$$m\angle BAD = 90° - m\angle ABD .$$

Looking at $\triangle AEC$, we notice that $m\angle ACE = 90°$ since $\angle ACE$ is inscribed in a semicircle. Further, we conclude that

$$m\angle CAE = 90° - m\angle AEC .$$

Thus, we conclude $m\angle BAD = m\angle CAE$.

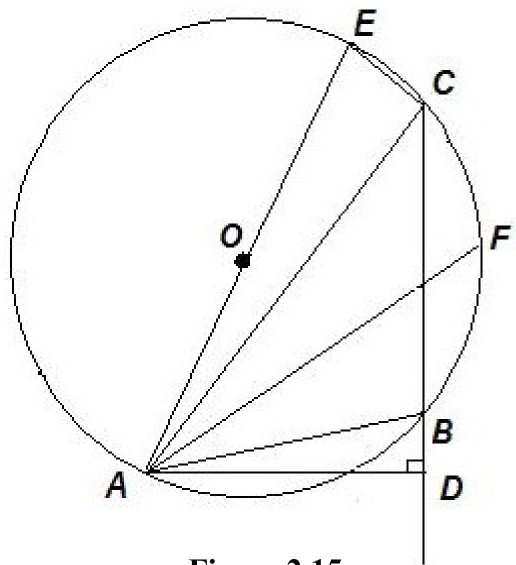

Figure 2.15

Next, since $\overline{AF}$ bisects $\angle BAC$, we get

$$m\angle BAF = m\angle CAF.$$

We also have

$$m\angle DAF = m\angle BAD + m\angle BAF \quad \text{and} \quad m\angle EAF = m\angle CAE + m\angle CAF.$$

Combining $m\angle BAD = m\angle CAE$ and $m\angle BAF = m\angle CAF$ from earlier, we have

$$m\angle DAF = m\angle BAD + m\angle BAF$$

$$m\angle EAF = m\angle BAD + m\angle BAF$$

so that

$$m\angle DAF = m\angle EAF,$$

which means that $\overline{AF}$ bisects $\angle DAO$, and Lemma 2.2 holds for Case 3.

By cases on $\angle ABC$, it follows that Lemma 2.2 holds overall.

∎

Lemma 2.3. **[EMB]**

Let $A, B, C > 0$ such that $A+B+C=180^{\circ}$, and let $a, b, c > 0$. Then the following holds:

$$b^2+c^2+2bc\cos A=(b\sin C+c\sin B)^2+(b\cos C-c\cos B)^2 .$$

Proof of Lemma 2.3.

This is an original proof.

Based on the trigonometric identity $\cos x=-\cos(180^{\circ}-x)$, we have:

$b^2+c^2+2bc\cos A$

$$= \quad b^2+c^2-2bc\cos(180^{\circ}-A)$$

Then, since $A+B+C=180^{\circ}$, it follows that $B+C=180^{\circ}-A$, which yields

$$= \quad b^2+c^2-2bc\cos(B+C)$$

And using the sum of angles identity for cosine, we have:

$$= \quad b^2+c^2-2bc[\cos B\cos C-\sin B\sin C]$$

Simplifying, we get

$$= \quad b^2+c^2-2bc\cos B\cos C+2bc\sin B\sin C$$

Recalling the Pythagorean Identity ($\sin^2 x+\cos^2 x=1$), we have:

$$= \ b^2\sin^2 C+b^2\cos^2 C+c^2\sin^2 B+c^2\cos^2 B-2bc\cos B\cos C+2bc\sin B\sin C$$

Rearranging terms, we have

$$= \quad b^2\sin^2 C+2bc\sin B\sin C+c^2\sin^2 B+b^2\cos^2 C-2bc\cos B\cos C+c^2\cos^2 B$$

$$= \quad (b\sin C+c\sin B)^2+(b\cos C-c\cos B)^2$$

Combining everything, we have

$$b^2+c^2+2bc\cos A=(b\sin C+c\sin B)^2+(b\cos C-c\cos B)^2 ,$$

which establishes Lemma 2.3. ∎

Lemma 2.4.

Let $\triangle A_1 A_2 A_3$ be any triangle, O be the center of its circumscribed circle, and R be the length of the radius of its circumscribed circle. Let a_i be the length of the side opposite A_i for each $1 \le i \le 3$. Finally, let α_i be the measure of the angle at vertex A_i of the original triangle. Then, for each $1 \le i \le 3$, we have:

$$a_i = 2R\sin(\alpha_i)$$

Proof of Lemma 2.4.

This result, based on the Law of Sines, can be found, for example, in **[LAW]**.

First, consider $i=1$, as shown in Figure 2.16.

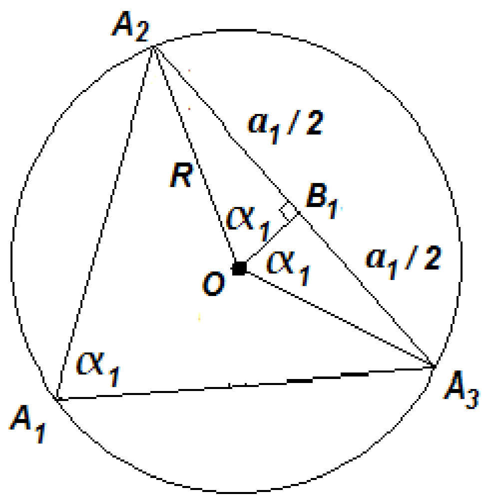

Figure 2.16

Since $\angle A_2 A_1 A_3$ is inscribed in circle O with corresponding central angle $\angle A_2 O A_3$, it follows that $m\angle A_2 O A_3 = 2\alpha_1$. Further, $\triangle A_2 O A_3$ is isosceles, so if B_1 denotes the foot of the perpendicular from O to $\overline{A_2 A_3}$, we have: $m\angle A_2 O B_1 = \alpha_1$ and $A_2 B_1 = a_1/2$. Considering $\triangle A_2 O B_1$, this gives:

$$\sin(\alpha_1) = \frac{a_1/2}{R} = \frac{a_1}{2R}.$$

Multiplying through by $2R$, we get $a_1 = 2R\sin(\alpha_1)$.

The result holds similarly for $i=2$ and $i=3$.

■

3 The Erdös-Mordell Inequality

This section centers on proofs of the Erdös-Mordell Inequality. We state the result again before proceeding.

Theorem 3.1. Erdös-Mordell Inequality [EMB]

Given $\triangle A_1 A_2 A_3$ and interior point *P* of $\triangle A_1 A_2 A_3$, let p_i denote the distance from *P* to the side of $\triangle A_1 A_2 A_3$ opposite vertex A_i, for each $1 \leq i \leq 3$. Then the following result holds:

$$PA_1 + PA_2 + PA_3 \geq 2(p_1 + p_2 + p_3).$$

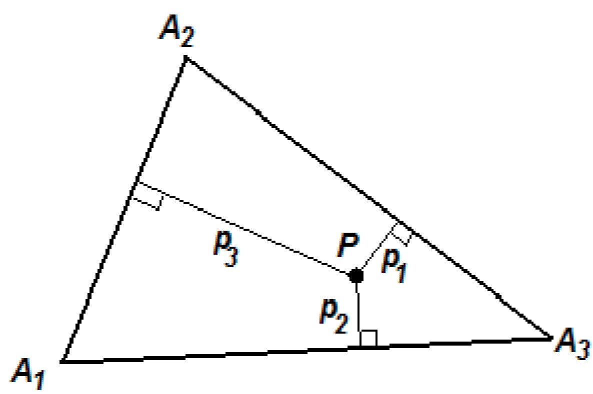

Figure 3.1

Comment.

In this section, we will show three proofs of the Erdös-Mordell Inequality. The first proof is based on the solution by L. J. Mordell **[EMB]**, the second proof, based on a solution by D. K. Kazarinoff, **[KAD]** includes a condition for equality and expands the location of *P*, and the third proof deals with a “signed” inequality based on the work of Clayton W. Dodge **[DOD]**.

Proof of Theorem 3.1. Based on **[EMB]**

This is based on the solution by L. J. Mordell, but it has been adapted for this paper.

First, we let H_i be the foot of the perpendicular from *P* to the side of $\triangle A_1A_2A_3$ opposite A_i, for each $1 \leq i \leq 3$. Additionally, We let α_i be the measure of the angle at the vertex A_i in the original triangle. This is shown in Figure 3.2.

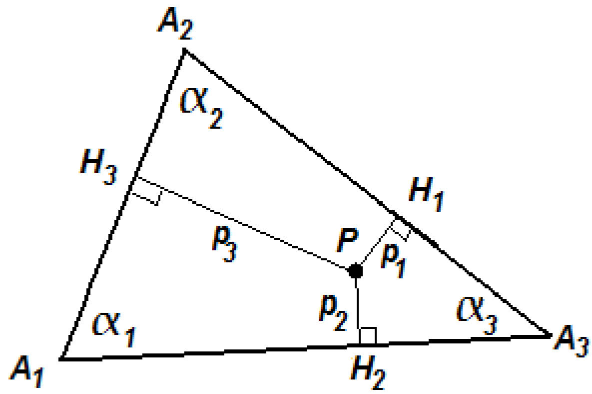

Figure 3.2

We notice that in quadrilateral $A_1H_2PH_3$, based on its interior angles summing to 360º, we have $m\angle H_2PH_3 = 180^\circ - \alpha_1$.

Additionally, $A_1H_2PH_3$ is a cyclic quadrilateral since two of its opposite angles are right angles (so it has a circumscribed circle). If we consider the circumscribed circle of $A_1H_2PH_3$, we note that $\overline{PA_1}$ would be a diameter of this circle (since $\angle PH_3A_1$ and $\angle PH_2A_1$ are both right angles). Applying the result of Lemma 2.4 specifically to $\triangle H_2A_1H_3$, and using $\overline{PA_1}$ as the diameter of the circumcircle:

$$H_2H_3 = PA_1 \sin\alpha_1 \quad \text{or} \quad PA_1 = \frac{H_2H_3}{\sin\alpha_1}.$$

Similarly, analyzing the other quadrilaterals, we conclude:

$$m\angle H_1PH_3 = 180^\circ - \alpha_2 \quad \text{and} \quad m\angle H_1PH_2 = 180^\circ - \alpha_3,$$

so that

$$PA_2 = \frac{H_1H_3}{\sin\alpha_2} \quad \text{and} \quad PA_3 = \frac{H_1H_2}{\sin\alpha_3}.$$

Using the Law of Cosines on $\triangle H_2PH_3$ as shown in Figure 3.3, (with the fact that $m\angle H_2PH_3 = 180° - \alpha_1$), we have

$$H_2H_3 = \sqrt{p_2^2 + p_3^2 - 2p_2p_3\cos(180° - \alpha_1)}$$

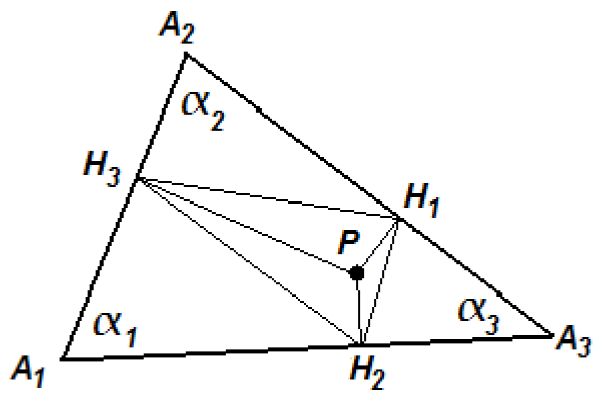

Figure 3.3

Recalling the trigonometric identity, $\cos x = -\cos(180° - x)$, we get

$$H_2H_3 = \sqrt{p_2^2 + p_3^2 + 2p_2p_3\cos\alpha_1}.$$

Similarly, when looking at $\triangle H_1PH_3$ and $\triangle H_1PH_2$, we get:

$$H_1H_3 = \sqrt{p_1^2 + p_3^2 + 2p_1p_3\cos\alpha_2} \quad \text{and} \quad H_1H_2 = \sqrt{p_1^2 + p_2^2 + 2p_1p_2\cos\alpha_3}.$$

Now, combining these gives:

$$PA_1 + PA_2 + PA_3$$

$$= \frac{H_2H_3}{\sin\alpha_1} + \frac{H_1H_3}{\sin\alpha_2} + \frac{H_1H_2}{\sin\alpha_3}$$

$$= \frac{\sqrt{p_2^2 + p_3^2 + 2p_2p_3\cos\alpha_1}}{\sin\alpha_1} + \frac{\sqrt{p_1^2 + p_3^2 + 2p_1p_3\cos\alpha_2}}{\sin\alpha_2} + \frac{\sqrt{p_1^2 + p_2^2 + 2p_1p_2\cos\alpha_3}}{\sin\alpha_3}$$

Using Lemma 2.3, this gives

$$= \frac{\sqrt{(p_2 \sin\alpha_3 + p_3 \sin\alpha_2)^2 + (p_2 \cos\alpha_3 - p_3 \cos\alpha_2)^2}}{\sin\alpha_1}$$

$$+ \frac{\sqrt{(p_1 \sin\alpha_3 + p_3 \sin\alpha_1)^2 + (p_1 \cos\alpha_3 - p_3 \cos\alpha_1)^2}}{\sin\alpha_2}$$

$$+ \frac{\sqrt{(p_1 \sin\alpha_2 + p_2 \sin\alpha_1)^2 + (p_1 \cos\alpha_2 - p_2 \cos\alpha_1)^2}}{\sin\alpha_3}$$

And since, for real values of x and y, $\sqrt{x^2 + y^2} \geq \sqrt{x^2}$, we get

$$\geq \frac{\sqrt{(p_2 \sin\alpha_3 + p_3 \sin\alpha_2)^2}}{\sin\alpha_1}$$

$$+ \frac{\sqrt{(p_1 \sin\alpha_3 + p_3 \sin\alpha_1)^2}}{\sin\alpha_2}$$

$$+ \frac{\sqrt{(p_1 \sin\alpha_2 + p_2 \sin\alpha_1)^2}}{\sin\alpha_3}$$

Simplifying yields

$$= \frac{p_2 \sin\alpha_3 + p_3 \sin\alpha_2}{\sin\alpha_1} + \frac{p_1 \sin\alpha_3 + p_3 \sin\alpha_1}{\sin\alpha_2} + \frac{p_1 \sin\alpha_2 + p_2 \sin\alpha_1}{\sin\alpha_3}$$

Rearranging terms provides

$$= \frac{p_1 \sin\alpha_3}{\sin\alpha_2} + \frac{p_1 \sin\alpha_2}{\sin\alpha_3} + \frac{p_2 \sin\alpha_3}{\sin\alpha_1} + \frac{p_2 \sin\alpha_1}{\sin\alpha_3} + \frac{p_3 \sin\alpha_2}{\sin\alpha_1} + \frac{p_3 \sin\alpha_1}{\sin\alpha_2}$$

$$= p_1 \cdot \left(\frac{\sin\alpha_3}{\sin\alpha_2} + \frac{\sin\alpha_2}{\sin\alpha_3} \right) + p_2 \cdot \left(\frac{\sin\alpha_3}{\sin\alpha_1} + \frac{\sin\alpha_1}{\sin\alpha_3} \right) + p_3 \cdot \left(\frac{\sin\alpha_2}{\sin\alpha_1} + \frac{\sin\alpha_1}{\sin\alpha_2} \right)$$

which, when using the Arithmetic Mean - Geometric Mean Inequality on each piece becomes

$$\geq 2\, p_1 \cdot \sqrt{\frac{\sin\alpha_3}{\sin\alpha_2} \cdot \frac{\sin\alpha_2}{\sin\alpha_3}} + 2\, p_2 \cdot \sqrt{\frac{\sin\alpha_3}{\sin\alpha_1} \cdot \frac{\sin\alpha_1}{\sin\alpha_3}} + 2\, p_3 \cdot \sqrt{\frac{\sin\alpha_2}{\sin\alpha_1} \cdot \frac{\sin\alpha_1}{\sin\alpha_2}}$$

$$= \quad 2p_1 + 2p_2 + 2p_3$$

$$= \quad 2(p_1 + p_2 + p_3).$$

Therefore, we have established

$$PA_1 + PA_2 + PA_3 \geq 2(p_1 + p_2 + p_3),$$

the Erdös-Mordell Inequality.

$\blacksquare$

Theorem 3.2. Modified Erdös-Mordell Inequality [KAD]

As proposed by D. K. Kazarinoff.

Given $\triangle A_1 A_2 A_3$ and interior or boundary point P of $\triangle A_1 A_2 A_3$, let p_i denote the distance from P to the side of $\triangle A_1 A_2 A_3$ opposite vertex A_i, for each $1 \le i \le 3$. Then the following result holds:

$$PA_1 + PA_2 + PA_3 \ge 2(p_1 + p_2 + p_3),$$

with equality happening only when $\triangle A_1 A_2 A_3$ is equilateral and P is its circumcenter.

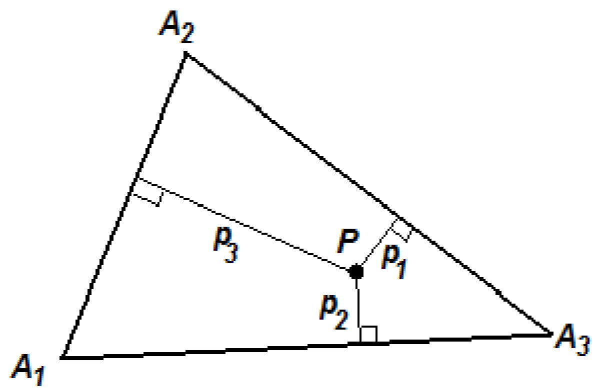

Figure 3.4

Proof of Theorem 3.2. Based on **[KAD]**

This is based on the solution by D. K. Kazarinoff, but it has been adapted for this paper.

We will prove this result by cases, contingent on the location of *P*. Before proceeding, let a_i be the length of the side of $\triangle A_1 A_2 A_3$ across from vertex A_i.

Case 1: *P* is interior to $\triangle A_1 A_2 A_3$.

Notice that our condition requires *P* to be interior to each of the three angles of the original triangle. We begin by focusing on $\angle A_3 A_1 A_2$.

Let *B* be the point where the bisector of $\angle A_3 A_1 A_2$ intersects $\overline{A_2 A_3}$, and let *D* be the foot of the altitude from A_1 to $\overline{A_2 A_3}$, as shown in Figure 3.5.

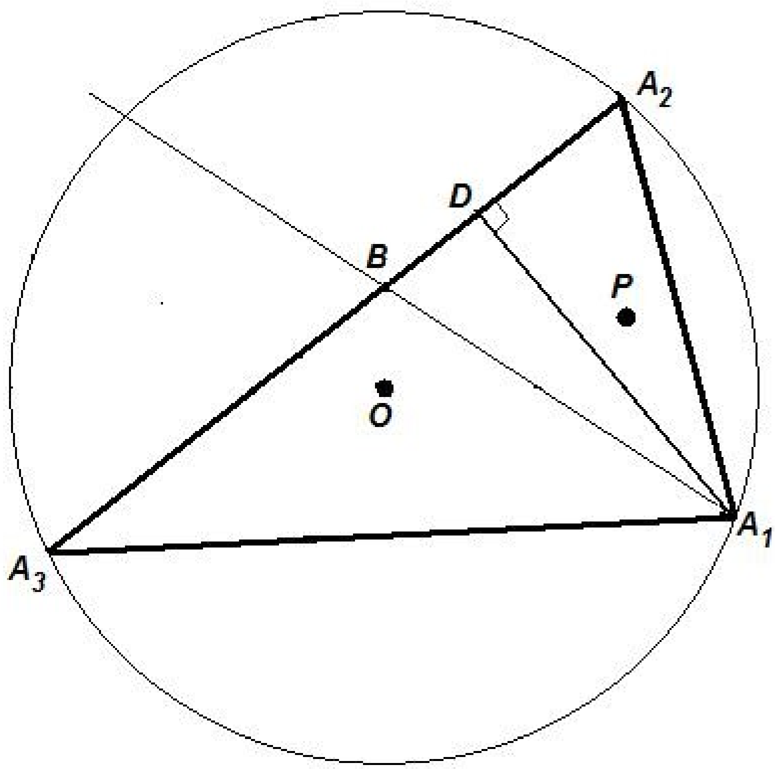

Figure 3.5

Now, we reflect $\triangle A_1 A_2 A_3$, including its altitude, across $\overline{A_1 B}$, calling the new figure $\triangle A_1 \tilde{A}_2 \tilde{A}_3$. Based on Lemma 2.2, we know that since $\overline{A_1 B}$ bisects $\angle A_3 A_1 A_2$ in this setup where O is the circumcenter of the circle, $\overline{A_1 B}$ must also bisect $\angle D A_1 O$.

Since $\overline{A_1 B}$ bisects $\angle D A_1 O$, it follows that the reflection of $\overline{A_1 D}$ must go through O.

Based on the reflection properties and the property that $\overline{A_1 D} \perp \overline{A_2 A_3}$, we let $\tilde{D}$ be the reflection of D, and we conclude $\overline{A_1 \tilde{D}} \perp \overline{\tilde{A}_2 \tilde{A}_3}$. This yields Figure 3.6.

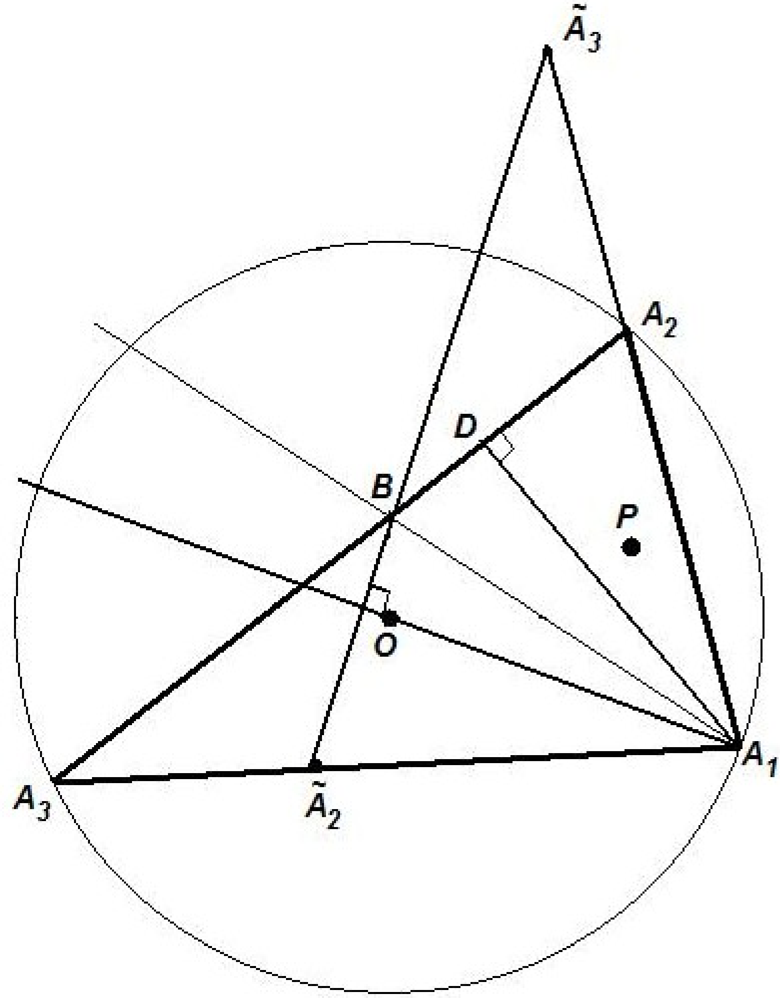

Figure 3.6

We now wish to apply the result from Theorem 2.1 (Pappus's Theorem) to $\triangle A_1 \tilde{A}_2 \tilde{A}_3$.

Pappus's Theorem applies for the following reason:

Since P is an interior point of $\angle A_3 A_1 A_2$ and we reflect across $\overline{A_1 B}$, the bisector of $\angle A_3 A_1 A_2$, P cannot change sides relative to $\overline{A_1 \tilde{A}_2}$ nor relative to $\overline{A_1 \tilde{A}_3}$.

Based on this, neither the parallelogram formed by the side $\overline{A_1 \tilde{A}_3}$ and P nor the parallelogram formed by the side $\overline{A_1 \tilde{A}_2}$ and P will fall completely outside $\triangle A_1 \tilde{A}_2 \tilde{A}_3$.

Essentially, P must remain interior to $\angle A_3 A_1 A_2 = \angle \tilde{A}_3 A_1 \tilde{A}_2$ throughout this process, as it is being reflected across the bisector of that angle.

Create parallelograms $A_1 \tilde{A}_3 W P$ and $A_1 \tilde{A}_2 X P$. Since P is interior to $\angle \tilde{A}_3 A_1 \tilde{A}_2$, it follows that $A_1 \tilde{A}_3 W P$ is not completely outside $\triangle A_1 \tilde{A}_2 \tilde{A}_3$ and $A_1 \tilde{A}_2 X P$ is not completely outside $\triangle A_1 \tilde{A}_2 \tilde{A}_3$ either. Hence, the conclusion of Theorem 2.1 (Pappus's Theorem) applies for $\triangle A_1 \tilde{A}_2 \tilde{A}_3$. That is, we create parallelogram $\tilde{A}_2 \tilde{A}_3 Y Z$ by using $\overrightarrow{PA_1}$ to create $\overrightarrow{\tilde{A}_3 Y}$. (We would not have a scenario where $\overrightarrow{PA_1} \| \overrightarrow{\tilde{A}_2 \tilde{A}_3}$, thereby not yielding a parallelogram, since this would require $\overline{PA_1} \| \overline{\tilde{A}_2 \tilde{A}_3}$, which can't happen since P is interior to $\angle \tilde{A}_3 A_1 \tilde{A}_2$.)

This is shown in Figure 3.7 on the next page.

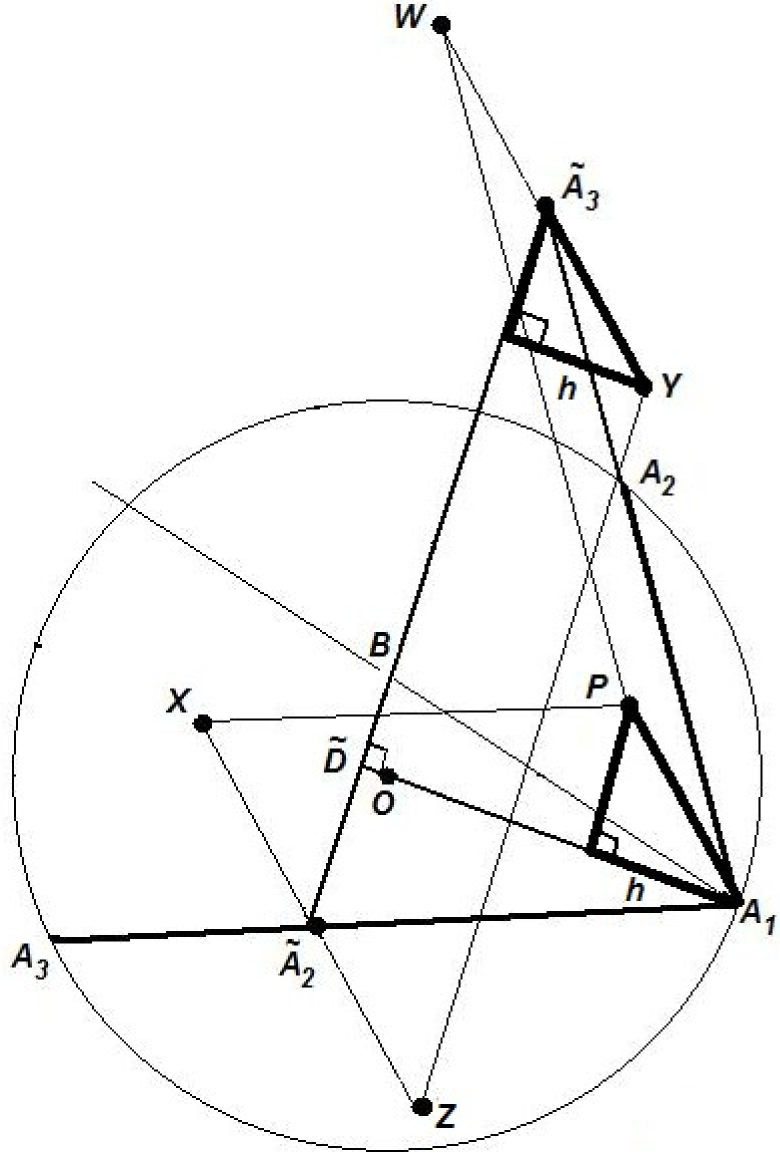

Figure 3.7

We will focus on the following properties of these parallelograms:

$A_1 \tilde{A}_3 W P$	$A_1 \tilde{A}_2 X P$	$\tilde{A}_2 \tilde{A}_3 Y Z$
Base: $A_1 \tilde{A}_3 = A_1 A_3 = a_2$	Base: $A_1 \tilde{A}_2 = A_1 A_2 = a_3$	Base: $\tilde{A}_2 \tilde{A}_3 = A_2 A_3 = a_1$
Height: p_3	Height: p_2	Height: $h = PA_1 \cdot \cos \angle P A_1 O$

Note: $\cos \angle P A_1 O > 0$ since $\overline{O A_1}$ is a radius of a circle, and we know that if *P* is interior to the circle, this angle must be acute. (It becomes a right angle if $\overline{P A_1}$ were tangent to the circle – which isn't the case, and therefore, it could only be obtuse if *P* were exterior to the circle – which isn't the case.)

So, by Theorem 2.1 (Pappus's Theorem),

$$\text{Area } \tilde{A}_2\tilde{A}_3YZ = \text{Area } A_1\tilde{A}_3WP + \text{Area } A_1\tilde{A}_2XP\,.$$

Or, when substituting the appropriate bases and heights listed earlier:

$$a_1 \cdot PA_1 \cos\angle PA_1O = a_2 p_3 + a_3 p_2$$

Since $1 \geq \cos x$, having $a_1 \cdot PA_1 \cos\angle PA_1O = a_2 p_3 + a_3 p_2$ means

$$a_1 \cdot PA_1 \geq a_2 p_3 + a_3 p_2 \qquad \text{so that} \qquad PA_1 \geq \frac{a_2 p_3 + a_3 p_2}{a_1}\,.$$

Similarly, we obtain

$$PA_2 \geq \frac{a_1 p_3 + a_3 p_1}{a_2} \qquad \text{and} \qquad PA_3 \geq \frac{a_2 p_1 + a_1 p_2}{a_3}$$

by focusing on $\angle A_1A_2A_3$ and $\angle A_1A_3A_2$ respectively.

(Note: In each of those two additional situations, *P* is interior to the original angle, and it would be interior to the angle formed after reflecting the triangle across the corresponding angle bisector. Thus, akin to what we saw, neither parallelogram formed by the reflected triangle and *P* would fall completely outside the reflected triangle, meaning Theorem 2.1 would apply for both of those situations as well.

Thus, we have

$$\begin{aligned} & PA_1 + PA_2 + PA_3 \\ & \geq \frac{a_2 p_3 + a_3 p_2}{a_1} + \frac{a_1 p_3 + a_3 p_1}{a_2} + \frac{a_2 p_1 + a_1 p_2}{a_3} \\ & = \frac{a_3 p_1}{a_2} + \frac{a_2 p_1}{a_3} + \frac{a_1 p_2}{a_3} + \frac{a_3 p_2}{a_1} + \frac{a_2 p_3}{a_1} + \frac{a_1 p_3}{a_2} \\ & = p_1\left(\frac{a_3}{a_2} + \frac{a_2}{a_3}\right) + p_2\left(\frac{a_1}{a_3} + \frac{a_3}{a_1}\right) + p_3\left(\frac{a_2}{a_1} + \frac{a_1}{a_2}\right) \end{aligned}$$

Using the Arithmetic Mean – Geometric Mean Inequality, we obtain

$$\geq \quad 2p_1\sqrt{\frac{a_3}{a_2}\cdot\frac{a_2}{a_3}}+2p_2\sqrt{\frac{a_1}{a_3}\cdot\frac{a_3}{a_1}}+2p_3\sqrt{\frac{a_2}{a_1}\cdot\frac{a_1}{a_2}}$$

$$= \quad 2p_1+2p_2+2p_3$$

$$= \quad 2(p_1+p_2+p_3)$$

and therefore, we have established

$$PA_1+PA_2+PA_3\geq 2(p_1+p_2+p_3)$$

for Case 1.

Additionally, since we had $a_1\cdot PA_1\cos\angle PA_1O=a_2p_3+a_3p_2$ and said that $a_1\cdot PA_1\geq a_2p_3+a_3p_2$, for equality to happen, we must have $\cos\angle PA_1O=1$, which requires $\angle PA_1O$ to be a straight angle. This means *P* would have to be on $\overline{A_1O}$.

From the repeated use of this property, we gather that *P* would likewise need to be on $\overline{A_2O}$ and $\overline{A_3O}$ for equality to hold. Thus, *P* must be the center of the circumscribed circle when equality is achieved.

Furthermore, when applying the Arithmetic Mean-Geometric Mean inequality, we have equality if and only if $a_1=a_2=a_3$, which forces the triangle to be equilateral.

Therefore, both the inequality and the condition for equality both hold in Case 1.

Case 2: P is on the boundary of $\triangle A_1 A_2 A_3$, but it is not a vertex point.

Without loss of generality, assume P is on $\overline{A_2 A_3}$. Draw semicircle m centered at P so that m is interior to $\triangle A_1 A_2 A_3$. Let ϵ_1 be the radius of this semicircle. Let $P_1 \in m$ such that P_1 is interior to $\triangle A_1 A_2 A_3$, as shown in Figure 3.8.

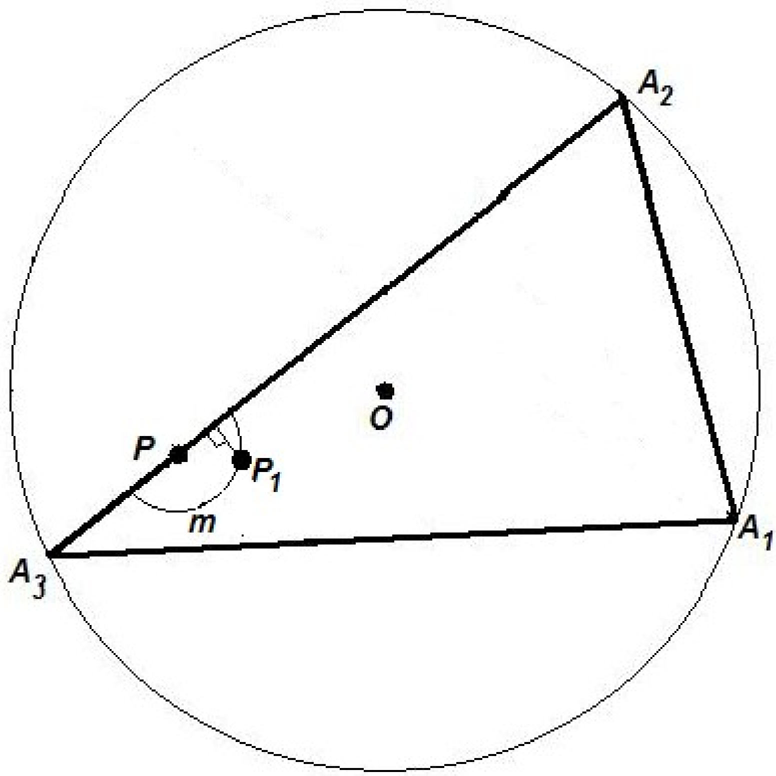

Figure 3.8

Let $\{\epsilon_n\}_{n=1}^{\infty}$ be such that ϵ_1 is as defined above and $\epsilon_{n+1} < \epsilon_n$ for all n. For each n, define P_n such that P_n is on the semicircle centered at P with radius ϵ_n interior to $\triangle A_1 A_2 A_3$ (with P_n also interior to the triangle).

By Case 1, the desired inequality (and its condition for equality) holds for each P_n, namely:

$$P_n A_1 + P_n A_2 + P_n A_3 \geq 2(p_{n,1} + p_{n,2} + p_{n,3}) .$$

From this, as $n \to \infty$, $\epsilon_n \to 0$, and $P_n \to P$, which means the inequality (and it's condition for equality) will also hold for P.

Thus, Case 2 holds.

Case 3: P is a vertex of the triangle.

Without loss of generality, assume P is A_1. In this case, we notice that $PA_1 = 0$, $p_2 = 0$, and $p_3 = 0$. Let D be the foot of the perpendicular from P to $\overline{A_2 A_3}$. This is shown in Figure 3.9.

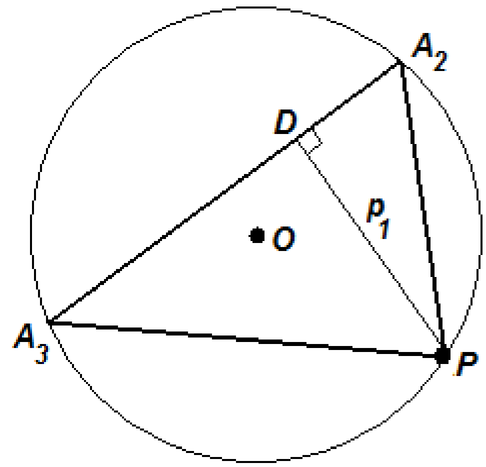

Figure 3.9

When looking at $\triangle P A_3 D$, we get

$$\sin \angle P A_3 D = \frac{p_1}{P A_3} \qquad \text{or} \qquad P A_3 \cdot \sin \angle P A_3 D = p_1 .$$

Similarly, looking at $\triangle P A_2 D$, we get

$$\sin \angle P A_2 D = \frac{p_1}{P A_2} \qquad \text{or} \qquad P A_2 \cdot \sin \angle P A_2 D = p_1 .$$

Putting these results together, we have

$$\begin{aligned} PA_1 + PA_2 + PA_3 &= 0 + PA_2 + PA_3 \\ &= PA_2 + PA_3 \\ &\geq PA_2 \cdot \sin \angle P A_2 D + PA_3 \sin \angle P A_3 D \\ &= p_1 + p_1 \\ &= 2\, p_1 \end{aligned}$$

$$= \quad 2(p_1+0+0)$$

$$= \quad 2(p_1+p_2+p_3).$$

Thus

$$PA_1+PA_2+PA_3 \geq 2(p_1+p_2+p_3),$$

and the inequality holds.

Notice, for equality, we are required to have $\sin\angle PA_2D=1$ and $\sin\angle PA_3D=1$, which would require both $\angle PA_2D$ and $\angle PA_3D$ to be right angles, but there cannot be two right angles in a single triangle. Thus, in this case, we cannot have equality.

Therefore, Case 3 holds.

It follows that the Erdös-Mordell Inequality holds, with equality occurring only when $\triangle A_1A_2A_3$ is equilateral and *P* is its circumcenter.

■

Comments:

We do consider what happens if the point *P* falls outside the triangle, and the next example applies.

Example 3.3. What happens when P is an exterior point?

Consider $\triangle A_1 A_2 A_3$ with $A_1=(0,0);\ A_2=(1,1);\ A_3=(2,0)$. We will investigate two choices of P, namely $P_1=(1,2);\ P_2=(0,1)$.

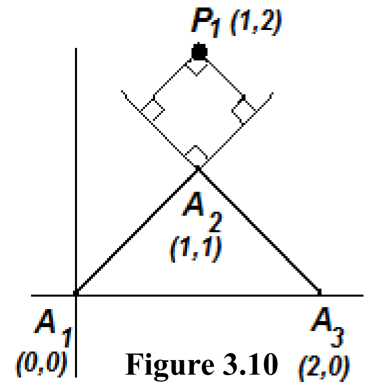

Figure 3.10

In the first case (Figure 3.10), we have:

$$P_1A_1=\sqrt{1^2+2^2}=\sqrt{5} \qquad p_1=\frac{\sqrt{2}}{2}$$

$$P_1A_2=1 \qquad p_2=2$$

$$P_1A_3=\sqrt{1^2+2^2}=\sqrt{5} \qquad p_3=\frac{\sqrt{2}}{2}$$

so that

$$P_1A_1+P_1A_2+P_1A_3=1+2\sqrt{5}\approx 5.47 \qquad \text{and} \qquad p_1+p_2+p_3=2+\sqrt{2}\approx 3.41\ .$$

We notice that $P_1A_1+P_1A_2+P_1A_3\geq 2(p_1+p_2+p_3)$ does not hold in this case, but if we consider p_1 to be "negative" since P_1 and A_1 are on different sides of $\overline{A_2A_3}$, p_2 to be "positive" since P_1 and A_2 are on the same side of $\overline{A_1A_3}$, and p_3 to be "negative" since P_1 and A_3 are on different sides of $\overline{A_1A_2}$, then we get

$$P_1A_1+P_1A_2+P_1A_3=1+2\sqrt{5}\approx 5.47\geq 1.18\approx 2(2-\sqrt{2})=2(p_1+p_2+p_3)$$

so that the inequality holds.

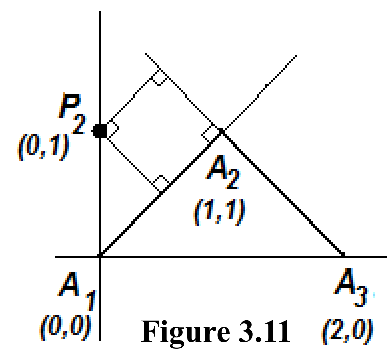

Figure 3.11

In the second case (Figure 3.11), we have:

$$P_2A_1=1 \qquad p_1=\frac{\sqrt{2}}{2}$$

$$P_2A_2=1 \qquad p_2=1$$

$$P_2A_3=\sqrt{2^2+1^2}=\sqrt{5} \qquad p_3=\frac{\sqrt{2}}{2}$$

so that

$$P_1A_1+P_1A_2+P_1A_3=2+\sqrt{5}\approx 4.24 \qquad \text{and} \qquad p_1+p_2+p_3=1+\sqrt{2}\approx 2.41\ .$$

Again, the inequality does not hold in this case. However, if we adopt the convention that p_1 is positive since P_2 and A_1 are on the same side of $\overline{A_2A_3}$, p_2 is positive since P_2 and A_2 are on the same side of $\overline{A_1A_3}$, and p_3 is negative since P_2 and A_3 are on different sides of $\overline{A_1A_2}$, then we get

$$P_2A_1+P_2A_2+P_2A_3=2+\sqrt{5}\approx 4.24>2=2(1)=2\left(\frac{\sqrt{2}}{2}+1-\frac{\sqrt{2}}{2}\right)=2(p_1+p_2+p_3)\,,$$

so the inequality holds.

Example 3.3 provides the motivation for a “signed” Erdös-Mordell Inequality, in order to account for what happens if P is an exterior point. Before getting such a result, we develop a precursor.

Theorem 3.4 **[DER]**

Given $\triangle A_1 A_2 A_3$, and let P be a point in the same plane.

Let p_i denote the signed distance from P to the side of $\triangle A_1 A_2 A_3$ opposite vertex A_i, for each $1 \le i \le 3$.

That is:

p_1 is positive if P and A_1 are on the same side of $\overline{A_2 A_3}$, p_1 is negative otherwise;

p_2 is positive if P and A_2 are on the same side of $\overline{A_1 A_3}$, p_2 is negative otherwise; and

p_3 is positive if P and A_3 are on the same side of $\overline{A_1 A_2}$, p_3 is negative otherwise.

Then the following result holds:

$$PA_1 + PA_2 + PA_3 \ge p_1\left(\frac{a_3}{a_2} + \frac{a_2}{a_3}\right) + p_2\left(\frac{a_1}{a_3} + \frac{a_3}{a_1}\right) + p_3\left(\frac{a_2}{a_1} + \frac{a_1}{a_2}\right).$$

Comment.

The statement of this theorem and its corresponding proof are based on the work of Nikolaos Dergiades in **[DER]**, but it has been adapted for this paper.

Proof of Theorem 3.4. Based on **[DER]**

We let h_1 denote the length of the altitude from A_1 to $\overline{A_2 A_3}$, we let a_i be the length of the side of $\triangle A_1 A_2 A_3$ opposite A_i, and we let *K* be the area of $\triangle A_1 A_2 A_3$.

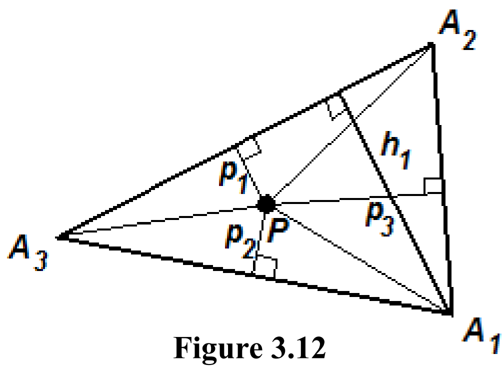

Figure 3.12

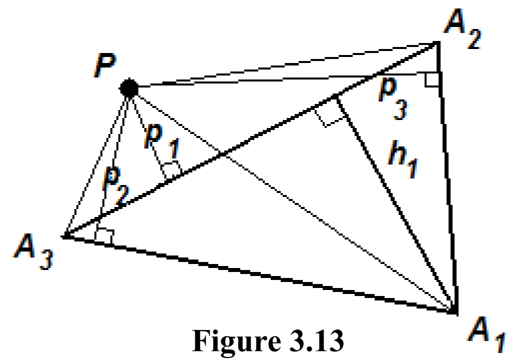

Figure 3.13

The first item we observe is that, regardless of the location of *P* relative to the triangle, as long as we have the signed distances defined above,

$$\text{Area} \triangle\ A_1 A_2 A_3 = \text{Area} \triangle\ A_1 P A_2 + \text{Area} \triangle\ A_2 P A_3 + \text{Area} \triangle\ A_1 P A_3 .$$

(When *P* is interior to the triangle, as shown in Figure 3.12, this is obvious. We notice that in the case where *P* is outside the triangle – one such example being shown in Figure 3.13 – we have to take $\triangle\ A_2 P A_3$ away from $\triangle\ A_1 P A_2$ and $\triangle\ A_1 P A_3$ to get $\triangle\ A_1 A_2 A_3$, which is exactly what we have with the signed value of $p_1 < 0$.)

Upon substitution, this equation becomes

$$K = \frac{a_1 h_1}{2} = \frac{a_1 p_1}{2} + \frac{a_2 p_2}{2} + \frac{a_3 p_3}{2} \qquad \text{or} \qquad 2K = a_1 h_1 = a_1 p_1 + a_2 p_2 + a_3 p_3 .$$

Next, we notice that $PA_1 + p_1 \geq h_1$, no matter the location of *P* (with the $p_1 < 0$ possibility). This is a simple consequence of the fact that the altitude is the shortest distance from a vertex of a triangle to its opposite side, and it is pictured in Figure 3.14 and Figure 3.15 for clarity.

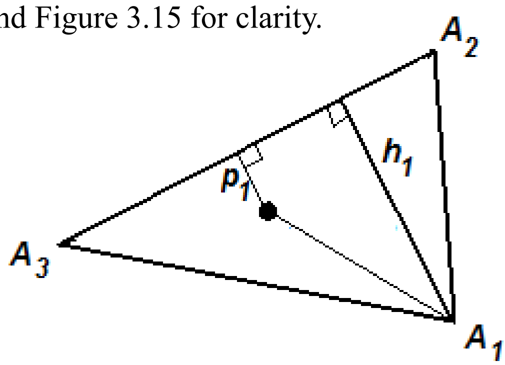

Figure 3.14

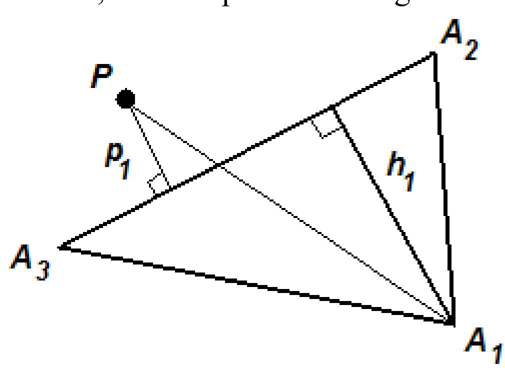

Figure 3.15

Also, it is clear that we get equality if and only if P is on the line containing the altitude.

Now, combining $a_1 h_1 = a_1 p_1 + a_2 p_2 + a_3 p_3$ with $PA_1 + p_1 \geq h_1$, we get

$$a_1 PA_1 + a_1 p_1 = a_1 (PA_1 + p_1) \geq a_1 h_1 = a_1 p_1 + a_2 p_2 + a_3 p_3 ,$$

which means $a_1 PA_1 + a_1 p_1 \geq a_1 p_1 + a_2 p_2 + a_3 p_3$, so that

$$a_1 PA_1 \geq a_2 p_2 + a_3 p_3 \quad \text{or equivalently} \quad PA_1 \geq \frac{a_2 p_2 + a_3 p_3}{a_1} . \qquad (3.4.A)$$

Similarly, we obtain

$$PA_2 \geq \frac{a_1 p_1 + a_3 p_3}{a_2} \quad \text{and} \quad PA_3 \geq \frac{a_1 p_1 + a_2 p_2}{a_3} .$$

We let $\triangle A_1 \tilde{A}_2 \tilde{A}_3$ be the image of $\triangle A_1 A_2 A_3$ when reflected across the bisector of $\angle A_2 A_1 A_3$. When doing this, we realize the bisector of $\angle A_2 A_1 A_3$ is also the bisector of the angle formed by the altitude from A_1 of the original triangle and the radius of the circumcircle of the original triangle. (See Lemma 2.2).

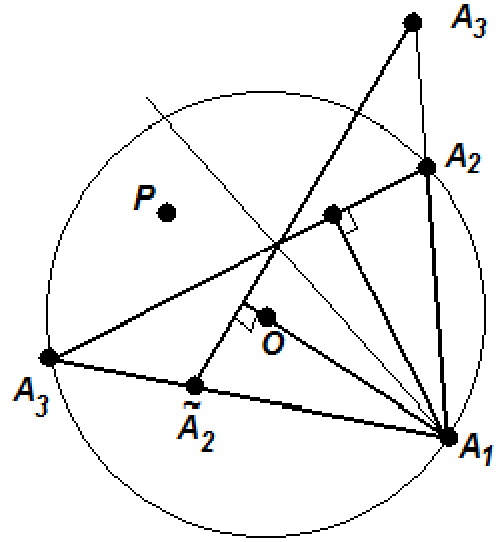

Figure 3.16

Applying the inequality $a_1 PA_1 \geq a_2 p_2 + a_3 p_3$ to $\triangle A_1 \tilde{A}_2 \tilde{A}_3$, (which essentially switches a_2 and a_3 in our inequality), we realize that $a_1 PA_1 \geq a_3 p_2 + a_2 p_3$, with equality being the case only if P is on the altitude through A_1 in $\triangle A_1 \tilde{A}_2 \tilde{A}_3$, which is the same line containing $\overline{OA_1}$.

The rationale for this is supported through Figure 3.19 and Figure 3.20 on the next page.

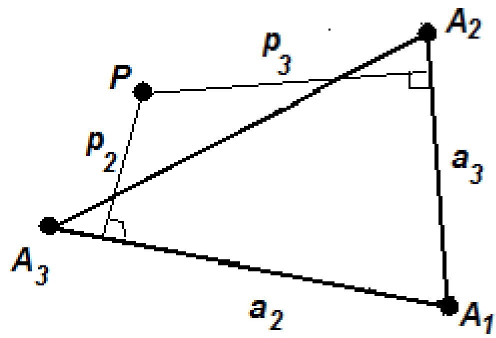

Figure 3.17

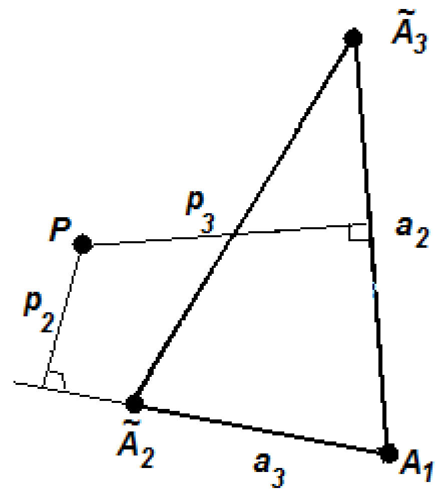

Figure 3.18

So now, like in Kazarinoff's proof, we have $a_1 PA_1 \geq a_3 p_2 + a_2 p_3$, which means

$$PA_1 \geq \frac{a_3 p_2}{a_1} + \frac{a_2 p_3}{a_1}. \qquad (3.4.B)$$

Similarly, we obtain

$$PA_2 \geq \frac{a_3 p_1}{a_2} + \frac{a_1 p_3}{a_2} \quad \text{and} \quad PA_3 \geq \frac{a_1 p_2}{a_3} + \frac{a_2 p_1}{a_3}.$$

(There is no problem with obtaining these results similarly, as we saw the location of *P* was not problematic, and each of the angles of the original triangle will have a bisector.)

Akin to the earlier proof, we get:

$$\begin{aligned} PA_1 + PA_2 + PA_3 &\geq \frac{a_3 p_2 + a_2 p_3}{a_1} + \frac{a_3 p_1 + a_1 p_3}{a_2} + \frac{a_1 p_2 + a_2 p_1}{a_3} \\ &= \frac{a_3 p_1}{a_2} + \frac{a_2 p_1}{a_3} + \frac{a_1 p_2}{a_3} + \frac{a_3 p_2}{a_1} + \frac{a_2 p_3}{a_1} + \frac{a_1 p_3}{a_2} \\ &= p_1\left(\frac{a_3}{a_2} + \frac{a_2}{a_3}\right) + p_2\left(\frac{a_1}{a_3} + \frac{a_3}{a_1}\right) + p_3\left(\frac{a_2}{a_1} + \frac{a_1}{a_2}\right). \end{aligned}$$

Thus, we have established

$$PA_1+PA_2+PA_3 \geq p_1\left(\frac{a_3}{a_2}+\frac{a_2}{a_3}\right)+p_2\left(\frac{a_1}{a_3}+\frac{a_3}{a_1}\right)+p_3\left(\frac{a_2}{a_1}+\frac{a_1}{a_2}\right),$$

our desired result.

∎

Corollary 3.5.

Under the premise of Theorem 3.4, we have (from *3.4.A*)

$$PA_1 \geq \frac{a_2 p_2+a_3 p_3}{a_1}, \quad PA_2 \geq \frac{a_1 p_1+a_3 p_3}{a_2}, \text{ and } \quad PA_3 \geq \frac{a_1 p_1+a_2 p_2}{a_3}$$

as well as (from *3.4.B*)

$$PA_1 \geq \frac{a_3 p_2+a_2 p_3}{a_1}, \quad PA_2 \geq \frac{a_3 p_1+a_1 p_3}{a_2}, \quad \text{and} \quad PA_3 \geq \frac{a_1 p_2+a_2 p_1}{a_3}.$$

Comment.

The second set of inequalities were also formed by D. K. Kazarinoff in **[KAD]** in the case where *P* is interior to the triangle.

Comment.

At the conclusion of Theorem 3.4, it is tempting to use the Arithmetic Mean – Geometric Mean Inequality on each piece to yield the following:

$$\begin{aligned}
PA_1+PA_2+PA_3 \quad &\geq \quad p_1\left(\frac{a_3}{a_2}+\frac{a_2}{a_3}\right)+p_2\left(\frac{a_1}{a_3}+\frac{a_3}{a_1}\right)+p_3\left(\frac{a_2}{a_1}+\frac{a_1}{a_2}\right) \\
&\geq \quad 2\,p_1\sqrt{\frac{a_3}{a_2}\cdot\frac{a_2}{a_3}}+2\,p_2\sqrt{\frac{a_1}{a_3}\cdot\frac{a_3}{a_1}}+2\,p_3\sqrt{\frac{a_2}{a_1}\cdot\frac{a_1}{a_2}} \\
&= \quad 2\,p_1+2\,p_2+2\,p_3 \\
&= \quad 2(p_1+p_2+p_3)
\end{aligned}$$

and therefore, claim $PA_1+PA_2+PA_3 \geq 2(p_1+p_2+p_3)$.

However, for example, if $p_1 < 0$, we cannot conclude $p_1\left(\frac{a_3}{a_2}+\frac{a_2}{a_3}\right)\geq 2\, p_1\sqrt{\frac{a_3}{a_2}\cdot\frac{a_2}{a_3}}$.

We want to show a "signed" Erdös-Mordell Inequality, namely, under the signed versions of $p_{1,}\, p_{2,}\, p_3$ described above, that we can conclude $PA_1+PA_2+PA_3\geq 2(p_1+p_2+p_3)$ for any point *P* in the same plane as $\triangle A_1 A_2 A_3$.

In fact, this exact problem appeared in the "Elementary Problems" section of the March 1974 issue of *The American Mathematical Monthly* as "E 2462" **[DEM]**.

One referee assigned to this problem was Clayton W. Dodge. In **[DOD]**, Dodge describes the situation emerging from this seemingly innocent problem. Three solutions were submitted in 1974. Each author used the methods of Kazarinoff and extended them for the signed values of p_i . Each author made the error referenced in the comment above: the incorrect application of the Arithmetic Mean – Geometric Mean inequality.

Dodge, working with the other referees, attempted to find a way around this hurdle; nothing immediately presented itself. Over the years, Dodge relates in the same article, he kept being drawn back to this problem until he finally devised a solution in 1984.

What we present next is the Signed Erdös-Mordell Inequality, which is based on the solution by Dodge **[DOD]**, involving ideas from Kazarinoff **[KAD]**.

Theorem 3.6. Signed Erdös-Mordell Inequality **[DOD]**

Given $\triangle A_1 A_2 A_3$, and let *P* be a point in the same plane.

Let p_i denote the signed distance from *P* to the side of $\triangle A_1 A_2 A_3$ opposite vertex A_i, for each $1 \le i \le 3$.

That is:

p_1 is positive if *P* and A_1 are on the same side of $\overline{A_2 A_3}$, p_1 is negative otherwise;

p_2 is positive if *P* and A_2 are on the same side of $\overline{A_1 A_3}$, p_2 is negative otherwise; and

p_3 is positive if *P* and A_3 are on the same side of $\overline{A_1 A_2}$, p_3 is negative otherwise.

Then the following result holds:

$$PA_1 + PA_2 + PA_3 \ge 2(p_1 + p_2 + p_3) .$$

Comment.

This result and its proof are based on the works of Clayton W. Dodge in **[DOD]**, incorporating ideas from Kazarinoff **[KAD]**.

Proof of Theorem 3.6. Based on **[DOD]**

We begin by stating that Theorem 3.2 handles the case where P is either interior to $\triangle A_1A_2A_3$ or on its boundary.

Thus, we need to consider all other possible locations of P.

As shown in Figure 3.19, we have the possibility that P is outside the triangle and:

(Case 1) P lies inside an angle vertical to one of the interior angles of $\triangle A_1A_2A_3$;

(Case 2) P is interior to only one of the interior angles of $\triangle A_1A_2A_3$; or

(Case 3) P is on the extension of one of the sides of $\triangle A_1A_2A_3$.

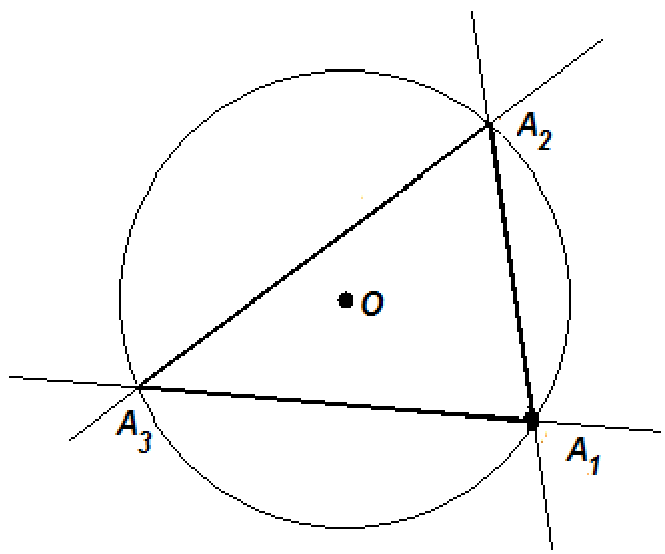

Figure 3.19

Before proceeding into cases, we establish some groundwork.

Let $d_i = |p_i|$.

Based on the Proof of Theorem 3.2 (based on **[KAD]**), we have, for *P* interior to $\triangle A_1 A_2 A_3$,

$$a_1 \cdot PA_1 \cos(\angle P A_1 O) = a_2 p_3 + a_3 p_2$$

$$a_2 \cdot PA_2 \cos(\angle P A_2 O) = a_1 p_3 + a_3 p_1 \qquad (3.6.A)$$

$$a_3 \cdot PA_3 \cos(\angle P A_3 O) = a_1 p_2 + a_2 p_1 \; .$$

From this, in that same proof, we said

$PA_1 + PA_2 + PA_3$

$$\geq \quad PA_1 \cos(\angle P A_1 O) + PA_2 \cos(\angle P A_2 O) + PA_3 \cos(\angle P A_3 O) \qquad (3.6.B)$$

$$= \quad \frac{a_2 p_3 + a_3 p_2}{a_1} + \frac{a_1 p_3 + a_3 p_1}{a_2} + \frac{a_1 p_2 + a_2 p_1}{a_3} \qquad (3.6.C)$$

$$= \quad \left(\frac{a_2}{a_3} + \frac{a_3}{a_2}\right) p_1 + \left(\frac{a_1}{a_3} + \frac{a_3}{a_1}\right) p_2 + \left(\frac{a_1}{a_2} + \frac{a_2}{a_1}\right) p_3 \qquad (3.6.D)$$

$$\geq \quad 2(p_1 + p_2 + p_3) \; . \qquad (3.6.E)$$

Of course, in that scenario, $p_i \geq 0$, so that the Arithmetic Mean – Geometric Mean Inequality applied to go from (*3.6.D)* to (*3.6.E)*.

Now, we base the proof for any location of *P* off this same basic concept.

Case 1: P is outside $\triangle A_1 A_2 A_3$ and P lies inside an angle vertical to one of the interior angles of $\triangle A_1 A_2 A_3$;

Without loss of generality, assume P lies inside the angle vertical to $\angle A_2 A_1 A_3$. Let F be the foot of the perpendicular from P to $\overline{A_2 A_3}$ as shown in Figure 3.20.

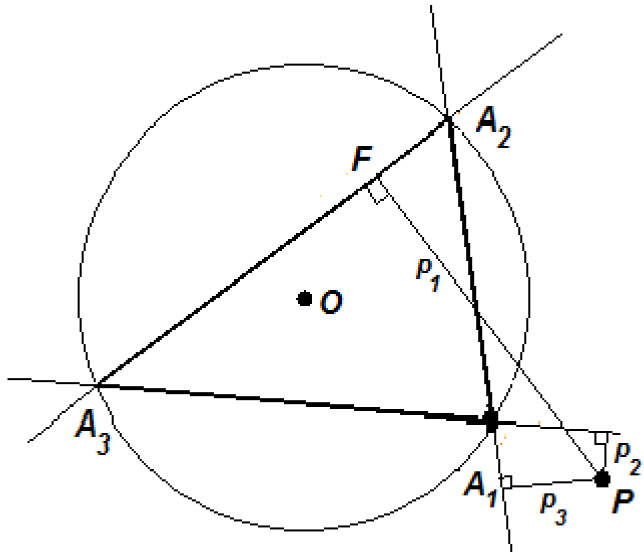

Figure 3.20

Now, in this scenario, $p_1 > 0$, $p_2 < 0$, and $p_3 < 0$, so that

$$p_1 = d_1, \quad p_2 = -d_2, \text{ and } \quad p_3 = -d_3.$$

Seeing that $\overline{PA_2}$ and d_1 are the hypotenuse and leg, respectively, of $\triangle PFA_2$, we have

$$PA_2 \geq d_1.$$

Similarly, since $\overline{PA_3}$ and d_1 are the hypotenuse and leg, respectively, of $\triangle PFA_3$, we have

$$PA_3 \geq d_1.$$

Thus $PA_1 + PA_2 + PA_3 \geq PA_2 + PA_3 \geq d_1 + d_1 \geq 2d_1 \geq 2(d_1 - d_2 - d_3)$.

In this case, we have $p_1 > 0$, $p_2 < 0$, and $p_3 < 0$, so that

$$PA_1 + PA_2 + PA_3 \geq 2(p_1 + p_2 + p_3),$$

as desired.

Case 2: P is outside $\triangle A_1 A_2 A_3$ and P is interior to only one of the interior angles of $\triangle A_1 A_2 A_3$.

Without loss of generality, assume P is interior to $\angle A_2 A_1 A_3$

Case 2.A: P is outside $\triangle A_1 A_2 A_3$ and P is interior to an interior angle of $\triangle A_1 A_2 A_3$, but it is far enough outside the triangle that the foot of the perpendicular F_i from P to the side of $\triangle A_1 A_2 A_3$ opposite A_i lies outside $\triangle A_1 A_2 A_3$ for either or both of the two vertex points to which P is not interior.

Under the conditions of Case 2.A, assume that P is far enough outside $\triangle A_1 A_2 A_3$ that both F_2 and F_3 do not lie on the triangle.

Then we have $p_1 = -d_1 < 0$.

Choose point A_2' such that F_3 is the midpoint of $\overline{A_2 A_2'}$, and let F_1' be the foot of the perpendicular from P to $\overline{A_2' A_3}$ as shown in Figure 3.21.

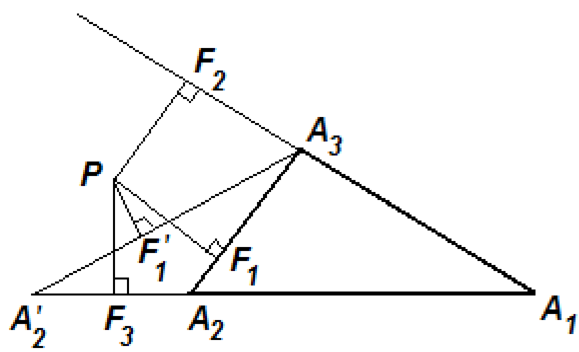

Figure 3.21

Then, we have $PA_2 = PA_2'$, and so the distances from P to the vertices of $\triangle A_1 A_2' A_3$ are the same as the distances from P to the vertices of $\triangle A_1 A_2 A_3$.

Additionally, we note that when considering $\triangle A_1 A_2' A_3$ compared to $\triangle A_1 A_2 A_3$, we have d_2 and d_3 remaining unchanged, but d_1 changes to $d_1' = PF_1'$.

If, as shown in Figure 3.23, P is outside $\triangle A_1 A_2' A_3$, then we get $d_1' < d_1$, so that $p_1 = -d_1 < -d_1' = p_1'$, or $p_1 < p_1'$.

If, however, P is inside $\triangle A_1 A_2' A_3$, then we get $p_1 < 0 < p_1'$, so that $p_1 < p_1'$.

Either way, $p_1 < p_1'$.

Thus,

$$p_1' + p_2 + p_3 > p_1 + p_2 + p_3 .$$

Similarly, we apply the same process on $\triangle A_1 A_2' A_3$:

Choose point A_3' such that F_2 is the midpoint of $\overline{A_3 A_3'}$, and let F_1'' be the foot of the perpendicular from P to $\overline{A_2' A_3'}$ as shown in Figure 3.22.

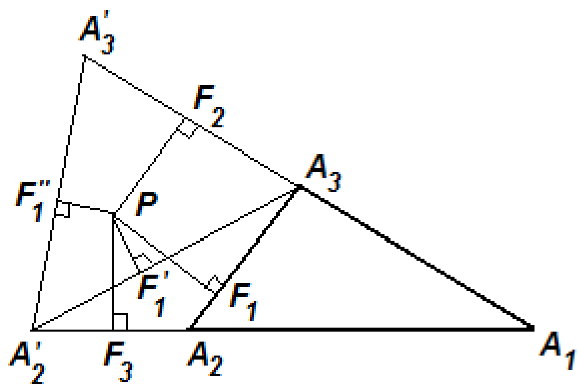

Figure 3.22

Then, we have $PA_3 = PA_3'$, and so the distances from P to the vertices of $\triangle A_1 A_2' A_3'$ are the same as the distances from P to the vertices of $\triangle A_1 A_2' A_3$.

Additionally, we note that when considering $\triangle A_1 A_2' A_3'$ compared to $\triangle A_1 A_2' A_3$, we have d_2 and d_3 remaining unchanged, but d_1' changes to $d_1'' = PF_1''$.

Akin to the earlier reasoning, $p_1' < p_1''$.

Thus, we get $p_1'' + p_2 + p_3 > p_1' + p_2 + p_3 > p_1 + p_2 + p_3$.

So, we have $p_1'' + p_2 + p_3 > p_1 + p_2 + p_3$, which means that if the Signed Erdös-Mordell result

$$PA_1 + PA_2 + PA_3 \geq 2(p_1 + p_2 + p_3)$$

holds in the newly constructed triangle whose feet of the perpendiculars from P are all on the sides of the triangle, it must hold for the original triangle.

Therefore, it suffices to reduce Case 2 to the situation where all of the feet of the perpendiculars from P to the sides of $\triangle A_1 A_2 A_3$ lie on $\triangle A_1 A_2 A_3$.

This is how we conclude Case 2.A.

A further consequence of the idea from Case 2.A is the idea that our point *P* must fall within the circumscribed circle of $\triangle A_1 A_2 A_3$.

To see this, consider Figure 3.23 below, where we have constructed *D* to be such that $\overline{A_1 D}$ is a diameter of the circumscribed circle.

Since we are only working with *P* outside the triangle, and within Case 2, we have assumed (without loss of generality) that *P* is interior to $\angle A_2 A_1 A_3$, it follows that we only need to consider *P* possibly being in the shaded region (that inside $\triangle D A_2 A_3$).

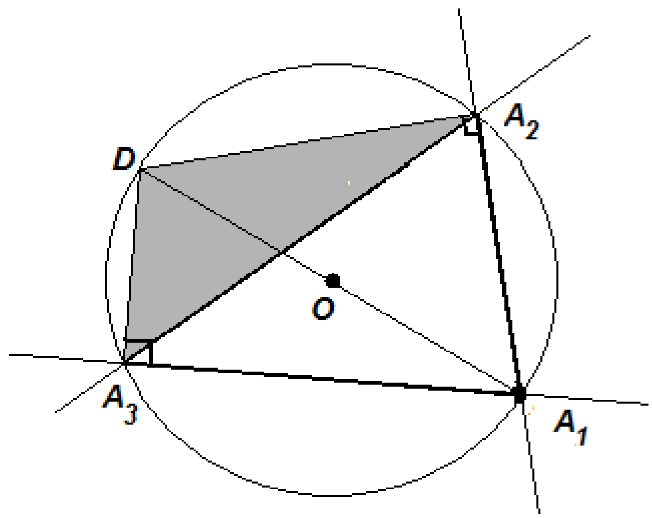

Figure 3.23

Given the constraints that *P* is outside $\triangle A_1 A_2 A_3$, interior to $\angle A_2 A_1 A_3$, and is such that the feet of the perpendiculars from *P* to the sides of $\triangle A_1 A_2 A_3$ are assumed to be on the triangle itself, these are the only possibilities for *P*.

Next, we seek to establish the validity of the results (*3.6.A*) – (*3.6.D*) for points in this region.

We let *P* be in the region specified, within $\triangle D A_2 A_3$.

We notice $p_1 < 0$, $p_2 > 0$, and $p_3 > 0$.

We consider reflecting $\triangle A_1 A_2 A_3$ over the bisector of $\angle A_2 A_1 A_3$ to obtain $\triangle A_1 \tilde{A}_2 \tilde{A}_3$ and parallelograms $A_1 \tilde{A}_3 X P$, $A_1 \tilde{A}_2 Y P$, and $\tilde{A}_2 \tilde{A}_3 X Y$ as in the Proof of Theorem 3.2. This is shown in Figure 3.24.

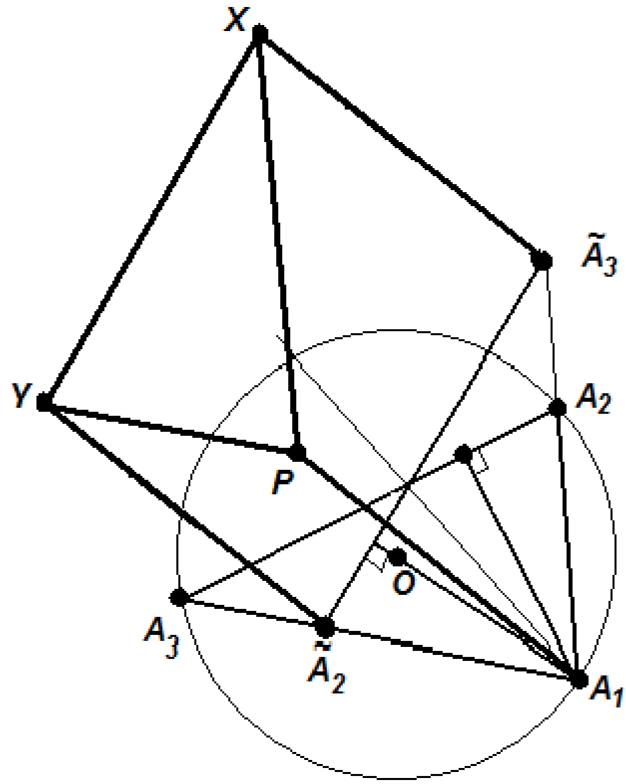

Figure 3.24

Here, we see that since P is interior to $\angle A_2 A_1 A_3$ and exterior to both $\angle A_1 A_2 A_3$ and $\angle A_1 A_3 A_2$, it follows that neither parallelogram $A_1 \tilde{A}_3 X P$ nor $A_1 \tilde{A}_2 Y P$ lies completely outside $\triangle A_1 \tilde{A}_2 \tilde{A}_3$. So Pappus's Theorem applies (akin to the Proof of Theorem 3.2).

We will focus on the following properties of these parallelograms:

$A_1 \tilde{A}_3 X P$	$A_1 \tilde{A}_2 Y P$	$\tilde{A}_2 \tilde{A}_3 X Y$
Base: $A_1 \tilde{A}_3 = A_1 A_3 = a_2$	Base: $A_1 \tilde{A}_2 = A_1 A_2 = a_3$	Base: $\tilde{A}_2 \tilde{A}_3 = A_2 A_3 = a_1$
Height: d_3	Height: d_2	Height: $h = PA_1 \cdot \cos \angle P A_1 O$

Note: $\cos \angle P A_1 O > 0$ since $\overline{O A_1}$ is a radius of a circle, and we know that if P is interior to the circle, this angle must be acute. (It becomes a right angle if $\overline{P A_1}$ were tangent to the circle – which isn't the case, and therefore, it could only be obtuse if P were exterior to the circle – which isn't the case.)

So, by Theorem 2.1 (Pappus's Theorem),

$$\text{Area } \tilde{A}_2 \tilde{A}_3 X Y = \text{Area } A_1 \tilde{A}_3 X P + \text{Area } A_1 \tilde{A}_2 Y P .$$

Or, when substituting the appropriate bases and heights listed earlier:

$$a_1 \cdot PA_1 \cos(\angle P A_1 O) = a_2 d_3 + a_3 d_2 .$$

Since $p_2 > 0$, and $p_3 > 0$, this gives

$$a_1 \cdot PA_1 \cos(\angle P A_1 O) = a_2 p_3 + a_3 p_2 .$$

Within the Proof of Theorem 3.2, we similarly obtained

$$a_2 \cdot PA_2 \cos(\angle P A_2 O) = a_1 d_3 + a_3 d_1$$

$$a_3 \cdot PA_3 \cos(\angle P A_3 O) = a_1 d_2 + a_2 d_1 ,$$

since *P* was interior to each of the angles. However, that is no longer the case.

Now, we consider reflecting $\triangle A_1 A_2 A_3$ over the bisector of $\angle A_1 A_2 A_3$ to obtain $\triangle \tilde{A}_1 A_2 \tilde{A}_3$, following the same notation and process as before. This is shown in Figure 3.25.

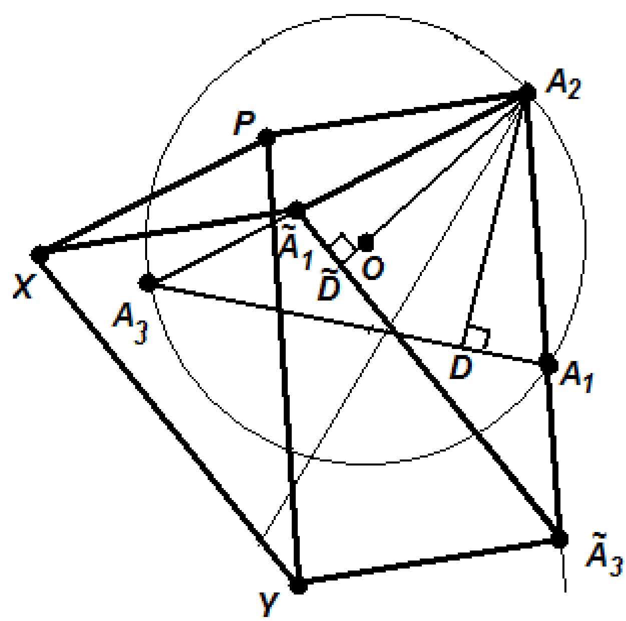

Figure 3.25

Here, we see that since *P* is interior to $\angle A_2 A_1 A_3$ and exterior to both $\angle A_1 A_2 A_3$ and $\angle A_1 A_3 A_2$, it follows that both parallelograms $A_2 \tilde{A}_1 X P$ and $\tilde{A}_1 \tilde{A}_3 Y X$ lie completely outside $\triangle \tilde{A}_1 A_2 \tilde{A}_3$. So Pappus's Theorem applies.

We will focus on the following properties of these parallelograms:

$A_2\tilde{A}_1XP$	$A_2\tilde{A}_3YP$	$\tilde{A}_1\tilde{A}_3YX$
Base: $A_2\tilde{A}_1=A_2A_1=a_3$	Base: $A_2\tilde{A}_3=A_2A_3=a_1$	Base: $\tilde{A}_1\tilde{A}_3=A_1A_3=a_2$
Height: d_1	Height: d_3	Height: $h=PA_2\cdot\cos\angle PA_2O$

Note: $\cos\angle PA_2O>0$ since $\overline{OA_2}$ is a radius of a circle, and we know that if *P* is interior to the circle, this angle must be acute. (It becomes a right angle if $\overline{PA_2}$ were tangent to the circle – which isn't the case, and therefore, it could only be obtuse if *P* were exterior to the circle – which isn't the case.)

So, by Theorem 2.1 (Pappus's Theorem), with $A_2\tilde{A}_1XP$ and $\tilde{A}_1\tilde{A}_3YX$ completely outside the triangle, we have:

$$\text{Area } A_2\tilde{A}_3YP=\text{Area } A_2\tilde{A}_1XP+\text{Area } \tilde{A}_1\tilde{A}_3YX$$

Or, when substituting the appropriate bases and heights listed earlier:

$$a_1d_3=a_3d_1+a_2\cdot PA_2\cos(\angle PA_2O),$$

or equivalently

$$a_2\cdot PA_2\cos(\angle PA_2O)=a_1d_3-a_3d_1.$$

Since $p_1<0$, and $p_3>0$, this gives

$$a_2\cdot PA_2\cos(\angle PA_2O)=a_1p_3+a_3p_1,$$

as desired.

When reflecting $\triangle A_1A_2A_3$ over the bisector of $\angle A_1A_3A_2$ to obtain $\triangle\tilde{A}_1\tilde{A}_2A_3$, we will have the same process as seen most recently, as *P* was interior to $\angle A_2A_1A_3$ and exterior to both $\angle A_1A_2A_3$ and $\angle A_1A_3A_2$.

So, similarly,

$$a_3\cdot PA_3\cos(\angle PA_3O)=a_1p_2+a_2p_1.$$

Thus, Kazarinoff's formulas (*3.6.A) – (3.6.D*) hold for points in the region under consideration, namely those inside of $\triangle DA_2A_3$.

Recalling that within this case, we are concerned with P being in the region formed by $\triangle DA_2A_3$ only, we may assume $m\angle DA_2A_3 < 90^o$ and $m\angle DA_3A_2 < 90^o$ (otherwise the shaded region in Figure 3.26 is empty).

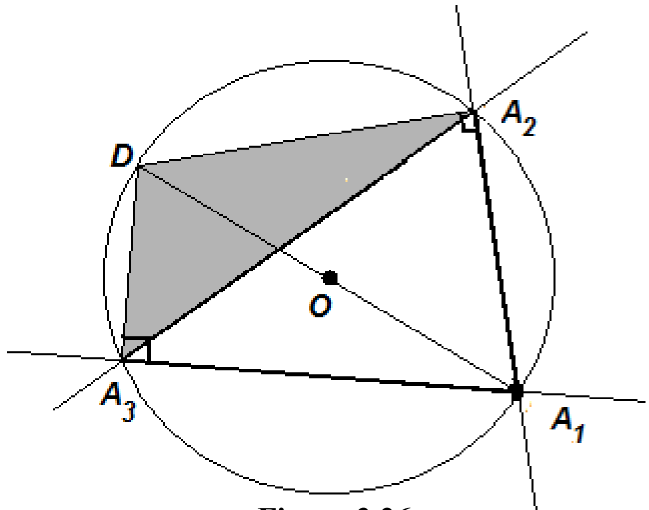

Figure 3.26

We have $p_1 < 0$, $p_2 > 0$, and $p_3 > 0$ in this region, so that we wish to show:

$$PA_1 + PA_2 + PA_3 \geq -2d_1 + 2d_2 + 2d_3 .$$

Case 2.B: P lies in $\triangle D A_2 A_3$ and at least one of $m\angle A_1 A_2 A_3$ or $m\angle A_1 A_3 A_2$ does not exceed 30º.

Without loss of generality, assume $m\angle A_1 A_2 A_3 \leq 30^{\circ}$.

Let $\alpha = m\angle A_1 A_2 A_3$ and $\epsilon = m\angle P A_2 A_3$, as noted in Figure 3.27.

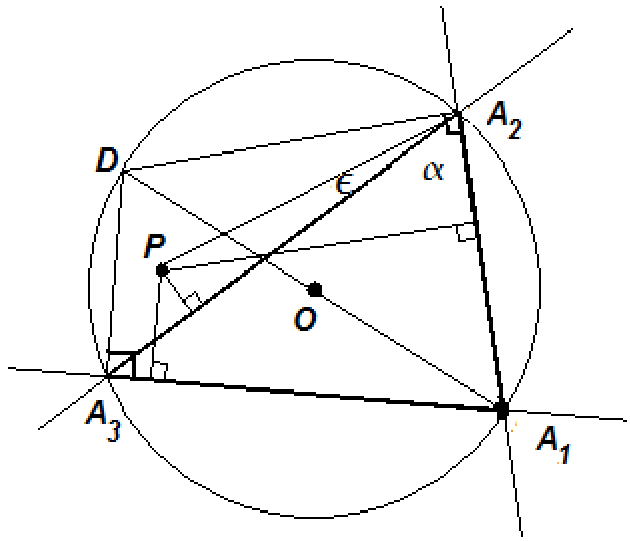

Figure 3.27

Then, since $\alpha \leq 30^{\circ}$, we know

$$\sin(\alpha) \leq \frac{1}{2} .$$

Additionally, based off Figure 3.27,

$$\sin(\alpha+\epsilon) = \frac{d_3}{PA_2} \quad \text{so that} \quad d_3 = (PA_2)\sin(\alpha+\epsilon)$$

and

$$\sin(\epsilon) = \frac{d_1}{PA_2} \quad \text{so that} \quad d_1 = (PA_2)\sin(\epsilon) .$$

We notice

$$\begin{aligned} \sin(\alpha+\epsilon) - \sin(\epsilon) &= \sin(\alpha)\cos(\epsilon) + \cos(\alpha)\sin(\epsilon) - \sin(\epsilon) \\ &= \sin(\alpha)\cos(\epsilon) + \sin(\epsilon)[\cos(\alpha) - 1] \end{aligned}$$

Since $\cos(\alpha) - 1 \leq 0$,

$$\leq \quad \sin(\alpha)\cos(\epsilon)$$

Since $\cos(\epsilon) \leq 1$

$$\leq \quad \sin(\alpha)$$

$$\leq \quad \frac{1}{2}.$$

So we have $\sin(\alpha+\epsilon) - \sin(\epsilon) \leq \frac{1}{2}$, which means $2\sin(\alpha+\epsilon) - 2\sin(\epsilon) \leq 1$.

Using this combined with $d_1 = (PA_2)\sin(\epsilon)$ and $d_3 = (PA_2)\sin(\alpha+\epsilon)$, we get

$$\begin{aligned} PA_2 &\geq PA_2[2\sin(\alpha+\epsilon) - 2\sin(\epsilon)] \\ &= 2PA_2\sin(\alpha+\epsilon) - 2PA_2\sin(\epsilon) \\ &= 2d_3 - 2d_1. \end{aligned}$$

Letting F_2 be the foot of the perpendicular from P to $\overline{A_1 A_3}$ as in Figure 3.28, we notice the following:

since $\overline{PA_3}$ and d_2 are the hypotenuse and leg, respectively, of $\triangle PA_3F_2$, we have

$$PA_3 \geq d_2$$

and since $\overline{PA_1}$ and d_2 are the hypotenuse and leg, respectively, of $\triangle PA_1F_2$,

$$PA_1 \geq d_2.$$

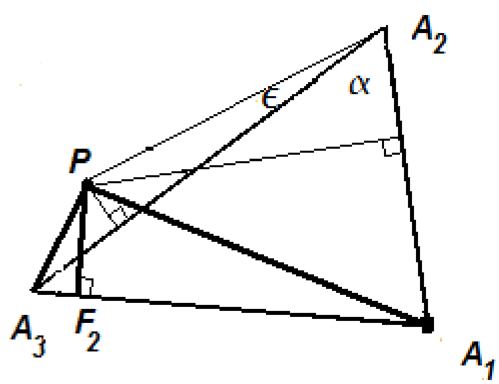

Figure 3.28

Putting everything together, we have (since $p_1<0$, $p_2>0$, and $p_3>0$)

$$\begin{aligned} PA_1+PA_2+PA_3 &\geq d_2+(2d_3-2d_1)+d_2 \\ &= -2d_1+2d_2+2d_3 \\ &= 2(p_1+p_2+p_3). \end{aligned}$$

Thus, we have $PA_1+PA_2+PA_3\geq 2(p_1+p_2+p_3)$, and Case 2.B holds.

Case 2.C: *P* lies in $\triangle DA_2A_3$ and $\angle A_2A_1A_3$ is the largest interior angle of $\triangle A_1A_2A_3$.

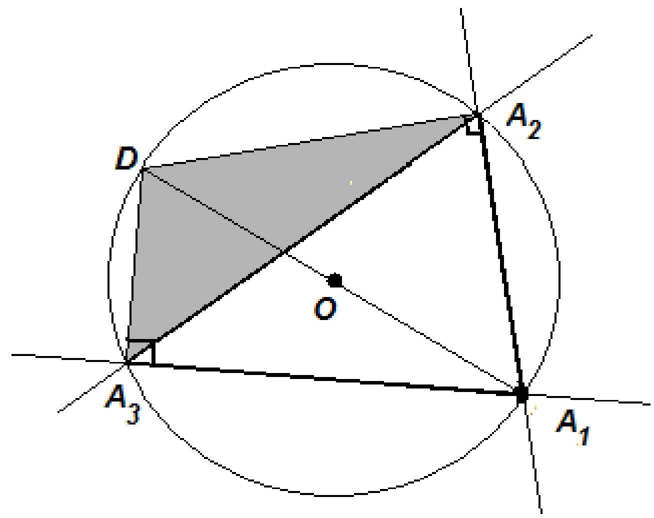

Figure 3.29

Without loss of generality, assume

$$m\angle A_2A_1A_3 \geq m\angle A_1A_2A_3 \geq m\angle A_1A_3A_2 .$$

This means we have $a_1 \geq a_2 \geq a_3$ and $d_1 \leq d_2$ (the former because of the assumption on the angles, the latter because of the location of *P* guarantees the distance from P to $\overline{A_2A_3}$ to be smaller than the distance from *P* to the extension of $\overline{A_1A_3}$).

Notice $a_1 \geq a_2 \geq a_3 > 0$ means $(a_1 - a_2)a_3^2 \leq a_1a_2(a_1 - a_2)$, so that

$$\frac{a_1a_2^2 + a_1a_3^2}{a_1a_2a_3} \leq \frac{a_2a_3^2 + a_1^2a_2}{a_1a_2a_3} .$$

Simplifying, we get

$$\frac{a_2}{a_3} + \frac{a_3}{a_2} \leq \frac{a_3}{a_1} + \frac{a_1}{a_3} .$$

Recalling the Arithmetic Mean – Geometric Mean Inequality, we have

$$2 \leq \frac{a_2}{a_3} + \frac{a_3}{a_2} \leq \frac{a_3}{a_1} + \frac{a_1}{a_3} .$$

Set $W=\frac{a_2}{a_3}+\frac{a_3}{a_2}\geq 2$ and $V=\frac{a_1}{a_3}+\frac{a_3}{a_1}\geq 2$.

For any number N such that $W d_1 \geq V d_2+N$, since $W\leq V$ we have

$$W d_1 \geq W d_2+N.$$

Realizing $W-2\geq 0$ and $d_1\leq d_2$, we also have

$$(W-2)d_1\leq (W-2)d_2.$$

Subtracting these inequalities yields

$$2d_1\geq 2d_2+N.$$

Thus, for any N making $\left(\frac{a_2}{a_3}+\frac{a_3}{a_2}\right)d_1\geq\left(\frac{a_1}{a_3}+\frac{a_3}{a_1}\right)d_2+N$, we have $2d_1\geq 2d_2+N$.

Now, by (*3.6.A – 3.6.D*), with $p_1<0$, $p_2>0$, and $p_3>0$, we have

$$\begin{aligned} PA_1+PA_2+PA_3 &\geq \left(\frac{a_2}{a_3}+\frac{a_3}{a_2}\right)p_1+\left(\frac{a_1}{a_3}+\frac{a_3}{a_1}\right)p_2+\left(\frac{a_1}{a_2}+\frac{a_2}{a_1}\right)p_3 \\ &= -\left(\frac{a_2}{a_3}+\frac{a_3}{a_2}\right)d_1+\left(\frac{a_1}{a_3}+\frac{a_3}{a_1}\right)d_2+\left(\frac{a_1}{a_2}+\frac{a_2}{a_1}\right)d_3 \end{aligned}$$

so that

$$\left(\frac{a_2}{a_3}+\frac{a_3}{a_2}\right)d_1\geq\left(\frac{a_1}{a_3}+\frac{a_3}{a_1}\right)d_2+\left[\left(\frac{a_1}{a_2}+\frac{a_2}{a_1}\right)d_3-PA_1-PA_2-PA_3\right].$$

By our most recent result, we must have

$$2d_1\geq 2d_2+\left[\left(\frac{a_1}{a_2}+\frac{a_2}{a_1}\right)d_3-PA_1-PA_2-PA_3\right],$$

or equivalently

$$PA_1+PA_2+PA_3\geq -2d_1+2d_2+\left(\frac{a_1}{a_2}+\frac{a_2}{a_1}\right)d_3.$$

But we can use the Arithmetic Mean – Geometric Mean Inequality on the last term to get

$$\left(\frac{a_1}{a_2}+\frac{a_2}{a_1}\right)d_3 \geq 2\sqrt{\frac{a_1}{a_2}\cdot\frac{a_2}{a_1}}\,d_3 = 2\,d_3\,,$$

so that

$$PA_1+PA_2+PA_3 \geq -2\,d_1+2\,d_2+2\,d_3\,.$$

Recall, we have $p_1<0$, $p_2>0$, and $p_3>0$ in this region, so that this gives us

$$PA_1+PA_2+PA_3 \geq 2\,p_1+2\,p_2+2\,p_3\,,$$

which is our desired inequality

$$PA_1+PA_2+PA_3 \geq 2(p_1+p_2+p_3)\,.$$

Thus, Case 2.C holds.

Before handling our last sub-case for Case 2, we consider two lemmas.

Lemma 3.6.1.

The function $f(x)=1-\cos(x)-\frac{1}{2}\sin(x-15^\circ)$ is positive on [15º, 90º].

Proof of Lemma 3.6.1.

Calculating the derivative of *f*, we have

$$f'(x)=\sin(x)-\frac{1}{2}\cos(x-15^\circ).$$

Finding critical points (setting $f'(x)=0$), we have

$$\begin{aligned} 2\sin(x) &= \cos(x-15^\circ) \\ &= \cos(x)\cos(15^\circ)+\sin(x)\sin(15^\circ). \end{aligned}$$

Thus $2\sin(x)=\cos(x)\cos(15^\circ)+\sin(x)\sin(15^\circ)$. Rearranging gives

$$2\sin(x)-\sin(x)\sin(15^\circ)=\cos(x)\cos(15^\circ)$$

which becomes

$$\sin(x)[2-\sin(15^\circ)]=\cos(x)\cos(15^\circ)$$

or

$$\tan(x)=\frac{\cos(15^\circ)}{2-\sin(15^\circ)}.$$

Based on properties of the tangent function, this only has one solution on [15º, 90º]. Namely, it is

$$x=\arctan\left[\frac{\cos(15^\circ)}{2-\sin(15^\circ)}\right]\approx 29.0194659^\circ.$$

Thus, to find the absolute minimum of *f* on [15º, 90º], we evaluate the function at the endpoints of the interval and its critical point:

$$f(15^\circ) = 1 - \cos(15^\circ) - \frac{1}{2}\sin(0^\circ) \approx 0.304074$$

$$f\left(\arctan\left[\cos\frac{(15^\circ)}{2 - \sin(15^\circ)}\right]\right) \approx 0.0044192876$$

$$f(90^\circ) = 1 - \cos(90^\circ) - \frac{1}{2}\sin(75^\circ) \approx 0.517037 \; .$$

It follows that *f* achieves its absolute minimum on [15º, 90º] at the point with approximate coordinates (29.019º, 0.0044192876).

Therefore, Lemma 3.6.1 holds as *f(x)* > 0 on this interval.

∎

Lemma 3.6.2.

Let $g(x) = x + \frac{1}{x}$. Then $g(x) \leq 2.5$ on the interval [1, 2].

Proof of Lemma 3.6.2.

We notice $g'(x) = 1 - \frac{1}{x^2} > 0$ for *x* on the interval [1, 2], so *g* is increasing on [1, 2].

It follows that *g* achieves its maximum at *x = 2*:

$$g(2) = 2 + \frac{1}{2} = 2.5 \; ,$$

and Lemma 3.6.2 holds.

∎

To finish off Case 2, we first realize that we have already considered the following:

Case 2.B: P lies in $\triangle DA_2A_3$ and at least one of $m\angle A_1A_2A_3$ or $m\angle A_1A_3A_2$ does not exceed 30º.

Case 2.C: P lies in $\triangle DA_2A_3$ and $\angle A_2A_1A_3$ is the largest interior angle of $\triangle A_1A_2A_3$.

Our last sub-case for Case 2 will be the following: P lies in $\triangle DA_2A_3$ where both $m\angle A_1A_2A_3 > 30^o$ and $m\angle A_1A_3A_2 > 30^o$, and $\angle A_2A_1A_3$ is not the largest interior angle of $\triangle A_1A_2A_3$.

Without loss of generality, we will assume $\angle A_1A_2A_3$ is the largest interior angle of $\triangle A_1A_2A_3$. Additionally, from Figure 3.30, we can tell that $m\angle A_1A_2A_3 < 90^o$.

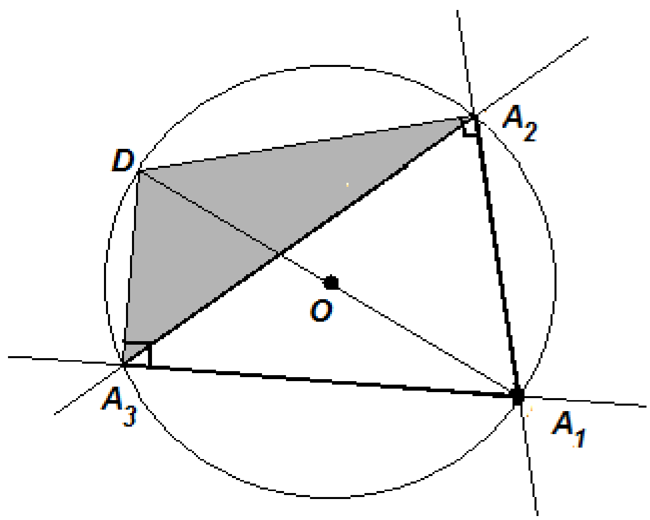

Figure 3.30

Case 2.D: *P* lies in $\triangle DA_2A_3$ where $30^o < m\angle A_1A_3A_2 \le m\angle A_1A_2A_3$ and $m\angle A_2A_1A_3 \le m\angle A_1A_2A_3 < 90^o$.

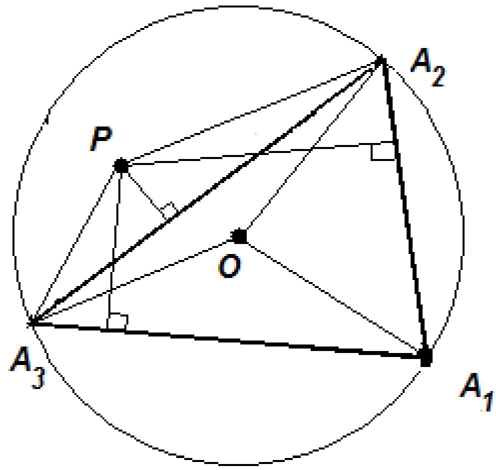

Figure 3.31

Since $30^o < m\angle A_1A_3A_2$, we know $m\angle A_2A_1A_3 + m\angle A_1A_2A_3 < 150^o$. With the requirement that $\angle A_1A_2A_3$ is the largest interior angle of $\triangle A_1A_2A_3$, that means we must have

$$m\angle A_2A_1A_3 < 75^o .$$

$\angle A_2A_1A_3$ is an inscribed angle in the circumscribed circle of $\triangle A_1A_2A_3$ having the corresponding central angle $\angle A_2OA_3$, so that

$$m\angle A_2OA_3 < 150^o .$$

Realizing that $\triangle A_2OA_3$ is isosceles with base angles summing to at least 30º, we have

$$m\angle OA_3A_2 > 15^o .$$

The Law of Sines gives

$$\frac{a_2}{\sin(\angle A_1A_2A_3)} = \frac{a_3}{\sin(\angle A_1A_3A_2)} ,$$

so that

$$\frac{a_2}{a_3} = \frac{\sin(\angle A_1A_2A_3)}{\sin(\angle A_1A_3A_2)} .$$

Coupling this with $30^{o} < m\angle A_1A_3A_2 \le m\angle A_1A_2A_3 < 90^{o}$, $a_2 \ge a_3$, and the notion that the sine function is increasing on [0º, 90º], we have:

$$1 \le \frac{a_2}{a_3} = \frac{\sin(\angle A_1A_2A_3)}{\sin(\angle A_1A_3A_2)} < \frac{\sin(90^{o})}{\sin(30^{o})} = \frac{1}{1/2} = 2 .$$

This means

$$1 \le \frac{a_2}{a_3} < 2 .$$

Applying Lemma 3.6.2, we have

$$\frac{a_2}{a_3} + \frac{a_3}{a_2} = g\left(\frac{a_2}{a_3}\right) \le 2.5 . \qquad (3.6.E)$$

Now, from Figure 3.32, we have

$$\sin(\angle PA_3A_2) = \frac{d_1}{PA_3} \quad \text{so that} \quad PA_3 \sin(\angle PA_3A_2) = d_1 .$$

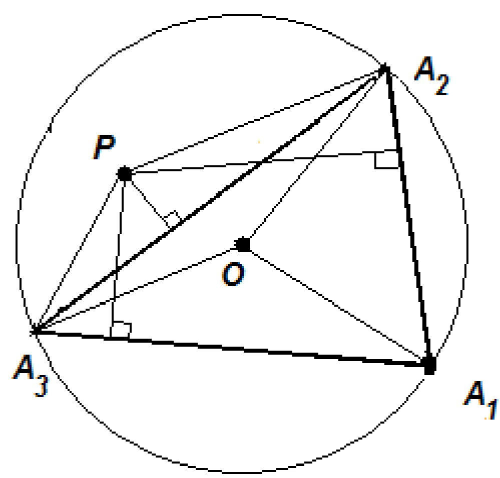

Figure 3.32

But

$$m\angle PA_3A_2 \quad = \quad m\angle PA_3O - m\angle OA_3A_2$$

Recalling $m\angle OA_3A_2 > 15^{o}$ gives

$$< \quad m\angle PA_3O - 15^{o} .$$

Let $\delta = m\angle PA_3O$. Then we have $m\angle PA_3A_2 < \delta - 15^\circ$.

Combining with $PA_3 \sin(\angle PA_3A_2) = d_1$, we have

$$d_1 < PA_3 \sin(\delta - 15^\circ).$$

Lemma 3.6.1 tells us (since $\delta = m\angle PA_3O > 15^\circ$)

$$1 - \cos(\delta) - \frac{1}{2}\sin(\delta - 15^\circ) > 0.$$

Equivalently,

$$PA_3(1 - \cos(\delta)) > \frac{1}{2} PA_3 \sin(\delta - 15^\circ) > \frac{1}{2} d_1.$$

Thus, we have

$$PA_3(1 - \cos(\delta)) > \frac{1}{2} d_1.$$

Now,

$PA_1 + PA_2 + PA_3 + 2d_1$

$$= \quad PA_1 + PA_2 + PA_3\cos(\delta) + PA_3(1 - \cos(\delta)) + 2d_1$$

$$> \quad PA_1 + PA_2 + PA_3\cos(\delta) + \frac{1}{2}d_1 + 2d_1$$

$$= \quad PA_1 + PA_2 + PA_3\cos(\delta) + 2.5d_1$$

$$\geq \quad PA_1\cos(\angle PA_1O) + PA_2\cos(\angle PA_2O) + PA_3\cos(\delta) + 2.5d_1$$

Since $\delta = m\angle PA_3O$

$$= \quad PA_1\cos(\angle PA_1O) + PA_2\cos(\angle PA_2O) + PA_3\cos(\angle PA_3O) + 2.5d_1$$

By *(3.6.A) – (3.6.D)*

$$= \quad -\left(\frac{a_2}{a_3} + \frac{a_3}{a_2}\right)d_1 + \left(\frac{a_1}{a_3} + \frac{a_3}{a_1}\right)d_2 + \left(\frac{a_1}{a_2} + \frac{a_2}{a_1}\right)d_3 + 2.5d_1$$

By (*3.6.E*)

$$> \quad -\left(\frac{a_2}{a_3}+\frac{a_3}{a_2}\right)d_1+\left(\frac{a_1}{a_3}+\frac{a_3}{a_1}\right)d_2+\left(\frac{a_1}{a_2}+\frac{a_2}{a_1}\right)d_3+\left(\frac{a_2}{a_3}+\frac{a_3}{a_2}\right)d_1$$

$$= \quad \left(\frac{a_1}{a_3}+\frac{a_3}{a_1}\right)d_2+\left(\frac{a_1}{a_2}+\frac{a_2}{a_1}\right)d_3$$

By the Arithmetic Mean – Geometric Mean Inequality

$$\geq \quad 2\sqrt{\frac{a_1}{a_3}\cdot\frac{a_3}{a_1}}d_2+2\sqrt{\frac{a_1}{a_2}\cdot\frac{a_2}{a_1}}d_3$$

$$= \quad 2d_2+2d_3 .$$

Overall, this means we must have

$$PA_1+PA_2+PA_3+2d_1>2d_2+2d_3 ,$$

or equivalently,

$$PA_1+PA_2+PA_3>-2d_1+2d_2+2d_3 .$$

Recalling that we have $p_1<0$, $p_2>0$, and $p_3>0$ in this region, this gives us

$$PA_1+PA_2+PA_3>2(p_1+p_2+p_3) ,$$

which certainly requires

$$PA_1+PA_2+PA_3\geq 2(p_1+p_2+p_3)$$

as desired.

So Case 2.D holds.

By the combined results of Case 2.A, Case 2.B, Case 2.C, and Case 2.D, we conclude that the Signed Erdös-Mordell Inequality holds in Case 2 overall.

Case 3: P is outside $\triangle A_1A_2A_3$ and P is on the extension of one of the sides of $\triangle A_1A_2A_3$.

This case parallels that of Case 2 of the Proof of Theorem 3.2.

Without loss of generality, assume P is on the extension of $\overline{A_2A_3}$. Draw circle m centered at P. Let ϵ_1 be the radius of this circle. Let $P_1 \in m$ such that P_1 is not on the extension of $\overline{A_2A_3}$, as shown in Figure 3.33.

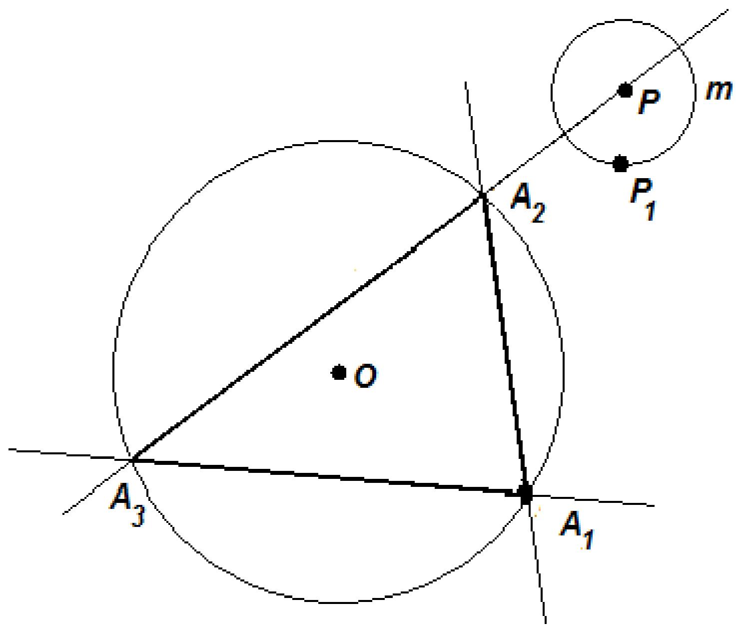

Figure 3.33

Let $\{\epsilon_n\}_{n=1}^{\infty}$ be such that ϵ_1 is as defined above and $\epsilon_{n+1} < \epsilon_n$ for all n. For each n, define P_n such that P_n is on the circle centered at P with radius ϵ_n but is not on the extension of $\overline{A_2A_3}$.

By earlier considerations in Case 2, the Signed Erdös-Mordell Inequality holds for each P_n, namely:

$$P_nA_1 + P_nA_2 + P_nA_3 \geq 2(p_{n,1} + p_{n,2} + p_{n,3}) .$$

From this, as $n \to \infty$, $\epsilon_n \to 0$, and $P_n \to P$, so the inequality will also hold for P, namely

$$PA_1 + PA_2 + PA_3 \geq 2(p_1 + p_2 + p_3) .$$

By Cases, we have concluded the Proof of Theorem 3.6. We have seen the following:

Theorem 3.2 handles the case where P is interior to $\triangle A_1A_2A_3$ or on its boundary.

Then, within this proof, for P outside $\triangle A_1A_2A_3$, we handled exhaustive cases:

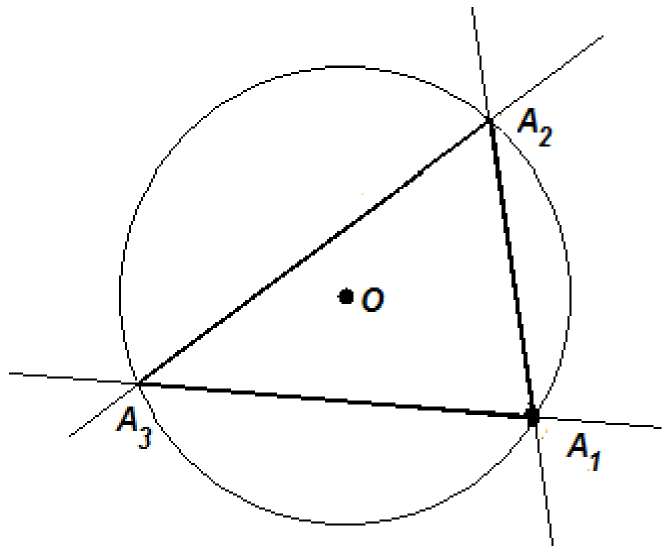

Figure 3.34

Case 1: P lies inside an angle vertical to one of the interior angles of $\triangle A_1A_2A_3$;

Case 2: P is interior to only one of the interior angles of $\triangle A_1A_2A_3$; or

Case 3: P is on the extension of one of the sides of $\triangle A_1A_2A_3$.

In each of these situations, the Signed Erdös-Mordell Inequality holds, so overall the Signed Erdös-Mordell Inequality holds, and Theorem 3.6 is proven.

∎

4 Twists on the Erdös-Mordell Inequality

This section involves new inequalities we get from slight changes to the inequality proposed by Erdös.

The question of whether weighting each of the sides would influence the inequality provides the motivation for the next result. Originally stated and proven by Seannie Dar and Shay Gueron in *The American Mathematical Monthly* **[DAR]**, this theorem is proven differently here; in this paper, we offer our own proof based off common ideas.

Theorem 4.1. Dar-Gueron Theorem. [DAR]

Let $\triangle A_1 A_2 A_3$ be given, let *P* be an interior point of the triangle, let p_i denote the distance from *P* to the side of $\triangle A_1 A_2 A_3$ opposite vertex A_i for each $1 \leq i \leq 3$, let a_i denote the length of the side of $\triangle A_1 A_2 A_3$ across from A_i for each $1 \leq i \leq 3$, and let $\lambda_1, \lambda_2, \lambda_3 > 0$. Then

$$\lambda_1 PA_1 + \lambda_2 PA_2 + \lambda_3 PA_3 \geq 2\left(\sqrt{\lambda_2 \lambda_3}\, p_1 + \sqrt{\lambda_1 \lambda_3}\, p_2 + \sqrt{\lambda_1 \lambda_2}\, p_3\right).$$

Equality holds if and only if $a_1 : a_2 : a_3 = \sqrt{\lambda_1} : \sqrt{\lambda_2} : \sqrt{\lambda_3}$ and *P* is the circumcenter of $\triangle A_1 A_2 A_3$.

Proof of Theorem 4.1.

We realize this is the same setup as we had for the Erdös-Mordell Inequality. Given this setup, we realize the inequalities paramount to proving the Erdös-Mordell Inequality apply, as stated in Corollary 3.5, namely:

$$PA_1 \geq \frac{a_2 p_3}{a_1} + \frac{a_3 p_2}{a_1}, \qquad PA_2 \geq \frac{a_1 p_3}{a_2} + \frac{a_3 p_1}{a_2}, \text{ and } \qquad PA_3 \geq \frac{a_1 p_2}{a_3} + \frac{a_2 p_1}{a_3}.$$

So that we have:

$$\lambda_1 PA_1 + \lambda_2 PA_2 + \lambda_3 PA_3$$

$$\geq \quad \lambda_1\left(\frac{a_2 p_3}{a_1} + \frac{a_3 p_2}{a_1}\right) + \lambda_2\left(\frac{a_1 p_3}{a_2} + \frac{a_3 p_1}{a_2}\right) + \lambda_3\left(\frac{a_1 p_2}{a_3} + \frac{a_2 p_1}{a_3}\right)$$

By regrouping terms:

$$= \quad \left(\frac{a_3}{a_2}\lambda_2 + \frac{a_2}{a_3}\lambda_3\right) p_1 + \left(\frac{a_3}{a_1}\lambda_1 + \frac{a_1}{a_3}\lambda_3\right) p_2 + \left(\frac{a_2}{a_1}\lambda_1 + \frac{a_1}{a_2}\lambda_2\right) p_3$$

By application of the Arithmetic Mean – Geometric Mean Inequality

$$\geq \qquad 2\sqrt{\frac{a_3}{a_2}\lambda_2 \cdot \frac{a_2}{a_3}\lambda_3}\, p_1 + 2\sqrt{\frac{a_3}{a_1}\lambda_1 \cdot \frac{a_1}{a_3}\lambda_3}\, p_2 + 2\sqrt{\frac{a_2}{a_1}\lambda_1 \cdot \frac{a_1}{a_2}\lambda_2}\, p_3$$

$$= \qquad 2\sqrt{\lambda_2\lambda_3}\, p_1 + 2\sqrt{\lambda_1\lambda_3}\, p_2 + 2\sqrt{\lambda_1\lambda_2}\, p_3 \,.$$

Thus, we have established

$$\lambda_1 PA_1 + \lambda_2 PA_2 + \lambda_3 PA_3 \geq 2\left(\sqrt{\lambda_2\lambda_3}\, p_1 + \sqrt{\lambda_1\lambda_3}\, p_2 + \sqrt{\lambda_1\lambda_2}\, p_3\right),$$

the desired result.

Additionally, based on the application of the Arithmetic Mean – Geometric Mean Inequality, equality holds if and only if

$$\frac{a_3}{a_2}\lambda_2 = \frac{a_2}{a_3}\lambda_3\,, \quad \frac{a_3}{a_1}\lambda_1 = \frac{a_1}{a_3}\lambda_3\,, \text{ and } \frac{a_2}{a_1}\lambda_1 = \frac{a_1}{a_2}\lambda_2\,,$$

which is equivalent to saying

$$\frac{a_3^2}{a_2^2} = \frac{\lambda_3}{\lambda_2}\,, \quad \frac{a_3^2}{a_1^2} = \frac{\lambda_3}{\lambda_1}\,, \text{ and } \frac{a_2^2}{a_1^2} = \frac{\lambda_2}{\lambda_1}\,,$$

which implies

$$\frac{a_3}{a_2} = \frac{\sqrt{\lambda_3}}{\sqrt{\lambda_2}}\,, \quad \frac{a_3}{a_1} = \frac{\sqrt{\lambda_3}}{\sqrt{\lambda_1}}\,, \text{ and } \frac{a_2}{a_1} = \frac{\sqrt{\lambda_2}}{\sqrt{\lambda_1}}\,,$$

or equivalently $a_1 : a_2 : a_3 = \sqrt{\lambda_1} : \sqrt{\lambda_2} : \sqrt{\lambda_3}$.

Also, by the application of the inequalities essential to proving the Erdös-Mordell Inequality, we know that *P* must be the circumcenter of $\triangle A_1 A_2 A_3$ for equality to hold. Thus, the criteria for equality are established, and Theorem 4.1 holds.

∎

After investigating Erdös's conjectured inequality, we might wonder if the inequality would apply when considering other aspects of the triangle. This brings us to Barrow's Inequality:

Theorem 4.2. Barrow's Inequality. [EMB and LEE]

Given $\triangle A_1 A_2 A_3$ and interior point *P* of $\triangle A_1 A_2 A_3$. Let W_i be the point on the side of $\triangle A_1 A_2 A_3$ opposite A_i such that $\overline{PW_i}$ bisects the angle whose vertex is at *P* and whose sides are formed by the two vertices of $\triangle A_1 A_2 A_3$ other than A_i. Further, let $w_i = PW_i$. Then

$$PA_1 + PA_2 + PA_3 \geq 2(w_1 + w_2 + w_3).$$

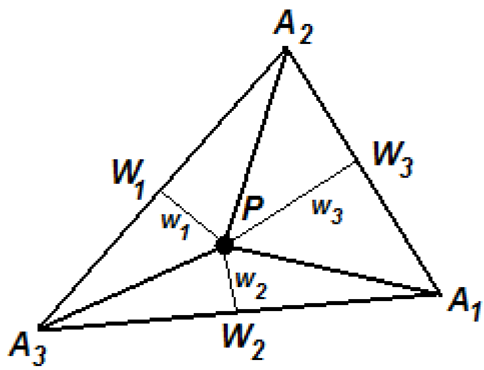

Figure 4.1

Comment.

We offer two proofs of Barrow's Inequality. The first is an adapted blend between the original proof by David F. Barrow **[EMB]** and that of Hojoo Lee **[LEE]**. The second is adapted from that of L. J. Mordell **[MOR]** and includes a condition for equality (the triangle must be equilateral and *P* must be its incenter).

Before proving Barrow's Inequality, we need a few results.

Lemma 4.2.1. **[LEE]**

Let $x, y, z, \theta_1, \theta_2, \theta_3 > 0$ such that $\theta_1+\theta_2+\theta_3=\pi$. Then

$$x^2+y^2+z^2 \geq 2(yz\cos\theta_1+xz\cos\theta_2+xy\cos\theta_3) .$$

Proof of Lemma 4.2.1. Based on **[LEE]**

We first aim to show that

$$\begin{aligned} & x^2+y^2+z^2-2[yz\cos(\theta_1)+xz\cos(\theta_2)+xy\cos(\theta_3)] \\ & \quad =[z-[x\cos(\theta_2)+y\cos(\theta_1)]]^2+[x\sin(\theta_2)-y\sin(\theta_1)]^2 . \end{aligned}$$

Notice $\theta_1+\theta_2+\theta_3=\pi$ means $\theta_3=\pi-[\theta_1+\theta_2]$.

$$\begin{aligned} & [z-[x\cos(\theta_2)+y\cos(\theta_1)]]^2+[x\sin(\theta_2)-y\sin(\theta_1)]^2 \\
= \quad & z^2-2z[x\cos(\theta_2)+y\cos(\theta_1)]+[x\cos(\theta_2)+y\cos(\theta_1)]^2 \\
+ \quad & x^2\sin^2(\theta_2)-2xy\sin(\theta_1)\sin(\theta_2)+y^2\sin^2(\theta_1) \\
= \quad & z^2-2xz\cos(\theta_2)-2yz\cos(\theta_1) \\
+ \quad & x^2\cos^2(\theta_2)+2xy\cos(\theta_1)\cos(\theta_2)+y^2\cos^2(\theta_1) \\
+ \quad & x^2\sin^2(\theta_2)-2xy\sin(\theta_1)\sin(\theta_2)+y^2\sin^2(\theta_1) \\
= \quad & z^2+[y^2\sin^2(\theta_1)+y^2\cos^2(\theta_1)]+[x^2\sin^2(\theta_1)+x^2\cos^2(\theta_2)] \\
+ \quad & -2xz\cos(\theta_2)-2yz\cos(\theta_1) \\
+ \quad & 2xy\cos(\theta_1)\cos(\theta_2)-2xy\sin(\theta_1)\sin(\theta_2) \\
= \quad & z^2+y^2[\sin^2(\theta_1)+\cos^2(\theta_1)]+x^2[\sin^2(\theta_1)+\cos^2(\theta_2)] \\
+ \quad & -2xz\cos(\theta_2)-2yz\cos(\theta_1) \\
+ \quad & 2xy[\cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2)] \\
= \quad & z^2+y^2+x^2-2xz\cos(\theta_2)-2yz\cos(\theta_1)+2xy\cos(\theta_1+\theta_2) \end{aligned}$$

Since $\pi-[\theta_1+\theta_2]=\theta_3$ and $\cos(\pi-x)=-\cos(x)$, we have $\cos(\theta_1+\theta_2)=-\cos(\theta_3)$ and

$$= \quad x^2+y^2+z^2-2xz\cos(\theta_2)-2yz\cos(\theta_1)-2xy\cos(\theta_3)$$

$$= \quad x^2+y^2+z^2-2[yz\cos(\theta_1)+xz\cos(\theta_2)+xy\cos(\theta_3)] .$$

Thus, we have shown

$$x^2+y^2+z^2-2[yz\cos(\theta_1)+xz\cos(\theta_2)+xy\cos(\theta_3)]$$

$$= \quad [z-[x\cos(\theta_2)+y\cos(\theta_1)]]^2+[x\sin(\theta_2)-y\sin(\theta_1)]^2$$

$$\geq \quad 0,$$

so that $$x^2+y^2+z^2\geq 2[yz\cos(\theta_1)+xz\cos(\theta_2)+xy\cos(\theta_3)] .$$

■

Lemma 4.2.2. **[LEE]**

Let $a, b, c, \theta_1, \theta_2, \theta_3 > 0$ such that $\theta_1+\theta_2+\theta_3=\pi$. Then

$$a\cos(\theta_1)+b\cos(\theta_2)+c\cos(\theta_3)\le\frac{1}{2}\left(\frac{bc}{a}+\frac{ac}{b}+\frac{ab}{c}\right).$$

Proof of Lemma 4.2.2. Based on **[LEE]**

We take $x=\sqrt{\frac{bc}{a}}$, $y=\sqrt{\frac{ac}{b}}$, and $z=\sqrt{\frac{ab}{c}}$ in Lemma 4.2.1. Then we have

$$\begin{aligned}
&2(a\cos\theta_1+b\cos\theta_2+c\cos\theta_3)\\
&= 2\left(\sqrt{a^2}\cos\theta_1+\sqrt{b^2}\cos\theta_2+\sqrt{c^2}\cos\theta_3\right)\\
&= 2\left(\sqrt{\frac{ac}{b}}\sqrt{\frac{ab}{c}}\cos\theta_1+\sqrt{\frac{bc}{a}}\sqrt{\frac{ab}{c}}\cos\theta_2+\sqrt{\frac{bc}{a}}\sqrt{\frac{ac}{b}}\cos\theta_3\right)\\
&\le \left(\sqrt{\frac{bc}{a}}\right)^2+\left(\sqrt{\frac{ac}{b}}\right)^2+\left(\sqrt{\frac{ab}{c}}\right)^2\\
&= \frac{bc}{a}+\frac{ac}{b}+\frac{ab}{c}.
\end{aligned}$$

Thus, we have

$$2(a\cos\theta_1+b\cos\theta_2+c\cos\theta_3)\le\frac{bc}{a}+\frac{ac}{b}+\frac{ab}{c},$$

or equivalently,

$$a\cos\theta_1+b\cos\theta_2+c\cos\theta_3\le\frac{1}{2}\left(\frac{bc}{a}+\frac{ac}{b}+\frac{ab}{c}\right),$$

which proves the lemma.

∎

Lemma 4.2.3. **[EMB and LEE]**

Given $\triangle ABC$, let W denote the intersection of the bisector of $\angle ABC$ with $\overline{AC}$, and let $\theta = m\angle ABC$. Then

$$BW = \frac{2(AB)(BC)}{AB+BC}\cos\left(\frac{\theta}{2}\right).$$

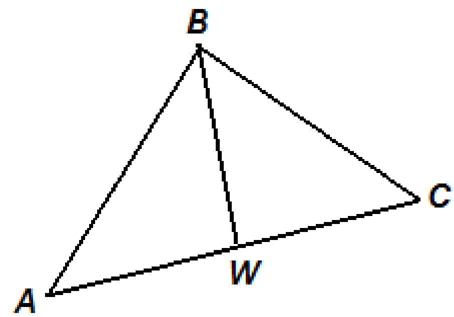

Figure 4.2

Proof of Lemma 4.2.3.

This is an original proof.

First, notice if $AB = BC$, then the result is trivial, as we would have $\triangle ABW$ is a right triangle and

$$BW = AB\cos\left(\frac{\theta}{2}\right) = \frac{2(AB)(AB)}{AB+AB}\cos\left(\frac{\theta}{2}\right).$$

Thus, we proceed assuming that $AB \neq BC$.

By the Law of Cosines, we have

$$(AW)^2 = (AB)^2 + (BW)^2 - 2(AB)(BW)\cos\left(\frac{\theta}{2}\right)$$

and

$$(CW)^2 = (BC)^2 + (BW)^2 - 2(BC)(BW)\cos\left(\frac{\theta}{2}\right).$$

Using the fact that the angle bisector of a triangle splits the opposite side of the triangle and the sides of the angle proportionally, we have

$$\frac{CW}{AW} = \frac{BC}{AB} \qquad \text{or equivalently} \qquad (AW)^2(BC)^2 = (CW)^2(AB)^2.$$

To apply this relation, we first note:

$$(AW)^2(BC)^2=(AB)^2(BC)^2+(BW)^2(BC)^2-2(AB)(BW)(BC)^2\cos\left(\frac{\theta}{2}\right)$$

and

$$(CW)^2(AB)^2=(BC)^2(AB)^2+(BW)^2(AB)^2-2(BC)(BW)(AB)^2\cos\left(\frac{\theta}{2}\right),$$

and setting their right hand sides equal, we get

$$(AB)^2(BC)^2+(BW)^2(BC)^2-2(AB)(BW)(BC)^2\cos\left(\frac{\theta}{2}\right)$$

$$= \quad (BC)^2(AB)^2+(BW)^2(AB)^2-2(BC)(BW)(AB)^2\cos\left(\frac{\theta}{2}\right),$$

which means

$$(BW)^2(BC)^2-2(AB)(BW)(BC)^2\cos\left(\frac{\theta}{2}\right)$$

$$= \quad (BW)^2(AB)^2-2(BC)(BW)(AB)^2\cos\left(\frac{\theta}{2}\right).$$

Dividing through by $(BW)^2$ gives

$$(BC)^2-\frac{2(AB)(BC)^2\cos\left(\frac{\theta}{2}\right)}{BW} = (AB)^2-\frac{2(BC)(AB)^2\cos\left(\frac{\theta}{2}\right)}{BW}$$

Manipulating this equation gives

$$(BC)^2-(AB)^2 = \frac{2(AB)(BC)^2\cos\left(\frac{\theta}{2}\right)}{BW}-\frac{2(BC)(AB)^2\cos\left(\frac{\theta}{2}\right)}{BW},$$

so that

$$(BC-AB)(BC+AB) = 2(AB)(BC)\cos\left(\frac{\theta}{2}\right)\cdot\frac{(BC-AB)}{BW}.$$

Multiplying both sides by $\dfrac{BW}{(BC-AB)(BC+AB)}$ yields

$$BW = \frac{2(AB)(BC)\cos\left(\frac{\theta}{2}\right)}{BC+AB}$$

$$= \frac{2(AB)(BC)\cos\left(\frac{\theta}{2}\right)}{AB+BC}.$$

Thus, we have established $BW = \dfrac{2(AB)(BC)}{AB+BC}\cos\left(\dfrac{\theta}{2}\right)$, and the lemma holds.

$\blacksquare$

First Proof of Theorem 4.2. Based on **[EMB and LEE]**

We let $2\theta_1 = m\angle A_2PA_3$, $2\theta_2 = m\angle A_1PA_3$, and $2\theta_3 = m\angle A_1PA_2$.

By Lemma 4.2.3, we have

$$w_1 = \frac{2(PA_2)(PA_3)}{PA_2+PA_3}\cos(\theta_1),$$

$$w_2 = \frac{2(PA_1)(PA_3)}{PA_1+PA_3}\cos(\theta_2), \text{ and}$$

$$w_3 = \frac{2(PA_1)(PA_2)}{PA_1+PA_2}\cos(\theta_3).$$

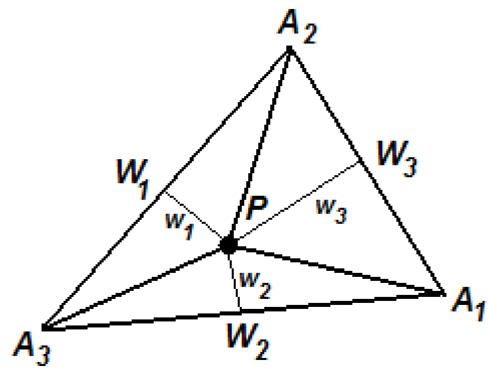

Figure 4.4

So, we obtain

$$\begin{aligned} w_1+w_2+w_3 &= \frac{2(PA_2)(PA_3)}{PA_2+PA_3}\cos(\theta_1)+\frac{2(PA_1)(PA_3)}{PA_1+PA_3}\cos(\theta_2)+\frac{2(PA_1)(PA_2)}{PA_1+PA_2}\cos(\theta_3) \\ &= \left(\frac{2}{PA_2+PA_3}\right)(PA_2)(PA_3)\cos(\theta_1) \\ &\quad + \left(\frac{2}{PA_1+PA_3}\right)(PA_1)(PA_3)\cos(\theta_2) \\ &\quad + \left(\frac{2}{PA_1+PA_2}\right)(PA_1)(PA_2)\cos(\theta_3) \end{aligned}$$

And by the Arithmetic Mean – Geometric Mean Inequality's reciprocal

$$\begin{aligned} &\le \frac{1}{\sqrt{(PA_2)(PA_3)}}(PA_2)(PA_3)\cos(\theta_1) \\ &\quad + \frac{1}{\sqrt{(PA_1)(PA_3)}}(PA_1)(PA_3)\cos(\theta_2) \\ &\quad + \frac{1}{\sqrt{(PA_1)(PA_2)}}(PA_1)(PA_2)\cos(\theta_3) \end{aligned}$$

$$= \sqrt{(PA_2)(PA_3)}\cos(\theta_1)$$

$$+ \sqrt{(PA_1)(PA_3)}\cos(\theta_2)$$

$$+ \sqrt{(PA_1)(PA_2)}\cos(\theta_3)$$

By Lemma 4.2.2, with $a=\sqrt{(PA_2)(PA_3)}$, $b=\sqrt{(PA_1)(PA_3)}$, and $c=\sqrt{(PA_1)(PA_2)}$ we get

$$\leq \frac{\sqrt{(PA_1)(PA_3)}\sqrt{(PA_1PA_2)}}{2\sqrt{(PA_2)(PA_3)}}$$

$$+ \frac{\sqrt{(PA_1)(PA_2)}\sqrt{(PA_2PA_3)}}{2\sqrt{(PA_1)(PA_3)}}$$

$$+ \frac{\sqrt{(PA_1)(PA_2)}\sqrt{(PA_1PA_3)}}{2\sqrt{(PA_2)(PA_3)}}$$

$$= \frac{PA_1}{2}+\frac{PA_2}{2}+\frac{PA_3}{2}.$$

Thus, we have

$$w_1+w_2+w_3 \leq \frac{PA_1}{2}+\frac{PA_2}{2}+\frac{PA_3}{2}.$$

Multiplying through by 2 gives the desired result,

$$PA_1+PA_2+PA_3 \geq 2(w_1+w_2+w_3).$$

∎

Comment.

It is worth noting that Barrow used this proof as his proof of the Erdös-Mordell Inequality, as we have $w_i \geq p_i$ for each *i*.

Second Proof of Theorem 4.2 Based on **[MOR]**

Let p_i denote the distance from *P* to the side of $\triangle A_1 A_2 A_3$ across from vertex A_i, and let $2\theta_1 = m\angle A_2 P A_3$, $2\theta_2 = m\angle A_1 P A_3$, and $2\theta_3 = m\angle A_1 P A_2$.

Before proceeding, we say that Mordell **[MOR]** adds a condition for equality, namely the triangle must be equilateral and *P* must be its incenter.

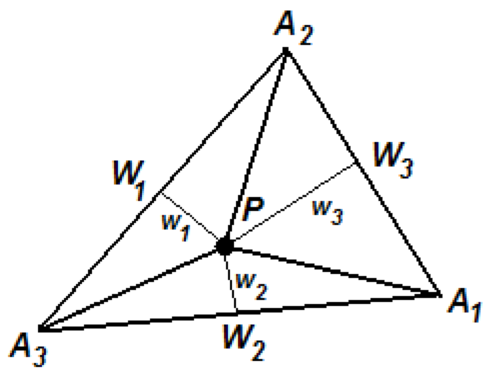

Figure 4.5

We begin by considering the area of $\triangle A_2 P A_3$. On the one hand, we have:

Area $\triangle A_2 P A_3$

$$= \frac{PA_2 \cdot PA_3 \sin(2\theta_1)}{2}$$

$$= \frac{2\, PA_2 \cdot PA_3 \sin(\theta_1)\cos(\theta_1)}{2}$$

$$= PA_2 \cdot PA_3 \sin(\theta_1)\cos(\theta_1).$$

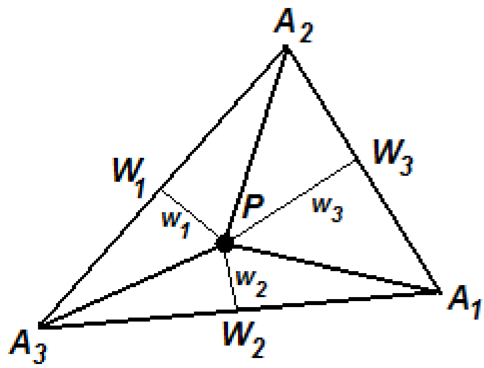

Figure 4.6

On the other hand, we obtain:

Area $\triangle A_2 P A_3$

$$= \quad \text{Area} \triangle A_2 P W_1 + \text{Area} \triangle A_3 P W_1$$

Since $\overline{PW_1}$ bisects $\angle A_2 P A_3$, we get

$$= \quad \frac{PA_2 \cdot w_1 \sin(\theta_1)}{2} + \frac{PA_3 \cdot w_1 \sin(\theta_1)}{2}$$

$$= \quad \frac{(PA_2 + PA_3) w_1 \sin(\theta_1)}{2}$$

And by the Arithmetic Mean – Geometric Mean Inequality, this becomes

$$\geq \quad \frac{2\sqrt{PA_2 PA_3}\, w_1 \sin(\theta_1)}{2}$$

$$= \quad w_1 \sqrt{PA_2\, PA_3} \sin(\theta_1),$$

with equality requiring $PA_2 = PA_3$. Thus, using our expressions for Area $\triangle A_2 P A_3$, we have

$$PA_2 \cdot PA_3 \sin(\theta_1) \cos(\theta_1) \geq w_1 \sqrt{PA_2 PA_3} \sin(\theta_1),$$

so that we conclude

$$w_1 \leq \sqrt{PA_2 PA_3} \cos(\theta_1). \qquad \textit{(4.2.A)}$$

Similarly, we obtain

$$w_2 \le \sqrt{PA_1 PA_3}\cos(\theta_2) \qquad \text{and} \qquad w_3 \le \sqrt{PA_1 PA_2}\cos(\theta_3), \qquad (4.2.A)$$

with $PA_1 = PA_3$ and $PA_1 = PA_2$ as necessary requirements for equality, respectively.

We notice

$$\begin{aligned}
0 \quad \le \quad & \left(\sqrt{PA_1} - \sqrt{PA_2}\cos(\theta_3) - \sqrt{PA_3}\cos(\theta_2)\right)^2 \qquad (4.2.B)\\
& + \quad \left(\sqrt{PA_2}\sin(\theta_3) - \sqrt{PA_3}\sin(\theta_2)\right)^2\\
= \quad & PA_1 + PA_2\cos^2(\theta_3) + PA_3\cos^2(\theta_2)\\
& + \quad -2\sqrt{PA_1 PA_2}\cos(\theta_3) - 2\sqrt{PA_1 PA_3}\cos(\theta_2)\\
& + \quad 2\sqrt{PA_2 PA_3}\cos(\theta_2)\cos(\theta_3)\\
& + \quad PA_2\sin^2(\theta_3) + PA_3\sin^2(\theta_2)\\
& + \quad -2\sqrt{PA_2 PA_3}\sin(\theta_2)\sin(\theta_3)\\
= \quad & PA_1 + PA_2\left[\sin^2(\theta_3) + \cos^2(\theta_3)\right]\\
& + \quad PA_3\left[\sin^2(\theta_2) + \cos^2(\theta_2)\right]\\
& + \quad -2\sqrt{PA_1 PA_2}\cos(\theta_3) - 2\sqrt{PA_1 PA_3}\cos(\theta_2)\\
& + \quad 2\sqrt{PA_2 PA_3}\cos(\theta_2)\cos(\theta_3)\\
& + \quad -2\sqrt{PA_2 PA_3}\sin(\theta_2)\sin(\theta_3)
\end{aligned}$$

And using both the Pythagorean Identity with the identity for the sum of angles, we have

$$\begin{aligned}
= \quad & PA_1 + PA_2 + PA_3\\
& + \quad -2\sqrt{PA_1 PA_2}\cos(\theta_3) - 2\sqrt{PA_1 PA_3}\cos(\theta_2)\\
& + \quad 2\sqrt{PA_2 PA_3}\cos(\theta_2 + \theta_3)
\end{aligned}$$

Since $2\theta_1+2\theta_2+2\theta_3=2\pi$, we have $\theta_1+\theta_2+\theta_3=\pi$, so that

$$= \quad PA_1+PA_2+PA_3$$

$$+ \quad -2\sqrt{PA_1\,PA_2}\cos(\theta_3)-2\sqrt{PA_1\,PA_3}\cos(\theta_2)$$

$$+ \quad 2\sqrt{PA_2\,PA_3}\cos(\pi-\theta_1)$$

And since $\cos(\pi-x)=-\cos(x)$, we get

$$= \quad PA_1+PA_2+PA_3$$

$$+ \quad -2\sqrt{PA_1\,PA_2}\cos(\theta_3)-2\sqrt{PA_1\,PA_3}\cos(\theta_2)$$

$$+ \quad -2\sqrt{PA_2\,PA_3}\cos(\theta_1)$$

Which from *(4.2.A)*

$$\leq \quad PA_1+PA_2+PA_3-2w_3-2w_2-2w_1 \,. \qquad \textit{(4.2.C)}$$

Thus, we have established $PA_1+PA_2+PA_3-2(w_1+w_2+w_3)\geq 0$, so that

$$PA_1+PA_2+PA_3\geq 2(w_1+w_2+w_3)\,,$$

and the inequality is proven.

For equality to hold overall, we need equality in *(4.2.C)*, which requires equality in *(4.2.A)* so that

$$PA_1=PA_2=PA_3\,.$$

For equality to then hold in *(4.2.B)* given the fact above, we need

$$\sqrt{PA_1}\sin(\theta_3)-\sqrt{PA_1}\sin(\theta_2)=0 \quad \text{which means} \quad \sin(\theta_3)=\sin(\theta_2)\,,$$

so that $\theta_2=\theta_3$ or $\theta_2=\pi-\theta_3$.

Additionally, for equality to then hold in *(4.2.B)*, we need

$$\sqrt{PA_1}-\sqrt{PA_1}\cos(\theta_3)-\sqrt{PA_1}\cos(\theta_2)=0 .$$

If $\theta_2=\pi-\theta_3$, then recalling $\cos(x)=-\cos(\pi-x)$, we have

$$\begin{aligned}
&\sqrt{PA_1}-\sqrt{PA_1}\cos(\theta_3)-\sqrt{PA_1}\cos(\theta_2)\\
&= \sqrt{PA_1}-\sqrt{PA_1}\cos(\theta_3)-\sqrt{PA_1}\cos(\pi-\theta_3)\\
&= \sqrt{PA_1}-\sqrt{PA_1}\cos(\theta_3)+\sqrt{PA_1}\cos(\theta_3)\\
&= \sqrt{PA_1}\\
&> 0,
\end{aligned}$$

so we cannot get equality in this situation. If, however, $\theta_2=\theta_3$, we have

$$\begin{aligned}
&\sqrt{PA_1}-\sqrt{PA_1}\cos(\theta_3)-\sqrt{PA_1}\cos(\theta_2)\\
&= \sqrt{PA_1}-\sqrt{PA_1}\cos(\theta_3)-\sqrt{PA_1}\cos(\theta_3)\\
&= \sqrt{PA_1}\left(1-2\cos(\theta_3)\right)
\end{aligned}$$

so that $\sqrt{PA_1}-\sqrt{PA_1}\cos(\theta_3)-\sqrt{PA_1}\cos(\theta_2)=0$ requires

$$\sqrt{PA_1}\left(1-2\cos(\theta_3)\right)=0 .$$

This means $\cos(\theta_3)=\frac{1}{2}$, so that $\theta_3=\frac{\pi}{3}$. It immediately follows that $\theta_2=\frac{\pi}{3}$.

Since $2\theta_1+2\theta_2+2\theta_3=2\pi$ (see Figure 4.7), it follows that $\theta_1=\frac{\pi}{3}$. Thus,

$$\frac{2\pi}{3}=m\angle A_2PA_3=m\angle A_1PA_3=m\angle A_1PA_2 .$$

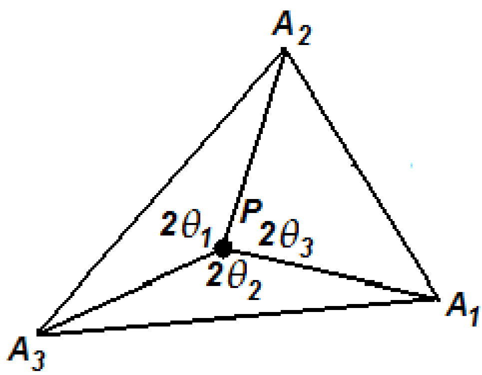

Figure 4.7

Therefore, equality overall requires both

$$PA_1 = PA_2 = PA_3$$

and

$$\frac{2\pi}{3} = m\angle A_2 P A_3 = m\angle A_1 P A_3 = m\angle A_1 P A_2 ,$$

so that the triangles $\triangle A_2 P A_3$, $\triangle A_1 P A_3$, and $\triangle A_1 P A_2$ are all congruent and isosceles.

Based on these triangles being congruent and isosceles, we have $A_1 A_2 = A_2 A_3 = A_1 A_3$ and

$$m\angle P A_3 A_1 = m\angle PA_3 A_2$$

$$m\angle P A_1 A_2 = m\angle PA_1 A_3$$

$$m\angle P A_2 A_1 = m\angle PA_2 A_3 .$$

This requires $\triangle A_1 A_2 A_3$ to be equilateral and *P* to be its incenter.

$\blacksquare$

Corollary 4.2.4.

Given $\triangle A_1A_2A_3$. Let W_i be the point on the side of $\triangle A_1A_2A_3$ opposite A_i such that $\overline{A_iW_i}$ bisects the interior angle of $\triangle A_1A_2A_3$ angle whose vertex is at A_i, let $w_i = A_iW_i$, let a_i denote the length of the side of $\triangle A_1A_2A_3$ across from vertex A_i, let α_i be the interior angle of $\triangle A_1A_2A_3$ with vertex A_i, and let h_i be the length of the altitude of $\triangle A_1A_2A_3$ from A_i. Then

$$h_1 \le \sqrt{a_2a_3}\cos\left(\frac{\alpha_1}{2}\right), \quad h_2 \le \sqrt{a_1a_3}\cos\left(\frac{\alpha_2}{2}\right), \quad \text{and} \quad h_3 \le \sqrt{a_1a_2}\cos\left(\frac{\alpha_3}{2}\right),$$

with $a_2 = a_3$, $a_1 = a_3$, and $a_1 = a_2$ being necessary conditions for equality in each, respectively.

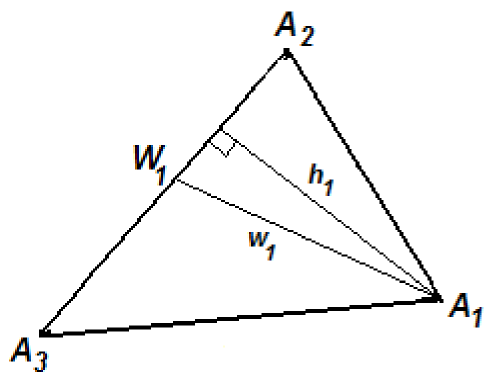

Figure 4.8

Proof of Corollary 4.2.4.

The Second Proof of Theorem 4.2 (see *4.2.A*) gives

$$w_1 \le \sqrt{a_2a_3}\cos\left(\frac{\alpha_1}{2}\right), \quad w_2 \le \sqrt{a_1a_3}\cos\left(\frac{\alpha_2}{2}\right), \quad \text{and} \quad w_3 \le \sqrt{a_1a_2}\cos\left(\frac{\alpha_3}{2}\right),$$

with $a_2 = a_3$, $a_1 = a_3$, and $a_1 = a_2$ being necessary conditions for equality in each, respectively.

This, coupled with $h_1 \le w_1$, $h_2 \le w_2$, and $h_3 \le w_3$ (as the altitude is the shortest distance from a vertex to the opposite side of a triangle) yields the desired result. We comment that in the situation where the triangle is isosceles, the angle bisector from the vertex angle and the altitude from the vertex angle coincide, establishing the condition for equality.

∎

Since we have investigated the distances involving *P* with the perpendiculars and *P* with the angle bisectors, one might wonder about *P* with the midpoints.

Example 4.3.

An Erdös-Mordell form inequality does not hold when pairing *P* with the midpoints of a triangle.

Consider $\triangle A_1 A_2 A_3$, an equilateral triangle with side lengths of 12. Let M_i be the midpoint of the side of $\triangle A_1 A_2 A_3$ opposite vertex A_i.

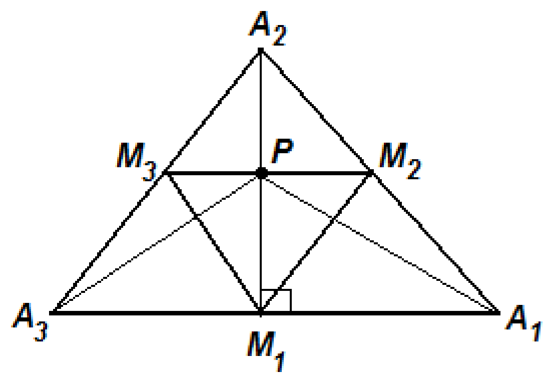

Figure 4.9

We note that $\triangle A_2 M_3 M_2$, $\triangle A_3 M_3 M_1$, $\triangle A_1 M_2 M_1$, and $\triangle M_1 M_2 M_3$ are all equilateral with side lengths of 6. This can be used to determine the values below:

$$PM_1 = 3\sqrt{3} \qquad PA_1 = \sqrt{(3\sqrt{3})^2 + 6^2} = \sqrt{63} = 3\sqrt{7}$$

$$PM_2 = 3 \qquad PA_2 = 3\sqrt{3}$$

$$PM_3 = 3 \qquad PA_3 = 3\sqrt{7}.$$

Thus, we have $PM_1 + PM_2 + PM_3 = 3\sqrt{3} + 3 + 3 = 3\sqrt{3} + 6 \approx 11.20$

and $PA_1 + PA_2 + PA_3 = 3\sqrt{7} + 3\sqrt{3} + 3\sqrt{7} = 6\sqrt{7} + 3\sqrt{3} \approx 21.07$.

From this, we gather that

$$2(PM_1 + PM_2 + PM_3) \approx 22.40 > 21.07 \approx PA_1 + PA_2 + PA_3,$$

so that an inequality of the form $2(PM_1 + PM_2 + PM_3) \leq PA_1 + PA_2 + PA_3$ does not hold for midpoints.

5 Inequalities Resembling the Erdös-Mordell Inequality

In this section, we investigate inequalities involving triangles whose general structure resembles that of the Erdös-Mordell Inequality.

We begin by outlining the examples, for ease of reference.

Given $\triangle A_1 A_2 A_3$ and interior point *P*, let a_i denote the length of the side of $\triangle A_1 A_2 A_3$ across from vertex A_i for each $1 \le i \le 3$, let p_i denote the distance from *P* to the side of $\triangle A_1 A_2 A_3$ opposite vertex A_i for each $1 \le i \le 3$, and let *K* be the area of $\triangle A_1 A_2 A_3$. Then the following inequalities hold:

Example 5.1. $a_1 PA_1 + a_2 PA_2 + a_3 PA_3 \ge 4K$

Example 5.2. $p_1 PA_1 + p_2 PA_2 + p_3 PA_3 \ge 2(p_1 p_2 + p_2 p_3 + p_1 p_3)$

Stated and proved in **[OP1]**.

Example 5.3. $PA_1 \cdot PA_2 \cdot PA_3 \ge 8 p_1 p_2 p_3$

Stated and proved in **[OP1; KAN pg 87 and 115]**.

Example 5.4. $PA_1 \cdot PA_2 \cdot PA_3 \ge (p_2 + p_3)(p_1 + p_3)(p_1 + p_2)$

Stated in **[MOR; OP2; KAN pg 88]**.

Proved in **[MOR; OP2]**.

Example 5.5. $PA_1 \cdot PA_2 + PA_1 \cdot PA_3 + PA_2 \cdot PA_3$

$\ge \quad (p_2 + p_3)(p_1 + p_3) + (p_2 + p_3)(p_1 + p_2) + (p_1 + p_3)(p_1 + p_2)$.

Stated in **[OP2; KAN pg 88]**.

Proved in **[OP2]**.

Example 5.1.

Given $\triangle A_1 A_2 A_3$ and interior point *P*, let a_i denote the length of the side of $\triangle A_1 A_2 A_3$ across from vertex A_i for each $1 \leq i \leq 3$, and let *K* be the area of $\triangle A_1 A_2 A_3$. Then

$$a_1 PA_1 + a_2 PA_2 + a_3 PA_3 \geq 4K.$$

Comment.

We present two solutions to this problem. Although neither is based off any solution in particular, they use concepts seen in numerous references, including those of Oppenheim **[OP 1]**.

Before beginning, we let p_i denote the distance from *P* to the side of $\triangle A_1 A_2 A_3$ across from vertex A_i.

First Solution to Example 5.1.

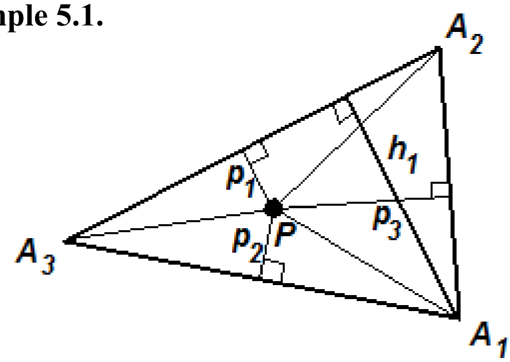

Figure 5.1

We first notice that $2K = a_1 p_1 + a_2 p_2 + a_3 p_3$.

It is clear that $PA_1 + p_1 \geq h_1$, since h_1 is the shortest distance from A_1 to $\overline{A_2 A_3}$. Thus, multiplying through the inequality by a_1, we get

$$a_1 PA_1 + a_1 p_1 \geq a_1 h_1 = 2K \quad \text{or} \quad a_1 PA_1 \geq 2K - a_1 p_1.$$

Similarly, we obtain

$$a_2 PA_2 \geq 2K - a_2 p_2 \quad \text{and} \quad a_3 PA_3 \geq 2K - a_3 p_3.$$

Summing these inequalities gives

$$a_1 PA_1 + a_2 PA_2 + a_3 PA_3 \geq 6K - (a_1 p_1 + a_2 p_2 + a_3 p_3) = 6K - 2K = 4K.$$

∎

Second Solution to Example 5.1.

With a slight modification from Corollary 3.5, we have

$$a_1 PA_1 \geq a_2 p_2 + a_3 p_3 , \quad a_2 PA_2 \geq a_1 p_1 + a_3 p_3 \text{, and} \quad a_3 PA_3 \geq a_1 p_1 + a_2 p_2 .$$

Thus,

$$\begin{aligned} a_1 PA_1 + a_2 PA_2 + a_3 PA_3 &\geq (a_2 p_2 + a_3 p_3) + (a_1 p_1 + a_3 p_3) + (a_1 p_1 + a_2 p_2) \\ &= 2(a_1 p_1 + a_2 p_2 + a_3 p_3) \\ &= 2(2K) \\ &= 4K , \end{aligned}$$

so the result holds.

■

Example 5.2. **[OP1]**

Given $\triangle A_1 A_2 A_3$ and interior point *P*, let a_i denote the length of the side of $\triangle A_1 A_2 A_3$ across from vertex A_i for each $1 \le i \le 3$ and let p_i denote the distance from *P* to the side of $\triangle A_1 A_2 A_3$ opposite vertex A_i for each $1 \le i \le 3$. Then:

$$p_1 P A_1 + p_2 P A_2 + p_3 P A_3 \ge 2(p_1 p_2 + p_2 p_3 + p_1 p_3).$$

Solution to Example 5.2. Based on **[OP1]**

By Corollary 3.5, we have

$$PA_1 \ge \frac{a_2 p_2 + a_3 p_3}{a_1}, \quad PA_2 \ge \frac{a_1 p_1 + a_3 p_3}{a_2}, \text{ and } PA_3 \ge \frac{a_1 p_1 + a_2 p_2}{a_3}.$$

So, we obtain

$$p_1 P A_1 + p_2 P A_2 + p_3 P A_3$$

$$\ge \quad p_1 \left(\frac{a_2 p_2 + a_3 p_3}{a_1} \right) + p_2 \left(\frac{a_1 p_1 + a_3 p_3}{a_2} \right) + p_3 \left(\frac{a_1 p_1 + a_2 p_2}{a_3} \right)$$

By rearranging terms

$$= \quad \left(\frac{a_2}{a_1} + \frac{a_1}{a_2} \right) p_1 p_2 + \left(\frac{a_3}{a_2} + \frac{a_2}{a_3} \right) p_2 p_3 + \left(\frac{a_1}{a_3} + \frac{a_3}{a_1} \right) p_1 p_3$$

Through the use of the Arithmetic Mean – Geometric Mean Inequality

$$\ge \quad 2\sqrt{\frac{a_2}{a_1} \cdot \frac{a_1}{a_2}}\, p_1 p_2 + 2\sqrt{\frac{a_3}{a_2} \cdot \frac{a_2}{a_3}}\, p_2 p_3 + 2\sqrt{\frac{a_1}{a_3} \cdot \frac{a_3}{a_1}}\, p_1 p_3$$

$$= \quad 2\, p_1 p_2 + 2\, p_2 p_3 + 2\, p_1 p_3,$$

so that we have established $p_1 P A_1 + p_2 P A_2 + p_3 P A_3 \ge 2(p_1 p_2 + p_2 p_3 + p_1 p_3)$.

∎

Example 5.3. **[OP1; KAN pg 87 and 115]**

Given $\triangle A_1 A_2 A_3$ and interior point *P*. Let a_i denote the length of the side of $\triangle A_1 A_2 A_3$ across from vertex A_i for each $1 \le i \le 3$ and let p_i denote the distance from *P* to the side of $\triangle A_1 A_2 A_3$ opposite vertex A_i, for each $1 \le i \le 3$. Then:

$$P A_1 \cdot P A_2 \cdot P A_3 \ge 8 p_1 p_2 p_3 .$$

Comment.

We offer two solutions, the first being more of an original solution.

First Solution to Example 5.3.

From Corollary 3.5, we have

$$PA_1 \ge \frac{a_2 p_3 + a_3 p_2}{a_1}, \quad PA_2 \ge \frac{a_1 p_3 + a_3 p_1}{a_2}, \text{ and } PA_3 \ge \frac{a_1 p_2 + a_2 p_1}{a_3} .$$

So

$$P A_1 \cdot P A_2 \cdot P A_3$$

$$\ge \left(\frac{a_2 p_3 + a_3 p_2}{a_1}\right) \cdot \left(\frac{a_1 p_3 + a_3 p_1}{a_2}\right) \cdot \left(\frac{a_1 p_2 + a_2 p_1}{a_3}\right)$$

$$= \frac{(a_1 a_3 p_2 p_3 + a_3^2 p_1 p_2 + a_1 a_2 p_3^2 + a_2 a_3 p_1 p_3)(a_1 p_2 + a_2 p_1)}{a_1 a_2 a_3}$$

$$= \frac{a_1^2 a_3 p_2^2 p_3}{a_1 a_2 a_3} + \frac{a_1 a_3^2 p_1 p_2^2}{a_1 a_2 a_3} + \frac{a_1^2 a_2 p_2 p_3^2}{a_1 a_2 a_3} + \frac{a_1 a_2 a_3 p_1 p_2 p_3}{a_1 a_2 a_3}$$

$$+ \frac{a_1 a_2 a_3 p_1 p_2 p_3}{a_1 a_2 a_3} + \frac{a_2 a_3^2 p_1^2 p_2}{a_1 a_2 a_3} + \frac{a_1 a_2^2 p_1 p_3^2}{a_1 a_2 a_3} + \frac{a_2^2 a_3 p_1^2 p_3}{a_1 a_2 a_3}$$

$$= \frac{a_1 p_2^2 p_3}{a_2} + \frac{a_3 p_1 p_2^2}{a_2} + \frac{a_1 p_2 p_3^2}{a_3} + p_1 p_2 p_3$$

$$+ p_1 p_2 p_3 + \frac{a_3 p_1^2 p_2}{a_1} + \frac{a_2 p_1 p_3^2}{a_3} + \frac{a_2 p_1^2 p_3}{a_1}$$

By regrouping

$$= \qquad 2\,p_1\,p_2\,p_3 + \left(\frac{a_1\,p_2^2\,p_3}{a_2} + \frac{a_2\,p_1^2\,p_3}{a_1}\right) + \left(\frac{a_3\,p_1\,p_2^2}{a_2} + \frac{a_2\,p_1\,p_3^2}{a_3}\right) + \left(\frac{a_1\,p_2\,p_3^2}{a_3} + \frac{a_3\,p_1^2\,p_2}{a_1}\right)$$

By the Arithmetic Mean – Geometric Mean Inequality

$$\geq \qquad 2\,p_1\,p_2\,p_3 + 2\sqrt{\frac{a_1\,p_2^2\,p_3}{a_2}\cdot\frac{a_2\,p_1^2\,p_3}{a_1}} + 2\sqrt{\frac{a_3\,p_1\,p_2^2}{a_2}\cdot\frac{a_2\,p_1\,p_3^2}{a_3}} + 2\sqrt{\frac{a_1\,p_2\,p_3^2}{a_3}\cdot\frac{a_3\,p_1^2\,p_2}{a_1}}$$

$$= \qquad 2\,p_1\,p_2\,p_3 + 2\sqrt{p_1^2\,p_2^2\,p_3^2} + 2\sqrt{p_1^2\,p_2^2\,p_3^2} + 2\sqrt{p_1^2\,p_2^2\,p_3^2}$$

$$= \qquad 2\,p_1\,p_2\,p_3 + 2\,p_1\,p_2\,p_3 + 2\,p_1\,p_2\,p_3 + 2\,p_1\,p_2\,p_3$$

$$= \qquad 8\,p_1\,p_2\,p_3\,.$$

Thus, we have established $\quad P\,A_1 \cdot P\,A_2 \cdot P\,A_3 \geq 8\,p_1\,p_2\,p_3$.

∎

Second Solution to Example 5.3. Based on **[OP1]**

Again, we start with the result of Corollary 3.5:

$$PA_1 \geq \frac{a_2 p_3 + a_3 p_2}{a_1}, \quad PA_2 \geq \frac{a_1 p_3 + a_3 p_1}{a_2}, \text{ and } PA_3 \geq \frac{a_1 p_2 + a_2 p_1}{a_3}.$$

Then we have

$$P A_1 \cdot P A_2 \cdot P A_3$$

$$\geq \left(\frac{a_2 p_3 + a_3 p_2}{a_1}\right) \cdot \left(\frac{a_1 p_3 + a_3 p_1}{a_2}\right) \cdot \left(\frac{a_1 p_2 + a_2 p_1}{a_3}\right)$$

$$= \frac{1}{a_1}(a_2 p_3 + a_3 p_2) \cdot \frac{1}{a_2}(a_1 p_3 + a_3 p_1) \cdot \frac{1}{a_3}(a_1 p_2 + a_2 p_1)$$

By the Arithmetic Mean – Geometric Mean Inequality, we have

$$\geq \frac{2}{a_1}\sqrt{a_2 p_3 a_3 p_2} \cdot \frac{2}{a_2}\sqrt{a_1 p_3 a_3 p_1} \cdot \frac{2}{a_3}\sqrt{a_1 p_2 a_2 p_1}$$

$$= \frac{8}{a_1 a_2 a_3}\sqrt{a_1^2 a_2^2 a_3^2 p_1^2 p_2^2 p_3^2}$$

$$= \frac{8}{a_1 a_2 a_3}(a_1 a_2 a_3 p_1 p_2 p_3)$$

$$= 8 p_1 p_2 p_3 .$$

Thus, we have established $P A_1 \cdot P A_2 \cdot P A_3 \geq 8 p_1 p_2 p_3$.

■

Example 5.4. **[MOR; OP2; KAN pg 88]**

Given $\triangle A_1 A_2 A_3$ and interior point *P*, let a_i denote the length of the side of $\triangle A_1 A_2 A_3$ across from vertex A_i for each $1 \le i \le 3$ and let p_i denote the distance from *P* to the side of $\triangle A_1 A_2 A_3$ opposite vertex A_i for each $1 \le i \le 3$. Then:

$$P A_1 \cdot P A_2 \cdot P A_3 \ge (p_2 + p_3)(p_1 + p_3)(p_1 + p_2) .$$

Comment.

We first need two lemmas.

Lemma 5.4.1. **[MOR]**

Let *a, b,* and *c* be positive, real numbers, and let $0 \le x \le 2\pi$. Then

$$a^2 + b^2 + 2ab\cos(x) = (a+b)^2 \cos^2\left(\frac{x}{2}\right) + (a-b)^2 \sin^2\left(\frac{x}{2}\right) .$$

Proof of Lemma 5.4.1.

This is an original proof.

$$(a+b)^2 \cos^2\left(\frac{x}{2}\right) + (a-b)^2 \sin^2\left(\frac{x}{2}\right)$$

$$= \quad (a^2 + 2ab + b^2)\left(\frac{1+\cos(x)}{2}\right) + (a^2 - 2ab + b^2)\left(\frac{1-\cos(x)}{2}\right)$$

$$= \quad \frac{a^2+2ab+b^2}{2} + \frac{a^2-2ab+b^2}{2} + \left(\frac{a^2+2ab+b^2}{2}\right)\cos(x) + \left(\frac{-a^2+2ab-b^2}{2}\right)\cos(x)$$

$$= \quad \frac{2a^2+2b^2}{2} + \left(\frac{4ab}{2}\right)\cos(x)$$

$$= \quad a^2 + b^2 + 2ab\cos x ,$$

which establishes the desired result:

$$a^2 + b^2 + 2ab\cos(x) = (a+b)^2 \cos^2\left(\frac{x}{2}\right) + (a-b)^2 \sin^2\left(\frac{x}{2}\right) .$$

∎

Lemma 5.4.2. **[MOR]**

Let $x, y, z > 0$ such that $x + y + z = \frac{\pi}{2}$. Then $\sin(x)\sin(y)\sin(z) \le \frac{1}{8}$.

Proof of Lemma 5.4.2. Based on **[MOR]**

We first note that $z = \frac{\pi}{2} - (x + y)$, and therefore $\sin(z) = \sin\left(\frac{\pi}{2} - (x + y)\right) = \cos(x + y)$.

Thus, we have $\sin(x)\sin(y)\sin(z) = \sin(x)\sin(y)\cos(x + y)$.

Now, by using the well-known identity

$$\sin(A)\cos(B) = \frac{1}{2}\sin(A + B) + \frac{1}{2}\sin(A - B)$$

with $A = y$ and $B = x + y$, we have

$$\sin(x)[2\sin(y)\cos(x + y)]$$

$$= \quad \sin(x)[\sin(x + 2y) + \sin(-x)]$$

and since sine is an odd function

$$= \quad \sin(x)[\sin(x + 2y) - \sin(x)] .$$

Now take $f(x, y) = \sin(x)[\sin(x + 2y) - \sin(x)]$.

To find the maximum of *f*, we investigate where its partial derivatives are zero:

$$f_y(x, y) = \sin(x)[2\cos(x + 2y)] = 0 ,$$

which means $\sin(x) = 0$ or $\cos(x + 2y) = 0$, so that on our interval of consideration,

$$x + 2y = \frac{\pi}{2} .$$

Now, we also find, under this condition

$$\begin{aligned}
f_x(x,y) &= \sin(x)[\cos(x+2y)-\cos(x)]+\cos(x)[\sin(x+2y)-\sin(x)] \\
&= \sin(x)[0-\cos(x)]+\cos(x)[1-\sin(x)] \\
&= -\sin(x)\cos(x)+\cos(x)-\sin(x)\cos(x) \\
&= \cos(x)-2\sin(x)\cos(x) \\
&= \cos(x)[1-2\sin(x)] .
\end{aligned}$$

So that $f_x(x,y)=0$ means $\cos(x)=0$ or $\sin(x)=\frac{1}{2}$.

Thus, on our interval of consideration, $\sin(x)=\frac{1}{2}$, as the other yields the boundary as solutions.

To find the absolute maximum of *f*, we check the boundary and this critical point. First, the boundary is the rectangle formed by $x=0$, $y=0$, $x=\frac{\pi}{2}$, and $y=\frac{\pi}{2}$:

Recalling $f(x,y)=\sin(x)[\sin(x+2y)-\sin(x)]$:

$$f(0,y)=\sin(0)[\sin(2y)-\sin(0)]=0$$

$$f(x,0)=\sin(x)[\sin(x)-\sin(x)]=0$$

$$f\left(\frac{\pi}{2,}y\right)=\sin\left(\frac{\pi}{2}\right)\left[\sin\left(\frac{\pi}{2}+2y\right)-\sin\left(\frac{\pi}{2}\right)\right]=1\left[\sin\left(\frac{\pi}{2}+2y\right)-1\right]\le 0$$

$$f\left(x,\frac{\pi}{2}\right)=2\sin(x)[\sin(x+\pi)-\sin(x)]=\sin(x)[-\sin(x)-\sin(x)]\le 0$$

So, on the boundary, the function never exceeds zero.

At its critical value, though, we have $\sin(x)=\frac{1}{2}$ and $x+2y=\frac{\pi}{2}$ so that

$$f(\text{Critical})=\frac{1}{2}\left[\sin\left(\frac{\pi}{2}\right)-\frac{1}{2}\right]=\frac{1}{2}\left[1-\frac{1}{2}\right]=\frac{1}{4}\,.$$

Thus, we know *f* attains its maximum value in our desired region at this critical point, and we have

$$\begin{aligned}\frac{1}{4} &\geq f(x,y)\\ &= \sin(x)[\sin(x+2y)-\sin(x)]\\ &= \sin(x)[2\sin(y)\cos(x+y)]\\ &= 2\sin(x)[\sin(y)\cos(x+y)]\\ &= 2\sin(x)\sin(y)\sin(z)\,,\end{aligned}$$

so that

$$2\sin(x)\sin(y)\sin(z)\leq\frac{1}{4}\,,$$

which means

$$\sin(x)\sin(y)\sin(z)\leq\frac{1}{8}\,,$$

and the lemma holds.

∎

First Solution to Example 5.4. Based on **[MOR]**

We begin by denoting P_2 and P_3 as the feet of the perpendiculars from P to $\overline{A_1 A_3}$ and $\overline{A_1 A_2}$ respectively. Also, let α_i be the interior angle of the triangle having vertex A_i.

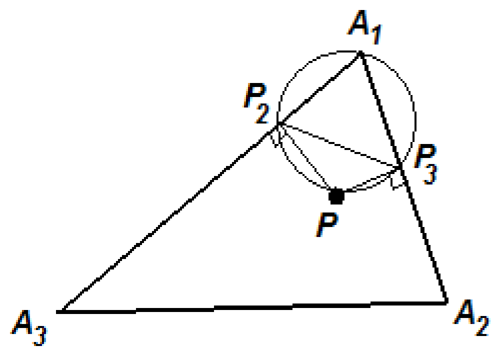

Figure 5.2

Noting that $P P_3 A_1 P_2$ has opposite angles that are both right angles, we realize that it must be cyclic. Additionally, $\overline{P A_1}$ is its diameter. By Lemma 2.4 applied to $\triangle A_1 P_2 P_3$, we get

$$P_2 P_3 = P A_1 \sin(\alpha_1) .$$

Now, we notice that $P P_3 A_1 P_2$ must have its interior angles add to 2π, so that $m\angle P_2 P P_3 = \pi - \alpha_1$. Recalling a trigonometric identity, we know $\cos(m\angle P_2 P P_3) = \cos(\pi - \alpha_1) = -\cos(\alpha_1)$.

When applying our the Law of Cosines to $\triangle P P_2 P_3$ to where we left off, we get:

$$(P A_1)^2 \sin^2(\alpha_1)$$

$$= (P_2 P_3)^2$$

$$= p_2^2 + p_3^2 - 2 p_2 p_3 \cos(\pi - \alpha_1)$$

$$= p_2^2 + p_3^2 + 2 p_2 p_3 \cos(\alpha_1)$$

By Lemma 5.4.1

$$= (p_2 + p_3)^2 \cos^2\left(\frac{\alpha_1}{2}\right) + (p_2 - p_3)^2 \sin^2\left(\frac{\alpha_1}{2}\right)$$

$$\geq \quad (p_2+p_3)^2\cos^2\left(\frac{\alpha_1}{2}\right),$$

since $(p_2-p_3)^2\sin^2\left(\frac{\alpha_1}{2}\right)$ is non-negative.

Thus, we have $(PA_1)^2\sin^2(\alpha_1)\geq(p_2+p_3)^2\cos^2\left(\frac{\alpha_1}{2}\right)$, or equivalently

$$(PA_1)\sin(\alpha_1)\geq(p_2+p_3)\cos\left(\frac{\alpha_1}{2}\right).$$

By the double-angle identity,

$$2(PA_1)\sin\left(\frac{\alpha_1}{2}\right)\cos\left(\frac{\alpha_1}{2}\right)\geq(p_2+p_3)\cos\left(\frac{\alpha_1}{2}\right),$$

which yields

$$2(PA_1)\geq\frac{p_2+p_3}{\sin\left(\frac{\alpha_1}{2}\right)}.$$

Similarly, we obtain

$$2(PA_2)\geq\frac{p_1+p_3}{\sin\left(\frac{\alpha_2}{2}\right)} \quad \text{and} \quad 2(PA_3)\geq\frac{p_1+p_2}{\sin\left(\frac{\alpha_3}{2}\right)}.$$

Thus, we get

$$\begin{aligned} & 8PA_1\cdot PA_2\cdot PA_3 \\ &= (2PA_1)(2PA_2)(2PA_3) \\ &\geq \left[\frac{p_2+p_3}{\sin\left(\frac{\alpha_1}{2}\right)}\right]\left[\frac{p_1+p_3}{\sin\left(\frac{\alpha_2}{2}\right)}\right]\left[\frac{p_1+p_2}{\sin\left(\frac{\alpha_3}{2}\right)}\right]. \end{aligned}$$

Now, $\frac{\alpha_1}{2}+\frac{\alpha_2}{2}+\frac{\alpha_3}{2}=\frac{\pi}{2}$ since α_1, α_2, and α_3 are the interior angles of the original triangle. Hence, we know Lemma 5.4.2 must apply to the term in the denominator, so that

$$\sin\left(\frac{\alpha_1}{2}\right)\sin\left(\frac{\alpha_2}{2}\right)\sin\left(\frac{\alpha_3}{2}\right)\le\frac{1}{8} \quad \text{or equivalently} \quad \frac{1}{\sin\left(\frac{\alpha_1}{2}\right)\sin\left(\frac{\alpha_2}{2}\right)\sin\left(\frac{\alpha_3}{2}\right)}\ge 8.$$

Therefore, we have

$$8\,PA_1\cdot PA_2\cdot PA_3=\left[\frac{p_2+p_3}{\sin\left(\frac{\alpha_1}{2}\right)}\right]\left[\frac{p_1+p_3}{\sin\left(\frac{\alpha_2}{2}\right)}\right]\left[\frac{p_1+p_2}{\sin\left(\frac{\alpha_3}{2}\right)}\right]\ge 8(p_2+p_3)(p_1+p_3)(p_1+p_2),$$

so that

$$8\,PA_1\cdot PA_2\cdot PA_3\ge 8(p_2+p_3)(p_1+p_3)(p_1+p_2),$$

which means

$$PA_1\cdot PA_2\cdot PA_3\ge(p_2+p_3)(p_1+p_3)(p_1+p_2),$$

and the result holds.

∎

Second Solution to Example 5.4. Based on **[OP2]**

Based on Corollary 3.5, we have

$$a_1 PA_1 \geq a_2 p_2 + a_3 p_3 , \quad a_2 PA_2 \geq a_1 p_1 + a_3 p_3 , \quad \text{and} \quad a_3 PA_3 \geq a_1 p_1 + a_2 p_2 ,$$

as well as

$$a_1 PA_1 \geq a_2 p_3 + a_3 p_2 , \quad a_2 PA_2 \geq a_1 p_3 + a_3 p_1 , \quad \text{and} \quad a_3 PA_3 \geq a_1 p_2 + a_2 p_1 .$$

Summing those inequalities involving PA_1 , we have

$$\begin{aligned} 2 a_1 PA_1 &\geq a_2 p_2 + a_3 p_3 + a_2 p_3 + a_3 p_2 \\ &= a_2 (p_2 + p_3) + a_3 (p_2 + p_3) \\ &= (a_2 + a_3)(p_2 + p_3) . \end{aligned}$$

So we get

$$2 a_1 PA_1 = (a_2 + a_3)(p_2 + p_3) .$$

Similarly, we obtain

$$2 a_2 PA_2 = (a_1 + a_3)(p_1 + p_3) \quad \text{and} \quad 2 a_3 PA_3 = (a_1 + a_2)(p_1 + p_2) .$$

Thus, we have

$$\begin{aligned} &8 a_1 a_2 a_3 PA_1 PA_2 PA_3 \\ &\quad = (2 a_1 PA_1)(2 a_2 PA_2)(2 a_3 PA_3) \\ &\quad \geq (a_2 + a_3)(p_2 + p_3)(a_1 + a_3)(p_1 + p_3)(a_1 + a_2)(p_1 + p_2) , \\ &\quad = (a_2 + a_3)(a_1 + a_3)(a_1 + a_2)(p_2 + p_3)(p_1 + p_3)(p_1 + p_2) \end{aligned}$$

and using the Arithmetic Mean – Geometric Mean inequality gives

$$\begin{aligned} &\geq 2\sqrt{a_2 a_3} \cdot 2\sqrt{a_1 a_3} \cdot 2\sqrt{a_1 a_2}(p_2 + p_3)(p_1 + p_3)(p_1 + p_2) \\ &= 8\sqrt{a_1^2 a_2^2 a_3^2}(p_2 + p_3)(p_1 + p_3)(p_1 + p_2) \\ &= 8 a_1 a_2 a_3 (p_2 + p_3)(p_1 + p_3)(p_1 + p_2) . \end{aligned}$$

So we have

$$8a_1a_2a_3PA_1PA_2PA_3 \geq 8a_1a_2a_3(p_2+p_3)(p_1+p_3)(p_1+p_2) \, ,$$

and when we divide each side by $8a_1a_2a_3$, we get our desired result:

$$PA_1 \cdot PA_2 \cdot PA_3 \geq (p_2+p_3)(p_1+p_3)(p_1+p_2) \, .$$

∎

Example 5.5. **[OP2]**

Given $\triangle A_1 A_2 A_3$ and interior point *P*, let p_i denote the distance from *P* to the side of $\triangle A_1 A_2 A_3$ across from vertex A_i. Then

$$PA_1 \cdot PA_2 + PA_1 \cdot PA_3 + PA_2 \cdot PA_3$$

$$\geq \quad (p_2+p_3)(p_1+p_3)+(p_2+p_3)(p_1+p_2)+(p_1+p_3)(p_1+p_2) .$$

Solution to Example 5.5. Based on **[OP2]**

Though this solution is based on Oppenheim's **[OP2]**, his omits many of the details, and does not discuss each case.

Let a_i denote the length of the side of $\triangle A_1 A_2 A_3$ across from vertex A_i. Without loss of generality, assume $a_1 \geq a_2 \geq a_3$. First, we notice that

$$(p_2+p_3)(p_1+p_3)+(p_2+p_3)(p_1+p_2)+(p_1+p_3)(p_1+p_2)$$

$$= \quad p_1 p_2 + p_2 p_3 + p_1 p_3 + p_3^2$$

$$+ \quad p_1 p_2 + p_2^2 + p_1 p_3 + p_2 p_3$$

$$+ \quad p_1^2 + p_1 p_2 + p_1 p_3 + p_2 p_3$$

$$= \quad p_1^2 + p_2^2 + p_3^2 + 3(p_1 p_2 + p_1 p_3 + p_2 p_3) .$$

Thus, to prove the desired inequality, it suffices to show

$$PA_1 \cdot PA_2 + PA_1 \cdot PA_3 + PA_2 \cdot PA_3$$

$$\geq \quad p_1^2 + p_2^2 + p_3^2 + 3(p_1 p_2 + p_1 p_3 + p_2 p_3) .$$

To do this, we use Corollary 3.5, which gives

$$PA_1 \geq \frac{a_2 p_2 + a_3 p_3}{a_1}, \quad PA_2 \geq \frac{a_1 p_1 + a_3 p_3}{a_2}, \text{ and } PA_3 \geq \frac{a_1 p_1 + a_2 p_2}{a_3},$$

in addition to

$$PA_1 \geq \frac{a_2 p_3 + a_3 p_2}{a_1}, \quad PA_2 \geq \frac{a_1 p_3 + a_3 p_1}{a_2}, \text{ and } PA_3 \geq \frac{a_1 p_2 + a_2 p_1}{a_3},$$

so that

$$PA_1 \geq \max\left\{\frac{a_2 p_2 + a_3 p_3}{a_1}, \frac{a_2 p_3 + a_3 p_2}{a_1}\right\};$$

$$PA_2 \geq \max\left\{\frac{a_1 p_1 + a_3 p_3}{a_2}, \frac{a_1 p_3 + a_3 p_1}{a_2}\right\}; \text{ and} \qquad (5.5.A)$$

$$PA_3 \geq \max\left\{\frac{a_1 p_1 + a_2 p_2}{a_3}, \frac{a_1 p_2 + a_2 p_1}{a_3}\right\}.$$

To complete this proof, we will consider cases based on the ordering of p_1, p_2, and p_3. Cases are handled similarly. In each case, we use the maximum option to pair the larger values of a_i and p_i together and the smaller values of a_i and p_i together.

Case 1. $p_1 \geq p_2 \geq p_3$.

Here, we choose

$$PA_1 \geq \frac{a_2 p_2 + a_3 p_3}{a_1}, \quad PA_2 \geq \frac{a_1 p_1 + a_3 p_3}{a_2}, \text{ and } PA_3 \geq \frac{a_1 p_1 + a_2 p_2}{a_3}.$$

so that

$$\begin{aligned} &PA_1 \cdot PA_2 + PA_1 \cdot PA_3 + PA_2 \cdot PA_3 \\ &\geq \left(\frac{a_2 p_2 + a_3 p_3}{a_1}\right)\left(\frac{a_1 p_1 + a_3 p_3}{a_2}\right) \\ &\quad + \left(\frac{a_2 p_2 + a_3 p_3}{a_1}\right)\left(\frac{a_1 p_1 + a_2 p_2}{a_3}\right) \\ &\quad + \left(\frac{a_1 p_1 + a_3 p_3}{a_2}\right)\left(\frac{a_1 p_1 + a_2 p_2}{a_3}\right) \end{aligned}$$

$$= \frac{a_1 a_2 p_1 p_2 + a_2 a_3 p_2 p_3 + a_1 a_3 p_1 p_3 + a_3^2 p_3^2}{a_1 a_2}$$

$$+ \frac{a_1 a_2 p_1 p_2 + a_2^2 p_2^2 + a_1 a_3 p_1 p_3 + a_2 a_3 p_2 p_3}{a_1 a_3}$$

$$+ \frac{a_1^2 p_1^2 + a_1 a_2 p_1 p_2 + a_1 a_3 p_1 p_3 + a_2 a_3 p_2 p_3}{a_2 a_3}$$

$$= p_1 p_2 + \frac{a_3}{a_1} p_2 p_3 + \frac{a_3}{a_2} p_1 p_3 + \frac{a_3^2}{a_1 a_2} p_3^2$$

$$+ \frac{a_2}{a_3} p_1 p_2 + \frac{a_2^2}{a_1 a_3} p_2^2 + p_1 p_3 + \frac{a_2}{a_1} p_2 p_3$$

$$+ \frac{a_1^2}{a_2 a_3} p_1^2 + \frac{a_1}{a_3} p_1 p_2 + \frac{a_1}{a_2} p_1 p_3 + p_2 p_3$$

And by rearranging terms, we get

$$= \frac{a_1^2}{a_2 a_3} p_1^2 + \frac{a_2^2}{a_1 a_3} p_2^2 + \frac{a_3^2}{a_1 a_2} p_3^2$$

$$+ \left(1 + \frac{a_2}{a_3} + \frac{a_1}{a_3}\right) p_1 p_2 + \left(1 + \frac{a_3}{a_2} + \frac{a_1}{a_2}\right) p_1 p_3 + \left(1 + \frac{a_3}{a_1} + \frac{a_2}{a_1}\right) p_2 p_3 \ .$$

We need to show this is at least $p_1^2 + p_2^2 + p_3^2 + 3(p_1 p_2 + p_1 p_3 + p_2 p_3)$.

To do this, we will show that

$$\frac{a_1^2}{a_2 a_3} p_1^2 + \frac{a_2^2}{a_1 a_3} p_2^2 + \frac{a_3^2}{a_1 a_2} p_3^2 \geq p_1^2 + p_2^2 + p_3^2$$

and

$$\left(1 + \frac{a_2}{a_3} + \frac{a_1}{a_3}\right) p_1 p_2 + \left(1 + \frac{a_3}{a_2} + \frac{a_1}{a_2}\right) p_1 p_3 + \left(1 + \frac{a_3}{a_1} + \frac{a_2}{a_1}\right) p_2 p_3 \geq 3(p_1 p_2 + p_1 p_3 + p_2 p_3) \ .$$

First, to show $\frac{a_1^2}{a_2 a_3} p_1^2 + \frac{a_2^2}{a_1 a_3} p_2^2 + \frac{a_3^2}{a_1 a_2} p_3^2 \geq p_1^2 + p_2^2 + p_3^2$:

Let

$$A = \frac{a_1^2}{a_2 a_3}, \quad B = \frac{a_2^2}{a_1 a_3}, \text{ and } \quad C = \frac{a_3^2}{a_1 a_2} .$$

Then, since $a_1 \geq a_2 \geq a_3$, we know $a_1^2 \geq a_2 a_3$ so that

$$A \geq 1 .$$

We also have $A + B = \frac{a_1^2}{a_2 a_3} + \frac{a_2^2}{a_1 a_3}$, and since $a_1 \geq a_2 \geq a_3$ meaning $a_1 a_2 \geq a_3^2$ coupled with the use of the Arithmetic Mean – Geometric Mean Inequality yields

$$A + B \geq 2\sqrt{\frac{a_1^2 a_2^2}{a_1 a_2 a_3^2}} = 2\sqrt{\frac{a_1 a_2}{a_3^2}} \geq 2 ,$$

so that

$$A + B \geq 2 .$$

We also have $A + B + C = \frac{a_1^2}{a_2 a_3} + \frac{a_2^2}{a_1 a_3} + \frac{a_3^2}{a_1 a_2}$, and the Arithmetic Mean – Geometric Mean Inequality tells us

$$A + B + C \geq 3\sqrt[3]{\frac{a_1^2 a_2^2 a_3^2}{a_1^2 a_2^2 a_3^2}} = 3 ,$$

so that

$$A + B + C \geq 3 .$$

This combines to mean

$$\frac{a_1^2}{a_2 a_3} p_1^2 + \frac{a_2^2}{a_1 a_3} p_2^2 + \frac{a_3^2}{a_1 a_2} p_3^2$$

$$= \quad A p_1^2 + B p_2^2 + C p_3^2$$

$$= \quad A p_1^2 - A p_2^2$$

$$+ \quad A p_2^2 + B p_2^2 - A p_3^2 - B p_3^2$$

$$+ \quad A p_3^2 + B p_3^2 + C p_3^2$$

$$= \quad A(p_1^2 - p_2^2) + (A+B)(p_2^2 - p_3^2) + (A+B+C) p_3^2$$

Since $p_1 \geq p_2 \geq p_3$, we know $p_1^2 - p_2^2 \geq 0$ and $p_2^2 - p_3^2 \geq 0$, so

$$\geq \quad 1(p_1^2 - p_2^2) + 2(p_2^2 - p_3^2) + 3 p_3^2$$

$$= \quad p_1^2 - p_2^2 + 2 p_2^2 - 2 p_3^2 + 3 p_3^2$$

$$= \quad p_1^2 + p_2^2 + p_3^2,$$

So we have

$$\frac{a_1^2}{a_2 a_3} p_1^2 + \frac{a_2^2}{a_1 a_3} p_2^2 + \frac{a_3^2}{a_1 a_2} p_3^2 \geq p_1^2 + p_2^2 + p_3^2 .$$

To show

$$\left(1 + \frac{a_2}{a_3} + \frac{a_1}{a_3}\right) p_1 p_2 + \left(1 + \frac{a_3}{a_2} + \frac{a_1}{a_2}\right) p_1 p_3 + \left(1 + \frac{a_3}{a_1} + \frac{a_2}{a_1}\right) p_2 p_3 \geq 3(p_1 p_2 + p_1 p_3 + p_2 p_3):$$

We begin by realizing that this case requires $p_1 p_2 \geq p_1 p_3 \geq p_2 p_3$, and we let

$$A = 1 + \frac{a_2}{a_3} + \frac{a_1}{a_3}, \qquad B = 1 + \frac{a_3}{a_2} + \frac{a_1}{a_2}, \qquad \text{and} \qquad C = 1 + \frac{a_3}{a_1} + \frac{a_2}{a_1}.$$

Then, we have $A=1+\frac{a_2}{a_3}+\frac{a_1}{a_3}$, and since $a_1 \geq a_2 \geq a_3$, we know

$$A \geq 3 .$$

We have $A+B=2+\frac{a_2}{a_3}+\frac{a_1}{a_3}+\frac{a_3}{a_2}+\frac{a_1}{a_2}$, and the Arithmetic Mean – Geometric Mean Inequality coupled with $a_1 \geq a_2 \geq a_3$, meaning $a_1^2 \geq a_2 a_3$, tells us

$$A+B \geq 2+4\sqrt[4]{\frac{a_1^2 a_2 a_3}{a_2^2 a_3^2}}=2+4\sqrt[4]{\frac{a_1^2}{a_2 a_3}} \geq 2+4=6 ,$$

so that

$$A+B \geq 6 .$$

We also have $A+B+C=3+\frac{a_2}{a_3}+\frac{a_1}{a_3}+\frac{a_3}{a_2}+\frac{a_1}{a_2}+\frac{a_3}{a_1}+\frac{a_2}{a_1}$, and the Arithmetic Mean – Geometric Mean Inequality tells us

$$A+B+C \geq 3+6\sqrt[6]{\frac{a_1^2 a_2^2 a_3^2}{a_1^2 a_2^2 a_3^2}}=3+6=9 ,$$

so that

$$A+B+C \geq 9 .$$

This combines to mean

$$\left(1+\frac{a_2}{a_3}+\frac{a_1}{a_3}\right) p_1 p_2+\left(1+\frac{a_3}{a_2}+\frac{a_1}{a_2}\right) p_1 p_3+\left(1+\frac{a_3}{a_1}+\frac{a_2}{a_1}\right) p_2 p_3$$

$$= \quad A p_1 p_2+B p_1 p_3+C p_2 p_3$$

$$= \quad A p_1 p_2-A p_1 p_3$$

$$+ \quad A p_1 p_3+B p_1 p_3-A p_2 p_3-B p_2 p_3$$

$$+ \quad A p_2 p_3+B p_2 p_3+C p_2 p_3$$

$$= \quad A(p_1 p_2-p_1 p_3)+(A+B)(p_1 p_3-p_2 p_3)+(A+B+C) p_2 p_3$$

Since $p_1 p_2 \geq p_1 p_3 \geq p_2 p_3$, we know $p_1 p_2-p_1 p_3 \geq 0$ and $p_1 p_3-p_2 p_3 \geq 0$, so

$$\geq \quad 3(p_1 p_2 - p_1 p_3) + 6(p_1 p_3 - p_2 p_3) + 9 p_2 p_3$$

$$= \quad 3 p_1 p_2 - 3 p_1 p_3 + 6 p_1 p_3 - 6 p_2 p_3 + 9 p_2 p_3$$

$$= \quad 3 p_1 p_2 + 3 p_1 p_3 + 3 p_2 p_3 ,$$

so that

$$\left(1 + \frac{a_2}{a_3} + \frac{a_1}{a_3}\right) p_1 p_2 + \left(1 + \frac{a_3}{a_2} + \frac{a_1}{a_2}\right) p_1 p_3 + \left(1 + \frac{a_3}{a_1} + \frac{a_2}{a_1}\right) p_2 p_3$$

$$\geq \quad 3(p_1 p_2 + p_1 p_3 + p_2 p_3) ,$$

as desired.

So Case 1 holds, as we have shown

$$P A_1 \cdot P A_2 + PA_1 \cdot PA_3 + PA_2 \cdot P A_3$$

$$\geq \quad \frac{a_1^2}{a_2 a_3} p_1^2 + \frac{a_2^2}{a_1 a_3} p_2^2 + \frac{a_3^2}{a_1 a_2} p_3^2$$

$$+ \quad \left(1 + \frac{a_2}{a_3} + \frac{a_1}{a_3}\right) p_1 p_2 + \left(1 + \frac{a_3}{a_2} + \frac{a_1}{a_2}\right) p_1 p_3 + \left(1 + \frac{a_3}{a_1} + \frac{a_2}{a_1}\right) p_2 p_3$$

$$\geq \quad p_1^2 + p_2^2 + p_3^2 + 3(p_1 p_2 + p_1 p_3 + p_2 p_3)$$

$$= \quad (p_2 + p_3)(p_1 + p_3) + (p_2 + p_3)(p_1 + p_2) + (p_1 + p_3)(p_1 + p_2) ,$$

which gives

$$P A_1 \cdot P A_2 + PA_1 \cdot PA_3 + PA_2 \cdot P A_3$$

$$\geq \quad (p_2 + p_3)(p_1 + p_3) + (p_2 + p_3)(p_1 + p_2) + (p_1 + p_3)(p_1 + p_2) .$$

Case 2. $p_2 \geq p_3 \geq p_1$.

Again, we use the maximum option to pair the larger values of a_i and p_i together and the smaller values of a_i and p_i together in (*5.5.A*):

$$PA_1 \geq \frac{a_2 p_2 + a_3 p_3}{a_1}, \quad PA_2 \geq \frac{a_1 p_3 + a_3 p_1}{a_2}, \text{ and } PA_3 \geq \frac{a_1 p_2 + a_2 p_1}{a_3} .$$

so that

$$PA_1 \cdot PA_2 + PA_1 \cdot PA_3 + PA_2 \cdot PA_3$$

$$\geq \left(\frac{a_2 p_2 + a_3 p_3}{a_1}\right)\left(\frac{a_1 p_3 + a_3 p_1}{a_2}\right)$$

$$+ \left(\frac{a_2 p_2 + a_3 p_3}{a_1}\right)\left(\frac{a_1 p_2 + a_2 p_1}{a_3}\right)$$

$$+ \left(\frac{a_1 p_3 + a_3 p_1}{a_2}\right)\left(\frac{a_1 p_2 + a_2 p_1}{a_3}\right)$$

$$= \frac{a_1 a_2 p_2 p_3 + a_2 a_3 p_1 p_2 + a_1 a_3 p_3^2 + a_3^2 p_1 p_3}{a_1 a_2}$$

$$+ \frac{a_1 a_2 p_2^2 + a_2^2 p_1 p_2 + a_1 a_3 p_2 p_3 + a_2 a_3 p_1 p_3}{a_1 a_3}$$

$$+ \frac{a_1^2 p_2 p_3 + a_1 a_2 p_1 p_3 + a_1 a_3 p_1 p_2 + a_2 a_3 p_1^2}{a_2 a_3}$$

$$= p_2 p_3 + \frac{a_3}{a_1} p_1 p_2 + \frac{a_3}{a_2} p_3^2 + \frac{a_3^2}{a_1 a_2} p_1 p_3$$

$$+ \frac{a_2}{a_3} p_2^2 + \frac{a_2^2}{a_1 a_3} p_1 p_2 + p_2 p_3 + \frac{a_2}{a_1} p_1 p_3$$

$$+ \frac{a_1^2}{a_2 a_3} p_2 p_3 + \frac{a_1}{a_3} p_1 p_3 + \frac{a_1}{a_2} p_1 p_2 + p_1^2$$

And by rearranging terms, we get

$$= \quad \frac{a_2}{a_3} p_2^2 + \frac{a_3}{a_2} p_3^2 + p_1^2$$

$$+ \quad \left(2+\frac{a_1^2}{a_2 a_3}\right) p_2 p_3 + \left(\frac{a_3}{a_1}+\frac{a_2^2}{a_1 a_3}+\frac{a_1}{a_2}\right) p_1 p_2 + \left(\frac{a_3^2}{a_1 a_2}+\frac{a_2}{a_1}+\frac{a_1}{a_3}\right) p_1 p_3 \ .$$

We need to show this is at least $p_1^2 + p_2^2 + p_3^2 + 3(p_1 p_2 + p_1 p_3 + p_2 p_3)$.

Again, to do this, we will show that

$$\frac{a_2}{a_3} p_2^2 + \frac{a_3}{a_2} p_3^2 + p_1^2 \geq p_1^2 + p_2^2 + p_3^2$$

and

$$\left(2+\frac{a_1^2}{a_2 a_3}\right) p_2 p_3 + \left(\frac{a_3}{a_1}+\frac{a_2^2}{a_1 a_3}+\frac{a_1}{a_2}\right) p_1 p_2 + \left(\frac{a_3^2}{a_1 a_2}+\frac{a_2}{a_1}+\frac{a_1}{a_3}\right) p_1 p_3$$

$$\geq \quad 3(p_1 p_2 + p_1 p_3 + p_2 p_3) \ .$$

First, to show $\frac{a_2}{a_3} p_2^2 + \frac{a_3}{a_2} p_3^2 + p_1^2 \geq p_1^2 + p_2^2 + p_3^2$:

Let

$$A = \frac{a_2}{a_3}, \quad B = \frac{a_3}{a_2}, \text{ and } \quad C = 1 \ .$$

Then, since $a_1 \geq a_2 \geq a_3$, we know

$$A \geq 1 \ .$$

We also have $A + B = \frac{a_2}{a_3} + \frac{a_3}{a_2}$, and the Arithmetic Mean – Geometric Mean Inequality yields

$$A + B \geq 2\sqrt{\frac{a_2 a_3}{a_2 a_3}} = 2 \ ,$$

so that

$$A+B\geq 2 \ .$$

We also have $A+B+C=1+\frac{a_2}{a_3}+\frac{a_3}{a_2}$, and the Arithmetic Mean – Geometric Mean Inequality tells us

$$A+B+C\geq 1+2\sqrt{\frac{a_2 a_3}{a_2 a_3}}=1+2=3 \ ,$$

so that

$$A+B+C\geq 3 \ .$$

This combines to mean

$$\frac{a_2}{a_3}p_2^2+\frac{a_3}{a_2}p_3^2+p_1^2$$

$$= \quad A\,p_2^2+B\,p_3^2+C\,p_1^2$$

$$= \quad A\,p_2^2-A\,p_3^2$$

$$+ \quad A\,p_3^2+B\,p_3^2-A\,p_1^2-B\,p_1^2$$

$$+ \quad A\,p_1^2+B\,p_1^2+C\,p_1^2$$

$$= \quad A(p_2^2-p_3^2)+(A+B)(p_3^2-p_1^2)+(A+B+C)\,p_1^2$$

Since $p_2\geq p_3\geq p_1$, we know $p_2^2-p_3^2\geq 0$ and $p_3^2-p_1^2\geq 0$, so

$$\geq \quad 1(p_2^2-p_3^2)+2(p_3^2-p_1^2)+3\,p_1^2$$

$$= \quad p_2^2-p_3^2+2\,p_3^2-2\,p_1^2+3\,p_1^2$$

$$= \quad p_1^2+p_2^2+p_3^2 \ ,$$

So we have

$$\frac{a_2}{a_3}p_2^2+\frac{a_3}{a_2}p_3^2+p_1^2\geq p_1^2+p_2^2+p_3^2 \ .$$

To show

$$\left(2+\frac{a_1^2}{a_2 a_3}\right)p_2 p_3+\left(\frac{a_3}{a_1}+\frac{a_2^2}{a_1 a_3}+\frac{a_1}{a_2}\right)p_1 p_2+\left(\frac{a_3^2}{a_1 a_2}+\frac{a_2}{a_1}+\frac{a_1}{a_3}\right)p_1 p_3$$

$$\geq \quad 3(p_1 p_2+p_1 p_3+p_2 p_3):$$

We begin by realizing that this case requires $p_2 p_3 \geq p_1 p_2 \geq p_2 p_3$, and we let

$$A=2+\frac{a_1^2}{a_2 a_3}, \quad B=\frac{a_3}{a_1}+\frac{a_2^2}{a_1 a_3}+\frac{a_1}{a_2}, \text{ and } \quad C=\frac{a_3^2}{a_1 a_2}+\frac{a_2}{a_1}+\frac{a_1}{a_3}.$$

Then, we have $A=2+\frac{a_1^2}{a_2 a_3}$, and since $a_1 \geq a_2 \geq a_3$, we know $a_1^2 \geq a_2 a_3$, so that

$$A \geq 3.$$

We have $A+B=2+\frac{a_1^2}{a_2 a_3}+\frac{a_3}{a_1}+\frac{a_2^2}{a_1 a_3}+\frac{a_1}{a_2}$, and the Arithmetic Mean – Geometric Mean Inequality coupled with $a_1 \geq a_2 \geq a_3$, meaning $a_1 \geq a_3$, tells us

$$A+B \geq 2+4\sqrt[4]{\frac{a_1^3 a_2^2 a_3}{a_1^2 a_2^2 a_3^2}}=2+4\sqrt[4]{\frac{a_1}{a_3}} \geq 2+4=6,$$

so that

$$A+B \geq 6.$$

We also have $A+B+C=2+\frac{a_1^2}{a_2 a_3}+\frac{a_3}{a_1}+\frac{a_2^2}{a_1 a_3}+\frac{a_1}{a_2}+\frac{a_3^2}{a_1 a_2}+\frac{a_2}{a_1}+\frac{a_1}{a_3}$, and the Arithmetic Mean – Geometric Mean Inequality tells us

$$A+B+C \geq 2+7\sqrt[7]{\frac{a_1^4 a_2^3 a_3^3}{a_1^4 a_2^3 a_3^3}}=2+7=9,$$

so that

$$A+B+C \geq 9.$$

This combines to mean

$$\left(2+\frac{a_1^2}{a_2a_3}\right)p_2p_3+\left(\frac{a_3}{a_1}+\frac{a_2^2}{a_1a_3}+\frac{a_1}{a_2}\right)p_1p_2+\left(\frac{a_3^2}{a_1a_2}+\frac{a_2}{a_1}+\frac{a_1}{a_3}\right)p_1p_3$$

$$= \quad A\,p_2p_3+B\,p_1p_2+C\,p_1p_3$$

$$= \quad A\,p_2p_3-A\,p_1p_2$$

$$+ \quad A\,p_1p_2+B\,p_1p_2-A\,p_1p_3-B\,p_1p_3$$

$$+ \quad A\,p_1p_3+B\,p_1p_3+C\,p_1p_3$$

$$= \quad A(p_2p_3-p_1p_2)+(A+B)(p_1p_2-p_1p_3)+(A+B+C)\,p_1p_3$$

Since $p_2p_3\geq p_1p_2\geq p_1p_3$, we know $p_2p_3-p_1p_2\geq 0$ and $p_1p_2-p_1p_3\geq 0$, so

$$\geq \quad 3(p_2p_3-p_1p_2)+6(p_1p_2-p_1p_3)+9\,p_1p_3$$

$$= \quad 3\,p_2p_3-3\,p_1p_2+6\,p_1p_2-6\,p_1p_3+9\,p_1p_3$$

$$= \quad 3\,p_1p_2+3\,p_1p_3+3\,p_2p_3\,,$$

so that

$$\left(2+\frac{a_1^2}{a_2a_3}\right)p_2p_3+\left(\frac{a_3}{a_1}+\frac{a_2^2}{a_1a_3}+\frac{a_1}{a_2}\right)p_1p_2+\left(\frac{a_3^2}{a_1a_2}+\frac{a_2}{a_1}+\frac{a_1}{a_3}\right)p_1p_3$$

$$\geq \quad 3(p_1p_2+p_1p_3+p_2p_3)\,,$$

as desired.

So Case 2 holds, as we have shown

$$PA_1\cdot PA_2+PA_1\cdot PA_3+PA_2\cdot PA_3$$

$$\geq \quad p_1^2+\frac{a_2}{a_3}p_2^2+\frac{a_3}{a_2}p_3^2$$

$$+ \quad \left(2+\frac{a_1^2}{a_2a_3}\right)p_2p_3+\left(\frac{a_3}{a_1}+\frac{a_2^2}{a_1a_3}+\frac{a_1}{a_2}\right)p_1p_2+\left(\frac{a_3^2}{a_1a_2}+\frac{a_2}{a_1}+\frac{a_1}{a_3}\right)p_1p_3$$

$$\geq \quad p_1^2 + p_2^2 + p_3^2 + 3(p_1 p_2 + p_1 p_3 + p_2 p_3)$$

$$= \quad (p_2 + p_3)(p_1 + p_3) + (p_2 + p_3)(p_1 + p_2) + (p_1 + p_3)(p_1 + p_2)\,,$$

which gives

$$PA_1 \cdot PA_2 + PA_1 \cdot PA_3 + PA_2 \cdot PA_3$$

$$\geq \quad (p_2 + p_3)(p_1 + p_3) + (p_2 + p_3)(p_1 + p_2) + (p_1 + p_3)(p_1 + p_2)\,.$$

Case 3. $p_3 \geq p_2 \geq p_1$.

Again, we use the maximum option to pair the larger values of a_i and p_i together and the smaller values of a_i and p_i together in (*5.5.A*):

$$PA_1 \geq \frac{a_2 p_3 + a_3 p_2}{a_1}, \quad PA_2 \geq \frac{a_1 p_3 + a_3 p_1}{a_2}, \text{ and } PA_3 \geq \frac{a_1 p_2 + a_2 p_1}{a_3} .$$

so that

$$\begin{aligned}
& PA_1 \cdot PA_2 + PA_1 \cdot PA_3 + PA_2 \cdot PA_3 \\
\geq \quad & \left(\frac{a_2 p_3 + a_3 p_2}{a_1}\right)\left(\frac{a_1 p_3 + a_3 p_1}{a_2}\right) \\
& + \left(\frac{a_2 p_3 + a_3 p_2}{a_1}\right)\left(\frac{a_1 p_2 + a_2 p_1}{a_3}\right) \\
& + \left(\frac{a_1 p_3 + a_3 p_1}{a_2}\right)\left(\frac{a_1 p_2 + a_2 p_1}{a_3}\right) \\
= \quad & \frac{a_1 a_2 p_3^2 + a_2 a_3 p_1 p_3 + a_1 a_3 p_2 p_3 + a_3^2 p_1 p_2}{a_1 a_2} \\
& + \frac{a_1 a_2 p_2 p_3 + a_2^2 p_1 p_3 + a_1 a_3 p_2^2 + a_2 a_3 p_1 p_2}{a_1 a_3} \\
& + \frac{a_1^2 p_2 p_3 + a_1 a_2 p_1 p_3 + a_1 a_3 p_1 p_2 + a_2 a_3 p_1^2}{a_2 a_3} \\
= \quad & p_3^2 + \frac{a_3}{a_1} p_1 p_3 + \frac{a_3}{a_2} p_2 p_3 + \frac{a_3^2}{a_1 a_2} p_1 p_2 \\
& + \frac{a_2}{a_3} p_2 p_3 + \frac{a_2^2}{a_1 a_3} p_1 p_3 + p_2^2 + \frac{a_2}{a_1} p_1 p_2 \\
& + \frac{a_1^2}{a_2 a_3} p_2 p_3 + \frac{a_1}{a_3} p_1 p_3 + \frac{a_1}{a_2} p_1 p_2 + p_1^2
\end{aligned}$$

And by rearranging terms, we get

$$= \quad p_1^2+p_2^2+p_3^2$$

$$+ \quad \left(\frac{a_3}{a_2}+\frac{a_2}{a_3}+\frac{a_1^2}{a_2 a_3}\right)p_2 p_3+\left(\frac{a_3}{a_1}+\frac{a_2^2}{a_1 a_3}+\frac{a_1}{a_3}\right)p_1 p_3+\left(\frac{a_3^2}{a_1 a_2}+\frac{a_2}{a_1}+\frac{a_1}{a_2}\right)p_1 p_2 \ .$$

We need to show this is at least $p_1^2+p_2^2+p_3^2+3(p_1 p_2+p_1 p_3+p_2 p_3)$.

In this case, this merely amounts to showing

$$\left(\frac{a_3}{a_2}+\frac{a_2}{a_3}+\frac{a_1^2}{a_2 a_3}\right)p_2 p_3+\left(\frac{a_3}{a_1}+\frac{a_2^2}{a_1 a_3}+\frac{a_1}{a_3}\right)p_1 p_3+\left(\frac{a_3^2}{a_1 a_2}+\frac{a_2}{a_1}+\frac{a_1}{a_2}\right)p_1 p_2$$

$$\geq \quad 3(p_1 p_2+p_1 p_3+p_2 p_3) \ .$$

We begin by realizing that this case requires $p_2 p_3 \geq p_1 p_3 \geq p_1 p_2$, and we let

$$A=\frac{a_3}{a_2}+\frac{a_2}{a_3}+\frac{a_1^2}{a_2 a_3}, \quad B=\frac{a_3}{a_1}+\frac{a_2^2}{a_1 a_3}+\frac{a_1}{a_3}, \text{ and } \quad C=\frac{a_3^2}{a_1 a_2}+\frac{a_2}{a_1}+\frac{a_1}{a_2} \ .$$

Then, we have $A=\frac{a_3}{a_2}+\frac{a_2}{a_3}+\frac{a_1^2}{a_2 a_3}$, and by the Arithmetic Mean – Geometric Mean Inequality and since $a_1 \geq a_2 \geq a_3$, we know $a_1^2 \geq a_2 a_3$, so that

$$A=\frac{a_3}{a_2}+\frac{a_2}{a_3}+\frac{a_1^2}{a_2 a_3} \geq 2\sqrt{\frac{a_2 a_3}{a_2 a_3}}+1=2+1=3 \ .$$

Thus

$$A \geq 3 \ .$$

We have $A+B=\frac{a_3}{a_2}+\frac{a_2}{a_3}+\frac{a_1^2}{a_2 a_3}+\frac{a_3}{a_1}+\frac{a_2^2}{a_1 a_3}+\frac{a_1}{a_3}$, and the Arithmetic Mean – Geometric Mean Inequality coupled with $a_1 \geq a_2 \geq a_3$, meaning $a_1 a_2 \geq a_3^2$, tells us

$$A+B\geq 6\sqrt[6]{\frac{a_1^3a_2^3a_3^2}{a_1^2a_2^2a_3^4}}=6\sqrt[6]{\frac{a_1a_2}{a_3^2}}\geq 6\ ,$$

so that

$$A+B\geq 6\ .$$

We also have $A+B+C=\frac{a_3}{a_2}+\frac{a_2}{a_3}+\frac{a_1^2}{a_2a_3}+\frac{a_3}{a_1}+\frac{a_2^2}{a_1a_3}+\frac{a_1}{a_3}+\frac{a_3^2}{a_1a_2}+\frac{a_2}{a_1}+\frac{a_1}{a_2}$, and the Arithmetic Mean – Geometric Mean Inequality tells us

$$A+B+C\geq 9\sqrt[9]{\frac{a_1^4a_2^4a_3^4}{a_1^4a_2^4a_3^4}}=9\ ,$$

so that

$$A+B+C\geq 9\ .$$

This combines to mean

$$\left(\frac{a_3}{a_2}+\frac{a_2}{a_3}+\frac{a_1^2}{a_2a_3}\right)p_2p_3+\left(\frac{a_3}{a_1}+\frac{a_2^2}{a_1a_3}+\frac{a_1}{a_3}\right)p_1p_3+\left(\frac{a_3^2}{a_1a_2}+\frac{a_2}{a_1}+\frac{a_1}{a_2}\right)p_1p_2$$

$$= \quad A\,p_2p_3+B\,p_1p_3+C\,p_1p_2$$

$$= \quad A\,p_2p_3-A\,p_1p_3$$

$$+ \quad A\,p_1p_3+B\,p_1p_3-A\,p_1p_2-B\,p_1p_2$$

$$+ \quad A\,p_1p_2+B\,p_1p_2+C\,p_1p_2$$

$$= \quad A(p_2p_3-p_1p_3)+(A+B)(p_1p_3-p_1p_2)+(A+B+C)\,p_1p_2$$

Since $p_2p_3\geq p_1p_3\geq p_1p_2$, we know $p_2p_3-p_1p_3\geq 0$ and $p_1p_3-p_1p_2\geq 0$, so

$$\geq \quad 3(p_2p_3-p_1p_3)+6(p_1p_3-p_1p_2)+9\,p_1p_2$$

$$= \quad 3\,p_2p_3-3\,p_1p_3+6\,p_1p_3-6\,p_1p_2+9\,p_1p_2$$

$$= \quad 3\,p_1p_2+3\,p_1p_3+3\,p_2p_3\ ,$$

so that

$$\left(\frac{a_3}{a_2}+\frac{a_2}{a_3}+\frac{a_1^2}{a_2 a_3}\right)p_2 p_3+\left(\frac{a_3}{a_1}+\frac{a_2^2}{a_1 a_3}+\frac{a_1}{a_3}\right)p_1 p_3+\left(\frac{a_3^2}{a_1 a_2}+\frac{a_2}{a_1}+\frac{a_1}{a_2}\right)p_1 p_2$$

$$\geq \quad 3(p_1 p_2+p_1 p_3+p_2 p_3),$$

as desired.

So Case 3 holds, as we have shown

$$PA_1\cdot PA_2+PA_1\cdot PA_3+PA_2\cdot PA_3$$

$$\geq \quad p_1^2+p_2^2+p_3^2$$

$$+ \quad \left(\frac{a_3}{a_2}+\frac{a_2}{a_3}+\frac{a_1^2}{a_2 a_3}\right)p_2 p_3+\left(\frac{a_3}{a_1}+\frac{a_2^2}{a_1 a_3}+\frac{a_1}{a_3}\right)p_1 p_3+\left(\frac{a_3^2}{a_1 a_2}+\frac{a_2}{a_1}+\frac{a_1}{a_2}\right)p_1 p_2$$

$$\geq \quad p_1^2+p_2^2+p_3^2+3(p_1 p_2+p_1 p_3+p_2 p_3)$$

$$= \quad (p_2+p_3)(p_1+p_3)+(p_2+p_3)(p_1+p_2)+(p_1+p_3)(p_1+p_2),$$

which gives

$$PA_1\cdot PA_2+PA_1\cdot PA_3+PA_2\cdot PA_3$$

$$\geq \quad (p_2+p_3)(p_1+p_3)+(p_2+p_3)(p_1+p_2)+(p_1+p_3)(p_1+p_2).$$

Case 4. $p_1 \geq p_3 \geq p_2$.

Again, we use the maximum option to pair the larger values of a_i and p_i together and the smaller values of a_i and p_i together in (*5.5.A*):

$$PA_1 \geq \frac{a_2 p_3 + a_3 p_2}{a_1}, \quad PA_2 \geq \frac{a_1 p_1 + a_3 p_3}{a_2}, \text{ and } PA_3 \geq \frac{a_1 p_1 + a_2 p_2}{a_3}.$$

so that

$$PA_1 \cdot PA_2 + PA_1 \cdot PA_3 + PA_2 \cdot PA_3$$

$$\geq \left(\frac{a_2 p_3 + a_3 p_2}{a_1}\right)\left(\frac{a_1 p_1 + a_3 p_3}{a_2}\right)$$

$$+ \left(\frac{a_2 p_3 + a_3 p_2}{a_1}\right)\left(\frac{a_1 p_1 + a_2 p_2}{a_3}\right)$$

$$+ \left(\frac{a_1 p_1 + a_3 p_3}{a_2}\right)\left(\frac{a_1 p_1 + a_2 p_2}{a_3}\right)$$

$$= \frac{a_1 a_2 p_1 p_3 + a_2 a_3 p_3^2 + a_1 a_3 p_1 p_2 + a_3^2 p_2 p_3}{a_1 a_2}$$

$$+ \frac{a_1 a_2 p_1 p_3 + a_2^2 p_2 p_3 + a_1 a_3 p_1 p_2 + a_2 a_3 p_2^2}{a_1 a_3}$$

$$+ \frac{a_1^2 p_1^2 + a_1 a_2 p_1 p_2 + a_1 a_3 p_1 p_3 + a_2 a_3 p_2 p_3}{a_2 a_3}$$

$$= p_1 p_3 + \frac{a_3}{a_1} p_3^2 + \frac{a_3}{a_2} p_1 p_2 + \frac{a_3^2}{a_1 a_2} p_2 p_3$$

$$+ \frac{a_2}{a_3} p_1 p_3 + \frac{a_2^2}{a_1 a_3} p_2 p_3 + p_1 p_2 + \frac{a_2}{a_1} p_2^2$$

$$+ \frac{a_1^2}{a_2 a_3} p_1^2 + \frac{a_1}{a_3} p_1 p_2 + \frac{a_1}{a_2} p_1 p_3 + p_2 p_3$$

And by rearranging terms, we get

$$= \quad \frac{a_1^2}{a_2 a_3} p_1^2 + \frac{a_3}{a_1} p_3^2 + \frac{a_2}{a_1} p_2^2$$

$$+ \quad \left(1 + \frac{a_2}{a_3} + \frac{a_1}{a_2}\right) p_1 p_3 + \left(\frac{a_3}{a_2} + 1 + \frac{a_1}{a_3}\right) p_1 p_2 + \left(\frac{a_3^2}{a_1 a_2} + \frac{a_2^2}{a_1 a_3} + 1\right) p_2 p_3 \ .$$

We need to show this is at least $p_1^2 + p_2^2 + p_3^2 + 3(p_1 p_2 + p_1 p_3 + p_2 p_3)$.

Again, to do this, we will show that

$$\frac{a_1^2}{a_2 a_3} p_1^2 + \frac{a_3}{a_1} p_3^2 + \frac{a_2}{a_1} p_2^2 \geq p_1^2 + p_2^2 + p_3^2$$

and

$$\left(1 + \frac{a_2}{a_3} + \frac{a_1}{a_2}\right) p_1 p_3 + \left(\frac{a_3}{a_2} + 1 + \frac{a_1}{a_3}\right) p_1 p_2 + \left(\frac{a_3^2}{a_1 a_2} + \frac{a_2^2}{a_1 a_3} + 1\right) p_2 p_3$$

$$\geq \quad 3(p_1 p_2 + p_1 p_3 + p_2 p_3) \ .$$

First, to show $\frac{a_1^2}{a_2 a_3} p_1^2 + \frac{a_3}{a_1} p_3^2 + \frac{a_2}{a_1} p_2^2 \geq p_1^2 + p_2^2 + p_3^2$:

Let

$$A = \frac{a_1^2}{a_2 a_3} \ , \qquad B = \frac{a_3}{a_1} \text{ , and } \ C = \frac{a_2}{a_1} \ ,$$

Then, we have $A = \frac{a_1^2}{a_2 a_3}$, and since $a_1 \geq a_2 \geq a_3$, we know $a_1^2 \geq a_2 a_3$, so that

$$A \geq 1 \ .$$

We have $A + B = \frac{a_1^2}{a_2 a_3} + \frac{a_3}{a_1}$, and the Arithmetic Mean – Geometric Mean Inequality coupled with $a_1 \geq a_2 \geq a_3$, meaning $a_1 \geq a_2$, tells us

$$A+B \geq 2\sqrt{\frac{a_1^2 a_3}{a_1 a_2 a_3}} = 2\sqrt{\frac{a_1}{a_2}} \geq 2 ,$$

so that

$$A+B \geq 2 .$$

We also have $A+B+C = \frac{a_1^2}{a_2 a_3} + \frac{a_3}{a_1} + \frac{a_2}{a_1}$, and the Arithmetic Mean – Geometric Mean Inequality tells us

$$A+B+C \geq 3\sqrt[3]{\frac{a_1^2 a_2 a_3}{a_1^2 a_2 a_3}} = 3 ,$$

so that

$$A+B+C \geq 3 .$$

This combines to mean

$$\frac{a_1^2}{a_2 a_3} p_1^2 + \frac{a_3}{a_1} p_3^2 + \frac{a_2}{a_1} p_2^2$$

$$= \quad A\, p_1^2 + B\, p_3^2 + C\, p_2^2$$

$$= \quad A\, p_1^2 - A\, p_3^2$$

$$+ \quad A\, p_3^2 + B\, p_3^2 - A\, p_2^2 - B\, p_2^2$$

$$+ \quad A\, p_2^2 + B\, p_2^2 + C\, p_2^2$$

$$= \quad A(p_1^2 - p_3^2) + (A+B)(p_3^2 - p_2^2) + (A+B+C)\, p_2^2$$

Since $p_1 \geq p_3 \geq p_2$, we know $p_1^2 - p_3^2 \geq 0$ and $p_3^2 - p_2^2 \geq 0$, so

$$\geq \quad 1(p_1^2 - p_3^2) + 2(p_3^2 - p_2^2) + 3\, p_2^2$$

$$= \quad p_1^2 - p_3^2 + 2\, p_3^2 - 2\, p_2^2 + 3\, p_2^2$$

$$= \quad p_1^2 + p_2^2 + p_3^2 ,$$

so we have

$$\frac{a_1^2}{a_2 a_3} p_1^2 + \frac{a_3}{a_1} p_3^2 + \frac{a_2}{a_1} p_2^2 \geq p_1^2 + p_2^2 + p_3^2 .$$

To show

$$\left(1+\frac{a_2}{a_3}+\frac{a_1}{a_2}\right) p_1 p_3 + \left(\frac{a_3}{a_2}+1+\frac{a_1}{a_3}\right) p_1 p_2 + \left(\frac{a_3^2}{a_1 a_2}+\frac{a_2^2}{a_1 a_3}+1\right) p_2 p_3$$

$$\geq \qquad 3(p_1 p_2 + p_1 p_3 + p_2 p_3):$$

We begin by realizing that this case requires $p_1 p_3 \geq p_1 p_2 \geq p_2 p_3$, and we let

$$A=1+\frac{a_2}{a_3}+\frac{a_1}{a_2}, \qquad B=\frac{a_3}{a_2}+1+\frac{a_1}{a_3}, \text{ and } C=\frac{a_3^2}{a_1 a_2}+\frac{a_2^2}{a_1 a_3}+1,$$

Then, we have $A=1+\frac{a_2}{a_3}+\frac{a_1}{a_2}$, and since $a_1 \geq a_2 \geq a_3$, we know

$$A \geq 3 .$$

We have $A+B=2+\frac{a_2}{a_3}+\frac{a_1}{a_2}+\frac{a_3}{a_2}+\frac{a_1}{a_3}$, and the Arithmetic Mean – Geometric Mean Inequality coupled with $a_1 \geq a_2 \geq a_3$, meaning $a_1^2 \geq a_2 a_3$, tells us

$$A+B \geq 2+4\sqrt[4]{\frac{a_1^2 a_2 a_3}{a_2^2 a_3^2}} = 2+4\sqrt[4]{\frac{a_1^2}{a_2 a_3}} \geq 2+4=6,$$

so that

$$A+B \geq 6 .$$

We also have $A+B+C=3+\frac{a_2}{a_3}+\frac{a_1}{a_2}+\frac{a_3}{a_2}+\frac{a_1}{a_3}+\frac{a_3^2}{a_1 a_2}+\frac{a_2^2}{a_1 a_3}$, and the Arithmetic Mean – Geometric Mean Inequality tells us

$$A+B+C \geq 3+6\sqrt[6]{\frac{a_1^2 a_2^3 a_3^3}{a_1^2 a_2^3 a_3^3}} = 3+6=9,$$

so that

$$A+B+C \geq 9 .$$

This combines to mean

$$\left(1+\frac{a_2}{a_3}+\frac{a_1}{a_2}\right)p_1 p_3+\left(\frac{a_3}{a_2}+1+\frac{a_1}{a_3}\right)p_1 p_2+\left(\frac{a_3^2}{a_1 a_2}+\frac{a_2^2}{a_1 a_3}+1\right)p_2 p_3$$

$$= \quad A\,p_1 p_3+B\,p_1 p_2+C\,p_2 p_3$$

$$= \quad A\,p_1 p_3-A\,p_1 p_2$$

$$+ \quad A\,p_1 p_2+B\,p_1 p_2-A\,p_2 p_3-B\,p_2 p_3$$

$$+ \quad A\,p_2 p_3+B\,p_2 p_3+C\,p_2 p_3$$

$$= \quad A(p_1 p_3-p_1 p_2)+(A+B)(p_1 p_2-p_2 p_3)+(A+B+C)\,p_2 p_3$$

Since $p_1 p_3\geq p_1 p_2\geq p_2 p_3$, we know $p_1 p_3-p_1 p_2\geq 0$ and $p_1 p_2-p_2 p_3\geq 0$, so

$$\geq \quad 3(p_1 p_3-p_1 p_2)+6(p_1 p_2-p_2 p_3)+9\,p_2 p_3$$

$$= \quad 3\,p_1 p_3-3\,p_1 p_2+6\,p_1 p_2-6\,p_2 p_3+9\,p_2 p_3$$

$$= \quad 3\,p_1 p_2+3\,p_1 p_3+3\,p_2 p_3\,,$$

so that

$$\left(1+\frac{a_2}{a_3}+\frac{a_1}{a_2}\right)p_1 p_3+\left(\frac{a_3}{a_2}+1+\frac{a_1}{a_3}\right)p_1 p_2+\left(\frac{a_3^2}{a_1 a_2}+\frac{a_2^2}{a_1 a_3}+1\right)p_2 p_3$$

$$\geq \quad 3(p_1 p_2+p_1 p_3+p_2 p_3)\,,$$

as desired.

So Case 4 holds, as we have shown

$$P A_1\cdot P A_2+PA_1\cdot PA_3+PA_2\cdot P A_3$$

$$\geq \quad \frac{a_1^2}{a_2 a_3}p_1^2+\frac{a_2}{a_1}p_2^2+\frac{a_3}{a_1}p_3^2$$

$$+ \quad \left(1+\frac{a_2}{a_3}+\frac{a_1}{a_2}\right)p_1 p_3+\left(\frac{a_3}{a_2}+1+\frac{a_1}{a_3}\right)p_1 p_2+\left(\frac{a_3^2}{a_1 a_2}+\frac{a_2^2}{a_1 a_3}+1\right)p_2 p_3$$

$$\geq \quad p_1^2+p_2^2+p_3^2+3(p_1 p_2+p_1 p_3+p_2 p_3)$$

$$= \quad (p_2+p_3)(p_1+p_3)+(p_2+p_3)(p_1+p_2)+(p_1+p_3)(p_1+p_2)\ ,$$

which gives

$$PA_1\cdot PA_2+PA_1\cdot PA_3+PA_2\cdot PA_3$$

$$\geq \quad (p_2+p_3)(p_1+p_3)+(p_2+p_3)(p_1+p_2)+(p_1+p_3)(p_1+p_2)\ .$$

Case 5. $p_2 \geq p_1 \geq p_3$.

Again, we use the maximum option to pair the larger values of a_i and p_i together and the smaller values of a_i and p_i together in (*5.5.A*):

$$PA_1 \geq \frac{a_2 p_2 + a_3 p_3}{a_1}, \quad PA_2 \geq \frac{a_1 p_1 + a_3 p_3}{a_2}, \text{ and } PA_3 \geq \frac{a_1 p_2 + a_2 p_1}{a_3}.$$

so that

$$\begin{aligned}
&PA_1 \cdot PA_2 + PA_1 \cdot PA_3 + PA_2 \cdot PA_3 \\
&\geq \left(\frac{a_2 p_2 + a_3 p_3}{a_1}\right)\left(\frac{a_1 p_1 + a_3 p_3}{a_2}\right) \\
&\quad + \left(\frac{a_2 p_2 + a_3 p_3}{a_1}\right)\left(\frac{a_1 p_2 + a_2 p_1}{a_3}\right) \\
&\quad + \left(\frac{a_1 p_1 + a_3 p_3}{a_2}\right)\left(\frac{a_1 p_2 + a_2 p_1}{a_3}\right) \\
&= \frac{a_1 a_2 p_1 p_2 + a_2 a_3 p_2 p_3 + a_1 a_3 p_1 p_3 + a_3^2 p_3^2}{a_1 a_2} \\
&\quad + \frac{a_1 a_2 p_2^2 + a_2^2 p_1 p_2 + a_1 a_3 p_2 p_3 + a_2 a_3 p_1 p_3}{a_1 a_3} \\
&\quad + \frac{a_1^2 p_1 p_2 + a_1 a_2 p_1^2 + a_1 a_3 p_2 p_3 + a_2 a_3 p_1 p_3}{a_2 a_3} \\
&= p_1 p_2 + \frac{a_3}{a_1} p_2 p_3 + \frac{a_3}{a_2} p_1 p_3 + \frac{a_3^2}{a_1 a_2} p_3^2 \\
&\quad + \frac{a_2}{a_3} p_2^2 + \frac{a_2^2}{a_1 a_3} p_1 p_2 + p_2 p_3 + \frac{a_2}{a_1} p_1 p_3 \\
&\quad + \frac{a_1^2}{a_2 a_3} p_1 p_2 + \frac{a_1}{a_3} p_1^2 + \frac{a_1}{a_2} p_2 p_3 + p_1 p_3
\end{aligned}$$

And by rearranging terms, we get

$$= \quad \frac{a_2}{a_3}p_2^2+\frac{a_1}{a_3}p_1^2+\frac{a_3^2}{a_1a_2}p_3^2$$

$$+ \quad \left(1+\frac{a_2^2}{a_1a_3}+\frac{a_1^2}{a_2a_3}\right)p_1p_2+\left(\frac{a_3}{a_1}+1+\frac{a_1}{a_2}\right)p_2p_3+\left(\frac{a_3}{a_2}+\frac{a_2}{a_1}+1\right)p_1p_3 \ .$$

We need to show this is at least $p_1^2+p_2^2+p_3^2+3(p_1p_2+p_1p_3+p_2p_3)$.

Again, to do this, we will show that

$$\frac{a_2}{a_3}p_2^2+\frac{a_1}{a_3}p_1^2+\frac{a_3^2}{a_1a_2}p_3^2\geq p_1^2+p_2^2+p_3^2$$

and

$$\left(1+\frac{a_2^2}{a_1a_3}+\frac{a_1^2}{a_2a_3}\right)p_1p_2+\left(\frac{a_3}{a_1}+1+\frac{a_1}{a_2}\right)p_2p_3+\left(\frac{a_3}{a_2}+\frac{a_2}{a_1}+1\right)p_1p_3$$

$$\geq \quad 3(p_1p_2+p_1p_3+p_2p_3)\ .$$

First, to show $\frac{a_2}{a_3}p_2^2+\frac{a_1}{a_3}p_1^2+\frac{a_3^2}{a_1a_2}p_3^2\geq p_1^2+p_2^2+p_3^2$:

Let

$$A=\frac{a_2}{a_3}\ , \quad B=\frac{a_1}{a_3}\ \text{, and} \quad C=\frac{a_3^2}{a_1a_2}\ .$$

Then, we have $A=\frac{a_2}{a_3}$, and since $a_1\geq a_2\geq a_3$, we know

$$A\geq 1\ .$$

We have $A+B=\frac{a_2}{a_3}+\frac{a_1}{a_3}$, and since $a_1\geq a_2\geq a_3$, we know

$$A+B\geq 2\ .$$

We also have $A+B+C=\frac{a_2}{a_3}+\frac{a_1}{a_3}+\frac{a_3^2}{a_1 a_2}$, and the Arithmetic Mean – Geometric Mean Inequality tells us

$$A+B+C\geq 3\sqrt[3]{\frac{a_1 a_2 a_3^2}{a_1 a_2 a_3^2}}=3 ,$$

so that

$$A+B+C\geq 3 .$$

This combines to mean

$$\frac{a_2}{a_3}p_2^2+\frac{a_1}{a_3}p_1^2+\frac{a_3^2}{a_1 a_2}p_3^2$$

$$\begin{aligned}
&= A p_2^2+B p_1^2+C p_3^2 \\
&= A p_2^2-A p_1^2 \\
&\quad + A p_1^2+B p_1^2-A p_3^2-B p_3^2 \\
&\quad + A p_3^2+B p_3^2+C p_3^2 \\
&= A(p_2^2-p_1^2)+(A+B)(p_1^2-p_3^2)+(A+B+C)p_3^2
\end{aligned}$$

Since $p_2\geq p_1\geq p_3$, we know $p_2^2-p_1^2\geq 0$ and $p_1^2-p_3^2\geq 0$, so

$$\begin{aligned}
&\geq 1(p_2^2-p_1^2)+2(p_1^2-p_3^2)+3 p_3^2 \\
&= p_2^2-p_1^2+2 p_1^2-2 p_3^2+3 p_3^2 \\
&= p_1^2+p_2^2+p_3^2 ,
\end{aligned}$$

So we have

$$\frac{a_2}{a_3}p_2^2+\frac{a_1}{a_3}p_1^2+\frac{a_3^2}{a_1 a_2}p_3^2\geq p_1^2+p_2^2+p_3^2 .$$

To show

$$\left(1+\frac{a_2^2}{a_1 a_3}+\frac{a_1^2}{a_2 a_3}\right)p_1 p_2+\left(\frac{a_3}{a_1}+1+\frac{a_1}{a_2}\right)p_2 p_3+\left(\frac{a_3}{a_2}+\frac{a_2}{a_1}+1\right)p_1 p_3$$

$$\geq \quad 3(p_1 p_2+p_1 p_3+p_2 p_3):$$

We begin by realizing that this case requires $p_1 p_2 \geq p_2 p_3 \geq p_1 p_3$, and we let

$$A=1+\frac{a_2^2}{a_1 a_3}+\frac{a_1^2}{a_2 a_3}, \quad B=\frac{a_3}{a_1}+1+\frac{a_1}{a_2}, \text{ and } \quad C=\frac{a_3}{a_2}+\frac{a_2}{a_1}+1 .$$

Then, we have $A=1+\frac{a_2^2}{a_1 a_3}+\frac{a_1^2}{a_2 a_3}$, and since $a_1 \geq a_2 \geq a_3$, we know by using the Arithmetic Mean – Geometric Mean Inequality and the fact that $a_1 a_2 \geq a_3^2$,

$$A \geq 1+2\sqrt{\frac{a_1^2 a_2^2}{a_1 a_2 a_3^2}}=1+2\sqrt{\frac{a_1 a_2}{a_3^2}} \geq 1+2=3 ,$$

so that

$$A \geq 3 .$$

We have $A+B=2+\frac{a_2^2}{a_1 a_3}+\frac{a_1^2}{a_2 a_3}+\frac{a_3}{a_1}+\frac{a_1}{a_2}$, and the Arithmetic Mean – Geometric Mean Inequality coupled with $a_1 \geq a_2 \geq a_3$ tells us

$$A+B \geq 2+4\sqrt[4]{\frac{a_1^3 a_2^2 a_3}{a_1^2 a_2^2 a_3^2}}=2+\sqrt[4]{\frac{a_1}{a_3}} \geq 2+4=6 ,$$

so that

$$A+B \geq 6 .$$

We also have $A+B+C=3+\frac{a_2^2}{a_1 a_3}+\frac{a_1^2}{a_2 a_3}+\frac{a_3}{a_1}+\frac{a_1}{a_2}+\frac{a_3}{a_2}+\frac{a_2}{a_1}+1$, and the Arithmetic Mean – Geometric Mean Inequality tells us

$$A+B+C \geq 3+6\sqrt[6]{\frac{a_1^3 a_2^3 a_3^2}{a_1^3 a_2^3 a_3^2}} = 3+6 = 9\ ,$$

so that

$$A+B+C \geq 9\ .$$

This combines to mean

$$\left(1+\frac{a_2^2}{a_1 a_3}+\frac{a_1^2}{a_2 a_3}\right) p_1 p_2 + \left(\frac{a_3}{a_1}+1+\frac{a_1}{a_2}\right) p_2 p_3 + \left(\frac{a_3}{a_2}+\frac{a_2}{a_1}+1\right) p_1 p_3$$

$$= \quad A\, p_1 p_2 + B\, p_2 p_3 + C\, p_1 p_3$$

$$= \quad A\, p_1 p_2 - A\, p_2 p_3$$

$$+ \quad A\, p_2 p_3 + B\, p_2 p_3 - A\, p_1 p_3 - B\, p_1 p_3$$

$$+ \quad A\, p_1 p_3 + B\, p_1 p_3 + C\, p_1 p_3$$

$$= \quad A(p_1 p_2 - p_2 p_3) + (A+B)(p_2 p_3 - p_1 p_3) + (A+B+C)\, p_1 p_3$$

Since $p_1 p_2 \geq p_2 p_3 \geq p_1 p_3$, we know $p_1 p_2 - p_2 p_3 \geq 0$ and $p_2 p_3 - p_1 p_3 \geq 0$, so

$$\geq \quad 3(p_1 p_2 - p_2 p_3) + 6(p_2 p_3 - p_1 p_3) + 9\, p_1 p_3$$

$$= \quad 3\, p_1 p_2 - 3\, p_2 p_3 + 6\, p_2 p_3 - 6\, p_1 p_3 + 9\, p_1 p_3$$

$$= \quad 3\, p_1 p_2 + 3\, p_1 p_3 + 3\, p_2 p_3\ ,$$

so that

$$\left(1+\frac{a_2^2}{a_1 a_3}+\frac{a_1^2}{a_2 a_3}\right) p_1 p_2 + \left(\frac{a_3}{a_1}+1+\frac{a_1}{a_2}\right) p_2 p_3 + \left(\frac{a_3}{a_2}+\frac{a_2}{a_1}+1\right) p_1 p_3$$

$$\geq \quad 3(p_1 p_2 + p_1 p_3 + p_2 p_3)\ ,$$

as desired.

So Case 5 holds, as we have shown

$$PA_1 \cdot PA_2 + PA_1 \cdot PA_3 + PA_2 \cdot PA_3$$

$$\geq \quad \frac{a_1}{a_3} p_1^2 + \frac{a_2}{a_3} p_2^2 + \frac{a_3^2}{a_1 a_2} p_3^2$$

$$+ \quad \left(1 + \frac{a_2^2}{a_1 a_3} + \frac{a_1^2}{a_2 a_3}\right) p_1 p_2 + \left(\frac{a_3}{a_1} + 1 + \frac{a_1}{a_2}\right) p_2 p_3 + \left(\frac{a_3}{a_2} + \frac{a_2}{a_1} + 1\right) p_1 p_3$$

$$\geq \quad p_1^2 + p_2^2 + p_3^2 + 3(p_1 p_2 + p_1 p_3 + p_2 p_3)$$

$$= \quad (p_2 + p_3)(p_1 + p_3) + (p_2 + p_3)(p_1 + p_2) + (p_1 + p_3)(p_1 + p_2)\ ,$$

which gives

$$PA_1 \cdot PA_2 + PA_1 \cdot PA_3 + PA_2 \cdot PA_3$$

$$\geq \quad (p_2 + p_3)(p_1 + p_3) + (p_2 + p_3)(p_1 + p_2) + (p_1 + p_3)(p_1 + p_2)\ .$$

Case 6. $p_3 \geq p_1 \geq p_2$.

Again, we use the maximum option to pair the larger values of a_i and p_i together and the smaller values of a_i and p_i together in (*5.5.A*):

$$PA_1 \geq \frac{a_2 p_3 + a_3 p_2}{a_1}, \quad PA_2 \geq \frac{a_1 p_3 + a_3 p_1}{a_2}, \text{ and } PA_3 \geq \frac{a_1 p_1 + a_2 p_2}{a_3} .$$

so that

$$PA_1 \cdot PA_2 + PA_1 \cdot PA_3 + PA_2 \cdot PA_3$$

$$\geq \left(\frac{a_2 p_3 + a_3 p_2}{a_1}\right)\left(\frac{a_1 p_3 + a_3 p_1}{a_2}\right)$$

$$+ \left(\frac{a_2 p_3 + a_3 p_2}{a_1}\right)\left(\frac{a_1 p_1 + a_2 p_2}{a_3}\right)$$

$$+ \left(\frac{a_1 p_3 + a_3 p_1}{a_2}\right)\left(\frac{a_1 p_1 + a_2 p_2}{a_3}\right)$$

$$= \frac{a_1 a_2 p_3^2 + a_2 a_3 p_1 p_3 + a_1 a_3 p_2 p_3 + a_3^2 p_1 p_2}{a_1 a_2}$$

$$+ \frac{a_1 a_2 p_1 p_3 + a_2^2 p_2 p_3 + a_1 a_3 p_1 p_2 + a_2 a_3 p_2^2}{a_1 a_3}$$

$$+ \frac{a_1^2 p_1 p_3 + a_1 a_2 p_2 p_3 + a_1 a_3 p_1^2 + a_2 a_3 p_1 p_2}{a_2 a_3}$$

$$= p_3^2 + \frac{a_3}{a_1} p_1 p_3 + \frac{a_3}{a_2} p_2 p_3 + \frac{a_3^2}{a_1 a_2} p_1 p_2$$

$$+ \frac{a_2}{a_3} p_1 p_3 + \frac{a_2^2}{a_1 a_3} p_2 p_3 + p_1 p_2 + \frac{a_2}{a_1} p_2^2$$

$$+ \frac{a_1^2}{a_2 a_3} p_1 p_3 + \frac{a_1}{a_3} p_2 p_3 + \frac{a_1}{a_2} p_1^2 + p_1 p_2$$

And by rearranging terms, we get

$$= \quad p_3^2+\frac{a_1}{a_2}p_1^2+\frac{a_2}{a_1}p_2^2$$

$$+ \quad \left(\frac{a_3}{a_1}+\frac{a_2}{a_3}+\frac{a_1^2}{a_2 a_3}\right)p_1 p_3+\left(\frac{a_3}{a_2}+\frac{a_2^2}{a_1 a_3}+\frac{a_1}{a_3}\right)p_2 p_3+\left(\frac{a_3^2}{a_1 a_2}+2\right)p_1 p_2 \ .$$

We need to show this is at least $p_1^2+p_2^2+p_3^2+3(p_1 p_2+p_1 p_3+p_2 p_3)$.

Again, to do this, we will show that

$$p_3^2+\frac{a_1}{a_2}p_1^2+\frac{a_2}{a_1}p_2^2 \geq p_1^2+p_2^2+p_3^2$$

and

$$\left(\frac{a_3}{a_1}+\frac{a_2}{a_3}+\frac{a_1^2}{a_2 a_3}\right)p_1 p_3+\left(\frac{a_3}{a_2}+\frac{a_2^2}{a_1 a_3}+\frac{a_1}{a_3}\right)p_2 p_3+\left(\frac{a_3^2}{a_1 a_2}+2\right)p_1 p_2$$

$$\geq \quad 3(p_1 p_2+p_1 p_3+p_2 p_3) \ .$$

First, to show $p_3^2+\frac{a_1}{a_2}p_1^2+\frac{a_2}{a_1}p_2^2 \geq p_1^2+p_2^2+p_3^2$:

Let

$$A=1 \ , \quad B=\frac{a_1}{a_2} \ , \text{and} \quad C=\frac{a_2}{a_1} \ .$$

Then, we clearly have $A=1$.

We also have $A+B=1+\frac{a_1}{a_2}$, and since $a_1 \geq a_2 \geq a_3$, we know

$$A+B \geq 2 \ .$$

We also have $A+B+C=1+\frac{a_1}{a_2}+\frac{a_2}{a_1}$, and the Arithmetic Mean – Geometric Mean Inequality tells us

$$A+B+C \geq 1+2\sqrt{\frac{a_1 a_2}{a_1 a_2}}=1+2=3 \text{ ,}$$

so that

$$A+B+C \geq 3 \text{ .}$$

This combines to mean

$$p_3^2+\frac{a_1}{a_2}p_1^2+\frac{a_2}{a_1}p_2^2$$

$$\begin{aligned}
&= A\,p_3^2+B\,p_1^2+C\,p_2^2 \\
&= A\,p_3^2-A\,p_1^2 \\
&\quad + A\,p_1^2+B\,p_1^2-A\,p_2^2-B\,p_2^2 \\
&\quad + A\,p_2^2+B\,p_2^2+C\,p_2^2 \\
&= A(p_3^2-p_1^2)+(A+B)(p_1^2-p_2^2)+(A+B+C)\,p_2^2
\end{aligned}$$

Since $p_3 \geq p_1 \geq p_2$, we know $p_3^2-p_1^2 \geq 0$ and $p_1^2-p_2^2 \geq 0$, so

$$\begin{aligned}
&\geq 1(p_3^2-p_1^2)+2(p_1^2-p_2^2)+3\,p_2^2 \\
&= p_3^2-p_1^2+2\,p_1^2-2\,p_2^2+3\,p_2^2 \\
&= p_1^2+p_2^2+p_3^2 \text{ ,}
\end{aligned}$$

So we have

$$p_3^2+\frac{a_1}{a_2}p_1^2+\frac{a_2}{a_1}p_2^2 \geq p_1^2+p_2^2+p_3^2 \text{ .}$$

To show

$$\left(\frac{a_3}{a_1}+\frac{a_2}{a_3}+\frac{a_1^2}{a_2 a_3}\right)p_1 p_3+\left(\frac{a_3}{a_2}+\frac{a_2^2}{a_1 a_3}+\frac{a_1}{a_3}\right)p_2 p_3+\left(\frac{a_3^2}{a_1 a_2}+2\right)p_1 p_2$$

$$\geq \quad 3(p_1 p_2+p_1 p_3+p_2 p_3):$$

We begin by realizing that this case requires $p_1 p_3 \geq p_2 p_3 \geq p_1 p_2$, and we let

$$A = \frac{a_3}{a_1} + \frac{a_2}{a_3} + \frac{a_1^2}{a_2 a_3}, \quad B = \frac{a_3}{a_2} + \frac{a_2^2}{a_1 a_3} + \frac{a_1}{a_3}, \text{ and } \quad C = \frac{a_3^2}{a_1 a_2} + 2 .$$

Then, we have $A = \frac{a_3}{a_1} + \frac{a_2}{a_3} + \frac{a_1^2}{a_2 a_3}$, and since $a_1 \geq a_2 \geq a_3$ coupled with the use of the Arithmetic Mean – Geometric Mean Inequality

$$A \geq 3 \sqrt[3]{\frac{a_1^2 a_2 a_3}{a_1 a_2 a_3^2}} = 3 \sqrt[3]{\frac{a_1}{a_3}} \geq 3 ,$$

so that

$$A \geq 3 .$$

We have $A + B = \frac{a_3}{a_1} + \frac{a_2}{a_3} + \frac{a_1^2}{a_2 a_3} + \frac{a_3}{a_2} + \frac{a_2^2}{a_1 a_3} + \frac{a_1}{a_3}$, and the Arithmetic Mean – Geometric Mean Inequality coupled with $a_1 \geq a_2 \geq a_3$, meaning $a_1 a_2 \geq a_3^2$, tells us

$$A + B \geq 6 \sqrt[6]{\frac{a_1^3 a_2^3 a_3^2}{a_1^2 a_2^2 a_3^4}} = 6 \sqrt[6]{\frac{a_1 a_2}{a_3^2}} \geq 6 ,$$

so that

$$A + B \geq 6 .$$

We also have $A + B + C = 2 + \frac{a_3}{a_1} + \frac{a_2}{a_3} + \frac{a_1^2}{a_2 a_3} + \frac{a_3}{a_2} + \frac{a_2^2}{a_1 a_3} + \frac{a_1}{a_3} + \frac{a_3^2}{a_1 a_2}$, and the Arithmetic Mean – Geometric Mean Inequality tells us

$$A + B + C \geq 2 + 7 \sqrt[7]{\frac{a_1^3 a_2^3 a_3^4}{a_1^3 a_2^3 a_3^4}} = 2 + 7 = 9 ,$$

so that

$$A + B + C \geq 9 .$$

This combines to mean

$$\left(\frac{a_3}{a_1}+\frac{a_2}{a_3}+\frac{a_1^2}{a_2 a_3}\right)p_1 p_3+\left(\frac{a_3}{a_2}+\frac{a_2^2}{a_1 a_3}+\frac{a_1}{a_3}\right)p_2 p_3+\left(\frac{a_3^2}{a_1 a_2}+2\right)p_1 p_2$$

$$= \quad A\, p_1 p_3+B\, p_2 p_3+C\, p_1 p_2$$

$$= \quad A\, p_1 p_3-A\, p_2 p_3$$

$$+ \quad A\, p_2 p_3+B\, p_2 p_3-A\, p_1 p_2-B\, p_1 p_2$$

$$+ \quad A\, p_1 p_2+B\, p_1 p_2+C\, p_1 p_2$$

$$= \quad A(p_1 p_3-p_2 p_3)+(A+B)(p_2 p_3-p_1 p_2)+(A+B+C)\, p_1 p_2$$

Since $p_1 p_3 \geq p_2 p_3 \geq p_1 p_2$, we know $p_1 p_3-p_2 p_3 \geq 0$ and $p_2 p_3-p_1 p_2 \geq 0$, so

$$\geq \quad 3(p_1 p_3-p_2 p_3)+6(p_2 p_3-p_1 p_2)+9\, p_1 p_2$$

$$= \quad 3\, p_1 p_3-3\, p_2 p_3+6\, p_2 p_3-6\, p_1 p_2+9\, p_1 p_2$$

$$= \quad 3\, p_1 p_2+3\, p_1 p_3+3\, p_2 p_3\,,$$

so that

$$\left(\frac{a_3}{a_1}+\frac{a_2}{a_3}+\frac{a_1^2}{a_2 a_3}\right)p_1 p_3+\left(\frac{a_3}{a_2}+\frac{a_2^2}{a_1 a_3}+\frac{a_1}{a_3}\right)p_2 p_3+\left(\frac{a_3^2}{a_1 a_2}+2\right)p_1 p_2$$

$$\geq \quad 3(p_1 p_2+p_1 p_3+p_2 p_3)\,,$$

as desired.

So Case 6 holds, as we have shown

$$PA_1 \cdot PA_2+PA_1 \cdot PA_3+PA_2 \cdot PA_3$$

$$\geq \quad \frac{a_1}{a_2}p_1^2+\frac{a_2}{a_1}p_2^2+p_3^2$$

$$+ \quad \left(\frac{a_3}{a_1}+\frac{a_2}{a_3}+\frac{a_1^2}{a_2 a_3}\right)p_1 p_3+\left(\frac{a_3}{a_2}+\frac{a_2^2}{a_1 a_3}+\frac{a_1}{a_3}\right)p_2 p_3+\left(\frac{a_3^2}{a_1 a_2}+2\right)p_1 p_2$$

$$\geq \quad p_1^2+p_2^2+p_3^2+3(p_1 p_2+p_1 p_3+p_2 p_3)$$

$$= \quad (p_2+p_3)(p_1+p_3)+(p_2+p_3)(p_1+p_2)+(p_1+p_3)(p_1+p_2) ,$$

which gives

$$PA_1 \cdot PA_2+PA_1 \cdot PA_3+PA_2 \cdot PA_3$$

$$\geq \quad (p_2+p_3)(p_1+p_3)+(p_2+p_3)(p_1+p_2)+(p_1+p_3)(p_1+p_2) .$$

Since all six cases hold, and these cases exhaust the possibilities for the ordering of the values p_1, p_2, and p_3, it follows that the inequality

$$PA_1 \cdot PA_2+PA_1 \cdot PA_3+PA_2 \cdot PA_3$$

$$\geq \quad (p_2+p_3)(p_1+p_3)+(p_2+p_3)(p_1+p_2)+(p_1+p_3)(p_1+p_2)$$

holds overall.

∎

6 Problem Solving with the Erdös-Mordell Inequality

This section pertains to problems that have arisen in journals and competitions involving the use of the Erdös-Mordell Inequality.

One interesting application of the Erdös-Mordell Inequality was presented as a problem in the 1991 International Mathematical Olympiad (IMO). The notation has been adapted to fit with this paper, and the problem is given in Example 6.1. A solution was given in **[IEQ]** that has been adapted to this paper.

Example 6.1. **[TIMO and IEQ]**

Let $A_1 A_2 A_3$ be a triangle and *P* an interior point of $\triangle A_1 A_2 A_3$. Show that at least one of the angles $\angle P A_1 A_2$, $\angle P A_2 A_3$, $\angle P A_3 A_1$ is less than or equal to 30º.

Solution to Example 6.1. **[IEQ]**

We adopt our familiar notation, with p_i denoting the distance from *P* to the side of $\triangle A_1 A_2 A_3$ opposite vertex A_i and a_i denoting the length of the side opposite vertex A_i for each $1 \leq i \leq 3$.

Suppose this is not true. Then $m\angle P A_1 A_2 > 30^\circ$, $m\angle P A_2 A_3 > 30^\circ$, and $m\angle P A_3 A_1 > 30^\circ$.

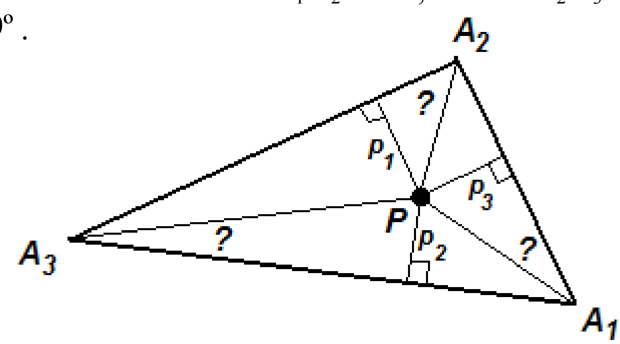

Figure 6.1

Now we have

$$\sin(m\angle P A_1 A_2) = \frac{p_3}{PA_1} \quad \text{so that} \quad PA_1 = \frac{p_3}{\sin(m\angle P A_1 A_2)},$$

$$\sin(m\angle P A_2 A_3) = \frac{p_1}{PA_2} \quad \text{so that} \quad PA_2 = \frac{p_1}{\sin(m\angle P A_2 A_3)}, \text{ and}$$

$$\sin(m\angle P A_3 A_1) = \frac{p_2}{PA_3} \quad \text{so that} \quad PA_2 = \frac{p_2}{\sin(m\angle P A_3 A_1)}.$$

We notice that if $m\angle PA_1A_2 \geq 150^{o}$, this would contradict that the sum of the measures of the interior angles of $\triangle A_1A_2A_3$ has to equal 180º, as we would have

$$m\angle A_3A_1A_2 + m\angle A_1A_2A_3 + m\angle A_2A_3A_1$$

$$> \quad m\angle PA_1A_2 + m\angle PA_2A_3 + m\angle PA_3A_1$$

$$> \quad 150^{o} + 30^{o} + 30^{o}$$

$$> \quad 180^{o}.$$

Thus, we conclude $m\angle PA_1A_2 < 150^{o}$, and similarly $m\angle PA_2A_3 < 150^{o}$ and $m\angle PA_3A_1 < 150^{o}$.

Also, since $\sin x$ is a continuous function strictly increasing on (0º, 90º) and strictly decreasing on (90º, 180º) with $\sin 30^{o} = \sin 150^{o}$, we conclude that

$$\sin(m\angle PA_1A_2) > \sin 30^{o} \quad \text{so that} \quad \frac{1}{\sin(m\angle PA_1A_2)} < \frac{1}{\sin 30^{o}},$$

$$\sin(m\angle PA_2A_3) > \sin 30^{o} \quad \text{so that} \quad \frac{1}{\sin(m\angle PA_2A_3)} < \frac{1}{\sin 30^{o}}, \text{ and}$$

$$\sin(m\angle PA_3A_1) > \sin 30^{o} \quad \text{so that} \quad \frac{1}{\sin(m\angle PA_3A_1)} < \frac{1}{\sin 30^{o}}.$$

So we have

$$PA_1 + PA_2 + PA_3$$

$$= \quad \frac{p_3}{\sin(m\angle PA_1A_2)} + \frac{p_1}{\sin(m\angle PA_2A_3)} + \frac{p_2}{\sin(m\angle PA_3A_1)}$$

$$< \quad \frac{p_3}{\sin 30^{o}} + \frac{p_1}{\sin 30^{o}} + \frac{p_2}{\sin 30^{o}}$$

$$= \quad \frac{p_3}{1/2} + \frac{p_1}{1/2} + \frac{p_2}{1/2}$$

$$= \quad 2(p_1 + p_2 + p_3).$$

This means

$$PA_1 + PA_2 + PA_3 < 2(p_1 + p_2 + p_3),$$

which contradicts the Erdös-Mordell Inequality.

Therefore, our original assumption was incorrect, and we conclude that at least one of the angles $\angle PA_1A_2$, $\angle PA_2A_3$, $\angle PA_3A_1$ has measure less than or equal to 30º.

∎

The next example was published in the "Problems and Solutions" section as 11491 of *The American Mathematical Monthly*, March 2010 **[ANG]**. Notation has been adapted to fit this paper.

Example 6.2. **[ANG]**

11491: *Proposed by Nicolae Anghel, University of North Texas, Denton, TX.*

Let P be an interior point of a triangle having vertices A_1, A_2, and A_3 opposite sides of length a_1, a_2, and a_3, respectively, and circumradius R. Show that

$$\frac{PA_1}{a_1^2}+\frac{PA_2}{a_2^2}+\frac{PA_3}{a_3^2}\geq\frac{1}{R}.$$

Comment.

We will offer two proofs to Example 6.2, both of which are original work and were submitted to *The American Mathematical Monthly.*

Comment.

Let p_i denote the distance from *P* to the side of $\triangle A_1A_2A_3$ opposite vertex A_i for each $1\leq i\leq 3$, and let θ_i be the measure of the interior angle of $\triangle A_1A_2A_3$ with vertex A_i for each $1\leq i\leq 3$.

We require a lemma first.

Lemma 6.2.1.

Under the conditions of Example 6.2, we have:

$$\frac{p_1}{\sin(\theta_2)\sin(\theta_3)}+\frac{p_2}{\sin(\theta_1)\sin(\theta_3)}+\frac{p_3}{\sin(\theta_1)\sin(\theta_2)}=2R\,.$$

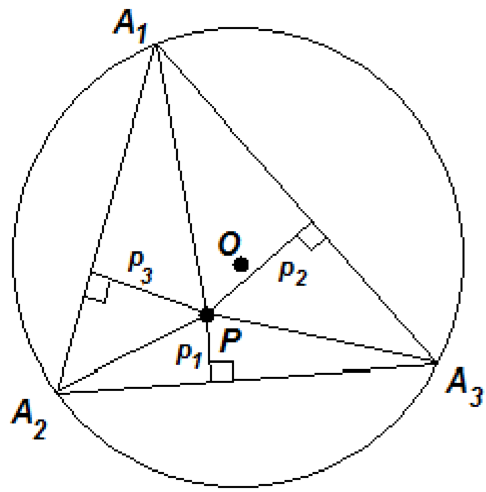

Figure 6.2

Proof of Lemma 6.2.1.

Notice that the area of $\triangle A_1 A_2 A_3$ is $\dfrac{a_2 a_3 \sin(\theta_1)}{2}$.

This area can also be found by taking the combined areas of $\triangle A_2 P A_3$, $\triangle A_1 P A_3$, and $\triangle A_1 P A_2$.

Using this concept, we have:

$$\frac{a_1 p_1}{2}+\frac{a_2 p_2}{2}+\frac{a_3 p_3}{2}=\frac{a_2 a_3 \sin(\theta_1)}{2}\quad .$$

Multiplying through by $\dfrac{2}{a_2 a_3 \sin(\theta_1)}$ gives

$$\frac{a_1 p_1}{a_2 a_3 \sin(\theta_1)}+\frac{a_2 p_2}{a_2 a_3 \sin(\theta_1)}+\frac{a_3 p_3}{a_2 a_3 \sin(\theta_1)}=1$$

or, when simplifying, we get

$$\frac{a_1}{a_2 \sin(\theta_1)} \cdot \frac{p_1}{a_3} + \frac{p_2}{a_3 \sin(\theta_1)} + \frac{p_3}{a_2 \sin(\theta_1)} = 1$$

Noticing that the Law of Sines $\left(\frac{\sin(\theta_2)}{a_2} = \frac{\sin(\theta_1)}{a_1}\right)$ implies $\frac{a_1}{a_2 \sin(\theta_1)} = \frac{1}{\sin(\theta_2)}$, we get

$$\frac{p_1}{\sin(\theta_2)\, a_3} + \frac{p_2}{\sin(\theta_1)\, a_3} + \frac{p_3}{\sin(\theta_1)\, a_2} = 1 \ .$$

By Lemma 2.4, we know $a_2 = 2R\sin(\theta_2)$ and $a_3 = 2R\sin(\theta_3)$, which gives

$$\frac{p_1}{\sin(\theta_2) \cdot 2R\sin(\theta_3)} + \frac{p_2}{\sin(\theta_1) \cdot 2R\sin(\theta_3)} + \frac{p_3}{\sin(\theta_1) \cdot 2R\sin(\theta_2)} = 1 \ .$$

Multiplying through by $2R$ gives

$$\frac{p_1}{\sin(\theta_2)\sin(\theta_3)} + \frac{p_2}{\sin(\theta_1)\sin(\theta_3)} + \frac{p_3}{\sin(\theta_1)\sin(\theta_2)} = 2R ,$$

which establishes Lemma 6.2.1.

■

First Solution to Example 6.2.

First, we have

$$\frac{PA_1}{a_1^2}+\frac{PA_2}{a_2^2}+\frac{PA_3}{a_3^2}$$

$$= \quad \frac{1}{a_1^2}PA_1+\frac{1}{a_2^2}PA_2+\frac{1}{a_3^2}PA_3$$

Seeing this, we apply Theorem 4.1 (Dar-Gueron) with $\lambda_1=\frac{1}{a_1^2}$, $\lambda_2=\frac{1}{a_2^2}$, and $\lambda_3=\frac{1}{a_3^2}$:

$$\geq \quad 2\left(\sqrt{\frac{1}{a_2^{\,2}}\cdot\frac{1}{a_3^{\,2}}}\,p_1+\sqrt{\frac{1}{a_1^{\,2}}\cdot\frac{1}{a_3^{\,2}}}\,p_2+\sqrt{\frac{1}{a_1^{\,2}}\cdot\frac{1}{a_2^{\,2}}}\,p_3\right)$$

$$= \quad 2\left(\frac{p_1}{a_2\,a_3}+\frac{p_2}{a_1\,a_3}+\frac{p_3}{a_1\,a_2}\right)$$

By Lemma 2.4, $a_1=2R\sin(\theta_1)$, $a_2=2R\sin(\theta_2)$ and $a_3=2R\sin(\theta_3)$, which gives

$$= \quad 2\left(\frac{p_1}{4R^2\sin(\theta_2)\sin(\theta_3)}+\frac{p_2}{4R^2\sin(\theta_1)\sin(\theta_3)}+\frac{p_3}{4R^2\sin(\theta_1)\sin(\theta_2)}\right)$$

$$= \quad \frac{2}{4R^2}\left(\frac{p_1}{\sin(\theta_2)\sin(\theta)_3}+\frac{p_2}{\sin(\theta_1)\sin(\theta)_3}+\frac{p_3}{\sin(\theta_1)\sin(\theta)_2}\right)$$

By Lemma 6.2.1

$$= \quad \frac{2}{4R^2}(2R)$$

$$= \quad \frac{1}{R}.$$

Thus, we have proven Example 6.2, namely $\frac{PA_1}{a_1^2}+\frac{PA_2}{a_2^2}+\frac{PA_3}{a_3^2}\geq\frac{1}{R}$.

∎

Second Solution to Example 6.2.

Let K be the area of $\triangle A_1 A_2 A_3$. Then, as we started the proof of Lemma 6.2.1, we have

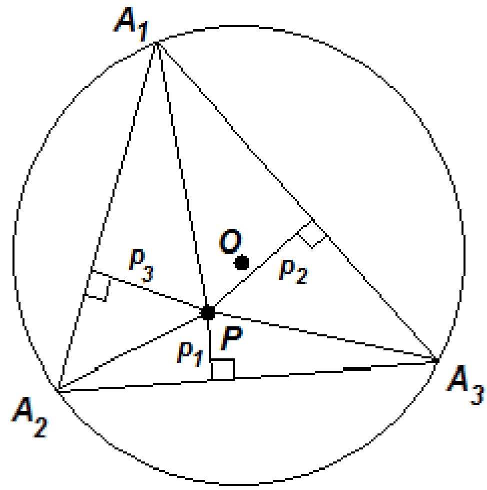

Figure 6.3

$$K = \frac{a_1 p_1}{2} + \frac{a_2 p_2}{2} + \frac{a_3 p_3}{2},$$

so that

$$2K = a_1 p_1 + a_2 p_2 + a_3 p_3 .$$

A common formula for area of a triangle says:

$$4RK = a_1 a_2 a_3 .$$

Now, we have

$$\frac{PA_1}{a_1^2} + \frac{PA_2}{a_2^2} + \frac{PA_3}{a_3^2} = \frac{1}{a_1^2} PA_1 + \frac{1}{a_2^2} PA_2 + \frac{1}{a_3^2} PA_3 .$$

From Corollary 3.5, we have

$$PA_1 \geq \frac{a_2 p_3 + a_3 p_2}{a_1}, \quad PA_2 \geq \frac{a_1 p_3 + a_3 p_1}{a_2} \quad \text{and} \quad PA_3 \geq \frac{a_2 p_1 + a_1 p_2}{a_3} .$$

We apply these inequalities here, to obtain

$$\frac{1}{a_1^2} PA_1 + \frac{1}{a_2^2} PA_2 + \frac{1}{a_3^2} PA_3$$

$$\geq \quad \frac{1}{a_1^2}\left(\frac{a_2 p_3 + a_3 p_2}{a_1}\right) + \frac{1}{a_2^2}\left(\frac{a_1 p_3 + a_3 p_1}{a_2}\right) + \frac{1}{a_3^2}\left(\frac{a_1 p_2 + a_2 p_1}{a_3}\right)$$

$$= \quad \left(\frac{a_2 p_3 + a_3 p_2}{a_1^3}\right) + \left(\frac{a_1 p_3 + a_3 p_1}{a_2^3}\right) + \left(\frac{a_1 p_2 + a_2 p_1}{a_3^3}\right)$$

Rearranging terms gives

$$= \quad \left(\frac{a_3}{a_2^3} + \frac{a_2}{a_3^3}\right) p_1 + \left(\frac{a_3}{a_1^3} + \frac{a_1}{a_3^3}\right) p_2 + \left(\frac{a_2}{a_1^3} + \frac{a_1}{a_2^3}\right) p_3$$

Applying the Arithmetic Mean – Geometric Mean Inequality on each of the three terms gives

$$\geq \quad 2\sqrt{\frac{a_3}{a_2^3}\cdot\frac{a_2}{a_3^3}}\,p_1+2\sqrt{\frac{a_3}{a_1^3}\cdot\frac{a_1}{a_3^3}}\,p_2+2\sqrt{\frac{a_2}{a_1^3}\cdot\frac{a_1}{a_2^3}}\,p_3$$

$$= \quad 2\left(\sqrt{\frac{1}{a_2^2\,a_3^2}}\,p_1+\sqrt{\frac{1}{a_1^2\,a_3^2}}\,p_2+\sqrt{\frac{1}{a_1^2\,a_2^2}}\,p_3\right)$$

$$= \quad 2\left(\frac{p_1}{a_2\,a_3}+\frac{p_2}{a_1\,a_3}+\frac{p_3}{a_1\,a_2}\right)$$

$$= \quad 2\left(\frac{a_1\,p_1}{a_1\,a_2\,a_3}+\frac{a_2\,p_2}{a_1\,a_2\,a_3}+\frac{a_3\,p_3}{a_1\,a_2\,a_3}\right)$$

$$= \quad 2\left(\frac{a_1\,p_1+a_2\,p_2+a_3\,p_3}{a_1\,a_2\,a_3}\right)$$

Recalling that $2\,K=a_1\,p_1+a_2\,p_2+a_3\,p_3$, this gives

$$= \quad 2\left(\frac{2\,K}{a_1\,a_2\,a_3}\right)$$

$$= \quad \frac{4\,K}{a_1\,a_2\,a_3}$$

And knowing $4\,R\,K=a_1\,a_2\,a_3$ as a well regarded formula, we have

$$= \quad \frac{4\,K}{4\,R\,K}$$

$$= \quad \frac{1}{R}\,.$$

Thus, we have established $\frac{PA_1}{a_1^2}+\frac{PA_2}{a_2^2}+\frac{PA_3}{a_3^2}\geq\frac{1}{R}$, as desired.

■

7 Extension to Quadrilaterals

We now examine the possibility of an Erdös-Mordell type inequality for quadrilaterals.

Theorem 7.1.

Let $A_1 A_2 A_3 A_4$ be a convex quadrilateral, and let *P* be an interior point of the quadrilateral.

Let

p_{ij} denote the (positive) distance from *P* to $\overline{A_i A_j}$, and let

$p_{ij\,;\,ijk}$ denote the "signed distance" – as defined in Theorem 3.6 – from *P* to $\overline{A_i A_j}$ when considering $\triangle A_i A_j A_k$.

Then we have

$$PA_1 + PA_2 + PA_3 + PA_4 \geq \frac{4}{3}\left(p_{12} + p_{23} + p_{34} + p_{14}\right) .$$

Comment.

Clayton W. Dodge discusses this result and its proof in **[DOD]**.

Figure 7.1 shows one possible scenario. In the proof, we regard *P* relative to each of the triangles $\triangle A_1 A_2 A_3$, $\triangle A_2 A_3 A_4$, $\triangle A_3 A_4 A_1$, and $\triangle A_4 A_1 A_2$ and apply the result of Theorem 3.6 – the Signed Erdös-Mordell Inequality.

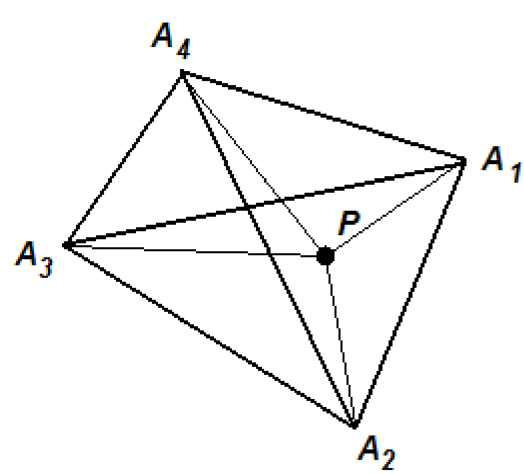

Figure 7.1

Proof of Theorem 7.1. Based on **[DOD]**

Applying the result of Theorem 3.6, to each specified triangle, we get the results below:

With $\triangle A_1 A_2 A_3$: $PA_1 + PA_2 + PA_3 \geq 2\left(p_{12\,;\,123} + p_{23\,;\,123} + p_{13\,;\,123}\right)$;

With $\triangle A_2 A_3 A_4$: $PA_2 + PA_3 + PA_4 \geq 2\left(p_{23\,;\,234} + p_{34\,;\,234} + p_{24\,;\,234}\right)$;

With $\triangle A_1 A_3 A_4$: $PA_1 + PA_3 + PA_4 \geq 2\left(p_{34\,;\,134} + p_{14\,;\,134} + p_{13\,;\,134}\right)$; and

With $\triangle A_1 A_2 A_4$: $PA_1 + PA_2 + PA_4 \geq 2\left(p_{14\,;\,124} + p_{12\,;\,124} + p_{24\,;\,124}\right)$.

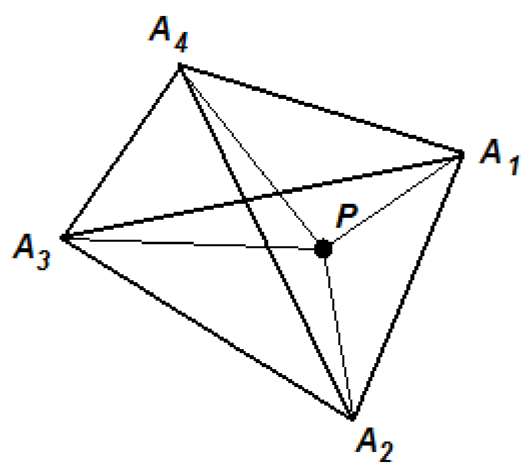

Figure 7.2

Though Figure 7.2 is merely one example of a possible location of *P*, the following relationships hold since *P* must be interior to $A_1 A_2 A_3 A_4$:

$p_{13\,;\,123} = -p_{13\,;\,134}$ since *P* can be interior to at most one of $\triangle A_1 A_2 A_3$ and $\triangle A_1 A_3 A_4$;

$p_{24\,;\,124} = -p_{24\,;\,234}$ since *P* can be interior to at most one of $\triangle A_1 A_2 A_4$ and $\triangle A_2 A_3 A_4$;

$p_{12\,;\,123} = p_{12\,;\,124} = p_{12}$ since *P* is must be on the same side of $\overline{A_1 A_2}$ as both A_3 and A_4;

$p_{23\,;\,123} = p_{23\,;\,234} = p_{23}$ since *P* is must be on the same side of $\overline{A_2 A_3}$ as both A_1 and A_4;

$p_{34\,;\,134} = p_{34\,;\,234} = p_{34}$ since *P* is must be on the same side of $\overline{A_3 A_4}$ as both A_1 and A_2;

$p_{14\,;\,124} = p_{14\,;\,134} = p_{14}$ since *P* is must be on the same side of $\overline{A_1 A_4}$ as both A_2 and A_3.

Thus, the four inequalities to start the proof become:

$$PA_1+PA_2+PA_3\geq 2\,p_{12}+2\,p_{23}-2\,p_{13;134}\,;$$

$$PA_2+PA_3+PA_4\geq 2\,p_{23}+2\,p_{34}+2\,p_{24;234}\,;$$

$$PA_1+PA_3+PA_4\geq 2\,p_{34}+2\,p_{14}+2\,p_{13;134}\text{ ; and}$$

$$PA_1+PA_2+PA_4\geq 2\,p_{14}+2\,p_{12}-2\,p_{24;234}\,.$$

Summing these inequalities gives

$$\begin{aligned}3(PA_1+PA_2+PA_3+PA_4)&\\ &= 3\,PA_1+3\,PA_2+3\,PA_3+3\,PA_4\\ &\geq 4\,p_{12}+4\,p_{23}+4\,p_{34}+4\,p_{14}\\ &= 4(p_{12}+p_{23}+p_{34}+p_{14})\,,\end{aligned}$$

so that we achieve

$$3(PA_1+PA_2+PA_3+PA_4)\geq 4(p_{12}+p_{23}+p_{34}+p_{14})\,,$$

or equivalently our desired result:

$$PA_1+PA_2+PA_3+PA_4\geq \frac{4}{3}(p_{12}+p_{23}+p_{34}+p_{14})\,.$$

∎

Comment.

One might wonder if this is as strong of an inequality as could be achieved for the quadrilateral. This question provides the motivation for Example 7.2 and Example 7.3.

Example 7.2.

Consider the situation where $A_1 A_2 A_3 A_4$ is a square of side length 2, as in Figure 7.3. Then we have $p_{12} = p_{23} = p_{34} = p_{14} = 1$, so that

$$PA_1 = PA_2 = PA_3 = PA_4 = \sqrt{2}\ .$$

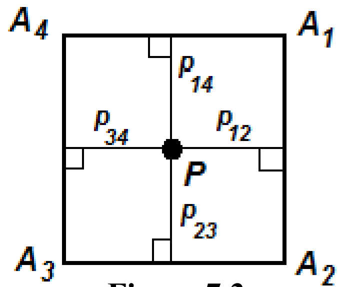

Figure 7.3

This gives $PA_1 + PA_2 + PA_3 + PA_4 = 4\sqrt{2}$ and $p_{12} + p_{23} + p_{34} + p_{14} = 4$, so that

$$PA_1 + PA_2 + PA_3 + PA_4 = \sqrt{2}\,(p_{12} + p_{23} + p_{34} + p_{14})\ .$$

Example 7.3.

Consider the situation with $A_1 A_2 A_3 A_4$ being a rectangle pictured in Figure 7.4. Then we have $p_{12} = 4$, $p_{23} = 10$, $p_{34} = 2$, and $p_{14} = 2$.

Also

$$PA_1 = \sqrt{2^2 + 4^2} = 2\sqrt{5}\ ,$$
$$PA_2 = \sqrt{4^2 + 10^2} = 2\sqrt{29}\ ,$$
$$PA_3 = \sqrt{2^2 + 10^2} = 2\sqrt{26}\ , \text{and}$$
$$PA_4 = \sqrt{2^2 + 2^2} = 2\sqrt{2}\ .$$

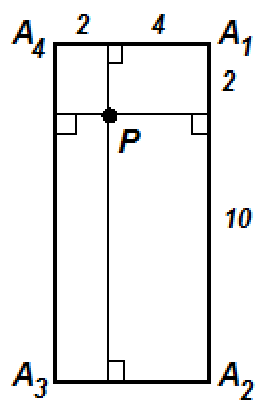

Figure 7.4

So

$$PA_1 + PA_2 + PA_3 + PA_4 = 2\sqrt{5} + 2\sqrt{29} + 2\sqrt{26} + 2\sqrt{2} \approx 28.27$$

and

$$p_{12} + p_{23} + p_{34} + p_{14} = 18\ .$$

Since $28.27 > 25.46 \approx 18\sqrt{2}$, this shows that, when regarding this example.

$$PA_1 + PA_2 + PA_3 + PA_4 \geq \sqrt{2}\,(p_{12} + p_{23} + p_{34} + p_{14})\ .$$

Comment.

Example 7.2 and Example 7.3 combine to provide the motivation for Theorem 7.4.

Theorem 7.4.

Let $A_1A_2A_3A_4$ be a convex quadrilateral, let P be an interior point of the quadrilateral, and let p_{ij} denote the (positive) distance from P to $\overline{A_iA_j}$. Then

$$PA_1+PA_2+PA_3+PA_4 \geq \sqrt{2}\left(p_{12}+p_{23}+p_{34}+p_{14}\right).$$

Equality requires $A_1A_2A_3A_4$ is a square and P is its center.

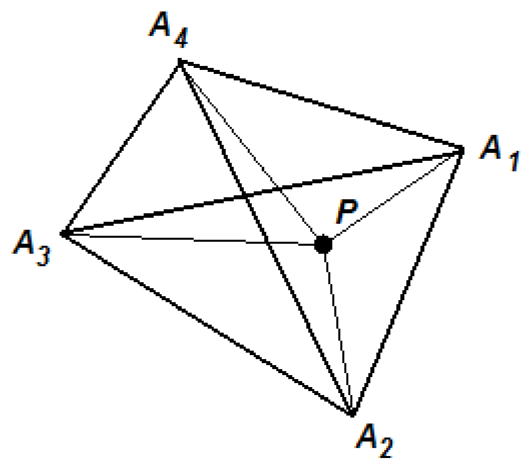

Figure 7.5

Comment.

This result has not been stated in many places. In fact, to our knowledge, the only place such a result is explicitly stated is by Shay Gueron and Itai Shafrir in **[GUE]**. While Gueron and Shafrir offer a more generalized result and associated proof, we confine ourselves to this situation.

We offer a proof that is not given explicitly in the literature (to our knowledge), but that is based off the ideas of Mordell in **[MOR]** involving finding a quadratic form.

Thus, our proof essentially extends Mordell's proof of Barrow's Inequality to the quadrilateral.

Comment.

Before proving this theorem, we must establish a few lemmas.

Lemma 7.4.1.

Given $\triangle A_1 A_2 A_3$. Let a_i denote the length of the side of $\triangle A_1 A_2 A_3$ across from vertex A_i, let α_i be the interior angle of $\triangle A_1 A_2 A_3$ with vertex A_i, and let h_i be the length of the altitude of $\triangle A_1 A_2 A_3$ from A_i. Then

$$h_1 \leq \sqrt{a_2 a_3} \cos\left(\frac{\alpha_1}{2}\right).$$

Equality requires $a_2 = a_3$.

Comment.

This is essentially Corollary 4.2.4, based on Mordell. For explanation, see that earlier result.

Lemma 7.4.2.

Let $A_1 A_2 A_3 A_4$ be a convex quadrilateral, let P be an interior point of the quadrilateral, and let p_{ij} denote the (positive) distance from P to $\overline{A_i A_j}$. Also, let

$\theta_{12} = m\angle A_1 P A_2$, $\theta_{23} = m\angle A_2 P A_3$, $\theta_{34} = m\angle A_3 P A_4$, and $\theta_{14} = m\angle A_1 P A_4$.

Then

$$p_{12} \le \sqrt{(PA_1)(PA_2)}\cos\left(\frac{\theta_{12}}{2}\right)\text{, with equality requiring } PA_1 = PA_2\text{;}$$
$$p_{23} \le \sqrt{(PA_2)(PA_3)}\cos\left(\frac{\theta_{23}}{2}\right)\text{, with equality requiring } PA_2 = PA_3\text{;}$$
$$p_{34} \le \sqrt{(PA_3)(PA_4)}\cos\left(\frac{\theta_{34}}{2}\right)\text{, with equality requiring } PA_3 = PA_4\text{; and}$$
$$p_{14} \le \sqrt{(PA_1)(PA_4)}\cos\left(\frac{\theta_{14}}{2}\right)\text{, with equality requiring } PA_1 = PA_4\text{.}$$

Proof of Lemma 7.4.2. Based on **[GUE]**

We apply the result of Lemma 7.4.1 to each of $\triangle A_1 P A_2$, $\triangle A_2 P A_3$, $\triangle A_3 P A_4$, and $\triangle A_1 P A_4$, as shown in Figure 7.6. The result follows immediately.

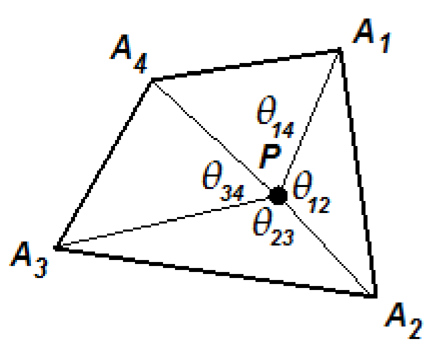

Figure 7.6

Proof of Theorem 7.4. Method Based on **[MOR]**

We base this proof off the methods Mordell employed in proving Barrow's Inequality (see Second Proof of Theorem 4.2), but Mordell never specifically used his methods to establish this result in the literature, to our knowledge. We additionally use the notation from Lemma 7.4.2.

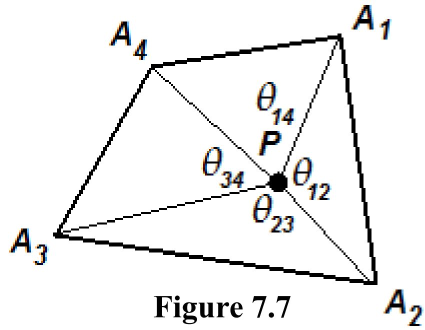

Figure 7.7

Begin by noting that since $\theta_{12}+\theta_{23}+\theta_{34}+\theta_{14}=2\pi$, we have $\dfrac{\theta_{12}+\theta_{14}}{2}=\pi-\dfrac{\theta_{23}+\theta_{34}}{2}$.

We have $-\cos(x)=\cos(\pi-x)$. Also, from Lemma 7.4.2, we can say (since $p_{ij}>0$)

$-\sqrt{2}\,p_{12}\geq-\sqrt{2\,PA_1\,PA_2}\cos\left(\dfrac{\theta_{12}}{2}\right)$ with equality requiring $PA_1=PA_2$; *(7.4.A)*

$-\sqrt{2}\,p_{23}\geq-\sqrt{2\,PA_2\,PA_3}\cos\left(\dfrac{\theta_{23}}{2}\right)$ with equality requiring $PA_2=PA_3$;

$-\sqrt{2}\,p_{34}\geq-\sqrt{2\,PA_3\,PA_4}\cos\left(\dfrac{\theta_{34}}{2}\right)$ with equality requiring $PA_3=PA_4$; and

$-\sqrt{2}\,p_{14}\geq-\sqrt{2\,PA_1\,PA_4}\cos\left(\dfrac{\theta_{14}}{2}\right)$ with equality requiring $PA_1=PA_4$.

Putting this together, we have

$$
\begin{aligned}
0 \quad \leq \quad & \left(\sqrt{PA_1}-\frac{\sqrt{PA_2}}{\sqrt{2}}\cos\left(\frac{\theta_{12}}{2}\right)-\frac{\sqrt{PA_4}}{\sqrt{2}}\cos\left(\frac{\theta_{14}}{2}\right)\right)^2 \\
+ \quad & \left(\frac{\sqrt{PA_2}}{\sqrt{2}}\sin\left(\frac{\theta_{12}}{2}\right)-\frac{\sqrt{PA_4}}{\sqrt{2}}\sin\left(\frac{\theta_{14}}{2}\right)\right)^2 \\
+ \quad & \left(\sqrt{PA_3}-\frac{\sqrt{PA_2}}{\sqrt{2}}\cos\left(\frac{\theta_{23}}{2}\right)-\frac{\sqrt{PA_4}}{\sqrt{2}}\cos\left(\frac{\theta_{34}}{2}\right)\right)^2 \\
+ \quad & \left(\frac{\sqrt{PA_2}}{\sqrt{2}}\sin\left(\frac{\theta_{23}}{2}\right)-\frac{\sqrt{PA_4}}{\sqrt{2}}\sin\left(\frac{\theta_{34}}{2}\right)\right)^2
\end{aligned}
\qquad (7.4.B)
$$

$$= \quad PA_1 + \frac{PA_2}{2}\cos^2\left(\frac{\theta_{12}}{2}\right) + \frac{PA_4}{2}\cos^2\left(\frac{\theta_{14}}{2}\right)$$

$$+ \quad -\frac{2\sqrt{PA_1 PA_2}}{\sqrt{2}}\cos\left(\frac{\theta_{12}}{2}\right) - \frac{2\sqrt{PA_1 PA_4}}{\sqrt{2}}\cos\left(\frac{\theta_{14}}{2}\right)$$

$$+ \quad \frac{2\sqrt{PA_2 PA_4}}{2}\cos\left(\frac{\theta_{12}}{2}\right)\cos\left(\frac{\theta_{14}}{2}\right)$$

$$+ \quad \frac{PA_2}{2}\sin^2\left(\frac{\theta_{12}}{2}\right) + \frac{PA_4}{2}\sin^2\left(\frac{\theta_{14}}{2}\right)$$

$$+ \quad -\frac{2\sqrt{PA_2 PA_4}}{2}\sin\left(\frac{\theta_{12}}{2}\right)\sin\left(\frac{\theta_{14}}{2}\right)$$

$$+ \quad PA_3 + \frac{PA_2}{2}\cos^2\left(\frac{\theta_{23}}{2}\right) + \frac{PA_4}{2}\cos^2\left(\frac{\theta_{34}}{2}\right)$$

$$+ \quad -\frac{2\sqrt{PA_2 PA_3}}{\sqrt{2}}\cos\left(\frac{\theta_{23}}{2}\right) - \frac{2\sqrt{PA_3 PA_4}}{\sqrt{2}}\cos\left(\frac{\theta_{34}}{2}\right)$$

$$+ \quad \frac{2\sqrt{PA_2 PA_4}}{2}\cos\left(\frac{\theta_{23}}{2}\right)\cos\left(\frac{\theta_{34}}{2}\right)$$

$$+ \quad \frac{PA_2}{2}\sin^2\left(\frac{\theta_{23}}{2}\right) + \frac{PA_4}{2}\sin^2\left(\frac{\theta_{34}}{2}\right)$$

$$+ \quad -\frac{2\sqrt{PA_2 PA_4}}{2}\sin\left(\frac{\theta_{23}}{2}\right)\sin\left(\frac{\theta_{34}}{2}\right)$$

$$= \quad PA_1 + \frac{PA_2}{2}\left[\cos^2\left(\frac{\theta_{12}}{2}\right) + \sin^2\left(\frac{\theta_{12}}{2}\right)\right] + \frac{PA_4}{2}\left[\cos^2\left(\frac{\theta_{14}}{2}\right) + \sin^2\left(\frac{\theta_{14}}{2}\right)\right]$$

$$+ \quad -\sqrt{2\,PA_1 PA_2}\cos\left(\frac{\theta_{12}}{2}\right) - \sqrt{2\,PA_1 PA_4}\cos\left(\frac{\theta_{14}}{2}\right)$$

$$+ \quad \sqrt{PA_2 PA_4}\cos\left(\frac{\theta_{12}}{2}\right)\cos\left(\frac{\theta_{14}}{2}\right) - \sqrt{PA_2 PA_4}\sin\left(\frac{\theta_{12}}{2}\right)\sin\left(\frac{\theta_{14}}{2}\right)$$

$$+ \quad PA_3 + \frac{PA_2}{2}\left[\cos^2\left(\frac{\theta_{23}}{2}\right) + \sin^2\left(\frac{\theta_{23}}{2}\right)\right] + \frac{PA_4}{2}\left[\cos^2\left(\frac{\theta_{34}}{2}\right) + \sin^2\left(\frac{\theta_{34}}{2}\right)\right]$$

$$+ \quad -\sqrt{2\,PA_2 PA_3}\cos\left(\frac{\theta_{23}}{2}\right) - \sqrt{2\,PA_3 PA_4}\cos\left(\frac{\theta_{34}}{2}\right)$$

$$+ \quad \sqrt{PA_2 PA_4}\cos\left(\frac{\theta_{23}}{2}\right)\cos\left(\frac{\theta_{34}}{2}\right) - \sqrt{PA_2 PA_4}\sin\left(\frac{\theta_{23}}{2}\right)\sin\left(\frac{\theta_{34}}{2}\right)$$

$$
\begin{aligned}
= \quad & PA_1+\frac{PA_2}{2}+\frac{PA_4}{2}+PA_3+\frac{PA_2}{2}+\frac{PA_4}{2} \\
+ \quad & -\sqrt{2\,PA_1\,PA_2}\cos\left(\frac{\theta_{12}}{2}\right)-\sqrt{2\,PA_1\,PA_4}\cos\left(\frac{\theta_{14}}{2}\right) \\
+ \quad & -\sqrt{2\,PA_2\,PA_3}\cos\left(\frac{\theta_{23}}{2}\right)-\sqrt{2\,PA_3\,PA_4}\cos\left(\frac{\theta_{34}}{2}\right) \\
+ \quad & \sqrt{PA_2 PA_4}\left[\cos\left(\frac{\theta_{12}}{2}\right)\cos\left(\frac{\theta_{14}}{2}\right)-\sin\left(\frac{\theta_{12}}{2}\right)\sin\left(\frac{\theta_{14}}{2}\right)\right] \\
+ \quad & \sqrt{PA_2 PA_4}\left[\cos\left(\frac{\theta_{23}}{2}\right)\cos\left(\frac{\theta_{34}}{2}\right)-\sin\left(\frac{\theta_{23}}{2}\right)\sin\left(\frac{\theta_{34}}{2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & PA_1+PA_2+PA_3+PA_4 \\
+ \quad & -\sqrt{2\,PA_1\,PA_2}\cos\left(\frac{\theta_{12}}{2}\right)-\sqrt{2\,PA_1\,PA_4}\cos\left(\frac{\theta_{14}}{2}\right) \\
+ \quad & -\sqrt{2\,PA_2\,PA_3}\cos\left(\frac{\theta_{23}}{2}\right)-\sqrt{2\,PA_3\,PA_4}\cos\left(\frac{\theta_{34}}{2}\right) \\
+ \quad & \sqrt{PA_2 PA_4}\cos\left(\frac{\theta_{12}+\theta_{14}}{2}\right)+\sqrt{PA_2\,PA_4}\cos\left(\frac{\theta_{23}+\theta_{34}}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & PA_1+PA_2+PA_3+PA_4 \\
+ \quad & -\sqrt{2\,PA_1\,PA_2}\cos\left(\frac{\theta_{12}}{2}\right)-\sqrt{2\,PA_1\,PA_4}\cos\left(\frac{\theta_{14}}{2}\right) \\
+ \quad & -\sqrt{2\,PA_2\,PA_3}\cos\left(\frac{\theta_{23}}{2}\right)-\sqrt{2\,PA_3\,PA_4}\cos\left(\frac{\theta_{34}}{2}\right) \\
+ \quad & \sqrt{PA_2 PA_4}\cos\left(\frac{\theta_{12}+\theta_{14}}{2}\right)+\sqrt{PA_2\,PA_4}\cos\left(\pi-\frac{\theta_{12}+\theta_{14}}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & PA_1+PA_2+PA_3+PA_4 \\
+ \quad & -\sqrt{2\,PA_1\,PA_2}\cos\left(\frac{\theta_{12}}{2}\right)-\sqrt{2\,PA_1\,PA_4}\cos\left(\frac{\theta_{14}}{2}\right) \\
+ \quad & -\sqrt{2\,PA_2\,PA_3}\cos\left(\frac{\theta_{23}}{2}\right)-\sqrt{2\,PA_3\,PA_4}\cos\left(\frac{\theta_{34}}{2}\right) \\
+ \quad & \sqrt{PA_2 PA_4}\cos\left(\frac{\theta_{12}+\theta_{14}}{2}\right)-\sqrt{PA_2\,PA_4}\cos\left(\frac{\theta_{12}+\theta_{14}}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & PA_1 + PA_2 + PA_3 + PA_4 \\
+ \quad & -\sqrt{2\,PA_1\,PA_2}\cos\left(\frac{\theta_{12}}{2}\right) - \sqrt{2\,PA_1\,PA_4}\cos\left(\frac{\theta_{14}}{2}\right) \\
+ \quad & -\sqrt{2\,PA_2\,PA_3}\cos\left(\frac{\theta_{23}}{2}\right) - \sqrt{2\,PA_3\,PA_4}\cos\left(\frac{\theta_{34}}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & PA_1 + PA_2 + PA_3 + PA_4 \\
+ \quad & -\sqrt{2\,PA_1\,PA_2}\cos\left(\frac{\theta_{12}}{2}\right) - \sqrt{2\,PA_2\,PA_3}\cos\left(\frac{\theta_{23}}{2}\right) \\
+ \quad & -\sqrt{2\,PA_3\,PA_4}\cos\left(\frac{\theta_{34}}{2}\right) - \sqrt{2\,PA_1\,PA_4}\cos\left(\frac{\theta_{14}}{2}\right).
\end{aligned}
$$

$$
\le \quad PA_1 + PA_2 + PA_3 + PA_4 - \sqrt{2}\,p_{12} - \sqrt{2}\,p_{23} - \sqrt{2}\,p_{34} - \sqrt{2}\,p_{14} \qquad \textit{(7.4.C)}
$$

$$
= \quad PA_1 + PA_2 + PA_3 + PA_4 - \sqrt{2}\,(p_{12} + p_{23} + p_{34} + p_{14})\,.
$$

So we have

$$
0 \le PA_1 + PA_2 + PA_3 + PA_4 - \sqrt{2}\,(p_{12} + p_{23} + p_{34} + p_{14})\;.
$$

This means

$$
PA_1 + PA_2 + PA_3 + PA_4 \ge \sqrt{2}\,(p_{12} + p_{23} + p_{34} + p_{14})\,,
$$

our desired inequality.

To establish the condition for equality:

For equality to hold in *(7.4.C)*, we need equality in *(7.4.A)*, which requires

$$
PA_1 = PA_2 = PA_3 = PA_4\;.
$$

For equality to then hold in *(7.4.B)* given the above criteria,

$$
\frac{\sqrt{PA_1}}{\sqrt{2}}\sin\left(\frac{\theta_{12}}{2}\right) - \frac{\sqrt{PA_1}}{\sqrt{2}}\sin\left(\frac{\theta_{14}}{2}\right) = 0 \quad \text{which means} \quad \sin\left(\frac{\theta_{12}}{2}\right) = \sin\left(\frac{\theta_{14}}{2}\right)
$$

so that $\frac{\theta_{12}}{2} = \frac{\theta_{14}}{2}$ or $\frac{\theta_{12}}{2} = \pi - \frac{\theta_{14}}{2}$.

Additionally, for equality to hold in *(7.4.B)* given the above requirements, we need

$$\sqrt{PA_1} - \frac{\sqrt{PA_1}}{\sqrt{2}}\cos\left(\frac{\theta_{12}}{2}\right) - \frac{\sqrt{PA_1}}{\sqrt{2}}\cos\left(\frac{\theta_{14}}{2}\right) = 0\ .$$

If $\frac{\theta_{12}}{2} = \pi - \frac{\theta_{14}}{2}$, then recalling $\cos(x) = -\cos(\pi - x)$, we have

$$\begin{aligned}
&\sqrt{PA_1} - \frac{\sqrt{PA_1}}{\sqrt{2}}\cos\left(\frac{\theta_{12}}{2}\right) - \frac{\sqrt{PA_1}}{\sqrt{2}}\cos\left(\frac{\theta_{14}}{2}\right) \\
&= \sqrt{PA_1} - \frac{\sqrt{PA_1}}{\sqrt{2}}\cos\left(\frac{\theta_{12}}{2}\right) - \frac{\sqrt{PA_1}}{\sqrt{2}}\cos\left(\pi - \frac{\theta_{12}}{2}\right) \\
&= \sqrt{PA_1} - \frac{\sqrt{PA_1}}{\sqrt{2}}\cos\left(\frac{\theta_{12}}{2}\right) + \frac{\sqrt{PA_1}}{\sqrt{2}}\cos\left(\frac{\theta_{12}}{2}\right) \\
&= \sqrt{PA_1} \\
&> 0,
\end{aligned}$$

so we cannot get equality in this situation. If, however, $\frac{\theta_{12}}{2} = \frac{\theta_{14}}{2}$, we have

$$\begin{aligned}
&\sqrt{PA_1} - \frac{\sqrt{PA_1}}{\sqrt{2}}\cos\left(\frac{\theta_{12}}{2}\right) - \frac{\sqrt{PA_1}}{\sqrt{2}}\cos\left(\frac{\theta_{14}}{2}\right) \\
&= \sqrt{PA_1} - \frac{\sqrt{PA_1}}{\sqrt{2}}\cos\left(\frac{\theta_{12}}{2}\right) - \frac{\sqrt{PA_1}}{\sqrt{2}}\cos\left(\frac{\theta_{12}}{2}\right) \\
&= \sqrt{PA_1}\left(1 - \frac{2}{\sqrt{2}}\cos\left(\frac{\theta_{12}}{2}\right)\right)
\end{aligned}$$

so that $\sqrt{PA_1} - \frac{\sqrt{PA_1}}{\sqrt{2}}\cos\left(\frac{\theta_{12}}{2}\right) - \frac{\sqrt{PA_1}}{\sqrt{2}}\cos\left(\frac{\theta_{14}}{2}\right) = 0$ requires

$$\sqrt{PA_1}\left(1 - \frac{2}{\sqrt{2}}\cos\left(\frac{\theta_{12}}{2}\right)\right) = 0\ .$$

This means $\cos\left(\frac{\theta_{12}}{2}\right)=\frac{\sqrt{2}}{2}$, so that $\theta_{12}=\frac{\pi}{2}$. It immediately follows that $\theta_{14}=\frac{\pi}{2}$.

Similarly, the other requirements for equality from *(7.4.B)*, namely

$$\sqrt{PA_3}-\frac{\sqrt{PA_2}}{\sqrt{2}}\cos\left(\frac{\theta_{23}}{2}\right)-\frac{\sqrt{PA_4}}{\sqrt{2}}\cos\left(\frac{\theta_{34}}{2}\right)=0$$

and

$$\frac{\sqrt{PA_2}}{\sqrt{2}}\sin\left(\frac{\theta_{23}}{2}\right)-\frac{\sqrt{PA_4}}{\sqrt{2}}\sin\left(\frac{\theta_{34}}{2}\right)=0$$

give rise to the conditions

$$\theta_{23}=\frac{\pi}{2} \quad \text{and} \quad \theta_{34}=\frac{\pi}{2}.$$

So, overall, for equality we require (see Figure 7.8)

$$PA_1=PA_2=PA_3=PA_4 \quad \text{and} \quad \theta_{12}=\theta_{23}=\theta_{34}=\theta_{14}=\frac{\pi}{2}.$$

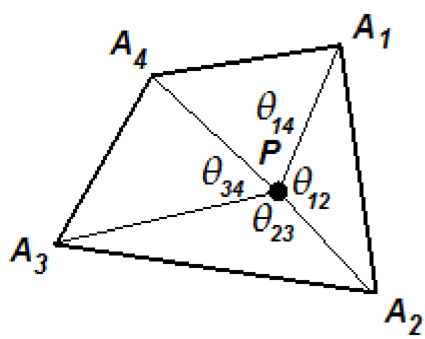

Figure 7.8

Therefore P must be on both $\overline{A_2A_4}$ and $\overline{A_1A_3}$, so that P is the point where the diagonals of $A_1A_2A_3A_4$ intersect.

Additionally, the diagonals intersect at P to form right angles with $PA_1=PA_2=PA_3=PA_4$, so $A_1A_2A_3A_4$ is a square.

Finally, since $A_1A_2A_3A_4$ is a square and P is the point where its diagonals intersect, P must be the center of $A_1A_2A_3A_4$.

∎

8 Conclusion

Throughout this exploration, we have seen how one conjectured inequality by Paul Erdös gave rise to numerous publications and results. This paper provided an overview of some of the extensions and applications of the Erdös-Mordell Inequality, and it shows just how far one conjecture can lead.

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