STRONG CONTINUITY ON PRODUCT SPACES

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ABSTRACT

Strong continuity as defined by O. Dzagnidze is explored. Strong continuity with respect to a variable is demonstrated to be continuity with respect to a variable in terms of A.L. Cauchy. New results concerning the relation between strong continuity and near continuity, quasi-continuity, and somewhat continuity on product spaces are given.

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1 Historical Context

The concept of continuity is among the most fundamental and important in all of mathematics. Augustin-Louis Cauchy attempted to formalize continuity back in 1821 as an integral component of his "Cours d'Analyse" in which he introduced modern rigor to analysis. According to Cauchy [1]:

> "...We also say that the function $f(x)$ is a continuous function of x in the neighborhood of a particular value assigned to the variable x, as long as it (the function) is continuous between those two limits of x, no matter how close together, which enclose the value in question..."

This first attempt at a rigorous definition of continuity surely leaves a bit to be desired. Indeed, we may well wish to ask Cauchy what it means for a function to be "continuous" between two points in order to understand his definition of continuity. This definition though, despite its vagueness and the necessary explanations from others on what exactly it entails for a function to be continuous between two points, has sufficed in earning Cauchy the bulk of the credit for defining continuity. The definition, of course, has evolved over time into our current definition.

Definition A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous at* x_0 if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. We say f is *continuous* if it is continuous for all $x \in \mathbb{R}$.

Definition Similarly to the one-dimensional case, a function $f : \mathbb{R}^n \to \mathbb{R}$ is *continuous at* x_0 or *jointly continuous at* x_0 if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x)-f(x_0)| < \epsilon$ whenever $d(x_0, x) < \delta$ where d represents the Euclidean distance between points in $\mathbb{R}^n$. Also, we say a function is *continuous* or *jointly continuous* if it is continuous at every $x \in \mathbb{R}^n$.

Cauchy goes on in his famous "Cours" to assert and that if a function of several variables is continuous in each one separately, it is a continuous function of all the variables. He even offers us a proof of this claim. Unfortunately, Cauchy never makes explicit what it means for a function of more than one variable to be a continuous function of a variable. Like continuity itself though, later mathematicians have formalized this idea into our current definition.

Definition A function $f : \mathbb{R}^n \to \mathbb{R}$ is *continuous with respect to a variable* if it is continuous when the other $n - 1$ variables are held constant. We say a function is *separately continuous* if it is continuous with respect to each variable.

We take this opportunity to interject a basic fact fundamental to the study of separate vs. joint continuity.

Theorem 1.1 *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous. Then f is separately continuous.*

Proof Let $(x_0, y_0) \times X \times Y$ and let $(x_n) \subset X \times \{y_0\}$ be a sequence such that $x_n \to x_0$. Then $f(x_n) \to f(x_0)$ by the continuity of f at (x_0, y_0). Thus, f is continuous at (x_0, y_0) when $y = y_0$. Since (x_0, y_0) was chosen arbitrarily, it follows that f is continuous with respect to x. Continuity with respect to y can be shown using the same argument, so f is separately continuous. $\blacksquare$

Our modern definitions of continuity and separate continuity are intended and are popularly believed to have descended from Cauchy's idea of continuity with respect to a variable. If in fact this is the case, then Cauchy's assertion that separately continuous functions (as they are now called) are continuous is an erroneous one, and his proof is appropriately regarded as one of his greatest blunders and a denigration on perhaps his most famous and influential work. That a separately continuous function need not be continuous has been known with certainty since 1873 when J. Thomae published a result by E. Heine which gave us a function that is separately continuous but not continuous.

Example [11] Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \sin 4 \arctan(x/y) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

We ought not concern ourselves too much with Heine's example except to acknowledge that it is the earliest such example to appear in the literature. A similar example serving the same purpose followed from Peano in 1884. This example, often referred to as the "classical example", is simpler than Heine's and is given below.

Classical Function [3] Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

If we let $y \neq 0$ be considered as a constant, then we see that the function becomes a fraction with a continuous numerator and a continuous denominator as a function of x, and so is continuous as a function of x. If we take $y = 0$, then the function is just the constant function $f(x, y) = 0$ and so is continuous as a function of x. Thus the function is continuous with respect to x. The function can be similarly seen to be continuous with respect to y by considering x as a constant, so we have that the function is separately continuous. Consider now the function as it approaches the origin along the line $y = x$. The function then becomes $f(x, y) = f(x, x) = \frac{2xx}{x^2+x^2} = \frac{2x^2}{2x^2} = 1$ when not at the origin and 0 at the origin. The function is therefore not continuous as a function of both variables. We thus see that the Classical Function is separately continuous but not continuous.

Keeping the classical example in mind and by looking at our modern definitions, we can see why Cauchy's proof fails to show that a separately continuous function is continuous. The proof, as translated by John Dalbec, reads as follows.

Cauchy's "Proof" [1] Now let $f(x, y, z, ...)$ be a function of several variables $x, y, z, ...$, and suppose that in the neighborhood of particular values $X, Y, Z, ...$ assigned to these variables, $f(x, y, z, ...)$ is at once a continuous function of x, a continuous function of y, a continuous function of z,.... We will prove easily that, if we designate by $\alpha, \beta, \gamma, ...$ infinitely small quantities, and if we assign to $x, y, z, ...$ the values $X, Y, Z, ...$ or values very close to them, the difference

$$f(x + \alpha, y + \beta, z + \gamma, ...) - f(x, y, z, ...)$$

will itself be infinitely small. Indeed, it is clear that, given the preceding hypothesis, the numerical values of the differences

$$f(x + \alpha, y, z, ...) - f(x, y, z, ...)$$
$$f(x + \alpha, y + \beta, z, ...) - f(x + \alpha, y, z, ...)$$
$$f(x + \alpha, y + \beta, z + \gamma, ...) - f(x + \alpha, y + \beta, z, ...)$$
$$...$$

will decrease indefinitely with those of the variables $\alpha, \beta, \gamma, ...$, that is, the numerical value of the first difference with the numerical value of α, that of the second difference with the numerical value of β, that of the third with the numerical value of γ, and so forth. We must conclude that the sum of all the differences, that is,

$$f(x + \alpha, y + \beta, z + \gamma, ...) - f(x, y, z, ...)$$

will converge to the limit zero, if $\alpha, \beta, \gamma, ...$ converge to this same limit. In other words,

$$f(x + \alpha, y + \beta, z + \gamma, ...)$$

will have the limit $f(x, y, z, ...)$.

The proposition we have just shown evidently remains valid in the case where we establish certain relations among the new variables $\alpha, \beta, \gamma, ...$ It suffices that these relations allow the new variables to converge all at the same time to the limit zero. ∎

To demonstrate why Cauchy has failed to show that a separately continuous function is continuous, we must consider the proof with our modern definitions of continuity and separate continuity. We consider the case $n = 2$ to allow for easy comparison with Peano's example. Note that the δ we choose is not only dependent on ϵ, but on the point x_0 as well. When Cauchy says that the numerical values of the difference

$f(x + \alpha, y) - f(x, y)$

will decrease indefinitely with α, this can be interpreted with our modern definitions to mean that for any $\epsilon > 0$ there exists some $\delta_1 > 0$ such that

$|f(x + \alpha, y) - f(x, y)| < \epsilon$ whenever $|(x + \alpha, y) - (x, y)| < \delta_1$ (i.e. whenever $|\alpha| < \delta_1$).

Similarly when Cauchy says that the numerical values of the difference

$f(x + \alpha, y + \beta) - f(x + \alpha, y)$

will decrease indefinitely with β, this can be interpreted with our modern definitions to mean that for any $\epsilon > 0$ there exists some $\delta_2 > 0$ such that

$|f(x + \alpha, y + \beta) - f(x + \alpha, y)| < \epsilon$ whenever $|(x + \alpha, y + \beta) - (x + \alpha, y)| < \delta_2$ (i.e. whenever $|\alpha| < \delta_2$).

Cauchy could then correctly claim that f was continuous as a function of two variables if it followed that there exists a $\delta > 0$ (say $\delta =$min(δ_1, δ_2)) such that

$|f(x + \alpha, y + \beta) - f(x, y)| < 2\epsilon$ whenever $|(x + \alpha, y + \beta) - (x, y)| < \delta$.

But notice that δ_2 depends on $(x + \alpha, y)$ where $(x + \alpha, y)$ is fixed. So if we take a point (x', y') within δ of (x, y) we may be able to guarantee that (x', y') is within δ_2 of (x', y), but this may not be enough. Using Cauchy's notation we would let $x' = x + \alpha'$ and would say

$|(x + \alpha', y + \beta) - (x + \alpha', y)| < \delta_2$.

Although this is true, it does not help us since δ_2 is dependent on $(x+\alpha, y)$, not $(x+\alpha', y)$. We would need

$|(x + \alpha', y + \beta) - (x + \alpha', y)| < \delta$.

for a δ dependent on $(x + \alpha', y)$. Since there need not be any relation between this δ and δ_2, we remain wanting.

Such is the difficulty that Peano's example presents us with: the value of δ needed to keep the value of $f(x, 0)$ or $f(0, y)$ within any given ϵ of $f(x, y)$ grows smaller as the function approaches the origin along the line $y = x$. We can therefore see the shortcoming of Cauchy's proof as an attempt to show that separately continuous functions are continuous.

While it is now clear that a function which is continuous in each variable separately need not be continuous, the basic idea Cauchy used in trying to prove otherwise still seems to ring true. As a matter of fact, none of the specific assertions given to us by Cauchy in his proof are erroneous. The only error would lie in a claim that these individual assertions amount to a proof that functions continuous in each variable separately are continuous. But can we be sure Cauchy ever intended to make such a claim. Notice that our critique of his proof required us to "translate" Cauchy's proof into a notation compatible with our modern definition of continuity a definition not given to us until after Cauchy's "Cour's". This thus begs the question: Did the vagueness of Cauchy's original definition contribute to an erroneous proof of something that is not true, or did it merely cause future mathematicians to misunderstand exactly what it was Cauchy meant to prove? Is there another way to interpret just what Cauchy meant by continuity with respect to a variable? Perhaps our modern notion of separate continuity is not a more rigorous definition Cauchy's continuity with respect to each variable, but rather an entirely different notion. If this is the case, then it would seem incumbent upon us to look for a true descendant of Cauchy's definition of continuity with respect to a variable. Our search takes us ahead almost two centuries from "Cours d'Analyse" to 1999 when Omar Dzagnidne introduced what he describes as a "new" type of separate continuity.

Definition [2] Let $X = Y = \mathbb{R}$. A function $f : X \times Y \to \mathbb{R}$ is *continuous in the strong sense with repsect to y (or x analogously) at* (x_0, y_0) if $\lim_{(x,y)\to(x_0,y_0)}[f(x, y) - f(x, y_0)] = 0$. We say f is *continuous in the strong sense with repsect to y (or x analogously)* if it is continuous in the strong sense with repsect to y at (x, y) for all $(x, y) \in X \times Y$.

Definition [2] Let $X = Y = \mathbb{R}$. A function $f : X \times Y \to \mathbb{R}$ is *separately continuous in the strong sense at* (x_0, y_0) if f is continuous in the strong sense with respect to both x and y at (x_0, y_0). We say f is *separately continuous in the strong sense* if it is separately continuous in the strong sense at (x, y) for all $(x, y) \in X \times Y$.

The simple fact that follows should come as no surprise.

Theorem 1.2 *Let $X = Y = \mathbb{R}$ and let $f : X \times Y \to \mathbb{R}$ be continuous in the strong sense with respect to y (or x). Then f is continuous with respect to y (or x)*

Proof Let $f : X \times Y \to \mathbb{R}$ be continuous in the strong sense with respect to y at (x_0, y_0). Choose a sequence $(x_0, y_n) \subset \{x_o\} \times Y$ such that $(x_0, y_n) \to (x_0, y_0)$. Then $\lim_{(x_0,y_n)\to(x_0,y_0)}[f(x_0, y_n) - f(x_0, y_0)] = 0$ by continuity in the strong sense with respect to y. Thus $\lim_{(x_0,y_n)\to(x_0,y_0)}[f(x_0, y_n)] = f(x_0, y_0)$, so f is continuous with respect to y. ∎

The converse however is not true, as we may use the Classical Function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

to demonstrate a function that is separately continuous but not separately continuous in the strong sense. We have already shown that the Classical Function is separately continuous. To see that it is not separately continuous in the strong sense, consider again the function as it approaches the origin along the line $y = x$. $\lim_{(x,y)\to(0,0)}[f(x,y) - f(0,y)] = \lim_{(x,y)\to(0,0)}[(\frac{2xy}{x^2+y^2} - \frac{2(0)y}{(0)^2+y^2})] = \lim_{(x,x)\to(0,0)}[\frac{2x^2}{2x^2} - 0] = 1 \neq 0$, so the function is not continuous in the strong sense with respect to x at the origin. This is enough to show that the Classical Function is not separately continuous in the strong sense at the origin, and naturally we could show that it is not continuous in the strong sense with respect to y at the origin in the same fashion.

We might now suspect, if we did not already, that separately continuous functions in the strong sense are continuous. Indeed this is the case. In fact, the converse holds as well.

Theorem 1.3 *[2] Let $X = Y = \mathbb{R}$. A function $f : X \times Y \to \mathbb{R}$ is continuous at the point $(x_0, y_0) \in X \times Y$ if and only if it is separately continuous in the strong sense at (x_0, y_0).*

Proof To show necessity, we write the equality

$$f(x,y) - f(x_0,y) = [f(x,y) - f(x_0,y_0)] + [f(x_0,y_0) - f(x_0,y)].$$

Since the function f is continuous at the point (x_0, y_0), the square bracketed expressions on the right-hand side of this equality are arbtrarily small. Therefore the left-hand side of this equality is also arbitrarily small. This is equivalent to

$$\lim_{(x,y)\to(x_0,y_0)}[f(x,y) - f(x_0,y)] = 0.$$

The function f is thus continuous in the strong sense with respect to x at (x_0, y_0). The same argument yields that f is continuous in the strong sense with respect to y at (x_0, y_0), so f is therefore separately continuous in the strong sense at (x_0, y_0).

To prove sufficiency, we first recall that the function f is continuous in the strong sense with respect to x at (x_0, y_0) and thus we have the equality

$$\lim_{(x,y)\to(x_0,y_0)}[f(x,y) - f(x_0,y)] = 0.$$

Likewise, continuity in the strong sense with respect to y at (x_0, y_0) is equivalent to the equality

$$\lim_{(x,y)\to(x_0,y_0)}[f(x,y) - f(x,y_0)] = 0,$$

which in the particular case $x = x_0$ takes the form

$$\lim_{(x,y)\to(x_0,y_0)}[f(x_0,y) - f(x_0,y_0)] = 0.$$

Summing up these two equalities we obtain

$$\lim_{(x,y)\to(0,0)}[f(x,y) - f(x_0,y_0)] = 0,$$

which is equivalent to

$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$. The theorem is proven. ∎

Let us look once more at Cauchy's proof. This time we will not restructure the proof to match the modern definition of continuity and separate continuity. Instead, we will merely change the theorem that we claim to prove to be the sufficiency part of Theorem 1.3. Again, we will look at the two-variable case. Consider when Cauchy tells us that "the numerical values of the differences

$f(x+\alpha, y) - f(x, y)$

$f(x+\alpha, y+\beta) - f(x+\alpha, y)$

will decrease indefinitely with those of the variables α, β". This can be rewritten (not "translated", just rewritten) to say

$\lim_{(x+\alpha,y+\beta)\to(x,y)}[f(x+\alpha, y) - f(x, y)] = 0$

$\lim_{(x+\alpha,y+\beta)\to(x,y)}[f(x+\alpha, y+\beta) - f(x+\alpha, y)] = 0.$

Consider now when Cauchy tells us that "the sum of the differences, that is,

$f(x+\alpha, y+\beta) - f(x, y)$

will converge to the limit zero, if α, β converge to this same limit." We can rewrite this sum to say

$\lim_{(x+\alpha,y+\beta)\to(x,y)}[f(x+\alpha, y+\beta) - f(x, y)] = 0.$

From this point, it is just a simple matter of changing variables to get the proof that Dzagnidze offers us for this same theorem.

So we see then that one could just as easily attribute Cauchy's famous error to the theorem he attempted to prove rather than the proof itself. When we consider that Cauchy never actually formalized exactly what it was he was trying to prove, it would seem harsh of us to assume that the "error" must have been in the proof. Certainly Cauchy can not be too critical of us for attacking a strawman given his own vagueness, but it would seem that Dzagnidze has offered us a much more viable candidate for Cauchy's continuity with respect to a variable than the one we had formerly attached to it. Rather than Dzagnidze's continuity in the strong sense with respect to a variable being a new sense of separate continuity as Dzagnidze himself supposes in his title, it seems entirely probable that it is nearly two centuries old!

One could argue, however, that by trying to rescue Cauchy's proof from fallacy, we in fact do him even greater injury. After all, the concept of continuity with respect to a variable as we define it today has been very bountiful in mathematics. If Cauchy's proof is indeed correct, then it must follow that this was not the concept he offered us in "Cours". But perhaps we now have an opportunity to slowly remedy this unfortunate circumstance. When one considers the profound impact Cauchy's concept of continuity has had in mathematics, we have good reason to suppose that his concept of continuity with respect to a variable would be fruitful as well. Mathematicians were wise to seek results stemming from continuity with respect to a variable as we have long known it, and we would now be wise to seek results from this concept as Cauchy had likely envisioned it.

2 Weak Continuities

To see how this new – or very old – notion of continuity with respect to a variable might be useful, we turn our attention to some generalized forms of continuity. As we shall see, each of these forms of continuity can be studied by comparing functions that possess the property with functions that possess the property in each variable separately. Indeed, "separate vs. joint continuity" of these weak continuities has proven to be a both fruitful and useful branch of Topology and Analysis, and so has understandably garnered great attention over the last century. In introducing these forms of generalized continuity we will take this opportunity to define continuity on general topological spaces.

For all the definitions that follow in this section, assume that X, Y, and Z are general topological spaces.

Definition A function $f : X \to Y$ is *continuous at* $x \in X$ if for any open set $V \subset Y$ containing $f(x)$ there exists an open set $U \subset X$ containing x such that $f^{\to}(U) \subset V$. We say a function f *continuous* if it is continuous at every $x \in X$.

Definition A function $f : X \times Y \to Z$ is *continuous with respect to a variable* if it is continuous when the other variable is taken to be constant. We say a function is *separately continuous* if it is continuous with respect to both x and y.

Definition A function $f : X \to Y$ is *nearly continuous at* $x \in X$ if for any open set $V \subset Y$ containing $f(x)$ the point x is in the interior of the closure of $f^{\leftarrow}(V)$. We say a function f *nearly continuous* if it is nearly continuous at every $x \in X$.

Definition A function $f : X \times Y \to Z$ is *nearly continuous with respect to a variable* if it is nearly continuous when the other variable is taken to be constant. We say a function is *separately nearly continuous* if it is nearly continuous with respect to both x and y.

Definition A function $f : X \to Y$ is *quasi-continuous at* $x \in X$ if for any open set $V \subset Y$ containing $f(x)$ and any open set $U \subset X$ containing x, there exists an open $G \subset U$ such that $f^{\to}(G) \subset V$. We say a function f *quasi-continuous* if it is quasi-continuous at every $x \in X$.

Definition A function $f : X \times Y \to Z$ is *quasi-continuous with respect to a variable* if it is quasi-continuous when the other variable is taken to be constant. We say a function is *separately quasi-continuous* if it is quasi-continuous with respect to both x and y.

Definition A function $f : X \to Y$ is *somewhat continuous* if for any open set $V \subset Y$ such that $f^{\leftarrow}(V)$ is nonempty we have the interior of $f^{\leftarrow}(V)$ is nonempty.

Definition A function $f : X \times Y \to Z$ is *somewhat continuous with respect to a variable* if it is somewhat continuous when the other variable is taken to be constant. We say a function is *separately somewhat continuous* if it is somewhat continuous with respect to both x and y.

It follows readily from the definitions that all continuous functions are both nearly continuous and quasi-continuous, and that all quasi-continuous functions are somewhat continuous.

Example Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x+1 & \text{if } x > 0 \end{cases}$$

This function is an example of a quasi-continuous function that is not nearly continuous. It is continuous for $x \neq 0$ since for any basic open set (a,b) containing $f(x)$ since $(a,\min(b,x/2))$ is an open set containing x such that $f^{\to}(a,\min(b,x/2) \subset (a,b)$ if $x < 0$, and $(\max(a,x/2+1),b)$ is an open set containing x such that $f^{\to}(\max(a,x/2+1),b) \subset (a,b)$ if $x > 0$. Thus f is quasi-continuous for $x \neq 0$. For any basic open set (a,b) containing $f(x)$ and any basic open set (c,d) containing x at $x=0$, $(\max(a,c),c/2) \subset (c,d)$ is an open set such that $f^{\to}(\max(a,c),c/2) \subset (a,b)$. Thus f is quasi-continuous at $x=0$ as well, so f is quasi-continuous.

Notice that f is not nearly continuous at $x=0$, for we can take the open set $(-1,1/2)$ which contains $f(x)$ and such that $f^{\to}(-1,1/2) = (-1,0]$. The closure of $f^{\to}(-1,1/2)$ is then $[-1,0]$ so the interior of the closure of $f^{\to}(-1,1/2)$ is $(-1,0)$ which does not contain $x=0$. It thus follows that f is also not continuous at $x=0$.

Example Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

This function is commonly rerered to as the "salt and pepper" function, and is an example of a nearly continuous function that is not quasi-continuous. For any $x \in \mathbb{Q}$ and any open set V containing $f(x) = 1$, the set $f^{\to}(V)$ contains all of $\mathbb{Q}$. Thus the closure of $f^{\to}(V)$ is all of $\mathbb{R}$ and the interior of the closure of $f^{\to}(V)$ is also all of $\mathbb{R}$, which of course contains x. Similarly, for any $x \in \mathbb{R} - \mathbb{Q}$ and any open set V containing $f(x) = 1$, the set $f^{\to}(V)$ contains all of $\mathbb{R} - \mathbb{Q}$. Thus the closure of $f^{\to}(V)$ is all of $\mathbb{R}$ and the interior of the closure of $f^{\rightarrow}(V)$ is also all of $\mathbb{R}$, which again naturally contains x. Thus f is nearly continuous.

Notice that f is not quasi-continuous at any point x. For say $x \in \mathbb{Q}$ and take $(1/2, 2)$ as our open set containing $f(x)$. Recall that for any open $G \subset \mathbb{R}$, G must contain some $g \in \mathbb{R} - \mathbb{Q}$. Thus for any open set $U \in X$ containing x there is no open subset $G \subset U$ such that $f^{\rightarrow}(G) \subset V$ since $f(g) = 0 \notin (1/2, 2)$ for some $g \in G$. Thus f is not quasi-continuous at x. This suffices to show that f is not quasi-continuous and therefore not continuous either, but note that a similar argument cna be made to show that f is not quasi-continuous at any $x \in \mathbb{R} - \mathbb{Q}$ as well.

Example [8] Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \text{ and } x \in \mathbb{Q} \\ 1 & \text{if } x < 0 \text{ and } x \in \mathbb{R} - \mathbb{Q} \\ 0 & \text{if } x \in [0, 1] \\ 1 & \text{if } x > 1 \end{cases}$$

This function is an example of a somewhat continuous function that is not quasi-continuous. Take any open set V such that $f^{\leftarrow}(V) \neq \emptyset$.. Then we must have $0 \in V$ or $1 \in V$. Therefore the interior of $f^{\leftarrow}(V)$ contains $(0, 1)$ or $(1, \infty)$ respectively, so is nonempty. We thus have that f is somewhat continuous.

Notice that for $x < 0$, f is just the salt and pepper function given in the previous example. The same argument given to show that function is not quasi-continuous for any x can therefore be used again here to show that f is not quasi-continuous at any $x < 0$. Thus, this f is not quasi-continuous

Given that it would make little sense to speak of the separate continuity of these various generalizations of continuity if it were equivalent to the respective generalized continuity of a function, we should now take this opportunity to demonstrate that such is not the case for any of the weakened continuities just introduced.

For a function $f : X \times Y \rightarrow Z$ the symbols f_x, f^y denote its x-section or y-section, respectively, i.e., f_x for any $x \in X$ is the function defined on Y such that $f_x(y) = f(x, y)$. The y-section is defined analogously.

Example [6] Let $f : (-1, 1) \times (-1, 1) \rightarrow \mathbb{R}$ be defined as

$$f(x, y) = \begin{cases} 1 & \text{if both } x \text{ and } y \text{ are irrational or } (x, y) = (0, 0) \\ 0 & \text{if at least one of } x, y \text{ is rational and } (x, y) \neq (0, 0) \end{cases}$$

Then f is nearly continuous at each point (x, y), but the sections f_{x_0}, f^{y_0} are not nearly continuous when $(x_0, y_0) = (0, 0)$, because neither of them is nearly continuous at the origin.

Example [6] On the interval $(-1, 1) \times (-1, 1)$ consider the set $S = \{(x, y) : 0 \leq x \leq 1, x/2 \leq y \leq x\}$.

Let $f : (-1, 1) \times (-1, 1) \to \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) \in S - \{(0,0)\} \\ 0 & \text{if both } x, y \text{ are simultaneously rational or irrational and } (x,y) \notin S \\ 1 & \text{if } x \text{ is rational, } y \text{ irrational or conversely and } (x,y) \notin S \end{cases}$$

The function f is not nearly continuous at $(0, 0)$. The near continuity of the sections f_{x_0}, f^{y_0} may be easily verified for each $x_0 \in X$, $y_0 \in Y$, respectively.

Example [4]

Let $f : (0, 1) \times (0, 1) \to \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} 1 & \text{if } 0 < x < 1/2 \text{ and } 0 < y < 1 \\ 0 & \text{if } 1/2 < x < 1 \text{ and } 0 < y < 1 \\ 1 & \text{if } x = 1/2 \text{ and } y \text{ rational in } (0,1) \\ 0 & \text{if } x = 1/2 \text{ and } y \text{ irrational in } (0,1) \end{cases}$$

The function f is quasi-continuous but $f_{1/2}$ defined by $f_{1/2}(y) = f(1/2, y)$ is not quasi-continuous.

Example [5] Define the functions f_1, f_2, f_3, f_4 on $[0, 1/2) \times [0, 1]$, $[1/2, 1] \times [0, 1]$, $[0, 1] \times [-1/2, 0)$, $[0, 1) \times [-1, -1/2)$ respectively.

Put $f_1(x, y) = \{1$ if y is rational, 0 if y is irrational

$f_2(x, y) = \{0$ if y is rational, 1 if y is irrational

$f_3(x, y) = \{1$ if x is rational, 0 if x is irrational

$f_4(x, y) = \{0$ if y is rational, 1 if y is irrational

Functions f_5, f_6, f_7, f_8 are defined on $[-1, 0) \times [1/2, 1]$, $[-1, 0) \times [0, 1/2)$, $[-1, 1/2) \times [-1, 0)$, $[-1/2, 0) \times [-1, 0)$ respectively as follows

$f_5(x, y) = f_4(-x, -y)$

$f_6(x, y) = f_3(-x, -y)$

$f_7(x, y) = f_2(-x, -y)$

$f_8(x,y) = f_1(-x,-y)$

Denote the interval $[-1,1]\times[-1,1]$ as I. Put $f(x,y) = f_i(x,y)$, where $1 \le i \le 8$. f is unambiguously defined on I by means of the functions f_i. It is easy to check that f is not somewhat continuous on I while the sections f_x and f^y are somewhat continuous for every $x \in [-1,1]$, $y \in [-1,1]$, respectively.

Example [9]

Let $f : [-1,1]\times[-1,1] \to \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} 0 & \text{if } -1 \le x \le -1/2 \text{ and } -1 \le y \le 1 \\ 0 & \text{if } x = 0 \text{ and } y \neq 0 \\ 0 & \text{if } x \in [-1/2, 1/2) \cap (\mathbb{Q} - \{0\}) \text{ and } y \in [-1,1] \cap \mathbb{Q} \\ 1 & \text{otherwise} \end{cases}$$

The function f is somewhat continuous. However it is not separately somewhat continuous.

From these examples, we see that near continuity and separate near continuity as well as somewhat continuity and separate somewhat continuity are independent of each other. There is, however, a relation between quasi-continuity and separate quasi-continuity. Perhaps surprisingly (although the surprise may have been spoiled by the previous examples), the relation is not analogous to that of continuity. It was S. Kempisty in 1932 who first proved that separately quasi-continuous functions were quasi-continuous (for $f : (0,1)\times(0,1) \to \mathbb{R}$). This result has been generalized by many authors, but we choose here to offer a proof by N.F.G. Martin. Martin proved his theorem for when X is Baire, Y is second countable, and M is metric; but since $\mathbb{R}$ is Baire, second countable, and metric, the theorem holds if we allow $X = Y = M = \mathbb{R}$ as well.

Theorem 2.1 *[4] Let X be a Baire space, Y be second countable, and M be a metric space with metric ρ. Let $f : X \times Y \to M$ be separately quasi-continuous. Then f is quasi-continuous.*

Proof For $p \in X$, f_p will denote the function on Y to M defined by $f_p(y) = f(p,y)$. Similarly for $p \in Y$.

Suppose there is a point $(p,q) \in X \times Y$ such that f is not quasi-continuous at (p,q). Then there is an $\epsilon > 0$ and a neighborhood $U \times V$ of (p,q) such that for every non-null open set $E \subset U \times V$ there is a point $(x,y) \in E$ such that $\rho(f(x,y) - f(p,q)) \ge \epsilon$.

Since f_q is quasi-continuous at p, there is a non-null open set $W \subset U$ such that for all $x \in W$, $\rho(f(x,q) - f(p,q)) < \epsilon/3$.

Let Ω be a countable base for Y and let $V_n : n = 1, 2, ...$ be those elements from Ω which are contained in V. For each positive integer n, let A_n denote the set of all points $x \in W$ such that for all $y \in V_n$, $\rho(f(x,y) - f(x,q)) < \epsilon/3$.

Let $x \in W$. Since f_x is quasi-continuous at q there is a non-null open set $E \subset V$ such that for all $y \in E$, $\rho(f(x,y) - f(x,q)) < \epsilon/3$. But there is a k such that $V_k \subset E$. Thus $x \in A_k$ and $W = \bigcup_{n=1}^{\infty} A_n$.

Let W' be any open subset of W and let n be a positive integer. Then $W' \times V_n \subset U \times V$ and there is a point $(x', y') \in W' \times V_n$ such that $\rho(f(x', y') - f(p,q)) \geq \epsilon$.

Since $f_{y'}$ is quasi-continuous at x', there is a non-null open set $W'' \subset W'$ such that for all $x \in W''$, $\rho(f(x,y') - f(x', y')) < \epsilon/3$. Let $x \in W''$. Then

$$\begin{aligned}\rho(f(x,y') - f(x,q)) &\geq \rho(f(x',y') - f(p,q)) - \rho(f(x',y') - f(x,y')) \\ &\quad - \rho(f(x,q) - f(p,q)) \\ &\geq \epsilon\end{aligned}$$

Since $y' \in V_n$, $x \notin A_n$ and we have $W'' \cap A_n = \emptyset$. Therefore A_n is nowhere dense and W is first category. A contradiction. ∎

Although we have demonstrated the independence of somewhat continuity and separate somewhat continuity, we can form a strengthening of somewhat continuity of a two-variable function by strengthening somewhat continuity with respect to one it its variables. Specifically, we have from T. Neubrunn the following theorem. Again, $\mathbb{R}$ is Baire, second countable and regular, so the theorem also holds for $X = Y = Z = \mathbb{R}$.

Theorem 2.2 *[5] Let X be a Baire space, Y second countable and Z regular. Let $f : X \times Y \to Z$ have all x-sections somewhat continuous and all y-sections quasi-continuous. Then f is somewhat continuous.*

Proof Denote by $\text{int}(K)$ and $\text{cl}(K)$ the interior and closure respectively of a set K. Let f not be somewhat continuous. There exists $G \neq \emptyset$ open such that $f^{\leftarrow}(G) \neq \emptyset$ and $\text{int}(f^{\rightarrow}(G)) = \emptyset$.

Let $(x_0, y_0) \in f^{\leftarrow}(G)$. Choose G_1 open such that $\text{cl}(G_1) \subset G$, $f(x_0, y_0) \in G_1$. This is possible because of the regularity of Z. Owing to the quasi-continuity and hence somewhat continuity of f_{y_0} at the point x_0 we have $\text{int}(f_{y_0}^{\leftarrow}(G_1)) \neq \emptyset$. Put $U = \text{int}(f_{y_0}^{\leftarrow}(G_1))$. For any $x \in U$ form $f_x^{\leftarrow}(G_1)$. Since $f_x(y_0) = f(x, y_0) \in G_1$, we have $f_x^{\leftarrow}(G_1) \neq \emptyset$. The somewhat continuity of f_x gives $\text{int}(f_x^{\leftarrow}(G_1)) \neq \emptyset$ for any $x \in U$.

Let $\{V_n\}$ be a countable basis of the space Y. Define A_n as the set of all $x \in u$ for which $V_n \subset \text{int}(f_x^{\leftarrow}(G_1))$. Evidently $\bigcup_{n=1}^{\infty} A_n = U$.

Let $S \subset U$ be any nonempty open set. Let us form $S \times V_n$ for given n. Because of the fact int$(f^{\leftarrow}(G))\neq \emptyset$ there exists $(x^*, y^*) \in S \times V_n$ such that $f(x^*, y^*) \notin G$.

Choose a neighborhood G^* of $f(x^*, y^*)$ such that $G^* \cap G_1 \neq \emptyset$. Using the quasi-continuity of f_{y^*} at x^* we obtain that there exists a nonempty set $S' \subset S$ such that $f(x, y^*) \in G^*$ for any $x \in S'$, hence $f(x, y^*) \notin G_1$. Thus $y^* \notin f_x^{\leftarrow}(G_1)$. This implies $V_n \not\subset f_x^{\leftarrow}(G_1)$, hence $x \notin A_n$. Thus $S' \cap A_n = \emptyset$. This means that A_n is nowhere dense and the set $U = \bigcup_{n=1} A_n$ is of the first category. This is a contradiction. ∎

3 New Results

While the distinction between continuity and continuity with respect to a variable (as well as among the weakened continuities) has been fortunate for the many mathematicians who have used it to produce deep and elegant results, it can prove a hindrance to many other mathematical endeavors. Specifically, it would be convenient to have the continuity (or weakened continuity) of a function guarantee the continuity (or same weakened condition) of that function with respect to a given variable and vice versa. As we have already seen, this condition can not be guaranteed as long as both variables are merely continuous (or are the same weakened condition). However, Dzagnidze is able to guarantee us that with the help of strong continuity with respect to a given variable, we can have this desirable property for the continuity of a function and its continuity with respect to the other variable.

Theorem 3.1 *[2] Let $X = Y = \mathbb{R}$ A function $f : X \times Y \to \mathbb{R}$ is continuous at (x_0, y_0) if and only if $f(x_0, y_0)$ is continuous in the strong sense at the point (x_0, y_0) with respect to one of the variables and continuous at (x_0, y_0) with respect to the other variable.*

Corollary 3.2 *Let $X = Y = \mathbb{R}$ and $f : X \times Y \to \mathbb{R}$ be continuous in the strong sense with respect to y. Then f is continuous with respect to x if and only if f is continuous.*

Having just been reminded of what it means for a function to be nearly continuous, quasi-continuous, and somewhat continuous, one might wonder whether continuity in the strong sense with respect to a variable might provide an analogous result for these weakened forms of continuity. Before we attack these questions, however, we must first determine whether continuity in the strong sense with respect to a variable is even needed, or whether continity with respect to a variable would do the job asked of it. We therefore require a few examples.

Example Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} \dfrac{y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

This function is continuous with respect to y and is quasi-continuous but is not quasi-continuous with respect to x along the $x - axis$.

Example Let $f : \mathbb{R} \times \mathbb{R} \to [0, 1]$ be defined as

$$f(x,y) = \begin{cases} 0 & \text{if } x \in \mathbb{R} - \mathbb{Q} \\ 1 & \text{if } x = 0 \\ 1 & \text{if } x = a/b \text{ where } a, b \in \mathbb{Z} \text{ such that } \gcd(a,b) = 1 \text{ and } y \geq 1/b \\ \text{y} & \text{if } x = a/b \text{ and } y < 1/b \end{cases}$$

This function is continuous with respect to y and is nearly continuous at the origin, but is not nearly continuous with respect to x on the x-axis at the origin.

Example As we have already demonstrated, the Classical Function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is a separately continuous function that is not continuous at $(0, 0)$. Thus, [10] guarantees us that f is not nearly continuous at $(0, 0)$. Therefore the Classical Function f is continuous with respect to y and nearly continuous with respect to x (since it is separately continuous), but is not nearly continuous.

We now see that continuity with respect to a variable is insufficient for our stated purpose with regard to near continuity and quasi-continuity, and we may now call on continuity in the strong sense with respect to a variable. We begin by offering a lemma that characterizes near continuity in terms of sequences.

Lemma 3.3 *Let X be first countable and Y a general topological space. Then $f : X \to Y$ is nearly continuous at $x \in X$ if and only if for any open $V \subset Y$ containing $f(x)$ there exists a dense set D in X such that $x_n \not\to x$ for any sequence $(x_n) \subset D \cap f^{\leftarrow}(Y - V)$.*

Proof We first show necessity. Let $f : X \to Y$ be nearly continuous at $x \in X$. Let V be an open set containing $f(x)$. Then $x \in int(cl(f^{\leftarrow}(V)))$ by the near continuity of f at x. Let $D = f^{\leftarrow}(V) \bigcup (X - int(cl(f^{\leftarrow}(V))))$. Since $f^{\leftarrow}(V)$ is dense in $cl(f^{\leftarrow}(V))$ it follows that $f^{\leftarrow}(V)$ is dense in $int(cl(f^{\leftarrow}(V)))$. Thus, D is dense in X. Choose a sequence $(d_n) \subset D \bigcap f^{\leftarrow}(Y - V)$. Seeking contradiction , assume $d_n \to x$. Then (d_n) is eventually in $int(cl(f^{\leftarrow}(V)))$. Since we've chosen $(d_n) \subset D$, it follows that (d_n) is eventually in $f^{\leftarrow}(V)$. But this contradicts $(d_n) \subset f^{\leftarrow}(Y - V)$, so (d_n) does not converge to x and necessity is proven.

For sufficiency, we now assume $f : X \to Y$ is not nearly continuous. Then $x \notin int(cl(f^{\leftarrow}(V)))$ for some open V containing $f(x)$. Thus there is no open set U containing x such that $U \subset cl(f^{\leftarrow}(V))$. So for any open U containing x there is no $U' \subset U$ such that $x \in U \subset cl(U') \subset cl(f^{\leftarrow}(V))$. It follows that for any open U containing x there is no $U' \subset U$ dense in U such that $f^{\rightarrow}(U') \subset V$. Let $D \subset X$ be dense in X and let $\{B_n : n \in \mathbb{N}\}$ be a descending countable base of open sets at x. Then for each n, we can take $G_n \subset B_n$ open such that $f^{\rightarrow}(G_n) \subset Y - V$. Since D is dense in X and G_n is open we have $G_n \cap D \neq \emptyset$ and can choose $d_n \in G_n \cap D$. We thus have a sequence (d_n) such that $(d_n) \subset D \cap f^{\leftarrow}(Y - V)$ and $d_n \to x$, from which sufficiency follows. ∎

We also provide, without proof, a simple lemma which follows readily from the definition of continuity in the strong sense.

Lemma 3.4 *Let $X = Y = \mathbb{R}$ and let $f : X \times Y \to R$ be continuous in the strong sense with respect to y at $(x_0, y_0) \in X \times Y$. Then for any $\epsilon > 0$, there exists an open $G \subset X \times Y$ containing (x_0, y_0) such that $|f(x, y) - f(x, y_0)| < \epsilon$ for any $(x, y) \in G$.*

We are now ready, with the help of separate continuity in the strong sense, to demonstrate the convenient property we had promised for near continuity.

Theorem 3.5 *Let $X = Y = \mathbb{R}$ and let $f : X \times Y \to \mathbb{R}$ be continuous in the strong sense with respect to y. Then f is nearly continuous with respect to x if and only if f is nearly continuous.*

Proof To show necessity, assume f is continuous in the strong sense with respect to y and is not nearly continuous at $(x_0, y_0) \in X \times Y$. Then by Lemma 3.3, there exists an open interval $B_r = (f(x_0, y_0) - r, f(x_0, y_0) + r)$ in $\mathbb{R}$ such that for any dense $D \subset X \times Y$ there is a sequence $\langle(x_n, y_n)\rangle \subset D \cap f^{\leftarrow}(\mathbb{R} - B_r)$ such that $(x_n, y_n) \to (x_0, y_0)$. Let D_{y_0} be dense in $X \times \{y_0\}$. Define $D^* = \{(x, y) : (x, y_0) \in D_{y_0}\}$. Then D^* is dense in $X \times Y$, so there is a sequence $\langle(x_n, y_n)\rangle \subset D^* \cap f^{\leftarrow}(\mathbb{R} - B_r)$ such that $(x_n, y_n) \to (x_0, y_0)$. We also have $\langle(x_n, y_0)\rangle \subset D_{y_0}$ and continuity in the strong sense with respect to y gives us $\lim_{(x_n, y_n) \to (x_0, y_0)}[f(x_n, y_n) - f(x_n, y_0)] = 0$. So for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|f(x_n, y_n) - f(x_n, y_0)| < \epsilon$. Thus there exists $N \in \mathbb{N}$ such that $|f(x_n, y_n) - f(x_n, y_0)| < \frac{r}{2}$ for any $n > N$. So $f(x_n, y_0) \in (\mathbb{R} - B_{\frac{r}{2}})$ for any $n > N$ where $B_r = (f(x_0, y_0) - \frac{r}{2}, f(x_0, y_0) + \frac{r}{2})$. So $(x_n, y_0) \to (x_0, y_0)$ and $\langle(x_n, y_0)\rangle \subset (D \cap f^{\leftarrow}(\mathbb{R} - B_{\frac{r}{2}}))$ for large n. So it follows from Lemma 3.3 that f is not nearly continuous with respect to x at (x_0, y_0). Necessity then follows from contraposition.

To show sufficiency, assume f is continuous in the strong sense with respect to y and nearly continuous but is not nearly continuous with respect to x and seek a contradiction. Since f is not nearly continuous with respect to x there exists $(x_0, y_0) \in X \times Y$ and open interval $B_r = (f(x_0, y_0) - r, f(x_0, y_0) + r)$ in $\mathbb{R}$ such that $(x_0, y_0) \notin int_{X\times\{y_0\}}(cl_{X\times\{y_0\}}(f^{\leftarrow}(B_r)\left|_{X\times\{y_0\}}\right.))$. Thus there is no open set U containing x such that $U\left|_{X\times\{y_0\}}\right. \subset cl_{X\times\{y_0\}}(f^{\leftarrow}(B_r))$. So for any open U containing x there is no $U' \subset U\left|_{X\times\{y_0\}}\right.$ such that

$x \in U\left|_{X\times\{y_0\}}\right. \subset cl_{X\times\{y_0\}}(U') \subset cl_{X\times\{y_0\}}(f^{\leftarrow}(B_r)\left|_{X\times\{y_0\}}\right.)$. It follows that for any open U containing x there is no $U' \subset U\left|_{X\times\{y_0\}}\right.$ dense in $U\left|_{X\times\{y_0\}}\right.$ such that $f^{\rightarrow}(U') \subset B_r$. So for any open $U \subset X \times Y$ containing (x_0, y_0) there exists $B \subset U\left|_{X\times\{y_0\}}\right.$ open in $X \times \{y_0\}$ such that $f^{\rightarrow}(A) \subset \mathbb{R} - B_r$. Take $B_{\frac{r}{2}} = (f(x_0, y_0) - \frac{r}{2}, f(x_0, y_0) + \frac{r}{2})$ By near continuity at (x_0, y_0) there is an open neighborhood U_1 of (x_0, y_0) and a dense set H in U_1 such that $f^{\rightarrow}(H) \subset B_{\frac{r}{2}}$. Take $A_1 \subset U_1\left|_{X\times\{y_0\}}\right.$ open in $X \times \{y_0\}$ such that $f^{\rightarrow}(A_1) \subset \mathbb{R} - B_r$. From Lemma 3.4 we get that there exists an open neighborhood G of (x_0, y_0) such that $|f(x, y) - f(x, y_0)| < \frac{r}{2}$. Without loss of generality, choose G such that $(x, y_0) \in A_1$ when $(x, y) \in G$. Then for any $(x, y) \in G$ we have $f(x, y_0) \in \mathbb{R} - B_r$ and $|f(x, y) - f(x, y_0)| < \frac{r}{2}$. So $f(x, y) \in \mathbb{R} - B_{\frac{r}{2}}$ for any $(x, y) \in G$, so $f^{\rightarrow}(G) \subset \mathbb{R} - B_{\frac{r}{2}}$. Thus $G \cap H = \emptyset$ so $(G \cap U_1) \cap H = \emptyset$. But this is a contradiction since $G \cap U_1$ is open in U_1 and H is dense in U_1. Sufficiency follows and the theorem is proven. ∎

In showing the analogous result which replaces near continuity with quasi-continuity, we will similarly wish to characterize quasi-continuity in terms of sequences. However, given the significant overlap between research dedicated to quasi-continuity and that dedicated to multifunctions, we will here take this opportunity to introduce the latter in route to a more general result.

Definition A *multifunction* $F : X \to Y$ from a topological space X to a topological space Y is a point to set correspondence and is assumed that $F(x) \neq \emptyset$ for all $x \in X$. If $F : X \to Y$ is a multifunction then for $A \subset Y$ we denote $F^+(A) = \{x \in X : F(x) \subset A\}$ and $F^-(A) = \{x \in X : F(x) \cap A \neq \emptyset\}$.

Definition [7] A multifunction $F : X \to Y$ is said to be *upper (lower) quasi-continuous at* $x_0 \in X$ (Neubrunn 1982) if for any open $V \subset Y$ such that $F(x_0) \subset V$ $(F(x_0) \cap V \neq \emptyset)$ and for any open U containing x_0 there exists an open nonempty $G \subset U$ such that $F(x_0) \subset V$ $(F(x) \cap V \neq \emptyset)$ for any $x \in G$. It is said to be upper (lower) quasi-continuous if it is upper (lower) quasi-continuous at any $x \in X$.

Lemma 3.6 *Let X and Y be first countable topological spaces. Then a multifunction $F : X \to Y$ is lower quasi-continuous at $x \in X$ if and only if for any dense set D in X and any $y \in F(x)$ there exists a sequence $(x_n) \subset D$ such that $x_n \to x$ and $y_n \to y$ for some sequence $(y_n) \subset Y$ where $y_n \in F(x_n)$ for each n.*

Proof To show necessity, let $F : X \times Y$ be lower quasi-continuous at $x_0 \in X$. Choose $y \in F(x_0)$ and D dense in X. Let $\{U_n\}_{n\in\mathbb{N}}$ be a descending countable base of open sets at x_0 and let $\{V_n\}_{n\in\mathbb{N}}$ be a descending countable base of open sets at y. Then there exists for each $n \in \mathbb{N}$ a nonempty open $G_n \subset U_n$ such that $F(x) \cap V_n \neq \emptyset$ for any $x \in G_n$. Since G_n is open and D is dense in X we may choose $x_n \in G_n \cap D$. Then $F(x_n) \cap V_n \neq \emptyset$ so we may choose $y_n \in F(x_n) \cap V_n$. Thus $(x_n) \subset D$, $x_n \to x_0$, $y_n \in F(x_n)$ for each n, and $y_n \to y$ from which necessity follows.

To show sufficiency, assume F is not lower quasi-continuous at $x_0 \in X$. Then there exists an open $V \subset Y$ such that $F(x_0) \cap V \neq \emptyset$ and an open U containing x_0 such that for any nonempty open $G \subset U$, $F(x) \cap V = \emptyset$ for some $x \in G$.

We claim that $F^+(Y - V)$ is dense in U. To prove our claim, let $G \subset U$ be open and nonempty. Then there is some $x \in G$ such that $F(x) \cap V = \emptyset$, so $F(x) \subset Y - V$. Then $x \in \{x \in X : F(x) \subset Y - V\} = F^+(Y - V)$. Thus $x \in G \cap F^+(Y - V)$ and so $G \cap F^+(Y - V) \neq \emptyset$ and the claim is proven.

It follows that $D = F^+(Y - V) \cup (X - U)$ is dense in X. Now choose $(x_n) \subset D$ such that $x_n \to x_0$ and choose some $y \in F(x_0) \cap V$. Then take an arbitrary $(y_n) \subset Y$ such that $y_n \in F(x_n)$ for each $n \in \mathbb{N}$. Since $x_n \to x_0$, x_n must eventually be in U so therefore must eventually be in $F^+(Y - V)$. Thus $y_n \in Y - V$ for large n, so y_n is eventually not in V. We thus have a dense set D in X and a $y \in F(x)$ such that $y_n \not\to y$ for any $x_n \to x$ where $(x_n) \subset D$ and any $(y_n) \subset Y$ where $y_n \in F(x_n)$ for each n. Sufficiency follows and the theorem is proven. ∎

It follows clearly from the definition that in the case of a single-valued mapping that upper quasi-continuity and lower quasi-continuity coincide with quasi-continuity. We thus have as an immediate consequence the following corrollary.

Corollary 3.7 *Let X and Y be first countable spaces. Then $f : X \to Y$ is quasi-continuous at $x \in X$ if and only if for any dense set D in X there exists some sequence $(x_n) \subset D$ such that $x_n \to x$ and $f(x_n) \to f(x)$.*

Note here that if we were to seek an analogous result to Lemma 3.6 for upper quasi-continuous functions that coincides with Corollary 3.7 in the case of single-valued functions, then we would need it to still hold for any dense set D in X and some sequence

$(x_n) \subset D$ such that $x_n \to x$. Since upper quasi-continuity is clearly independent of lower quasi-continuity this would mean that the condition given in Lemma 3.6 must be changed to read for *some* $y \in F(x)$ or for *any* sequence $(y_n) \subset Y$ where $y_n \in F(x_n)$ for each n. However, if either change were to be made without the other, then this would imply that upper quasi-continuity is either a weakening or a strengthening of lower quasi-continuity respectively (which again, it is not). The only viable option for such a condition must therefore read: for any dense set D in X and *some* $y \in F(x)$ there exists a sequence $(x_n) \subset D$ such that $x_n \to x$ and $y_n \to y$ for *any* sequence $(y_n) \subset Y$ where $y_n \in F(x_n)$ for each n. Since any number of examples will show that this condition is not equivalent to upper quasi-continuity, we must conclude that Lemma 3.6 has no analogous result for upper quasi-continuous functions that coincides with Corollary 3.7 in the case of single-valued functions.

We are now ready to demonstrate for quasi-continuity the same convenient property we have already seen for continuity and near continuity.

Theorem 3.8 *Let $X = Y = \mathbb{R}$ and let $f : X \times Y \to \mathbb{R}$ be continuous in the strong sense with respect to y. Then f is quasi-continuous with respect to x if and only if f is quasi-continuous.*

Proof For necessity, assume that $f : X \times Y \to \mathbb{R}$ is continuous in the strong sense with respect to y and is quasi-continuous with respect to x. Continuity in the strong sense with respect to y implies continuity with respect to y which in turn implies quasi-continuity with respect to y. We therefore have that f is separately quasi-continuous. The quasi-continuity of f follows from [4] and sufficiency is proven.

For sufficiency, assume that f is continuous in the strong sense with respect to y and quasi-continuous. Take an arbitrary x-section $X \times \{y_0\}$ and let $(x_0, y_0) \in X \times \{y_0\}$. Let D be dense in $X \times \{y_0\}$ and define $D^* = \{(x, y) : (x, y_0) \in D\}$. D^* is then dense in $X \times Y$ so we have from Corollary 3.7 that there exists $(x_n, y_n) \subset D^*$ such that

$$(x_n, y_n) \to (x_0, y_0) \text{ and } f(x_n, y_n) \to f(x_0, y_0).$$

Also,

$$\lim_{(x_n,y_n)\to(x_0,y_0)}[f(x_n, y_n) - f(x_n, y_0)] = 0$$

by continuity in the strong sense with respect to y.

Therefore

$$\lim_{(x_n,y_0)\to(x_0,y_0)}[f(x_n, y_n) - f(x_n, y_0)] = 0$$

and so

$$\lim_{(x_n,y_0)\to(x_0,y_0)}[f(x_n, y_0) - f(x_0, y_0)] = 0.$$

Thus $(x_n, y_0) \subset D$, $(x_n, y_0) \to (x_0, y_0)$, and $f(x_n, y_0) \to f(x_0, y_0)$, so $X \times \{y_0\}$ is quasi-continuous by Corollary 3.7. Sufficiency follows and the theorem is proven. ∎

Although continuity in the strong sense with respect to a variable is clearly a useful tool to have, we should not be concerned that it is "too useful" since we know from Dzagnidze that separately continuous functions in the strong sense are equivalent to continuous function. Furthermore, while it may still be debatable exactly what Cauchy meant to give us when he offered his version of continuity with respect to a variable, we can be certain that he was either referring to strong continuity with respect to a variable or that he thought (albeit erroneously) the two concepts were equivalent when held by a function with respect to each variable. If we accept that Cauchy was a very gifted mathematician, this should largely remove those worries in either case. If there does still remain any concern about the power of continuity in the strong sense with respect to a variable though, we offer this example to show its limitations.

Example Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} 0 & \text{if } x \leq 0 \text{ and } x \text{ is irrational} \\ 1 & \text{if } x \leq 0 \text{ and } x \text{ is rational} \\ \text{y} & \text{if } x > 0 \end{cases}$$

Then f is continuous in the strong sense with respect to y and is somewhat continuous, but is not somewhat continuous with respect to x.

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