Friendly and Unfriendly $k-$partitions

by

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ABSTRACT

A friendly partition of a graph is a partition of the vertices into two sets so that every vertex has at least as many neighbors (adjacent vertices) in its own set as in the other set. An unfriendly partition of a graph is a partition of the vertices into two sets so that every vertex has at least as many neighbors in the other set as in its own set. In this paper we extend these concepts to $k-$partitions of vertices. We define and explore friendly and unfriendly edge partitions and extend these concepts to $k-$partitions of edges. In extending these concepts to the edges of a graph, we will show that one type of a friendly vertex partition of a $K_{m,n}$ graph can be used to produce a friendly edge partition. We will also look at partitions that are both friendly and unfriendly (dual). We will investigate these properties for several types of graphs (star, tree, K_n, C_n, $K_{m,n}$).

I would like to thank my wife, Joetta, for her unfailing love and support through this seemingly long and unending process, Dr. O'Mellan for her support, encouragement and guidance, and Dr.'s Fabrykowski and Wingler for the comments and suggestions.

I would like to dedicate this Clara M. Edmonds, who would have said: "I should have done this sooner."

Contents

1 Introduction

A friendly partition of a graph is a partition of the vertices into two sets so that every vertex has at least as many neighbors (adjacent vertices) in its own set as in the other set. An unfriendly partition of a graph is a partition of the vertices into two sets so that every vertex has at least as many neighbors in the other set as in its own set. Friendly partitions are also known as satisfactory [3, 4, 6], internal [2], and strong defensive alliance [9]. Unfriendly partitions are also known as co-satisfactory [4] and external [2]. A special case of unfriendly partitions is when the vertices have more neighbors in the other set, then it is strictly unfriendly and the partition is an alliance free partition [9]. Others have explored the complexity of finding a partition from a given graph [3, 4, 6, 9].

A graph G is an ordered pair (V, E), where V is a set of vertices and E is a set of edges. In this paper a graph is undirected and contains no loops or parallel edges. Other notation is taken from [7], and/or will be defined within this paper.

2 K_n and $K_{m,n}$ partitions

In this section we will be considering complete graphs. The first type are K_n, where each vertex is adjacent to every other vertex. For the graph denoted $K_{m,n}$, it is called a complete bipartite graph, and the m vertices are only adjacent to all of the n vertices, but not to any of the other m vertices. A special type of a $K_{m,n}$ graph is a star graph; this is where $m = 1$. For a multipartite graph, $K_{m_1,m_2,\cdots m_j}$, the vertices are only adjacent to vertices of the other sets.

Notation: In a $K_{m,n}$ graph, $m \leq n$, let $M = \{m_i\}_{i=1}^{m}$ be the set of vertices that correspond to the first subscript, and let $N = \{n_i\}_{i=1}^{n}$ be the set of vertices that correspond to the second subscript. For the general case: Let $M_i = \{m_{i_l}\}_{l=1}^{k_i}$ be the set of vertices that correspond to the i^{th} subscript, $K_{m_1,m_2,\cdots,m_k}$, where $m_i \leq m_{i+1}$ $1 \leq i < k$.

The partition of a graph is where the vertices or edges are distributed amoung k pairwise disjoint sets. The standard partition of a complete bipartite graph is where all of the m vertices are in one set of the partition and the n vertices are in the other set.

2.1 Definitions

Definition 1 (Neighbors).

Adjacent vertices are neighbors.

Definition 2 (Self-neighbor).

For a given vertex in a given partition, a self-neighbor is any other vertex in the same set of the partition that is a neighbor to the given vertex. The number of self-neighbors of a vertex v is denoted by $n_s(v) = a$.

Definition 3 (Other-neighbor).

For a given vertex in a given partition, an other-neighbor is a vertex in any other set of the partition that is a neighbor to the given vertex. The number of other-neighbors of a vertex v in a particular set is denoted by $n_{o_i}(v) = a$, for $i = 1$ to one less than the number of sets in the partition. If the partition consists of only two sets, then it is denoted by $n_o(v) = a$.

Definition 4 (Friendly vertex).

For a given vertex in a given partition, the vertex is friendly if it has at least as many *neighbors in its own set as it has in any other set of the partition. Thus the number of self-neighbors is greater than or equal to the number of other-neighbors for each other set in the partition. A partition is friendly if and only if all the vertices are friendly.*

Definition 5 (Unfriendly vertex)**.**

For a given vertex in a given partition, the vertex is unfriendly if it has at most as many neighbors in its own set as it has in any other set of the partition. Thus the number of self-neighbors is less than or equal to the number of other-neighbors for each other set in the partition. A partition is unfriendly if and only if all the vertices are unfriendly.

Definition 6 (Dual vertex)**.**

For a given vertex in a given partition, a dual vertex has the same number of self-neighbors as other-neighbors for each other set in the partition, thus it is both friendly and unfriendly. A partition is dual if and only if all the vertices are dual.

Definition 7 (Standard partition)**.**

For a $K_{m,n}$ graph, a standard partition is formed by placing the m vertices in one set and the n vertices in the other set. This can be extended to graphs of the form $K_{m_1,m_2,\cdots,m_k}$, where $m_i \leq m_{i+1}$ for $1 \leq i < k$, where there are k sets of vertices, and the i^{th} set contains the m_i vertices.

Definition 8 (Singleton)**.**

In a given partition, a set with one and only one vertex is called a singleton. This vertex is always unfriendly, since it has no self-neighbors.

Definition 9 (Singleton partition)**.**

This is a partition with only singletons. This partition is always unfriendly.

Definition 10 (Ginsu partition)**.**

For the graph $K_{m_1,m_2,\cdots,m_k}$, where $m_i \leq m_{i+1}$ for $1 \leq i < k$, and $1 \neq d \mid m_i$, for all $i \leq k$, this partition consists of the d sets that have $\dfrac{m_i}{d}$ vertices from each of the M_i sets where $1 \leq i \leq k$.

2.2 Partitions of $\mathbf{K}_n$

In this section we will first show that a K_n graph has no friendly partitions and then proceed to show the conditions under which a K_n graph has an unfriendly partition and the number of such partitions.

Theorem 11. *A K_n graph has no friendly 2-partition.*

Proof. Let $P = \{S_1, S_2\}$ be a friendly partition of a K_n graph. Then for every vertex of K_n, $n_s(v) \geq n_o(v)$. Let m be the cardinality of the smaller set, then $m \leq \left\lfloor \dfrac{n}{2} \right\rfloor$, and $S_1 = \{v_i\}_{i=1}^{m}$, $S_2 = V \setminus S_1$. Note that $|S_2| = n - m$. Now for $v \in S_1$, $n_s(v) = m - 1$ and $n_o(v) = n - m$, thus

$$m - 1 \geq n - m$$

$$2m \geq n + 1$$

This is a contradiction, since $m \leq \left\lfloor \dfrac{n}{2} \right\rfloor$. Hence P is not friendly.
Therefore a K_n graph has no friendly 2−partitions. □

Theorem 12. *A K_n graph has no friendly $k-$partition.*

Proof. Let $P = \{S_i\}_{i=1}^{k}$ be a friendly partition of a K_n graph. Then for every vertex of K_n, $n_s(v) \geq n_{o_l}(v)$. Let $m = \min\{|S_i|\}_{i=1}^{k}$. Without loss of generality we may assume $m = |S_1|$ and $|S_i| \geq m$ for $2 \leq i \leq k$. Let $v \in S_1$. Then $n_s(v) = m - 1$ and $n_{o_i}(v) \geq m$. Since P is friendly, then $m - 1 \geq m$, which is a contradiction. Hence P is not friendly.

Therefore a K_n graph has no friendly $k-$partitions.

$\square$

In the chart below, each set represents a partition of the indicated complete graph. Each number is the cardinality of the set of vertices in the partition. Note that the order of the cardinalities does not matter, $\{2, 2, 1\} \approx \{2, 1, 2\}$. So I have chosen to display the cardinalities in descending order. Note that the cardinalities in a partition always sum to n, and that for any two cardinalities in a partition, c_i and c_j, $|c_i - c_j| \leq 1$.

	2 sets	3 sets	4 sets	5 sets	6 sets	7 sets
K_2	$\{1,1\}$					
K_3	$\{2,1\}$	$\{1,1,1\}$				
K_4	$\{2,2\}$	$\{2,1,1\}$	$\{1,1,1,1\}$			
K_5	$\{3,2\}$	$\{2,2,1\}$	$\{2,1,1,1\}$	$\{1,1,1,1,1\}$		
K_6	$\{3,3\}$	$\{2,2,2\}$	$\{2,2,1,1\}$	$\{2,1,1,1,1\}$	$\{1,1,1,1,1,1\}$	
K_7	$\{4,3\}$	$\{3,2,2\}$	$\{2,2,2,1\}$	$\{2,2,1,1,1\}$	$\{2,1,1,1,1,1\}$	$\{1,1,1,1,1,1,1\}$

Lemma 13. *A partition of a K_n graph, into m sets of vertices, $2 \leq m \leq n$, is unfriendly if and only if the cardinalities of any 2 of the sets differ by at most 1.*

Proof. (⇒) We need to show that $|c_i - c_j| \leq 1$.

Let K_n be partitioned into m sets of vertices, $2 \leq m \leq n$, such that the partition is unfriendly. Since the partition is unfriendly, then for all $v \in V(K_n)$, $n_s(v) \leq n_{o_i}(v)$ for $i = 1, 2, \cdots, m-1$. By way of a contradiction, assume that there exists 2 sets of the partition whose cardinalities differ by at least 2. Let the sets be S_1 and S_2 with $j = |S_1|$, $j + 2 \leq |S_2|$. Then for all $v \in S_2$, $n_s(v) \geq j + 1$ and $n_{o_1}(v) = j$, and since $n_s(v) > n_{o_1}(v)$, then v is not an unfriendly vertex, which is a contradiction. Thus this partition must be such that the cardinalities of any two sets of vertices differ by at most 1. ✓

(⇐) We need to show that the partition is unfriendly.

Let K_n be partitioned into m sets of vertices, $2 \leq m \leq n$, such that the cardinality of any two sets of vertices differs by at most 1. Let q be the cardinality of the smaller sets, r be the number of sets with a cardinality of $q + 1$.

Case 1: $r = 0$

Now if $r = 0$, then $|S_i| = q$, for all i, so for $v \in S_i$, $n_s(v) = q - 1$ and $n_{o_j}(v) = q$, for all $j \neq i$. Thus $n_s(v) < n_{o_j}(v)$, for all j implies that v is an unfriendly vertex.

Case 2: $r \geq 1$

Let $|S_j| = q+1$, $1 \leq j \leq r$, and $|S_i| = q$, $r+1 \leq i \leq m$. Then for $v \in S_i$, $r+1 \leq i \leq m$, we have $n_s(v) = q - 1$ and $n_{o_d}(v) = q + 1$ or $n_{o_d}(v) = q$. Thus $n_s(v) \leq n_{o_d}(v)$, which implies that v is an unfriendly vertex. Now for $u \in S_j$, $1 \leq j \leq r$, $n_s(u) = q$ and $n_{o_d}(u) = q$ or $n_{o_d}(u) = q + 1$. Thus $n_s(u) \leq n_{o_d}(u)$, which implies u is an unfriendly vertex.

Now since the choices of v and u were arbitrary in both cases above, the partition is unfriendly. ✓

Therefore any partition of a K_n graph, into $2 \leq m \leq n$ sets of vertices is unfriendly if and only if the cardinalities of any 2 sets of vertices differ by at most 1.

□

Theorem 14. *The number of unfriendly partitions of a K_n graph is $n-1$.*

Proof. Let the number of sets in an unfriendly partition of a K_n graph be m, $2 \leq m \leq n$. From lemma 13 we know the cardinalities of these m sets differ by at most 1, thus the cardinalities are either q or $q+1$. Let m_1 be the number of sets with cardinality of q and m_2 be the number of sets with cardinality $q+1$.

Then

$$n = m_1 q + m_2(q+1) = (m_1+m_2)q + m_2.$$

Note that $0 \leq m_1 + m_2 \leq n$, so by the Division Algorthim we know that m_2 and q are unique. So for each m there exist unique q_m and r_m such that $n = mq_m + r_m$. Since there are $n-1$ unique values for m, then there are $n-1$ unfriendly partitions of K_n. □

2.3 Star $K_{1,n}$ graphs

Now we will consider the partitions of the star $K_{1,n}$ graphs. These graphs can not have a friendly partition because to be friendly each vertex has to have at least the same number of neighbors in its own set as in each of the other sets of the partition. This implies that the lone M vertex is in all sets, which is a contradiction. Thus there are no friendly star partitions. So we will explore some of the conditions that provide for unfriendly partitions.

Lemma 15. *For a vertex v of a given graph, if $n_s(v) = 0$, then v is unfriendly.*

Proof. Let v be a vertex of a graph such that $n_s(v) = 0$. Thus $n_s(v) \leq n_{o_i}(v)$, for all i, and hence v is unfriendly.

$\square$

Theorem 16. *A partition of a star graph is unfriendly if and only if the center vertex is a singleton.*

Proof. ($\Rightarrow$) We need to show that $|S_1| = 1$.

Let P be an unfriendly partition of a $K_{1,n}$ graph, into k sets, and let S_1 be the set containing the center vertex, C. Since the graph is unfriendly, then for all $v \in V(K_{1,n})$, $n_s(v) \leq n_{o_i}(v)$, for all $i \leq k-1$. By way of a contradiction, assume that $|S_1| > 1$, and let $u \in S_1$, $u \neq C$. Now since $u \in S_1$, $n_s(u) = 1$ and $n_{o_j}(u) = 0$, for all $j \leq k-1$. Thus u is friendly and not unfriendly which is a contradiction, because the partition is unfriendly. Hence $|S_1| = 1$. ✓

($\Leftarrow$) We need to show that the partition is unfriendly.

Let P be a partition of a $K_{1,n}$ graph, into k sets, and let $S_1 = \{C\}$ where C is the center vertex. Then $|S_1| = 1$, and $n_s(C) = 0$, then by lemma 15 C is unfriendly. Now for all $v \in V(K_{1,n})$, $n_s(v) = 0$. Then by lemma 15, v is unfriendly.

Hence since the choice of v was arbitrary, P is an unfriendly partition of $K_{1,n}$. $\checkmark$

Therefore a partition of a star graph is unfriendly if and only if the set containing the center vertex has cardinality 1. $\square$

Theorem 17. *If G is a star graph, then there are $p(n)$ unfriendly partitions of G, where $p(n)$ is the number of unrestricted partitions of n. This is the number of ways to distribute n things into k boxes.[1]*

Proof. Let G be a $K_{1,n}$ graph, with center vertex C, and $P = \{S_i\}_{i=1}^{k+1}$ be a partition of $K_{1,n}$, where $1 \leq k \leq n$. By theorem 16, if P is unfriendly, then the set containing the center vertex has cardinality 1. Suppose $S_1 = \{C\}$. Now the remaining n vertices can be distributed among the other k sets. This is equivalent to distributing n ones among k boxes, which is equivalent to an unrestricted partition of n, denoted $p(n)$ [1]. Hence a star graph has $p(n)$ unfriendly partitions.

$\square$

2.4 Partitions of $K_{m,n}$

In this section we will explore the conditions under which a $K_{m,n}$ graph has a friendly and / or an unfriendly partition. We will also show that some of these concepts can be extended to $K_{m_1,m_2,\cdots,m_n}$ graphs. We will also develop the concept of a dual partition, that is, a partition that is both friendly and unfriendly.

Theorem 18. *The standard partition of the graph $K_{m_1,m_2,\cdots,m_k}$, where $m_i \leq m_{i+1}$ for $1 \leq i < k$, is unfriendly.*

Proof. Let the partition be the sets M_i, $1 \leq i \leq k$. Then for $m_{ij} \in M_i$, m_{ij} has no neighbors in M_i so $n_s(m_{ij}) = 0$; thus by lemma 15 m_{ij} is unfriendly. Since the choice of m_{ij} was arbitrary, this partition is unfriendly. $\square$

Theorem 19. *The ginsu partition of a $K_{m,n}$ graph, such that* $\gcd(m,n) \neq 1$ *is dual.*

Proof. Let $K_{m,n}$ be a graph such that $\gcd(m,n) \neq 1$. Then for all $d \in \mathbb{N}$ such that $d \mid m$ and $d \mid n$, let d be the number of sets in the partition, and let each set have $\frac{m}{d}$ elements from M, and $\frac{n}{d}$ elements from N. Now for each $m_i \in M$, $n_s(m_i) = \frac{n}{d}$ and $n_{o_i}(m_i) = \frac{n}{d}$ in each of the other $1 \leq i \leq d-1$ sets. Thus each m_i is a dual vertex. Now for each $n_i \in N$ $n_s(n_i) = \frac{m}{d}$ and $n_{o_j}(n_i) = \frac{m}{d}$ in each of the other $1 \leq i \leq d-1$ sets. Thus each n_i is a dual vertex, and hence the partition is dual. $\square$

Theorem 20. *The ginsu partition of a $K_{m_1,m_2,\cdots,m_k}$ graph, such that* $\gcd(m_1, m_2, \cdots, m_k) \neq 1$ *is dual.*

Proof. Let $K_{m_1,m_2,\cdots,m_k}$ be a graph such that $\gcd(m_1, m_2, \cdots, m_k) \neq 1$. Then for all $d \in \mathbb{N}$ such that $d \mid m_i$, $i = 1, 2, \cdots k$, let d be the number of sets in the partition. Then each set has $\frac{m_i}{d}$ elements from each of the M_i, $1 \leq i \leq k$ sets. For each $m_{ij} \in M_i$, $n_s(m_{ij}) = \frac{m_i}{d}$ and $n_{o_i}(m_{ij}) = \frac{m_i}{d}$ in each of the other $1 \leq i \leq d-1$ sets. Thus m_{ij} is a dual vertex, and since the choice of m_{ij} was arbitrary, the partition is dual. $\square$

Lemma 21. *A $K_{m,n}$ graph has a friendly partition into 2 sets, if and only if both m and n are even.*

Proof. ($\Rightarrow$) We need to show that both m and n are even.

Let $P = \{S_1, S_2\}$ be a friendly partition of $K_{m,n}$. Since P is friendly, then for all $v \in V(K_{m,n})$, $n_s(v) \geq n_o(v)$. Then $m_i \in S_j \cap M$, $j = 1, 2$, implies $\deg(m_i) = |N|$ and $n_i \in S_j \cap N$, $j = 1, 2$, implies $\deg(n_i) = |M|$. Suppose that $2 \nmid |M|$. Then

$|S_1 \cap M| \neq |S_2 \cap M|$. We may assume that $|S_1 \cap M| > |S_2 \cap M|$, which implies S_1 has at least 1 more vertex from M than S_2. This implies that for $u \in S_2 \cap N$, $n_s(u) < n_o(u)$ which is a contradiction because the partition is friendly. Hence $2 \mid |M|$. Similarly $2 \mid |N|$. Thus $\gcd(m, n) \geq 2 \neq 1$. ✓

($\Leftarrow$) We need to show that the partition is friendly.
Let $P = \{S_1, S_2\}$ be the Ginsu partition of $K_{m,n}$ with k parts where $\gcd(m, n) > 1$. Now by theorem 19, P is dual which implies that P is friendly. ✓

Therefore, a $K_{m,n}$ graph has a friendly partition of 2 sets, if and only if both m and n are even.

□

Lemma 22. *A $K_{m,n}$ graph has a friendly partition into k sets, if and only if $k \mid m$ and $k \mid n$.*

Proof. ($\Rightarrow$) We need to show that $k \mid m$ and $k \mid n$.
Let $P = \{S_1, S_2, \cdots, S_k\}$ be a friendly partition of $K_{m,n}$. Since P is friendly, then for all $v \in V(K_{m,n})$, $n_s(v) \geq n_o(v)$. Then $m_i \in S_j \cap M$, $j = 1, 2, \cdots, k$, implies $\deg(m_i) = |N|$ and $n_i \in S_j \cap N$, $j = 1, 2, \cdots, k$, implies $\deg(n_i) = |M|$. Suppose that $k \nmid |M|$, then there exist α, β, such that $|S_\alpha \cap M| \neq |S_\beta \cap M|$. We may assume that $|S_\alpha \cap M| > |S_\beta \cap M|$, which implies S_α has at least 1 more vertex from M than S_β. This implies that for $u \in S_\beta \cap N$, $n_s(u) < n_o(u)$, which is a contradiction because the partition is friendly. Hence $k \mid |M|$. Similarly $k \mid |N|$. Thus $\gcd(m, n) \geq k \neq 1$.
✓

($\Leftarrow$) We need to show that the partition is friendly.

Let $P = \{S_1, S_2, \cdots, S_k\}$ be the Ginsu partition of $K_{m,n}$with k parts where $\gcd(m,n) > 1$. Now by theorem 19, P is dual which implies that P is friendly. ✓

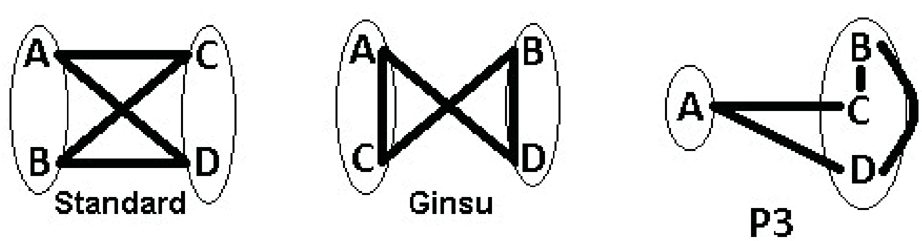
Therefore, a $K_{m,n}$ graph has a friendly partition of k sets, if and only if $k \mid m$ and $k \mid n$.

□

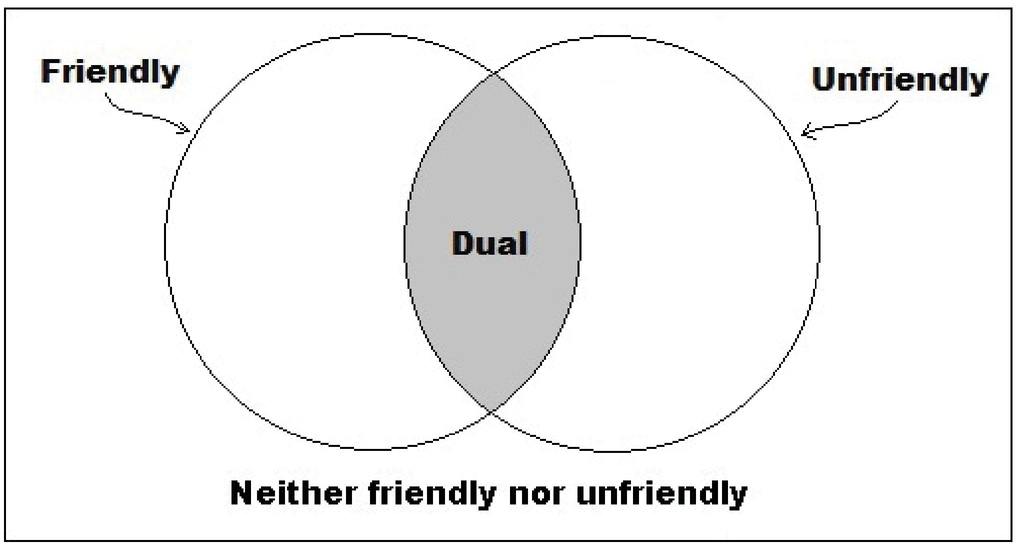
Corollary 23. *The only dual partitions of a $K_{m,n}$ graph are the Ginsu partitions.*

2.5 Partition diagram

Example 24. For the graph $K_{2,2}$:
Let $M = \{A, B\}$ and $N = \{C, D\}$. There are three non isomorphic partitions into two sets of this graph:



For the **P3** partition, create a singleton set by picking one vertex from the set $M \cup N$, and placing the remaining vertices in the other partition. Without loss of generality let the singleton vertex be m_1. Since there are no other vertices in its partition, it has zero self-neighbors, and it has n other-neighbors. Now consider the remain vertices of the M set of vertices. Since $0 < n$, this is an unfriendly vertex and not a friendly vertex. For the other set of the partition, the other m_i, $2 \leq i \leq m$, vertices all have n self-neighbors and zero other-neighbors, thus these vertices have more self-neighbors than other-neighbors which would make them friendly vertices. Hence this partition is neither, since not all vertices are of the same type, friendly or unfriendly.



This Venn diagram represents the friendly (on the left side) and unfriendly (on the right side) partitions. The area in the middle represents the partitions that are both friendly and unfriendly (dual). If the partition is not friendly, then it might be unfriendly, but it also might be neither friendly nor unfriendly. Therefore you can't show that a partition is unfriendly by showing that it is not friendly.

An example of each type of partition follows: A Ginsu partition of a $K_{m,n}$ graph is dual. We will show that the 2-partition of a C_5 graph, with 2 adjacent vertices in one set and the other 3 vertices in the other set, is friendly and not unfriendly. A 2-partition of a K_n graph where the cardinality of the two sets differ by at most 1 is unfriendly and not friendly. A partition of $K_{2,2}$ where one vertex is a singleton and the other three vertices are in the other set is none of the above types.

3 C_n partitions

In this section we will consider cycle graphs of n vertices, denoted C_n. This is a graph where each vertex has 2 adjacent edges, and the edges form a closed path.

We will now explore the conditions under which a C_n graph has friendly and / or unfriendly partitions.

Theorem 25. *A partition of C_n, $n \geq 4$ is friendly if and only if for all $v \in V(C_n)$, $n_s(v) > 0$.*

Proof. ($\Rightarrow$) We need to show that $n_s(v) > 0$.

Let P be a friendly partition of the graph C_n, into k sets. Since P is friendly, then for all $v \in V(C_n)$, $n_s(v) \geq n_{o_i}(v)$, for all $i \leq k-1$. Since the graph is a cycle, $\deg(v) = 2$, for all $v \in V(C_n)$. Thus

$$2 = n_s(v) + \sum_{i=1}^{k-1} n_{o_i}(v) \leq kn_s(v).$$

Which implies that $n_s(v) > 0$, for all $v \in V(C_n)$. ✓

($\Leftarrow$) We need to show that the partion is friendly.

Let P be a partition of the graph C_n into k sets, where for all $v \in V(C_n)$, $n_s(v) \geq 1$. Since $\deg(v) = 2$, for all $v \in V(C_n)$, and $n_s(v) \geq 1$, then $n_{o_i}(v) \leq 1$, for all $i \leq k-1$.

Thus $n_{o_i}(v) \leq 1 \leq n_s(v)$. Thus v is friendly, and since the choice of v was arbitrary, the partition P is friendly. ✓

□

Theorem 26. *A partition of C_n, $n \geq 3$ into 2 sets is unfriendly if and only if for all $v \in V(C_n)$, $n_s(v) < \deg(v) = 2$.*

Proof. ($\Rightarrow$) We need to show that for all $v \in V(C_n)$, $n_s(v) < \deg(v) = 2$.

Let P be an unfriendly partition of the graph C_n into 2 sets. Since the partition is unfriendly, then $n_s(v) \leq n_o(v)$, for all $v \in V(C_n)$. Since the graph is a cycle, then $\deg(v) = 2$, for all $v \in V(C_n)$. Now

$$2 = \deg(v) = n_s(v) + n_o(v) \geq 2n_s(v).$$

Which implies that $n_s(v) \leq 1$. Thus no set of the partition can have a vertex with $n_s(v) = 2$, and since the choice of v was arbitrary, we have for all $v \in V(C_n)$, $n_s(v) < \deg(v) = 2$. ✓

($\Leftarrow$) We need to show that the partition is unfriendly

Let P be a partition of the graph C_n, $n \geq 2$, into 2 sets such that for all $v \in V(C_n)$, $n_s(v) < \deg(v) = 2$. Thus for all $v \in V(C_n)$, $n_s(v) = 0, 1$. If $n_s(v) = 0$, then $n_s(v) \leq n_o(v) = 2$. If $n_s(v) = 1$, then $n_s(v) \leq n_o(v) = 1$. Thus $n_s(v) \leq n_o(v)$. Hence v is unfriendly and since the choice of v was arbitrary, the partition is unfriendly. ✓

□

4 Tree partitions

In this section we will consider tree graphs of n vertices, denoted T_n. This is a graph where for any two vertices, it is possible to find a path between them, and the graph contains no cycles.

For trees, I will consider the center as the root of the tree. If the diameter of the tree is even, then there is only one root, denoted $v_{0,1}$. If the diameter of the tree is odd, then there are two roots, denoted $v_{0,1}$ and $v_{0,2}$. This is called a double root tree.

4.1 Definitions

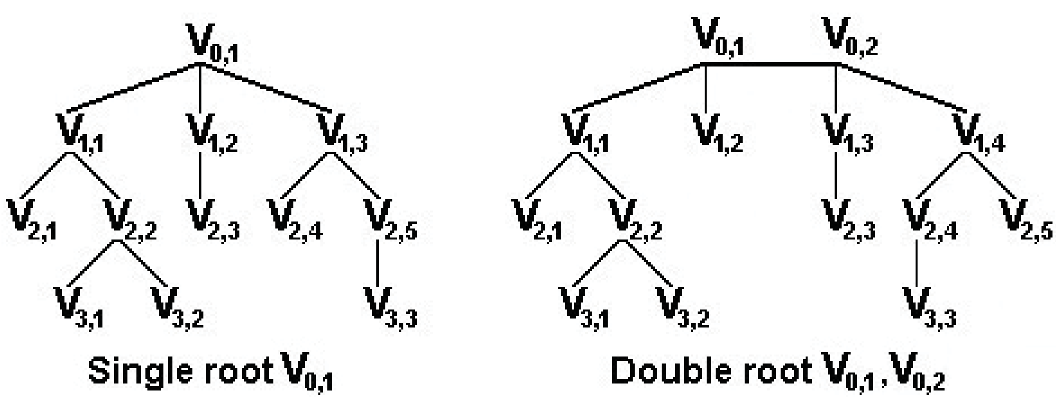
Definition 27. *(Level)*

For a given vertex v in a given tree, the distance from v to the root is its level. The root is on level 0. The level is the first subscript of the vertex.

Definition 28. *(Height)*

For a given tree T, where $diam(T) = 2k + r$, $r \in \{0, 1\}$, let the height of T is k, denoted $ht(T) = k$.

Example 29. *Both trees shown have height 3.*



Definition 30. *(Level partition) For a given tree, the vertices in a level partition are grouped into sets based on their distance from the root(s). The number of sets in this partition is equal to the height of the tree plus one. The sets listed below are the sets for the level partitions of the above single and double root trees.*

Single root sets

$$\begin{aligned} S_0 &= \{V_{0,1}\} \\ S_1 &= \{V_{1,1}, V_{1,2}, V_{1,3}\} \\ S_2 &= \{V_{2,1}, V_{2,2}, V_{2,3}, V_{2,4}, V_{2,5}\} \\ S_3 &= \{V_{3,1}, V_{3,2}, V_{3,3}\} \end{aligned}$$

Double root sets

$$\begin{aligned} S_0 &= \{V_{0,1}, V_{0,2}\} \\ S_1 &= \{V_{1,1}, V_{1,2}, V_{1,3}, V_{1,4}\} \\ S_2 &= \{V_{2,1}, V_{2,2}, V_{2,3}, V_{2,4}, V_{2,5}\} \\ S_3 &= \{V_{3,1}, V_{3,2}, V_{3,3}\} \end{aligned}$$

Definition 31. *(Pruned partition) For a given tree, this is the partition that consists of the sets created by deleting the edges $v_{0,x}v_{1,j}$, for all $x \in \{1,2\}$ and for all j, and placing each of the subtrees, whose roots are the $v_{1,j}$ vertices, into sets, and placing the root(s) into one of the sets that contains a level 1 vertex that the root is adjacent to. The sets listed below are the sets for the pruned partitions of the above single and double root trees. The sets are shown without the roots.*

Single root sets

$$\begin{aligned} S_1 &= \{V_{1,1}, V_{2,1}, V_{2,2}, V_{3,1}, V_{3,2}\} \\ S_2 &= \{V_{1,2}, V_{2,3}\} \\ S_3 &= \{V_{1,3}, V_{2,4}, V_{2,5}, V_{3,3}\} \end{aligned}$$

Double root sets

$$\begin{aligned} S_1 &= \{V_{1,1}, V_{2,1}, V_{2,2}, V_{3,1}, V_{3,2}\} \\ S_2 &= \{V_{1,2}\} \\ S_3 &= \{V_{1,3}, V_{2,3}\} \\ S_4 &= \{V_{1,4}, V_{2,4}, V_{2,5}, V_{3,3}\} \end{aligned}$$

4.2 Single root trees

Theorem 32. *A level partition of a single root tree is unfriendly.*

Proof. A single root in a level partition has $n_s(v_{0,1}) = 0$. Let T be a tree of height h. Now $n_{o_i}(v_{0,1}) = \deg(v_{0,1})$ for some $i \leq h$, and $n_{o_k}(v_{0,1}) = 0$ for all $k \neq i$. Thus $v_{0,1}$ is unfriendly. Let $v_{i,j} \in V(T)$, $i \geq 1$, such that $v_{i,j}$ is a vertex from the i^{th} level. It is sufficient to show that $n_s(v) = 0$, because $n_{o_i}(v) \geq 0$. Since each set of the partition is the i^{th} level, then $n_s(v_{i,j}) = 0$, for all $v_{i,j} \in V(T)$. Thus this partition is unfriendly. □

Theorem 33. *A pruned partition of a single root tree T is friendly, if and only if there is at most 1 level one leaf, and the root is placed in this set.*

Proof. ($\Leftarrow$) We need to show that the partition is friendly.

Let $P = \{S_1, S_2, \cdots, S_k\}$, be a pruned partition of the graph T, with $\deg(v_{1,i}) \geq 1$ where $v_{0,1}$ and $v_{1,i}$ are in the same set, and for all $j \neq i$, $\deg(v_{1,j}) \geq 2$. Since each S_i of a pruned partition is a nontrival tree, then every vertex has at least one self neighbor. The only vertices that have other neighbors are the root and the level 1 vertices. The root has exactly one other neighbor in each of the other sets of the partition and one self neighbor, thus the root is friendly. Since the level one vertices are connected to the root and all level one vertices have degree greater than or equal to 1, then all level one vertices are friendly. All the other vertices (level 2 and greater) have 0 other neighbors, thus they are friendly. Hence P is friendly. ✓

($\Rightarrow$) Let $P = \{S_i\}_{i=1}^{\alpha}$ be a friendly pruned partition of the tree T, and let $v_{0,1}$ and $v_{1,i}$ be in the same set, where $\deg(v_i) \geq 1$. Since P is friendly, then for all $v \in V(T)$,

$n_s(v) \geq n_{o_k}(v)$, for all k. Now suppose there exists $j \neq i$, such that $\deg(v_{1,j}) = 1$, then $v_{1,j}$ is only adjacent to $v_{0,1}$. Since $j \neq i$, $v_{0,1}$ is not in the same set of the partition as $v_{1,j}$. Then $n_s(v_{1,j}) = 0$; thus $v_{1,j}$ is not friendly which is a contradiction because $v_{1,j}$ is friendly. Hence $\deg(v_{i,j}) \geq 2$.

Hence the friendliness of a pruned partition of T implies that $\deg(v_{1,i}) \geq 1$ where $v_{0,1}$ and $v_{1,i}$ are in the same set, and for all $j \neq i$, $\deg(v_{1,j}) \geq 2$. ✓

Therefore, a pruned partition is friendly, if and only if $\deg(v_{1,i}) \geq 1$ where $v_{0,1}$ and $v_{1,i}$ are in the same set, and for all $j \neq i$, $\deg(v_{1,j}) \geq 2$.

□

4.3 Double root trees

The following theorem extends theorem 33 to a double root tree. The proof uses similar techniques and is omitted.

Theorem 34. *A pruned partition of a double root tree T is friendly, if and only if there are at most 2 level 1 vertices that are leaves, these level one leaves (if any) are placed with roots, and all other level 1 vertices have degree greater than 1.*

5 Friendly and unfriendly edge partitions

In this section we will extend the ideas of friendly and unfriendly partitions to the edges of a graph. First we extend the basic definitions from the vertex partitions. We then look at the conditions for a C_n graph to have friendly and / or unfriendly edge partitions. We then define new types of partitions called constant partitions. In working with these edge partitions I placed an artificial construct on the vertices of the graph, and some interesting patterns emerged: star polygons and star figures. [10]

Definition 35 (Neighbors)**.**

Adjacent edges are neighbors.

Definition 36 (Self-neighbor)**.**

For a given edge in a given partition, a self-neighbor is any other edge in the same set that is a neighbor to the given edge; i.e. the two edges share a vertex. The number of self-neighbors of an edge is denoted by $n_s(uv)$*.*

Definition 37 (Other-neighbor)**.**

For a given edge in a given partition, an other-neighbor is an edge in any other set that is a neighbor to the given edge. The number of other-neighbors of an edge in a particular set is denoted by $n_{o_i}(uv)$*, for* $i = 1$ *to one less than the number of sets in the partition. If the partition consists of only two sets, then it is denoted by* $n_o(uv)$*.*

Definition 38 (Friendly edge)**.**

For a given edge in a given partition, the edge is friendly if it has at least as many neighbors in its own set as it has in any other set of the partition. Thus the number of self-neighbors is greater than or equal to the number of other-neighbors for each other set in the partition. A partition is friendly if and only if all the edges are friendly.

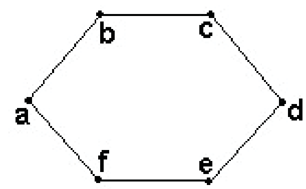
Definition 39 (Unfriendly edge)**.**
For a given edge in a given partition, the edge is unfriendly if it has at most as many neighbors in its own set as it has in any other set of the partition. Thus the number of self-neighbors is less than or equal to the number of other-neighbors for each other set in the partition. A partition is unfriendly if and only if all the edges are unfriendly.

Definition 40 (Dual edge)**.**
For a given edge in a given partition, a dual edge has the same number of self-neighbors as other-neighbors for each other set in the partition; thus it is both friendly and unfriendly. A partition is dual if and only if all the edges are dual.

5.1 C_n edge partitions

In this section we will consider the properties that make an edge partition of a C_n graph friendly and / or unfriendly. We will show that friendly edge partitions require continuous paths, while an unfriendly partition is composed of isolated edges. We will also see why only C_{4n} graphs have dual edge partitions.

Example 41.



For this C_6 graph, two edge partitions are listed below, one friendly and one unfriendly.

$$P_1 = \{\{ab, bc, cd\}, \quad \{de, ef, fa\}\}, \quad \textit{friendly}$$

$$P_2 = \{\{ab, cd, ef\}, \quad \{bc, de, fa\}\}, \quad \textit{unfriendly}$$

Theorem 42. *An edge partition of 2 sets of C_n is friendly, if and only if each set of the partition consists of a union of paths where each path has length at least two.*

Proof. ($\Rightarrow$) We need to show that the partition consists of the union of paths.
Let $P = \{S_1, S_2\}$ be a friendly edge partition of C_n. Since P is friendly, then for all $e \in E(C_n)$, $n_s(e) \geq n_o(e)$. Now the degree of every vertex of C_n is 2, so if $n_o(e) = 2$, then $n_s(e) = 0$, which implies that e is not friendly which is a contradiction. Thus $n_o(e) \leq 1$, and so $n_s(e) \geq 1$. So each edge in a set of the partition has an adjacent edge in the set. Thus e is part of a path whose length is at least 2. Since the choice of S_i was arbitrary, both sets consist of a union of paths where each path has length at least two. ✓

($\Leftarrow$) We need to show that the partition is friendly.
Let $P = \{S_1, S_2\}$ be an edge partition of C_n, such that each set of the partition

consists of a union of paths where each path has length at least two. Let $e \in S_i$, since e is in a path whose length is greater than 1, then $n_s(e) \geq 1$, which implies that $n_o(e) \leq 1$. Thus e is friendly. Since the choice of e was arbitrary, then P is friendly. ✓

Therefore an edge partition of 2 sets of C_n is friendly, if and only if each set of the partition consists of a union of paths where each path has length at least two.

□

The following theorem extends theorem 42 to a partition of k sets. The proof uses similar techniques and is omitted.

Theorem 43. *An edge partition of C_n into k sets is friendly, if and only if each set of the partition consists of a union of paths where each path has length at least two.*

Theorem 44. *An edge partition, $P = \{S_1, S_2\}$, of C_n is unfriendly, if and only if for all $e \in E(C_n)$, $n_s(e) \leq 1$.*

Proof. ($\Rightarrow$) We need to show that for all $e \in E(C_n)$, $n_s(e) \leq 1$.
Let $P = \{S_1, S_2\}$ be a unfriendly edge partition of C_n. Since the partition is unfriendly, then for all $e \in E(C_n)$, $n_s(e) \leq n_o(e)$. Now since $n_s(e) + n_o(e) = 2$, then $n_o(e) = 2 - n_s(e)$. If $n_s(e) = 2$, then $n_o(e) = 0$, and $n_s(e) > n_o(e)$ which is a contradiction, because the partition is unfriendly. Thus $n_s(e) \neq 2$. So $n_s(e) \leq 1$. Hence for all $e \in E(C_n)$, $n_s(e) \leq 1$. ✓

($\Leftarrow$) We need to show that the partition is unfriendly.

Let $P = \{S_1, S_2\}$ be a partition of C_n such that for all $e \in E(C_n)$, $n_s(e) \leq 1$. Now for $e \in E(C_n)$, $n_s(e) + n_o(e) = 2$. Since $n_s(e) \leq 1$, then $1 \leq n_o(e)$. Thus $n_s(e) \leq n_o(e)$. Since the choice of e was arbitrary, P is unfriendly. ✓

Therefore an edge partition of 2 sets of C_n is unfriendly, if and only if for all $e \in E(C_n)$, $n_s(e) \leq 1$.

□

Theorem 45. *An edge partition of C_n is dual, if and only if*

(1) it consists of exactly 2 sets, and

(2) for all $e \in S_i$, $i = 1, 2$, $n_s(e) = 1$.

Proof. ($\Rightarrow$) We need to show conditions (1) and (2).

(1)

Without loss of generality we may assume that the number of sets in the partition is $k \geq 2$. If $k = 1$ then it is just the set $E(C_n)$. We need to show that $k \leq 2$. By way of contradiction, assume that $k > 2$. Since the partition is dual, then $n_s(e) = n_{o_j}(e)$, for all j. Thus for $e \in E(C_n)$, $n_s(e) + \sum_{j=1}^{k-1} n_{o_j}(e) = 2$, which implies there exists $m < k$ such that $n_{o_m}(e) = 0$. This is a contradiction, because the partition is dual. Hence $k = 2$. ✓

(2)

Let $P = \{S_1, S_2\}$ be a dual edge partition of C_n. Since P is dual, then for all $e \in E(C_n)$, $n_s(e) = n_o(e)$. Since $n_s(e) + n_o(e) = 2$, then $n_s(e) + n_s(e) = 2$ which implies $n_s(e) = 1$. Thus for all $e \in E(C_n)$, $n_s(e) = 1$. ✓

($\Leftarrow$) We need to show that the partition is dual.
Let $P = \{S_1, S_2\}$ be an edge partition of C_n such that for all $e \in S_i$, $i = 1, 2$, $n_s(e) = 1$. Since $n_s(e) = 1$, then $n_o(e) = 1$, for all e. Thus $n_s(e) = n_o(e)$, for all e and so the edge partition is dual. ✓

Hence an edge partition of C_n is dual, if and only if (1) it consists of exactly 2 sets, and (2) for all $e \in S_i$, $i = 1, 2$, $n_s(e) = 1$.

□

Corollary 46. *A C_n graph has a dual edge partition, if and only if $4 \mid n$.*

Proof. Pick a vertex and label it v_1, proceeding in a clockwise direction label the remaining vertices $v_2, v_3, \cdots, v_n$.
($\Rightarrow$) We need to show that $4 \mid n$.
Let P be a dual edge partition of C_n. Since P is a dual edge partition, $P = \{S_1, S_2\}$ and for all $e \in E(C_n)$, $n_s(e) = 1 = n_o(e)$. Thus each set of the partition consists of only edges that are adjacent to one and only one other edge. Thus the sets consist of paths of length 2, whose vertices are either v_{4k+1}, v_{4k+2}, and v_{4k+3}; or v_{4k-1}, v_{4k}, and v_{4k+1}. Thus $2 \mid |S_i| = \dfrac{n}{2}$, $i = 1, 2$. Hence $4 \mid n$.

($\Leftarrow$) We need to show that the partition is a dual edge partition.
Let $S_1 = \{v_1v_2, v_2v_3, v_5v_6, v_6v_7, \cdots, v_{n-3}v_{n-2}, v_{n-2}v_{n-1}, \}$, and $S_2 = \{v_3v_4, v_4v_5, v_7v_8, v_8v_9, \cdots, v_{n-1}v_n$
By construction, for all $e \in E(C_n)$, $n_s(e) = 1$ and the partition consists of exactly 2 sets. Hence the edge partition is dual.

Therefore a C_n graph has a dual edge partition, if and only if $4 \mid n$.

□

5.2 Description of constant partition ideas to be presented.

We are going to look at edge partitions where the number of self-neighbors is the same for all $e \in E(G)$, and the number of other-neighbors is the same for all $e \in E(G)$, i.e. for all $e \in E(G)$, then $n_s(e) = c_1$ and $n_{o_i}(e) = c_2$ where c_1 and c_2 are constants. We call such partitions constant partitions. There are several types of constant edge partitions: Homogeneous, Star, Combined, and Set. The homogeneous type produces Hamitonian cycles under certain conditions, otherwise multiple cycles are generated. These sets when combined with a fixed sequencial vertex labeling produce star polygons from the Hamiltonian cycles and star figures are produced by the multiple cycles.

I define a measure called the "hop length" that is applied to a sequencial vertex labeling, that gives a measure of "distance" between two given vertices. The hop length of an edge is the length of the path between the two vertices by going along the perimeter of the graph, when the vertices are placed sequencially around a circle. This perimeter distance is key to creating the star polygons/figures.

We will also see how to take a vertex Ginsu partition and create a friendly constant edge partition. To make the process cleaner, I defined the operation "vertex cross product." This operation takes two sets of disjoint vertices and creates a set of edges. These edges are incident to a vertex from each set, and each vertex from one set is paired with every vertex of the other set.

There is also a way to combine the homogeneous sets to form either a set constant partition or a combined constant partition. The combined constant partition is a constant edge partition; while the set constant partition is only constant within each set. That is, in a set constant partition, the number of self or other neighbors differs from set to set, and the partition is neither friendly nor unfriendly.

Definition 47. *($l(e)$ or Hop Length)*

For a K_n graph, label the vertices v_0, v_1, $\cdots$,v_{n-1}. Consider a drawing of the graph where the vertices are placed sequencially around the perimeter of a circle. Then the function $l(e)$, where $e = v_i v_l$, is the length of the trail $v_i v_{i+1} \cdots v_l$, then

$$l(e) = l(v_i v_l) = (l - i) \mod n.$$

Definition 48 (Constant edge partition)**.**

For a given graph G, this is an edge partition, where for all $e_i, e_j \in E(G)$, $n_s(e_i) = n_s(e_j)$ and $n_{o_k}(e_i) = n_{o_k}(e_j)$, such that $i \neq j$. There are several types of constant edge partitions.

Definition 49 (Homogeneous Constant partition)**.**

This is a constant edge partition where each set of the partition consists of n edges. Also each vertex is incident to two different edges of the set, and for all $e_k, e_j \in S_i$, $l(e_k) = l(e_j)$. This is the partition $P = \{S_i\}_{i=1}^{(n-1)/2}$, where $|S_i| = n = |V(G)|$, for all i.

Definition 50 (Star Constant partition)**.**

This is a constant edge partition where one set of the partition consist of all the edges incident to a given vertex, v. Now the remaining sets of the partition form a homogeneous constant partition of K_{n-1}*. This is the partition* $P = \{S_i\}_{i=1}^{n/2}$*, where* $|S_i| = n-1$*, for all* i*.*

Definition 51. *(Combined constant partition)*

For a homogeneous constant partition $P = \{S_i\}_{i=1}^{k}$ *where* k *is composite, a combined constant partition can be formed. Since* k *is composite, there exists* d*, such that* $d \mid k$*,* $k = \dfrac{n-1}{2}$ *and* $d \leq \dfrac{n-1}{4}$*, then the combined constant partition is* $P_C = \{S_h\}_{h=1}^{d}$*, where each* S_h *is the union of* $\dfrac{k}{d}$ *sets of the* $S_i \in P$*, and* $|S_h| = kn/d$*, for all* h*.*

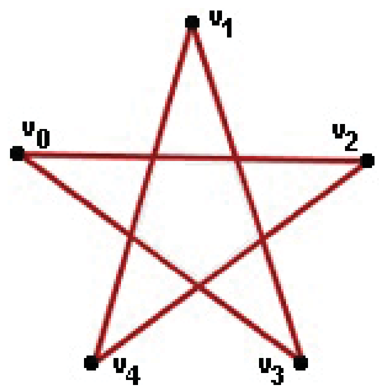
Definition 52. *(Star Polygon/figure)[10]*

A Star Polygon/figure is a set of n *vertices evenly spaced around the circumference of a circle, and drawing the following edges:*

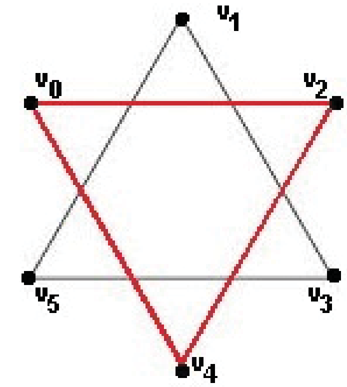
$$n_i n_{(i+d)(\mod n)}, n_{(i+d)(\mod n)} n_{(i+2d)(\mod n)}, \cdots ,$$

$$n_{(i+(n-2)d)(\mod n)} n_{(i+(n-1)d)(\mod n)}.$$

Denoted as $\left\{\dfrac{n}{d}\right\}$*. If* $d \mid n$*,* $1 \leq d \leq \dfrac{n}{2}$*, then once the path is closed, start over at the next isolated vertex and repeat until all the vertices have been included. This results in a star figure, since* $d \mid n$*; otherwise it results in a star polygon. A star figure consist* *of $\dfrac{n}{d}$ component cycles.*



The above star polygon is a $\left\{\dfrac{5}{2}\right\}$



The above star figure is a $\left\{\dfrac{6}{2}\right\}$

For $P = \{S_i\}_{i=1}^{k}$, a homogeneous constant partition of K_n, then each S_i describes a star polygon $\left\{\dfrac{n}{d}\right\}$, where $d = l(e)$, $e \in S_i$.

Pick a vertex and label it v_0, proceeding clockwise label the next vertex v_1, continue in this fashion until all the vertices are labeled. Let $P = \{S_i\}_{i=1}^{k}$, be a homogeneous constant partition of K_n, where for each pair of edges $e_m, e_j \in S_i$, $l(e_m) = i = l(e_j)$. Note that each edge is incident to vertices that are i apart. Now by the definition of a star polygon, the edges of the set S_i form the star polygon $\left\{\dfrac{n}{i}\right\}$. Hence for $P = \{S_i\}_{i=1}^{k}$, a homogeneous constant partition of K_n, each S_i, $i > 1$ is a star polygon $\left\{\dfrac{n}{d}\right\}$, where $d = l(e)$, $e \in S_i$.

Definition 53. *(vertex cross product)*

Let $A = \{a_i\}_{i=1}^{k}$ and $B = \{b_j\}_{j=1}^{l}$ be two disjoint sets of vertices of a graph. Then the vertex cross product of A and B, denoted $A \times B$, is the set of edges where each vertex *of A is joined to every vertex of B by an edge. Thus*

$$A \times B = \{a_1b_1, a_2b_1, \cdots, a_kb_1, a_1b_2, a_2b_2, \cdots, a_kb_2, \cdots a_1b_l, a_2b_l, \cdots, a_kb_l, \}.$$

Definition 54. *(partition cross product)*
Let $A = \{a_i\}_{i=1}^k$ and $B = \{b_j\}_{j=1}^l$ be two disjoint sets of vertices of a graph. Then the partition cross product of A and B, where A is partitioned into d sets of k/d vertices each is the vertex cross product between each set of the partition of A and all of B.

Example 55. *Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ and $B = \{b_1, b_2, b_3, b_4, b_5, b_6\}$, where A is partitioned as:*

$$\{\{a_1, a_2\}, \{a_3, a_4\}, \{a_5, a_6\}\},$$

then the partition cross product is:

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \times \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix} = \begin{matrix} \begin{bmatrix} a_1 & a_2 \end{bmatrix} \times \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix}, \\ \begin{bmatrix} a_3 & a_4 \end{bmatrix} \times \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix}, \\ \begin{bmatrix} a_5 & a_6 \end{bmatrix} \times \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix} \end{matrix}$$

Example 56. *(combined constant partitions)*
The homogeneous constant partition of K_9 is:

$$\{ \;\; S_1 = \{ab, bc, cd, de, ef, fg, gh, hi, ai\}, \;\; S_2 = \{ac, bd, ce, df, eg, fh, gi, ah, bi\},$$

$$S_3 = \{ad, be, cf, dg, eh, fi, ag, bh, ci\},\ S_4 = \{ae, bf, cg, dh, ei, af, bg, ch, di\}\ \}$$

The hop length in each set of the partition is a constant, thus:

$$ab, bc \in S_1 \quad l(ab) = l(bc) = 1, \qquad ac, bd \in S_2 \quad l(ac) = l(bd) = 2$$

$$ad, be \in S_3 \quad l(ab) = l(bc) = 3, \qquad ae, bf \in S_4 \quad l(ac) = l(bd) = 4.$$

There are three different combined constant partitions of K_9, they are:

$$P_1 = \{S_1 \cup S_2, S_3 \cup S_4\}$$

$$P_2 = \{S_1 \cup S_3, S_2 \cup S_4\}$$

$$P_3 = \{S_1 \cup S_4, S_2 \cup S_3\}$$

For K_{13} there are 25 different combined constant partitions; fifteen of these combine two S_i's and ten combine three S_i's.

The next example is a more general case of the previous example, and the next theorem shows what conditions are needed to create a Homogeneous constant partition.

Example 57. *(set constant partitions)*
If the sets of the K_9 partition are combined thusly,

$$P = \{S_1 \cup S_2 \cup S_3, S_4\}$$

then the edges in the larger set are all friendly, while the single set S_4, is unfriendly. For edges e_i and e_j, $i \neq j$, from the same set of the partition, then $n_s(e_i) = n_s(e_j)$ and $n_o(e_i) = n_o(e_j)$. However for e_i and e_j not from the same set of the partition, then $n_s(e_i) \neq n_s(e_j)$ and $n_o(e_i) \neq n_o(e_j)$.

5.3 Constant partitions

Theorem 58. *For any homogeneous constant partition, of K_n, such that $n = 4k+1$, $k \geq 2$, there exists a combined constant partition $P = \{C_i\}_{i=1}^{j}$, such that $|C_i| = dn$, for all i, where $d \mid \dfrac{n-1}{2}$ and $j = \dfrac{n-1}{2d}$.*

Proof. Let $N = \{1, 2, 3, \cdots, \dfrac{n-1}{2}\}$ and since $n = 4k+1$, then $2 \mid \dfrac{n-1}{2}$. Let $d = 2$. Now choose $i_1, i_2 \in N$, such that $i_1 \neq i_2$. Now let $C_1 = S_{i_1} \cup S_{i_2}$. Now choose $i_3, i_4 \in N \setminus \{i_1, i_2\}$, such that $i_3 \neq i_4$ and let $C_2 = S_{i_3} \cup S_{i_4}$. Continuing in this fashion we construct the set $P_C = \{C_j\}_{j=1}^{(n-1)/4}$. Now for all j, $|C_j| = 2n$, and so by definition, P_C is a combined constant partition.

Similarly for $d > 2$, when $d \mid \dfrac{n-1}{2}$, d of the S_i sets may be combined to form a combined constant partition.

Hence for any homogeneous constant partition, of K_n, such that $n = 4k+1$, $k \geq 2$, there exists a combined constant partition $P = \{C_i\}_{i=1}^{j}$, such that $|C_i| = dn$, for all i, where $d \mid \dfrac{n-1}{2}$ and $j = \dfrac{n-1}{2d}$.

□

Example 59. *Complete graphs, with $n \geq 4$, have a Constant Partition. For K_{2n} it is a star constant partition, and for K_{2n-1} it is a homogeneous constant partition* *involving Hamiltonian cycles. In 1890, Walecki proved that every K_{2n-1}, or $K_{2n} - I$, where I is a $1-$factor, can be decomposed into Hamiltonian cycles [5, 8]. The partitions for K_4, K_5, and K_6 are shown below:*

$$
\begin{aligned}
K_4 &: \{\{ab, ac, ad\}, \{bc, bd, cd\}\} \\
K_5 &: \{\{ab, bc, cd, de, ae\}, \{ac, ad, bd, be, ce\}\} \\
K_6 &: \{\{ab, bc, cd, de, ae\}, \{ac, ad, bd, be, ce\}, \{af, bf, cf, df, ef\}\}
\end{aligned}
$$

Example 60. *Here are some examples of homogeneous constant partitions (n odd) and the cycles they contain. Note: that all of the cycles listed below are Hamiltonian cycles, except the third one from K_9, it contains 3 cycles.*

$$
\begin{array}{lllll}
K_5 & \{ & \{ab, bc, cd, de, ae\}, & & \text{1 cycle: } abcdea \\
 & & \{ac, bd, ce, ae, be\} & \} & \text{1 cycle: } acebda
\end{array}
$$

$$
\begin{array}{lllll}
K_7 & \{ & \{ab, bc, cd, de, ef, fg, ag\}, & & \text{1 cycle: } abcdefga \\
 & & \{ac, bd, ce, df, eg, af, bg\}, & & \text{1 cycle: } acegbdfa \\
 & & \{ad, be, cf, dg, ae, bf, cg\} & \} & \text{1 cycle: } adgcfbea
\end{array}
$$

$$
\begin{array}{lllll}
K_9 & \{ & \{ab, bc, cd, de, ef, fg, gh, hi, ai\}, & & \text{1 cycle: } abcdefghia \\
 & & \{ac, bd, ce, df, eg, fh, gi, ah, bi\}, & & \text{1 cycle: } acegibdfha \\
 & & \{ad, be, cf, dg, eh, fi, ag, bh, ci\}, & & \text{3 cycles: } adga, behb, cfic \\
 & & \{ae, bf, cg, dh, ei, af, bg, ch, di\} & \} & \text{1 cycle: } aeidhcgbfa
\end{array}
$$

Theorem 61. *For a given Ginsu vertex partition of a $K_{m,n}$ graph where $n \geq m \geq 2$, a friendly constant edge partition can be found.*

Proof. Let $P = \{S_i\}_{i=1}^{d}$ be a Ginsu vertex partition of $K_{m,n}$, such that $d \mid m$ and $d \mid n$, then there exists $k_m, k_n \in \mathbb{N}$ such that $dk_m = m$ and $dk_n = n$. Choose either the m or n vertices of S_1. Without loss of generality we may choose m. Let the set $M_1 = \{m_i\}_{i=1}^{k_m}$, let $M_2 = \{m_i\}_{i=k_m+1}^{2k_m}$, continue in this fashion until $M \setminus \bigcup_{i=1}^{d} M_i = \emptyset$. Now let $Q = \{R_i\}_{i=1}^{d}$, where $R_i = M_i \times N$, $1 \leq i \leq d$. Now Q is an edge partition of $K_{m,n}$, since $R_i \cap R_j = \emptyset$, for all $i \neq j$, and every edge is included.

We need to show that the partition is a friendly partition.
Let $e \in E(K_{m,n})$, then $n_s(e) = n + k_m - 2$ and $n_{o_j}(e) = k_m$. Since $n \geq 2$, then $k_m \leq n + k_m - 2$, and so e is friendly, and since the choice of e was arbitrary, the partition is friendly.

We need to show that the partition is a constant partition.
Let e_i, $e_j \in E(K_{m,n})$, $i \neq j$. By the last part, $n_s(e_i) = n + k_m - 2 = n_s(e_j)$ and $n_{o_l}(e_i) = k_m = n_{o_l}(e_j)$. Since the choices of e_i and e_j were arbitrary, Q is a constant partition.

Therefore, from a given Ginsu vertex partition of a $K_{m,n}$, $m \leq n$ graph, a friendly constant edge partition can be found.

□

Example 62. *The Ginsu vertex partition of* $K_{2,4}$ *is* $\{\{m_1, n_1, n_2\}, \{m_2, n_3, n_4\}\}$*, the following friendly constant edge partitions can be made:*
$P_1 = \{\{m_1, m_2\} \times \{n_1, n_2\}, \{m_1, m_2\} \times \{n_3, n_4\}\}$ *is dual thus friendly, and*

$P_2 = \{\{m_1\} \times \{n_1, n_2, n_3, n_4\}, \{m_2\} \times \{n_1, n_2, n_3, n_4\}\}$ *is friendly, but not dual.*

For a given Ginsu partition of a $K_{m,n}$ graph, where $d \mid \gcd(m,n)$, where $dk_m = m$ and $dk_n = n$, then at least two friendly edge partitions can be created. Let $P = \{S_1, S_2, \cdots, S_d\}$, where $S_j = \{m_i\}_{i=(j-1)k_m+1}^{jk_m} \times \{n_i\}_{i=1}^{|N|}$, or $S_j = \{m_l\}_{l=1}^{|M|} \times \{n_i\}_{i=(j-1)k_n+1}^{jk_n}$ $1 \leq j \leq d$.

Theorem 63. *A K_n, $n \geq 4$ graph has a homogeneous constant partition if and only if n is odd.*

Proof. ($\Rightarrow$) We need to show that n is odd.

Let $P = \{S_1, S_2, \cdots, S_k\}$ be a homogeneous constant partition of a K_n graph. Now each $|S_i| = n$, $i = 1, 2, \cdots, k$. So the number of edges in the graph is $kn = \dfrac{n(n-1)}{2}$, which implies that $2k + 1 = n$. Hence n is odd. ✓

($\Leftarrow$) We need to show that the partition is a homogeneous constant partition. Pick a vertex and label it v_0, proceeding clockwise label the next vertex v_1, continue in this fashion until all the vertices are labeled. Let $P = \{S_i\}_{i=1}^{k}$, $k = \dfrac{n-1}{2}$, and $S_i = \{v_j v_{(j+i) \mod n} \mid 0 \leq j \leq n\}\}$. Now by construction for all $v \in V(K_n)$, v is incident to two different edges, and for all $e_a, e_b \in S_i$, $l(e_a) = l(e_b)$, and $|S_i| = n$, for all i. Hence by definition, P is a homogeneous constant partition. ✓

□

Theorem 64. *A K_n, $n \geq 4$ graph has a star constant partition if and only if n is even.*

Proof. ($\Rightarrow$) We need to show that n is even.

Let $P = \{S_i\}_{i=1}^{k}$ be a star constant partition of a K_n graph. Now each $|S_i| = n - 1$, for all i. So the number of edges in the graph is $k(n-1) = \dfrac{n(n-1)}{2}$, which implies that $2k = n$. Hence n is even. ✓

($\Leftarrow$) We need to show that the partition is a star constant partition. Let $P = \{S_i\}_{i=1}^k$, where $S_1 = \{v_i v_j \mid \text{ for all } j \neq i, i \text{ fixed}\}$, and $\{S_l\}_{l=2}^{n/2}$, such that $\{S_l\}_{l=2}^{n/2}$ forms a homogeneous constant partition of K_{n-1}. Then $|S_1| = n-1 = |S_l|$, for all $l \geq 2$. Hence P is a star constant partition. ✓ □

Theorem 65. *For a constant partition of a graph K_n, $n \geq 4$, $n_s(e) + \sum_{j=1}^k n_{o_j}(e) = 2(n-2)$, for all $e \in E(K_n)$.*

Proof. In a K_n graph, every vertex has $n-1$ edges incident to it. So given an edge, there are $n-2$ edges incident to each end vertex of the edge. Thus the total number of neighbors to an edge is $2(n-2)$.

□

Theorem 66. *A $K_{2,n}$, $n \geq 2$ has a dual edge partition.*

Proof. Consider $K_{2,n}$, $n \geq 2$. Let $P = \{S_i\}_{i=1}^n$, where $S_i = \{m_1 n_i, m_2 n_i\}_{i=1}^n$. Thus for $e \in S_i$, $1 \leq i \leq n$, $n_s(e) = 1 = n_{o_l}(e)$, $1 \leq l < n$. Hence P is dual. ✓

□

Conjecture 67. *A $K_{m,n}$, $1 < m \leq n$ has a dual edge partition if and only if $m = 2$.*

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