# Normal $p$-Complement Theorems

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## Abstract

Finite group theory deals with classifying groups. One characteristic that a group may have is the possession of a normal $p$-complement. A normal $p$-complement is a normal subgroup $N$ of a group $G$ such that the order of $N$ is relatively prime to $p$ and the order of $G/N$ is a power of $p$. This result means that a group can be written as a product a Sylow $p$-subgroup and the normal $p$-complement. The objective of this thesis is to discuss three major results, one each by William Burnside, Ferdinand Georg Frobenius, and John G. Thompson, which state when a group has this property. The necessary background leading up to these theorems is included, with corresponding examples where possible.

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# Contents

# 1 Introduction

The primary purpose of this thesis is to explore several important normal $p$-complement theorems in the context of finite groups. An important goal of group theory is the classification of groups and their various properties. A normal $p$-complement is a normal subgroup $N$ of a group $G$ such that the order of $N$ is relatively prime to $p$ and the order of $G/N$ is a power of $p$. When a group has normal $p$-complement, the group may be broken down as the product of a Sylow $p$-subgroup and the normal $p$-complement.

Section Two presents background material that is necessary for understanding the group theory material presented. Section Three is broken down into four main components. The first subsection introduces the transfer map and presents examples to help clarify the material. The second subsection introduces various subgroups and definitions in building towards Burnside's Normal $p$-Complement Theorem. The section also contains several examples and other interesting results. The next subsection begins building towards Frobenius' Normal $p$-Complement Theorem, which generalizes Burnside's Theorem in the sense that it broadens the requirements for a group to have a normal $p$-complement. Again, this section explores various subgroups and properties thereof, and how these attributes relate to one another. The fourth and final subsection introduces $p$-solvability and the Puig subgroup, which are necessary in proving Thompson's Normal $p$-Complement Theorem.

Except where noted otherwise, the following is an adaption of Dr. Stephen Gagola's Advanced Group Theory notes at Kent State University.[1] All groups considered within this thesis are finite.

# 2 Background

We begin by defining the commutator subgroup, and present some of its various properties. Note that for elements $x$ and $y$ of a group $G$ that $[x,y] = x^{-1}y^{-1}xy$.

**Definition 2.1.** Let $G$ be a group. The *commutator subgroup of* $G$, denoted by $G'$, is defined by $G' = \langle [x,y] \mid x,y \in G \rangle = \left\langle x^{-1}y^{-1}xy \mid x,y \in G \right\rangle$. For subgroups $H$ and $K$ of $G$, denote $[H,K] = \langle [h,k] \mid h \in H, k \in K \rangle$.

**Proposition 2.1.** *The commutator subgroup $G' \leq G$ satisfies the following properties:*

*(a)* $G' \trianglelefteq G$

*(b)* $G/G'$ *is abelian*

*(c)* *$G$ is abelian if and only if $G' = \{e\}$.[3]*

No group theory paper would be complete without mentioning Sylow's Theorems. The three theorems are stated together for simplicity.

**Theorem 2.1.** *Suppose $G$ is a group and let $p$ be a prime number. Assume $|G| = p^n m$, where $n \geq 1$ and $p \nmid m$.*

***(First Sylow Theorem)*** *$G$ contains at least one subgroup of order $p^k$ for all $1 \leq k \leq n$.*

***(Second Sylow Theorem)*** *A $p$-subgroup of order $p^n$ is a Sylow $p$-subgroup of $G$. The Sylow $p$-subgroups are conjugate to one another, and all $p$-subgroups are contained in a conjugate of a Sylow $p$-subgroup.*

***(Third Sylow Theorem)*** *Denote the number of Sylow $p$-subgroups in $G$ by $n_p$. Then:*

$(i)$ $n_p \equiv 1 \bmod p$, *and*

$(ii)$ $n_p$ *divides $m$.*

Proofs of the Sylow Theorems are ubiquitous. A particularly nice version can be found in *Algebra: Pure and Applied* by Aigli Papantonopoulou.[3] A $p'$-group is a group whose order is not divisible by prime $p$. An important concept in group theory is that of a group action.

**Definition 2.2.** Let $G$ be a group and $\Omega$ be a set. A *right action* of $G$ on $\Omega$ is a function $\cdot : \Omega \times G \to \Omega$ such that:

1. $(\alpha \cdot g) \cdot h = \alpha \cdot (gh)$ for all $\alpha \in \Omega$, and $g, h \in G$
2. $\alpha \cdot e = \alpha$ for all $\alpha \in \Omega$, where $e$ is the identity element of $G$.

A left action is defined similarly. Actions are used to prove the result that normalizers grow in $p$-groups. Note that $H < G$ is used to mean $H \leq G$ but $H \neq G$.

**Corollary 2.1.** *Let $H < G$ where G is a p-group. Then $H < \mathbf{N}_G(H)$.*

*Proof.* Let $H$ act on the set $S = \{Hg \mid g \in G \text{ and } Hg \neq H\}$ by right multiplication. Then $|S| = \dfrac{|G|}{|H|} - 1$. Since $G$ is a $p$-group, $p$ does not divide $|S|$. Since $H$ is a $p$-group, there exists $Hg \in S$ such that $H_{Hg} = H$. Now $x \in H_{Hg}$ if and only if $Hgx = Hg$, which is true if only if $gxg^{-1} \in H$, which implies $x \in g^{-1}Hg$. But $H_{Hg} = g^{-1}Hg$, so $g^{-1}Hg = H$. This implies $g \in \mathbf{N}_G(H)$. Since $Hg \in S$, $Hg \neq H$. Therefore, $g \notin H$, and so $H < \mathbf{N}_G(H)$. □

**Definition 2.3.** Let $G$ be a group. Then $G$ is *simple* if it has no nontrivial proper normal subgroups.

**Definition 2.4.** Let $M < G$ be groups. Then $M$ is a *maximal subgroup of* $G$ if there does not exist $K < G$ such that $M < K < G$.

**Definition 2.5.** Let $H \leq G$. Then $H$ is a *characteristic subgroup* of $G$ if $\phi(H) = H$ for all automorphisms $\phi$ of $G$.

Before turning to background material more specific to normal $p$-complement theorems, the Isomorphism Theorems are needed.

**Theorem 2.2.** *(First Isomorphism Theorem) Let G and $G'$ be groups, and suppose $\phi : G \to G'$ is a homomorphism. Denote the kernel by $K = ker(\phi)$. Then $G/K \cong \phi(G)$.[3]*

**Theorem 2.3.** *(Second Isomorphism Theorem) Suppose* $H \trianglelefteq G$ *and* $K \leq G$ *are groups. Then* $KH/H \cong K/(K \cap H)$*.[3]*

**Theorem 2.4.** *(Third Isomorphism Theorem) Suppose* $H \trianglelefteq G$ *and* $K \trianglelefteq G$ *such that* $H \leq K$*. Then* $G/K \cong (G/H)/(K/H)$*.[3]*

Next are a couple definitions involving Sylow $p$-subgroups of $G$, denoted $Syl_p(G)$.

**Definition 2.6.** Suppose $G$ is a group and $P \in Syl_p(G)$. Then $\mathbf{O}_p(G) = \bigcap_{P \in Syl_p(G)} P$.

Note that $\mathbf{O}_p(G)$ is the largest normal $p$-subgroup of $G$. This leads to the next definition.

**Definition 2.7.** Let $G$ be a group and $P \in Syl_p(G)$ such that $P = \mathbf{O}_p(G)$ (i.e., $P \trianglelefteq G$). Then $G$ is called *p-closed*.

The next theorem makes use of $\mathbf{O}_p(G)$.

**Theorem 2.5.** *(Brodkey's Theorem) Let G be a group such that P is abelian for all Sylow p-subgroups of G. Then there exists* $P, Q \in Syl_p(G)$ *such that* $P \cap Q = \mathbf{O}_p(G)$*.*

*Proof.* Choose $P, Q \in Syl_p(G)$ with $P \cap Q$ minimal. Since the Sylow $p$-subgroups are abelian, $P \cap Q \trianglelefteq P$ and $P \cap Q \trianglelefteq Q$. Now $\mathbf{O}_p(G) \leq P \cap Q$ by definition. It is only necessary to show that $P \cap Q$ is contained in every Sylow $p$-subgroup of $G$. Let $R \in Syl_p(G)$. Now $\mathbf{N}_G(P \cap Q) \leq \mathbf{N}_G(P \cap Q)$ and $P \in Syl_p(\mathbf{N}_G(P \cap Q))$. Since $\mathbf{N}_R(P \cap Q)$ is a $p$-group, there exists an $x \in \mathbf{N}_G(P \cap Q)$ such that $\mathbf{N}_R(P \cap Q) \leq x^{-1}Px$ by Sylow, Theorem 2.1. Now $Q \leq \mathbf{N}_G(P \cap Q)$, so $x^{-1}Qx \leq x^{-1}\mathbf{N}_G(P \cap Q)x = \mathbf{N}_{x^{-1}Gx}(P \cap Q) = \mathbf{N}_G(P \cap Q)$ since $x \in \mathbf{N}_G(P \cap Q)$. Now $R \cap x^{-1}Qx = R \cap x^{-1}Qx \cap \mathbf{N}_G(P \cap Q) = \mathbf{N}_R(P \cap Q) \cap x^{-1}Qx \leq x^{-1}Px \cap x^{-1}Qx = x^{-1}(P \cap Q)x = P \cap Q$. Hence $R \cap x^{-1}Qx \leq P \cap Q$ and so $P \cap Q = R \cap x^{-1}Qx$ by minimality of $P \cap Q$. Hence $P \cap Q \leq R$ and so $P \cap Q \leq \mathbf{O}_p(G)$. Therefore, $P \cap Q = \mathbf{O}_p(G)$. $\square$

This thesis focuses on developing normal $p$-complement theorems. The following definitions outline the difference between normal complements and normal $p$-complements.

**Definition 2.8.** Let $H \leq G$ be groups. Then $G$ has a *normal complement* if there exists $N \trianglelefteq G$ such that $G = HN$ and $H \cap N = \{e\}$.

**Definition 2.9.** Let $N \trianglelefteq G$ be groups. Suppose $|N|$ is relative prime to $p$ and $|G:N|$ is a power of $p$. Then $N$ is a *normal p-complement* in $G$ if $G = NH$ and $N \cap H = \{e\}$ for some $H \leq G$.

The following definition and statements make use of a nilpotent group.

**Definition 2.10.** Let $G$ be a group. Define $Z_0(G) = \{e\}$ and $Z_1(G) = \mathbf{Z}(G)$, and let $Z_{i+1}(G)$ be the subgroup such that $Z_{i+1}(G)/Z_i(G) \cong Z(G/Z_i(G))$. Note that $Z_i(G) \leq Z_{i+1}(G)$. Consider the series $\{e\} = Z_0 \leq Z_1 \leq \cdots$. Then $G$ is *nilpotent* if $Z_n(G) = G$ for some $n$.[3]

**Lemma 2.1.** *Let G be a p-group. Then G is nilpotent.[4]*

**Theorem 2.6.** *Suppose G is a group. Then the following are equivalent:*

*(a) G is nilpotent*

*(b) If $H < G$, then $H < \mathbf{N}_G(H)$, and*

*(c) G is p-closed for all primes p.[4]*

The next set of definitions and theorems make use of the Frattini subgroup.

**Definition 2.11.** Let $G$ be a group. The intersection of all the maximal subgroups of $G$ is called the *Frattini subgroup*, and is denoted by $\Phi(G)$.

Note that $\Phi(G)$ is a characteristic subgroup of $G$ since maximal subgroups are permuted by automorphisms. It is also useful to state that elements of $\Phi(G)$ can be removed from a generating set since they are non-generators. The next theorem discusses the Frattini subgroup in the context of $p$-groups. A $p$-group $E$ is called *elementary abelian* if it is abelian and $x^p = e$ for all $x \in E$.

**Theorem 2.7.** *Suppose $G$ is a $p$-group. Then $G/\Phi(G)$ is elementary abelian. Furthermore, suppose $N \trianglelefteq G$. Then $G/N$ is elementary abelian if and only if $\Phi(G) \leq N$.[4]*

The next theorem is referred to as Fitting's Lemma. It is stated in full, although only part of it is necessary for the purposes of this thesis.

**Theorem 2.8.** *(Fitting's Lemma) Let $H$ and $N$ be groups, where $N$ is abelian, and such that the gcd $(|H|, |N|) = 1$. Suppose $H$ acts on $N$ by automorphisms. Then*

*(a)* $N = \mathbf{C}_N(H) \times [N,H]$,

*(b) If $H, N \leq G$ for some group $G$ where $N \trianglelefteq G$ and $G = HN$, and the action in $G$ is conjugation, then $G = \mathbf{C}_N(H) \times H[N,H]$, and*

*(c) If part $(b)$ is satisfied, then $\mathbf{C}_N(H)$ and $[N,H]$ are characteristic subgroups of $G$.*

The last theorem is the Schur-Zassenhaus Theorem. It provides a basis for finding a complement in a group.

**Theorem 2.9.** *(Schur-Zassenhaus Theorem) Suppose $N \trianglelefteq G$ are groups such that $\gcd(|N|, |G:N|) = 1$. Then there exists $H \leq G$ such that $G = HN$ and $H \cap N = \{e\}$ (i.e., $N$ is complemented in $G$).[4]*

# 3 $p$-Complement Theorems

## 3.1 The Transfer Map

Before exploring the transfer map, it is useful to define a transversal for a subgroup.

**Definition 3.1.** Let $G$ be a group with $H \leq G$ and $T \subseteq G$. We say that $T$ is a *right transversal* for $H$ in $G$ if $G = HT = \bigcup_{t \in T}$ and $|T| = |G : H|$.

A transversal then is a set of coset representatives. An important restriction is placed on the action in the transfer map. The group $G$ acts on the right cosets of $H$ in $G$ by right multiplication, i.e. since $H/G = \{Hg | g \in G\}$ the action is given by $Hg \cdot x = Hgx$ where $x \in G$. The map $T \to H/G$ is onto, since the transversal is defined to be a set satisfying $G = \bigcup_{t \in T} Ht$. Furthermore, since $|T| = |G : H|$, the map $T \to H/G$ is a bijection. Thus, the action $t \cdot g$, for $t \in T$ and $g \in G$, is the unique element of $T$ such that $H(t \cdot g) = Htg$. From this, it can be seen that $G$ is acting on $T$. Also, $tg(t \cdot g) \in H$ for $t \in T$ and $g \in G$.

**Example 3.1.** Let $G = D_4$ be the group of symmetries of the square. Now $D_4 = \{ e, p, p^2, p^3, f_1, f_2, \tau_1, \tau_2 \}$, where $p, p^2, p^3$ are 90 degree, 180 degree, and 270 degree counterclockwise rotations, respectively, $f_1$ and $f_2$ are flips along the negative and positive diagonals respectively, and $\tau_1$ and $\tau_2$ are flips about the vertical axis and horizontal axis respectively. Suppose $T = \{e, p\}$. Consider $t \cdot g$ for $t = p$ and $g = p^2$. Now $t \cdot g = p \cdot p^2 = p^3 = p \in T$ since $H(t \cdot g) = Hp^3 = Hp$.

**Definition 3.2.** Suppose $H_0 \trianglelefteq H \leq G$ such that $|G : H| = n$ and $H/H_0$ is abelian. Furthermore, suppose $T$ is a right transversal for $H$ in $G$. Then the transfer from $G$ to $H/H_0$ is the map $V : G \to H/H_0$ defined by $V(g) = \prod_{t \in T} tg(t \cdot g)^{-1} H_0$.

Before applying this map to other concepts, it is crucial to check that the transfer map is both well-defined and a homomorphism.

**Lemma 3.1.** *The transfer map is well-defined.*

*Proof.* Assume $g_1 = g_2$. Now $V(g_1) = \prod_{t\in T} tg_1 (t\cdot g_1)^{-1} H_0$ and $V(g_2) = \prod_{t\in T} tg_2(t\cdot g_2)^{-1}H_0$. Since $H/H_0$ is abelian, the order of $t \in T$ does not matter, and so $\prod_{t\in T} tg_1 (t\cdot g_1)^{-1} H_0 = \prod_{t\in T} tg_2 (t\cdot g_2)^{-1} H_0$ which implies $V(g_1) = V(g_2)$. Therefore, the transfer is well-defined. □

**Proposition 3.1.** *Suppose $H_0 \trianglelefteq H \leq G$ are groups with $|G:H| = n$ and $H/H_0$ abelian, and let $T$ be a right transversal for $H$ in $T$. Then $V : G \to H/H_0$ defined by $V(g) = \prod_{t\in T} tg(t\cdot g)^{-1} H_0$ is a homomorphism.*

*Proof.* Let $x, y \in G$.

$$\begin{aligned}
\text{Now } V(xy) &= \prod_{t\in T} txy\,(t\cdot(xy))^{-1} H_0 \\
&= \prod_{t\in T} tx\,(t\cdot x)^{-1}(t\cdot x)\,y\,(t\cdot(xy))^{-1} H_0 \\
&= \left[\prod_{t\in T} tx\,(t\cdot x)^{-1} H_0\right]\left[\prod_{t\in T} (t\cdot x)\,y\,(t\cdot(xy))^{-1} H_0\right] \text{ by definition of cosets} \\
&= \left[\prod_{t\in T} tx\,(t\cdot x)^{-1} H_0\right]\left[\prod_{t\in T} (t\cdot x)\,y\,((t\cdot x)\cdot y)^{-1} H_0\right] \text{ by right action} \\
&= V(x)\left[\prod_{t\in T} (t\cdot x)\,y\,((t\cdot x)\cdot y)^{-1} H_0\right] \text{ by definition of the transfer map} \\
&= V(x)\left[\prod_{t_2\in T} t_2 y\,(t_2\cdot y)^{-1} H_0\right] \text{ since } t\cdot x \in T \\
&= V(x)\,V(y) \text{ since we are multiplying over all elements of } t \in T.
\end{aligned}$$

Therefore, the transfer map is a homomorphism. □

It is useful to study an example.

**Example 3.2.** Let $G = D_4$ be the group of symmetries of the square as defined in Example 3.1. Suppose $H = \{e, p^2, \tau_1, \tau_2\}$, $H_0 = \{e\}$, and $T = \{e, p\}$. Then $H_0 \trianglelefteq H \leq G$, $H/H_0$ is abelian, and $T$ is a right transversal for $H$ in $G$.

Now $V(e) = \left[ee\,(e\cdot e)^{-1}\right]\left[pe\,(p\cdot e)^{-1}\right] H_0 = [e]\left[pp^{-1}\right] H_0 = pp^3 H_0 = eH_0 = H_0$. Thus $V(e) = H_0$.

Next, $V(p) = \left[ep\,(e\cdot p)^{-1}\right]\left[pp\,(p\cdot p)^{-1}\right] H_0 = \left[pp^{-1}\right]\left[ppe^{-1}\right] H_0 = pp^3 ppH_0 = p^2 H_0$. So $V(p) = p^2 H_0$.

Now $V\left(p^2\right) = \left[ep^2\left(e\cdot p^2\right)^{-1}\right]\left[pp^2\left(p\cdot p^2\right)^{-1}\right]H_0 = \left[p^2e^{-1}\right]\left[pp^2p^{-1}\right]H_0 = p^2pp^2p^3H_0$ $= eH_0 = H_0$. Hence, $V\left(p^2\right) = H_0$.

Fourth, $V\left(p^3\right) = \left[ep^3\left(e\cdot p^3\right)^{-1}\right]\left[pp^3\left(p\cdot p^3\right)^{-1}\right]H_0 = \left[p^3p^{-1}\right]\left[ee^{-1}\right]H_0 = p^3p^3H_0 =$ $p^2H_0$. Thus, $V\left(p^3\right) = p^2H_0$.

Next, $V\left(f_1\right) = \left[ef_1\left(e\cdot f_1\right)^{-1}\right]\left[pf_1\left(p\cdot f_1\right)^{-1}\right]H_0 = \left[f_1p^{-1}\right]\left[pf_1e^{-1}\right]H_0 = f_1p^3pf_1H_0 =$ $f_1p^3\tau_2H_0 = f_1f_1H_0 = eH_0 = H_0$. So $V\left(f_1\right) = H_0$.

Now, $V\left(f_2\right) = \left[ef_2\left(e\cdot f_2\right)^{-1}\right]\left[pf_2\left(p\cdot f_2\right)^{-1}\right]H_0 = \left[f_2p^{-1}\right]\left[pf_2e^{-1}\right]H_0 = f_2p^3pf_2H_0 =$ $f_2p^3\tau_1H_0 = f_2f_2H_0 = eH_0 = H_0$. Hence, $V\left(f_2\right) = H_0$.

Also, $V\left(\tau_1\right) = \left[e\tau_1\left(e\cdot \tau_1\right)^{-1}\right]\left[p\tau_1\left(p\cdot \tau_1\right)^{-1}\right]H_0 = \left[\tau_1e^{-1}\right]\left[p\tau_1p^{-1}\right]H_0 = \tau_1p\tau_1p^3H_0 =$ $\tau_1pf_1H_0 = \tau_1\tau_2H_0 = p^2H_0$. Thus, $V\left(\tau_1\right) = p^2H_0$.

Finally, $V\left(\tau_2\right) = \left[e\tau_2\left(e\cdot \tau_2\right)^{-1}\right]\left[p\tau_2\left(p\cdot \tau_2\right)^{-1}\right]H_0 = \left[\tau_2e^{-1}\right]\left[p\tau_2p^{-1}\right]H_0 = \tau_2p\tau_2p^3H_0 =$ $\tau_2pf_2H_0 = \tau_2\tau_1H_0 = p^2H_0$. Therefore, $V\left(\tau_2\right) = p^2H_0$.

The next question would be whether the transfer map varies based on the choice of transversal and Proposition 3.2 shows that the map is independent of the choice of transversal.

**Proposition 3.2.** *Assume $H_0 \trianglelefteq H \leq G$ and $H/H_0$ is abelian. Then the transfer map is independent of the choice of transversal.*

*Proof.* Let $T$ and $S$ be right transversals for $H$ in $G$, and $V_T$ and $V_S$ be the respective transfer maps from $G \to H/H_0$. Since $T$ and $S$ are both transversals, $G = HT = \cup_{t\in T}Ht$ and $G = HS = \cup_{s\in S}Hs$. Based on the properties of transversals, $S = \{h_tt \mid t \in T\}$ for some function $h \to h_t$ from $T$ to $H$. Let $*$ denote the transport of structure action of $G$ on $S$, so $(h_tt) * g = h_{t\cdot g}(t\cdot g)$. Fix $x \in G$.

$$\begin{aligned}
\text{Then } V_S(x) &= \prod_{s\in S} sx\,(s\cdot x)^{-1}H_0 \\
&= \prod_{t\in T}(h_tt)\,x\,((h_tt)*x)^{-1}H_0 \\
&= \prod_{t\in T}(h_tt)\,x\,(h_{t\cdot x}(t\cdot x))^{-1}H_0 \\
&= \prod_{t\in T}h_ttx\,(t\cdot x)^{-1}h_{t\cdot x}^{-1}H_0
\end{aligned}$$

$$
\begin{aligned}
&= \left[\prod_{t\in T} h_t H_0\right] \left[\prod_{t\in T} tx\,(t\cdot x)^{-1} H_0\right] \left[\prod_{t\in T} h_{t\cdot x}^{-1} H_0\right] \\
&= \prod_{t\in T} tx\,(t\cdot x)^{-1} H_0 \text{ since } H/H_0 \text{ is abelian} \\
&= V_T\,(x).
\end{aligned}
$$

□

## 3.2 Burnside's Normal $p$-Complement Theorem

To prove Burnside's Normal $p$-Complement Theorem, several other results must first be established. We begin by defining some new subgroups. The group $\mathbf{O}_p\,(G)$ was defined in Section 2. The following quotient groups are also of note:

- $\mathbf{O}^p\,(G)$ is the unique normal subgroup of $G$ minimal with respect to $G/\mathbf{O}^p\,(G)$ being a $p$-group.
- $\mathbf{A}^p\,(G)$ is the unique normal subgroup of $G$ minimal with respect to $G/\mathbf{A}^p\,(G)$ being an abelian $p$-group.
- $\mathbf{E}^p\,(G)$ is the unique normal subgroup of $G$ minimal with respect to $G/\mathbf{E}^p\,(G)$ being an elementary abelian $p$-group.

$\mathbf{O}^p\,(G)$ is generated by all elements of order prime to $p$ in $G$. The above factor groups build in their respective requirements, which leads to the conclusion that $\mathbf{O}^p\,(G) \leq \mathbf{A}^p\,(G) \leq \mathbf{E}^p\,(G) \leq G$. Strict containment in $G$ of these subgroups occurs in certain circumstances. For instance, it can be shown that $\mathbf{A}^p\,(G) < G$ if and only if $p \mid |G:G'|$. It can also be shown that $\mathbf{O}^p\,(G) < G$ if and only if $\mathbf{E}^p\,(G) < G$. The next result shows that $\mathbf{A}^p\,(G)$ is the kernel of the transfer map.

**Theorem 3.1.** *Let $G$ be a group and $P \in Syl_P\,(G)$. Then $\mathbf{A}^p\,(G)$ is the kernel of the transfer map $V : G \to P/P'$, where $P'$ is the commutator subgroup of $P$.*

*Proof.* Suppose $T$ is a right transversal for $P$ in $G$. Let $V : G \to P/P'$ be the transfer map $V\,(g) = \prod_{t\in T} tg\,(t\cdot g)^{-1} P'$. Now $P/P'$ is an abelian $p$-group. By the First Isomorphism

Theorem, $G/ker(V) \cong P/P'$, and so $G/ker(V)$ is an abelian $p$-group. Since $\mathbf{A}^p(G)$ is the minimal normal subgroup of $G$ such that its quotient group is an abelian $p$-group, $\mathbf{A}^p(G) \leq ker(V)$. Thus, it is necessary to show $ker(V) \leq \mathbf{A}^p(G)$.

Since $\mathbf{A}^p(G)$ contains the full $p'$-power of $|G|$, and $P \in Syl_p(G)$, then $G = P\mathbf{A}^p(G)$. Define an element $g \in ker(V)$ by $g = xy$ where $x \in P$ and $y \in \mathbf{A}^p(G)$. To show $g \in \mathbf{A}^p(G)$, it is sufficient to show $x \in \mathbf{A}^p(G)$. Since $ker(V)$ is a group, and $\mathbf{A}^p(G) \leq ker(V)$, $x \in ker(V)$. Thus, $V(x) = \prod\limits_{t \in T} tx(t \cdot x)^{-1} P' = eP' \in P/P'$. Consider just the product $\prod\limits_{t \in T} tg(t \cdot g)^{-1}$. Depending on the ordering of the elements, different elements of $P'$ will result. But $P' \leq G' \leq \mathbf{A}^p(G)$, so the element is in $\mathbf{A}^p(G)$.

Now consider a new map. Let $\phi : G \to G/\mathbf{A}^p(G)$ be the natural homomorphism from $G$ onto $G/\mathbf{A}^p(G)$. Now $\phi\left(\prod\limits_{t \in T} \overline{tg(t \cdot g)}^{-1}\right) = \prod\limits_{t \in T} \overline{tg(t \cdot g)}^{-1} \mathbf{A}^p(G) = \mathbf{A}^p(G) = e \in G/\mathbf{A}^p(G)$ since the product is contained in $\mathbf{A}^p(G)$. Since $G/\mathbf{A}^p(G)$ is abelian, we have $e = \prod\limits_{t \in T} \bar{t}\bar{x}\overline{(t \cdot x)}^{-1} = \left(\prod\limits_{t \in T} \bar{t}\right)\left(\prod\limits_{t \in T} \bar{x}\right)\left(\prod\limits_{t \in T} \overline{(t \cdot x)}^{-1}\right)$. The first and third products cancel since $G/\mathbf{A}^p(G)$ is abelian. Thus $e = \prod\limits_{t \in T} \bar{x} = \bar{x}^{|T|} = \bar{x}^{|G:P|}$. Since $x \in P$, $x$ is a $p$-element, so $\bar{x}^{|P|} = e$. Therefore, $\bar{x} = e$ and $x \in \mathbf{A}^p(G)$. □

Despite the utility of the transfer map, actual computations can be time consuming. The following lemma provides some useful characteristics of the transfer map, and a convenient method of calculating values.

**Lemma 3.2.** *Suppose $H_0 \trianglelefteq H \leq G$ with $H/H_0$ abelian. Let $T$ be a right transversal for $H$ in $G$. For $g \in G$, let $T_0 \subseteq T$ be a set of representatives of the $\langle g \rangle$-orbits on $T$, and denote the size of the $\langle g \rangle$-orbit containing $t \in T_0$ by $n_t$. Then:*

*(a)* $n_t$ *divides* $|g|$ *for all* $t \in T_0$

*(b)* $\sum\limits_{t \in T_0} n_t = |T| = |G:H|$

*(c)* $tg^{n_t}t^{-1} \in H$ *for all* $t \in T_0$

*(d)* $V(g) = \prod\limits_{t \in T_0} tg^{n_t}t^{-1}H_0$ *for* $V : G \to H/H_0$.

*Proof.* As the order of an orbit divides the order of the group, part $a$ is proved. Likewise, since orbits partition a group, part $b$ is shown.

Part $c$: Fix $t \in T_0$. Since $n_t$ denotes the size of the orbit containing $t$, $t \cdot g^{n_t} = t$. Thus, the orbit containing $t$ is the set of elements $\left\{t, t \cdot g, t \cdot g^2, \ldots, t \cdot g^{n_t - 1}\right\}$. Now consider the product

$$\begin{aligned}
&\left(tg\,(t \cdot g)^{-1} H_0\right)\left((t \cdot g)\,g\left(t \cdot g^2\right)^{-1} H_0\right)\cdots\left(\left(t \cdot g^{n_t-1}\right) g\,(t \cdot g^{n_t})^{-1} H_0\right) \\
&\qquad = \left(tg\,(t \cdot g)^{-1}\right)\left((t \cdot g)\,g\left(t \cdot g^2\right)^{-1}\right)\cdots\left(\left(t \cdot g^{n_t-1}\right) g\,(t \cdot g^{n_t})^{-1}\right) H_0 \\
&\qquad = tg^{n_t}t^{-1}H_0.
\end{aligned}$$

Now $g^{n_t}$ is itself an element of $G$. Thus, by the earlier results of the transfer, we have that $tg^{n_t}t^{-1} \in H$ for all $t \in T_0$.

Part $d$: From part $c$, we have that the product over the orbit of an individual element of $t \in T_0$ results in $tg^{n_t}t^{-1}$. Thus, taking the product over all $t \in T_0$, $V(g) = \prod_{t \in T_0} tg^{n}t^{-1}H_0$.

□

The next subgroup to consider is the focal subgroup of $H$ in $G$. Its relationship to other subgroups is a necessary component in proving Burnside's Normal $p$-Complement Theorem.

**Definition 3.3.** Let $H \leq G$. The *focal subgroup of $H$ in $G$* is $Foc_G(H) = \langle\, h^{-1}g^{-1}hg \mid h \in H, g \in G, g^{-1}hg \in H \,\rangle$ .

Consider $[H,G] = \langle [h,g] \mid h \in H, g \in G \rangle$, and the commutator subgroups $H' \leq H$ and $G' \leq G$. The following relationships can be seen:

$$\begin{aligned}
H' &= \left\{x^{-1}y^{-1}xy \mid x, y \in H\right\} \\
&\leq \left\langle h^{-1}g^{-1}hg \mid h \in H, g \in G, g^{-1}hg \in H\right\rangle \\
&\leq \left\langle h^{-1}g^{-1}hg \mid h \in H, g \in G\right\rangle \cap H \\
&\leq \left\langle m^{-1}n^{-1}mn \mid m, n \in G\right\rangle = G'.
\end{aligned}$$

Hence, the relationships between commutators and focal subgroups is summarized as $H' \leq Foc_G(H) \leq [H,G] \cap H \leq G'$. Let us now consider Hall subgroups, and their relation to focal subgroups.

**Definition 3.4.** A subgroup whose index is relatively prime to its order is called a *Hall subgroup*.

Since the index of a Sylow $p$-subgroup will be relatively prime to its order, it will be a Hall subgroup.

**Theorem 3.2.** *Let $H \leq G$ be a Hall subgroup, and $V : G \to H/H'$ be the transfer map. Then $ker(V) \cap H = Foc_G(H)$.*

*Proof.* Let $k \in ker(V) \cap H$ and $T$ be a right transversal for $H$ in $G$. Choose $T_0 \subseteq T$ and $n_t \in \mathbb{Z}$ satisfying Lemma 3.2. Then $V(k) = \prod_{t \in T_0} t k^{n_t} t^{-1} H'$. Since $k \in ker(V)$, $V(k) = eH'$. Now consider $\sum_{t \in T_0} n_t = n = |T|$, and suppose $T_0$ has $m$ elements. Since $k \in H$ and $H/H'$ is abelian, the product

$$\begin{aligned}
k^n H' \cdot \prod_{t \in T_0} k^{-n_t} t k^{n_t} t^{-1} H' &= k^n H' \cdot \left[\left(k^{-n_t} t_1 k^{n_t} t_1^{-1}\right) H' \left(k^{-n_t} t_2 k^{n_t} t_2^{-1}\right) H' \cdots \left(k^{-n_t} t_m k^{n_t} t_m^{-1}\right) H'\right] \\
&= k^n H' \cdot \left[\left(k^{-n_t} t_1 k^{n_t} t_1^{-1}\right) \left(k^{-n_t} t_2 k^{n_t} t_2^{-1}\right) \cdots \left(k^{-n_t} t_m k^{n_t} t_m^{-1}\right) H'\right] \\
&= k^n H' \cdot \left[\left(k^{-n_t} k^{-n_t} \cdots k^{-n_t}\right) \left(t_1 k^{n_t} t_1^{-1}\right) \left(t_2 k^{n_t} t_2^{-1}\right) \cdots \left(t_m k^{n_t} t_m^{-1}\right)\right] H' \\
&= k^n H' \cdot \left[k^{-n} H' \cdot \prod_{t \in T_0} t k^{n_t} t^{-1} H'\right] \\
&= eH' \cdot \textstyle\prod_{t \in T_0} t k^{n_t} t^{-1} H' \\
&= eH' \in H/H'.
\end{aligned}$$

Hence, $k^n \cdot \prod_{t \in T_0} k^{-n_t} t k^{n_t} t^{-1} \in H' \leq Foc_G(H)$ for a fixed ordering of $t \in T_0$. Further, for each $t \in T_0$, the product $k^{-n_t} t k^{n_t} t^{-1} \in Foc_G(H)$, so $k^n \in Foc_G(H)$. Now $|k|$ divides $|H|$, and $n = |G : H|$, and $gcd(|H|, |G : H|) = 1$ since $H$ is a Hall subgroup. Thus, $gcd(|k|, n) = 1$. Therefore, $k \in \langle k^n \rangle \leq Foc_G(H)$.

$\Leftarrow$: From Definition 3.3, $Foc_G(H) \leq H$, and from the notes following Definition 3.3, $Foc_G(H) \leq G'$. Hence, $Foc_G(H) \leq G' \cap H$. From Theorem 3.1, $G' \leq ker(V)$.

Thus, $Foc_G(H) \leq G' \cap H \leq ker(V) \cap H$. □

From Theorems 3.1 and 3.2, we immediately get the following result.

**Corollary 3.1.** *If $P \in Syl_p(G)$, then $Foc_G(P) = P \cap \mathbf{A}^p(G)$.*

*Proof.* From Theorem 3.1, $\mathbf{A}^P(G) = ker(V)$. From the notes following Definition 3.4, $P$ is a Hall subgroup of $G$. Therefore, $Foc_G(P) = P \cap \mathbf{A}^P(G)$ by Theorem 3.2. $\square$

The next set of relationships to consider are the conjugacy classes of groups, subgroups, and elements, and how they relate to one another. Any group can be partitioned into conjugacy classes. Suppose $H \leq G$. Then $G$ can be split into $G$-conjugacy classes, $H$ can be split into $H$-conjugacy classes, and each $H$-conjugacy class is contained in only one $G$-conjugacy class. This leads to the next definition.

**Definition 3.5.** Let $H \leq G$, with $G$ split into $G$-conjugacy classes, and $H$ split into $H$-conjugacy classes. If $K$ and $L$ are two different $H$-conjugacy classes, but lie in the same $G$-conjugacy class, then $K$ and $L$ are said to be *fused*.

Elements may also be referred to as fused. Since certain elements are lost when looking at subgroups, $G$-conjugacy classes may be split up when looking at $H$-conjugacy classes. Now $Foc_G(H) = \langle h^{-1}g^{-1}hg \mid h \in H, g \in G, g^{-1}hg \in H \rangle$. Consider $h \in H$, and pick $g \in G$ such that $g^{-1}hg \in H$. Now $h$ and $g^{-1}hg$ are conjugate in $G$, so they will be in the same $G$-conjugacy class. However, $h$ and $g^{-1}hg$ may be in different $H$-conjugacy classes. If so, then these two $H$-conjugacy classes will be fused. Thus, fusion of $H$-conjugacy classes determines the $Foc_G(H)$.

**Example 3.3.** Let $G = D_4$, as defined in Example 3.1. Now $G$ has five $G$-conjugacy classes: $\{e\}$, $\{p, p^3\}$, $\{p^2\}$, $\{f_1, f_2\}$ and $\{\tau_1, \tau_2\}$. Let $H = \{e, p^2, \tau_1, \tau_2\}$, so $H \leq G$. Then $H$ has four $H$-conjugacy classes: $\{e\}$, $\{p^2\}$, $\{\tau_1\}$ and $\{\tau_2\}$. Set $K = \{\tau_1\}$ and $L = \{\tau_2\}$. Then $K$ and $L$ are distinct conjugacy classes of $H$, but lie in the same $G$-conjugacy class. Therefore, $K$ and $L$ are fused.

Now turn to $Foc_G(H)$, and consider $h \in H$ and $g^{-1}hg \in H$. If $h = e$, then $g^{-1}eg = e \in H$ for all $g \in G$. So $h^{-1}g^{-1}hg = ee = e \in Foc_G(H)$. If $h = p^2$, then $g^{-1}p^2g = p^2 \in H$ for all $g \in G$ since $p^2$ is in its own $G$-conjugacy class. Thus, $h^{-1}g^{-1}hg = p^2p^2 = e$.

If $h = \tau_1$, then $g^{-1}\tau_1 g = \tau_1$ or $g^{-1}\tau_1 g = \tau_2$ based on the $G$-conjugacy classes. Hence, $h^{-1}g^{-1}hg = \tau_1\tau_1 = e \in Foc_G(H)$ or $h^{-1}g^{-1}hg = \tau_1\tau_2 = p^2 \in Foc_G(H)$. If $h = \tau_2$, then once again $g^{-1}hg = \tau_1$ or $g^{-1}hg = \tau_2$. Therefore, $h^{-1}g^{-1}hg = \tau_2\tau_1 = p^2 \in Foc_G(H)$ or $h^{-1}g^{-1}hg = \tau_2\tau_2 = e \in Foc_G(H)$. Thus, $Foc_G(H) = \{e, p^2\}$.

Fusion of $H$-conjugacy classes need not exist. In this case, $Foc_G(H) = H'$. If $P \in Syl_P(G)$, and no fusion exists, then by Corollary 3.1, $\mathbf{A}^P(G) < G$. Furthermore, in any non-abelian simple group $G$, fusion must take place for all $P \in Syl_P(G)$, for every prime $p$ that divides the order of $G$.

**Lemma 3.3.** *Let $H \leq G$ be groups such that there is no fusion of $H$-conjugacy classes in $G$. Then $Foc_G(H) = H'$.*

*Proof.* $\Rightarrow$: Now $Foc_G(H) = \langle\, h^{-1}g^{-1}hg \mid h \in H, g \in G, g^{-1}hg \in H \,\rangle = \langle\, h^{-1}h_2 \mid h, h_2 \in H, h_2 = g^{-1}hg \quad for \quad some \quad g \in G \,\rangle$. So $Foc_G(H)$ is generated by elements of $H$, and elements of $H$ that are conjugate in $G$. But there is no fusion of $H$-conjugacy classes, so $h$ and $h_2$ are in the same $H$-conjugacy class. Thus, if $g^{-1}hg = h_2 \in H$, then there must exist $h_3 \in H$ such that $g^{-1}hg = h_3^{-1}hh_3$ since $h$ and $h_2$ must be conjugate in $H$. So then $Foc_G(H) = \langle\, h^{-1}h_3hh_3 \mid h, h_3 \in H, g^{-1}hg = h_3^{-1}hh_3 \quad for \quad some \quad g \in G \leq H' \,\rangle$.

$\Leftarrow$: This is immediate from the notes following Definition 3.3, so $H' \leq Foc_G(H)$.

□

We can apply Lemma 3.3 in the following example.

**Example 3.4.** Let $G = D_4$ as defined in Example 3.1. Suppose $H = D_4$. Thus, $H$ has the same five conjugacy classes as $G$, and therefore no fusion takes place. Now $Foc_G(H) = \langle h^{-1}g^{-1}hg \mid h \in H, g \in G, g^{-1}hg \in H \rangle$. If $h = e$, then $g^{-1}eg = e$ so $h^{-1}g^{-1}hg = ee = e \in Foc_G(H)$. If $h = p$, then $g^{-1}hg = p$ or $g^{-1}hg = p^3$, so $h^{-1}g^{-1}hg = p^3p = e$ or $h^{-1}g^{-1}hg = p^3p^3 = p^2$. If $h = p^2$, then $g^{-1}hg = p$ or $g^{-1}hg = p^2$, so $h^{-1}g^{-1}hg = p^2p^2 = e$. If $h = p^3$, then $g^{-1}hg = p$ or $g^{-1}hg = p^3$, so $h^{-1}g^{-1}hg = pp = p^2$ or $h^{-1}g^{-1}hg = pp^3 = p = e$. If $h = \tau_1$, then $g^{-1}hg = \tau_1$ or $g^{-1}hg = \tau_2$, so $h^{-1}g^{-1}hg = \tau_1\tau_1 = e$ or $h^{-1}g^{-1}hg = \tau_1\tau_2 = p^2$.

If $h = \tau_2$, then $g^{-1}hg = \tau_1$ or $g^{-1}hg = \tau_2$, so $h^{-1}g^{-1}hg = \tau_2\tau_1 = p^2$ or $h^{-1}g^{-1}hg = \tau_2\tau_2 = e$. If $h = f_1$, then $g^{-1}hg = f_1$ or $g^{-1}hg = f_2$, so $h^{-1}g^{-1}hg = f_1 f_1 = e$ or $h^{-1}g^{-1}hg = f_1 f_2 = p^2$. If $h = f_2$, then $g^{-1}hg = f_1$ or $g^{-1}hg = f_2$, so $h^{-1}g^{-1}hg = f_2 f_1 = p^2$ or $h^{-1}g^{-1}hg = f_2 f_2 = e$. Thus, $Foc_G(H) = \langle e, p^2 \rangle = \{e, p^2\}$.

Furthermore, $P' = \langle x^{-1}y^{-1}xy \mid x, y \in P \rangle = \langle e, p^2 \rangle = \{e, p^2\}$. Proof of this is left to the reader. Therefore, $Foc_G(H) = H'$.

Since $|G| = 2^3 = |H|$, we have that $H \in Syl_P(G)$. By Corollary 3.1, $Foc_G(P) = P \cap \mathbf{A}^P(G) = \{e, p^2\}$. Thus, $\mathbf{A}^P(G) = \{e, p^2\}$, and $\mathbf{A}^P(G) < G$.

We can condense these ideas and consider the relationship between different groups and fusion.

**Definition 3.6.** Let $H \leq X \leq G$ be groups. Then $X$ *controls fusion in* $H$ *with respect to* $G$ if $x, y \in H$ are conjugate in $G$, then $x$ and $y$ are conjugate in $X$.

Suppose $K \leq H \leq X \leq G$ are groups. If $X$ controls fusion in $H$ with respect to $G$, then $X$ controls fusion in $K$ with respect to $G$. Fusion control is useful for calculating focal subgroups. Consider the following lemma.

**Lemma 3.4.** *Let $H \leq X \leq G$ be groups, and suppose $X$ controls fusion in $H$ with respect to $G$. Then $Foc_G(H) = Foc_X(H)$.*

*Proof.* $\Rightarrow$: Let $a \in Foc_G(H)$. Then $a = h^{-1}g^{-1}hg$ for some $h \in H$, $g \in G$, where $g^{-1}hg \in H$. Suppose $g^{-1}hg = b$. Since $X$ controls fusion in $H$ with respect to $G$, there exists $x \in X$ such that $x^{-1}hx = b$, and $x^{-1}hx \in H$. Thus, $a = h^{-1}g^{-1}hg = h^{-1}x^{-1}hx \in Foc_X(H)$. Therefore, $Foc_G(H) \leq Foc_X(H)$.

$\Leftarrow$: Let $a \in Foc_X(H)$. Then $a = h^{-1}x^{-1}hx$ for some $h \in H, x \in X$, where $x^{-1}hx \in H$. But $X \leq G$, so $x \in G$, so $a = h^{-1}x^{-1}hx \in Foc_G(H)$. Thus, $Foc_X(H) \leq Foc_G(H)$. □

We can continue our ongoing example to explore these new definitions and lemmas.

**Example 3.5.** Let $G = D_4$ as defined in Example 3.1, $X = \left\{e, p^2, \tau_1, \tau_2\right\}$, and $H = \{e, \tau_1\}$. Now $\tau_1$ is conjugate to itself in $G$ as $\tau_2 \tau_1 \tau_2 = \tau_1$. But $\tau_2 \in X$ as well, so $\tau_1$ is also conjugate to itself in $X$. As $e$ is clearly conjugate to itself in both $G$ and $X$, we have the $X$ controls fusion in $H$ with respect to $G$. Now, $Foc_G(H) = \left\langle h^{-1}g^{-1}hg \mid h \in H, g \in G, g^{-1}hg \in H\right\rangle$. If $h = e$, then $g^{-1}eg = e$, so $h^{-1}g^{-1}hg = e \in Foc_G(H)$ . If $h = \tau_1$, then $g^{-1}\tau_1 g = \tau_1$ or $g^{-1}\tau_1 g = \tau_2$. But $g^{-1}hg \in H$, so only the first conjugation holds. Then $h^{-1}g^{-1}hg = \tau_1\tau_1 = e \in Foc_G(H)$. Thus, $Foc_G(H) = \{e\}$. Now, $Foc_X(H) = \langle\, h^{-1}x^{-1}hx \mid h \in H, x \in X, x^{-1}hx \in H\,\rangle$. For $h = e$, similarly, $e \in Foc_X(H)$. Since $x^{-1}hx \in H$, we must have $h^{-1}x^{-1}hx = \tau_1\tau_1 = e \in Foc_X(H)$ when $h = \tau_1$. Hence, $Foc_X(H) = \{e\}$ and $Foc_G(H) = Foc_X(H)$.

The next two corollaries provide useful examples of fusion control in a more general group setting.

**Corollary 3.2.** *Let $P \in Syl_P(G)$. Then $\mathbf{N}_G(P)$ controls fusion in $\mathbf{C}_G(P)$ with respect to $G$.*

*Proof.* Let $x, y \in \mathbf{C}_G(P)$ and suppose $g^{-1}xg = y$ for some $g \in G$. Now $\mathbf{C}_G(y)$ consists of elements that commute with $y$, but $y \in \mathbf{C}_G(P)$, so $P \leq \mathbf{C}_G(y)$. Similarly, $P \leq \mathbf{C}_G(x)$, so $g^{-1}Pg \leq g^{-1}\mathbf{C}_G(x)g = \mathbf{C}_G\left(g^{-1}xg\right) = \mathbf{C}_G(y)$. Since $P \in Syl_p(G)$, we must have that $P, g^{-1}Pg \in Syl_p(\mathbf{C}_G(y))$ . By Theorem 2.1, there exists a $c \in \mathbf{C}_G(y)$ such that $c^{-1}g^{-1}Pgc = P$. Then $gc \in \mathbf{N}_G(P)$ and $c^{-1}g^{-1}xgc = c^{-1}\left(g^{-1}xg\right)c = c^{-1}yc = c^{-1}cy = y$ since $c \in \mathbf{C}_G(y)$. Therefore, elements of $\mathbf{C}_G(P)$ that are conjugate in $G$ are conjugate in $\mathbf{N}_G(P)$, and so $\mathbf{N}_G(P)$ controls fusion in $\mathbf{C}_G(P)$. $\square$

**Corollary 3.3.** *Let $P \in Syl_p(G)$. Then $\mathbf{N}_G(P)$ controls fusion in $\mathbf{Z}(P)$. If $P$ is abelian, then $\mathbf{N}_G(P)$ controls fusion in $P$.*

*Proof.* Now, $\mathbf{Z}(P) \leq \mathbf{C}_G(P) \leq \mathbf{N}_G(P) \leq G$ and $\mathbf{N}_G(P)$ controls fusion in $\mathbf{C}_G(P)$ by Corollary 3.2. From the notes following Definition 3.6, $\mathbf{N}_G(P)$ controls fusion in $\mathbf{Z}(P)$. If $P$ is abelian, then $\mathbf{Z}(P) = P$, and $\mathbf{N}_G(P)$ controls fusion in P. $\square$

We may now prove Burnside's Normal $p$-Complement Theorem.

**Theorem 3.3.** ***(Burnside's Normal p-Complement Theorem)*** *Let $P \in Syl_p(G)$. Suppose $P \leq \mathbf{Z}(\mathbf{N}_G(P))$. Then $G$ has a normal p-complement.*

*Proof.* Let $x, y \in P$. Suppose there exists $g \in G$ such that $y = g^{-1}xg$. Since $P \leq \mathbf{Z}(\mathbf{N}_G(P))$, $P$ is abelian. By Corollary 3.3, $\mathbf{N}_G(P)$ controls fusion in $P$. So there exists $h \in \mathbf{N}_G(P)$ such that $y = h^{-1}xh$. Since $x \in P$, $x \in \mathbf{Z}(\mathbf{N}_G(P))$, so $y = h^{-1}xh = h^{-1}hx = x$. Therefore, distinct elements of $P$ are not conjugate in $G$. Hence, $Foc_G(P) = \{e\}$. Now $\mathbf{A}^p(G) \trianglelefteq G$, so $P \cap \mathbf{A}^p(G) \in Syl_p(\mathbf{A}^p(G))$. By Corollary 3.1, $Foc_G(P) = P \cap \mathbf{A}^p(G) = \{e\}$, so $\mathbf{A}^p(G)$ must be a $p'$ group. By definition, $G/\mathbf{A}^p(G)$ is a $p$-group. Therefore, $\mathbf{A}^p(G)$ is a normal $p$-complement in $G$. □

## 3.3 Frobenius' Normal $p$-Complement Theorem

We now start to build the material required to prove Frobenius' Normal $p$-Complement Theorem. If $P \leq N \leq G$, and $P \in Syl_p(G)$, then $G = P\mathbf{A}^p(G)$, so $G = N\mathbf{A}^p(G)$. Now $N \cap \mathbf{A}^p(G) \trianglelefteq N$, and $N/(N \cap \mathbf{A}^p(G))$ is an abelian $p$-group. By definition of $\mathbf{A}^p(N)$, $\mathbf{A}^p(N) \leq N \cap \mathbf{A}^p(G)$. Now $\mathbf{A}^p(N)$ may be a strict subgroup, or equal to $N \cap \mathbf{A}^p(G)$, which leads to the first definition.

**Definition 3.7.** Let $P \in Syl_p(G)$ and $P \leq N$. Then $N$ *controls p-transfer* if $\mathbf{A}^p(G) \cap N = \mathbf{A}^p(N)$.

From this definition, we get that if $N$ controls $p$-transfer and $\mathbf{O}^p(N) < N$, then $\mathbf{O}^p(G) < G$. The next theorem connects $\mathbf{A}^p(G)$ with $\mathbf{O}^p(G)$.

**Theorem 3.4.** ***(Tate's Theorem)*** *Let $P \in Syl_p(G)$ and suppose $P \leq N \leq G$. Then $N \cap \mathbf{A}^p(G) = \mathbf{A}^p(N) \Longleftrightarrow N \cap \mathbf{O}^p(G) = \mathbf{O}^p(N)$.*

The proof is omitted here. Tate's Theorem can be proved using group cohomology, character theory, crossed-homomorphisms, or the transfer map itself. The next result provides equivalencies for controlling $p$-transfer.

**Lemma 3.5.** *Let $P \leq N \leq G$ be groups and $P \in Syl_p(G)$. Then the following are equivalent.*

*(a)* *N controls p-transfer* $\mathbf{A}^p(G) \cap N = \mathbf{A}^p(N)$

*(b)* $G/\mathbf{A}^p(G) \cong N/\mathbf{A}^p(N)$

*(c)* $|G : \mathbf{A}^p(G)| = |N : \mathbf{A}^p(N)|$

*(d)* $P \cap \mathbf{A}^p(G) = P \cap \mathbf{A}^p(N)$.

*Proof.* $(a) \Rightarrow (b)$: By Theorem 2.3, $N\mathbf{A}^p(G)/\mathbf{A}^p(G) \cong N/(N \cap \mathbf{A}^p(G))$. Since $G = P\mathbf{A}^p(G)$ and $P \leq N \leq G$, then $G = N\mathbf{A}^p(G)$. Since $N$ controls $p$-transfer, $\mathbf{A}^p(G) \cap N = \mathbf{A}^p(N)$. Thus, rewrite the isomorphism as $G/\mathbf{A}^p(G) \cong N/\mathbf{A}^p(N)$.

$(b) \Rightarrow (c)$: Since $G/\mathbf{A}^p(G) \cong N/\mathbf{A}^p(N)$, we must have that $|G : \mathbf{A}^p(G)| = |N : \mathbf{A}^p(N)|$.

$(c) \Rightarrow (d)$: Let $x \in P \cap \mathbf{A}^p(G)$. Then $x \in P$ and it suffices to show $x \in \mathbf{A}^p(N)$. Now $\mathbf{A}^p(N) \leq \mathbf{A}^p(G)$, and have the same $p$-power. Since $x$ is a $p$-element, this forces $x \in \mathbf{A}^p(N)$.

Let $x \in P \cap \mathbf{A}^p(N)$. Then $x \in P$ and $x \in \mathbf{A}^p(N) \leq \mathbf{A}^p(G)$, and so $x \in P \cap \mathbf{A}^p(G)$.

$(d) \Rightarrow (a)$: To show $N$ controls $p$-transfer, we need $\mathbf{A}^p(G) \cap N = \mathbf{A}^p(N)$.

Let $x \in \mathbf{A}^p(G) \cap N$. Suppose $|G| = p^n m$, $|N| = p^n k$, $|\mathbf{A}^p(G)| = p^i m$, and $|\mathbf{A}^p(N)| = p^i k$. Thus, the order of $x$ is either a power of $p$, or a $p'$ element. If $x$ is a power of $p$, then $x \in P$. By assumption, $x \in P \cap \mathbf{A}^p(G) = P \cap \mathbf{A}^p(N)$. Therefore, $x \in \mathbf{A}^p(N)$. If $x$ is a $p'$-element, then $x \in \mathbf{A}^p(N)$ since $\mathbf{A}^p(N)$ contains all of the $p'$-elements of $N$, which are a subset of the $p'$-elements of $G$.

Let $x \in \mathbf{A}^p(N)$. Now $\mathbf{A}^p(N) \leq \mathbf{A}^p(G)$, and $\mathbf{A}^p(N) \leq N$, so $x \in \mathbf{A}^p(G) \cap N$. □

Next, we can explore the connection between fusion control and controlling $p$-transfer.

**Theorem 3.5.** *Let $P \leq N \leq G$, $P \in Syl_p(G)$, and suppose $N$ controls fusion in $P$ with respect to $G$. Then $N$ controls $p$-transfer.*

*Proof.* By Lemma 3.5, we can show $N$ controls $p$-transfer by showing that $P \cap \mathbf{A}^p(G) = P \cap \mathbf{A}^p(N)$. Now, $P \cap \mathbf{A}^p(G) = Foc_G(P)$ by Corollary 3.1. Similarly, since $P \in Syl_p(N)$, $P \cap \mathbf{A}^p(N) = Foc_N(P)$. Thus, we can show $Foc_G(P) = Foc_N(P)$. Now $Foc_G(P) = \langle\, p^{-1}g^{-1}pg \mid p \in P, g \in G, g^{-1}pg \in P \,\rangle$ and $Foc_N(P) = \langle\, p^{-1}n^{-1}pn \mid p \in P, n \in N, n^{-1}pn \in P \,\rangle$. But $N$ controls fusion in $P$ with respect to $G$, so if $x, y \in P$ are conjugate in $G$, then they are conjugate in $N$. Therefore, $Foc_G(P) = Foc_N(P)$. □

The next result gives a scenario for the existence of a normal complement.

**Corollary 3.4.** *Let $P \in Syl_p(G)$ with $P$ abelian. Then $P \cap \mathbf{Z}(\mathbf{N}_G(P))$ has a normal complement.*

*Proof.* Now $P \trianglelefteq N_G(P)$ and $gcd(|P|, |\mathbf{N}_G(P) : P|) = 1$, so there exists $H \leq \mathbf{N}_G(P)$ such that $\mathbf{N}_G(P) = PH$ and $P \cap H = \{e\}$ by Theorem 2.9. Let $H$ act on $P$ by automorphisms under conjugation. Then $\mathbf{N}_G(P) = \mathbf{C}_P(H) \times H[P,H]$ by Theorem 2.8. But $\mathbf{C}_P(H) = \{\, p \in P \mid hp = ph \quad for \quad all \quad h \in H \,\} = P \cap \mathbf{Z}(\mathbf{N}_G(P))$. So we have $\mathbf{N}_G(P) = (P \cap \mathbf{Z}(\mathbf{N}_G(P))) \times H[P,H]$. Furthermore, $H[P,H] \trianglelefteq \mathbf{N}_G(P)$, and $\mathbf{N}_G(P)/(H[P,H]) \cong P \cap \mathbf{Z}(\mathbf{N}_G(P))$. Now $P \cap \mathbf{Z}(\mathbf{N}_G(P))$ is an abelian $p$-group, so $\mathbf{A}^p(\mathbf{N}_G(P)) \leq H[P,H]$. Additionally, $P = (\, P \cap \mathbf{Z}(\mathbf{N}_G(P))\,) \times [P,H]$ by Theorem 2.8.

By Corollary 3.3, $\mathbf{N}_G(P)$ controls fusion in $P$. Then, by Theorem 3.5, $\mathbf{N}_G(P)$ controls $p$-transfer. By Definition 3.7, $\mathbf{A}^p(G) \cap \mathbf{N}_G(P) = \mathbf{A}^p(\mathbf{N}_G(P)) \leq H[P,H]$. Hence, $\mathbf{A}^p(G) \cap (P \cap \mathbf{Z}(\mathbf{N}_G(P))) = \{e\}$. Now, $[P, \mathbf{N}_G(P)] \leq \mathbf{N}_G(P)' \leq \mathbf{A}^p(\mathbf{N}_G(P))$, so $P = (\, P \cap \mathbf{Z}(N_G(P))\,)[P, \mathbf{N}_G(P)] \leq (\, P \cap \mathbf{Z}(N_G(P))\,)\mathbf{A}^p(\mathbf{N}_G(P))$. Since $\mathbf{A}^p(\mathbf{N}_G(P)) \leq \mathbf{A}^p(G)$, we have $G = P\mathbf{A}^p(G) \leq (P \cap \mathbf{Z}(\mathbf{N}_G(P)))\mathbf{A}^p(N)\mathbf{A}^p(G) = P \cap \mathbf{Z}(\mathbf{N}_G(P))\mathbf{A}^p(G)$. Therefore, $\mathbf{A}^p(G)$ is a normal complement for $P \cap \mathbf{Z}(\mathbf{N}_G(P))$ in $G$. □

The next two corollaries provide examples for when a group is not simple. They follow immediately from the previous corollary.

**Corollary 3.5.** *Let $P \in Syl_p(G)$. Suppose $P$ is abelian and $P \cap \mathbf{Z}(\mathbf{N}_G(P)) \neq \{e\}$. Then $G$ is not simple. (We assume $G$ is not cyclic of order $p$.)*

*Proof.* By Corollary 3.4, $\mathbf{A}^p(G)$ is a nontrivial normal subgroup of $G$. Then $G$ is not simple. □

**Corollary 3.6.** *Let $P \in Syl_p(G)$. Suppose $P$ is abelian and $[P, \mathbf{N}_G(P)] < P$. Then $G$ is not simple.*

*Proof.* By Theorem 2.9, $P$ has a complement in $\mathbf{N}_G(P)$. Let's say $H \leq \mathbf{N}_G(P)$ such that $\mathbf{N}_G(P) = PH$ and $P \cap H = \{e\}$. Let $H$ act on $P$ by automorphisms. By Theorem 2.8, $P = \mathbf{C}_P(H) \times [P, H]$. Notice that $[P, H] \leq [P, \mathbf{N}_G(P)] < P$. Thus, $P \cap \mathbf{Z}(\mathbf{N}_G(P)) = \mathbf{C}_P(H) \neq \{e\}$. Therefore, by Corollary 3.5, $G$ is not simple. □

We move on to another definition.

**Definition 3.8.** Let $W \leq H \leq G$. Then $W$ is *weakly closed in $H$ with respect to $G$* if the only $G$-conjugate of $W$ contained in $H$ is $W$ itself *(i.e., for all $g \in G$ such that $g^{-1}Wg \leq H$, then $g^{-1}Wg = W$).*

Suppose $W \leq H \leq G$, $W$ is weakly closed in $H$ with respect to $G$, and consider $N_G(H)$. Now $W \leq H \trianglelefteq N_G(H)$. Now for $x \in N_G(H)$ we have $x^{-1}Wx \leq x^{-1}Hx = H$. Since $W$ is weakly closed, we must have that $W = x^{-1}Wx \leq H$, and $W \trianglelefteq N_G(H)$.

**Example 3.6.** Let $G = D_4$ as defined in Example 3.1, $H = \{e, p^2, \tau_1, \tau_2\}$, and $W = \{e, \tau_1\}$, and consider $g = p$. Now $W \neq g^{-1}Wg = \{e, \tau_2\} \leq H$. So we have an example where $W$ is not weakly closed in $H$ with respect to $G$.

What is more interesting is when $W$ is weakly closed. The next lemma provides an example.

**Lemma 3.6.** *Let $W \leq P \leq G$ where $P \in Syl_p(G)$. Then $W$ is weakly closed in $P$ with respect to $G$ if and only if*

*(a)* $W \trianglelefteq \mathbf{N}_G(P)$, *and*

*(b)* *If $W \leq x^{-1}Px$ then $W \trianglelefteq x^{-1}Px$ for $x \in G$.*

*Proof.* $\Rightarrow$: Suppose that $W$ is weakly closed in $P$ with respect to $G$. Part $(a)$ follows immediately, and is shown in the notes following Definition 3.8. Now suppose that $x \in G$ and $W \leq x^{-1}Px$. Through appropriate multiplication, $xWx^{-1} \leq P$. Since $W$ is weakly closed in $P$, $xWx^{-1} = W$. Furthermore, $W \leq P \trianglelefteq \mathbf{N}_G(P)$, and $W \trianglelefteq \mathbf{N}_G(P)$ by *Part (a)*, so $W \trianglelefteq P$. This implies that $x^{-1}Wx \trianglelefteq x^{-1}Px$. Hence $W = x^{-1}Wx \trianglelefteq x^{-1}Px$.

$\Leftarrow$: We now assume that $W \trianglelefteq \mathbf{N}_G(P)$ and if $W \leq x^{-1}Px$ then $W \trianglelefteq x^{-1}Px$ for $x \in G$. Let $g \in G$ such that $g^{-1}Wg \leq P$. Then $W \leq gPg^{-1}$, and by $(b)$ $W \trianglelefteq gPg^{-1}$. Now $W \trianglelefteq \mathbf{N}_G(W)$, and by part $(a)$ $W \trianglelefteq \mathbf{N}_G(P)$, so $\mathbf{N}_G(P) \leq \mathbf{N}_G(W)$. Thus $P \leq \mathbf{N}_G(P) \leq \mathbf{N}_G(W)$. Since $P \in Syl_p(G)$, $P \in Syl_p(\mathbf{N}_G(W))$. So $gPg^{-1} \in Syl_p(\mathbf{N}_G(W))$ as well. By Theorem 2.1, there must exist $h \in G$ such that $h^{-1}Ph = gPg^{-1}$. Hence, $g^{-1}h^{-1}Phg = P$, so $hg \in \mathbf{N}_G(P) \leq \mathbf{N}_G(W)$. Thus, $g \in \mathbf{N}_G(W)$, Therefore, if $g^{-1}Wg \leq P$, then $g^{-1}Wg = W$, and $W$ is weakly closed in $P$ with respect to $G$. □

The next result provides an interesting link between a subgroup being weakly closed and fusion control, and in turn controlling $p$-transfer.

**Theorem 3.6.** *Suppose $W$ is weakly closed in $P$ with respect to $G$ where $P \in Syl_p(G)$. Furthermore, assume that $W \leq \mathbf{Z}(P)$. Then $\mathbf{N}_G(W)$ controls fusion in $P$ with respect to $G$, and therefore controls $p$-transfer.*

*Proof.* Let $x, y \in P$ and suppose that $g^{-1}xg = y$ for some $g \in G$. We want to show that $x$ and $y$ are conjugate in $\mathbf{N}_G(W)$. Now $W \leq \mathbf{Z}(P)$ by assumption, so we must have that $W \leq \mathbf{Z}(P) \leq \mathbf{C}_G(x)$ and $W \leq \mathbf{Z}(P) \leq \mathbf{C}_G(y)$ since $x, y \in P$. Now $g^{-1}Wg \leq g^{-1}\mathbf{C}_G(x)g = \mathbf{C}_G\left(g^{-1}xg\right) = \mathbf{C}_G(y)$. Since $W \leq P$, it consists of $p$-elements. There exists $Q \in Syl_p(\mathbf{C}_G(y))$ by Theorem 2.1 such that $W \leq Q$. Also by Theorem 2.1, there exists $a \in G$ such that $a^{-1}Qa \leq P$. Combining these, $a^{-1}Wa \leq a^{-1}Qa \leq P$. Similarly, there exists $c \in \mathbf{C}_G(y)$ such that $c^{-1}\left(g^{-1}Wg\right)c \leq Q$. So $a^{-1}\left(c^{-1}g^{-1}Wgc\right)a \leq a^{-1}Qa \leq P$.

By assumption $W$ is weakly closed in $P$ with respect to $G$, and we have that $a^{-1}Wa \leq P$ and $(gca)^{-1}W(gca) \leq P$. Thus, $a^{-1}Wa = W = (gca)^{-1}W(gca)$. This implies that $W =$

$(gc)^{-1}W(gc)$, and that $gc \in \mathbf{N}_G(W)$. Now $(gc)^{-1}x(gc) = c^{-1}g^{-1}xgc = c^{-1}yc = c^{-1}cy = y$ since $c \in \mathbf{C}_G(y)$. Hence, $x$ and $y$ are conjugate in $\mathbf{N}_G(W)$, so $\mathbf{N}_G(W)$ controls fusion in $P$ with respect to $G$ by Definition 3.6. By Theorem 3.5, $\mathbf{N}_G(W)$ controls $p$-transfer. □

This result occurs for a subgroup of $\mathbf{Z}(P)$ where $P \in Syl_p(G)$. But the $\mathbf{Z}(P)$ itself could be weakly closed in $P$ with respect to $G$. The following definition and theorem explore this idea.

**Definition 3.9.** Now $\mathbf{Z}(P) \leq P \leq G$ where $P \in Syl_p(G)$. If for all $S \in Syl_p(G)$ such that $\mathbf{Z}(P) \leq S$, $\mathbf{Z}(P) = \mathbf{Z}(S)$ then $G$ is called *p-normal*.

**Lemma 3.7.** *Suppose $P \in Syl_p(G)$. Then $G$ is p-normal if and only if $\mathbf{Z}(P)$ is weakly closed in $P$ with respect to $G$.*

*Proof.* $\Rightarrow$: Assume first that $G$ is $p$-normal. There exists $g \in G$ so that $g^{-1}\mathbf{Z}(P)g \leq P$. This implies that $\mathbf{Z}(P) \leq gPg^{-1} = S$ for some $S \in Syl_p(G)$. Since $G$ is $p$-normal, $\mathbf{Z}(P)$ is the center of $S$. If $g$ acts on both terms, then $g^{-1}\mathbf{Z}(P)g$ is the center of $g^{-1}Sg$. But $g^{-1}Sg = P$. Therefore, $\mathbf{Z}(P) = g^{-1}\mathbf{Z}(P)g$ and $\mathbf{Z}(P)$ is weakly closed in $P$ with respect to $G$.

$\Leftarrow$: Suppose now that $\mathbf{Z}(P)$ is weakly closed in $P$ with respect to $G$. Suppose $\mathbf{Z}(P)$ is contained in some Sylow $p$-subgroup $S$. By Sylow, Theorem 2.1, there exists a $g \in G$ such that $P = g^{-1}Sg$. Now $g^{-1}\mathbf{Z}(P)g$ is contained in $P$. Then $\mathbf{Z}(P) = g^{-1}\mathbf{Z}(P)g$ since $\mathbf{Z}(P)$ is weakly closed in $P$. If $\mathbf{Z}(S)$ is the center of $S$, then $g^{-1}\mathbf{Z}(S)g$ is the center of $g^{-1}Sg = P$. Hence $\mathbf{Z}(P) = g^{-1}\mathbf{Z}(P)g = g^{-1}\mathbf{Z}(S)g$. So $\mathbf{Z}(P) = \mathbf{Z}(S)$, and $G$ is $p$-normal.[2] □

The next theorem, by Burnside, provides another approach for determining weak closure. In this case, however, we consider $W$ when it is not weakly closed.

**Theorem 3.7.** *Let $P \in Syl_p(G)$. Suppose $W \trianglelefteq P$ and $W$ is not weakly closed in $P$ with respect to $G$. Then there exists a p-group $H$ such that:*

*(a) $W \trianglelefteq H$, and*

(b) *there exists an* $x \in \mathbf{N}_G(H)$, *where* $x$ *is a* $p'$*-element, but* $x^{-1}Wx \neq W$.

*Proof.* There are two cases to consider.

*Case 1:* Suppose $W \ntrianglelefteq \mathbf{N}_G(P)$. Now $\mathbf{N}_G(P)$ is generated by $P$ and its $p'$-elements. But $W \trianglelefteq P$, so there must exist a $p'$ element $x \in \mathbf{N}_G(P)$ such that $x^{-1}Wx \neq W$. Therefore, set $H = P$.

*Case 2:* Now suppose $W \trianglelefteq \mathbf{N}_G(P)$. By assumption, $W$ is not weakly closed in $P$ with respect to $G$ so there exists a $P_1 \in Syl_p(G)$ such that $W \leq P$ but $W \ntrianglelefteq P_1$. Thus, we can define a nonempty set $S$, where

$$S = \left\{Q,R \mid Q,R \in Syl_p(G), W \leq Q \cap R, W \trianglelefteq Q, W \ntrianglelefteq R\right\}.$$

Select a pair $(Q,R) \in S$ of maximal order. We will show that $H = Q \cap R$ is a $p$-group that satisfies both Part $(a)$ and $(b)$.

Since $Q,R \in Syl_p(G)$, $Q \cap R$ is a $p$-group. Let $h \in Q \cap R$. So $h \in Q$ and $h \in R$. Now $W \trianglelefteq Q$, so $x^{-1}Wx = W$. Hence, $W \trianglelefteq (Q \cap R)$, and Part $(a)$ is done.

Now $Q \cap R < Q$ and $Q \cap R < R$ by definition of $S$. So $Q \cap R < \mathbf{N}_Q(Q \cap R)$ and $Q \cap R < \mathbf{N}_R(Q \cap R)$ since normalizers grow in $p$-groups. Hence, $Q \cap R < [\mathbf{N}_G(Q \cap R)] \cap Q$ and $Q \cap R < [\mathbf{N}_G(Q \cap R)] \cap R$. Now select $Q^* \in Syl_p(\mathbf{N}_G(Q \cap R))$ such that $[\mathbf{N}_G(Q(\cap R))] \cap Q \leq Q^*$ and $R^* \in Syl_p(\mathbf{N}_G(Q \cap R))$ such that $[\mathbf{N}_G(Q(\cap R))] \cap R \leq R^*$. Applying Theorem 2.1 in $N$, there exists $n \in \mathbf{N}_G(Q \cap R)$ such that $n^{-1}Q^*n = R^*$. Again using Theorem 2.1, but in $G$, there exists $Q_0 \in Syl_p(G)$ such that $Q^* \leq Q_0$. Fix $R_0 = n^{-1}Q_0n$, and so $R^* \leq R_0$.

Now $Q \cap R < [\mathbf{N}_G(Q \cap R)] \cap Q \leq Q \cap Q^* \leq Q \cap Q_0$. Using the definition of $S$, $W \leq Q \cap R \leq Q \cap Q_0$, so $W \leq Q_0$ and $W \trianglelefteq Q$. Now $Q \cap R$ is chosen maximal, but $Q \cap R \leq Q \cap Q_0$, so $(Q,Q_0) \notin S$. Since $W \leq Q \cap Q_0$, and $W \trianglelefteq Q$, it must be that $W \trianglelefteq Q_0$. Since $Q^* \leq Q_0$, $W \trianglelefteq Q^*$. Using a similar argument, $W \leq Q \cap R \leq R \cap R_0$. Contradicting the maximality of $(Q,R)$ again, $(R,R_0) \notin S$. Here, however, $W \ntrianglelefteq R$, so $W \ntrianglelefteq R_0 = n^{-1}Q_0n$. But $Q_0$ normalizes $W$, so $n$ does not normalize $W$. Since $n \in \mathbf{N}_G(Q \cap R)$, $\mathbf{N}_G(Q \cap R)$ does not normalize $W$.

Putting it all together, $W \trianglelefteq Q^*$, where $Q^* \in Syl_p(\mathbf{N}_G(Q \cap R))$, but then $W \ntrianglelefteq \mathbf{N}_G(Q \cap R)$. Since $\mathbf{N}_G(Q \cap R)$ is generated by $Q^*$ and its $p'$-elements, there must exist $x \in \mathbf{N}_G(Q \cap R)$, where $x$ has $p'$ order, but $x^{-1}Wx \neq W$. □

Theorem 3.7 says there exists $x \in \mathbf{N}_G(P \cap Q)$ such that $x^{-1}Wx \neq W$ where $W$, $P$, and $Q$ satisfy the Theorem. Consider the map $\phi : (P \cap Q) \to (P \cap Q)$ defined by $\phi(g) = x^{-1}gx$ for all $g \in (P \cap Q)$. The kernel of this map consists of the elements $k$ such that $\phi(k) = x^{-1}kx = e$. Rearranging this equation gives that $k = e$ and so the map is one-to-one. To show onto, there needs to exist an element $h \in (P \cap Q)$ such that $\phi(h) = x^{-1}hx = g$ for every $g \in (P \cap Q)$. But $x \in \mathbf{N}_G(P \cap Q)$, so $x^{-1}(P \cap Q)x = (P \cap Q)$, so the element $h$ must exist. Furthermore, since $x^{-1}Wx \neq W$, $W \trianglelefteq (P \cap Q)$, but $W \trianglelefteq \mathbf{N}_G(P \cap Q)$, this map must be nontrivial. Thus, the map $\phi$ induces a non-trivial $p'$-automorphism of $P \cap Q$. Combining the fact that $\mathbf{C}_G(P \cap Q) \leq \mathbf{N}_G(P \cap Q)$ and the existence of the $p'$-element $x \in \mathbf{N}_G(P \cap Q)$, $\mathbf{N}_G(P \cap Q)/\mathbf{C}_G(P \cap Q)$ is not a $p$-group. We refer to $\mathbf{N}_G(P \cap Q)/\mathbf{C}_G(P \cap Q)$ as the *automizer* of $P \cap Q$. More generally, $\mathbf{N}_G(H)/\mathbf{C}_G(H)$ is the automizer of $H$ where $H \leq G$, and is the group of automorphisms of $H$, $Aut(H)$.

This gives a sufficient condition for weak closure. If $W \trianglelefteq P^*$ and the automizer of $P \cap Q$ is a $p$-group, where $P$ and $Q$ satisfy the $S$ of Theorem 3.7 and are of maximal order, then $W$ is weakly closed in $P^*$ with respect to $G$. Frobenius' Normal $p$-Complement Theorem does not require we consider this specific automizer be used. One last lemma is required to prove Frobenius' Theorem.

**Lemma 3.8.** *Suppose $G$ is a group that has a normal $p$-complement. Then $H \leq G$ also has a normal $p$-complement.*

*Proof.* Since $G$ has a normal $p$-complement, there exists $N \trianglelefteq G$ such that the order of $N$ is relatively prime to $p$, and $|G : N|$ is a power of $p$. Let $H \leq G$. We need a group $K$ such that $K \trianglelefteq H$, the order of $K$ is relatively prime to $p$, and $|H : K|$ is a power of $p$. Consider $H/(H \cap N)$. Since $N \trianglelefteq G$, $(H \cap N) \trianglelefteq H$. Now $H \leq G$, so $H$ could be a $p$-group, a $p'$-group, or neither. If $H$ is a $p$-group, then $|H \cap N| = 1$ since $N$ is comprised only of $p'$-elements. Hence, $|H \cap N|$ is relatively prime to $p$, and $|H : (H \cap N)|$ is a power of $p$. If $H$ is a $p'$-group, then $|H \cap N| = |H|$ since $N$ contains every $p'$-element. Thus, $|H \cap N|$ is relatively prime to $p$, and $|H : (H \cap N)| = 1$ which is equivalent to $p^0$. Lastly, if $H$ is a combination of $p$-elements and $p'$-elements, then $|H \cap N|$ is equal to the $p'$ order of $H$. So $|H \cap N|$ is relatively prime to $p$, and $|H : (H \cap N)|$ is a power of $p$. In any situation, $H \cap N$ is a normal $p$-complement in $H$. $\square$

Note that this result can also be shown for quotient groups. Frobenius' Theorem removes the restriction that the Sylow $p$-subgroup be abelian, and accordingly can be considered a generalization of Theorem 3.3, Burnside's Normal $p$-Complement Theorem.

**Theorem 3.8.** ***(Frobenius' Normal p-Complement Theorem)*** *Let G be a group. The following are equivalent:*

*(a) G has a normal p-complement*

*(b)* $\mathbf{N}_G(H)$ *has a normal p-complement for all p-subgroups* $H \leq G$ *such that* $H \neq \{e\}$

*(c)* $\mathbf{N}_G(H)/\mathbf{C}_G(H)$ *is a p-group for every p-subgroup* $H \leq G$.

*Proof.* $(a) \Rightarrow (b)$ : By Lemma 3.8, $\mathbf{N}_G(H)$ has a normal $p$-complement since $\mathbf{N}_G(H) \leq G$ for all $H \leq G$ for $H \neq \{e\}$.

$(b) \Rightarrow (c)$ : If $H = \{e\}$, then $\mathbf{N}_G(H)/\mathbf{C}_G(H)$ is trivial. Suppose then that $H$ is a nonidentity $p$-subgroup of $G$. Let $K$ be the normal $p$-complement of $\mathbf{N}_G(H)$. Now $H$ is normalized by $K$ since $K \leq \mathbf{N}_G(H)$, and $K$ is normalized by $H$ since $K \trianglelefteq \mathbf{N}_G(H)$.

Consider $[H,K] = \left\langle h^{-1}k^{-1}hk \mid h \in H, k \in K \right\rangle$ and $H \cap K$. Since $K$ is the normal $p$-complement, $H \cap K = \{e\}$. So if $x \in H \cap K$, $x = e$, and so $H \cap K \leq [H,K]$. Let $x \in [H,K]$. Then there exists $h \in H$ and $k \in K$ such that $x = h^{-1}k^{-1}hk$. Now $h^{-1}\left(k^{-1}hk\right) \in H$ since $H$ is normalized by $K$ and $H$ is a group. Also, $\left(h^{-1}k^{-1}h\right)k \in K$ since $K$ is normalized by $H$ and $K$ is a group. Hence, $x \in H \cap K$. Thus, $[H,K] = H \cap K$=$\{e\}$.

Now $h^{-1}k^{-1}hk = e$. Rearranging gives $hk = kh$, and so $H$ and $K$ centralize each other. Thus $K \leq \mathbf{C}_G(H) \leq \mathbf{N}_G(H)$. Now $K$ must contain all $p'$-elements of $\mathbf{N}_G(H)$ since it is the normal $p$-complement, so $\mathbf{C}_G(H)$ must contain all the $p'$-elements as well. Thus $\mathbf{N}_G(H)/\mathbf{C}_G(H)$ is a $p$-group for every $p$-subgroup $H \leq G$.

$(c) \Rightarrow (a)$ : Assume $p$ divides the order of $G$, and all automizers, $\mathbf{N}_G(H)/\mathbf{C}_G(H)$, are $p$-subgroups where $H \leq G$ is a $p$-subgroup. Proceed by induction on the order of $G$.

**Trivial Case:** For small orders of $G$, the statement is true vacuously.

**Initial Induction:** Consider $|G| = pq$, where $p$ and $q$ are distinct primes such that $p < q$. The possible number of Sylow $q$-subgroups, $n_q$, is $1, p, q$, or $pq$. But $q$ divides $n_q - 1$, so $n_q = 1$, and $Q \trianglelefteq G$. Thus, $G$ has a normal $p$-complement, $Q$.

**Automizer Condition Holds for All Subgroups:** Let $H$ be a $p$-group such that $H \leq K \leq G$. Let $\phi : \mathbf{N}_K(H) \to \mathbf{N}_G(H)$ be defined by $\phi(x) = x$ for $x \in \mathbf{N}_K(H)$. Define $\psi : \mathbf{N}_G(H) \to \mathbf{N}_G(H)/\mathbf{C}_G(H)$ by $\psi(n) = n\mathbf{C}_G(H)$ where $n \in \mathbf{N}_G(H)$. Composing these functions gives a map $\psi(\phi) : \mathbf{N}_K(H) \to \mathbf{N}_G(H)/\mathbf{C}_G(H)$ defined by $\psi(\phi(x)) = x\mathbf{C}_G(H)$ where $x \in \mathbf{N}_K(H)$. Let $k$ be in the kernel of the composed map, so that $\psi(\phi(k)) = \mathbf{C}_G(H)$. Thus, $k \in \mathbf{C}_K(H)$. Let $x$ be an element of the intersection $\mathbf{N}_K(H) \cap \mathbf{C}_G(H)$. Then $x$ centralizes the elements of $H$, but is contained in $K$, so $x \in \mathbf{C}_K(H)$. Suppose instead that $x \in \mathbf{C}_K(H)$. Then $x \in \mathbf{C}_G(H)$, and $x \in \mathbf{N}_G(H)$ since it will normalize $H$. So $\mathbf{N}_K(H) \cap \mathbf{C}_G(H) = \mathbf{C}_K(H)$. Using the definitions of the respective groups, $\mathbf{N}_K(H)/\mathbf{C}_K(H)$ is a subgroup of $\mathbf{N}_G(H)/\mathbf{C}_G(H)$, and so is itself a $p$-group. By the inductive hypothesis, $K$ has a normal $p$-complement, and therefore all proper subgroups of $G$ have a normal $p$-complement.

With these cases verified, the induction can be built up to the order of $G$ starting with quotient groups.

**Automizer Condition Holds for Quotient Groups of Normal $p$-Subgroups:** Let $U$ be a $p$-subgroup such that $U \trianglelefteq G$. Let $H/U$ be a $p$-subgroup of $G/U$ for some $p$-subgroup $H \leq G$. Consider the groups $\mathbf{C}_{G/U}(H/U)$ and $C/U$ for some subgroup $C$. If $\mathbf{C}_{G/U}(H/U) = C/U$ then $C$ contains at least those elements that commute with elements of $H$ and normalize $U$, so $\mathbf{C}_G(H) \leq C$. Now, $\mathbf{N}_{G/U}(H/U)$ consists of elements from $G/U$ that normalize $H/U$, and $\mathbf{N}_G(H)/U$ consists of cosets of $U$ from elements that normalize $H$. So $\mathbf{N}_{G/U}(H/U) = \mathbf{N}_G(H)/U = \mathbf{N}_G(U)/U$, with the last equality holding since $U \trianglelefteq G$. Using the above equalities $\mathbf{N}_{G/U}(H/U)/\mathbf{C}_{G/U}(H/U) = (\mathbf{N}_G(U)/U)/(C/U)$. Furthermore, $C \trianglelefteq \mathbf{N}_G(U)$, so by Theorem 2.4 $(\mathbf{N}_G(U)/U)/(C/U) \cong \mathbf{N}_G(U)/C$. Now a map $\tau : \mathbf{N}_G(U)/\mathbf{C}_G(H) \to \mathbf{N}_G(U)/C$ defined by $\tau(n\mathbf{C}_G(U)) = nC$ is onto since $\mathbf{C}_G(H) \leq C$. Thus $\mathbf{N}_G(U)/C$ is a $p$-group since it is the homomorphic image of a $p$-group. By the inductive hypothesis, $\mathbf{N}_G(U)/C$ has a normal $p$-complement. Thus, proper homomorphic images of a normal $p$-group has a normal $p$-complement, and the automizer condition is inherited by $G/U$.

**Quotient Groups of Prime Power Order:** Let $K \trianglelefteq G$ be a proper subgroup of $G$ such that $G/K$ is a $p$-group. By the inductive hypothesis, $K$ has a normal $p$-complement. Call this normal $p$-complement $N$. Suppose $\sigma \in Aut(K)$ and let $n \in N$. Then $|n| = |\sigma(n)| = |\langle \sigma(n) \rangle|$. Now consider $|\sigma(n)N|$. Now $|\sigma(n)N|$ divides $|\sigma(n)|$, which in turn divides $|n|$, which divides $|N|$. So $|\sigma(n)N|$ divides $|N|$. But $\sigma(n) \in K$, so $|\sigma(n)N|$ also divides $|K : N|$. But $gcd(|N|, |K : N| = 1)$ since $N$ is a normal $p$-complement, so $|\sigma(n)N| = 1$. This implies $\sigma \in Aut(N)$, and so $N$ is characteristic in $K$. Consider $\psi \in Aut(G)$ defined by $\psi(x) = g^{-1}xg$. Now $K \trianglelefteq G$, so $\psi(K) = g^{-1}Kg = K$, and this implies $\psi \in Aut(K)$. But $N$ is characteristic in $K$, so $\psi(N) = g^{-1}Ng = N$, and thus $N \trianglelefteq G$. Now the order of $K$ contains all the $p'$-elements of $G$, and some $p$-elements as well. $N$ is a normal $p$-complement in $K$, so it contains all the $p'$-elements of $K$, and thus $G$, so $N$ is a normal $p$-complement in $G$.

What remains to be shown is the $G$ will always contain a normal $p$-complement.

**The Group $G$:** Let $P \in Syl_p(G)$ and consider $\mathbf{Z}(P)$. $P/\mathbf{Z}(P)$ is a $p$-group, so by Theorem 3.7, and the notes following the proof, $\mathbf{Z}(P)$ is weakly closed in $P$. Using Theorem 3.6, $\mathbf{N}_G(\mathbf{Z}(P))$ controls fusion in $P$ with respect to $G$, and thus controls $p$-transfer. There are two cases to consider. The first is where $\mathbf{N}_G(\mathbf{Z}(P)) < G$ and the second where $\mathbf{N}_G(\mathbf{Z}(P)) = G$.

If $\mathbf{N}_G(\mathbf{Z}(P)) < G$ then $\mathbf{N}_G(\mathbf{Z}(P))$ has a normal $p$-complement by the induction. Say $\mathbf{N}_G(\mathbf{Z}(P)) = KN$ where $\mathbf{N}_G(\mathbf{Z}(P))/K$ is a $p$-group and $gcd(|K|, |\mathbf{N}_G(\mathbf{Z}(P)) : K|) = 1$. By Sylow, Theorem 2.1, $\mathbf{N}_G(\mathbf{Z}(P))/K$ has an abelian $p$-subgroup $V/K$. Now $V/K \leq \mathbf{N}_G(\mathbf{Z}(P))/\mathbf{A}^p(\mathbf{N}_G(\mathbf{Z}(P)))$, of which the latter is the largest abelian $p$-group. So now $\mathbf{A}^p(\mathbf{N}_G(\mathbf{Z}(P))) \leq K < G$. Since $\mathbf{N}_G(\mathbf{Z}(P))$ controls $p$-transfer, $\mathbf{A}^p(\mathbf{N}_G(\mathbf{Z}(P))) = \mathbf{A}^p(G)$ $\cap \mathbf{N}_G(\mathbf{Z}(P))$. Now $\mathbf{A}^p(\mathbf{N}_G(\mathbf{Z}(P))) < \mathbf{N}_G(\mathbf{Z}(P)) < G$ which implies $\mathbf{A}^p(G) < G$, so the quotient group is proper, and by the last paragraph, $G$ has a normal $p$-complement.

If $\mathbf{N}_G(\mathbf{Z}(P)) = G$, then $\mathbf{Z}(P) \trianglelefteq G$ and is nontrivial proper $p$-subgroup since $P$ is a Sylow $p$-subgroup. From the earlier induction, $G/\mathbf{Z}(P)$ has a normal $p$-complement of the form $K/\mathbf{Z}(P)$ for some $K \trianglelefteq G$. Note that $K/\mathbf{Z}(P)$ has $p$-power index since it a normal $p$-complement. By quotient groups of power $p$ section, if $K < G$ then $G$ has a normal $p$-complement. Consider the case where $K = G$. If $\mathbf{Z}(P)$ is not a Sylow $p$-subgroup $G$, then the order of $G/\mathbf{Z}(P)$ is divisible by both $p$ and the $p'$ power of $G$. Now $K/\mathbf{Z}(P)$ contains only the full $p'$ power of $G$, so $K/\mathbf{Z}(P) < G/\mathbf{Z}(P)$, but this is a contradiction since $K = G$. Thus $\mathbf{Z}(P) \in Syl_p(G)$. This means $\mathbf{N}_G(\mathbf{Z}(P))/\mathbf{C}_G(\mathbf{Z}(P)) = G/\mathbf{C}_G(\mathbf{Z}(P))$ is a $p$-group by hypothesis. But $\mathbf{Z}(P) \leq \mathbf{C}_G(\mathbf{Z}(P))$, so $\mathbf{C}_G(\mathbf{Z}(P))$ contains the full $p$-power of $G$, and so $p$ does not divide the order of $G/\mathbf{C}_G(\mathbf{Z}(P))$. Thus $G = \mathbf{C}_G(\mathbf{Z}(P))$. Now $\mathbf{Z}(P) \leq \mathbf{Z}(G) = \mathbf{Z}(\mathbf{N}_G(\mathbf{Z}(P)))$. Therefore, by Theorem 3.3, $G$ has a normal $p$-complement. □

Following Frobenius' Theorem, there are several more normal complement theorems to explore.

**Corollary 3.7.** *Let $G$ be a group with order $|G| = p^a m$ where $p$ is a prime not dividing $m$, and that no prime divisor of $m$ divides $p^j - 1$ for $1 \leq j \leq a$. Then $G$ has a normal $p$-complement.*

*Proof.* Assume $|G| = p^a m$ so that the proposition is satisfied. Suppose that $\mathbf{N}_G(H)/\mathbf{C}_G(H)$ is not a $p$-group for some $H \leq G$ where $H$ is a $p$-group. Then there exists an element $x$ with

$p'$-order such that $x \in \mathbf{N}_G(H)$ but $x \notin \mathbf{C}_G(H)$. So $|x\mathbf{C}_G(H)| = q$ for some prime $q \neq p$. Let $\sigma \in Aut(H)$ such that $\sigma$ is effected by conjugation of $x$. Then $|\langle\sigma\rangle| = q$, and so all orbits have order 1 or $q$. The elements in $\mathbf{C}_H(\sigma)$ are those that have order 1. Now $|H|$ is the sum of the all the orbits. Suppose $|H| = p^u$ and $|\mathbf{C}_H(\sigma)| = p^v$. So $q \mid |H| - |\mathbf{C}_H(\sigma)| = p^u - p^v = p^v(p^{u-v} - 1)$. Thus, $q$ divides $p^{u-v} - 1$, which is a contradiction. Therefore, by Theorem 3.8, $G$ has a normal $p$-complement. □

The latter normal complement theorems require a new definition, which makes use of the focal subgroup from Definition 3.3.

**Definition 3.10.** Let $H \leq G$ be groups. The *focal series of H in G is the sequence defined by:*

$$
\begin{aligned}
H_1 &= H \\
H_2 &= Foc_G(H_1) \\
&\vdots \\
H_n &= Foc_G(H_{n-1}).
\end{aligned}
$$

$H$ is called *hyperfocal in G* if there exists an $n$ such that $H_n = \{e\}$. The notation $H_{k+1} = Foc_G^k(H)$ is the composition of the function $Foc_G(H)$ with itself $k$ times.

Consider the following example.

**Example 3.7.** Let $G = D_4$ as defined in Example 3.1 and $H = \left\{e, p^2, \tau_1, \tau_2\right\}$. Now $H_1 = H = \left\{e, p^2, \tau_1, \tau_2\right\}$. Using Example 3.3, $H_2 = Foc_G(H_1) = \left\{e, p^2\right\}$. Next, determine $H_3 = Foc_G(H_2)$. If $h = e$, then $g^{-1}eg = e$, so $h^{-1}g^{-1}hg = e^{-1}e = e \in Foc_G(H_2)$. Alternatively, if $h = p^2$, then $g^{-1}p^2g = p^2$ since $p^2$ is in its own conjugacy class from Example 3.3, so $h^{-1}g^{-1}hg = p^2p^2 = e \in Foc_G(H_2)$. Thus, $H_3 = \{e\}$, and the focal series of $H$ in $G$ is complete. Furthermore, since $H_3 = \{e\}$, $H$ is hyperfocal in $G$.

The next result shows that hyperfocal Hall subgroups have normal complements.

**Theorem 3.9.** *Suppose $H$ is a hyperfocal Hall subgroup of $G$. Then $H$ has a normal complement in $G$.*

*Proof.* This proof proceeds by induction on $|H|$.

For $|H| = |H_1| = 1$, the case is trivial. So assume $|H| > 1$. The induction hypothesis states that if $H$ is hyperfocal in a group, then the group has a normal complement.

Suppose $M$ is the kernel of the transfer map $V : G \to H/H'$. Since $M$ is the kernel, $M \trianglelefteq G$. By Theorem 3.2, $M \cap H = Foc_G(H)$. Building the focal series, $H_1 = H$, and $H_2 = Foc_G(H_1) = M \cap H$. Since $H$ is hyperfocal and nontrivial, $H_2 < H$. Consider indices to show $G = HM$. Let $|G| = mn$, where $m$ and $n$ are relatively prime, and suppose $|H| = m$ as a Hall subgroup. By the First Isomorphism Theorem, Theorem 2.2, $G/M$ is isomorphic to the range of $V$, which is a subset of $H/H'$, and so will have order $\frac{m}{d}$ for some divisor $d$ of $m$. So $G/M$ will have the same order, and this implies $|M| = nd$. Thus, $G = HM$ and there are $d$ elements that overlap in the two subgroups. Now $H_2 = M \cap H$, which has order $d$, and so $H_2$ is Hall subgroup of $M$.

Now $Foc_M^k(H_2) \leq Foc_G^k(H_2) \leq Foc_G^{k+1}(H) = \{e\}$ since $M \leq G$ and $H$ is hyperfocal in $G$. By induction, $H_2$ is hyperfocal in $M$. Thus $H_2$ has a normal complement in $M$, call it $K$. Now $K \trianglelefteq M$, $M = H_2K$, and $H_2 \cap K = \{e\}$ by definition, and using the orders above, $|K| = n$. This means $K$ is a normal Hall subgroup of $M$. Now $K$ is a characteristic subgroup of $M$ based on order arguments from being a Hall subgroup, as in Theorem 3.8. As before, $K \trianglelefteq G$.

Lastly, verify that $K$ is a normal complement in $G$. Now $KH = KH_2H = MH = G$, since $H_2 \leq H$ and $M = H_2K$ . Thus $G = HK$. Next, $K \cap H = K \cap M \cap H = K \cap H_2 = \{e\}$, for the same reasons of the previous line, and since $K$ complements $H_2$ in $M$. Since $K \trianglelefteq G$ from the previous paragraph, $G$ has a normal $p$-complement. □

The next Theorem continues to make use of Hall subgroups, and builds off the previous result.

**Theorem 3.10.** *Suppose $H$ is a nilpotent Hall subgroup of $G$ such that $H$ controls fusion in itself with respect to $G$. Then $G$ has a normal complement.*

*Proof.* If $H$ is hyperfocal in $G$ then by Theorem 3.9 $G$ has a normal $p$-complement. Begin with the focal series of $H$ in $G$, denoted by $H = H_1 \geq H_2 \geq H_3 \geq \ldots$. Proceed by induction to prove $H_k \leq [H,H,\ldots H]$ ($k$ times). For $k = 1$, the containment $H_1 \leq H = [H]$ is trivial. Assume $H_k \leq [H,H,\ldots,H]$ ($k$ times).

Consider now $H_{k+1} = Foc_G(H_k) = \left\langle x^{-1}g^{-1}xg \mid x \in H_k, g \in G, g^{-1}xg \in H_k \right\rangle$. Now $x \in H$, so by hypothesis $g^{-1}xg = y = h^{-1}xh$ for some $h \in H$.

Hence, $H_{k+1} = \left\langle x^{-1}g^{-1}xg \mid x \in H_k, g \in G, g^{-1}xg \in H_k \right\rangle$

$$= \left\langle x^{-1}h^{-1}xh \mid x \in H_k, h \in H, g^{-1}xg \in H_k \right\rangle$$

$$\leq \langle [k,h] \mid k \in H_k, h \in H \rangle$$

$$= [H_k, K]$$

$$\leq [H,H,\ldots,H,H] \; k+1 \text{ times, based on the induction.}$$

Now $H$ is nilpotent, so its central series converges to the trivial group, which implies $H_k = \{e\}$ for some $k$. Therefore, $H$ is hyperfocal in $G$ as desired, so $G$ has a normal complement.

□

The following Corollary is immediate from the prior Theorem.

**Corollary 3.8.** *Suppose $P \in Syl_p(G)$ and that $P$ controls fusion in itself with respect to $G$. Then $G$ has a normal $p$-complement.*

*Proof.* Since $P \in Syl_p(G)$, it is a Hall subgroup based on order. From Lemma 2.1, $P$ is nilpotent. By Theorem 3.10, $G$ has a normal $p$-complement. □

The following Theorem gives another situation where $G$ has a normal complement. Now $H \leq G$ need only satisfy certain conditions, rather than be a particular type of group.

**Theorem 3.11.** *Suppose $H \leq G$. Assume that for every prime $p$ that divides the order of $H$, where $P \in Syl_p(H)$ that $H = \mathbf{N}_G(P)$. Suppose as well that $H$ is not a $p$-group. Then $G$ has a normal $p$-complement.*

*Proof.* Assume the order of $H$ is at least comprised of two distinct primes, as otherwise the proof is trivial. Suppose $P \notin Syl_p(G)$. Then $P < Q$ for some $Q \in Syl_p(G)$. Since normalizers grow in $p$-groups, $P < \mathbf{N}_Q(P)$. But $P \trianglelefteq H = \mathbf{N}_G(P)$. This is a contradiction based on the orders of the groups. Therefore $P \in Syl_p(G)$, and $P$ is a Hall subgroup of $G$. But $P \trianglelefteq H$ for all $p$, so $H$ is $p$-closed. By Theorem 2.6, $H$ is also nilpotent.

Let $P_i \in Syl_{p_i}(H)$ for all primes that divide the order of $H$. Then denote $H = P_1 \times P_2 \times \cdots \times P_3$. Let $x, y \in H$ such that $g^{-1}xg = y$ for some $g \in G$. Now $x = x_1x_2 \ldots x_k$ and $y = y_1y_2 \ldots y_k$ where each $x_i, y_i \in P_i$. Since each prime $p$ is distinct, each $x_i$ is a fixed power of $x$, and each $y_i$ is the same fixed power of $y$. Say this power is $m$. Then $g^{-1}x_ig = g^{-1}x^mg = g^{-1}xg \ldots g^{-1}xg(m \quad times) = \left(g^{-1}xg\right)^m = y^m = y_i$ for all $i$. By Corollary 3.2, $H = \mathbf{N}_G(P_i)$ controls fusion in $\mathbf{C}_G(P_i)$ for all $i$. Let $x \in P_i$, and $y \in P_j$ for $i \neq j$. Now $x \in \mathbf{N}_G(P_i) = H = \mathbf{N}_G\left(P_j\right)$, and similarly $y \in \mathbf{N}_G(P_i)$. So $y^{-1}x^{-1}yx \in P_i$ since $y$ normalizes $P_i$ and similarly $y^{-1}x^{-1}yx \in P_j$. But $P_i \cap P_j = \{e\}$ so $yx = xy$. Therefore $P_j \leq \mathbf{C}_G(P_i) \leq \mathbf{N}_G(P_i)$. By the notes following Definition 3.6, $H$ controls fusion in each $P_j$ as well. Thus, there exists $h_i \in H$ such that $h_i^{-1}x_ih_i = y_i$ for all $i$. Since $h_j$ will centralize $x_i$ and $y_i$ for $i \neq j$, it be assumed that each $h_i \in P_i$. Repeating this process for all $i$, can write $h = h_1h_2 \ldots h_k \in H$.

Now $h^{-1}x_ih = h_k^{-1} \cdots h_1^{-1}x_ih_1 \cdots h_k$

$= h_k^{-1} \cdots h_{i+1}^{-1}h_i^{-1}x_ih_ih_{i+1} \cdots h_k$ since $h_j$ centralizes $x_i$ for $i \neq j$

$= h_k^{-1} \cdots h_{i+1}^{-1}y_ih_{i+1} \cdots h_k$ since $h_i^{-1}x_ih_i = y_i$

$= y_i$ since $h_j$ centralizes $y_i$ for $i \neq j$.

Thus, $h^{-1}xh = y$. Hence, $H$ controls fusion $H$ with respect to $G$. By Theorem 3.10, $G$ has a normal complement. □

## 3.4 Thompson's Normal $p$-Complement Theorem

The last $p$-complement theorem to consider is Thompson's Normal $p$-Complement Theorem, which generalizes Frobenius' Theorem, Theorem 3.8. Thompson's original proof is simplified by making use of $p$-solvable groups and Puig subgroups. We define $\pi$-groups and $\pi$-solvable groups first.

**Definition 3.11.** Let $G$ be a group and $\pi$ be a set of primes. Then $G$ is a *$\pi$-group* if its order is divisible only by the primes in $\pi$.

**Definition 3.12.** Let $G$ be a group and $\pi$ be a set of primes. Then $G$ is *$\pi$-separable* if it has a chain of subgroups $\{K_i\}$ which satisfy the following:

$$\{e\} = K_0 \leq K_1 \leq \cdots \leq K_n = G$$
$$K_i \trianglelefteq G \text{ for } 0 \leq i \leq n, \text{ and}$$
$$K_{i+1}/K_i \text{ is either a } \pi\text{-group or a } \pi'\text{-group for } 0 \leq i < n.$$

The number of strict inclusions in the series of a $\pi$-group is referred to as the $\pi$-length. A $\pi'$-group is a group whose order is not divisible by any primes in the set $\pi$. Now $\mathbf{O}_{\pi'}(G)$ represents the largest normal $\pi'$-subgroup of $G$, and that $\mathbf{O}_{\pi',\pi}(G)$ is the complete inverse image of $G$ in $\mathbf{O}_{\pi}(G/\mathbf{O}_{\pi'}(G))$. Note that $\mathbf{O}_{\pi'}(G)$ is a characteristic subgroup of $G$. The next result relates $\mathbf{C}_G(\mathbf{O}_{\pi}(G))$ to $\mathbf{O}_{\pi}(G)$ in a $\pi$-separable group. The special case to consider is where $\pi$ consists of only one prime $p$. In this situation $\pi$-solvable means $p$-solvable, and is the interest of this thesis.

**Theorem 3.12.** *Suppose $G$ is a $\pi$-separable group and $\mathbf{O}_{\pi'}(G) = \{e\}$. Then $\mathbf{C}_G(\mathbf{O}_{\pi}(G)) \leq \mathbf{O}_{\pi}(G)$.*

*Proof.* Say $C = \mathbf{C}_G(\mathbf{O}_{\pi}(G))$. Suppose to the contrary that $C \nsubseteq \mathbf{O}_{\pi}(G)$. This implies that $C$ is not a $\pi$-group. Let $Z = C \cap \mathbf{O}_{\pi}(G)$. Now $Z = \mathbf{O}_{\pi}(C)$ and $Z = \mathbf{Z}(\mathbf{O}_{\pi}(G))$ based on the definition of the different groups. Let $K = \mathbf{O}_{\pi,\pi'}(C)$. Now $K$ consists of the elements that map to $\mathbf{O}_{\pi,\pi'}(C)$, and $Z$ is the largest normal $\pi$-group of $C$. This implies $Z < K$. Since $Z = \mathbf{Z}(\mathbf{O}_{\pi}(G))$, it is normal in $K$, and is a Hall subgroup of $K$ as well. Then by Theorem 2.9, there exists $H \leq K$ such that $K = ZH$ and $Z \cap H = \{e\}$. Now $Z = \mathbf{Z}(\mathbf{O}_{\pi}(G)) \leq \mathbf{O}_{\pi}(G)$ and $H \leq K \leq C$, so $[Z,H] \leq [\mathbf{O}_{\pi}(G), C] = \{e\}$. This implies $K = Z \times H$. Since $H$ is a complement in $K$, $H = \mathbf{O}_{\pi'}(K)$. But $\mathbf{O}_{\pi'}(K)$ is characteristic in $K$, so $H$ is characteristic in $K$, so $H \trianglelefteq G$. Since $H$ is a $\pi'$-group, $H \leq \mathbf{O}_{\pi'}(G) = \{e\}$. This is a contradiction, so $C \leq \mathbf{O}_{\pi}(G)$. □

Next consider the definition of a $p$-constrained group, and its relation to a $p$-solvable group.

**Definition 3.13.** Suppose $G$ is a group and let $p$ be any prime. Let $\overline{G} = G/\mathbf{O}_{p'}(G)$. Then $G$ is *p-constrained* if $\mathbf{C}_{\overline{G}}\left(\mathbf{O}_p\left(\overline{G}\right)\right) \leq \mathbf{O}_p\left(\overline{G}\right)$.

**Theorem 3.13.** *Assume $G$ is a $p$-solvable group for a prime $p$. Then $G$ is $p$-constrained.*

*Proof.* Consider $\overline{G} = G/\mathbf{O}_{p'}(G)$. Now $\mathbf{O}_{p'}\left(G/\mathbf{O}_{p'}(G)\right) = \{e\}$. By Theorem 3.12, $\mathbf{C}_{\overline{G}}\left(\mathbf{O}_p\left(\overline{G}\right)\right) \leq \mathbf{O}_p\left(\overline{G}\right)$. □

There is one last definition and theorem to consider before discussing the Puig Subgroup. These define a strongly solvable $p$-subgroup, and describe its action on a $p$-group. First, the definition of what it means for a group to be involved in another group.

**Definition 3.14.** Let $G$ and $A$ be groups. Then $A$ is involved in $G$ if there exists $H \leq G$ and $N \trianglelefteq H$ such that $H/N \cong A$.

**Definition 3.15.** Let $G$ be a $p$-solvable group. Then $G$ is *strongly p-solvable* if $p \geq 5$, or if $p = 3$, then $SL_2(3)$ is not involved in $G$.

**Theorem 3.14.** *Let $G$ be a strongly $p$-solvable group, where $p$ is an odd prime. Let $H \leq G$ be a $p$-subgroup. Then $G$ acts $p$-stably on $H$ if the action is by automorphisms.*

The next concept to explore is a special relationship between subgroups of $G$. Suppose $G$ is a group and $X$ and $Y$ are subgroups of $G$. If $Y$ is generated by abelian subgroups of $G$, which are each normalized by $X$, then $X \rightsquigarrow Y$. The unique largest subgroup $Y$ which satisfies this condition for $X$, is denoted by $L_G(X)$. By definition, $L_G(X) = \langle A \leq G \mid A \quad is \quad abelian, \quad X \leq \mathbf{N}_G(A) \rangle$.

Construct a sequence of subgroups $\{L_i\}$ of $G$. First define $L_0(G) = \{e\}$. Then $L_{n+1}(G) = L_G(L_n(G))$. The Puig Subgroup can now be defined.

**Definition 3.16.** The *Puig Subgroup of* $G$ is denoted by $L(G)$, and $L(G) = \bigcap_{n \geq 0} L_{2n+1}(G)$, i.e., it is the intersection of the $L_n(G)$ subgroups where $n$ is odd.

Note that $\mathbf{ZL}(\mathbf{X}) = \mathbf{Z}(L(X))$ for notation simplicity. Two final theorems are necessary to prove Thompson's Normal $p$-Complement Theorem.

**Theorem 3.15.** *Let $H \leq G$, and suppose $L(G) \leq H$. Then $L(G) = L(H)$, and $L(G)$ is a characteristic subgroup of $H$.*

**Theorem 3.16.** ***(Puig's** $Z(L)$ **Theorem)*** *Suppose $G$ is a $p$-constrained group such that $\mathbf{O}_{p'}(G) = \{e\}$. Let $G'$s action on every normal $p$-subgroup be $p$-stable. If $P \in Syl_p(G)$ then $\mathbf{ZL}(P) \trianglelefteq G$.*

At last, Thompson's Normal $p$-Complement Theorem.

**Theorem 3.17.** ***(Thompson's Normal** $p$**-Complement Theorem)*** *Suppose $G$ is a group, and let $P \in Syl_p(G)$ where $p$ is an odd prime. If $\mathbf{N}_G(\mathbf{ZL}(P))$ has a normal $p$-complement, then $G$ has a normal $p$-complement as well.*

*Proof.* This proof proceeds by contradiction. Suppose $G$ is the minimal group that does not satisfy the theorem. Then $\mathbf{N}_G(\mathbf{ZL}(P))$ has a normal $p$-complement, but $G$ does not. Fix $P \in Syl_p(G)$.

Define the collection

$$\mathscr{H} = \left\{ H \;\middle|\; \begin{array}{l} H \leq G \text{ is a } p\text{-subgroup of } G, H \neq \{e\}, \text{ and} \\ \mathbf{N}_G(H) \text{ does not have a normal } p\text{-complement} \end{array} \right\}.$$

Suppose $\mathscr{H} = \emptyset$. Then for every $H$ satisfying the above, $\mathbf{N}_G(H)$ has a normal complement. By Frobenius, Theorem 3.8, $G$ has a normal $p$-complement as well. Hence, $\mathscr{H} \neq \emptyset$. Thus, select $H \in \mathscr{H}$ satisfying

(i) $|\mathbf{N}_G(H)|_p$ is maximal, and

(ii) $|H|$ is maximal, subject to $(i)$.

The elements of $\mathscr{H}$ have an implied ordering, and the chosen $H$ is a maximal element with regards to this ordering.

**Step One:** Show that $|\mathbf{N}_G(H)|_p = |G|_p = |P|$.

Let $Q \in Syl_p(\mathbf{N}_G(H))$. Note that $Q$ and $H$ can be replaced by conjugates, if necessary, so that $Q \leq P$. Assume that $Q < P$. Since normalizers grow in $p$-groups, $Q < \mathbf{N}_P(Q) \leq \mathbf{N}_G(Q) \leq \mathbf{N}_G(\mathbf{ZL}(Q))$, the last containment holds since $\mathbf{ZL}(Q)$ is characteristic in $Q$. Now $|\mathbf{N}_G(H)|_p = |Q| < |\mathbf{N}_G(\mathbf{ZL}(Q))|$. Given the choice of $H \in \mathscr{H}$, $\mathbf{ZL}(Q) \notin \mathscr{H}$, which implies $\mathbf{N}_G(\mathbf{ZL}(Q))$ has a normal $p$-complement. Since normal $p$-complements are inherited by subgroups, Lemma 3.8, $\mathbf{N}_{\mathbf{N}_G(H)}(\mathbf{ZL}(Q))$ has a normal $p$-complement also. Now $\mathbf{N}_G(H)$ does not have a normal $p$-complement by definition, so $\mathbf{N}_G(H)$ is a counterexample to the theorem statement. But $Q < P$ implies $\mathbf{N}_G(H) \neq G$. This contradicts $G$ being the minimal counterexample. Thus $Q = P$ and Step One is done.

**Step Two:** Show that $H = \mathbf{O}_p(G)$, $p \mid |G : H|$, and $\mathbf{O}_{p'}(G) = \{e\}$.

Continue from Step One above, and replace $Q$ with $P$ where necessary. Since $\mathbf{N}_G(H)$ is a counterexample to the theorem, and $G$ is the minimal counterexample, $\mathbf{N}_G(H) = G$. Now $H \trianglelefteq \mathbf{N}_G(H) = G$, so $H \trianglelefteq G$. Since $H$ is a $p$-group, $H \leq \mathbf{O}_p(G)$ since $\mathbf{O}_p(G)$ is the largest normal $p$-subgroup of $G$. Based on the maximality of the the chosen $H$, $H = \mathbf{O}_p(G)$. Thus, $\mathbf{O}_p(G) \in \mathscr{H}$.

Assume that $H = P$. Then $\mathbf{ZL}(P) = \mathbf{ZL}(H) \trianglelefteq G$ since $\mathbf{ZL}(H)$ is characteristic in $H \trianglelefteq G$. So $\mathbf{ZL}(P) \trianglelefteq G$, which implies $\mathbf{N}_G(\mathbf{ZL}(P)) = G$. Since $\mathbf{N}_G(\mathbf{ZL}(P))$ has a normal $p$-complement, $G$ has a normal $p$-complement, which is a contradiction, so $H < P$. Hence $p \mid |G : H|$.

Set $U = \mathbf{O}_{p'}(G)$, and suppose that $U$ is nontrivial. Denote $\overline{G} = G/U$. The natural map from $P \to \overline{P} = PU/U$ is isomorphic. This implies that $\mathbf{ZL}(P) \cong \mathbf{ZL}\left(\overline{P}\right)$, and in turn signifies that $\overline{\mathbf{ZL}(P)} = \mathbf{ZL}\left(\overline{P}\right)$. Hence, $\mathbf{N}_{\overline{G}}\left(\mathbf{ZL}\left(\overline{P}\right)\right) = \mathbf{N}_{\overline{G}}\left(\overline{\mathbf{ZL}(P)}\right) = \mathbf{N}_G(\mathbf{ZL}(P)U)/U$. Now $\mathbf{N}_G(\mathbf{ZL}(P)U) = \mathbf{N}_G(\mathbf{ZL}(P))U$, by the Frattini argument. Hence $\mathbf{N}_{\overline{G}}\left(\mathbf{ZL}\left(\overline{P}\right)\right) = \mathbf{N}_G(\mathbf{ZL}(P))U/U$. The latter is the homomorphic image of $\mathbf{N}_G(\mathbf{ZL}(P))$, and so has a normal $p$-complement. This means $\mathbf{N}_{\overline{G}}\left(\mathbf{ZL}\left(\overline{P}\right)\right)$ has a normal $p$-complement as well, and $\left|\overline{G}\right| < |G|$ since $U \neq \{e\}$. Therefore, $\overline{G}$ is not a counterexample, and so it must have a normal $p$-complement. But $p \nmid |U|$ and $\overline{G} = G/U$, so $G$ must have a normal $p$-complement. This is a contradiction, so $\mathbf{O}_{p'}(G) = \{e\}$.

Step Two has been proved.

**Step Three:** Show that $G/H$ has a normal $p$-complement, and so $G$ is $p$-solvable with $p$-length two.

Let $\mathbf{ZL}(P/H) = W/H$. Now $W/H \neq \{e\}$, so $H < W$. Now $W \trianglelefteq P$, so $P \leq \mathbf{N}_G(W)$. This implies $|\mathbf{N}_G(W)|_p = |P|$, and $|H| < |W|$. Thus, $W \notin \mathscr{H}$ based on choice of $H$, so $\mathbf{N}_G(W)$ has a normal $p$-complement. Now $\mathbf{N}_G(W)/H$ has a normal $p$-complement as well. So $\mathbf{N}_{G/H}(\mathbf{ZL}(P/H))$ has a normal $p$-complement since $\mathbf{N}_{G/H}(\mathbf{ZL}(P/H)) = \mathbf{N}_G(W/H) = \mathbf{N}_G(W)/H$. But $|G/H| < |G|$, so $G/H$ is not a counterexample to the theorem. Thus $G/H$ has a normal $p$-complement. This means $G/H = (P/H)(K/H)$, where $K/H$ is the normal $p$-complement. So there is a series $H \leq K \leq G$ satisfying Definition 3.12. Therefore $G$ is $p$-solvable with $p$-length two.

Step Three is done.

**Step Four:** Show that $P$ is a maximal subgroup of $G$, and $\mathbf{O}_{p,p'}(G)/\mathbf{O}_p(G)$ is an abelian $q$ group, for a prime $q$.

Let $K = \mathbf{O}_{p,p'}(G)$. Now $K/H$ is the normal $p$-complement of $G/H$ by Step Three above, since $K$ is complete inverse image in $G$ of $\mathbf{O}_{p'}(G/\mathbf{O}_p(G))$.

Suppose there exists a group $G_0$ such that $P < G_0 < G$. Thus $P \in Syl_p(G_0)$, and $\mathbf{N}_{G_0}(\mathbf{ZL}(P)) \leq \mathbf{N}_G(\mathbf{ZL}(P))$. Since the latter has a normal $p$-complement, by Lemma 3.8, $\mathbf{N}_{G_0}(\mathbf{ZL}(P))$ has a normal $p$-complement as well. Since $G_0 < G$ and $G$ is the minimal counterexample to the theorem, $G_0$ must have a normal $p$-complement. Denote this $p$-complement by $M$. Since $P < G_0$, $G_0$ must contain some $p'$-elements, and so $M \neq \{e\}$. Now $H \trianglelefteq G$ from Step Two, so $H \trianglelefteq G_0$, and $M \trianglelefteq G_0$ by definition. Thus, $[H,M] = \left\langle h^{-1}m^{-1}hm \mid h \in H, m \in M \right\rangle \leq H \cap M$. But $H \cap M = \{e\}$ since $H$ is a $p$-group and $M$ is a $p'$-group. Thus $h^{-1}m^{-1}hm = e$ which implies that $M \leq C_G(H)$. Now $G$ is $p$-solvable by Step Three, and $\mathbf{O}_{p'}(G) = \{e\}$ by Step Two, so by Theorem 3.12, $\mathbf{C}_G(\mathbf{O}_\pi(G)) \leq \mathbf{O}_\pi(G)$. Here, $H = \mathbf{O}_p(G) = \mathbf{O}_\pi(G)$, so $C_G(H) \leq H$. Therefore, $M \leq \mathbf{C}_G(H) \leq H$ which is a contradiction based on the orders of $H$ and $M$. Hence, $G_0$ does not exist, which implies that $P$ is maximal in $G$.

Suppose $q$ is a prime such that $q \mid |K/H|$, and let $P$ act on $Syl_q(K/H)$. By Sylow, Theorem 2.1, $p \nmid \left|Syl_q(K/H)\right|$, so $P$ must stabilize an element of $K/H$. Say this element is $L/H$, and let $\mathbf{Z}(L/H) = V/H$. Note $H < V$. Now $V/H \leq L/H$ and $P$ stabilizes $L/H$, so $P$ must normalize $V$. So $PV$ is a group, and $P < PV$. This implies $G = PV$ since $P$ is maximal in $G$. Thus, $V = L = K$, and so combining the above properties means $K/H$ is an abelian $q$-group.

Since $K = \mathbf{O}_{p,p'}(G)$ and $H = \mathbf{O}_p(G)$, Step Four has been shown.

**Step Five:** Show that $G$ is strongly $p$-solvable.

From Step Three, $G$ is $p$-solvable. If $p \geq 5$ then $G$ is strongly $p$-solvable by definition. Suppose then that $p = 3$, and consider $q$ and $K/H$ above. If $q$ is odd, then $K/H$ is trivial, and $SL_2(3)$ is not involved in $G$. If $q = 2$, then $K/H$ is abelian by Step Four. The Sylow 2-subgroup of $SL_2(3)$ is not abelian, and so $SL_2(3)$ is not involved. In any case, $G$ is strongly $p$-solvable.

Step Five is complete.

**Step Six:** Show that $G = \mathbf{N}_G(\mathbf{ZL}(P))$, a final contradiction.

By Theorem 3.13, $G$ is $p$-constrained. By Theorem 3.14, $G$ acts by automorphism $p$-stably on $p$-groups. Hence, $\mathbf{ZL}(P) \trianglelefteq G$ by Theorem 3.16 since $\mathbf{O}_{p'}(G) = \{e\}$. This implies that $\mathbf{N}_G(\mathbf{ZL}(P)) = G$, a contradiction. Therefore, $G$ must have a normal $p$-complement. □

This concludes Thompson's Normal $p$-Complement Theorem, the last of the three big theorems in this thesis. All of the theorems require powerful results in order to be proved, and it is interesting to see how they build upon one another.

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