PRINCIPAL SERIES REPRESENTATIONS OF GL(2) OVER FINITE FIELDS

by

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ABSTRACT

The goal of this thesis is to construct the principal series representations of $GL(2)$. We do this in three parts. The aim of Part II is to become acquinted with representation theory. We observe rudimentary results and examples using prerequisite knowledge from linear algebra and group theory. We begin Part III by inducing new representations from old ones. A key component in the classification of induced representations is Mackey's theorem. A generous portion of this thesis is dedicated to the proof of Mackey's theorem. At last is Part IV where we construct the principal series representations. This constuction is motivated by the Bruhat decomposition of $GL(2)$ into the Borel subgroup and is achieved by counting the conjugacy classes of $GL_2(\mathbb{F}_q)$.

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Contents

List of Tables **vi**

I Introduction **1**

II Preliminaries of Representation Theory **2**

1 Review of Linear Algebra **2**

2 Basic Definitions and Examples **4**

3 Maschke's Theorem **11**

4 Characters **18**

III Induced Representation **30**

5 Basis of Induced Representation **33**

6 Frobenius Reciprocity **36**

7 Mackey's Theorem **39**

IV Representations of $GL_2(\mathbb{F}_q)$ **45**

8 The Borel Subgroup **45**

9 The Main Result **48**

10 Conjugacy Classes of $GL_2(\mathbb{F}_q)$ **52**

References **58**

List of Tables

1 Analogies between groups, vector spaces, and representations 10

2 Character Table of S_4 25

3 Character Table of the Induced Representation of S_4 39

Part I

Introduction

Representation theory studies the ways in which groups act on vector spaces. By studying actions on vector spaces, we obtain further information about a group by means of linear transformations and various properties of matrices. In general, the subject of representation theory is significant because it reduces a probem in abstract algebra to a problem in linear algebra.

One of the problems in representation theory is to construct and classify all irreducible representations. This problem is welcomed by Theorem 3. The eventual goal of this thesis is to partially solve this problem. We do this by constructing principal series representations of $GL(2)$.

Part II comprises of the building blocks of representation theory and is heavily based on [2]. We are formally introduced to the concept of construction in Example 2.8 where we construct all irreducible representations of $\mathbb{Z}_n$. We complete this part of this thesis with a brief encounter of character theory which is essential in the study of representation theory.

Part III details the way in which we construct representations of a group by the representations of its subgroup. An important component in induced representations is Mackey's theorem 7.2. We will see that an induced irreducible representation need not be irreducible; we employ Mackey's theorem to determine its irreducibility. The proof of Mackey's theorem provided in this thesis is based on exercises in the published lecture notes [1].

Part IV concludes this thesis. Based on [1], we constuct the principal series representations of $GL_2(\mathbb{F}_q)$. We do this by the Bruhat decomposition of $GL_2(\mathbb{F}_q)$, the Borel subgroup, and our knowledge of double cosets. Lastly, we will illustrate that the classification the conjugacy classes of $GL_2(\mathbb{F}_q)$ is the indeed the classification of irreducible representations of $GL_2(\mathbb{F}_q)$.

Part II

Preliminaries of Representation Theory

1 Review of Linear Algebra

In this section, we review the linear algebra which shall be assumed or needed for future reference.

Definition 1.1. Let V be a complex vector space. An *inner product* on V is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$$

satisfying that for each $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\lambda \in \mathbb{C}$,

1. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$,

   i.e., $\langle \cdot, \cdot \rangle$ is positive definite;

2. $\langle \mathbf{v}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{v} \rangle}$,

   i.e., $\langle \cdot, \cdot \rangle$ is conjugate symmetric;

3. $\langle \mathbf{v} + \mathbf{w}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{w}, \mathbf{u} \rangle$,

   i.e., $\langle \cdot, \cdot \rangle$ is additive in the first argument;

4. $\langle \lambda \mathbf{v}, \mathbf{w} \rangle = \lambda \langle \mathbf{v}, \mathbf{w} \rangle$,

   i.e., $\langle \cdot, \cdot \rangle$ is homogeneous in the first argument.

The pair $(V, \langle \cdot, \cdot \rangle)$ is an *inner product space*.

Definition 1.2. For $\begin{bmatrix} v_1, \ldots, v_n \end{bmatrix}^T, \begin{bmatrix} w_1, \ldots, w_n \end{bmatrix}^T \in \mathbb{C}^n$, the *standard inner product* on $\mathbb{C}^n$ is

$$\left\langle \begin{bmatrix} v_1, \ldots, v_n \end{bmatrix}^T, \begin{bmatrix} w_1, \ldots, w_n \end{bmatrix}^T \right\rangle = \sum_{k=1}^{n} v_k \overline{w_k}.$$

Definition 1.3. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. If $\mathbf{u}, \mathbf{v} \in V$ with $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, then $\mathbf{u}, \mathbf{v}$ are *orthogonal* and we write $\mathbf{u} \perp \mathbf{v}$. For any $\mathbf{w} \in V$, the *orthogonal complement* to $\mathbf{w}$ is the set $\mathbf{w}^{\perp} = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0\}$. If W is a subspace of V, then the *orthogonal complement* of W is the subspace $W^{\perp} = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0 \ \forall \ \mathbf{w} \in W\}$.

Definition 1.4. A matrix $A \in M_n(\mathbb{C})$ is *orthogonal* if the column vectors of $A = \{\mathbf{a_1}, \ldots, \mathbf{a_n}\}$ are orthonormal, that is, if

$$\mathbf{a_j} \cdot \mathbf{a_k} = \mathbf{a_j}^T \mathbf{a_k} = \delta_{jk},$$

where

$$\delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k. \end{cases}$$

The right-most equality says that $A^T = A^{-1}$. Therefore, the *orthogonal group* is the collection of orthogonal matrices $O_n(\mathbb{C}) = \left\{A \in GL_n(\mathbb{C}) : A^T = A^{-1}\right\}$ and is a subgroup of $GL_n(\mathbb{C})$.

Definition 1.5. A matrix $A \in GL_n(\mathbb{C})$ is *unitary* if $A^* = A^{-1}$. Here, A^* is the *adjoint*, or *conjugate-transpose*, of A, defined by $A^* = \overline{A^T}$. The *unitary group* is $U_n(\mathbb{C}) = \left\{A \in GL_n(\mathbb{C}) : A^* = A^{-1}\right\}$.

Proposition 1.6. *Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{C}^n$. Then $A \in U_n(\mathbb{C})$ exactly when $\langle A\mathbf{v}, A\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$.*

Proof. Let $\langle \cdot, \cdot \rangle$ be the usual inner product on $\mathbb{C}^n$ and let $A \in U_n(\mathbb{C})$. For $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$,

$$\begin{aligned} \langle \mathbf{x}, A\mathbf{y} \rangle &= \sum_{j=1}^{n} \overline{x_j} \sum_{k=1}^{n} a_{jk} y_k \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} \overline{\overline{a_{jk}} x_j} y_k \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} \overline{a_{kj}^* x_j} y_k \\ &= \langle A^* \mathbf{x}, \mathbf{y} \rangle . \end{aligned}$$

So for each $A \in M_n(\mathbb{C})$, the adjoint A^* of A has the property that $\langle \mathbf{x}, A\mathbf{y} \rangle = \langle A^*\mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$.

Suppose $A \in U_n(\mathbb{C})$ and $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$. Then

$$\langle A\mathbf{v}, A\mathbf{w} \rangle = \langle A^* A\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle .$$

Thus, $\langle A\mathbf{v}, A\mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle$ for each $A \in U_n(\mathbb{C})$ and $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$.

Now, let $A \in GL_n(\mathbb{C})$. For each $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$, suppose $\langle A\mathbf{v}, A\mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle$. Let $B \in GL_n(\mathbb{C})$ such that $B = A^*A$. Considering the standard basis $\{\mathbf{e_1}, \dots, \mathbf{e_n}\}$ of $\mathbb{C}^n$,

$$\left\langle A\mathbf{e_i}, A\mathbf{e_j}\right\rangle = \left\langle A^*A\mathbf{e_i}, \mathbf{e_j}\right\rangle = \left\langle B\mathbf{e_i}, \mathbf{e_j}\right\rangle$$

where

$$\left\langle \mathbf{e_i}, \mathbf{e_j}\right\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

If $i = j$, then

$$1 = \langle \mathbf{e_i}, \mathbf{e_i}\rangle = \langle B\mathbf{e_i}, \mathbf{e_i}\rangle = \left\langle \left[b_{1i}, \dots, b_{ii}, \dots, b_{ni}\right]^T, \left[0, \dots, 1, \dots, 0\right]^T \right\rangle = b_{ii}.$$

If $i \neq j$,

$$0 = \left\langle B\mathbf{e_i}, \mathbf{e_j}\right\rangle = \left\langle \left[b_{1i}, \dots, b_{ji}, \dots, b_{ni}\right]^T, \left[0, \dots, 1, \dots, 0\right]^T \right\rangle = b_{ji}.$$

We have that $A^*A = B = I$ since

$$b_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

So for $A \in GL_n(\mathbb{C})$ and every $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$, if $\langle A\mathbf{v}, A\mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle$, then $A \in U_n(\mathbb{C})$.

Hence, $A \in U_n(\mathbb{C})$ exactly when $\langle A\mathbf{v}, A\mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle$. □

2 Basic Definitions and Examples

Assume G to be a finite group, and V to be a complex vector space.

Definition 2.1. The pair (φ, V) is a *representation* of G on V if $\varphi : G \to GL(V)$ is a homomorphism. The *degree* of φ is the dimension of V.

Rather than having a map from a group to the set of invertible linear transformations, we often have examples of a map from a group into the general linear group. The following lemma enables this abuse of notation.

Lemma 2.2. $GL(\mathbb{C}^n) \simeq GL_n(\mathbb{C})$ *(as groups).*

Proof. Define $\phi : GL_n(\mathbb{C}) \to GL(\mathbb{C}^n)$ by $\phi(A) = T_A$, where $T_A : \mathbb{C}^n \to \mathbb{C}^n$ is given by $T_A(\mathbf{v}) = A\mathbf{v}$ for every $A \in GL_n(\mathbb{C})$ and $\mathbf{v} \in \mathbb{C}^n$. Here, we note that T_A is linear and bijective. To show that $GL(\mathbb{C}^n) \simeq GL_n(\mathbb{C})$, we will show that ϕ is a bijective homomorphism. To begin, let A, $B \in GL_n(\mathbb{C})$ and $\mathbf{v} \in \mathbb{C}^n$. Because $T_A, T_B \in GL(\mathbb{C}^n)$, we have

$$\phi(A)\,\phi(B)(\mathbf{v}) = T_A(\mathbf{v})\,T_B(\mathbf{v}) = T_{AB}(\mathbf{v}) = \phi(AB)(\mathbf{v}).$$

Therefore, $\phi(A)\,\phi(B) = \phi(AB)$ and ϕ is a group homomorphism.

By way of definition, ϕ is surjective. To see that ϕ is injective, suppose $\phi(A) = \phi(B)$. So $\phi(A)(\mathbf{v}) = \phi(B)(\mathbf{v})$ for each $\mathbf{v} \in \mathbb{C}^n$. Then $T_A(\mathbf{v}) = T_B(\mathbf{v})$. So $A\mathbf{v} = B\mathbf{v}$ for each $\mathbf{v} \in \mathbb{C}^n$. Thus, $A = B$ and ϕ is injective.

Hence, since ϕ is a bijective homomorphism, $GL(\mathbb{C}^n) \simeq GL_n(\mathbb{C})$. □

Definition 2.3. Define $\varphi : S_n \to GL_n(\mathbb{C})$ on the standard basis by $\varphi(\sigma)(\mathbf{e_i}) = \mathbf{e}_{\sigma(\mathbf{i})}$. This is the *standard representation of* S_n. The matrix for $\varphi(\sigma)$ is obtained by permuting the rows of the identity matrix according to σ.

Example 2.4. Consider $\varphi : S_3 \to GL(\mathbb{C}^3)$. By Lemma 2.2, $GL(\mathbb{C}^3) \cong GL_3(\mathbb{C})$ and so we have the standard representation $\varphi : S_3 \to GL_3(\mathbb{C})$. Because the pair of cycles (1 2) and

$(1\ 2 \cdots n)$ generate S_n, we have that $(1\ 2)$ and $(1\ 2\ 3)$ generate S_3. Thus,

$$\varphi(1\ 2) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\varphi(1\ 2)^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \varphi(e)$$

$$\varphi(1\ 2\ 3) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\varphi(1\ 2\ 3)^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \varphi(1\ 3\ 2)$$

$$\varphi(1\ 2)\varphi(1\ 2\ 3) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \varphi(2\ 3)$$

$$\varphi(1\ 2)\varphi(1\ 2\ 3)^2 = \varphi(1\ 2)\varphi(1\ 3\ 2) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \varphi(1\ 3)$$

Definition 2.5. Let (φ, V) be a representation of G. A subspace W of V is *G-invariant* if $\varphi(g)(\mathbf{w}) \in W$ for all $g \in G$ and $\mathbf{w} \in W$.

More precisely, if W is a G-invariant subspace of V, then W is a *subrepresentation* of V. In this thesis, the two terms will be used interchangeably.

Definition 2.6. A representation is *irreducible* if it contains no proper and nontrivial subrepresentations.

Example 2.7. Define $\varphi : GL_2(\mathbb{C}) \to GL(\mathbb{C}^2)$ by $\varphi(A)(\mathbf{v}) = A\mathbf{v}$. Let W be a nontrivial $GL_2(\mathbb{C})$-invariant subspace of $\mathbb{C}^2$ and let $\mathbf{w} \in W$. We will show that φ is irreducible by showing that $W = \mathbb{C}^2$. We have three possible cases for $\mathbf{w}$.

Case 1. Say $\mathbf{w} = \begin{bmatrix} a \\ 0 \end{bmatrix}$, $a \neq 0$. Then $\frac{1}{a}\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W$. Because W is $GL_2(\mathbb{C})$-invariant, $\varphi(A)\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W$ for every $A \in GL_2(\mathbb{C})$. As $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a basis element of $GL_2(\mathbb{C})$, we have that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in W$. Therefore, $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ are two linearly independent vectors in W. Since $\dim(W) \leq \dim\left(\mathbb{C}^2\right) = 2$, $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is a basis for W. Hence, $W = \mathbb{C}^2$.

Case 2. Say $\mathbf{w} = \begin{bmatrix} 0 \\ b \end{bmatrix}$, $b \neq 0$. Then $\frac{1}{b}\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in W$. Since W is $GL_2(\mathbb{C})$-invariant, $\varphi(A)\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = A\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in W$ for every $A \in GL_2(\mathbb{C})$. And because $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a basis element of $GL_2(\mathbb{C})$, we have that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W$. Thus $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W$. But $\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ are two linearly independent vectors in W and therefore $\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ is a basis for W. Hence $W = \mathbb{C}^2$.

Case 3. Say $\mathbf{w} = \begin{bmatrix} a \\ b \end{bmatrix}$, $a, b \neq 0$ and consider $\begin{bmatrix} b & -a \\ 0 & 1 \end{bmatrix} \in GL_2(\mathbb{C})$. We have that $\begin{bmatrix} b & -a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix} \in W$ and thus $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} 0 \\ b \end{bmatrix} \in W$. Since $\begin{bmatrix} 0 \\ b \end{bmatrix} \in W$, then by the second case, $W = \mathbb{C}^2$.

Thus if W is a nontrivial $GL_2(\mathbb{C})$-invariant subspace of $\mathbb{C}^2$, then $W = \mathbb{C}^2$. Because $\mathbb{C}^2$ contains no proper and nontrivial $GL_2(\mathbb{C})$-invariant subspaces, φ is irreducible.

We will now find all irreducible representations of $\mathbb{Z}_n$.

Example 2.8. Suppose $\phi : \mathbb{Z}_n \to GL(V)$ is an irreducible representation of $\mathbb{Z}_n$. We will prove in Corollary 3.8 that the degree of an irreducible representation of an abelian group is one. So, since $\mathbb{Z}_n$ is an abelian group, $\deg(\phi) = 1$, and by the definition of degree,

$\dim(V) = 1$. Since $GL_1(V) \simeq \mathbb{C}^{\times}$, we will find all irreducible representations of $\mathbb{Z}_n$ by finding all homomorphisms of $\phi : \mathbb{Z}_n \to \mathbb{C}^{\times}$.

Let $\phi : \mathbb{Z}_n \to \mathbb{C}^{\times}$ be the homomorphism given by $\phi(0) = 1$. Put $\phi(1) = \lambda \in \mathbb{C}^{\times}$. Then

$$1 = \phi(0) = \phi(n) = \phi(1 + 1 + \cdots + 1) = \phi(1) + \phi(1) + \cdots + \phi(1) = \phi(1)^n = \lambda^n,$$

and thus $1 = \lambda^n$. However, since $1 = e^{2\pi i}$, then $\lambda^n = e^{2\pi i}$ and so $\lambda = \exp\left(\frac{2\pi i}{n}\right)$. Let $\lambda_0 = \exp\left(\frac{2\pi i}{n}\right)$. Following the above list of equalities, $\phi(1)$ is an n^{th} root of unity and thus we get n solutions: $\lambda_0^1, \lambda_0^2, \lambda_0^3, \ldots, \lambda_0^{n-1}, 1$. Therefore, for $m = 1, 2, \ldots, n-1, n$, we may define $\phi_m : \mathbb{Z}_n \to \mathbb{C}^{\times}$ by $\phi_m(1) = \phi(1)^m = \lambda^m$ where $\lambda^m = \exp\left(\frac{2\pi i m}{n}\right)$. Hence, $\phi_1, \phi_2, \ldots, \phi_{n-1}, \phi_n$ are all of the irreducible representations of $\mathbb{Z}_n$.

Definition 2.9. An *intertwiner* (*intertwining operator*) between two representations (φ, V) and (ψ, W) is a linear map $T : V \to W$ such that $\psi(g)(T(\mathbf{v})) = T(\varphi(g)(\mathbf{v}))$. The set of all intertwining operators is the subset $\mathrm{Hom}_G(V, W) \subseteq \mathrm{Hom}(V, W)$.

In other words, the linear map T commutes with the group action on G and we have the following commutative diagram.

$$\begin{array}{ccc} V & \xrightarrow{\varphi(g)} & V \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ W & \xrightarrow[\psi(g)]{} & W \end{array}$$

Proposition 2.10. *Let $T \in Hom_G(V, W)$. Then $Ker(T)$ is a subrepresentation of V and $T(V) = Im(T)$ is a subrepresentation of W.*

Proof. Let (φ, V) and (ψ, W) be representations of G and let $T \in \mathrm{Hom}_G(V, W)$. For each $\mathbf{v} \in \mathrm{Ker}(T)$ and $g \in G$,

$$T(\varphi(g)(\mathbf{v})) = \psi(g)(T(\mathbf{v})) = \psi(g)(\mathbf{0}) = \mathbf{0}.$$

So $\varphi(g)(\mathbf{v}) \in \mathrm{Ker}(T)$ and $\mathrm{Ker}(T)$ is G-invariant.

Now, say $\mathbf{w} \in \operatorname{Im}(T)$ such that $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v} \in V$. Then

$$\psi(g)(\mathbf{w}) = \psi(g)(T(\mathbf{v})) = T(\varphi(g)(\mathbf{v})),$$

and so $T(\varphi(g)(\mathbf{v})) = \psi(g)(\mathbf{w}) \in \operatorname{Im}(T)$. Thus $\operatorname{Im}(T)$ is G-invariant, and furthermore, $T(V) = \operatorname{Im}(T)$. □

Although $\operatorname{Hom}_G(V, W)$ is a subset of $\operatorname{Hom}(V, W)$ by definition, we have yet to describe $\operatorname{Hom}_G(V, W)$ as a vector space. We do this in the proposition below.

Proposition 2.11. *Let (φ, V) and (ψ, W) be representations of G. Then $Hom_G(V, W)$ is a subspace of $Hom(V, W)$.*

Proof. Let (φ, V) and (ψ, W) be representations of G. For each $g \in G$, $T_1, T_2 \in \operatorname{Hom}_G(V, W)$, and $\alpha_1, \alpha_2 \in \mathbb{C}$,

$$(\alpha_1 T_1 + \alpha_2 T_2)\varphi(g) = \alpha_1 T_1 \varphi(g) + \alpha_2 T_2 \varphi(g) = \alpha_1 \psi(g) T_1 + \alpha_2 \psi(g) T_2 = \psi(g)(\alpha_1 T_1 + \alpha_2 T_2).$$

Thus $(\alpha_1 T_1 + \alpha_2 T_2)\varphi(g) = \psi(g)(\alpha_1 T_1 + \alpha_2 T_2)$ and so $\alpha_1 T_1 + \alpha_2 T_2 \in \operatorname{Hom}_G(V, W)$. That is, $\operatorname{Hom}_G(V, W)$ has the additional structure of a vector space and hence $\operatorname{Hom}_G(V, W)$ is a subspace of $\operatorname{Hom}(V, W)$. □

Definition 2.12. Two representations (φ, V) and (ψ, W) are *equivalent*, and we write $\varphi \sim \psi$, if there exists $T \in \operatorname{Hom}_G(V, W)$ such that T is invertible.

Definition 2.13. Let (φ_1, V_1) and (φ_2, V_2) be representations of G. Then their direct sum

$$\varphi_1 \oplus \varphi_2 : G \to GL(V_1 \oplus V_2)$$

is given by

$$(\varphi_1 \oplus \varphi_2)(g)(\mathbf{v_1}, \mathbf{v_2}) = (\varphi_1(g)(\mathbf{v_1}), \varphi_2(g)(\mathbf{v_2})).$$

Table 1 briefly summarizes our current understanding of representation theory. In other words, the information we have obtained by studying group actions on a vector space are disguised in the following manner.

Table 1: Analogies between groups, vector spaces, and representations

Groups	Vector spaces	Representations
subgroup	subspace	G-invariant subspace
simple group	one-dimensional subspace	irreducible representation
direct product	direct sum	direct sum
isomorphism	isomorphism	equivalence

Definition 2.14. A representation (φ, V) of G is *completely reducible* if $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$, where each V_i is a nontrivial G-invariant subspace and φ_{V_i} is irreducible for all $i = 1, \ldots, n$.

Equivalently, φ is completely reducible if $\varphi \sim \varphi_1 \oplus \varphi_2 \oplus \cdots \oplus \varphi_n$, where each φ_i is an irreducible representation.

Definition 2.15. A representation (φ, V) of G is *decomposable* if $V = W_1 \oplus W_2$ where W_1 and W_2 are nontrivial G-invariant subspaces. Otherwise, V is *indecomposable.*

Lemma 2.16. *Let (φ, V) be equivalent to a decomposable representation. Then φ is decomposable.*

Proof. Let $\varphi : G \to GL(V)$ and $\psi : G \to GL(W)$ be two representations of G such that $\varphi \sim \psi$. Then there exists an isomorphism $T : V \to W$ such that $\psi(g) = T\varphi(g)T^{-1}$ for every $g \in G$. Assume W_1, W_2 are nontrivial G-invariant subspaces of W, and let V_1, V_2 be subspaces of V. Suppose ψ is decomposable. Thus, $W = W_1 \oplus W_2$. Define $T(V_i) = W_i$ for $i = 1, 2$.

Let $\mathbf{v} \in V$. Since $T : V \to W$, there exists $\mathbf{w} \in W$ such that $T(\mathbf{v}) = \mathbf{w}$. Since $W = W_1 \oplus W_2$, let $\mathbf{w_1} \in W_1$ and $\mathbf{w_2} \in W_2$ with $\mathbf{w} = \mathbf{w_1} + \mathbf{w_2}$. Therefore, $T(\mathbf{v}) = \mathbf{w_1} + \mathbf{w_2}$. Recall that for $i = 1, 2$, the preimage of W_i is $T^{-1}(W_i) = \{\mathbf{x} \in V_i : T(\mathbf{x}) \in W_i\}$. Moreover, $V_i = T^{-1}(W_i)$. So, since $T(\mathbf{v}) = \mathbf{w_1} + \mathbf{w_2}$, we have that $\mathbf{v} = T^{-1}(\mathbf{w_1}) + T^{-1}(\mathbf{w_2}) \in V_1 + V_2$. Thus, $\mathbf{v} \in V_1 + V_2$. Because $\mathbf{v} \in V$ was arbitrary, it follows that $V = V_1 + V_2$.

Now, suppose $\mathbf{v} \in V_1 \cap V_2$. By definition of T, $T(V_1 \cap V_2) = W_1 \cap W_2$, and so $T(\mathbf{v}) \in W_1 \cap W_2$. But $W_1 \cap W_2 = \{\mathbf{0}\}$ since $W = W_1 \oplus W_2$. So, $T(\mathbf{v}) = \mathbf{0}$ and $\mathbf{v} \in \ker(T)$. Because $T : V \to W$ is injective, $\ker(T) = \{\mathbf{0}\}$. Thus, $\mathbf{v} = \mathbf{0}$. Because $\mathbf{v} \in V_1 \cap V_2$ was arbitrary, it follows that $V_1 \cap V_2 = \{\mathbf{0}\}$. Therefore, $V = V_1 + V_2$ and $V_1 \cap V_2 = \{\mathbf{0}\}$ and hence $V = V_1 \oplus V_2$.

Let $\mathbf{v} \in V_i$. Because $\varphi \sim \psi$, we have that $\varphi(g)(\mathbf{v}) = T^{-1}\psi(g)T(\mathbf{v})$ for every $g \in G$. Recall that for $i = 1, 2$, $T(V_i) = W_i$, and thus $T(\mathbf{v}) \in W_i$. Say $T(\mathbf{v}) = \mathbf{w}$ for some $\mathbf{w} \in W_i$. So, $\varphi(g)(\mathbf{v}) = T^{-1}\psi(g)T(\mathbf{v}) = T^{-1}\psi(g)(\mathbf{w})$. Since W_i is G-invariant, $\psi(g)(\mathbf{w}) \in W_i$. Say $\psi(g)(\mathbf{w}) = \mathbf{x}$ for some $\mathbf{x} \in W_i$. So, $\varphi(g)(\mathbf{v}) = T^{-1}\psi(g)(\mathbf{w}) = T^{-1}(\mathbf{x})$. Because $T(V_i) = W_i$, $T^{-1}(\mathbf{x}) \in V_i$. Therefore, $\varphi(g)(\mathbf{v}) = T^{-1}(\mathbf{x}) \in V_i$, and thus $\varphi(g)(\mathbf{v}) \in V_i$ for every $\mathbf{v} \in V_i$. Hence, V_1 and V_2 are nontrivial G-invariant subspaces of V. Since $V = V_1 \oplus V_2$ and V_1 and V_2 are nontrivial G-invariant subspaces of V, it follows that φ is decomposable.

Hence, if $\varphi : G \to GL(V)$ is equivalent to a decomposable representation, then φ itself must be decomposable. □

Other types of representations generate similar results. For this reason, the following two proofs have been omitted.

Lemma 2.17. *Let (φ, V) be equivalent to an irreducible representation. Then φ is irreducible.*

Lemma 2.18. *Let (φ, V) be equivalent to a completely reducible representation. Then φ is completely reducible.*

3 Maschke's Theorem

An alternative approach when studying finite groups is to consider the classification of groups. With this in mind, it is natural to inspect the decomposition of representations. This objective is put forth and met by Maschke's Theorem.

Recall the definition of inner product in Definition 1.1 and the result of Proposition 1.6.

Definition 3.1. A representation (φ, V) of G is *unitary* if for each $g \in G$ and $\mathbf{v}, \mathbf{w} \in V$, $\langle \varphi(g)\mathbf{v}, \varphi(g)\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$.

We are interested in unitary representations for the reason brought to light in the following lemma. That is, every indecomposable unitary representation is irreducible.

Proposition 3.2. *Let (φ, V) be a unitary representation of a group. Then φ is either irreducible or decomposable.*

Proof. Suppose (φ, V) is a reducible unitary representation of G. Let W be a subspace of V such that W is a nontrivial proper subrepresentation of V. Because W is a proper subspace of V, then $W^{\perp}$ is a proper subspace of V as well, and $V = W \oplus W^{\perp}$.

The definition of orthogonal complement ensures that for each $\mathbf{w} \in W$ and $\mathbf{w}' \in W^{\perp}$, $\langle \mathbf{w}', \mathbf{w} \rangle = 0$. And since φ is unitary,

$$\left\langle \varphi(g)\left(\mathbf{w}'\right), \varphi(g)(\mathbf{w}) \right\rangle = \left\langle \mathbf{w}', \mathbf{w} \right\rangle = 0,$$

for all $g \in G$.

As W is a subrepresentation of V, $\varphi(g)(\mathbf{w}) \in W$ for every $g \in G$. Thus, for the sake of simplicity, denote $\varphi(g)(\mathbf{w})$ as the arbitrary vector $\mathbf{u}$ in W. Substituting $\mathbf{u}$ into $\varphi(g)(\mathbf{w})$ yields $\langle \varphi(g)(\mathbf{w}'), \mathbf{u} \rangle = 0$ for each $g \in G$, $\mathbf{w}' \in W^{\perp}$, and $\mathbf{u} \in W$. It follows that $\varphi(g)(\mathbf{w}') \in W^{\perp}$ for all $g \in G$, and so $W^{\perp}$ is a subrepresentation of V. Therefore, V decomposes into two nontrivial subrepresentations, particularly $V = W \oplus W^{\perp}$. Hence, V is decomposable. □

Proposition 3.3. *Every representation of a finite group is equivalent to a unitary representation.*

Proof. Let $\varphi : G \to GL(V)$ be a representation of G where $\dim(V) = n$. For some basis $\mathcal{B}$ of V, let $T : V \to \mathbb{C}^n$ be the isomorphism taking coordinates with respect to $\mathcal{B}$. Then setting $\psi(g) = T\varphi(g)T^{-1}$ for each $g \in G$ yields a representation $\psi : G \to GL_n(\mathbb{C})$ equivalent to φ.

Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{C}^n$. For every $\mathbf{v}, \mathbf{w} \in V$, define

$$(\mathbf{v}, \mathbf{w}) = \sum_{g \in G} \langle \psi(g)(\mathbf{v}), \psi(g)(\mathbf{w}) \rangle$$

as a new inner product product on $\mathbb{C}^n$, denoted by $(\cdot, \cdot)$. Note that the above summation requires G to be finite.

Since for each $\mathbf{v} \in V$, $\langle \psi(g)(\mathbf{v}), \psi(g)(\mathbf{v}) \rangle \geq 0$, then clearly

$$(\mathbf{v}, \mathbf{v}) = \sum_{g \in G} \langle \psi(g)(\mathbf{v}), \psi(g)(\mathbf{v}) \rangle \geq 0.$$

Also, for every $\mathbf{v} \in V$, $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$. Say

$$0 = (\mathbf{v}, \mathbf{v}) = \sum_{g \in G} \langle \psi(g)(\mathbf{v}), \psi(g)(\mathbf{v}) \rangle .$$

Since we are summing over all of G and $\langle \psi(g)(\mathbf{v}), \psi(g)(\mathbf{v}) \rangle \geq 0$, then $\langle \psi(g)(\mathbf{v}), \psi(g)(\mathbf{v}) \rangle = 0$ for every $g \in G$. Hence, $\langle \psi(1)(\mathbf{v}), \psi(1)(\mathbf{v}) \rangle = \langle \mathbf{v}, \mathbf{v} \rangle$ and so $\mathbf{v} = \mathbf{0}$. Thus, $(\cdot, \cdot)$ is positive definite.

Let $\mathbf{v}, \mathbf{w} \in V$. Then

$$\begin{aligned}(\mathbf{v}, \mathbf{w}) &= \sum_{g \in G} \langle \psi(g)(\mathbf{v}), \psi(g)(\mathbf{w}) \rangle \\ &= \sum_{g \in G} \overline{\langle \psi(g)(\mathbf{w}), \psi(g)(\mathbf{v}) \rangle} \\ &= \overline{(\mathbf{w}, \mathbf{v})}.\end{aligned}$$

Thus, $(\cdot, \cdot)$ is conjugate symmetric.

Let $\mathbf{v}, \mathbf{w}, \mathbf{x} \in V$ and $\alpha, \beta \in \mathbb{C}$. Then

$$\begin{aligned}(\alpha\mathbf{v} + \beta\mathbf{w}, \mathbf{u}) &= \sum_{g \in G} \langle \psi(g)(\alpha\mathbf{v} + \beta\mathbf{w}), \psi(g)(\mathbf{u}) \rangle \\ &= \sum_{g \in G} [\alpha \langle \psi(g)(\mathbf{v}), \psi(g)(\mathbf{u}) \rangle + \beta \langle \psi(g)(\mathbf{w}), \psi(g)(\mathbf{u}) \rangle] \\ &= \alpha \sum_{g \in G} \langle \psi(g)(\mathbf{v}), \psi(g)(\mathbf{u}) \rangle + \beta \sum_{g \in G} \langle \psi(g)(\mathbf{w}), \psi(g)(\mathbf{u}) \rangle \\ &= \alpha(\mathbf{v}, \mathbf{u}) + \beta(\mathbf{w}, \mathbf{u}) .\end{aligned}$$

Thus, $(\cdot, \cdot)$ is linear. Therefore, $(\cdot, \cdot)$ is indeed an inner product on $\mathbb{C}^n$.

We must now show that φ is unitary. Let $h \in G$ and $\mathbf{v}, \mathbf{w} \in V$. As $\psi : G \to GL_n(\mathbb{C}) \simeq \mathbb{C}^n$, $\psi(h)(\mathbf{v}), \psi(h)(\mathbf{w}) \in \mathbb{C}^n$, and thus by the definition of $(\cdot, \cdot)$,

$$(\psi(h)(\mathbf{v}), \psi(h)(\mathbf{w})) = \sum_{g \in G} \langle \psi(g)\psi(h)(\mathbf{v}), \psi(g)\psi(h)(\mathbf{w}) \rangle = \sum_{g \in G} \langle \psi(gh)(\mathbf{v}), \psi(gh)(\mathbf{w}) \rangle .$$

Take $x \in G$ such that $x = gh$. Let $k \in G$. Since g ranges over all of G, then when $g = kh^{-1}$,

$x = k$. So x ranges over all of G and thus,

$$(\psi(h)(\mathbf{v}), \psi(h)(\mathbf{w})) = \sum_{g \in G} \langle \psi(gh)(\mathbf{v}), \psi(gh)(\mathbf{w}) \rangle = \sum_{x \in G} \langle \psi(x)(\mathbf{v}), \psi(x)(\mathbf{w}) \rangle = (\mathbf{v}, \mathbf{w}).$$

Therefore, $(\psi(h)(\mathbf{v}), \psi(h)(\mathbf{w})) = (\mathbf{v}, \mathbf{w})$ for all $h \in G$ and $v, w \in V$, and so ψ is unitary.

Hence, every representation of a finite group G is equivalent to a unitary representation.

□

Corollary 3.4. *Let $\varphi : G \to GL(V)$ be a nontrivial representation. Then φ is either irreducible or decomposable.*

Proof. Let $\varphi : G \to GL(V)$ be a nontrivial representation of G. By Proposition 3.3, φ is equivalent to a unitary representation, say ψ. And by Proposition 3.2, ψ is either irreducible or decomposable. If ψ is irreducible, then by Lemma 2.17, φ is irreducible. If ψ is decomposable, then by Lemma 2.16, φ is decomposable.

Therefore, every nontrivial representation is either irreducible or decomposable. □

This corollary is applicable so long as the indicated group is finite. In other words, for some nontrivial representation (φ, V) of an infinite group G, φ may be both reducible and indecomposable.

Example 3.5. Define $\varphi : \mathbb{Z} \to GL_2(\mathbb{C})$ by

$$\varphi(n) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

Show that φ is neither irreducible nor decomposable.

Proof. Let $\varphi : \mathbb{Z} \to GL_2(\mathbb{C}) \simeq GL(\mathbb{C}^2)$ be as defined above. Then $(\varphi, \mathbb{C}^2)$ is a representation of $\mathbb{Z}$ since for each $a, b \in \mathbb{Z}$,

$$\varphi(a)\varphi(b) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = \varphi(a+b).$$

Consider the standard basis $\{\mathbf{e_1}, \mathbf{e_2}\}$ of $\mathbb{C}^2$. Since $\varphi(n)(\mathbf{e_1}) = \mathbf{e_1}$, $\langle \mathbf{e_1} \rangle$ is a $\mathbb{Z}$-invariant subspace of $\mathbb{C}^2$. Because $\langle \mathbf{e_1} \rangle$ is a proper nontrivial subrepresentation of $\mathbb{Z}$, then $(\varphi, \mathbb{C}^2)$ is not irreducible. Note that $\varphi(n)(\mathbf{e_2}) = \begin{bmatrix} n \\ 1 \end{bmatrix} \notin \langle \mathbf{e_2} \rangle$, and so $\langle \mathbf{e_2} \rangle$ is not a $\mathbb{Z}$-invariant subspace of $\mathbb{C}^2$.

Assume $(\varphi, \mathbb{C}^2)$ is decomposable. Then there exists a nontrivial subrepresentation V of $\mathbb{C}^2$ such that $\mathbb{C}^2 = \langle \mathbf{e_1} \rangle \oplus V$. Suppose vectors in V are of the form $\mathbf{v} = a\mathbf{e_1} + b\mathbf{e_2} \in V$, $b \neq 0$. So,

$$\begin{aligned}
\varphi(1)(\mathbf{v}) &= \varphi(1)(a\mathbf{e_1} + b\mathbf{e_2}) \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\
&= \begin{bmatrix} a+b \\ b \end{bmatrix} \\
&= \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix},
\end{aligned}$$

and

$$\varphi(1)(\mathbf{v}) - \mathbf{v} = \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} - \left(\begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} \right) = \begin{bmatrix} b \\ 0 \end{bmatrix} = b\mathbf{e_1} + 0\mathbf{e_2} = b\mathbf{e_1} \in V.$$

Therefore, $\mathbf{e_1} \in V$ and $\mathbf{e_2} = \frac{\mathbf{v} - a\mathbf{e_1}}{b} \in V$, or in other words, $V = \mathbb{C}^2$. It follows that $\langle \mathbf{e_1} \rangle$ is the only nontrivial subrepresentation of $\mathbb{C}^2$. Hence, φ is not decomposable. □

We are now ready to prove our first fundamental result, Maschke's theorem, which enables us to reduce all finite representations into irreducible representations. The proof of this theorem is by induction and is parallel to the proof of the existence of a prime factorization of an integer or of a factorization of polynomials into irreducibles.

Theorem 3.6 (Maschke's)**.** *Every representation is completely reducible.*

Proof. Let (φ, V) be a representation of G.

Base Case. Suppose $\dim(V) = 1$. Then V has no nontrivial proper subspaces. So φ is irreducible.

Inductive Hypothesis. Assume every representation of degree less than or equal to n of a finite group is completely reducible.

Inductive Step. Suppose $\dim V = n+1$. If φ is irreducible, then we are done. If φ is not irreducible, then φ is decomposable by Corollary 3.4. That is, there exists nontrivial G-invariant subspaces V_1, V_2 of V such that $V = V_1 \oplus V_2$. Because the degrees of φ_{V_1} and φ_{V_2} are strictly less than the degree of φ, then by our inductive hypothesis, φ_{V_1} and φ_{V_2} are completely reducible. Therefore, $V_1 = U_1 \oplus \cdots \oplus U_s$ and $V_2 = W_1 \oplus \cdots \oplus W_r$ where U_i, W_j are G-invariant subspaces of V_1 and V_2, respectively, and the subrepresentations $\varphi_{U_i}, \varphi_{W_j}$ are irreducible for each $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, r\}$. Therefore,

$$V = V_1 \oplus V_2 = U_1 \oplus \cdots \oplus U_s \oplus W_1 \oplus \cdots \oplus W_r,$$

and so V is completely reducible. Hence, every representation of a finite group is completely reducible. □

Lemma 3.7 (Schur's lemma). *Let (φ, V) and (ψ, W) be irreducible representations of G and let $T \in Hom_G(V, W)$. Then either T is invertible or $T = 0$. Consequently:*

(a) If $\varphi \nsim \psi$, then $Hom_G(V, W) = 0$,

i.e., if $\varphi \nsim \psi$, then there exists no intertwining operator;

(b) If $\varphi = \psi$, then $T = \lambda I$ for some $\lambda \in \mathbb{C}$,

i.e., if $\varphi \sim \psi$, then T is multiplication by a scalar.

Proof. Let (φ, V) and (ψ, W) be irreducible representations of G and $T : V \to W$ be in $\mathrm{Hom}_G(V, W)$.

If $T = 0$, we are done. Assume that $T \neq 0$. By Proposition 2.10, $\mathrm{Ker}(T)$ is a G-invariant subspace of V, and so either $\mathrm{Ker}(T) = V$ or $\mathrm{Ker}(T) = 0$. As $\mathrm{Ker}(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}_W\}$, then $\mathrm{Ker}(T) = V$ implies that $T = 0$. But $T \neq 0$ and so $\mathrm{Ker}(T) \neq V$. Thus, $\mathrm{Ker}(T) = 0$ and hence T is injective.

By the same proposition, $\mathrm{Im}(T)$ is a G-invariant subspace of W, and so either $\mathrm{Im}(T) = W$ or $\mathrm{Im}(T) = 0$. As $\mathrm{Im}(T) = \{\mathbf{w} \in W : T(\mathbf{v}) = \mathbf{w} \text{ for some } \mathbf{v} \in V\}$, then $\mathrm{Im}(T) = 0$ implies that

$T = 0$. But $T \neq 0$ and so $\mathrm{Im}(T) \neq 0$. Thus, $\mathrm{Im}(T) = W$ and hence T is surjective. Therefore, T is bijective and thus, T is invertible.

(a) Assume $\mathrm{Hom}_G(V, W) \neq 0$. Then there exists $T \neq 0$ in $\mathrm{Hom}_G(V, W)$. Because $T \in \mathrm{Hom}_G(V, W)$ and $T \neq 0$, then as proven above, T is invertible. Since $T \in \mathrm{Hom}_G(V, W)$ is invertible, it follows that T is an isomorphism, i.e., $\varphi \sim \psi$. Hence, if $\mathrm{Hom}_G(V, W) \neq 0$, then $\varphi \sim \psi$. By the contrapositive, if $\varphi \nsim \psi$, then $\mathrm{Hom}_G(V, W) = 0$.

(b) Suppose $\varphi = \psi$. Then $T : V \to V$. Since we are working over the algebraically closed field $\mathbb{C}$, T must have at least one eigenvalue. So, let $\lambda \in \mathbb{C}$ such that λ is an eigenvalue of T. But λ is an eigenvalue of T if and only if $\det(T - \lambda I) = 0$. Because $\det(T - \lambda I) = 0$, then $T - \lambda I$ is not invertible. As $I : V \to V$ is always an element of $\mathrm{Hom}_G(V, V)$, we have that $T, I \in \mathrm{Hom}_G(V, V)$. Now, using Proposition 2.11, set $\alpha_1 = 1$ and $\alpha_2 = -\lambda$. Then $T - \lambda I \in \mathrm{Hom}_G(\varphi, \varphi)$. And because $T - \lambda I \in \mathrm{Hom}_G(V, V)$, then by the initial statement, either $T - \lambda I$ is invertible or $T - \lambda I = 0$. Since the former is not true by the definition of eigenvalue, we have that $T - \lambda I = 0$. Therefore, $T = \lambda I$.

□

It follows from Schur's lemma that if (φ, V) and (ψ, W) are equivalent irreducible representations of G, then $\dim \mathrm{Hom}_G(V, W) = 1$.

Corollary 3.8. *Let G be an abelian group. Then any irreducible representation of G has degree one.*

Proof. Suppose G is an abelian group and let (φ, V) be an irreducible representation of G. Fix $h \in G$ and set $T = \varphi(h)$. Then for all $g \in G$,

$$T\varphi(g) = \varphi(h)\varphi(g) = \varphi(hg) = \varphi(gh) = \varphi(g)\varphi(h) = \varphi(g)T.$$

So $T = \varphi(h)$ is an intertwining operator. Because T is a nontrivial intertwiner, Schur's lemma implies that $\varphi(h) = \lambda_h I$, where $\lambda_h \in \mathbb{C}$ depends on $h \in G$. If W is a subspace of V, then for every $\mathbf{w} \in W$,

$$\varphi(h)(\mathbf{w}) = \lambda_h I\mathbf{w} = \lambda \mathbf{w} \in W.$$

Therefore, every subspace of V is G-invariant. Because φ is irreducible, the only G-invariant subspaces of V are $\{\mathbf{0}\}$ and V itself. Therefore $\dim(V) = 1$.

Hence, any irreducible representation of an abelian group has degree one. □

4 Characters

We've established that the significance of representation theory is being able to study abstract groups more clearly by representing their elements as linear transformations of vector spaces, which we may view as matrices. By representing a group in such a way, we obtain concrete information about its abstract structure. Now, we will take this idea one step further. The character function will muster this information, thereby allowing us to study the group in an even more compact and understandable way.

Definition 4.1. The *character* of a representation (φ, V) of G is the function $\chi_\varphi : G \to \mathbb{C}$ defined by setting $\chi_\varphi(g) = \mathrm{Tr}(\varphi(g))$. The character of an irreducible representation is called an *irreducible character*.

Recall the definition of the standard representation of S_n in Definition 2.3.

Example 4.2. Let $\varphi : S_n \to GL(\mathbb{C}^n)$ be the standard representation of S_n. We will first find the conjugacy classes of S_n, and then find the character $\chi_\varphi : S_n \to \mathbb{C}$.

We begin by finding the conjugacy classes of S_3.

	σ^{-1}	$\sigma(1)\sigma^{-1}$	$\sigma(1\,2)\sigma^{-1}$	$\sigma(1\,2\,3)\sigma^{-1}$
(1)	(1)	(1)	$(1\,2)$	$(1\,2\,3)$
$(1\,2)$	$(1\,2)$	(1)	$(1\,2)$	$(1\,3\,2)$
$(1\,3)$	$(1\,3)$	(1)	$(2\,3)$	$(1\,3\,2)$
$(2\,3)$	$(2\,3)$	(1)	$(1\,3)$	$(1\,3\,2)$
$(1\,2\,3)$	$(1\,3\,2)$	(1)	$(2\,3)$	$(1\,2\,3)$
$(1\,3\,2)$	$(1\,2\,3)$	(1)	$(1\,3)$	$(1\,2\,3)$

The three conjugacy classes of S_3 are $\{(1)\}$, $\{(1\,2), (1\,3), (2\,3)\}$, and $\{(1\,2\,3), (1\,3\,2)\}$. Notice that each conjugacy class consists of one type of cycle. Lets see if this is true for the conjugacy classes of S_4.

	σ^{-1}	$\sigma(1)\sigma^{-1}$	$\sigma(1\,2)\sigma^{-1}$	$\sigma(1\,2\,3)\sigma^{-1}$	$\sigma(1\,2\,3\,4)\sigma^{-1}$	$\sigma(1\,2)(3\,4)\sigma^{-1}$
(1)	(1)	(1)	(1 2)	(1 2 3)	(1 2 3 4)	(1 2)(3 4)
(1 2)	(1 2)	(1)	(1 2)	(1 3 2)	(1 3 4 2)	(1 2)(3 4)
(1 3)	(1 3)	(1)	(2 3)	(1 3 2)	(1 4 3 2)	(1 4)(3 2)
(1 4)	(1 4)	(1)	(2 4)	(2 3 4)	(1 4 2 3)	(1 3)(2 4)
(2 3)	(2 3)	(1)	(1 3)	(1 3 2)	(1 3 2 4)	(1 3)(2 4)
(2 4)	(2 4)	(1)	(1 4)	(1 4 3)	(1 4 3 2)	(1 4)(2 3)
(3 4)	(3 4)	(1)	(1 2)	(1 2 4)	(1 2 4 3)	(1 2)(3 4)
(1 2 3)	(1 3 2)	(1)	(2 3)	(1 2 3)	(1 4 2 3)	(1 4)(2 3)
(1 3 2)	(1 2 3)	(1)	(1 3)	(1 2 3)	(1 2 4 3)	(1 3)(2 4)
(2 3 4)	(2 4 3)	(1)	(1 3)	(1 3 4)	(1 3 4 2)	(1 3)(2 4)
(2 4 3)	(2 3 4)	(1)	(1 4)	(1 4 2)	(1 4 2 3)	(1 4)(2 3)
(1 3 4)	(1 4 3)	(1)	(2 3)	(2 4 3)	(1 3 2 4)	(1 4)(2 3)
(1 4 3)	(1 3 4)	(1)	(2 4)	(1 4 2)	(1 3 4 2)	(1 3)(2 4)
(1 2 4)	(1 4 2)	(1)	(2 4)	(2 4 3)	(1 2 4 3)	(1 3)(2 4)
(1 4 2)	(1 2 4)	(1)	(1 4)	(1 3 4)	(1 3 2 4)	(1 4)(2 3)
(1 2 3 4)	(1 4 3 2)	(1)	(2 3)	(2 3 4)	(1 2 3 4)	(1 4)(2 3)
(1 2 4 3)	(1 3 4 2)	(1)	(2 4)	(1 2 4)	(1 3 2 4)	(1 3)(2 4)
(1 3 2 4)	(1 4 2 3)	(1)	(3 4)	(2 3 4)	(1 3 4 2)	(1 2)(3 4)
(1 3 4 2)	(1 2 4 3)	(1)	(1 3)	(1 4 3)	(1 4 2 3)	(1 3)(2 4)
(1 4 2 3)	(1 3 2 4)	(1)	(3 4)	(1 4 3)	(1 2 4 3)	(1 2)(3 4)
(1 4 3 2)	(1 2 3 4)	(1)	(1 4)	(1 2 4)	(1 2 3 4)	(1 4)(2 3)
(1 2)(3 4)	(1 2)(3 4)	(1)	(1 2)	(1 4 2)	(1 4 3 2)	(1 2)(3 4)
(1 3)(2 4)	(1 3)(2 4)	(1)	(3 4)	(1 3 4)	(1 2 3 4)	(1 2)(3 4)
(1 4)(2 3)	(1 4)(2 3)	(1)	(3 4)	(2 4 3)	(1 4 3 2)	(1 2)(3 4)

The five conjugacy classes of S_4 are $\{(1)\}$, $\{(1\,2), (1\,3), (1\,4), (2\,3), (2\,4), (3\,4)\}$, $\{(1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$, $\{(1\,2\,3), (1\,3\,2), (2\,3\,4), (2\,4\,3), (3\,4\,1), (3\,1\,4), (4\,1\,2), (4\,2\,1)\}$, and $\{(1\,2\,3\,4), (1\,2\,4\,3), (1\,3\,2\,4), (1\,3\,4\,2), (1\,4\,2\,3), (1\,4\,3\,2)\}$. Again, each conjugacy class of S_4 consists of one type of cycle.

We (prematurely) conclude that for all $n \in \mathbb{N}$, each type of cycle of S_n has its own conjugacy class.

Next, we find the character of the standard representation of S_n. First consider S_3, then consider S_4.

$$\chi_\varphi((1)) = \mathrm{Tr}\left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right) = 3 \qquad \chi_\varphi((1)) = \mathrm{Tr}\left(\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\right) = 4$$

$$\chi_\varphi((1\,2)) = \mathrm{Tr}\left(\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\right) = 1 \qquad \chi_\varphi((1\,2)) = \mathrm{Tr}\left(\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\right) = 2$$

$$\chi_\varphi((1\,3)) = \mathrm{Tr}\left(\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\right) = 1 \qquad \chi_\varphi((1\,2\,3)) = \mathrm{Tr}\left(\begin{bmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{bmatrix}\right) = 1$$

$$\chi_\varphi((2\,3)) = \mathrm{Tr}\left(\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\right) = 1 \qquad \chi_\varphi((1\,2\,3\,4)) = \mathrm{Tr}\left(\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}\right) = 0$$

$$\chi_\varphi((1\,2\,3)) = \mathrm{Tr}\left(\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\right) = 0 \qquad \chi_\varphi((1\,2)(3\,4)) = \mathrm{Tr}\left(\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\right) = 0$$

$$\chi_\varphi((1\,3\,2)) = \mathrm{Tr}\left(\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\right) = 0$$

We conclude that the character $\chi_\varphi : S_n \to \mathbb{C}$ is defined by $\chi_\varphi(\sigma) = |\mathrm{Fix}(\sigma)|$ for each $\sigma \in S_n$.

Proposition 4.3. *Let (φ, V) be a representation of G. Then $\chi_\varphi(1) = deg(\varphi)$.*

Proof. Suppose (φ, V) is a representation of G. Then

$$\chi_\varphi(1) = \text{Tr}(\varphi(1)) = \text{Tr}(I) = \dim(V) = \deg(\varphi).$$

Therefore, $\chi_\varphi(1) = \deg(\varphi)$. □

Proposition 4.4. *If φ and ψ are equivalent representations, then $\chi_\varphi = \chi_\psi$.*

Proof. Assume $\varphi, \psi : G \to GL_n(\mathbb{C})$ such that $\varphi \sim \psi$. Then there exists an invertible matrix $T \in GL_n(\mathbb{C})$ such that $\varphi(g) = T\psi(g)T^{-1}$ for all $g \in G$. Since $Tr(AB) = Tr(BA)$,

$$\chi_\varphi(g) = Tr(\varphi(g)) = Tr\left(T\psi(g)T^{-1}\right) = Tr\left(TT^{-1}\psi(g)\right) = Tr(\psi(g)) = \chi_\psi(g).$$

Therefore, if φ and ψ are equivalent representations, then $\chi_\varphi = \chi_\psi$. □

Proposition 4.5. *For all $g, h \in G$, the equality $\chi_\varphi(g) = \chi_\varphi\left(hgh^{-1}\right)$ holds.*

Proof. For all $g, h \in G$,

$$\begin{aligned}\chi_\varphi\left(hgh^{-1}\right) = \text{Tr}\left(\varphi\left(hgh^{-1}\right)\right) &= \text{Tr}\left(\varphi(h)\varphi(g)\varphi\left(h^{-1}\right)\right) \\ &= \text{Tr}\left(\varphi(h)\varphi\left(h^{-1}\right)\varphi(g)\right) = \text{Tr}(\varphi(g)) = \chi_\varphi(g).\end{aligned}$$

Hence, characters are constant on conjugacy classes. □

Definition 4.6. The *group algebra* of a group G is the set

$$L(G) = \{f \mid f : G \to \mathbb{C}\}.$$

where $L(G)$ is an inner product space with addition and scalar multiplication given by

$$(f_1 + f_2)(g) = f_1(g) + f_2(g)$$
$$(\alpha f)(g) = \alpha \cdot f(g)$$

and with the innder product defined by

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

Definition 4.7. A *class function* is a function $f : G \to \mathbb{C}$ where $f(g) = f\left(hgh^{-1}\right)$ for all $g, h \in G$. We denote by $Z(L(G))$ the space of class functions.

Proposition 4.8. *Let G be a group of order n and define $\delta_i : G \to \mathbb{C}$ by $\delta_i(g_j) = 1$ if $i = j$ and $\delta_i(g_j) = 0$ if $i \neq j$. Then $\{\delta_1, \ldots, \delta_n\}$ is a basis for $L(G)$.*

Proof. Let G is a group of order n with $G = \{g_1, \ldots, g_n\}$, and define $\delta_i : G \to \mathbb{C}$ by

$$\delta_i(g_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $\alpha_1, \ldots, \alpha_n$ be scalars. Assume $f \in L(G)$ such that $f(g_i) = \alpha_i$, and $h \in L(G)$ with $h = \alpha_1\delta_1 + \cdots + \alpha_n\delta_n$. Then

$$\begin{aligned} h(g_j) &= \alpha_1\delta_1(g_j) + \cdots + \alpha_n\delta_n(g_j) \\ &= f(g_1)\delta_1(g_j) + \cdots + f(g_n)\delta_n(g_j) \\ &= f(g_j)\delta_j(g_j) \\ &= f(g_j). \end{aligned}$$

Therefore, $h = f$ and so $f = \alpha_1\delta_1 + \cdots + \alpha_n\delta_n$. Thus, $L(G) = \text{Span}\{\delta_1, \ldots, \delta_n\}$.

Now, let $\alpha_1, \ldots, \alpha_n$ be scalars such that $\alpha_1\delta_1 + \cdots + \alpha_n\delta_n = 0$, and let $f = \alpha_1\delta_1 + \cdots + \alpha_n\delta_n$. Therefore, $f(g) = 0$ for every $g \in G$. Then for every $j = 1, \ldots, n$,

$$f(g_j) = \alpha_1\delta_1(g_j) + \cdots + \alpha_n\delta_n(g_j) = \alpha_j\delta_j(g_j) = \alpha_j = 0.$$

Therefore, $\alpha_1 = \cdots = \alpha_n = 0$ and so $\{\delta_1, \ldots, \delta_n\}$ is linearly independent.

Hence, $\{\delta_1, \ldots, \delta_n\}$ is a basis of $L(G)$. □

Theorem 4.9 (Schur orthogonality relations)**.** *Let* $\varphi : G \to U_n(\mathbb{C})$ *and* $\psi : G \to U_m(\mathbb{C})$ *be inequivalent irreducible unitary representations. Then*

1. $\langle \varphi_{ij}, \psi_{kl} \rangle = 0$,
2. $\langle \varphi_{ij}, \psi_{kl} \rangle = \begin{cases} \frac{1}{n} & \text{if } i = k \text{ and } j = l \\ 0 & \text{else} \end{cases}$

With this result, we are able to prove the following fundamental theorem.

Theorem 4.10 (First orthogonality relations)**.** *Let* φ, ψ *be irreducible representations of* G*. Then*

$$\langle \chi_\varphi, \chi_\psi \rangle = \begin{cases} 1 & \varphi \sim \psi \\ 0 & \varphi \nsim \psi \end{cases}.$$

Thus the irreducible characters of G *form an orthonormal set of class functions.*

Proof. Let φ, ψ be irreducible representations of G. From Proposition 3.3, every representation is equivalent to a unitary representation. So, assume $\varphi : G \to U_n(\mathbb{C})$ and $\psi : G \to U_m(\mathbb{C})$ are unitary. We have

$$\begin{aligned}
\langle \chi_\varphi, \chi_\psi \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_\varphi(g) \overline{\chi_\psi(g)} \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{n} \varphi_{ii}(g) \sum_{j=1}^{m} \overline{\psi_{jj}(g)} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{1}{|G|} \varphi_{ii}(g) \overline{\psi_{jj}(g)} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \varphi_{ii}(g), \psi_{jj}(g) \rangle .
\end{aligned}$$

Following from Theorem 4.9, $\langle \chi_\varphi, \chi_\psi \rangle = 0$ if $\varphi \nsim \psi$. And following from Proposition 4.4, if φ and ψ are equivalent representations, then $\chi_\varphi = \chi_\psi$. Thus, if $\varphi \sim \psi$, then we may assume that $\varphi = \psi$. So by Theorem 4.9,

$$\langle \varphi_{ii}, \varphi_{ii} \rangle = \begin{cases} \frac{1}{n} & i = j \\ 0 & i \neq j, \end{cases}$$

and thus

$$\langle \chi_\varphi, \chi_\varphi \rangle = \sum_{i=1}^{n} \langle \varphi_{ii}, \varphi_{ii} \rangle = \sum_{i=1}^{n} \frac{1}{n} = 1.$$

Hence, the irreducible characters of G form an orthonormal set of class functions. □

Note that Theorem 4.10 implies inequivalent irreducible repesentations have distinct characters.

Corollary 4.11. *There are at most* $|cl\,(G)|$ *equivalence classes of irreducible representations of* G*.*

If V is a vector space, φ is a representation and $m > 0$, then

$$mV = \underbrace{V \oplus \cdots \oplus V}_{m \text{ times}} \text{ and } m\varphi = \underbrace{\varphi \oplus \cdots \oplus \varphi}_{m \text{ times}}.$$

Let $\varphi_1, \ldots, \varphi_s$ be a complete set of irreducible unitary representations of G, up to equivalence. Set $d_i = \deg(\varphi_i)$.

Definition 4.12. If $\psi \sim m_1\varphi_1 \oplus m_2\varphi_2 \oplus \cdots \oplus m_s\varphi_s$, then m_i is the *multiplicity* of φ_i in ψ. If $m_i > 0$, then φ_i is an *irreducible constituent* of ψ.

If $\psi \sim m_1\varphi_1 \oplus m_2\varphi_2 \oplus \cdots \oplus m_s\varphi_s$, then

$$\deg(\psi) = m_1 d_1 + \cdots + m_s d_s.$$

The following lemma is as short as it is important. Because of this lemma, we have that each character is an integral linear combination of irreducible characters.

Lemma 4.13. *Let* $\phi \sim \varphi \oplus \psi$*. Then* $\chi_\phi = \chi_\varphi + \chi_\psi$*.*

Proof. Suppose $\varphi : G \to GL_n(\mathbb{C})$ and $\psi : G \to GL_m(\mathbb{C})$ are irreducible representations of G. Then $\phi : G \to GL_{n+m}(\mathbb{C})$ and

$$\phi(g) = \begin{bmatrix} \varphi(g) & 0 \\ 0 & \psi(g) \end{bmatrix}.$$

Therefore, for each $g \in G$,

$$\chi_\phi(g) = \mathrm{Tr}(\phi(g)) = \mathrm{Tr}(\varphi(g)) + \mathrm{Tr}(\psi(g)) = \chi_\varphi(g) + \chi_\psi(g).$$

Hence, $\chi_\phi = \chi_\varphi + \chi_\psi$. □

Theorem 4.14. *Let $\varphi_1, \ldots, \varphi_s$ be a complete set of representatives of the equivalence classes of irreducible representations of G and let*

$$\psi \sim m_1\varphi_1 \oplus m_2\varphi_2 \oplus \cdots \oplus m_s\varphi_s.$$

Then $m_i = \langle \chi_\psi, \chi_{\varphi_i} \rangle$. Consequently, the decomposition of ψ into irreducible constituents is unique and ψ is determined up to equivalence by its character.

Example 4.15. Let $\varphi : S_4 \to GL_4(\mathbb{C})$ be the standard representation of S_4. As shown in Example 4.2, $\chi_\varphi : S_4 \to \mathbb{C}$ is given by $\chi_\varphi(\sigma)$ = the number of fixed points of σ.

	(1)	(12)	(123)	(1234)	(12)(34)
χ_φ	4	2	1	0	0

Conjugacy Class	Size
(1)	1
(12)	6
(123)	8
(1234)	6
(12)(34)	3

Because S_4 has five conjugacy classes, then S_4 has five irreducible representations. The character table of S_4 is below.

Table 2: Character Table of S_4

S_4	(1)	(12)	(123)	(1234)	(12)(34)
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	3	1	0	-1	-1
χ_4	3	-1	0	1	-1
χ_5	2	0	-1	0	2

Per usual, we know that χ_1 is the trivial character and χ_2 the sign character.

That is, $\chi_1 : S_4 \to \mathbb{C}^\times$ is defined to be $\chi_1(\sigma) = 1$ for every $\sigma \in S_4$, and $\chi_2 : S_4 \to \mathbb{C}^\times$ is defined to be

$$\chi_2(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ even} \\ -1 & \text{if } \sigma \text{ odd} \end{cases}$$

for every $\sigma \in S_4$.

Since

$$\begin{aligned} \langle \chi_\varphi, \chi_1 \rangle &= \frac{1}{24}(1 \cdot 4 \cdot 1 + 6 \cdot 2 \cdot 1 + 8 \cdot 1 \cdot 1 + 6 \cdot 0 \cdot 1 + 3 \cdot 0 \cdot 1) \\ &= \frac{1}{24}(4 + 12 + 8) \\ &= 1 \\ \langle \chi_\varphi, \chi_2 \rangle &= \frac{1}{24}(1 \cdot 4 \cdot 1 + 6 \cdot 2 \cdot -1 + 8 \cdot 1 \cdot 1 + 6 \cdot 0 \cdot -1 + 3 \cdot 0 \cdot 1) \\ &= \frac{1}{24}(4 - 12 + 8) \\ &= 0, \end{aligned}$$

then by Theorem 4.10, $\varphi \sim \chi_1$ and $\varphi \nsim \chi_2$. So φ contains one copy of the trivial representation and zero copies of the sign representation. Thus, to find the third irreducible representation of S_4, we subtract χ_1 from χ_φ. Define $\varphi_3 : S_4 \to GL_3(\mathbb{C})$ by $\varphi_3(\sigma) = \varphi(\sigma) - 1$. So $\chi_3 : S_4 \to \mathbb{C}$ is given by $\chi_3(\sigma) = \chi_\varphi(\sigma) - \chi_1(\sigma) = \chi_\varphi(\sigma) - 1$.

	(1)	(12)	(123)	(1234)	(12)(34)
χ_3	3	1	0	-1	-1

Moreover,

$$\begin{aligned}\langle \chi_3, \chi_3 \rangle &= \frac{1}{24}\left(1 \cdot 3 \cdot 3 + 6 \cdot 1 \cdot 1 + 8 \cdot 0 \cdot 0 + 6 \cdot -1 \cdot -1 + 3 \cdot -1 \cdot -1\right) \\ &= \frac{1}{24}\left(9 + 6 + 6 + 3\right) \\ &= 1,\end{aligned}$$

and so φ_3 is indeed irreducible.

We obtain the fourth irreducible representation by taking the tensor product of φ_2 and φ_3. So the map $\varphi_4 : S_4 \to GL_3(\mathbb{C})$ is given by $\varphi_4 = \varphi_2 \otimes \varphi_3$. Define $\chi_4 : S_4 \to \mathbb{C}$ by $\chi_4(\sigma) = \chi_2(\sigma) \otimes \chi_3(\sigma)$.

	(1)	(12)	(123)	(1234)	(12)(34)
χ_4	3	-1	0	1	-1

Since

$$\begin{aligned}\langle \chi_4, \chi_4 \rangle &= \frac{1}{24}\left(1 \cdot 3 \cdot 3 + 6 \cdot -1 \cdot -1 + 8 \cdot 0 \cdot 0 + 6 \cdot 1 \cdot 1 + 3 \cdot -1 \cdot -1\right) \\ &= \frac{1}{24}\left(9 + 6 + 6 + 3\right) \\ &= 1,\end{aligned}$$

φ_4 is indeed an irreducible representation of S_4.

Our characte is nearly complete, with the exception of the last irreducible representation, φ_5.

Since the sum of the first column of the character table is $|S_4| = 24$,

$$\begin{aligned}24 &= 1^2 + 1^2 + 3^2 + 3^2 + \deg(\varphi_5)^2 \\ 24 &= 20 + \deg(\varphi_5)^2 \\ 4 &= \deg(\varphi_5)^2 \\ 2 &= \deg(\varphi_5).\end{aligned}$$

Therefore, φ_5 is an irreducible representation of degree two. So we have some mapping from

S_4 to $GL_2(\mathbb{C})$. To complete the table, we apply Theorem 4.10 to each column.

We first find that $\chi_5((12)) = 0$:

$$\begin{aligned} 0 &= \chi_1((1))\chi_1((12)) + \chi_2((1))\chi_2((12)) + \chi_3((1))\chi_3((12)) + \chi_4((1))\chi_4((12)) \\ &\quad + \chi_5((1))\chi_5((12)) \\ &= 1 \cdot 1 + 1 \cdot -1 + 3 \cdot 1 + 3 \cdot -1 + 2 \cdot \chi_5((12)) \\ &= 2 \cdot \chi_5((12)). \end{aligned}$$

Next, $\chi_5((123)) = 0$:

$$\begin{aligned} 0 &= \chi_1((1))\chi_1((123)) + \chi_2((1))\chi_2((123)) + \chi_3((1))\chi_3((123)) + \chi_4((1))\chi_4((123)) \\ &\quad + \chi_5((1))\chi_5((123)) \\ &= 1 \cdot 1 + 1 \cdot 1 + 3 \cdot 0 + 3 \cdot 0 + 2 \cdot \chi_5((123)) \\ &= 2 + 2 \cdot \chi_5((123)). \end{aligned}$$

Next, $\chi_5((1234)) = 0$:

$$\begin{aligned} 0 &= \chi_1((1))\chi_1((1234)) + \chi_2((1))\chi_2((1234)) + \chi_3((1))\chi_3((1234)) + \chi_4((1))\chi_4((1234)) \\ &\quad + \chi_5((1))\chi_5((1234)) \\ &= 1 \cdot 1 + 1 \cdot -1 + 3 \cdot -1 + 3 \cdot 1 + 2 \cdot \chi_5((1234)) \\ &= 2 \cdot \chi_5((1234)). \end{aligned}$$

Lastly, $\chi_5((12)(34)) = 0$:

$$\begin{aligned} 0 &= \chi_1((1))\chi_1((12)(34)) + \chi_2((1))\chi_2((12)(34)) + \chi_3((1))\chi_3((12)(34)) + \chi_4((1))\chi_4((12)(34)) \\ &\quad + \chi_5((1))\chi_5((12)(34)) \\ &= 1 \cdot 1 + 1 \cdot 1 + 3 \cdot -1 + 3 \cdot -1 + 2 \cdot \chi_5((12)(34)) \\ &= -4 + 2 \cdot \chi_5((12)(34)). \end{aligned}$$

	(1)	(12)	(123)	(1234)	(12)(34)
χ_5	2	0	-1	0	2

Part III

Induced Representation

Definition 4.16. Let H be a subgroup of G. If $\pi : G \to GL(U)$ is a representation of G, then we can restrict π to H to obtain a map $\mathrm{Res}_H^G \pi : H \to GL(U)$ called the *restriction* of π. So $\mathrm{Res}_H^G \pi(h) = \pi(h)$ for all $h \in H$.

Definition 4.17. Let H be a subgroup of G and let (ϕ, V) be a representation of H. The *induced representation* of G is the representation $\left(\pi, \mathrm{Ind}_H^G(\phi)\right)$ where

$$\mathrm{Ind}_H^G(\phi) = \{f : G \to V \mid f(hg) = \phi(h) f(g) \text{ for all } h \in H,\ g \in G\}.$$

The action of G on $\mathrm{Ind}_H^G(\phi)$ is given by

$$(\pi(g) f)\left(g'\right) = f\left(g'g\right).$$

In this thesis, we also write $\left(\pi^G, V^G\right)$ to mean the representation $\left(\pi, \mathrm{Ind}_H^G(\phi)\right)$.

The induced representation is indeed a vector space under the usual addition and scalar multiplication. For example, let $f_1, f_2 \in \mathrm{Ind}_H^G(\phi)$, $g \in G$, and $h \in H$. Then

$$\begin{aligned}(f_1 + f_2)(hg) &= f_1(hg) + f_2(hg) \\ &= \phi(h) f_1(g) + \phi(h) f_2(g) \\ &= \phi(h)[f_1(g) + f_2(g)] \\ &= \phi(h)(f_1 + f_2)(g).\end{aligned}$$

Since $(f_1 + f_2)(hg) = \phi(h)(f_1 + f_2)(g)$ for each $f_1, f_2 \in \mathrm{Ind}_H^G(\phi)$, $g \in G$, $h \in H$, we have that $f_1 + f_2 \in \mathrm{Ind}_H^G(\phi)$. Therefore, $\mathrm{Ind}_H^G(\phi)$ is closed under addition. The remaining seven axioms can be checked in a similar way.

Theorem 4.18. *Let $\pi(g) : Ind_H^G(\phi) \to Ind_H^G(\phi)$ be given by $(\pi(g) f)(g') = f(g'g)$. Then $\left(\pi, Ind_H^G(\phi)\right)$ is a representation of G induced by ϕ.*

Proof. Let $\pi(g)$ be defined as above. To show that $\left(\pi, \operatorname{Ind}_H^G(\phi)\right)$ is a representation of G induced by ϕ, we must verify the action of G onto $\operatorname{Ind}_H^G(\phi)$. Suppose $g, g', x \in G$, $h \in H$, and $f \in \operatorname{Ind}_H^G(\phi)$. Then π is a homomorphism since

$$\pi\left(gg'\right) f(x) = f\left(xgg'\right) = \pi\left(g'\right) f(xg) = \pi(g)\,\pi\left(g'\right) f(x),$$

and $\pi(g) f \in \operatorname{Ind}_H^G(\phi)$ since

$$\pi(g) f\left(hg'\right) = f\left(hg'g\right) = \phi(h) f\left(g'g\right) = \phi(h)\,\pi(g) f\left(g'\right).$$

Therefore, $\left(\pi, \operatorname{Ind}_H^G(\phi)\right)$ is the representation of G induced by ϕ. □

Proposition 4.19. *Let H be a subgroup of G and χ be the character of a representation of H. Define $\dot{\chi} : G \to \mathbb{C}$ by*

$$\dot{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}.$$

Then the character of the induced representation is

$$Ind_H^G(\chi) = \frac{1}{|H|} \sum_{x \in G} \dot{\chi}(x^{-1} g x).$$

Proof. Let H be a subgroup of G and $\phi : H \to GL_n(\mathbb{C})$. Say $[G : H] = d$ with $G = Hg_1 \uplus Hg_2 \uplus \cdots \uplus Hg_d$. Then we have the induced representation $\operatorname{Ind}_H^G(\phi) : G \to GL_{dn}(\mathbb{C})$. For $g \in G$, denote the matrix representation of $\operatorname{Ind}_H^G(\phi(g))$ as $\phi^G(g)$. Define $\phi^G(g)$ as the $d \times d$ block matrix with $n \times n$ blocks by setting $[\phi^G(g)]_{ij} = \dot{\phi}\left(g_i^{-1} g g_j\right)$ for $1 \le i, j \le d$ and where

$$\dot{\phi}(x) = \begin{cases} \phi(x) & \text{if } x \in H \\ 0 & \text{if } x \notin H \end{cases}.$$

Recall that the character $\chi_\varphi : G \to \mathbb{C}$ of ϕ is defined by setting $\chi_\phi(g) = \mathrm{Tr}(\phi(g))$. Set

$$\dot{\chi_\phi}(x) = \begin{cases} \chi_\phi(x) & \text{if } x \in H \\ 0 & \text{if } x \notin H \end{cases}.$$

Denote the character of $\mathrm{Ind}_H^G(\phi)$ as $\mathrm{Ind}_H^G(\chi_\phi)$. Then

$$\begin{aligned}
\mathrm{Ind}_H^G(\chi_\phi)(g) &= \mathrm{Tr}\left(\phi^G(g)\right) \\
&= \mathrm{Tr}\left(\begin{bmatrix} \dot{\phi}(g_1^{-1}gg_1) & \dot{\phi}(g_1^{-1}gg_2) & \cdots & \dot{\phi}(g_1^{-1}gg_d) \\ \dot{\phi}(g_2^{-1}gg_1) & \dot{\phi}(g_2^{-1}gg_2) & \cdots & \dot{\phi}(g_2^{-1}gg_d) \\ \vdots & \vdots & \cdots & \vdots \\ \dot{\phi}(g_d^{-1}gg_1) & \dot{\phi}(g_d^{-1}gg_2) & \cdots & \dot{\phi}(g_d^{-1}gg_d) \end{bmatrix}\right) \\
&= \sum_{i=1}^{d} \mathrm{Tr}\left(\dot{\phi}\left(g_i^{-1}gg_i\right)\right) \\
&= \sum_{i=1}^{d} \dot{\chi}_\phi\left(g_i^{-1}gg_i\right).
\end{aligned}$$

So $\mathrm{Ind}_H^G(\chi_\phi)(g) = \sum_{i=1}^{d} \dot{\chi}_\phi\left(g_i^{-1}gg_i\right)$.and since χ is a class function on H, it follows that $\dot{\chi}_\phi\left(g_i^{-1}gg_i\right) = \dot{\chi}_\phi\left(g_i^{-1}h^{-1}ghg_i\right)$ for all $h \in H$. So, by summing $\dot{\chi}_\phi\left(g_i^{-1}h^{-1}ghg_i\right)$ for $1 \le i \le d$, we have $|H|$ copies of each $\{g_i\}$ and thus to maintain equality of $\mathrm{Ind}_H^G(\chi_\phi)(g)$ we must divide by $|H|$. That is,

$$\begin{aligned}
\mathrm{Ind}_H^G(\chi_\phi)(g) &= \sum_{i=1}^{d} \dot{\chi}_\phi(g_i^{-1}gg_i) \\
&= \frac{1}{|H|} \sum_{h \in H} \sum_{i=1}^{d} \dot{\chi}_\phi(g_i^{-1}h^{-1}ghg_i) \\
&= \frac{1}{|H|} \sum_{h \in H} \sum_{i=1}^{d} \dot{\chi}_\phi((hg_i)^{-1}g(hg_i)).
\end{aligned}$$

Since $G = Hg_1 \dot\cup Hg_2 \dot\cup \cdots \dot\cup Hg_d$,

$$\begin{aligned}\mathrm{Ind}_H^G(\chi_\phi)(g) &= \frac{1}{|H|}\sum_{h\in H}\sum_{i=1}^{d}\dot\chi_\phi((hg_i)^{-1}g(hg_i)) \\ &= \frac{1}{|H|}\sum_{x\in G}\dot\chi_\phi(x^{-1}gx).\end{aligned}$$

□

5 Basis of Induced Representation

Theorem 5.1. *Let G be a group and let H be a subgroup of G such that $|G\backslash H| = d$, where $\{g_1, g_2, \ldots, g_d\}$ are the coset representatives of H in G. Also, let (φ, W) be a representation of H and let $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$ be a basis of W. Then for $1 \le i, k \le m$ and $1 \le j \le n$,*

$$f_{ij}(g_k) = \begin{cases}\mathbf{e_j} & \text{if } k = i\\ \mathbf{0} & \text{if } k \neq i\end{cases}$$

is a basis of $\mathrm{Ind}_H^G(\varphi)$.

Proof. Let H be a subgroup of G, W be a vector space with $W = \{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$, (φ, W) be a representation of H, and $\{g_1, g_2, \ldots, g_d\}$ be coset representatives of H in G. Then $G = Hg_1 \dot\cup Hg_2 \dot\cup \cdots \dot\cup Hg_d$, and so every $g \in G$ can be written uniquely as $g = hg_i$ for some $g_i \in \{g_1, \ldots, g_d\}$ and $h \in H$. Note that if $f \in \mathrm{Ind}_H^G(\varphi)$ and $g = hg_i$ in G is fixed, then we have that

$$f(g) = f(hg_i) = \varphi(h)\, f(g_i).$$

Because the group action of H on W extends each d coset representatives to its entire coset, each $f \in \mathrm{Ind}_H^G(\varphi)$ is determined by its value of $\{f(g_i)\}_{i=1}^{d}$. Therefore, we need only consider the representatives $\{g_1, g_2, \ldots, g_d\}$ of the transversals of H in G.

Consider $g_k \in G$ such that g_k is a coset representative of H in G. Let $f \in \mathrm{Ind}_H^G(\varphi)$. So $f : G \to W$ and so $f(g_k) \in W$. Therefore $f(g_k) \in \mathrm{Span}\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$. Say

$$f(g_k) = \alpha_{k1}\mathbf{e_1} + \alpha_{k2}\mathbf{e_2} + \cdots + \alpha_{kn}\mathbf{e_n}$$

for some scalars $\alpha_{k1}, \alpha_{k2}, \ldots, \alpha_{kn}$. By definition of f_{ij},

$$\mathbf{e_1} = f_{k1}(g_k),\ \mathbf{e_2} = f_{k2}(g_k), \ldots,\ \mathbf{e_n} = f_{kn}(g_k)$$

and therefore

$$\begin{aligned} f(g_k) &= \alpha_{k1}\mathbf{e_1} + \alpha_{k2}\mathbf{e_2} + \cdots + \alpha_{kn}\mathbf{e_n} \\ &= \alpha_{k1} f_{k1}(g_k) + \alpha_{k2} f_{k2}(g_k) + \cdots + \alpha_{kn} f_{kn}(g_k) \\ &= \sum_{i=1}^{d}\sum_{j=1}^{n} \alpha_{ij} f_{ij}(g_k), \end{aligned} \tag{1}$$

where the last equality holds by definition of f_{ij}. Thus for each $g_k \in \{g_1, g_2, \ldots, g_d\}$, $f(g_k) = \sum_{i=1}^{d}\sum_{j=1}^{n} \alpha_{ij} f_{ij}(g_k)$.

Now consider $g \in G$ where g need not be a coset representative of H in G. Because $G = Hg_1 \uplus Hg_2 \uplus \cdots \uplus Hg_d$, then for some unique $1 \le k \le d$, $g \in Hg_k$. Hence for some unique $h \in H$, $g = hg_k$. Thus,

$$f(g) = f(hg_k) = \varphi(h)\, f(g_k) = \varphi(h)\left(\sum_{j=1}^{n} \alpha_{kj}\mathbf{e_j}\right) \in W.$$

Therefore $\varphi(h)\left(\sum_{j=1}^{n} \alpha_{kj}\mathbf{e_j}\right) \in \operatorname{Span}\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$. Thus there exist unique scalars $\beta_{k1}, \beta_{k2}, \ldots, \beta_{kn}$ such that

$$\varphi(h)\left(\sum_{j=1}^{n} \alpha_{kj}\mathbf{e_j}\right) = \beta_{k1}\mathbf{e_1} + \beta_{k2}\mathbf{e_2} + \cdots + \beta_{kn}\mathbf{e_n}. \tag{2}$$

As before, by applying the definition of f_{ij}, we have

$$\begin{aligned} f(g) &= \varphi(h)\left(\sum_{j=1}^{n} \alpha_{kj}\mathbf{e_j}\right) \\ &= \beta_{k1}\mathbf{e_1} + \beta_{k2}\mathbf{e_2} + \cdots + \beta_{kn}\mathbf{e_n} \\ &= \beta_{k1} f_{k1}(g_k) + \beta_{k2} f_{k2}(g_k) + \cdots + \beta_{kn} f_{kn}(g_k) \\ &= \sum_{i=1}^{d}\sum_{j=1}^{n} \beta_{ij} f_{ij}(g). \end{aligned}$$

Therefore for every $g \in G$, $f(g) = \sum_{i=1}^{d}\sum_{j=1}^{n}\beta_{ij}f_{ij}(g)$, i.e., $f = \sum_{i=1}^{d}\sum_{j=1}^{n}\beta_{ij}f_{ij}$. Hence $\operatorname{Ind}_H^G(\varphi) = \operatorname{Span}\{f_{ij}\}_{\substack{1\le i\le d\\ 1\le j\le n}}$.

Now, let $f(g_k)$ and $f(g)$ be defined as in (1) and (2), respectively. Suppose $f = 0$. Therefore

$$\begin{aligned} f(g_k) &= \alpha_{k1}\mathbf{e_1} + \alpha_{k2}\mathbf{e_2} + \cdots + \alpha_{kn}\mathbf{e_n} \\ &= \alpha_{k1}f_{k1}(g_k) + \alpha_{k2}f_{k2}(g_k) + \cdots + \alpha_{kn}f_{kn}(g_k) \\ &= \sum_{i=1}^{d}\sum_{j=1}^{n}\alpha_{ij}f_{ij}(g_k) \\ &= \mathbf{0}, \end{aligned}$$

and

$$\begin{aligned} f(g) &= \varphi(h)\left(\sum_{j=1}^{n}\alpha_{kj}\mathbf{e_j}\right) \\ &= \beta_{k1}\mathbf{e_1} + \beta_{k2}\mathbf{e_2} + \cdots + \beta_{kn}\mathbf{e_n} \\ &= \beta_{k1}f_{k1}(g_k) + \beta_{k2}f_{k2}(g_k) + \cdots + \beta_{kn}f_{kn}(g_k) \\ &= \sum_{i=1}^{d}\sum_{j=1}^{n}\beta_{ij}f_{ij}(g) \\ &= \mathbf{0}. \end{aligned}$$

Because $W = \{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$ and

$$f(g_k) = \sum_{i=1}^{d}\sum_{j=1}^{n}\alpha_{ij}f_{ij}(g_k) = f(g) = \sum_{i=1}^{d}\sum_{j=1}^{n}\beta_{ij}f_{ij}(g) = \mathbf{0},$$

it follows that each $\alpha_{ij}, \beta_{ij} = 0$. Therefore, if $f = 0$, then each scalar of every f_{ij} must be equal to 0. Hence $\{f_{ij}\}_{\substack{1\le i\le d\\ 1\le j\le n}}$ is linearly independent.

Hence, for $1 \le i, k \le d$ and $1 \le j \le n$,

$$f_{ij}(g_k) = \begin{cases} \mathbf{e_j} & \text{if } k = i \\ \mathbf{0} & \text{if } k \ne i \end{cases}$$

is a basis of $\mathrm{Ind}_H^G(\varphi)$. □

6 Frobenius Reciprocity

Theorem 6.1 (Frobenius reciprocity). *Suppose that H is a subgroup of G and let α be a class function on H and β be a class function on G. Then the formula*

$$\left\langle Ind_H^G(\alpha), \beta \right\rangle = \left\langle \alpha, Res_H^G(\beta) \right\rangle$$

holds.

Proof. We have

$$\begin{aligned}
\left\langle \mathrm{Ind}_H^G(\alpha), \beta \right\rangle &= \frac{1}{|G|} \sum_{g \in G} \mathrm{Ind}_H^G \alpha(g) \overline{\beta(g)} && \text{definition of group algebra} \\
&= \frac{1}{|G|} \sum_{g \in G} \frac{1}{|H|} \sum_{x \in G} \dot{\alpha}(x^{-1}gx) \overline{\beta(g)} && \text{definition of } \mathrm{Ind}_H^G \alpha(g) \\
&= \frac{1}{|G|} \frac{1}{|H|} \sum_{x \in G} \sum_{g \in G} \dot{\alpha}(x^{-1}gx) \overline{\beta(g)}.
\end{aligned}$$

Recall that

$$\dot{\alpha}(x^{-1}gx) = \begin{cases} \alpha(x^{-1}gx) & \text{if } x^{-1}gx \in H \\ 0 & \text{if } x^{-1}gx \notin H \end{cases}$$

and $x^{-1}gx \in H$ if and only if $x^{-1}gx = h$ for some $h \in H$, i.e., $g = xhx^{-1}$. By substituting such

g and h, we have

$$\begin{aligned}\left\langle \mathrm{Ind}_H^G(\alpha), \beta \right\rangle &= \frac{1}{|G|}\frac{1}{|H|}\sum_{x\in G}\sum_{g\in G}\dot{\alpha}(x^{-1}gx)\overline{\beta(g)} \\ &= \frac{1}{|G|}\frac{1}{|H|}\sum_{x\in G}\sum_{h\in H}\alpha(h)\overline{\beta(xhx^{-1})} \\ &= \frac{1}{|G|}\frac{1}{|H|}\sum_{x\in G}\sum_{h\in H}\alpha(h)\overline{\beta(h)} \\ &= \frac{1}{|G|}\sum_{x\in G}\left\langle \alpha, \mathrm{Res}_H^G(\beta)\right\rangle \\ &= \left\langle \alpha, \mathrm{Res}_H^G(\beta)\right\rangle.\end{aligned}$$

Therefore, $\left\langle \mathrm{Ind}_H^G(\alpha), \beta \right\rangle = \left\langle \alpha, \mathrm{Res}_H^G(\beta)\right\rangle$. □

We also have an analogous theorem of Frobenius, but in terms of vector spaces.

Theorem 6.2 (Frobenius Reciprocity)**.** *Let H be a subgroup of G, (π, V) be a representation of H, and (τ, U) be a representation of G. Then*

$$Hom_H\left(Res_H^G(U), V\right) \simeq Hom_G\left(U, Ind_H^G(V)\right).$$

Proof. Suppose $\phi \in \mathrm{Hom}_G\left(U, \mathrm{Ind}_H^G(V)\right)$ and let $\tilde{\phi} : \mathrm{Res}_H^G(U) \to V$ be given by $\tilde{\phi}(\mathbf{u}) = \phi(\mathbf{u})(1)$ for each $\mathbf{u} \in U$. For every $h \in H$,

$$\begin{aligned}\tilde{\phi}(\tau(h)\mathbf{u}) &= \phi(\tau(h)\mathbf{u})(1) \\ &= \left(\pi^G(h)\phi(\mathbf{u})\right)(1) && \because \phi \in \mathrm{Hom}_G\left(U, \mathrm{Ind}_H^G(V)\right) \\ &= \phi(\mathbf{u})(h) \\ &= \pi(h)\phi(\mathbf{u})(1). && \because \phi(\mathbf{u}) \in \mathrm{Ind}_H^G(V)\end{aligned}$$

Therefore, $\tilde{\phi} \in \mathrm{Hom}_H\left(\mathrm{Res}_H^G(U), V\right)$.

Suppose $\psi \in \mathrm{Hom}_H\left(\mathrm{Res}_H^G(U), V\right)$ and $\tilde{\psi} : U \to \mathrm{Ind}_H^G(V)$ by $\tilde{\psi}(\mathbf{u})(x) = \psi(\tau(x)\mathbf{u})$ for each $\mathbf{u} \in U$ and $x \in G$. For every $h \in H$, $x \in G$, $\mathbf{u} \in U$,

$$\begin{aligned}\tilde{\psi}(\mathbf{u})(hx) &= \psi(\tau(hx)\mathbf{u}) \\ &= \psi(\tau(h)\tau(x)\mathbf{u}) \\ &= \pi(h)\psi(\tau(x)\mathbf{u}) \\ &= \pi(h)\tilde{\psi}(\mathbf{u})(x).\end{aligned}$$

Thus $\tilde{\psi}(\mathbf{u}) \in \mathrm{Ind}_H^G(V)$. And for every $g, x \in G$, $\mathbf{u} \in U$,

$$\begin{aligned}\tilde{\psi}(\tau(g)\mathbf{u})(x) &= \psi(\tau(x)\tau(g)\mathbf{u}) \\ &= \psi(\tau(xg)\mathbf{u}) \\ &= \tilde{\psi}(\mathbf{u})(xg) \\ &= \left(\pi^G(g)\tilde{\psi}(\mathbf{u})\right)(x). \qquad \because \tilde{\psi}(\mathbf{u}) \in \mathrm{Ind}_H^G(V)\end{aligned}$$

Therefore, $\tilde{\psi} \in \mathrm{Hom}_G\left(U, \mathrm{Ind}_H^G(V)\right)$.

Now, define $\Psi : \mathrm{Hom}_G\left(U, \mathrm{Ind}_H^G(V)\right) \to \mathrm{Hom}_H\left(\mathrm{Res}_H^G(U), V\right)$ by $\Psi(\phi) = \tilde{\phi}$. Also, define $\Phi : \mathrm{Hom}_H\left(\mathrm{Res}_H^G(U), V\right) \to \mathrm{Hom}_G\left(U, \mathrm{Ind}_H^G(V)\right)$ by $\Phi(\psi) = \tilde{\psi}$.

Suppose $\phi \in \mathrm{Hom}_G\left(U, \mathrm{Ind}_H^G(V)\right)$, $\mathbf{u} \in U$, and $x \in G$. Then

$$\begin{aligned}\Phi(\Psi(\phi))(\mathbf{u})(x) &= \Psi(\phi)(\tau(x)(\mathbf{u})) \\ &= \phi(\tau(x)\mathbf{u})(1) \\ &= (\pi(x)\phi(\mathbf{u}))(1) \\ &= \phi(\mathbf{u})(x).\end{aligned}$$

Therefore, $\Phi(\Psi(\phi)) = \phi$.

In a similar manner, $\Psi(\Phi(\psi)) = \psi$. We have thus shown that the maps $\phi \mapsto \tilde{\phi}$ and $psi \mapsto \tilde{\psi}$ are inverses.

Hence, $\mathrm{Hom}_H\left(\mathrm{Res}_H^G(U), V\right) \simeq \mathrm{Hom}_G\left(U, \mathrm{Ind}_H^G(V)\right)$. $\square$

7 Mackey's Theorem

If H is a subgroup of G and χ is an irreducible character of H, then $\mathrm{Ind}_H^G\chi$ need not be an irreducible character of G. We thus employ Mackey's theorem, which describes when an induced character is irreducible.

Example 7.1. The character table for the induced representation of S_4 from S_3 is shown below.

Table 3: Character Table of the Induced Representation of S_4

	(1)	(12)	(123)	(1234)	(12)(34)
$\mathrm{Ind}_{S_3}^{S_4}(\chi_1)$	4	2	1	0	0
$\mathrm{Ind}_{S_3}^{S_4}(\chi_2)$	4	-2	1	0	0
$\mathrm{Ind}_{S_3}^{S_4}(\chi_3)$	8	0	-1	0	0

To constuct this character table for $\mathrm{Ind}_{S_3}^{S_4}(\chi_\varphi)$, we use the proven formula from Proposition 4.19.

We have that

$$(1)(1)(1) = (1) \in S_3$$
$$(14)(1)(14) = (1) \in S_3$$
$$(24)(1)(24) = (1) \in S_3$$
$$(34)(1)(34) = (1) \in S_3$$

and so $\mathrm{Ind}_{S_3}^{S_4}(\chi_1)((1)) = 4$, $\mathrm{Ind}_{S_3}^{S_4}(\chi_2)((1)) = 4$, and $\mathrm{Ind}_{S_3}^{S_4}(\chi_3)((1)) = 8$. Next,

$$(1)(12)(1) = (12) \in S_3$$
$$(14)(12)(14) = (24) \notin S_3$$
$$(24)(12)(24) = (14) \notin S_3$$
$$(34)(12)(34) = (12) \in S_3$$

and so $\text{Ind}_{S_3}^{S_4}(\chi_1)((12)) = 2$, $\text{Ind}_{S_3}^{S_4}(\chi_2)((12)) = -2$, and $\text{Ind}_{S_3}^{S_4}(\chi_3)((12)) = 0$. Lastly,

$$\begin{aligned}(1)(123)(1) &= (123) \in S_3\\(14)(123)(14) &= (234) \notin S_3\\(24)(123)(24) &= (143) \notin S_3\\(34)(123)(34) &= (124) \notin S_3\end{aligned}$$

and so $\text{Ind}_{S_3}^{S_4}(\chi_1)((123)) = 1$, $\text{Ind}_{S_3}^{S_4}(\chi_2)((123)) = 1$, and $\text{Ind}_{S_3}^{S_4}(\chi_3)((123)) = -1$.

Here, $\text{Ind}_{S_3}^{S_4}(\chi_1)(1) = 4$. Because there is no irreducible representation of degree 4, this induced representation is not irreducible.

The statement and proof of Mackey's theorem are largely outlined in [1] by exercises left for the reader. In accordance with Prasad's lecture notes, a successful proof of Mackey's theorem has been constructed and provided in this section. The exercises provided by Prasad have been revised into a string of lemmas from which the proof follows.

Let H_1 and H_2 be subgroups of G. Let (π_1, V_1) and (π_2, V_2) be representations of H_1 and H_2, respectively. For $f : G \to V_1$ and $\Delta : G \to \text{Hom}_{\mathbb{C}}(V_1, V_2)$, define a convolution $\Delta * f : G \to V_2$ by

$$(\Delta * f)(x) = \frac{1}{|G|} \sum_{g \in G} \Delta\left(xg^{-1}\right)(f(g)).$$

Define the set of functions

$$D = \left\{ \Delta : G \to \text{Hom}_{\mathbb{C}}(V_1, V_2) \;\middle|\; \begin{array}{l} \Delta(h_2 g h_1) = \pi_2(h_2) \circ \Delta(g) \circ \pi_1(h_1) \\ \text{for all } g \in G,\ h_1 \in H_1,\ h_2 \in H_2 \end{array} \right\}.$$

We now have the necessary definitions to make sense of Mackey's theorem.

Theorem 7.2 (Mackey's Theorem)**.** *The map $\Delta \mapsto L_\Delta$ is an isomorphism from $D \to Hom_G\left(V_1^G, V_2^G\right)$.*

Providing that we have each of the following lemmas, the proof of Mackey's theorem is straightforward.

Lemma 7.3. *If $\Delta \in D$ and $f_1 \in V_1^G$, then $\Delta * f_1 \in V_2^G$.*

Proof. Suppose $\Delta \in D$ and $f_1 \in V_1^G$. For each $g \in G$ and $h_2 \in H$, our goal is to show that $(\Delta * f_1)(h_2 g) = (\pi_2(h_2)(\Delta * f_1))(g)$. Indeed,

$$\begin{aligned}
(\Delta * f_1)(h_2 g) &= \frac{1}{|G|}\sum_{x\in G}\Delta\left(h_2 g x^{-1}\right) f_1(x) && \text{by definition of } \Delta * f_1\\
&= \frac{1}{|G|}\sum_{x\in G}\pi_2(h_2)\circ\Delta\left(g x^{-1}\right) f_1(x) && \text{by definition of } \Delta\\
&= \pi_2(h_2)\circ\frac{1}{|G|}\sum_{x\in G}\Delta\left(g x^{-1}\right) f_1(x) && \text{since } \pi_2 \text{ is linear}\\
&= \pi_2(h_2)\circ(\Delta * f_1)(g) && \text{by definition of } \Delta * f_1\\
&= (\pi_2(h_2)(\Delta * f_1))(g). && \text{by definition of function composition}
\end{aligned}$$

Therefore, $\Delta * f_1 : G \to V_2$ such that for each $h_2 \in H_2$ and $g \in G$, $(\Delta * f_1)(h_2 g) = (\pi_2(h_2)(\Delta * f_1))(g)$. Hence $\Delta * f_1 \in V_2^G$. □

Lemma 7.4. *Let $L_\Delta : V_1^G \to V_2^G$ be given by $L_\Delta(f_1) = \Delta * f_1$. Then $L_\Delta \in Hom_G\left(V_1^G, V_2^G\right)$.*

Proof. Define $L_\Delta : V_1^G \to V_2^G$ by $L_\Delta(f_1) = \Delta * f_1$. Then for each $g, g' \in G$ and $f_1 \in V_1^G$,

$$\begin{aligned}
\pi_2^G(g)(L_\Delta(f_1))\left(g'\right) &= (L_\Delta(f_1))\left(g' g\right)\\
&= (\Delta * f_1)\left(g' g\right)\\
&= \frac{1}{|G|}\sum_{x\in G}\Delta\left(g' g x^{-1}\right) f_1(x) && \text{definition of } \Delta * f_1\\
&= \frac{1}{|G|}\sum_{y\in G}\Delta\left(g' y^{-1}\right) f_1(yg) && \text{change of variables } x = yg\\
&= \frac{1}{|G|}\sum_{y\in G}\Delta\left(g' y^{-1}\right)\left(\pi_1^G(g) f_1\right)(y) && \text{action of } G \text{ since } f_1 \in V_1^G\\
&= \left(\Delta * \pi_1^G(g) f_1\right)\left(g'\right) && \text{definition of } \Delta * \pi_1^G(g) f_1\\
&= L_\Delta\left(\pi_1^G(g) f_1\right)\left(g'\right).
\end{aligned}$$

Therefore, $\pi_2^G(g)(L_\Delta f_1) = L_\Delta\left(\pi_1^G(g) f_1\right)$ and so $L_\Delta \in \mathrm{Hom}_G\left(V_1^G, V_2^G\right)$. □

Define $L : D \to \mathrm{Hom}_G\left(V_1^G, V_2^G\right)$ by $L(\Delta) = L_\Delta$.

Lemma 7.5. *The map L is a linear transformation.*

Proof. Let $g \in G, \Delta_1, \Delta_2 \in D,\ f \in V_1^G, \alpha \in \mathbb{C}$. Then L is linear since

$$
\begin{aligned}
L(\Delta_1 + \Delta_2)(f)(g) &= L_{\Delta_1+\Delta_2}(f)(g) \\
&= ((\Delta_1 + \Delta_2) * f)(g) && \text{by definition of } L_\Delta \\
&= \frac{1}{|G|} \sum_{x \in G} (\Delta_1 + \Delta_2)\left(gx^{-1}\right) f(x) && \text{by definition of } \Delta * f_1 \\
&= \frac{1}{|G|} \sum_{x \in G} \left(\Delta_1\left(gx^{-1}\right) + \Delta_2\left(gx^{-1}\right)\right) f(x) \\
&= \frac{1}{|G|} \sum_{x \in G} \Delta_1\left(gx^{-1}\right) f(x) + \Delta_2\left(gx^{-1}\right) f(x) \\
&= \frac{1}{|G|} \sum_{x \in G} \Delta_1\left(gx^{-1}\right) f(x) + \frac{1}{|G|} \sum_{x \in G} \Delta_2\left(gx^{-1}\right) f(x) \\
&= (\Delta_1 * f)(g) + (\Delta_2 * f)(g) && \text{by definition of } \Delta * f \\
&= L_{\Delta_1}(f)(g) + L_{\Delta_2}(f)(g) && \text{by definition of } L_\Delta \\
&= L(\Delta_1)(f)(g) + L(\Delta_2)(f)(g),
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha L(\Delta)(f)(g) &= \alpha L_\Delta(f)(g) \\
&= \alpha(\Delta * f)(g) \\
&= \alpha \frac{1}{|G|} \sum_{x \in G} \Delta\left(gx^{-1}\right) f(x) \\
&= \frac{1}{|G|} \sum_{x \in G} \alpha\Delta\left(gx^{-1}\right) f(x) \\
&= (\alpha\Delta * f)(g) \\
&= L_{\alpha\Delta}(f)(g) \\
&= L(\alpha\Delta)(f)(g).
\end{aligned}
$$

□

For any $g \in G$ and $\mathbf{v} \in V_1$, define a collection $f_{g,\mathbf{v}}$ of elements in V_1^G as

$$f_{g,\mathbf{v}}(x) = \begin{cases} \pi_1(h)\mathbf{v} & \text{if } x = hg, h \in H_1 \\ 0 & \text{if } x \notin H_1 g \end{cases}.$$

Lemma 7.6. *Let $J \in Hom_G\left(V_1^G, V_2^G\right)$. Define $\Delta_J : G \to Hom_{\mathbb{C}}(V_1, V_2)$ by*

$$\Delta_J(g)(\mathbf{v}) = \frac{|G|}{|H_1|} J\left(f_{g^{-1},\mathbf{v}}\right)(1)$$

for every $g \in G$, $\mathbf{v} \in V_1$. Then $\Delta_J \in D$.

Lemma 7.7. *The map L is a bijection.*

Proof. We will first show that L is injective. Suppose $L(\Delta) = 0$ for some $\Delta \in D$. Let $x \in G$. Then

$$0 = (L(\Delta)(f_{g,\mathbf{v}}))(x) = (L_\Delta(f_{g,\mathbf{v}}))(x) = (\Delta * f_{g,\mathbf{v}})(x) = \frac{1}{|G|}\sum_{y \in G} \Delta\left(xy^{-1}\right) f_{g,\mathbf{v}}(y).$$

Notice that if $y \notin H_1 g$, then $f_{g,\mathbf{v}}(y) = 0$. Thus, we need only consider $y \in G$ such that $y = hg$ for some unique $h \in H_1$. Continuing from above, we have

$$\begin{aligned} 0 &= \frac{1}{|G|}\sum_{y \in G} \Delta\left(xy^{-1}\right) f_{g,\mathbf{v}}(y) \\ &= \frac{1}{|G|}\sum_{h \in H_1 g} \Delta\left(x(hg)^{-1}\right) \pi_1(h)\mathbf{v} \\ &= \frac{1}{|G|}\sum_{h \in H_1 g} \Delta\left(xg^{-1}h^{-1}\right) \pi_1(h)\mathbf{v} \\ &= \frac{1}{|G|}\sum_{h \in H_1 g} \Delta\left(xg^{-1}h^{-1}h\right) \mathbf{v} \\ &= \frac{1}{|G|}\sum_{h \in H_1 g} \Delta\left(xg^{-1}\right) \mathbf{v} \\ &= \frac{|H_1|}{|G|}\Delta\left(xg^{-1}\right)\mathbf{v}. \end{aligned}$$

Because $\frac{|H_1|}{|G|}\Delta\left(xg^{-1}\right)\mathbf{v} = 0$ for each $g, x \in G$ and $\mathbf{v} \in V_1$, it follows that $L(\Delta) = 0$ if and only if $\Delta = 0$. That is, Ker $L = \{0\}$ and L is injective.

We will now show that L is surjective. Let J be as defined in Lemma 7.6. Then for $f \in V_1^G$,

$$(L(\Delta_J) f)(g) = (\Delta_J * f)(g) = \frac{1}{|G|} \sum_{x \in G} \Delta_J \left(gx^{-1}\right) f(x) = \frac{1}{|G|} \sum_{x \in G} \Delta_J \left(gx^{-1}\right) \mathbf{v},$$

where $f(x) = \mathbf{v} \in V_1$.

Then continuing from above,

$$\begin{aligned}
(L(\Delta_J) f)(g) &= \frac{1}{|G|} \sum_{x \in G} \Delta_J \left(gx^{-1}\right) \mathbf{v} \\
&= \frac{1}{|G|} \sum_{x \in G} \frac{|G|}{|H_1|} J\left(f_{(gx^{-1})^{-1}, \mathbf{v}}\right)(1) \\
&= \frac{1}{|H_1|} \sum_{x \in G} J\left(f_{(gx^{-1})^{-1}, \mathbf{v}}\right)(1) \\
&= J\left(\frac{1}{|H_1|} \sum_{x \in G} f_{(gx^{-1})^{-1}, \mathbf{v}}\right)(1) && J \text{ linear} \\
&= J\left(\frac{1}{|H_1|} \sum_{x \in H_1 g} \pi_1\left(gx^{-1}\right)(\mathbf{v})\right) && 1 \in H_1(gx^{-1}) \to xg^{-1} \in H_1 \\
&= J\left(\frac{1}{|H_1|} \sum_{x \in H_1 g} \pi_1\left(gx^{-1}\right) f(x)\right) && \because f(x) = \mathbf{v} \\
&= J\left(\frac{1}{|H_1|} \sum_{x \in H_1 g} f\left(gx^{-1}x\right)\right) && \because f = V_1^G \\
&= J\left(\frac{1}{|H_1|} \sum_{x \in H_1 g} f(g)\right) \\
&= J\left(\frac{1}{|H_1|} |H_1| f(g)\right) \\
&= J(f(g)).
\end{aligned}$$

Therefore, for all $g \in G$ and $f \in V_1^G$, $(L(\Delta_J) f)(g) = J(f(g))$. That is, for each $J \in \mathrm{Hom}_G\left(V_1^G, V_2^G\right)$, there exists $\Delta_J \in D$ such that $L(\Delta_J) = J$. Hence L is surjective. □

Finally, we have the proof of Mackey's theorem.

Theorem 7.8 (Mackey's Theorem)**.** *The map* $\Delta \mapsto L_\Delta$ *is an isomorphism from* D *into* $Hom_G\left(V_1^G, V_2^G\right)$.

Proof. We begin with an arbitrary map $\Delta \mapsto L_\Delta$. As we previously defined, $L_\Delta : V_1^G \to V_2^G$ such that $L_\Delta(f_1) = \Delta * f_1$. It was shown by Lemma 7.4 that $L_\Delta \in \mathrm{Hom}_G\left(V_1^G, V_2^G\right)$. Also recall that $L : D \to \mathrm{Hom}_G\left(V_1^G, V_2^G\right)$ where $L(\Delta) = L_\Delta$. By Lemma 7.5 and 7.7, L is linear and bijective, respectively. Hence L is an isomorphism from D into $\mathrm{Hom}_G\left(V_1^G, V_2^G\right)$. □

Part IV

Representations of $GL_2(\mathbb{F}_q)$

Assume G to be the group $GL_2(\mathbb{F}_q)$.

Definition 7.9. The *Borel subgroup* of G is the group $B = \left\{ \begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix} : \alpha, \delta \in \mathbb{F}_q^\times, \beta \in \mathbb{F}_q \right\}$.

A subgroup of B is the group $T = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix} : \alpha, \delta \in \mathbb{F}_q^\times \right\}$. Another subgroup of B is the group $N = \left\{ \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} : \beta \in \mathbb{F}_q \right\}$.

8 The Borel Subgroup

Proposition 8.1. *1. Every $b \in B$ can be written uniquely as $b = tn$, where $t \in T$ and $n \in N$;*

2. N is a normal subgroup of B;

3. $B/N \simeq T$.

Proof. 1. Let $t \in T$ and $n \in N$ such that $t = \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}$ and $n = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}$. The product $t \cdot n$ is

$$\begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix} \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & \alpha\beta \\ 0 & \delta \end{bmatrix}$$

and so $t \cdot n \in B$. Setting $t \cdot n$ equal to an arbitrary element of B, say $\begin{bmatrix} \alpha' & \beta' \\ 0 & \delta' \end{bmatrix} = \begin{bmatrix} \alpha & \alpha\beta \\ 0 & \delta \end{bmatrix}$, yields $\alpha' = \alpha$, $\beta' = \alpha\beta$, and $\delta' = \delta$. It is now clear that each matrix in B is unique with respect to its factor from T.

2. For some $\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{bmatrix} \in B$ with $\alpha_{12} \neq 0$ and $\begin{bmatrix} 1 & \beta_{12} \\ 0 & 1 \end{bmatrix} \in N$, we have

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{bmatrix} \begin{bmatrix} 1 & \beta_{12} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{11}\beta_{12} + \alpha_{12} \\ 0 & \alpha_{22} \end{bmatrix},$$
$$\begin{bmatrix} 1 & \beta_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{22}\beta_{12} + \alpha_{12} \\ 0 & \alpha_{22} \end{bmatrix}.$$

Since $bN = Nb$ for all $b \in B$, N is a normal subgroup of B.

Alternatively, we can show that N is a normal subgroup of B merely using the properties of determinants. That is, for every $b \in B$ and $n \in N$,

$$\det\left(bnb^{-1}\right) = \det(b)\det(n)\det\left(b^{-1}\right) = \det(b)\det\left(b^{-1}\right)\det(n) = \det(n) = 1.$$

Now, since $bnb^{-1} \in N$ for each $b \in B$ and $n \in N$, we have that N is a normal subgroup of B.

3. Define $\varphi : B \to T$ by $\varphi\left(\begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix}\right) = \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}$. For each $\begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix}, \begin{bmatrix} \alpha' & \beta' \\ 0 & \delta' \end{bmatrix} \in B$,

$$\varphi\left(\begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix}\begin{bmatrix} \alpha' & \beta' \\ 0 & \delta' \end{bmatrix}\right) = \varphi\left(\begin{bmatrix} \alpha\alpha' & \alpha\beta' + \beta\gamma' \\ 0 & \delta\delta' \end{bmatrix}\right) = \begin{bmatrix} \alpha\alpha' & 0 \\ 0 & \delta\delta' \end{bmatrix}$$

and

$$\varphi\left(\begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix}\right)\varphi\left(\begin{bmatrix} \alpha' & \beta' \\ 0 & \delta' \end{bmatrix}\right) = \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}\begin{bmatrix} \alpha' & 0 \\ 0 & \delta' \end{bmatrix} = \begin{bmatrix} \alpha\alpha' & 0 \\ 0 & \delta\delta' \end{bmatrix}.$$

So φ is a homomorphism. And since $N = \operatorname{Ker}(\varphi)$, then by the First Isomorphism Theorem, $B/N \simeq \varphi(B)$. Moreover, for each $\begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix} \in T$ there exists $\begin{bmatrix} t_1 & * \\ 0 & t_2 \end{bmatrix} \in B$ such that

$\varphi\left(\begin{bmatrix} t_1 & * \\ 0 & t_2 \end{bmatrix}\right) = \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix}$. So φ is surjective and hence $B/N \simeq T$.

□

Proposition 8.2. *The order of the group $GL_n(\mathbb{F}_q)$ is defined to be*

$$|GL_n(\mathbb{F}_q)| = \prod_{i=0}^{n-1} \left(q^n - q^i\right).$$

Proof. Let $A \in GL_n(\mathbb{F}_q)$ such that $A = \begin{bmatrix} \mathbf{a_1} \, \mathbf{a_2} \cdots \mathbf{a_n} \end{bmatrix}$. Consider first the column vector. There are q possible choices for each of the n entries. As a column vector, however, at least one entry must be non-zero. Therefore there are $q^n - 1$ choices for $\mathbf{a_1}$.

Now consider the second column vector, $\mathbf{a_2}$. We begin with $q^n - 1$ possibilities for $\mathbf{a_2}$. However, A is invertible and so each column vector of A must be linearly independent. That is, $\mathbf{a_2} \neq \lambda \mathbf{a_1}$. Thus we have that at least one entry in $\mathbf{a_2}$ not equivalent to its corresponding entry in $\mathbf{a_1}$. Thus there are $q^n = q$ choices for $\mathbf{a_2}$.

We may begin with having $q^n - q$ choices for $\mathbf{a_3}$, but because each column vector is linearly independent, at least one more entry in $\mathbf{a_3}$ is distinct. Thus there are $q^n = q^2$ choices for $\mathbf{a_3}$.

Continuing this way, there are $q^n - q^{n-1}$ choices for $\mathbf{a_n}$.

Hence there are $\prod_{i=0}^{n-1} \left(q^n - q^i\right)$ possible matrices for some $A \in GL_n(\mathbb{F}_q)$. That is, $|GL_n(\mathbb{F}_q)| = \prod_{i=0}^{n-1} \left(q^n - q^i\right)$. □

Fix $s = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Proposition 8.3 (Bruhat decomposition)**.** *The (disjoint) union of $B \backslash G / B$ is all of G. That is, $G = B \dot{\cup} BsB$.*

Proof. Consider $\begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \in G$. If $\gamma_{21} = 0$, then $\begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \in B$. So, assume $\gamma_{21} \neq 0$ and let

$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{bmatrix}, \begin{bmatrix} \beta_{11} & \beta_{12} \\ 0 & \beta_{22} \end{bmatrix} \in B$. Then

$$BsB \ni \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \beta_{11} & \beta_{12} \\ 0 & \beta_{22} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{bmatrix} \begin{bmatrix} 0 & -\beta_{22} \\ \beta_{11} & \beta_{12} \end{bmatrix} = \begin{bmatrix} \alpha_{12}\beta_{11} & \alpha_{12}\beta_{12} - \alpha_{11}\beta_{22} \\ \alpha_{22}\beta_{11} & \alpha_{22}\beta_{12} \end{bmatrix}.$$

As $\begin{bmatrix} \alpha_{12}\beta_{11} & \alpha_{12}\beta_{12} - \alpha_{11}\beta_{22} \\ \alpha_{22}\beta_{11} & \alpha_{22}\beta_{12} \end{bmatrix} \in G$, its determinant is nonzero. That is,

$$\alpha_{12}\beta_{11}\alpha_{22}\beta_{12} - \alpha_{12}\beta_{11}\alpha_{22}\beta_{12} + \alpha_{11}\beta_{22}\alpha_{22}\beta_{12} = \alpha_{11}\beta_{22}\alpha_{22}\beta_{12} \neq 0$$

and thus $\alpha_{11} \neq 0$, $\alpha_{22} \neq 0$, $\beta_{12} \neq 0$, and $\beta_{22} \neq 0$. We now have that the entry $\alpha_{22}\beta_{11}$ is nonzero and so we may put $\begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} = \begin{bmatrix} \alpha_{12}\beta_{11} & \alpha_{12}\beta_{12} - \alpha_{11}\beta_{22} \\ \alpha_{22}\beta_{11} & \alpha_{22}\beta_{12} \end{bmatrix}$. This illustrates that whenever $g \in G$ but $g \notin B$, then $g \in BsB$. Hence $G = B \mathbin{\dot\cup} BsB$. □

9 The Main Result

For characters χ_1 and χ_2 of $\mathbb{F}_q^\times$, define the character χ of T by

$$\chi \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix} = \chi_1(\alpha)\chi_2(\delta).$$

We extend χ to a character of B by letting N lie in the kernel. Thus,

$$\chi \begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix} = \chi_1(\alpha)\chi_2(\delta).$$

The representation of G induced from this character χ of B is

$$I(\chi_1, \chi_2) = \left\{ f : G \to \mathbb{C} \;\middle|\; \begin{array}{l} f\left(\begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix} g\right) = \chi_1(\alpha)\chi_2(\delta)\chi(g) \\ \text{for all } \alpha, \delta \in \mathbb{F}_q^\times,\ \beta \in \mathbb{F}_q,\ g \in G \end{array} \right\}.$$

Theorem 9.1. *Let $\chi_1, \chi_2, \mu_1, \mu_2$ be characters of $\mathbb{F}_q^\times$. Then*

$$dim\ Hom_G\left(I\left(\chi_1, \chi_2\right), I\left(\mu_1, \mu_2\right)\right) = e_1 + e_s,$$

where

$$e_1 = \begin{cases} 1 & \text{if } \chi_1 = \mu_1 \text{ and } \chi_2 = \mu_2 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$e_s = \begin{cases} 1 & \text{if } \chi_1 = \mu_2 \text{ and } \chi_2 = \mu_1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose $\chi_1, \chi_2, \mu_1, \mu_2$ are characters of $\mathbb{F}_p^\times$ and $\Delta \in D$. Then for each $g \in G$ and $b_1, b_2 \in B$, $\Delta\left(b_2 g b_1\right) = \chi\left(b_2\right) \Delta\left(g\right) \mu\left(b_1\right)$. Following from the Bruhat decomposition, each $g \in G$ is of the form $Bs'B$ where $s' \in \{1, s\}$. So each function Δ in the set D is determined by its values at 1 and s, and moreover, the dimension of D is no more than two.

To begin, let $g = 1$ and $t = \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}$. Note $\Delta\left(t\right) = \Delta\left(t \cdot 1\right) = \chi\left(t\right) \Delta\left(1\right)$ and $\Delta\left(t\right) = \Delta\left(1 \cdot t\right) = \Delta\left(1\right) \mu\left(t\right)$. Therefore, $\Delta\left(1\right) \mu\left(t\right) = \chi\left(t\right) \Delta\left(1\right)$ and we have

$$\Delta\left(1\right) \mu_1\left(\alpha\right) \mu_2\left(\beta\right) = \chi_1\left(\alpha\right) \chi_2\left(\beta\right) \Delta\left(1\right).$$

If $\mu \neq \chi$ and $t = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}$,

$$\Delta\left(1\right) \mu_1\left(\alpha\right) \mu_2\left(1\right) = \chi_1\left(\alpha\right) \chi_2\left(1\right) \Delta\left(1\right).$$

So $\mu_1 \neq \chi_1$ and $\Delta\left(1\right) = 0$. Similarly, if $t = \begin{bmatrix} 1 & 0 \\ 0 & \delta \end{bmatrix}$ then $\mu_2 \neq \chi_2$ and $\Delta\left(1\right) = 0$. Therefore, if $\mu \neq \chi$ then $\Delta\left(1\right) = 0$. In the case that $\mu = \chi$, define the function

$$\Delta_1\left(b\right) = \begin{cases} \chi\left(b\right) & \text{if } b \in B \\ 0 & \text{if } b \in BsB, \end{cases}$$

and take $\Delta_1 \equiv 0$ whenever $e_1 = 0$.

Now, let $g = s$ and $t = \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}$. So we have

$$\mu(t)\,\Delta(s) = \Delta(ts) = \Delta\left(s\left(s^{-1}ts\right)\right) = \Delta(s)\,\chi\left(s^{-1}ts\right).$$

Since

$$s^{-1}ts = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -\alpha \\ \delta & 0 \end{bmatrix} = \begin{bmatrix} \delta & 0 \\ 0 & \alpha \end{bmatrix},$$

we can write $\mu(t)\,\Delta(s) = \Delta(s)\,\chi\left(s^{-1}ts\right)$ as

$$\mu_1(\alpha)\,\mu_2(\delta)\,\Delta(s) = \Delta(s)\,\chi_1(\delta)\,\chi_2(\alpha).$$

Therefore, if $\mu \neq \chi$ then $\Delta(s) = 0$. However, if $\mu = \chi$, then let the function Δ_s be given by

$$\Delta_s(b_2 s b_1) = \begin{cases} \chi(b_1)\,\mu(b_2) & \text{for all } b_1, b_2 \in B \\ 0 & \text{otherwise,} \end{cases}$$

and define $\Delta_s \equiv 0$ whenever $e_s = 0$.

Notice that any $\Delta \in D$, $\Delta = \lambda_1\Delta_1 + \lambda_2\Delta_2$. We have thus constructed a basis for D, namely $\{\Delta_1, \Delta_s\}$, and hence $\dim \operatorname{Hom}_G(I(\chi_1,\chi_2), I(\mu_1,\mu_2)) = e_1 + e_s$. □

Corollary 9.2. *Let $\chi_1, \chi_2, \mu_1, \mu_2$ be characters of $\mathbb{F}_q^\times$. Then $I(\chi_1,\chi_2)$ is an irreducible representation of degree $q+1$ of $GL_2(\mathbb{F}_p)$ unless $\chi_1 = \chi_2$, in which case it is a direct sum of two irreducible representations having degrees 1 and q. We have*

$$I(\chi_1,\chi_2) \simeq I(\mu_1,\mu_2)$$

if and only if either

$$\chi_1 = \mu_1 \text{ and } \chi_2 = \mu_2$$

or else

$$\chi_1 = \mu_2 \text{ and } \chi_2 = \mu_1.$$

Proof. Following Theorem 9.1, $I(\chi_1, \chi_2)$ is an irreducible representation if and only if $\chi_1 \neq \chi_2$. By Proposition 8.2, $|G| = (q^2 - 1)(q^2 - q)$. Observing some arbitrary matrix in B gives us $|B| = q(q-1)^2$. So $[G : H] = q + 1$. Thus, $\dim I(\chi_1, \chi_2) = [G : H] \dim \chi = q + 1$ since the dimension of χ is one.

We now concern ourselves with the case $\chi_1 = \chi_2$. We have by Theorem 9.1 that $\dim \text{End}_G I(\chi_1, \chi_2) = 2$. Recall that if $I(\chi_1, \chi_2) \sim \pi_1 \oplus \cdots \oplus \pi_d$ where each π_i is an irreducible representation, then $\dim \text{End}_G I(\chi_1, \chi_2) = \sum_{i=1}^d m_i d_i^2$ where m_i is the multiplicaticy of π_i and d_i is the degree of π_i. Applying this definition gives us $2 = \sum_{i=1}^d m_i d_i^2$. Because $2 = 1^2 + 1^2$ is the only way to satisy the former equation, we conclude that $I(\chi_1, \chi_2)$ is the direct sum of two irreducible representations.

Courtesy of Prasad [1], let $f : G \to \mathbb{C}$ be defined by $f(g) = \chi_1(\det(g))$. For any $g \in G$ and $\begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix} \in B$,

$$\begin{aligned} f\left(\begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix} g\right) &= \chi_1\left(\det\left(\begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix} g\right)\right) \\ &= \chi_1\left(\det\left(\begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix}\right) \det(g)\right) \\ &= \chi_1\left(\det\left(\begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix}\right)\right) \chi_1(\det(g)) \\ &= \chi_1(\alpha\beta) f(g) \\ &= \chi_1(\alpha) \chi_1(\beta) f(g) \\ &= \chi_1(\alpha) \chi_2(\beta) f(g). \end{aligned}$$

Thus $f\left(\begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix} g\right) = \chi_1(\alpha) \chi_2(\beta) f(g)$ and so $f \in I(\chi_1, \chi_2)$.

Also, for each $g, g' \in G$,

$$\begin{aligned}(g.f)(g') &= f(g'g) \\ &= \chi_1(\det(g'g)) \\ &= \chi_1(\det(g')\det(g)) \\ &= \chi_1(\det(g')\det(g)) \\ &= \chi_1(\det(g'))\chi_1(\det(g)) \\ &= f(g')\chi_1(\det(g))\end{aligned}$$

and so the subspace $\langle f \rangle$ is G-invariant.

Without loss of generality, let $\pi_1 = \langle f \rangle$ and note the degree of π_1 is thus one. By process of elimination, the irreducible representation π_2 must be of degree q.

Last but not least, we now establish the explicit cases in which $I(\chi_1, \chi_2) \simeq I(\mu_1, \mu_2)$. Clearly, if $\chi_1 = \mu_1$ and $\chi_2 = \mu_2$, then $I(\chi_1, \chi_2) \simeq I(\mu_1, \mu_2)$. Also, $I(\chi_1, \chi_2) \simeq I(\mu_1, \mu_2)$ if and only if $\dim \mathrm{Hom}_G(I(\chi_1, \chi_2), I(\mu_1, \mu_2)) = 1$. If $\dim \mathrm{Hom}_G(I(\chi_1, \chi_2), I(\mu_1, \mu_2)) = 1$, then either $e_1 = 1$ or $e_s = 1$, but not both due to the functions' conditions. Because $e_1 = 1$ is parallel to this paragraph's first scenerio, take $e_s = 1$ and recall $e_s = 1$ only when $\chi_1 = \mu_2$ and $\chi_2 = \mu_1$. Thus if $I(\chi_1, \chi_2) \simeq I(\mu_1, \mu_2)$, then either $\chi_1 = \mu_1$ and $\chi_2 = \mu_2$ or $\chi_1 = \mu_2$ and $\chi_2 = \mu_1$. On the contrary, if either $\chi_1 = \mu_1$ and $\chi_2 = \mu_2$ or $\chi_1 = \mu_2$ and $\chi_2 = \mu_1$, then $I(\chi_1, \chi_2) \simeq I(\mu_1, \mu_2)$

This completes the corollary. □

10 Conjugacy Classes of $GL_2(\mathbb{F}_q)$

To count the number of conjugacy classes in a group, we first recall several results from linear algebra.

Theorem 10.1 (Orbit-Stabilizer Theorem)**.** *Let G be a group and let X be a set on which G acts. Let $x \in X$. Then the number of cosets of the stabilizers of x is the number of elements in the orbit of x.*

Corollary 10.2. *Let $x \in G$ and $Z(x)$ be the centralizer of x. The number of conjugates of x is $|G : Z(x)|$.*

Proof. Let G be a group and denote the conjugacy class of $x \in G$ as

$$cl(x) = \left\{gxg^{-1} : g \in G\right\},$$

and the centralizer of x in G as

$$\begin{aligned} Z(x) &= \left\{g \in G : gxg^{-1} = x\right\} \\ &= \{g \in G : gx = xg\}. \end{aligned}$$

Define $\varphi : G/Z(x) \to cl(x)$ by $\varphi(gZ(x)) = gxg^{-1}$.

Let $g_1Z(x), g_2Z(x) \in G/Z(x)$ such that $g_1Z(x) = g_2Z(x)$. Then

$$\begin{aligned} g_1Z(x) = g_2Z(x) &\Leftrightarrow g_2^{-1}g_1 \in Z(x) \\ &\Leftrightarrow g_2^{-1}g_1x(g_2^{-1}g_1)^{-1} = x \\ &\Leftrightarrow g_2^{-1}g_1xg_1^{-1}g_2 = x \\ &\Leftrightarrow g_2g_2^{-1}g_1xg_1^{-1}g_2g_2^{-1} = g_2xg_2^{-1} \\ &\Leftrightarrow g_1xg_1^{-1} = g_2xg_2^{-1} \\ &\Leftrightarrow \varphi(g_1Z(x)) = \varphi(g_2Z(x)). \end{aligned}$$

Therefore if $g_1Z(x) = g_2Z(x)$ then $\varphi(g_1Z(x)) = \varphi(g_2Z(x))$ and if $\varphi(g_1Z(x)) = \varphi(g_2Z(x))$ then $g_1Z(x) = g_2Z(x)$. Hence φ is well-defined and injective, respectively.

Suppose $k \in cl(x)$. Then there exists $g \in G$ such that $gxg^{-1} = k$. By definition of φ, $\varphi(gZ(x)) = gxg^{-1} = k$. Thus φ is surjective.

Because φ is bijective, it follows that $|cl(x)| = |G/Z(x)|$. Using Lagrange's Theorem, $|G/Z(x)| = |G : Z(x)|$. Thus $|cl(x)| = |G : Z(x)|$. □

Definition 10.3. Two matrices $A, B \in M_n(\mathbb{C})$ are *similar* if there is an invertible matrix C such that $A = CBC^{-1}$. A matrix $A \in M_n(\mathbb{C})$ is diagonalizable if and only if there exist n

linearly independent eigenvectors for A. Specifically, if $\mathbf{v_1}, \dots, \mathbf{v_n}$ are linearly independent eigenvectors, let C be the matrix whose k th column is $\mathbf{v_k}$. Then C is invertible and we have

$$A = C \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} C^{-1},$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues associated to the eigenvectors $\mathbf{v_1}, \dots, \mathbf{v_n}$, respectively.

Proposition 10.4. *If $A, B \in M_n(\mathbb{C})$ are similar, then the characteristic polynomial of A is the characteristic polynomial of B.*

Proof. Suppose $A, B \in M_n(\mathbb{C})$ are similar matrices. Let $C \in GL_n(\mathbb{C})$ such that $A = CBC^{-1}$. We know that the characteristic polynomial of A is $p_A(\lambda) = \det(A - \lambda I)$, and that the characteristic polynomial of B is $p_B(\lambda) = \det(B - \lambda I)$. Therefore,

$$\begin{aligned} p_A(\lambda) &= \det(A - \lambda I) \\ &= \det\left(CBC^{-1} - \lambda I\right) \\ &= \det\left(CBC^{-1} - C\lambda IC^{-1}\right) \\ &= \det\left(C(B - \lambda I)C^{-1}\right) \\ &= \det(C)\det(B - \lambda I)\det\left(C^{-1}\right) \\ &= \det(C)\det\left(C^{-1}\right)\det(B - \lambda I) \\ &= \det(C)\det(C)^{-1}\det(B - \lambda I) \\ &= \det(B - \lambda I) \\ &= p_B(\lambda). \end{aligned}$$

Hence, similar matrices have the same characteristic polynomials. □

We are now ready to count the number of conjugacy classes in G, i.e., the number of irreducible representations. We will find the distinct conjugacy classes of G by finding all of the possible roots of a characteristic polynomial over $\mathbb{F}_q$.

1. Consider matrices $g_1 \in G$ of the form $g_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $\lambda_1 \neq \lambda_2$. As $\lambda_1, \lambda_2 \in \mathbb{F}_q^\times$ and $\lambda_1 \neq \lambda_2$, there are $(q-1)(q-2)$ such matrices. And since

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix} = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

there are $\frac{(q-1)(q-2)}{2}$ (distinct) such conjugacy classes.

If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is conjugate to g_1, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} a\lambda_1 & b\lambda_2 \\ c\lambda_1 & d\lambda_2 \end{bmatrix} = \begin{bmatrix} a\lambda_1 & b\lambda_1 \\ c\lambda_2 & d\lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Therefore, $b = c = 0$ and every matrix in G of the form $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ is conjugate to g_1. As $a, d \in \mathbb{F}_q^\times$,

$$|Z(g_1)| = (q-1)^2$$

and

$$|cl(g_1)| = \frac{|G|}{|Z(g_1)|} = \frac{q(q-1)(q^2-1)}{(q-1)^2} = \frac{q(q-1)(q-1)(q+1)}{(q-1)^2} = q(q+1).$$

2. Consider matrices $g_2 \in G$ of the form $g_2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}$. Because $\lambda_1 \in \mathbb{F}_q^\times$, there are $(q-1)$ such conjugacy classes.

If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is conjugate to g_2, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix} = \begin{bmatrix} a\lambda_1 & b\lambda_1 \\ c\lambda_1 & d\lambda_1 \end{bmatrix} = \begin{bmatrix} a\lambda_1 & b\lambda_1 \\ c\lambda_1 & d\lambda_1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Therefore, every matrix in G is conjugate to g_2. So

$$|Z(g_2)| = |G|$$

and

$$|cl(g_2)| = \frac{|G|}{|G|} = 1.$$

3. Consider matrices $g_3 \in G$ of the form $g_3 = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$. Because $\lambda_1 \in \mathbb{F}_q^\times$, there are $(q-1)$ such conjugacy classes.

   If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is conjugate to g_3, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} = \begin{bmatrix} a\lambda_1 & a + b\lambda_1 \\ c\lambda_1 & c + d\lambda_1 \end{bmatrix} = \begin{bmatrix} a\lambda_1 + c & b\lambda_1 + d \\ c\lambda_1 & d\lambda_1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

   Therefore, $a = d$ and $c = 0$, and every matrix in G of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ is conjugate to g_3. As $a \in \mathbb{F}_q^\times$ and $b \in \mathbb{F}_q$,

$$|Z(g_3)| = q(q-1)$$

   and

$$|cl(g_2)| = \frac{|G|}{|Z(g_3)|} = \frac{q(q-1)(q^2-1)}{q(q-1)} = (q+1)^2.$$

4. Consider matrices $g_4 \in G$ of the form $g_4 = \begin{bmatrix} 0 & -a_0 \\ 1 & a_1 \end{bmatrix}$. The characteristic polynomial of g_4 is $p_{g_4}(\lambda) = \lambda^2 - a_1\lambda + a_0$. Note that the cardinality of $\mathbb{F}_q$ is q, and the total number of quadratic monic polynomials over $\mathbb{F}_q$ is q^2. If $p_{g_4}(\lambda)$ is reducible in $\mathbb{F}_q[\lambda]$, then either $p_{g_4}(\lambda) = (\lambda - p)^2$ or $p_{g_4}(\lambda) = (\lambda - p_1)(\lambda - p_2)$. As there are q ways of choosing

p and $\begin{pmatrix} q \\ 2 \end{pmatrix}$ ways of choosing $\{p_1, p_2\}$, the number of reducible polynomials over $\mathbb{F}_q$ is

$$q + \begin{pmatrix} q \\ 2 \end{pmatrix} = q + \frac{q!}{2!\,(q-2)!} = \frac{2q}{2} + \frac{q\,(q-1)}{2} = \frac{2q + q^2 - q}{2} = \frac{q^2 + q}{2}.$$

Therefore, the number of irreducible polynomials over $\mathbb{F}_q$ is

$$q^2 - \frac{q^2 + q}{2} = \frac{2q^2}{2} - \frac{q^2 + q}{2} = \frac{2q^2 - q^2 - q}{2} = \frac{q^2 - q}{2}.$$

Equivalently, there are $\frac{q^2-q}{2}$ (distinct) conjugacy classes.

It can be shown that $\mid Z\,(g_4) \mid = (q-1)\,(q+1)$ and $\mid cl\,(g_4) \mid = q\,(q-1)$.

We claim to have found all of the conjugacy classes of G. To check, we add the number in each case times the order of the conjugacy class of that type. By doing so, we have

$$\begin{aligned}
(q-1)\cdot 1 + (q-1)\cdot(q-1)\,(q+1) + \frac{(q-1)\,(q-2)}{2}\cdot q\,(q+1) + \frac{q^2-q}{2}\cdot q\,(q-1) \\
&= q - 1 + (q-1)\left(q^2-1\right) + \frac{\left(q^2 - 3q + 2\right)\left(q^2+q\right)}{2} + \frac{\left(q^2-q\right)\left(q^2-q\right)}{2} \\
&= q - 1 + q^3 - q - q^2 + 1 + \frac{q^4 + q^3 - 3q^3 - 3q^2 + 2q^2 + 2q}{2} + \frac{q^4 - q^3 - q^3 + q^2}{2} \\
&= \frac{2q^3 - 2q^2 + q^4 - 2q^3 - q^2 + 2q + q^4 - 2q^3 + q^2}{2} \\
&= \frac{2q^4 - 2q^3 - 2q^2 + 2q}{2} \\
&= q^4 - q^3 - q^2 + q \\
&= \left(q^2-1\right)\left(q^2-q\right) \\
&= \mid G \mid.
\end{aligned}$$

Therefore, we have found all of the conjugacy classes of $GL_2\,(\mathbb{F}_q)$.

References

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