A NONLINEAR THEORY FOR THIN ELASTIC SHELLS

by

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ABSTRACT

A NONLINEAR THEORY OF THIN ELASTIC SHELLS

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The purpose of this thesis is to derive a nonlinear theory of thin elastic shells including the effects of transverse shear stress, in-plane stability forces, and transverse and rotary inertia.

Using a variational theorem due to E. Reissner, the equations of motion, the stress-strain relationships, and the associated boundary conditions are simultaneously determined. The resulting equations may be applied to a particular group of shell problems where the applied statics and dynamic loads produce deformations which are of such an order that only appropriate nonlinear theory accounts for them.

The resulting equations are simplified for the special case of a thin circular plate subjected to the above stress and loading conditions.

An analogy is made with the problem of a thin rectangular elastic plate using rectangular cartesian co-ordinate analyzed by R. D. Mindlin.

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LIST OF SYMBOLS

SYMBOL	DEFINITION
x, y, z	Surface and normal co-ordinates
X, Y, Z	Rectangular co-ordinates
α, β, γ	Lame's coefficient
r_1, r_2	Radii of curvature in direction of x, y respectively
$\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}$	Normal strain components in direction of x, y, z respectively
$\gamma_{xz}, \gamma_{yz}, \gamma_{xy}$	Shearing strain components
$\tau_{xx}, \tau_{yy}, \tau_{zz}$	Normal stress components
$\tau_{xy}, \tau_{xz}, \tau_{yz}$	Shearing stress components
U, V, W	Displacement components in direction of x, y, z respectively for any arbitrary point in the shell
u, v, w	Displacement components at the middle surface of the shell
ϕ, ψ	Change of slope of the normal to the middle surface
ρ	Mass density per unit mass
μ	Poisson's ratio
z	Normal co-ordinate in z-direction of any arbitrary point in the shell
h	Thickness of the shell
D	Flexural rigidity of the shell
E	Modulus of elasticity
F_x, F_y, F_z	Body force per unit volume in direction of x, y, z respectively

SYMBOL	DEFINITION
G	Shear modulus of elasticity
Nxx, Nyy, Nxy, Nyx	Normal stress resultants in unit of force per unit length
Mxx, Myy, Mxy, Myx	Bending stress couples in unit of moment per unit length
Qxz, Qyz	Shear stress resultants in z-direction
Px, Py, Pz	External shear and normal stress components in the direction of x, y, z respectively

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CHAPTER I

INTRODUCTION

A wide range of nonlinear theories for thin elastic shells as derived by using the finite displacements, differ greatly depending on the restricting assumptions placed on the resulting deformations.

In Naghdi[4] presents a linear theory for thin elastic shells which includes the effects of transverse normal stress, transverse shear stress, and rotary inertia. A number of existing theories is summarized by Sanders[6] where he derives a class of nonlinear theories and analyzed the Donnell-Mushtari-Vlosov theory as a special case.

Numbers of practical shell problems involving dynamic loads and displacments are of such an effective order that they must be taken into consideration by means of an appropriate nonlinear theory. Archer[1] presents the nonlinear theory of Donnell-type including shear deformations, transverse and rotary inertia effects in-plane stability forces. The effect of transverse normal stress is neglected.

In wave propagation problems, the effects of transverse shear stress are of prime importance. In elastic stability problems the effect of the interaction of transverse shear stress with the in-plane stability forces is of greatest significance.

The objective of this thesis is to derive a nonlinear theory of thin elastic shells including the effects of transverse shear stress, in-plane stability forces, and transverse and rotary inertia, using a variational theorem due to E. Reissner.[5]

The equations from the resulting theory are applied to the special case of a thin circular plate. An analogy is made with the problem of thin elastic plate in cartesian co-ordinate analyzed by R. D. Mindlin[3].

CHAPTER 2

ANALYSIS

The nonlinear theory is derived based upon the following assumptions:

1. The thickness of shell is considered uniform and is small compared to the least radius of curvature, that is, terms of the order of $(h/r)^2$ are retained in comparison to unity.

2. The component of stress, normal to the middle surface, is negligible in comparison with other five stress components.

   Linear elastic stress-strain relationships are assumed to hold.

3. Points on the lines normal to the middle surface before deformation do not coincide with points on lines normal to middle surface (i.e., shear deformations are accounted for).

2.1 THE CO-ORDINATE SYSTEM AND NOTATIONS

The notation used throughout the paper is comparable with that used by Langhaar.[2] A point on the middle surface of the shell is defined by the rectangular cartesian co-ordinates given as $X = X\ (x,y)$, $Y = Y\ (x,y)$, and $Z = Z\ (x,y)$ where the parameters x and y are called "curvilinear surface co-ordinates" (see Fig. 2.1). The normal distance from the middle surface to an arbitrary point in the shell is denoted by z. The unit tangent vectors to the curves of constant x and y are defined by $\overline{\gamma_x}$ and $\overline{\gamma_y}$, respectively. If the co-ordinate lines on middle surface of constant x and constant y are orthogonal, these co-ordinate lines coincide with curves of principal curvacture of the middle surface.

The distance ds between any two points in the shell is given by the equation:

$$ds^2 = \alpha^2 dx^2 + \beta^2 dy^2 + \gamma^2 dz^2, \qquad \cdots (1)$$

where

$$\left.\begin{aligned} \alpha &= A\left(1+\frac{z}{r_1}\right). \\ \beta &= B\left(1+\frac{z}{r_2}\right). \\ \gamma &= 1, \\ A^2 &= (\bar{r}_x \cdot \bar{r}_x) \quad \text{and} \\ B^2 &= (\bar{r}_y \cdot \bar{r}_y). \end{aligned}\right] \cdots (2)$$

Also, the parameters $\frac{1}{r_1}$ and $\frac{1}{r_2}$ are defined as the principal curvatures of the middle surface, and the parameters α, β, and γ are known as Lame's coefficients.

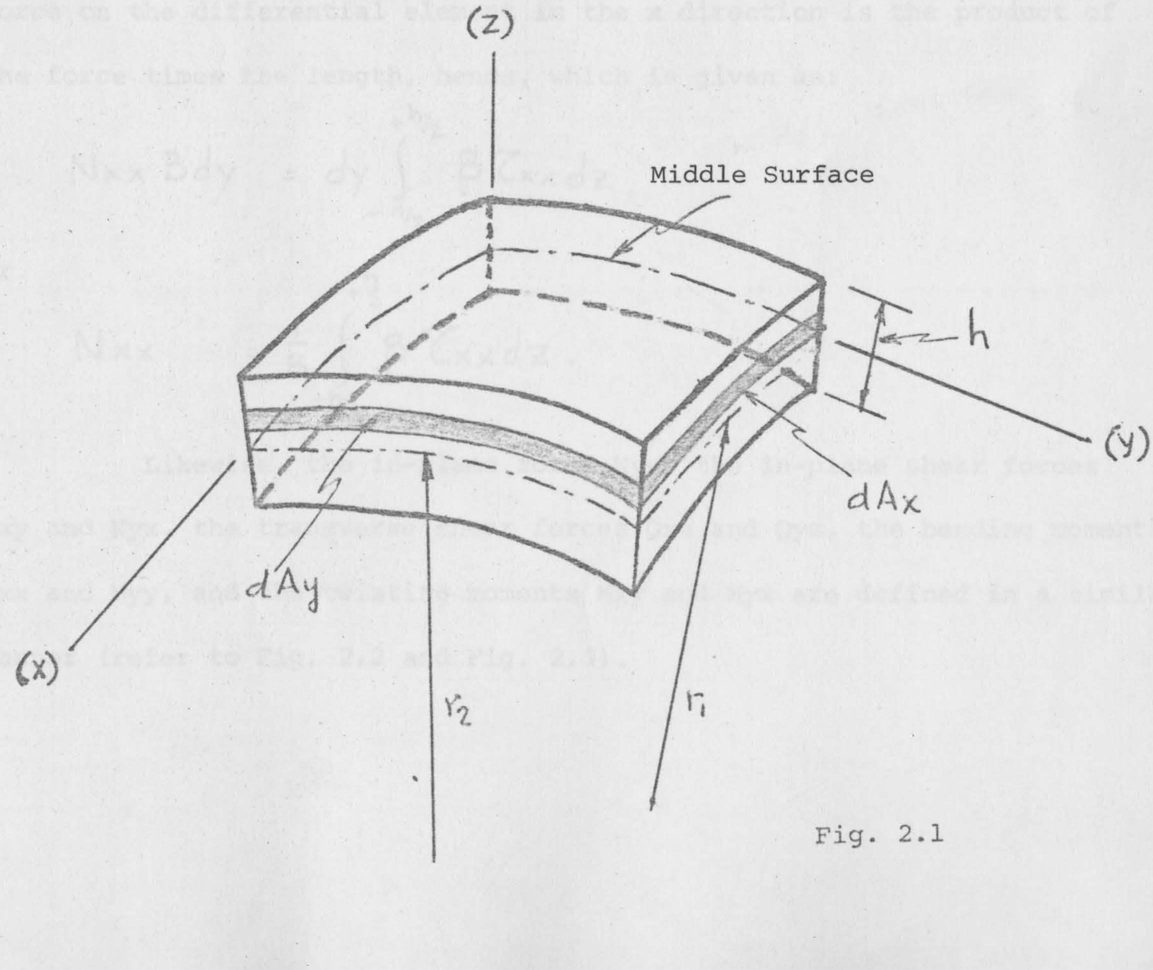


Fig. 2.1

2.2 STRESS RESULTANTS AND STRESS COUPLES

Stress resultants and stress couples are applied to a differential shell element as shown in Fig. 2.2 and Fig. 2.3. These stress resultants and stress couples are defined as total forces and moments acting per unit length of the middle surface, respectively. From equations (1) and (2), the areas of the cross-sectional elements are defined as follows:

$$dA_x = \alpha\,dx\,dz = A\left(1+\frac{z}{r_1}\right)dx\,dz$$

and $dA_y = \beta\,dy\,dz = B\left(1+\frac{z}{r_2}\right)dy\,dz$,

where r_1 and r_2 are the principal radii of curvature of the middle surface.

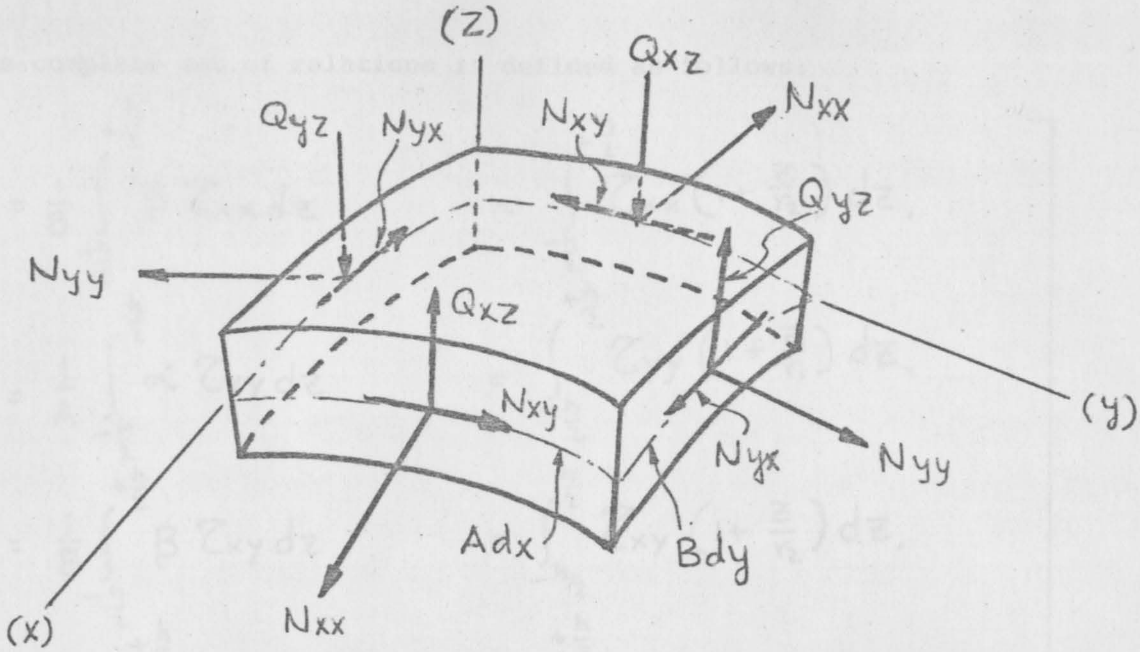
Nxx is defined as the in-plane force on a cross-section per unit length along the co-ordinate direction. The total in-plane axial force on the differential element in the x direction is the product of the force times the length, hence, which is given as:

$$N_{xx}\,B\,dy = dy\int_{-h/2}^{+h/2} \beta\,\tau_{xx}\,dz,$$

or

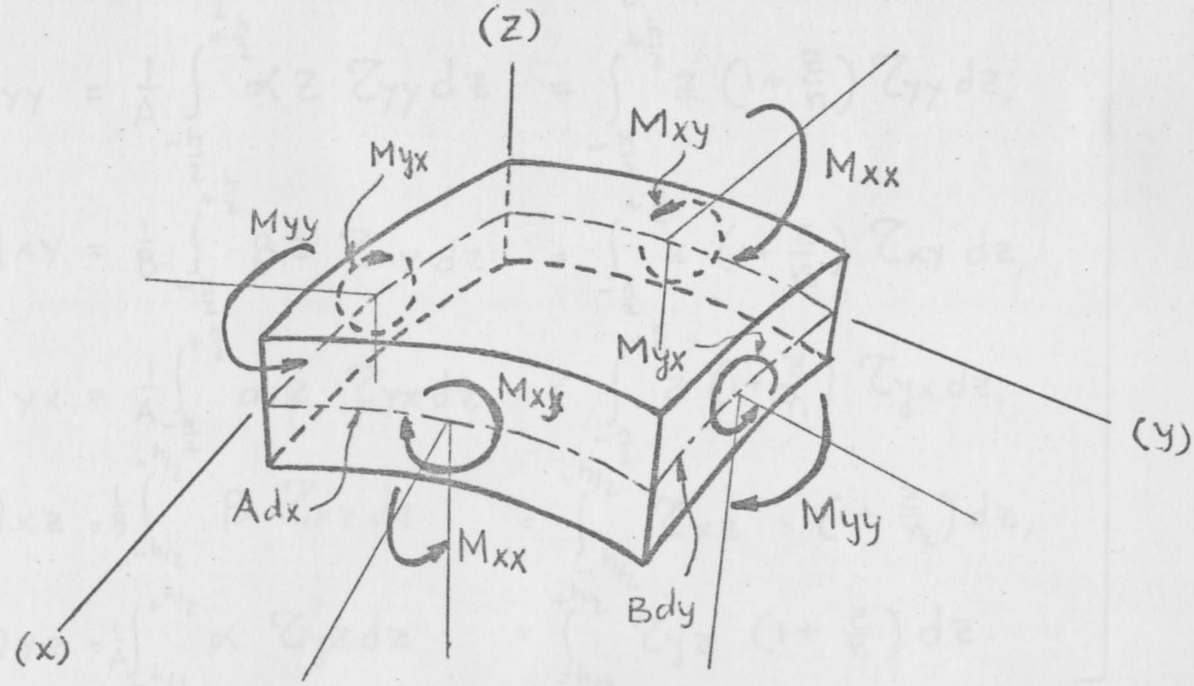
$$N_{xx} = \frac{1}{B}\int_{-h/2}^{+\frac{h}{2}} \beta\,\tau_{xx}\,dz.$$

Likewise, the in-plane force Nyy, the in-plane shear forces Nxy and Nyx, the transverse shear forces Qxz and Qyz, the bending moments Mxx and Myy, and the twisting moments Mxy and Myx are defined in a similar manner (refer to Fig. 2.2 and Fig. 2.3).



(Z)
Qxz
Nxx
Qyz
Nyx
Nxy
Qyz
Nyy
Qxz
Nxy
(y)
Nyx
Nyy
Adx
Bdy
(x)
Nxx

Fig. 2.2



(Z)
Mxy
Mxx
Myx
Myy
Myx
Mxy
(y)
Adx
Myy
Mxx
Bdy
(x)

Fig. 2.3

This complete set of relations is defined as follows:

$$
\left.
\begin{aligned}
N_{xx} &= \frac{1}{B}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \beta\,\tau_{xx}\,dz &&= \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{xx}\left(1+\frac{z}{r_2}\right)dz,\\
N_{yy} &= \frac{1}{A}\int_{-\frac{h}{2}}^{\frac{h}{2}} \alpha\,\tau_{yy}\,dz &&= \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yy}\left(1+\frac{z}{r_1}\right)dz,\\
N_{xy} &= \frac{1}{B}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \beta\,\tau_{xy}\,dz &&= \int_{-h/2}^{\frac{h}{2}} \tau_{xy}\left(1+\frac{z}{r_2}\right)dz,\\
N_{yx} &= \frac{1}{A}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \alpha\,\tau_{xy}\,dz &&= \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{xy}\left(1+\frac{z}{r_1}\right)dz,\\
M_{xx} &= \frac{1}{B}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \beta z\,\tau_{xx}\,dz &&= \int_{-\frac{h}{2}}^{+\frac{h}{2}} z\left(1+\frac{z}{r_2}\right)\tau_{xx}\,dz,\\
M_{yy} &= \frac{1}{A}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \alpha z\,\tau_{yy}\,dz &&= \int_{-\frac{h}{2}}^{+\frac{h}{2}} z\left(1+\frac{z}{r_1}\right)\tau_{yy}\,dz,\\
M_{xy} &= \frac{1}{B}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \beta z\,\tau_{xy}\,dz &&= \int_{-\frac{h}{2}}^{+\frac{h}{2}} z\left(1+\frac{z}{r_2}\right)\tau_{xy}\,dz,\\
M_{yx} &= \frac{1}{A}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \alpha z\,\tau_{yx}\,dz &&= \int_{-\frac{h}{2}}^{+\frac{h}{2}} z\left(1+\frac{z}{r_1}\right)\tau_{yx}\,dz,\\
Q_{xz} &= \frac{1}{B}\int_{-h/2}^{+h/2} \beta\,\tau_{xz}\,dz &&= \int_{-h/2}^{+h/2} \tau_{xz}\left(1+\frac{z}{r_2}\right)dz,\\
Q_{yz} &= \frac{1}{A}\int_{-h/2}^{+h/2} \alpha\,\tau_{yz}\,dz &&= \int_{-h/2}^{+h/2} \tau_{yz}\left(1+\frac{z}{r_1}\right)dz.
\end{aligned}
\right\} \cdots (3)
$$

From the above equations, the following equality is obtained,

$$
N_{xy} + \frac{M_{xy}}{r_1} = N_{yx} + \frac{M_{yx}}{r_2} \quad \text{------------} \cdots (4)
$$

2.3 STRAIN DISPLACEMENT RELATIONS

The general three dimensional nonlinear strain displacement equations are defined by Langhaar as follows:

$$\begin{aligned}
\epsilon_{xx} &= \frac{1}{\alpha}\left[U_x + \frac{\alpha_y}{\beta}V + \frac{\alpha_z}{\gamma}W + \frac{1}{2\alpha}\left(U_x + \frac{\alpha_y}{\beta}V + \frac{\alpha_z}{\gamma}W\right)^2 \right. \\
&\quad \left. + \frac{1}{2\alpha}\left(V_x - \frac{\alpha_y}{\beta}U\right)^2 + \frac{1}{2\alpha}\left(W_x - \frac{\alpha_z}{\gamma}U\right)^2\right], \\
\epsilon_{yy} &= \frac{1}{\beta}\left[V_y + \frac{\beta_z}{\gamma}W + \frac{\beta_x}{\alpha}U + \frac{1}{2\beta}\left(V_y + \frac{\beta_z}{\gamma}W + \frac{\beta_x}{\alpha}U\right)^2 \right. \\
&\quad \left. + \frac{1}{2\beta}\left(W_x - \frac{\beta_z}{\gamma}V\right)^2 + \frac{1}{2\beta}\left(U_y - \frac{\beta_x}{\alpha}V\right)^2\right], \\
\epsilon_{zz} &= \frac{1}{\gamma}\left[W_z + \frac{\gamma_x}{\alpha}U + \frac{\gamma_y}{\beta}V + \frac{1}{2\gamma}\left(W_z + \frac{\gamma_x}{\alpha}U + \frac{\gamma_y}{\beta}V\right)^2 \right. \\
&\quad \left. + \frac{1}{2\gamma}\left(U_z - \frac{\gamma_x}{\alpha}W\right)^2 + \frac{1}{2\gamma}\left(V_z - \frac{\gamma_y}{\beta}W\right)^2\right], \\
\gamma_{xy} &= \left[\frac{U_y}{\beta} + \frac{V_x}{\alpha} - \frac{\beta_x V}{\alpha\beta} - \frac{\alpha_y U}{\alpha\beta} + \frac{1}{2\alpha}\left(U_x + \frac{\alpha_y}{\beta}V + \frac{\alpha_z}{\gamma}U\right)\cdot \right. \\
&\quad \left(U_y - \frac{\beta_x}{\gamma}V\right) + \frac{1}{\alpha\beta}\left(V_y + \frac{\beta_x}{\alpha}U + \frac{\beta_z}{\gamma}W\right)\left(V_x - \frac{\alpha_y}{\beta}U\right) \\
&\quad \left. + \frac{1}{\alpha\beta}\left(W_x - \frac{\alpha_z}{\gamma}U\right)\left(W_y - \frac{\beta_z}{\gamma}V\right)\right], \\
\gamma_{yz} &= \left[\frac{V_z}{\gamma} + \frac{W_y}{\beta} - \frac{\gamma_y W}{\beta\gamma} - \frac{\beta_z V}{\beta\gamma} + \frac{1}{\beta\gamma}\left(V_y + \frac{\beta_z}{\gamma}W + \frac{\beta_x}{\alpha}U\right)\left(V_z - \frac{\gamma_y}{\beta}W\right) \right. \\
&\quad \left. + \frac{1}{\beta\gamma}\left(W_z + \frac{\gamma_y}{\beta}V + \frac{\gamma_x}{\alpha}U\right)\left(W_y - \frac{\beta_z}{\gamma}V\right) + \frac{1}{\beta\gamma}\left(U_y - \frac{\beta_x}{\alpha}V\right)\left(U_z - \frac{\gamma_x}{\alpha}W\right)\right], \\
&\text{and,} \\
\gamma_{xz} &= \left[\frac{W_x}{\alpha} + \frac{U_z}{\gamma} - \frac{\alpha_z U}{\gamma\alpha} - \frac{\gamma_x W}{\gamma\alpha} + \frac{1}{\gamma\alpha}\left(W_z + \frac{\gamma_x U}{\alpha} + \frac{\gamma_y V}{\beta}\right)\left(W_x - \frac{\alpha_z}{\gamma}U\right) \right. \\
&\quad \left. + \frac{1}{\gamma\alpha}\left[\left(U_x + \frac{\alpha_z}{\gamma}W + \frac{\alpha_y}{\beta}V\right)\left(U_z - \frac{\gamma_x}{\alpha}W\right) + \frac{1}{\gamma\alpha}\left(V_z - \frac{\gamma_y}{\beta}W\right)\left(V_x - \frac{\alpha_y}{\beta}U\right)\right]\right].
\end{aligned} \tag{5}$$

Retaining all the linear terms and rotation terms of the second order as Wx^2, Wy^2 and $WxWy$, the preceding equations reduce to the following:

$$
\left.
\begin{aligned}
\epsilon_{xx} &= \frac{1}{\alpha}\left[U_x + \frac{\alpha_y}{\beta} V + \alpha_z W + \frac{1}{2\alpha} W_x^2\right], \\
\epsilon_{yy} &= \frac{1}{\beta}\left[\frac{\beta_x}{\alpha} U + V_y + \beta_z W + \frac{1}{2\beta} W_y^2\right], \\
\epsilon_{zz} &= W_z \simeq 0, \quad \text{By assumption}, \\
\gamma_{xy} &= \frac{U_y}{\beta} + \frac{V_x}{\alpha} - \frac{\beta_x}{\alpha\beta} V - \frac{\alpha_y}{\alpha\beta} U + \frac{W_x W_y}{\alpha\beta}, \\
\gamma_{yz} &= V_z + \frac{W_y}{\beta} - \frac{\beta_z}{\beta} V, \\
\gamma_{xz} &= U_z + \frac{W_x}{\alpha} - \frac{\alpha_z}{\alpha} U.
\end{aligned}
\right] \quad \cdots (6)
$$

For the special case of orthogonal curvilinear shell co-ordinates, the Codazzi's equations apply and take the following form:

$$
\left.
\begin{aligned}
\left(\frac{A}{r_1}\right)_y &= \frac{1}{r_2} A_y, \quad \text{and} \\
\left(\frac{B}{r_2}\right)_x &= \frac{1}{r_1} B_x.
\end{aligned}
\right] \quad \cdots (7)
$$

Also, the following equalities are obtained using equations (2) and (7):

$$\left.\begin{aligned} \frac{A_y}{B} &= \frac{\alpha_y}{\beta}, \\ \frac{B_x}{A} &= \frac{\beta_x}{\alpha}, \\ \frac{A}{r_1} &= \alpha_z, \\ \text{and} \quad \frac{B}{r_2} &= \beta_z. \end{aligned}\right] \quad \cdots (8)$$

Substituting the above equations in the equation (6), the following reduced forms of strain displacement relationships are obtained:

$$\left.\begin{aligned} \epsilon_{xx} &= \frac{1}{\alpha}\left[U_x + \frac{A_y}{B}V + \frac{A}{r_1}W + \frac{1}{2\alpha}W_x^2\right], \\ \epsilon_{yy} &= \frac{1}{\beta}\left[V_y + \frac{B_x U}{A} + \frac{B}{r_2}W + \frac{1}{2\beta}W_y^2\right], \\ \epsilon_{zz} &= 0, \\ \gamma_{xy} &= \frac{1}{\beta}U_y + \frac{1}{\alpha}V_x - \frac{B_x}{AB}V - \frac{A_y U}{B\alpha} + \frac{W_x W_y}{\alpha\beta}, \\ \gamma_{xz} &= U_z + \frac{W_x}{\alpha} - \frac{\alpha_z}{\alpha}U, \quad \text{and} \\ \gamma_{yz} &= V_z + \frac{W_y}{\beta} - \frac{\beta_z}{\beta}V. \end{aligned}\right] \quad \cdots (9)$$

The following displacement approximations are introduced:

$$\left.\begin{aligned} U &= u\,(x,y) + z\,\phi(x,y), \\ V &= v\,(x,y) + z\,\psi(x,y), \text{ and} \\ W &= w\,(x,y). \end{aligned}\right] \quad \cdots (10)$$

The parameters u, v and w are the components of displacement of the middle surface in the x, y and z directions, respectively. The functions Φ (x,y) and Ψ (x,y) are the change of slope of the normal to the middle surface along the x and y co-ordinates, respectively.

Substituting equation (10) into equation (9), the following expressions for strain-displacement are obtained:

$$
\left.
\begin{aligned}
\epsilon_{xx} &= \frac{1}{\alpha}\Big[U_x + z\Phi_x + \frac{A_y}{B}(V + z\Psi) + \frac{A}{r_1}W + \frac{1}{2\alpha}W_x^2\\
\epsilon_{yy} &= \frac{1}{\beta}\Big[V_y + z\Psi_y + \frac{B_x}{A}(U + z\Phi) + \frac{B}{r_2}W + \frac{1}{2\beta}W_y^2\\
\epsilon_{zz} &= 0,\\
\gamma_{xy} &= \frac{1}{\alpha\beta}\Big[\alpha(U_y + z\Phi_y) + \beta(V_x + z\Psi_x) - \frac{B_x}{A}\alpha(V + z\Psi)\\
&\qquad - \frac{A_y}{B}\beta(U + z\Phi)\Big],\\
\gamma_{xz} &= U_z + \frac{W_x}{\alpha} - \frac{\alpha_z}{\alpha}, \quad \text{and}\\
\gamma_{yz} &= V_z + \frac{W_y}{\beta} - \frac{\beta_z}{\beta}V.
\end{aligned}
\right] \quad \ldots(11)
$$

Replacing the parameters $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ by $\frac{1}{A}$ and $\frac{1}{B}$, respectively, for the nonlinear terms in the first and second equations above, the equations reduce to the form:

$$
\left.
\begin{aligned}
\left(1 + \frac{z}{r_1}\right)\epsilon_{xx} &= \varepsilon^{\circ}_{xx} + z k_x,\\
\left(1 + \frac{z}{r_2}\right)\epsilon_{yy} &= \varepsilon^{\circ}_{yy} + z k_y,\\
\left(1 + \frac{z}{r_1}\right)\left(1 + \frac{z}{r_2}\right)\gamma_{xy} &= \left(1 + \frac{z}{r_2}\right)\left(\gamma^{\circ}_x + z\delta_x\right) + \left(1 + \frac{z}{r_1}\right)\left(\gamma^{\circ}_y + z\delta_y\right)\\
&\quad + \frac{1}{AB}W_x W_y,\\
\left(1 + \frac{z}{r_1}\right)\gamma_{xz} &= \gamma^{\circ}_{xz}, \quad \text{and}\\
\left(1 + \frac{z}{r_2}\right)\gamma_{yz} &= \gamma^{\circ}_{yz},
\end{aligned}
\right] \quad \ldots(12)
$$

where,

$$
\left.
\begin{aligned}
\varepsilon_{xx}^{\circ} &= \frac{1}{A}\left(U_x + \frac{V}{B}A_y\right) + \frac{W}{r_1} + \frac{1}{2A^2}W_x^2 , \\
k_x &= \frac{1}{A}\left(\Phi_x + \frac{\Psi}{B}A_y\right), \\
\varepsilon_{yy}^{\circ} &= \frac{1}{B}\left(V_y + \frac{U}{A}B_x\right) + \frac{W}{r_2} + \frac{1}{2B^2}W_y^2 , \\
k_y &= \frac{1}{B}\left(\Psi_y + \frac{\Phi}{A}B_x\right), \\
\overset{\circ}{\gamma}_x &= \frac{1}{A}\left(V_x - \frac{U}{B}A_y\right), \\
\delta_x &= \frac{1}{A}\left(\Psi_x - \frac{\Phi}{B}A_y\right), \\
\overset{\circ}{\gamma}_y &= \frac{1}{B}\left(U_y - \frac{V}{A}B_x\right), \\
\delta_y &= \frac{1}{B}\left(\Phi_y - \frac{\Psi}{A}B_x\right), \\
\overset{\circ}{\gamma}_{xz} &= \frac{W_x}{A} + \Phi - \frac{U}{r_1} , \quad \text{and} \\
\overset{\circ}{\gamma}_{yz} &= \frac{W_y}{B} + \Psi - \frac{V}{r_2} .
\end{aligned}
\right] \quad \ldots(13)
$$

2.4 COMPONENTS OF STRESS

The usual definitions of the stress resultants are applicable and take the following form:

$$
\left.
\begin{aligned}
N_{xx} &= \int_{-\frac{h}{2}}^{h/2} \tau_{xx}\left(1+\frac{z}{r_2}\right) dz, \\
M_{xx} &= \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{xx}\left(1+\frac{z}{r_2}\right) dz, \\
Q_{xz} &= \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{xz}\left(1+\frac{z}{r_2}\right) dz, \text{ and} \\
N_{yx} &= \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{xy}\left(1+\frac{z}{r_1}\right) dz.
\end{aligned}
\right] \quad \ldots (14)
$$

Assuming a linear variation in stress distribution, the following stress distributions are defined:

$$
\left.
\begin{aligned}
\left(1+\frac{z}{r_2}\right)\tau_{xx} &= \frac{N_{xx}}{h} + \frac{12 M_{xx} z}{h^3}, \\
\left(1+\frac{z}{r_1}\right)\tau_{yy} &= \frac{N_{yy}}{h} + \frac{12 M_{yy} z}{h^3}, \\
\left(1+\frac{z}{r_2}\right)\tau_{xy} &= \frac{N_{xy}}{h} + \frac{12 M_{xy} z}{h^3}, \\
\left(1+\frac{z}{r_2}\right)\tau_{yx} &= \frac{N_{yx}}{h} + \frac{12 M_{yx} z}{h^3}, \text{ and} \\
\tau_{xy} &= \tau_{yx}.
\end{aligned}
\right] \quad \ldots (15)
$$

The three differential equations of stress equilibrium are given in curvilinear coordinate form as:

$$\left.\begin{aligned}
&\frac{\partial}{\partial x}(\beta\gamma\tau_{xx})+\frac{\partial}{\partial y}(\alpha\gamma\tau_{yx})+\frac{\partial}{\partial z}(\alpha\beta\tau_{xz})+\gamma\alpha_y\tau_{xy}\\
&\quad+\beta\alpha_z\tau_{xz}-\gamma\beta_x\tau_{yy}-\beta\gamma_x\tau_{zz}+\rho\alpha\beta\gamma F_x=0,\\
&\frac{\partial}{\partial x}(\beta\gamma\tau_{xy})+\frac{\partial}{\partial y}(\alpha\gamma\tau_{yy})+\frac{\partial}{\partial z}(\alpha\beta\tau_{xy})+\gamma\beta_x\tau_{yx}\\
&\quad+\alpha\beta_z\tau_{yz}-\gamma\alpha_y\tau_{xx}-\alpha\gamma_y\tau_{zz}+\rho\alpha\beta\gamma F_y=0,\\
&\frac{\partial}{\partial x}(\beta\gamma\tau_{xz})+\frac{\partial}{\partial y}(\alpha\gamma\tau_{yz})+\frac{\partial}{\partial z}(\alpha\beta\tau_{zz})+\beta\gamma_x\tau_{zx}\\
&\quad+\alpha\gamma_y\tau_{zy}-\beta\alpha_z\tau_{xx}-\alpha\beta_z\tau_{yy}+\rho\alpha\beta\gamma F_z=0.
\end{aligned}\right]\quad\cdots(16)$$

Neglecting body forces and substituting the condition $\gamma = 1$ for thin shell theory, the previous equations reduces to:

$$\left.\begin{aligned}
&\frac{\partial}{\partial x}(\beta\tau_{xx})+\frac{\partial}{\partial y}(\alpha\tau_{yx})+\frac{\partial}{\partial z}(\alpha\beta\tau_{zx})+\alpha_y\tau_{xy}\\
&\qquad+\beta\alpha_z\tau_{xz}-\beta_x\tau_{yy}=0,\\
&\frac{\partial}{\partial x}(\beta\tau_{xy})+\frac{\partial}{\partial y}(\alpha\tau_{yy})+\frac{\partial}{\partial z}(\alpha\beta\tau_{xy})+\beta_x\tau_{yx}\\
&\qquad+\alpha\beta_z\tau_{yz}-\alpha_y\tau_{xx}=0,\\
&\frac{\partial}{\partial x}(\beta\tau_{xz})+\frac{\partial}{\partial y}(\alpha\tau_{yz})+\frac{\partial}{\partial z}(\alpha\beta\tau_{zz})+\alpha\gamma_y\tau_{zy}\\
&\qquad-\beta\alpha_z\tau_{xx}-\alpha\beta_z\tau_{yy}=0.
\end{aligned}\right]\quad\cdots(17)$$

Substituting the stress variations given in equation (15) into the first two equations of (17), yields the following stress distributions:

$$\left.\begin{aligned}
&\left(1+\frac{z}{r_2}\right)\tau_{xz}=\frac{3}{2}\frac{Q_{xx}}{h}\left[1-\frac{4z^2}{h^2}\right],\\
&\left(1+\frac{z}{r_1}\right)\tau_{yz}=\frac{3}{2}\frac{Q_{yy}}{h}\left[1-\frac{4z^2}{h^2}\right].
\end{aligned}\right]\quad\cdots(18)$$

Noting the assumption that $\tau_{zz}=0$, equations (15) and equations (18) define the five components of the assumed stress field.

2.5 REISSNER'S VARIATIONAL THEOREM

Using the Variational Theorem of E. Reissner, a derivation of appropriate stress-strain relations, equilibrium equations, and associated natural boundary conditions for a given set of stress components (see equations "15" and "18") and strain displacement relationships (see equations "12" and "13") is carried out.

The following form of the variational theorem for three dimensional elasticity is written:

$$\delta I = \delta \int_{t_1}^{t_2} \Big\{ \Big\{ \iiint_{-\frac{h}{2}}^{+\frac{h}{2}} \Big[\tau_{xx}\varepsilon_{xx} + \tau_{yy}\varepsilon_{yy} + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz} \Big]$$

$$- \frac{1}{2E}\Big[\tau_{xx}^2 + \tau_{yy}^2 - 2\mu\,\tau_{xx}\tau_{yy} + 2(1+\mu)(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2) \Big]$$

$$+ \frac{\rho}{2}\Big[u_t^2 + v_t^2 + w_t^2 \Big] \Big\} \Big(1+\frac{z}{r_1}\Big)\Big(1+\frac{z}{r_2}\Big) AB\, dx\,dy\,dz$$

$$- \iint \Big[p_x u + p_y v + q w \Big] AB\, dx\,dy$$

$$- \oint \Big[\int_{-\frac{h}{2}}^{+\frac{h}{2}} (\sigma_n u_n + \tau_{nt} u_t + \tau_{nz} w)\Big(1+\frac{z}{r_t}\Big) dz \Big] A_t\, ds \Big\}\, ds = 0. \qquad \cdots (19)$$

Substituting equations (11) - (15) into equation (19) and integrating over the thickness of the shell yields:

$$
\begin{aligned}
\delta I = \delta \int_{t_1}^{t_2} \iint \Bigg\{ & \Big[\frac{N_{xx}}{h}\left(h\varepsilon^{\circ}_{xx}\right) + \frac{12M_{xx}}{h^3}\left(\frac{h^3}{12}k_x\right)\Big] + \Big[\frac{N_{yy}}{h}\left(h\cdot\varepsilon^{\circ}_{yy}\right) \\
& + \frac{12M_{yy}}{h^3}\left(\frac{h^3}{12}k_y\right)\Big] \\
& + \Big\{ \frac{N_{xy}}{h}\Big[h\gamma^{\circ}_x + \Big(h + \frac{h^3}{12r_2^2}\Big)\frac{W_xW_y}{AB}\Big] + \frac{12M_{xy}}{h^3}\Big[\frac{h^3}{12}\delta_x - \frac{h^3}{12r_2}\frac{W_xW_y}{AB}\Big] \\
& \quad + N_{yx}\gamma^{\circ}_{yx} + M_{yx}\delta_{yx}\Big\} \\
& + \big\{Q_{xz}\gamma^{\circ}_{xz}\big\} + \big\{Q_{yz}\gamma^{\circ}_{yz}\big\} \\
& - \frac{1}{2E}\Big\{\frac{N_{xx}^2}{h^2}\Big[h + \frac{h^3}{12}\Big(\frac{1}{r_2^2} - \frac{1}{r_1r_2}\Big)\Big] + \frac{24N_{xx}M_{xx}}{h^4}\Big[\frac{h^3}{12}\Big(\frac{1}{r_1} - \frac{1}{r_2}\Big)\Big] \\
& \quad + \frac{144M_{xx}^2}{h^6}\Big[\frac{h^3}{12} + \frac{h^5}{5\cdot 16}\Big(\frac{1}{r_2^2} - \frac{1}{r_1r_2}\Big)\Big]\Big\} \\
& - \frac{1}{2E}\Big\{\frac{N_{yy}^2}{h^2}\Big[h + \frac{h^3}{12}\Big(\frac{1}{r_1^2} - \frac{1}{r_1r_2}\Big)\Big] + \frac{24N_{yy}M_{yy}}{h^4}\Big[\frac{h^3}{12}\Big(\frac{1}{r_2} - \frac{1}{r_1}\Big)\Big] \\
& \quad + \frac{144M_{yy}^2}{h^6}\Big[\frac{h^3}{12} + \frac{h^5}{5\cdot 16}\Big(\frac{1}{r_1^2} - \frac{1}{r_1r_2}\Big)\Big]\Big\} \\
& + \frac{\mu}{E}\Big[\frac{N_{xx}N_{yy}}{h} + \frac{12M_{xx}M_{yy}}{h^3}\Big] \\
& - \frac{(1+\mu)}{E}\Big\{\frac{N_{xy}^2}{h^2}\Big[h + \frac{h^3}{12}\Big(\frac{1}{r_2^2} - \frac{1}{r_1r_2}\Big)\Big] + \frac{24N_{xy}M_{xy}}{h^4}\Big[\frac{h^3}{12}\Big(\frac{1}{r_1} - \frac{1}{r_2}\Big)\Big] \\
& \quad + \frac{144M_{xy}^2}{h^6}\Big[\frac{h^3}{12} + \frac{h^5}{5\cdot 16}\Big(\frac{1}{r_2^2} - \frac{1}{r_1r_2}\Big)\Big]\Big\} \\
& - \frac{(1+\mu)}{E}\Big\{\Big(\frac{3}{2}\frac{Q_{xz}}{h}\Big)^2\Big[\frac{8}{15}h + \Big(\frac{1}{r_2^2} - \frac{1}{r_1r_2}\Big)\frac{2}{105}h^3\Big]\Big\} \\
& - \frac{(1+\mu)}{E}\Big\{\Big(\frac{3}{2}\frac{Q_{yz}}{h}\Big)^2\Big[\frac{8}{15}h + \frac{2}{105}h^3\Big(\frac{1}{r_1^2} - \frac{1}{r_1r_2}\Big)\Big]\Big\} \\
& - \frac{\rho}{2}\Big\{u_t^2\Big[h + \frac{h^3}{12r_1r_2}\Big] + 2u_t\phi_t\Big[\frac{h^3}{12}\Big(\frac{1}{r_1} + \frac{1}{r_2}\Big)\Big] + \phi_t^2\Big[\frac{h^3}{12} + \frac{h^5}{5\cdot 16}\frac{1}{r_1r_2}\Big]\Big\} \\
& - \frac{\rho}{2}\Big\{v_t^2\Big[h + \frac{h^3}{12r_1r_2}\Big] + 2v_t\psi_t\Big[\frac{h^3}{12}\Big(\frac{1}{r_1} + \frac{1}{r_2}\Big)\Big] + \psi_t^2\Big[\frac{h^3}{12} + \frac{h^5}{5\cdot 16}\frac{1}{r_1r_2}\Big]\Big\} \\
& - \frac{\rho}{2}\Big\{w_{t(x,y)}^2\Big[h + \frac{h^3}{12r_1r_2}\Big]\Big\}\Bigg\} AB\,dx\,dy \\
& - \iint \Big[\tau_{xz}(u + z\phi + \tau_{yz}(v + z\psi) + \tau_{zz}w\Big]\Big(1 + \frac{z}{r_1}\Big)\Big(1 + \frac{z}{r_2}\Big)\Big|_{-\frac{h}{2}}^{+\frac{h}{2}} AB\,dx\,dy \\
& - \int_{t_1}^{t_2}\Big\{\oint\Big[\int_{-\frac{h}{2}}^{\frac{h}{2}}(\sigma_n u_n + \tau_{nt}u_t + \tau_{nz}w)\Big(1 + \frac{z}{r_t}\Big)dz\Big]A_t\,ds\Big\}dt = 0 \qquad \cdots(20)
\end{aligned}
$$

Before carrying out the variation of the preceding equation, the following approximations are introduced:

$$\left.\begin{aligned}
&\int_{-\frac{h}{2}}^{+\frac{h}{2}} \left(1+\frac{z}{r_2}\right)\left(1+\frac{z}{r_1}\right)^{-1} dz \cong h\left[1+\frac{h^2}{12r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right], \text{ and} \\
&\int_{-\frac{h}{2}}^{+\frac{h}{2}} \left(1+\frac{z}{r_2}\right)\left(1+\frac{z}{r_1}\right)^{-1} z^2 dz \cong \frac{h^3}{12}\left[1+\frac{3}{20}\frac{h^2}{r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right].
\end{aligned}\right] \cdots (21)$$

Taking the variational operations on the displacement and rotation parameters in the sequence u, v, w, ϕ, and ψ, and also on the force and moment components in the order, Nxx, Nyy, Nxy, Mxx, Myy, Mxy, Qxz, and Qyz, yields the resulting equation,

$$\begin{aligned}
\int_t \Big\{ \iint &-\Big[\frac{\partial}{\partial x}(BN_{xx}) + \frac{\partial}{\partial y}(AN_{yx}) + A_y N_{xy} - B_x N_{yy} \\
&+ \frac{ABQ_{xz}}{r_1} + \frac{ABP_x}{1} - \rho hAB\Big\{\Big(1+\frac{h^2}{12r_1r_2}\Big)u_{tt} \\
&+ \frac{h^2}{12}\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)\phi_{tt}\Big\}\Big]\delta u \\
&-\Big[\frac{\partial}{\partial x}(BN_{xy}) + \frac{\partial}{\partial y}(AN_{yy}) - A_y N_{xx} + B_x N_{yx} \\
&+ \frac{ABQ_{yz}}{r_2} + \frac{ABP_y}{1} - \rho hAB\Big\{\Big(1+\frac{h^2}{12r_1r_2}\Big)v_{tt} \\
&+ \frac{h^2}{12}\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big)\psi_{tt}\Big\}\Big]\delta v \\
&-\Big[\frac{\partial}{\partial x}\Big\{BQ_{xz} + \frac{B}{A}N_{xx}w_x + N_{xy}w_y\Big(1+\frac{h^2}{12r_2^2}\Big) - \frac{M_{xy}}{r_2}w_y\Big\} \\
&+ \frac{\partial}{\partial y}\Big\{AQ_{yz} + \frac{A}{B}N_{yy}w_y + N_{xy}w_x\Big(1+\frac{h^2}{12r_2^2}\Big) - \frac{M_{xy}}{r_2}w_x\Big\} \\
&+ ABP_z + AB\Big\{\frac{N_{xx}}{r_1} + \frac{N_{yy}}{r_2}\Big\} - \rho hAB\Big\{\Big(1+\frac{h^2}{12r_1r_2}\Big)w_{tt}\Big\}\Big]\delta w
\end{aligned}$$

$$-\Big[\frac{\partial}{\partial x}(B M_{xx}) + \frac{\partial}{\partial y}(A M_{yx}) + A_y M_{xy} - B_x M_{yy} - ABQ_{xz}$$

$$+ ABR_y - \frac{\rho h^3}{12} AB\Big\{\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big) u_{tt} + \Big(1+\frac{3h^2}{20 r_1 r_2}\Big)\phi_{tt}\Big\}\Big]\delta\phi$$

$$-\Big[\frac{\partial}{\partial x}(B M_{xy}) + \frac{\partial}{\partial y}(A M_{yy}) - A_y M_{xx} + B_x M_{yx} - ABQ_{yz}$$

$$+ ABR_x - \frac{\rho h^3}{12} AB\Big\{\Big(\frac{1}{r_1}+\frac{1}{r_2}\Big) v_{tt} + \Big(1+\frac{3h^2}{20 r_1 r_2}\Big)\psi_{tt}\Big\}\Big]\delta\psi$$

$$+\Big[\frac{AB}{Eh}\Big\{N_{xx}\Big(1+\frac{h^2}{12r_2}\Big(\frac{1}{r_2}-\frac{1}{r_1}\Big)\Big) - \mu N_{yy} + M_{xx}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big\} - \varepsilon_x^{\circ}\Big]\delta N_{xx}$$

$$+\Big[\frac{AB}{Eh}\Big\{N_{yy}\Big(1+\frac{h^2}{12r_2}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big) - \mu N_{xx} + M_{yy}\Big(\frac{1}{r_2}-\frac{1}{r_1}\Big)\Big\} - \varepsilon_y^{\circ}\Big]\delta N_{yy}$$

$$+\Big[\frac{2(1+\mu)}{Eh}\Big\{N_{xy}\Big(1+\frac{h^2}{12r_2}\Big(\frac{1}{r_2}-\frac{1}{r_1}\Big)\Big) + M_{xy}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big\} - \mathring{\gamma}_x$$

$$-\mathring{\gamma}_y\Big[1+\frac{h^2}{12r_2}\Big(\frac{1}{r_2}-\frac{1}{r_1}\Big)\Big] - \delta_y\frac{h^2}{12}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big) - \frac{W_x W_y}{AB}\Big(1+\frac{h^2}{12r_2^2}\Big)\Big]\delta N_{xy}$$

$$+\Big[\frac{12}{Eh^3}M_{xx}\Big\{1+\frac{3h^2}{20r_2}\Big(\frac{1}{r_2}-\frac{1}{r_1}\Big) - \mu M_{yy}\Big\} + \frac{N_{xx}}{Eh}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big) - k_x\Big]\delta M_{xx}$$

$$+\Big[\frac{12}{Eh^3}M_{yy}\Big\{1+\frac{3h^2}{20r_1}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big) - \mu M_{xx}\Big\} + \frac{N_{yy}}{Eh}\Big(\frac{1}{r_2}-\frac{1}{r_1}\Big) - k_y\Big]\delta M_{yy}$$

$$+\Big[\frac{24(1+\mu)}{Eh^3}M_{xy}\Big\{1+\frac{3h^2}{20r_2}\Big(\frac{1}{r_2}-\frac{1}{r_1}\Big)\Big\} + \frac{2(1+\mu)}{Eh}N_{xy}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)$$

$$-\delta_x - \delta_y\Big\{1+\frac{3h^2}{20r_2}\Big(\frac{1}{r_2}-\frac{1}{r_1}\Big)\Big\} - \mathring{\gamma}_y\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big) + \frac{W_x W_y}{ABr_2}\Big]\delta M_{xy}$$

$$+\Big[\frac{12(1+\mu)}{5Eh}Q_{xz}\Big\{1+\frac{h^2}{28r_2}\Big(\frac{1}{r_2}-\frac{1}{r_1}\Big)\Big\} - \mathring{\gamma}_{xz}\Big]\delta Q_{xz}$$

$$+\Big[\frac{12(1+\mu)}{5Eh}Q_{yz}\Big\{1+\frac{h^2}{28r_1}\Big(\frac{1}{r_1}-\frac{1}{r_2}\Big)\Big\} - \mathring{\gamma}_{yz}\Big]\delta Q_{yz}\Big\}AB\,dx\,dy\Big\}\,dt$$

$$\ldots(22)$$

2.6 EQUATIONS OF EQUILIBRIUM AND STRESS-STRAIN RELATIONS

The following equations of equilibrium are obtained from the variational operations on the independent variables u, v, w, ϕ and ψ:

$$
\left.
\begin{aligned}
&\frac{\partial}{\partial x}\left[B N_{xx}\right] + \frac{\partial}{\partial y}\left[A N_{yx}\right] + N_{xy}A_y - N_{yy}B_x + \frac{Q_{xz}}{r_1}AB + AB P_x \\
&\qquad = \rho h AB\left[u_{tt}\left(1+\frac{h^2}{12 r_1 r_2}\right) + \phi_{tt}\frac{h^2}{12}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right]; \\[2ex]
&\frac{\partial}{\partial x}\left[B N_{xy}\right] + \frac{\partial}{\partial y}\left[A N_{yy}\right] + N_{yx}B_x - N_{xx}A_y + \frac{AB Q_{yz}}{r_2} + AB P_y \\
&\qquad = \rho h AB\left[v_{tt}\left(1+\frac{h^2}{12 r_1 r_2}\right) + \psi_{tt}\frac{h^2}{12}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right]; \\[2ex]
&\frac{\partial}{\partial x}\left[B M_{xx}\right] + \frac{\partial}{\partial y}\left[A M_{yx}\right] + M_{xy}A_y - M_{yy}B_x - AB Q_{xz} + AB R_y \\
&\qquad = \frac{\rho h^3}{12}AB\left[u_{tt}\left(\frac{1}{r_1}+\frac{1}{r_2}\right) + \phi_{tt}\left(1+\frac{3h^2}{20 r_1 r_2}\right)\right]; \\[2ex]
&\frac{\partial}{\partial x}\left[B M_{xy}\right] + \frac{\partial}{\partial y}\left[A M_{yy}\right] + M_{yx}B_x - M_{xx}A_y - AB Q_{xz} + AB R_x \\
&\qquad = \frac{\rho h^3}{12}AB\left[v_{tt}\left(\frac{1}{r_1}+\frac{1}{r_2}\right) + \psi_{tt}\left(1+\frac{3h^2}{20 r_1 r_2}\right)\right]; \\[2ex]
&\frac{\partial}{\partial x}\left[B Q_{xz} + \frac{B}{A}N_{xx}W_x + N_{xy}W_y\left[1+\frac{h^2}{12 r_2^2}\right] - \frac{M_{xy}}{r_2}W_y\right] \\
&\quad + \frac{\partial}{\partial y}\left[A Q_{yz} + \frac{A}{B}N_{yy}W_y + N_{xy}W_x\left[1+\frac{h^2}{12 r_2^2}\right] - \frac{M_{xy}}{r_2}W_x\right] \\
&\quad - AB\left(\frac{N_{xx}}{r_1} + \frac{N_{yy}}{r_2}\right) + AB P_z = \rho h AB\left[W_{tt}\left(1+\frac{h^2}{12 r_1 r_2}\right)\right].
\end{aligned}
\right] \quad \ldots(23)
$$

Equating the co-efficients of variational terms δNxx, δNyy, δNxy, δMyy, δMxx, δMxy, δQxz and δQyz to zero, one obtains the following set of eight stress-strain relationships:

$$\frac{1}{Eh}\left[N_{xx}\left\{1+\frac{h^2}{12r_2}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\right\}-\mu N_{yy}+M_{xx}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]$$
$$=\frac{u_x}{A}+\frac{v}{AB}A_y+\frac{w}{r_1}+\frac{1}{2A^2}w_x^2=\varepsilon_{xx}^{\circ},$$

$$\frac{1}{Eh}\left[N_{yy}\left\{1+\frac{h^2}{12r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right\}-\mu N_{xx}+M_{yy}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\right]$$
$$=\frac{v_y}{B}+\frac{u}{AB}B_x+\frac{w}{r_2}+\frac{1}{2B^2}w_y^2=\varepsilon_{yy}^{\circ},$$

$$\frac{2(1+\mu)}{Eh}\left[N_{xy}\left\{1+\frac{h^2}{12r_2}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\right)+M_{xy}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]$$
$$=\left[\frac{v_x}{A}-\frac{u}{AB}A_y\right]+\left[\frac{v_y}{B}-\frac{v}{AB}B_x\right]\left[1+\frac{h^2}{12r_2}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\right]$$
$$+\left[\frac{\Phi_y}{B}-\frac{\Psi}{AB}B_x\right]\left[\frac{h^2}{12}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]+\frac{w_xw_y}{AB}\left[1+\frac{h^2}{12r_2^2}\right]$$
$$=\overset{\circ}{\gamma}_x+\overset{\circ}{\gamma}_y\left[1+\frac{h^2}{12r_2}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\right]+\delta_y\frac{h^2}{12}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$
$$+\frac{w_xw_y}{AB}\left[1+\frac{h^2}{12r_2^2}\right],$$

$$\frac{12}{Eh^3}\left[M_{xx}\left\{1+\frac{3h^2}{20r_2}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\right\}-\mu M_{yy}\right]+\frac{N_{xx}}{Eh}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$
$$=\frac{\Phi_x}{A}+\frac{\Psi}{AB}A_y=k_x,$$

$$\frac{12}{Eh^3}\left[M_{yy}\left\{1+\frac{3h^2}{20r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right\}-\mu M_{xx}\right]+\frac{N_{yy}}{Eh}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)$$
$$=\frac{\Psi_y}{B}+\frac{\Phi}{AB}B_x=k_y,$$

$$\frac{24(1+\mu)}{Eh^3}M_{xy}\left[1+\frac{3h^2}{20r_2}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\right]+\frac{2(1+\mu)}{Eh}N_{xy}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$
$$=\left[\frac{\Psi_x}{A}-\frac{\Phi}{AB}A_y\right]+\left[\frac{\Phi_y}{B}-\frac{\Psi}{AB}B_x\right]\left[1+\frac{3h^2}{20r_2}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\right]$$
$$+\left[\frac{v_y}{B}-\frac{v}{AB}B_x\right]\left[\frac{1}{r_1}-\frac{1}{r_2}\right]-\frac{w_xw_y}{ABr_2}$$
$$=\delta_x+\delta_y\left[1+\frac{3h^2}{20r_2}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\right]+\overset{\circ}{\gamma}_y\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$
$$-\frac{w_xw_y}{ABr_2}, \quad \cdots(24)$$

$$\frac{12(1+\mu)}{5Eh} Q_{xz}\left[1+\frac{h^2}{28r_2}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\right] = \frac{W_x}{A} + \Phi - \frac{u}{r_1} = \overset{\circ}{\gamma}_{xz},$$

$$\frac{12(1+\mu)}{5Eh} Q_{yz}\left[1+\frac{h^2}{28r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right] = \frac{W_y}{B} + \Psi - \frac{v}{r_2} = \overset{\circ}{\gamma}_{yz}. \quad \cdots(24) \text{ CONT.}$$

Additional algebraic operation on preceding set of equations (see appendix) yields a set of ten stress-strain relationships as follows:

$$N_{xx} = \frac{Eh}{(1-\mu^2)}\overset{\circ}{\varepsilon}_{xx} + \frac{\mu Eh}{(1-\mu^2)}\overset{\circ}{\varepsilon}_{yy} - D\left(\frac{1}{r_1}-\frac{1}{r_2}\right)k_x,$$

$$N_{yy} = \frac{\mu Eh}{(1-\mu^2)}\overset{\circ}{\varepsilon}_{xx} + \frac{Eh}{(1-\mu^2)}\overset{\circ}{\varepsilon}_{yy} - D\left(\frac{1}{r_2}-\frac{1}{r_1}\right)k_y,$$

$$N_{xy} = \frac{Eh}{2(1+\mu)}\left[\overset{\circ}{\gamma}_x + \overset{\circ}{\gamma}_y + \delta_y\frac{h^2}{12}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) + \frac{W_x W_y}{AB}\right] + \frac{Gh^3}{12}\left[\frac{1}{r_2}-\frac{1}{r_1}\right]\left[\delta_x + \delta_y + \overset{\circ}{\gamma}_y\left(\frac{1}{r_1}-\frac{1}{r_2}\right) - \frac{W_x W_y}{ABr_2}\right],$$

$$N_{yx} = \frac{Eh}{2(1+\mu)}\left[\overset{\circ}{\gamma}_x + \overset{\circ}{\gamma}_y + \delta_x\frac{h^2}{12}\left(\frac{1}{r_2}-\frac{1}{r_1}\right) + \frac{W_x W_y}{AB}\right] + \frac{Gh^3}{12}\left[\frac{1}{r_1}-\frac{1}{r_2}\right]\left[\delta_x + \delta_y + \overset{\circ}{\gamma}_x\left(\frac{1}{r_2}-\frac{1}{r_1}\right) - \frac{W_x W_y}{ABr_1}\right],$$

$$M_{xx} = -D\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\overset{\circ}{\varepsilon}_{xx} + Dk_x + D\mu k_y,$$

$$M_{yy} = -D\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\overset{\circ}{\varepsilon}_{yy} + D\mu k_x + Dk_y,$$

$$M_{xy} = \frac{D(1-\mu)}{2}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\left[\overset{\circ}{\gamma}_x + \overset{\circ}{\gamma}_y + \delta_y\frac{h^2}{12}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) + \frac{W_x W_y}{AB}\right] + \frac{D(1-\mu)}{2}\left[\delta_x + \delta_y + \overset{\circ}{\gamma}_y\left(\frac{1}{r_1}-\frac{1}{r_2}\right) - \frac{W_x W_y}{ABr_2}\right],$$

$$M_{yx} = \frac{D(1-\mu)}{2}\left[\frac{1}{r_1}-\frac{1}{r_2}\right]\left[\overset{\circ}{\gamma}_x + \overset{\circ}{\gamma}_y + \delta_x\frac{h^2}{12}\left(\frac{1}{r_2}-\frac{1}{r_1}\right) + \frac{W_x W_y}{AB}\right] + \frac{D(1-\mu)}{2}\left[\delta_x + \delta_y + \overset{\circ}{\gamma}_x\left(\frac{1}{r_2}-\frac{1}{r_1}\right) - \frac{W_x W_y}{ABr_1}\right],$$

$$Q_{xz} = \frac{5}{6}Gh\overset{\circ}{\gamma}_{xz}, \text{ and}$$

$$Q_{yz} = \frac{5}{6}Gh\overset{\circ}{\gamma}_{yz}. \quad \cdots(25)$$

The $(4)^{th}$ and $(8)^{th}$ equations of the above set of ten equations are defined by noting equation (4).

The associated forced and natural boundary conditions are determined from the following terms which result from the variational operation:

$$\int_{t_1}^{t_2}\int_y \left| B N_{xx}\,\delta u \right|_{x_1}^{x_2} dy\,dt\,; \quad \int_t\int_x \left| A N_{yx}\,\delta u \right|_{y_1}^{y_2} dx\,dt\,;$$

$$-\int_x\int_y \left| \rho h AB\left\{ u_t\left[1+\frac{h^2}{12r_1r_2}\right] + \Phi_t\left[\frac{h^2}{12}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right]\right\}\delta u \right|_{t_1}^{t_2} dx\,dy\,;$$

$$\int_t\int_x \left| A N_{yy}\,\delta v \right|_{y_1}^{y_2} dx\,dt\,; \quad \int_t\int_y \left| B N_{xy}\,\delta v \right|_{x_1}^{x_2} dy\,dt\,;$$

$$-\int_x\int_y \left| \rho h AB\left\{ v_t\left[1+\frac{h^2}{12r_1r_2}\right] + \Psi_t\left[\frac{h^2}{12}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right]\right\}\delta v \right|_{t_1}^{t_2} dx\,dy\,;$$

$$\int_t\int_y \left| \left\{\left[\frac{B}{A}N_{xx}w_x + BQ_{xz}\right] + \left[\frac{N_{xy}}{h}\left(h+\frac{h^3}{12r_2^2}\right)w_y - \frac{M_{xy}}{r_2}w_y\right]\right\}\delta w \right|_{x_1}^{x_2} dy\,dt\,;$$

$$\int_t\int_x \left| \left\{\left[\frac{A}{B}N_{yy}w_y + AQ_{yz}\right] + \left[\frac{N_{xy}}{h}\left(h+\frac{h^3}{12r_2^2}\right)w_x - \frac{M_{xy}}{r_2}w_x\right]\right\}\delta w \right|_{y_1}^{y_2} dx\,dt\,;$$

$$-\int_x\int_y \left| \left\{ABh\left[1+\frac{h^2}{12r_1r_2}\right]w_t\,\delta w \right|_{t_1}^{t_2} dx\,dy\,; \quad \int_t\int_y \left| B M_{xx}\,\delta\Phi \right|_{x_1}^{x_2} dy\,dt\,;$$

$$\int_t\int_y \left| B M_{xy}\,\delta\Psi \right|_{x_1}^{x_2} dy\,dt\,; \quad \int_t\int_x \left| A M_{yx}\,\delta\Phi \right|_{y_1}^{y_2} dx\,dt\,;$$

$$\int_t\int_x \left| A M_{yy}\,\delta\Psi \right|_{y_1}^{y_2} dx\,dt\,;$$

$$-\int_x\int_y \left\{ \left| AB\rho h u_t\left[\frac{h^2}{12}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right] + AB\rho\Phi_t\left[\frac{h^3}{12}+\frac{h^5}{80r_1r_2}\right]\delta\Phi\right\} \right|_{t_1}^{t_2} dx\,dy\,;$$

$$-\int_x\int_y \left| \left\{ AB\rho h v_t\left[\frac{h^2}{12}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right] + AB\rho\Psi_t\left[\frac{h^3}{12}+\frac{h^5}{80r_1r_2}\right]\delta\Psi\right\} \right|_{t_1}^{t_2} dx\,dy\,;$$

$$\ldots(26)$$

CHAPTER III

A SAMPLE EXAMPLE

Using the theory developed in the previous section, the equations of motion, the stress-strain relationships, and the natural and forced boundary conditions are formulated for the special case of a thin circular plate subjected to a constant in-plane stability force Nrr as shown in Fig. (3.1).

The internal forces and moments acting on an arbitrary element of the plate are shown in Figs. (3.2a, b, ~~c~~).

Using cylindrical polar co-ordinates, equation (1) which describes distance between two points in the plate reduces to the form:

$$ds^2 = A^2 dr^2 + B^2 d\theta^2 \quad \cdots (27)$$

~~where~~,

$$\left.\begin{aligned} A &= 1, \quad \text{and} \\ B &= r. \end{aligned}\right] \quad \cdots (28)$$

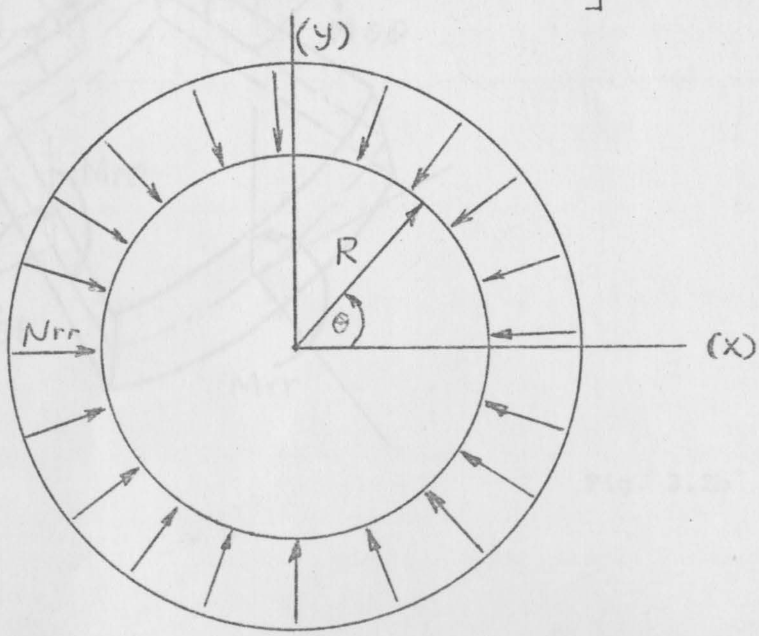


Fig. 3.1

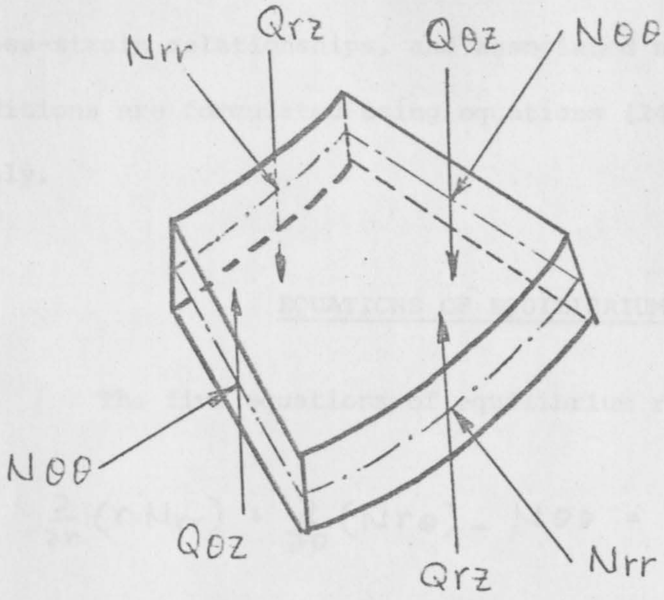


Fig. 3.2a

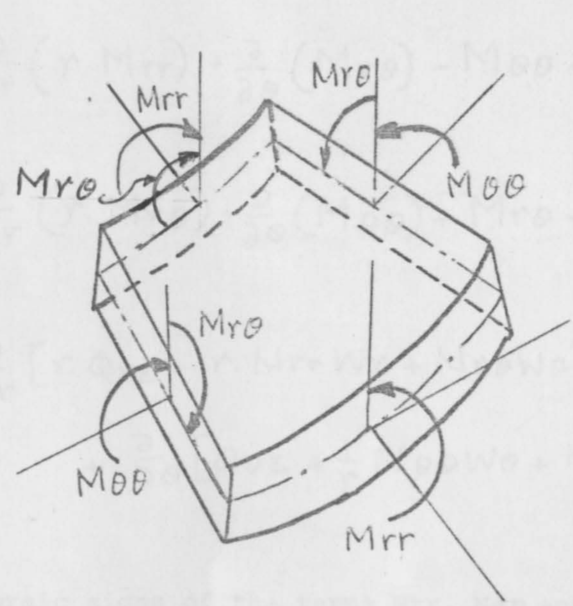


Fig. 3.2b

Also, since the plate has no curvature, the following conditions hold:

$$\frac{1}{r_1} = \frac{1}{r_2} = 0 .$$

Using the above information, the five equations of equilibrium, ten stress-strain relationships, and associated natural and free boundary conditions are formulated using equations (24), (25) and (26) respectively.

3.1 EQUATIONS OF EQUILIBRIUM

The five equations of equilibrium reduce to the form:

$$\left.\begin{aligned}
&\frac{\partial}{\partial r}(r.N_{rr}) + \frac{\partial}{\partial \theta}(N_{r\theta}) - N_{\theta\theta} = \rho h r \frac{\partial}{\partial t}(U_t), \\
&\frac{\partial}{\partial r}(r.N_{r\theta}) + \frac{\partial}{\partial \theta}(N_{\theta\theta}) + N_{r\theta} = \rho h r \frac{\partial}{\partial t}(V_t), \\
&\frac{\partial}{\partial r}(r.M_{rr}) + \frac{\partial}{\partial \theta}(M_{r\theta}) - M_{\theta\theta} + r.Q_{rz} = \frac{\rho h^3}{12} r \frac{\partial}{\partial t}(\phi_t), \\
&\frac{\partial}{\partial r}(r.M_{r\theta}) + \frac{\partial}{\partial \theta}(M_{\theta\theta}) + M_{r\theta} - r.Q_{\theta z} = \frac{\rho h^3}{12} r \frac{\partial}{\partial t}(\psi_t), \\
&\frac{\partial}{\partial r}\left[r.Q_{rz} + r.N_{rr}W_r + N_{r\theta}W_\theta\right] \\
&\qquad + \frac{\partial}{\partial \theta}\left[Q_{\theta z} + \frac{1}{r}N_{\theta\theta}W_\theta + N_{r\theta}W_r\right] = \rho h AB \frac{\partial}{\partial t}(W_t).
\end{aligned}\right] \quad \dots (29)$$

The algebraic signs of the terms Nrr, Nθθ and Nrθ must be reversed so that the effect of the stability forces is taken into account.

3.2 Stress-Strain Relationships

The stress-strain relationships given by equations (25) reduce to the cylindrical polar co-ordinates form defined as:

$$\left.\begin{aligned}
&\frac{1}{Eh}\left[N_{rr} - \mu N_{\theta\theta}\right] = \overset{\circ}{\varepsilon}_{rr}\,, \\
&\frac{1}{Eh}\left[N_{\theta\theta} - \mu N_{yy}\right] = \overset{\circ}{\varepsilon}_{\theta\theta}\,, \\
&\frac{2(1+\mu)}{Eh}\left[N_{r\theta}\right] = \overset{\circ}{\gamma}_{r} + \overset{\circ}{\gamma}_{\theta}\left[\frac{W_r W_\theta}{r}\right], \\
&\frac{12}{Eh^3}\left[M_{rr} - \mu M_{\theta\theta}\right] = k_r\,, \\
&\frac{12}{Eh^3}\left[M_{\theta\theta} - \mu M_{rr}\right] = k_\theta\,, \\
&\frac{24(1+\mu)}{Eh^3} M_{r\theta} = \delta_r + \delta_\theta\,, \\
&\frac{12(1+\mu)}{5Eh} Q_{rz} = \overset{\circ}{\gamma}_{rz}\,, \text{ and} \\
&\frac{12(1+\mu)}{5Eh} Q_{\theta z} = \overset{\circ}{\gamma}_{\theta z}\,,
\end{aligned}\right] \quad \cdots(30)$$

where,

$$\left.\begin{aligned}
&\overset{\circ}{\varepsilon}_{rr} = U_r + \frac{W_r^2}{2}\,, && \overset{\circ}{\varepsilon}_{\theta\theta} = \frac{V_\theta}{r} + \frac{U}{r} + \frac{W_\theta^2}{2r^2}\,, \\
&\overset{\circ}{\gamma}_{r} = V_r\,, && \overset{\circ}{\gamma}_{\theta} = \frac{1}{r}\left[U_\theta - V\right], \\
&k_r = \Phi_r\,, && k_\theta = \frac{1}{r}\left[\Psi_\theta + \Phi\right], \\
&\delta_r = \Psi_r\,, && \delta_\theta = \frac{1}{r}\left[\Phi_\theta - \Psi\right], \\
&\overset{\circ}{\gamma}_{rz} = W_r + \Phi\,, && \overset{\circ}{\gamma}_{\theta z} = \frac{W_\theta}{r} + \Psi\,.
\end{aligned}\right] \quad \cdots(31)$$

Simultaneous solution of equations (30) yield the following definitions for the in-plane forces and shear forces, bending and twisting moments, and transverse shear forces in the following form respectively:

$$\left.\begin{aligned}
N_{rr} &= \frac{Eh}{(1-\mu^2)}\left[u_r + \frac{w_r^2}{2} + \mu\left(\frac{v_\theta}{r} + \frac{u}{r} + \frac{w_\theta^2}{2r^2}\right)\right],\\
N_{\theta\theta} &= \frac{Eh}{(1-\mu^2)}\left[\mu\left(u_r + \frac{w_r^2}{2}\right) + \frac{v_\theta}{r} + \frac{u}{r} + \frac{w_\theta^2}{2r^2}\right],\\
N_{\theta r} = N_{r\theta} &= \frac{Eh}{2(1+\mu)}\left[v_r + \frac{u_\theta}{r} - \frac{v}{r} + \frac{w_r w_\theta}{r}\right],\\
M_{rr} &= \frac{Eh^3}{12(1-\mu^2)}\left[\phi_r + \mu\left(\frac{\psi_\theta}{r} + \frac{\phi}{r}\right)\right],\\
M_{\theta\theta} &= \frac{Eh^3}{12(1-\mu^2)}\left[\mu\phi_r + \frac{\psi_\theta}{r} + \frac{\phi}{r}\right],\\
M_{\theta r} = M_{r\theta} &= \frac{Eh^3}{24(1+\mu)}\left[\psi_r + \frac{\phi_\theta}{r} - \frac{\psi}{r}\right],\\
Q_{rz} &= \frac{5}{12}\frac{Eh}{(1+\mu)}\left[w_r + \phi\right],\\
Q_{\theta z} &= \frac{5}{12}\frac{Eh}{(1+\mu)}\left[\frac{w_\theta}{r} + \psi\right],\\
N_{r\theta} &= N_{\theta r}, \text{ and}\\
M_{r\theta} &= M_{\theta r}.
\end{aligned}\right] \quad \cdots(32)$$

3.3 ASSOCIATED BOUNDARY CONDITIONS

The natural and forced boundary conditions obtained from the reduction of equations (26) take the following form for a circular plate, on boundary r = R.

$$\left.\begin{array}{ll} \text{Either} & \text{or} \\ N_{rr}=0, & \delta u=0, \\ N_{r\theta}=0, & \delta v=0, \\ N_{rr}W_r+N_{r\theta}\frac{W_\theta}{R}+Q_{rz}=0, & \delta w=0, \\ M_{rr}=0, & \delta\phi=0, \\ M_{r\theta}=0, & \delta\psi=0. \end{array}\right] \quad \cdots(33)$$

Also at r = o, the displacements u, v, and w and the rotations ϕ and ψ must be finite.

It is expedient at this point to reduce the 5 equations of equilibrium to a combined set of three equations. This operation is performed by combining the first, second, and the fifth equations of equilibrium given by equations (29) into a single equation. This resulting equation is written,

$$\left.\begin{array}{l} \left[\rho h r(u_{tt})+N_{\theta\theta}\right]W_r+\left[\rho h r(v_{tt})-N_{r\theta}\right]\frac{W_\theta}{r} \\ +\left[r\cdot N_{rr}\right]\frac{\partial}{\partial r}(W_r)+\left[N_{r\theta}\right]\frac{\partial}{\partial\theta}(W_r)+\left[r\cdot N_{r\theta}\right]\frac{\partial}{\partial r}\left(\frac{W_\theta}{r}\right) \\ +\left[N_{\theta\theta}\right]\frac{\partial}{\partial\theta}\left(\frac{W_\theta}{r}\right)+\frac{\partial}{\partial r}\left[r\cdot Q_{rz}\right]+\frac{\partial}{\partial\theta}\left[Q_{\theta z}\right] \\ \quad = \rho h r\, w_{tt}-r\cdot P_z \quad \ldots \end{array}\right] \quad \cdots(34)$$

Applying the proper algebraic signs to the function Nrr and $N_{\theta\theta}$ and taking the special case when $N_{r\theta} = 0$, the three equations of equilibrium reduce to the form:

$$
\left.
\begin{aligned}
&\left[\rho h r(u_{tt}) - N_{\theta\theta}\right] W_r + \left[\rho h r(v_{tt})\right] \frac{W_\theta}{r} - \left[r \cdot N_{rr}\right] \frac{\partial}{\partial r}(W_r) \\
&\quad - \left[N_{\theta\theta}\right] \frac{\partial}{\partial \theta}\left(\frac{W_\theta}{r}\right) + \frac{\partial}{\partial r}\left[r \cdot Q_{rz}\right] + \frac{\partial}{\partial \theta}\left[Q_{\theta z}\right] = \rho h r W_{tt} - r P_z, \\
&\frac{\partial}{\partial r}\left[r \cdot M_{rr}\right] + \frac{\partial}{\partial \theta}\left(M_{r\theta}\right) - M_{\theta\theta} - r Q_{rz} = \frac{\rho h^3}{12} r \cdot \phi_{tt}, \\
&\frac{\partial}{\partial r}\left[r \cdot M_{r\theta}\right] + \frac{\partial}{\partial \theta}\left(M_{\theta\theta}\right) + M_{r\theta} - r \cdot Q_{\theta z} = \frac{\rho h^3 r}{12} \cdot \psi_{tt}.
\end{aligned}
\right] \quad \ldots(35)
$$

Substituting the stress-strain relationships given by equation (32) into the above equations yield

$$
\left.
\begin{aligned}
&\frac{D}{2}\left[(1-\mu)\left\{\nabla^2 \phi - \left(\frac{\phi}{r^2} + \frac{2\psi_\theta}{r^2}\right)\right\} + (1+\mu)\frac{\partial}{\partial r}\Phi\right] \\
&\qquad - k^2 G h\left(\phi + W_r\right) = \frac{\rho h^3}{12} \phi_{tt}, \\
&\frac{D}{2}\left[(1-\mu)\left\{\nabla^2 \psi - \frac{\psi}{r^2} + \frac{2\phi_\theta}{r^2}\right\} + (1+\mu)\frac{1}{r}\frac{\partial}{\partial \theta}\Phi\right] \\
&\qquad - k^2 G h\left(\psi + \frac{W_\theta}{r}\right) = \frac{\rho h^3}{12} \psi_{tt}, \text{ and} \\
&k^2 G h\left[\nabla^2 W + \Phi\right] + P_z + \left[N_{rr} \cdot W_{rr} + N_{\theta\theta}\left(\frac{W_r}{r} + \frac{W_{\theta\theta}}{r^2}\right)\right] \\
&\qquad = \rho h W_{tt},
\end{aligned}
\right] \quad \ldots(36)
$$

where,

$$\left.\begin{aligned} \Phi &= \left[\Phi_r + \frac{\Phi}{r} + \frac{\Psi_\theta}{r} \right], \text{ and} \\ \nabla^2 \Phi &= \Phi_{rr} + \frac{\Phi_r}{r} + \frac{\Phi_{\theta\theta}}{r^2}. \end{aligned}\right] \quad \cdots (37)$$

Differentiating the first and second equation of equations (36), performing proper operations, and re-arranging terms, the following equation is obtained:

$$\left[D\nabla^2 - k^2 G h - \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2} \right] \Phi = k^2 G h \nabla^2 W \quad \cdots (38)$$

Combining equation (38) with third equation of (36) and eliminating the parameter Φ yields the equation,

$$\left.\begin{aligned} &\left[\nabla^2 - \frac{\rho h}{k^2 G h} \frac{\partial^2}{\partial t^2} \right] \left[D\nabla^2 - \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2} \right] W + \frac{\rho h \, \partial^2 W}{\partial t^2} \\ &= \left[1 - \frac{D\nabla^2}{k^2 G h} + \frac{\rho h^3}{12 k^2 G h} \frac{\partial^2}{\partial t^2} \right]. \\ &\qquad \left[P_z + \left\{ N_{rr} W_{rr} + N_{\theta\theta} \left(\frac{W_r}{r} + \frac{W_{\theta\theta}}{r^2} \right) \right\} \right]. \end{aligned}\right] \quad \cdots (39)$$

The solution of this 4th order partial equation defines the transverse deflection of the middle surface of circular plate. This resulting equation compares to that given by R. D. Mindlin in (6) if the stability forces are neglected.

SUMMARY

Using E. Reissner's variational theorem, the equations of motions, stress-strain relationships, associated and natural boundary conditions for thin elastic shells are derived for the special case of orthogonal curvilinear co-ordinates. The resulting equations are applicable to cylindrical and spherical shells as well as circular and rectangular plates.

If the theory is reduced to the special case of circular plate and the resulting equations are combined to yield the differential equation for the displacement of the middle surface of the plate in operative form, the results coincide with existing classical theory of thin elastic plates.

If the non-linear terms are dropped from the strain displacement equations, the resulting equations derived in this thesis correspond to the same equations given by Naghdi[4] if the effect of transverse normal stress is neglected from his work.

The theory derived in this paper corresponds directly to that given by Archer.[1] However, in addition, the natural and forced boundary conditions for the general case of orthogonal curvilinear shell theory is derived. In addition, certain items of the resulting theory in both the equations of equilibrium and stress-strain relationship differ significantly to those given by Archer. These conflicting terms are in the minority.

CONCLUSION

The use of E. Reissner's variational theorem is proven extremely efficient in deriving equations of equilibrium, stress-strain relationships, natural and forced boundary conditions for the case of orthogonal curvilinear shell theory. The efficiency of this method is based on the fact that the complete set of resulting equations is obtained without the use of free body diagrams which in the past have proven to be extremely misleading, especially in the case of non-linear type problems, since this method requires one's experience and judgment.

The theory is developed in orthogonal curvilinear coordinates and includes the inertia terms so that the results may be applicable to spherical and cylindrical shells as well as circular and rectangular plates. Since the effect of transverse shear stress is included, the equations are well developed for use in wave propagation theory.

The equations are reduced to the special case of circular plate theory as a sample example. The critical buckling load as well as the dynamic response of the plate due to arbitrary dynamic loading may be obtained.

APPENDIX A

The eight equations of stress-strain relationships obtained in (24) are written in matrix form as follows:

$$\begin{bmatrix} A_{11} & A_{12} & 0 & A_{14} & 0 & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & -A_{14} & 0 & 0 & 0 \\ 0 & 0 & A_{33} & 0 & 0 & A_{36} & 0 & 0 \\ A_{14} & 0 & 0 & A_{44} & A_{45} & 0 & 0 & 0 \\ 0 & -A_{14} & 0 & A_{45} & A_{55} & 0 & 0 & 0 \\ 0 & 0 & A_{36} & 0 & 0 & A_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{77} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{88} \end{bmatrix} \begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \\ Q_{xz} \\ Q_{yz} \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{bmatrix} \quad \cdots (A_1)$$

where,

$$A_{11} = \frac{1}{Eh}\left[1 + \frac{h^2}{12r_2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)\right],$$

$$A_{12} = -\frac{\mu}{Eh},$$

$$A_{14} = \frac{1}{Eh}\left(\frac{1}{r_1} - \frac{1}{r_2}\right),$$

$$A_{22} = \frac{1}{Eh}\left[1 + \frac{h^2}{12r_1}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\right],$$

$$A_{25} = -A_{14} = \frac{1}{Eh}\left(\frac{1}{r_2} - \frac{1}{r_1}\right), \quad \cdot (A_2)$$

$$A_{33} = \frac{2(1+\mu)}{Eh}\left[1 + \frac{h^2}{12r_2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)\right],$$

$$A_{36} = \frac{2(1+\mu)}{Eh}\left(\frac{1}{r_1} - \frac{1}{r_2}\right),$$

$$A_{44} = \frac{12}{Eh^3}\left[1 + \frac{3h^2}{20r_2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)\right],$$

$$A_{45} = \frac{-12\mu}{Eh^3},$$

$$A_{55} = \frac{12}{Eh^3}\left[1 + \frac{3h^2}{20r_1}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\right],$$

$$A_{66} = \frac{24(1+\mu)}{Eh^3}\left[1 + \frac{3h^2}{20r_2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)\right],$$

$$A_{77} = \frac{12(1+\mu)}{5Eh}\left[1 + \frac{h^2}{28r_2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)\right], \text{ and}$$

$$A_{88} = \frac{12}{5}\frac{(1+\mu)}{Eh}\left[1 + \frac{h^2}{28r_1}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\right],$$

...(A2) CONTD.

and also,

$$A = \varepsilon^{o}_{xx},$$
$$B = \varepsilon^{o}_{yy},$$
$$C = \gamma^{o}_{x} + \gamma^{o}_{y}\left[1 + \frac{h^2}{12r_2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)\right] + \delta_y\left[\frac{h^2}{12}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\right] + \frac{W_x W_y}{AB}\left(1 + \frac{h^2}{12r_2^2}\right),$$
$$D = k_x,$$
$$E = k_y,$$
$$F = \delta_x + \delta_y\left[1 + \frac{3h^2}{20r_2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)\right] + \gamma^{o}_{y}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - \frac{W_x W_y}{AB r_2},$$
$$G = \gamma^{o}_{xz}, \text{ and}$$
$$H = \gamma^{o}_{yz}.$$

...(A3)

Inversion of the A matrix yields the following inverse solutions for the stress-strain relationships:

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \\ Q_{xz} \\ Q_{yz} \end{bmatrix} = \begin{bmatrix} \overline{A_{11}} & \overline{A_{12}} & 0 & \overline{A_{14}} & \overline{A_{15}} & 0 & 0 & 0 \\ \overline{A_{12}} & \overline{A_{22}} & 0 & \overline{A_{24}} & \overline{A_{25}} & 0 & 0 & 0 \\ 0 & 0 & \overline{A_{33}} & 0 & 0 & \overline{A_{36}} & 0 & 0 \\ \overline{A_{14}} & \overline{A_{24}} & 0 & \overline{A_{44}} & \overline{A_{45}} & 0 & 0 & 0 \\ \overline{A_{15}} & \overline{A_{25}} & 0 & \overline{A_{45}} & \overline{A_{55}} & 0 & 0 & 0 \\ 0 & 0 & \overline{A_{36}} & 0 & 0 & \overline{A_{66}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \overline{A_{77}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \overline{A_{88}} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{bmatrix} \quad \ldots(A4)$$

$$\overline{A_{11}} = \frac{12Eh\left\{12\left[1+\frac{h^2}{12}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\left(\frac{14}{5r_1}-\frac{9}{5r_2}\right)\right]-12\mu^2\left[1+\frac{h^2}{12r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]-h^2\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right\}}{(1-\mu^2)\left[144(1-\mu^2)+\frac{48h^2}{5}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right]},$$

$$\overline{A_{12}} = \frac{\mu E h}{(1-\mu^2)}$$

$$\overline{A_{14}} = \frac{-12Eh^3\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}{\left[144(1-\mu^2)+\frac{48h^2}{5}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right]},$$

$$\overline{A_{15}} = -\frac{12\mu Eh^3\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\left[h^2\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\left(\frac{3}{20r_2}+\frac{1}{12r_1}\right)\right]}{(1-\mu^2)\left[144(1-\mu^2)+\frac{48h^2}{5}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right]},$$

$$\overline{A_{22}} = \frac{Eh\left\{144(1-\mu^2)\left[1-\frac{h^2}{12r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)+\frac{48h^2}{5}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right]\right\}}{(1-\mu^2)\left[144(1-\mu^2)+\frac{48h^2}{5}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right]},$$

$\ldots(A5)$

$$\overline{A_{24}} = \frac{-12\mu E h^3 \left(\frac{1}{r_1}-\frac{1}{r_2}\right)\left[h^2\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\left(\frac{1}{12r_2}+\frac{3}{20r_1}\right)\right]}{(1-\mu^2)\left[144(1-\mu^2)+\frac{48h^2}{5}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right]},$$

$$\overline{A_{25}} = \frac{E h^3 \left(\frac{1}{r_1}-\frac{1}{r_2}\right)\left[12(1-\mu^2)-\frac{7}{5}\frac{h^2}{r^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)-\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2 h^2\right]}{(1-\mu^2)\left[144(1-\mu^2)+\frac{48h^2}{5}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right]},$$

$$\overline{A_{33}} = \frac{6Eh\left[1-\frac{3h^2}{20r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]}{(1+\mu)\left[12-\frac{14h^2}{5r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)-h^2\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right]},$$

$$\overline{A_{36}} = \frac{-Eh^3\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}{2(1+\mu)\left[12\left\{1-\frac{7h^2}{30r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right\}-h^2\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right]},$$

$$\overline{A_{44}} = \frac{Eh^3\left[12+12h^3\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\left(\frac{7}{30r_1}-\frac{1}{12r_2}\right)-h^2\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2-12\mu^2-\frac{36\mu^2h^2}{20r_1}\left[\frac{1}{r_1}-\frac{1}{r_2}\right]\right]}{(1-\mu^2)\left[144(1-\mu^2)+\frac{48h^2}{5}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right]}, \quad \ldots (A5)$$

CONT.

$$\overline{A_{45}} = \frac{12\mu E h^3}{144(1-\mu^2)+\frac{48h^2}{5}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2},$$

$$\overline{A_{55}} = \frac{12Eh^3\left[1-\frac{3h^2}{20r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]}{144(1-\mu^2)+\frac{48h^2}{5}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2},$$

$$\overline{A_{66}} = \frac{Eh^3\left[1-\frac{h^2}{12r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]}{2(1+\mu)\left[12-\frac{12h^2}{r_2}\cdot\frac{7}{30}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)-h^2\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right]},$$

$$\overline{A_{77}} = \frac{5Eh}{12(1+\mu)\left[1-\frac{h^2}{28r_2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right]}, \text{ and}$$

$$\overline{A_{88}} = \frac{5Eh}{12(1+\mu)\left[1+\frac{h^2}{28r_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2\right]},$$

and A, B, C, D, E, F, G and H are defined by equations (A3).

Neglecting the terms of order $\frac{h^2}{r^2}$ and above in comparison to unity in the preceding matrix form, the elements of the matrix reduce to:

$$\left.\begin{aligned}
\overline{B_{11}} &= \frac{Eh}{(1-\mu^2)}, \\
\overline{B_{22}} &= \frac{Eh}{(1-\mu^2)}, \\
\overline{B_{33}} &= \frac{Eh}{2(1+\mu)}, \\
\overline{B_{44}} &= \frac{Eh^3}{12(1-\mu^2)}, \\
\overline{B_{55}} &= \frac{Eh^3}{12(1-\mu^2)}, \\
\overline{B_{66}} &= \frac{Eh^3}{24(1+\mu)}, \\
\overline{B_{77}} &= \frac{5Eh}{12(1+\mu)}, \\
\overline{B_{88}} &= \frac{5Eh}{12(1+\mu)}, \\
\overline{B_{12}} &= \frac{\mu Eh}{(1-\mu^2)}, \\
\overline{B_{14}} &= -\frac{Eh^3}{12(1-\mu^2)}\left[\frac{1}{r_1}-\frac{1}{r_2}\right], \\
\overline{B_{15}} &= 0, \\
\overline{B_{24}} &= 0, \\
\overline{B_{25}} &= \frac{Eh^3}{12(1-\mu^2)}\left[\frac{1}{r_1}-\frac{1}{r_2}\right], \\
\overline{B_{36}} &= -\frac{Eh^3}{24(1+\mu)}\left[\frac{1}{r_1}-\frac{1}{r_2}\right], \text{ and} \\
\overline{B_{45}} &= \frac{\mu Eh^3}{12(1-\mu^2)}.
\end{aligned}\right] \quad \cdots (A6)$$

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \\ Q_{xz} \\ Q_{yz} \end{bmatrix} = \begin{bmatrix} \overline{B_{11}} & \overline{B_{12}} & 0 & \overline{B_{14}} & \overline{B_{15}} & 0 & 0 & 0 \\ \overline{B_{12}} & \overline{B_{22}} & 0 & \overline{B_{24}} & \overline{B_{25}} & 0 & 0 & 0 \\ 0 & 0 & \overline{B_{33}} & 0 & 0 & \overline{B_{36}} & 0 & 0 \\ \overline{B_{14}} & \overline{B_{24}} & 0 & \overline{B_{44}} & \overline{B_{45}} & 0 & 0 & 0 \\ \overline{B_{15}} & \overline{B_{25}} & 0 & \overline{B_{45}} & \overline{B_{55}} & 0 & 0 & 0 \\ 0 & 0 & \overline{B_{36}} & 0 & 0 & \overline{B_{66}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \overline{B_{77}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \overline{B_{88}} \end{bmatrix} \begin{bmatrix} \bar{A} \\ \bar{B} \\ \bar{C} \\ \bar{D} \\ \bar{E} \\ \bar{F} \\ \bar{G} \\ \bar{H} \end{bmatrix} \quad \cdots (A$$

where,

$$\left.\begin{aligned} \bar{A} &= \overset{\circ}{\varepsilon}_{xx}, \\ \bar{B} &= \overset{\circ}{\varepsilon}_{yy}, \\ \bar{C} &= \overset{\circ}{\gamma}_{x} + \overset{\circ}{\gamma}_{y} + \delta_{y} \frac{h^2}{12}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{W_x W_y}{AB}, \\ \bar{D} &= k_x, \\ \bar{E} &= k_y, \\ \bar{F} &= \delta_x + \delta_y + \overset{\circ}{\gamma}_{y}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - \frac{W_x W_y}{AB r_2}, \\ \bar{G} &= \overset{\circ}{\gamma}_{xz}, \text{ and} \\ \bar{H} &= \overset{\circ}{\gamma}_{yz}. \end{aligned}\right] \quad \cdots (A8)$$

and Exx, Eyy, $\overset{\circ}{\gamma}_x$, $\overset{\circ}{\gamma}_y$, δ_x, δ_y, k_x, k_y, $\overset{\circ}{\gamma}_{xz}$ and $\overset{\circ}{\gamma}_{yz}$ are defined on page (13), equations (14).

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